ECE569/AI539 Convex Optimization - Course Project Optimal Multiuser Transmit Beamforming: A Difficult Problem with a Simple Solution Structure

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Abstract

Overall, I read and almost understood the paper, *Optimal Multiuser Transmit Beamforming: A Difficult Problem with a Simple Solution Structure* [1]. In this report, the detailed problem statement and the solution methods of multiuser transmit beamforming are presented. Especially, I designed a new convex feasibility problem and run some corresponding experiments. In addition, I also duplicated some results from the original paper. These numerical results are analyzed and evaluated.

1 Introduction

1.1 Background

Transmit beamforming is a strong technique for transmitting the signal from multiple antenna arrays to multiple users [2]. Regarding the designing of beamforming vectors (describe the amplitudes and phases), the target is to have large inner products with the vectors describing the intended channels and small inner products with non-intended user channels.

While designing a beamforming vector that maximizes the signal power at the intended user is straightforward, it is difficult to derive a perfect balance between minimizing the interference and maximizing the signal power. In fact, the optimization of multiuser transmit beamforming is generally a non-deterministic polynomial-time (NP)-hard problem [3].

1.2 System Model

The paper [1] considers a downlink channel where a base station (BS) equipped with N antennas transmits the signal to K single-antenna users. The data signal to user k is described as $s_k \in \mathbb{C}$, while the vector $h_k \in \mathbb{C}^{N\times 1}$ denotes the corresponding channel. The K different data signals are separated spatially using the linear beamforming vectors $w_1, ..., w_K \in \mathbb{C}^{N\times 1}$. The normalized version (i.e., $w_k/\left\|w_k\right\|$) is called the beamforming direction. The received signal $r_k \in \mathbb{C}$ at user k is modeled as

$$r_k = h_k^H \left(\sum_{i=1}^{K} w_i s_i\right) + n_k, \tag{1}$$

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where n_k is additive receiver noise with zero mean and variance σ^2. Consequently, the signal-to-interference-and-noise ratio (SINR) at the user is as follows

$$\mathrm{SINR}_k = \frac{\left|h_k^H w_k\right|^2}{\sum_{i \neq k} \left|h_k^H w_i\right|^2 + \sigma^2} = \frac{\frac{1}{\sigma} \left|h_k^H w_k\right|^2}{\sum_{i \neq k} \frac{1}{\sigma^2} \left|h_k^H w_i\right|^2 + 1} \tag{2}$$

1.3 Problem (P1): Power Minimization with SINR Constraints

The transmit beamforming can be optimized to maximize some performance metrics, which is a function of the SINRs. Firstly, the power minimization problem (P1) can be expressed as

$$\begin{aligned} (P_1) \min_{w_1,\dots,w_K} \quad & \sum_{k=1}^{K} \|w_k\|^2 \\ \text{subject to} \quad & \mathrm{SINR}_k \geq \gamma_k, \end{aligned} \tag{3}$$

The parameters $\gamma_1, ..., \gamma_K$ are the SINRs that each user can achieve at the optimum of (P1) with minimized transmit power.

1.4 Problem (P2): General Transmit Beamforming Optimization

For this problem, authors want to maximize some arbitrary utility function $f\left(\mathrm{SINR}_1, ..., \mathrm{SINR}_K\right)$ ($\mathrm{SINR}_1 > \mathrm{SINR}_2 > ... > \mathrm{SINR}_K$), while the total transmit power is limited by P. The problem is represented as follows

$$\begin{aligned} (P_2) \max_{w_1,\dots,w_K} \quad & f\left(\mathrm{SINR}_1, ..., \mathrm{SINR}_K\right) \\ \text{subject to} \quad & \textstyle\sum_{k=1}^{K} \|w_k\|^2 \leq P, \end{aligned} \tag{4}$$

Generally, (P2) is NP-hard for many common utility functions. For instance, the sum rate as $f\left(\mathrm{SINR}_1, ..., \mathrm{SINR}_K\right) = \sum_{k=1}^{K} \log_2\left(1 + \mathrm{SINR}_k\right)$. However, the authors showed that the structure of the optimal solution to (P2) is easily obtained (in section 2.2).

1.5 Problem (P3): Multiuser Transmit Beamforming as Convex Feasibility Problem

In this section, I will propose a new transmit beamforming problem. The feasibility ratio of this problem will be considered in the next section. The problem (P1) is not convex due to the complex vectors in SINR constraints. To transform it into SOCP, the problem needs to be reformulate as below

$$\begin{aligned} \min_{w_1,\dots,w_K} \quad & \sum_{k=1}^{K} \|w_k\|_2^2 \\ \text{subject to} \quad & h_k^H w_k \geq \sqrt{\gamma_k \textstyle\sum_{i \neq k} \left\|h_k^H w_i\right\|_2^2 + \gamma_k \sigma^2} \\ & \Im(h_k^H w_k = 0) \end{aligned} \tag{5}$$

where $\Im(.)$ is the imaginary part.

In fact, the absolute values in the SINRs in (2) make w_k and $e^{j\theta_k} w_k$ completely equivalent for any common phase rotation $\theta_k \in \mathbb{R}$. Hence, we exploit this phase ambiguity to rotate the phase such that the inner product $h_k^H w_k$ is real-valued and positive. This implies, $\sqrt{\left|h_k^H w_k\right|^2} = h_k^H w_k \geq 0$. Therefore, we can check whether a given set of SINR constraints can be met under the given transmit sum-power constraint P to form a convex feasibility problem. This is stated mathematically as follows

$$
\begin{array}{rl}
(P3) & w_1, ..., w_K \\
\text{subject to} & h_k^H w_k \geq \sqrt{\gamma_k \sum_{i \neq k} \left\| h_k^H w_i \right\|_2^2 + \gamma_k \sigma^2} \\
& \Im(h_k^H w_k = 0) \\
& \sum_{k=1}^K \|w_k\|_2^2 \leq P
\end{array}
\tag{6}
$$

In the next section, I will use the convex optimization package, CVX in MATLAB, to implement this problem.

2 Proposed Method

2.1 Problem (P1) Solution

The first step is to reformulate (P1) as a convex problem. The cost function $\sum_{k=1}^K \|w_k\|^2$ is clearly a convex function. Let $\Re(.)$ denote the real part. The constraint $\text{SINR}_k \geq \gamma_k$ can be rewritten as

$$
\frac{1}{\gamma_k \sigma^2} \left| h_k^H w_k \right|^2 \geq \sum_{i \neq k} \frac{1}{\sigma^2} \left| h_k^H w_i \right|^2 + 1 \tag{7}
$$

Therefore,

$$
\frac{1}{\sqrt{\gamma_k \sigma^2}} \Re(h_k^H w_k) \geq \sqrt{\sum_{i \neq k} \frac{1}{\sigma^2} \left| h_k^H w_i \right|^2 + 1} \tag{8}
$$

The reformulated SINR constraint in (8) is a second-order cone constraint, which is a convex type of constraint [4]–[6]. Following the strong duality and KKT conditions for (P1), the Lagrangian function is expressed as

$$
\mathcal{L}\left(w_1, ..., w_K, \lambda_1, ..., \lambda_K\right) = \sum_{k=1}^K \|w_k\|^2 + \sum_{k=1}^K \lambda_k \left(\sum_{i \neq k} \frac{1}{\sigma^2} \left| h_k^H w_i \right|^2 + 1 - \frac{1}{\gamma_k \sigma^2} \left| h_k^H w_k \right|^2 \right), \tag{9}
$$

where $\lambda_k \geq 0$ is the Lagrange multiplier associated with the kth SINR constraint. The stationarity KKT conditions, which say that $\partial \mathcal{L} / \partial w_k = 0$ for $k = 1, ..., K$. This implies

$$
w_k + \sum_{i \neq k} \frac{\lambda_i}{\sigma^2} h_i h_i^H w_k - \frac{\lambda_k}{\gamma_k \sigma^2} h_k h_k^H w_k = 0 \tag{10}
$$

$$
\Leftrightarrow \left(\mathbf{I}_N + \sum_{i=1}^K \frac{\lambda_i}{\sigma^2} h_i h_i^H \right) w_k = \frac{\lambda_k}{\sigma^2} \left(1 + \frac{1}{\gamma_k} \right) h_k h_k^H w_k \tag{11}
$$

$$
\Leftrightarrow w_k = \left(\mathbf{I}_N + \sum_{i=1}^K \frac{\lambda_i}{\sigma} h_i h_i^H \right)^{-1} h_k \frac{\lambda_k}{\sigma^2} \left(1 + \frac{1}{\gamma_k} \right) h_k^H w_k, \tag{12}
$$

where $\mathbf{I}_N$ denotes the $N \times N$ identity matrix. Since $\frac{\lambda_k}{\sigma^2} \left(1 + \frac{1}{\gamma_k} \right) h_k^H w_k$ is a scalar, (12) shows that the optimal w_k must be parallel to $\left(\mathbf{I}_N + \sum_{i=1}^K \frac{\lambda_i}{\sigma^2} h_i h_i^H \right)^{-1} h_k$. Hence, the optimal beamforming vectors $w_1^*, ..., w_K^*$ as follows

$$
w_k^* = \sqrt{p_k} \frac{\left(\mathbf{I}_N + \sum_{i=1}^K \frac{\lambda_i}{\sigma^2} h_i h_i^H \right)^{-1} h_k}{\left\| \left(\mathbf{I}_N + \sum_{i=1}^K \frac{\lambda_i}{\sigma^2} h_i h_i^H \right)^{-1} h_k \right\|} \tag{13}
$$

where $\sqrt{p_k}$ is beamforming power. $\tilde{w_k}^* = \frac{\left(\mathbf{I}_N + \sum_{i=1}^K \frac{\lambda_i}{\sigma^2} h_i h_i^H\right)^{-1} h_k}{\left\| \left(\mathbf{I}_N + \sum_{i=1}^K \frac{\lambda_i}{\sigma^2} h_i h_i^H\right)^{-1} h_k \right\|}$ is unit-norm beamforming direction for user k. It is noted that the SINR constraints (8) hold with equality at the optimal solution. This implies $\frac{1}{\gamma_k} p_k \left| h_k^H \tilde{w}_k^* \right|^2 - \sum_{i \neq k} p_i \left| h_k^H \tilde{w}_i^* \right|^2 = \sigma^2$ for $k = 1, ..., K$. And, K linear equations and obtain the K powers are expressed as

$$\begin{bmatrix} p_1 \\ ... \\ p_K \end{bmatrix} = \mathbf{M}^{-1} \begin{bmatrix} \sigma^2 \\ ... \\ \sigma^2 \end{bmatrix}, \tag{14}$$

where

$$[\mathbf{M}]_{i,j} = \begin{cases} \frac{1}{\gamma_i \left| h_i^H \tilde{w}_i^* \right|^2} & i = j, \\ -\left| h_i^H \tilde{w}_j^* \right| & i \neq j, \end{cases} \tag{15}$$

and $[\mathbf{M}]_{i,j}$ denotes the (i, j)th element of the matrix $\mathbf{M} \in \mathbb{R}^{K \times K}$.

Consequently, the structure of optimal beamforming as a function of the Lagrange multipliers $\lambda_1, ..., \lambda_K$ can be obtained by combining (13) and (14). It is noted that the Lagrange multipliers can be computed by convex optimization [4] or from the fixed-point equations [5], [6] as follows

$$\lambda_k = \frac{\sigma^2}{\left(1 + \frac{1}{\gamma_k}\right) h_k^H \left(\mathbf{I}_N + \sum_{i=1}^K \frac{\lambda_i}{\sigma^2} h_i h_i^H\right)^{-1} h_k} \tag{16}$$

.

2.2 Problem (P2) Solution

Suppose that $\text{SINR}_1^*, ..., \text{SINR}_K^*$ are achieved by the optimal solution to (P2). Let set $\gamma_k = \text{SINR}_k^*$, for $k = 1, ..., K$, and solve (P1) for these particular γ-parameters. The answer is that the beamforming vectors that solve (P1) will also solve (P2) [7]. The difficulty of (P2) is that we need to find the optimal SINR values along with the beamforming vectors. Hence, the optimal beamforming for (P2) as

$$w_k^* = \sqrt{p_k} \frac{\left(\mathbf{I}_N + \sum_{i=1}^K \frac{\lambda_i}{\sigma^2} h_i h_i^H\right)^{-1} h_k}{\left\| \left(\mathbf{I}_N + \sum_{i=1}^K \frac{\lambda_i}{\sigma^2} h_i h_i^H\right)^{-1} h_k \right\|} \tag{17}$$

for $k = 1, ..., K$

Since the matrix inverse in (17) is the same for all users, the matrix with the optimal beamforming vectors is denoted as $\mathbf{W}^* = [w_1^* ... w_K^*] \in \mathbb{C}^{N \times K}$. It is noted that $\sum_{i=1}^K \frac{\lambda_i}{\sigma^2} h_i h_i^H = \frac{1}{\sigma^2} \mathbf{H} \Lambda \mathbf{H}^H$ where $\mathbf{H} = [h_1 ... h_K] \in \mathbb{C}^{N \times K}$ contains the channels and $\Lambda = \text{diag}(\lambda_1, ... \lambda_K)$ is a diagonal matrix. By gathering the power allocation in a matrix $\mathbf{P}$, the optimal beamforming matrix can be derived as

$$\mathbf{W}^* = \left(\mathbf{I}_N + \frac{1}{\sigma^2 \mathbf{H} \Lambda \mathbf{H}^H}\right)^{-1} \mathbf{H} \mathbf{P}^{1/2}, \tag{18}$$

where $\mathbf{P} = \text{diag}\left(\frac{p_1}{\left\| (\mathbf{I}_N + \frac{1}{\sigma^2} \mathbf{H} \Lambda \mathbf{H}^H)^{-1} h_1 \right\|^2}, ..., \frac{p_K}{\left\| (\mathbf{I}_N + \frac{1}{\sigma^2} \mathbf{H} \Lambda \mathbf{H}^H)^{-1} h_K \right\|^2}\right)$.

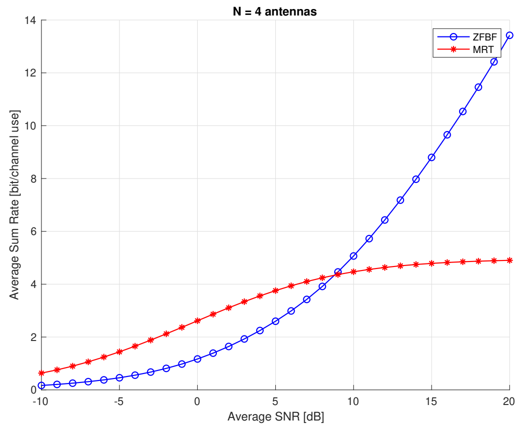

Figure 1: The average sum rate versus the average SNR, $N = 4$ antennas.

3 Beamforming Techniques

3.1 Maximum Ratio Transmission (MRT)

The optimal beamforming direction in (17) consists of two main parts: 1) the channel vector h_k between the BS and the intended user k; 2) the matrix $\left(\mathbf{I}_N + \sum_{i=1}^{K} \frac{\lambda_i}{\sigma^2} h_i h_i^H\right)^{-1}$. Beamforming in the same direction as the channel (i.e., $\tilde{w_k}^{\mathrm{MRT}} = \frac{h_k}{\|h_k\|}$) is known as *maximum ratio transmission (MRT)* [8]. This selection maximizes the recived signal power $p_k \left|h_k^H \tilde{w}_k\right|^2$ at the intended user as

$$\mathrm{argmax}_{\tilde{w}_k:\|\tilde{w}_k\|^2=1} \left|h_k^H \tilde{w}_k\right|^2 = \frac{h_k}{\|h_k\|} \tag{19}$$

This is the optimal beamforming direction for $K = 1$, but not when there are multiple users since the interuser interference is unaccounted for in MRT.

3.2 Zero-Forcing Beamforming (ZFBF)

At high SNRs, $\sigma^2 \to 0$, the system is interference-limited. To avoid singularity in the inverse when σ^2 is small, using the identity $(\mathbf{I} + \mathbf{AB})^{-1}\mathbf{A} = \mathbf{A}(\mathbf{I} + \mathbf{BA})^{-1}$ and rewriting (18) as $\mathbf{W}^* = \mathbf{H}(\sigma^2\mathbf{I}_K + \Lambda\mathbf{H}^H\mathbf{H})^{-1}\tilde{\mathbf{P}}^{1/2}$, where $\tilde{\mathbf{P}}$ is the corresponding rewritten power allocation matrix.

$$\tilde{\mathbf{P}} = \mathrm{diag}\left(\frac{p_1}{\left\|\left(\sigma^2\mathbf{I}_K + \Lambda\mathbf{H}^H\mathbf{H}\right)^{-1} h_1\right\|^2}, \ldots, \frac{p_K}{\left\|\left(\sigma^2\mathbf{I}_K + \Lambda\mathbf{H}^H\mathbf{H}\right)^{-1} h_K\right\|^2}\right) \tag{20}$$

It follows that

$$\mathbf{W}^*_{\sigma^2\to 0} = \mathbf{H}(0\mathbf{I}_k + \Lambda\mathbf{H}^H\mathbf{H})^{-1}\tilde{\mathbf{P}}_{\sigma^2\to 0} = \mathbf{H}(\mathbf{H}^H\mathbf{H})^{-1}\Lambda^{-1}\tilde{\mathbf{P}}_{\sigma^2\to 0}, \tag{21}$$

This solution is known as *zero-forcing beamforming (ZFBF)* [9], because it contains the pseudoinverse $\mathbf{H}(\mathbf{H}^H\mathbf{H})^{-1}$ of the channel matrix $\mathbf{H}^H$. Hence, $\mathbf{H}^H\mathbf{W}^*_{\sigma^2\to 0} = \Lambda^{-1}\tilde{\mathbf{P}}_{\sigma^2\to 0}$ is a diagonal matrix. Since the off-diagonal elements are of the form $h_i^H w_k^* = 0$ for $i \neq k$, this beamforming causes zero inter-user interference by projecting h_k onto the subspace that is orthogonal to the co-user channels.

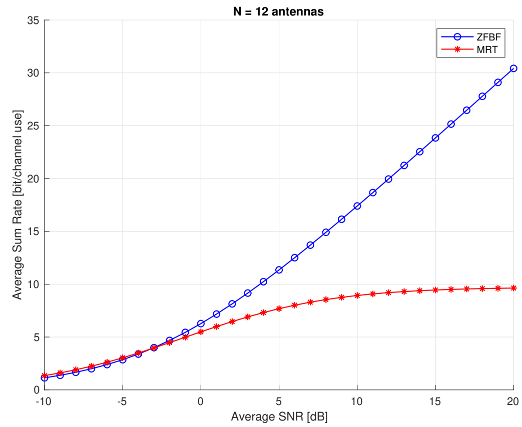

Figure 2: The average sum rate versus the average SNR, $N = 12$ antennas.

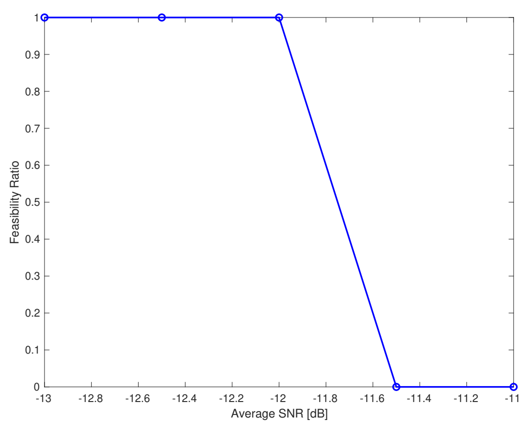

Figure 3: The feasibility ratio versus the average SNR, $K = 50$ users, $N = 3$ antennas (Other experimental setting).

4 Experiment

This section represents analytical results to determine the system performance (the average sum rate and the feasibility rate). I also compare the system performance under two methods: (i) Maximum Ratio Transmission (MRT); (ii) Zero-Forcing Beamforming (ZFBF). In additions, I investigate $K = 4$ users and problem (P2) with the sum rate as utility function: $f(\text{SINR}_1, ..., \text{SINR}_4) = \sum_{k=1}^{4} \log_2(1+\text{SINR}_k)$. The channel realizations are random circularly symmetric complex Gaussian, $h_k \sim \mathcal{CN}(0, \mathbf{I}_N)$ and the SNR is measured as P/σ^2.

Figure 1 and 2 show the average sum rate versus the average SNR for $N = 4$ and $N = 12$ transmit antennas correspondingly. Firstly, figure 1 demonstrates that while MRT beamforming is optimal at low SNRs (i.e., SNR < 8 dB), ZFBF is near optimal at high SNRs.

In the case of $N = 12$ (i.e., the number of antennas is many more than the number of users), the average sum rate of ZFBF and MRT are the same at low SNRs (i.e., SNR < -1 dB). However, ZFBF is near-optimal in the entire high SNRs regime.

Figure 3 investigates the feasibility of radio versus the average SNR, $K = 50$ users, $N = 3$ antennas (Another experimental setting). The feasibility rate is high when $SNR < -12$ dB.

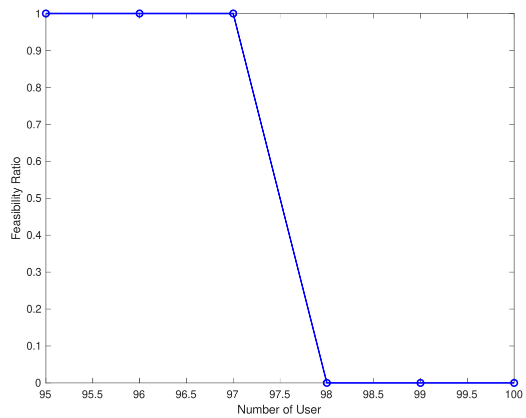

Figure 4: The feasibility ratio versus the number of user, $N = 3$ antennas (Other experimental setting).

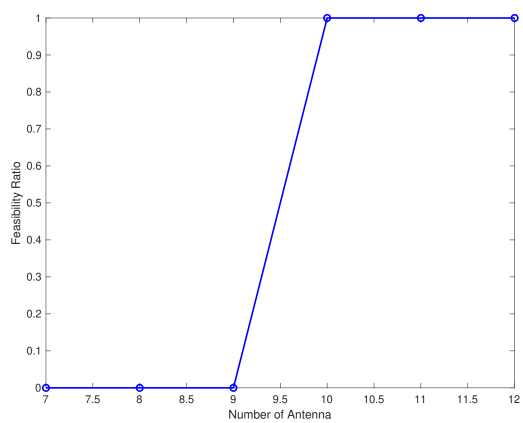

Figure 5: The feasibility ratio versus the number of antenna, $K = 100$ users (Other experimental setting).

Figure 4 depicts the feasibility radio versus the number of users, $N = 3$ antennas (Another experimental setting). The feasibility rate is high when $SNR < -12$ dB.The feasibility rate is high when $K < 97$ users.

Figure 5 shows the feasibility ratio versus the number of antennas, $K = 100$ users (Another experimental setting). The feasibility rate is high when $SNR < -12$ dB.The feasibility rate is high when $N > 10$ antennas.

In general, relaxed target SINR, fewer users, and more antennas will make the problem more feasible, since the constraints are easier to be met.

5 Conclusion

Maximum Ratio Transmission and Zero-Forcing Beamforming methods were investigated to solve the optimal multiuser transmit beamforming problems. Especially, performance analysis of these methods was shown in terms of the average sum rate and the feasibility rate.

By doing this project, I have learned something: (1) Maximum Ratio Transmission and Zero-Forcing Beamforming techniques; (2) Have an understanding of some multiuser transmit beamforming optimization algorithms (3) Try to understand how authors manipulate the optimization problems (e.g., convexity, KKT conditions); (4) Get some hand-on experiences on implementing some algorithms on Matlab.

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