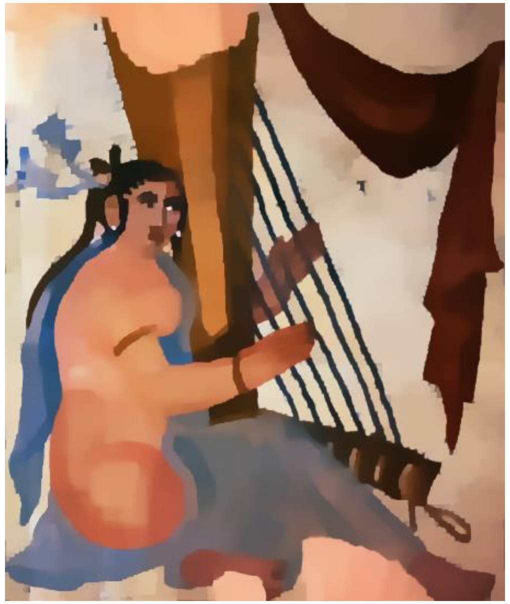
KOOC W4

Variational optimization in Image Understanding

Changick Kim, EE, KAIST

Outline

- Introduction
  - Examples
  - Why variational approaches?
  - Goal
  - Background and Categorization of image filtering
- Variational approaches for image denoising
  - Review of calculus of variation
  - Tikhonov regularization vs TV regularization
- Implementation of TV regularization models
  - ROF model (1992)
  - Linearization for EL equation (1996)
  - Duality-based algorithm (2004)
  - ROF dual: residual vs #iterations
  - TV denoising

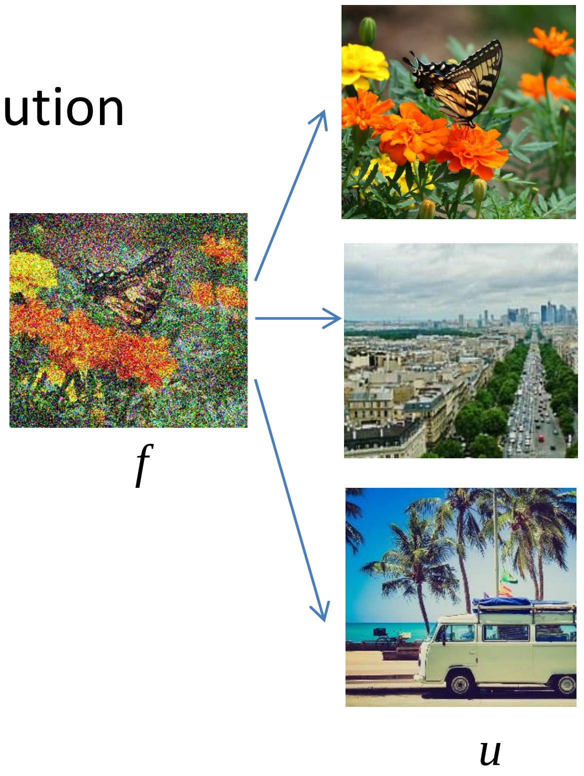
Introduction

- Image understanding is highly ambiguous
  - It is in general not possible to solve inverse problems by means of direct methods
  - Attach some probability to the quality of the solution
  - The Bayesian formula tells us

$$u^* = \max\left\{ p(u|f) = \frac{p(f|u)p(u)}{p(f)} \right\}$$

  - MAP: Select such μ that maximizes posterior
  - Energy minimization

$$u^* = \min\left\{ \int_\Omega R(u)dx + \int_\Omega D(u,f)dx \right\}$$



$$p(f|u) = \prod_{(x,y)\in D} \frac{1}{\sqrt{2\pi\mu}} e^{-\frac{(f(x,y)-u(x,y))^2}{2\mu^2}}, \quad p(u) = \prod_{(x,y)\in D} \frac{1}{\sqrt{2\pi\nu}} e^{-\frac{|\nabla u(x,y)|^2}{2\nu^2}}$$

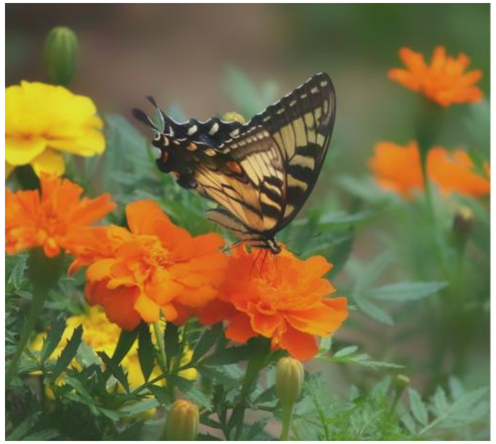
Examples of variational approaches

- Image Denoising

$$\min_u \left\{ \int_\Omega |\nabla u| dx + \frac{\lambda}{2} \int_\Omega (u-f)^2 dx \right\}$$

Input

Noisy



Denoised

Slide courtesy of Unger and Werlberger

Examples of variational approaches

- Image Restoration
  - Restoration = denoising + deblurring

$$\min_u \left\{ \int_\Omega |\nabla u| dx + \frac{\lambda}{2} \int_\Omega (Au - f)^2 dx \right\}$$

Input

Blurred + Noisy

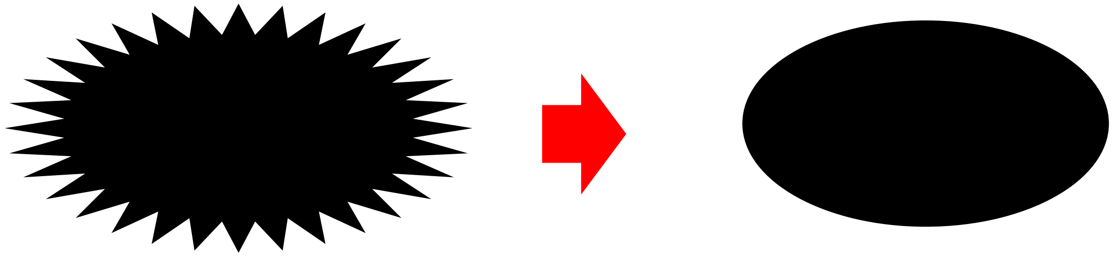
Deblurred

Slide courtesy of Unger and Werlberger

Examples of variational approaches

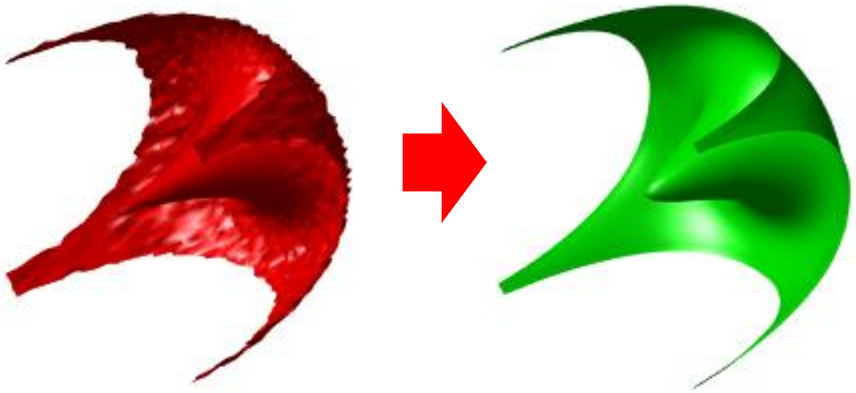
- Shape Denoising

$$\min_u \left\{ \int_\Omega |\nabla u| dx + \lambda \int_\Omega |u - f| dx \right\}$$



Noisy binary image

Denoised binary image



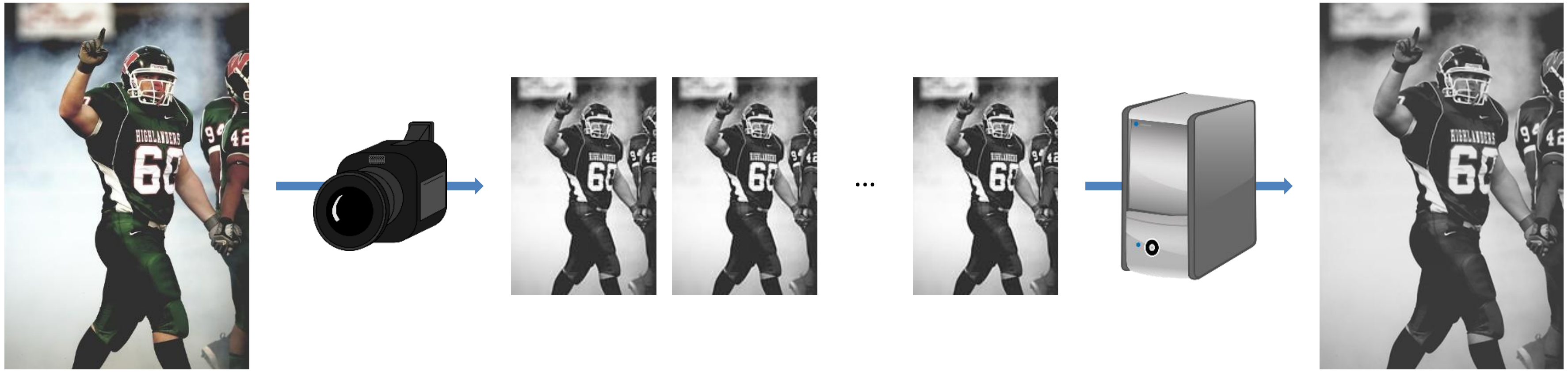
Slide courtesy of Unger and Werlberger

Examples of variational approaches

- Super-resolution

$$\min_{u} \left\{ \int_{\Omega} \lambda |\nabla u|_{\epsilon} dx + \sum_{k=1}^{K} \int_{\Omega} |DBW_{k} u - f_{k}| dx \right\}$$

Real world | Captured video | Super-Resolution



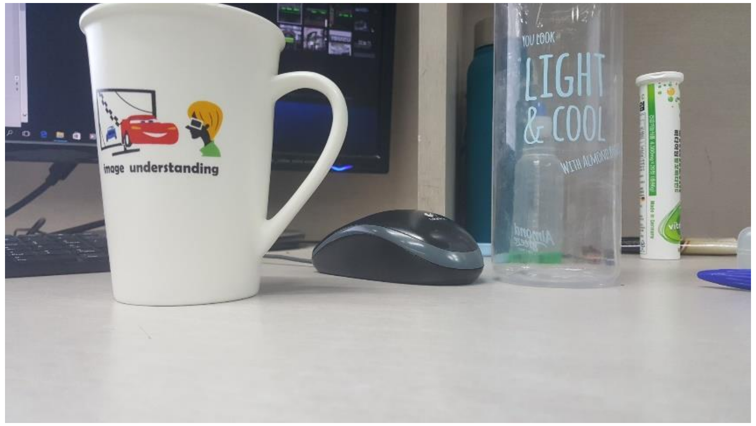
continuous | coarse sampling | fine sampling

Slide courtesy of Unger and Werlberger

Examples of variational approaches

- Optical flow

$$\min_u \left\{ \int_\Omega |\nabla u| \, dx + \lambda \int_\Omega \Big| I_0(x) - I_1(x + u(x)) \Big| \, dx \right\}$$



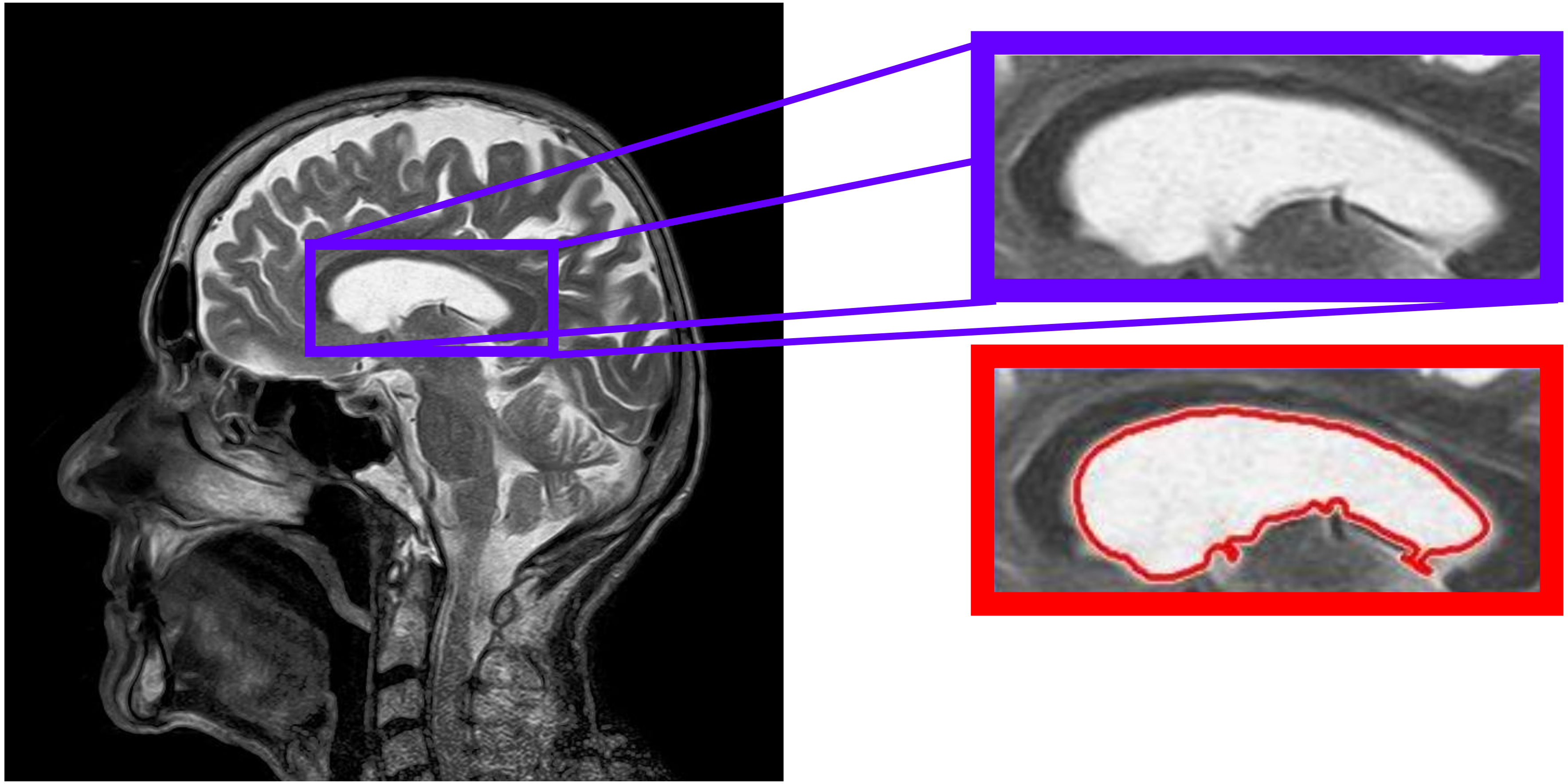
Lots of Applications:

- Driver assistance systems
- Action recognition
- Tracking
- Video restoration ,,,,,,

Slide courtesy of Unger and Werlberger

Examples of variational approaches

- Segmentation using active contour model



Caselles,Kimmel, Sapiro ICCV'95

$$E = \int_0^1 \frac{1}{2}\left[\alpha|\mathbf{v}'(s)|^2 + \beta|v''(s)|^2\right] + E_{ext}(v(s))ds$$

Why variational approaches?

- Perfectly suited for ill-posed problems
- Powerful optimization frameworks available
- Highly parallel (implementation on GPU using CUDA)

Goal

- This lecture will focus on image denoising (deblurring is out of focus in this lecture).

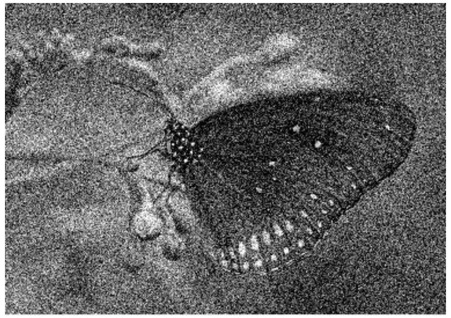
$$\boldsymbol{f} = \boldsymbol{H}\boldsymbol{u} + \boldsymbol{n}.$$

- There are two goals in this lecture.
  - to share the trend in variation approaches for image restoration
  - to see the minimization techniques(implementations) of energy functionals

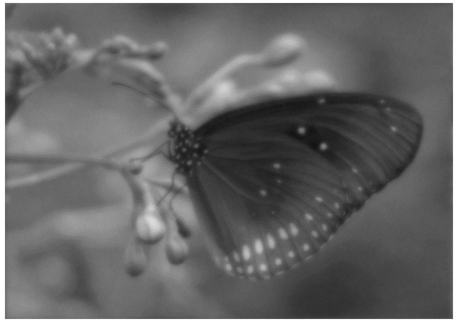
Background and Categorization of image filtering

- Image restoration using the Tikhonov model (1980's)

True Image



Observed Image
(Gaussian noise)



Restored Image

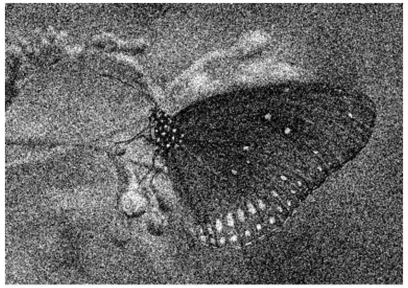
$$\min_u \left\{ E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 d\Omega + \frac{\lambda}{2}\int_\Omega (f-u)^2 d\Omega \right\}$$

Background and Categorization of image filtering

- Total variation regularization (ROF model, 1992)

$$\min_u \left\{ \int_\Omega |\nabla u| \, d\Omega + \frac{\lambda}{2} \int_\Omega (u - f)^2 \, d\Omega \right\}$$

- Structure-preserving image smoothing
  - Which edge is preserved and suppress?
  - Preserve important and salient edges (image structure)
  - Suppress insignificant details (noise and texture)
  - Difficult to classify structure and texture



Noisy image

Restored (TV reg.)

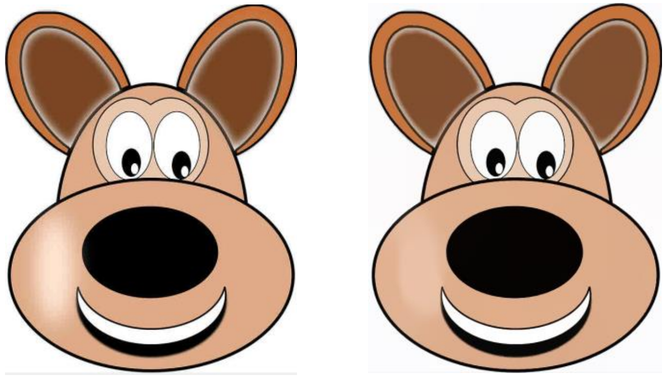
Structure-preserving image smoothing

Background and Categorization of image filtering

- Applications
  - Detail manipulation
  - Artifact removal
  - Edge enhancement and extraction
  - Image segmentation

Detail manipulation

Edge enhancement and extraction



Artifact removal

Background and Categorization of image filtering

- Edge-aware filters (structure-preserving filters)
  - Average-based methods
    : anisotropic diffusion, bilateral filter, guided filter, geodesic filter
  - Optimization-based methods (i.e., variational methods)
    : Tikhonov regularization, Total variation (TV), TV-L1, weighted least squares (WLS, 2008), L0 smoothing (2011)
- Scale-aware filters:
  - Texture Smoothing: Relative TV (RTV, 2012))
  - Rolling guidance filter (2014)
- Others
  - Mode/Median filters
  - Iterated Nonlocal Means: by Brox and Cremers (2007)

Variational approaches for image denoising

Variational approaches for image denosing

$$\min_u \left\{ E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 d\Omega + \frac{\lambda}{2}\int_\Omega (f-u)^2 d\Omega \right\}$$

Tikhonov regularization

$$\min_u \left\{ \int_\Omega |\nabla u|\, d\Omega + \frac{\lambda}{2}\int_\Omega (u-f)^2\, d\Omega \right\}$$

TV regularization

$$\min_u \left\{ \int_\Omega |\nabla u|\, d\Omega + \lambda \int_\Omega |u-f|\, d\Omega \right\}$$

TV-L1 model

$$\min_u \sum_p \left((u_p - g_p)^2 + \lambda \left(a_{x,p}(g)\left(\frac{\partial u}{\partial x}\right)_p^2 + a_{y,p}(g)\left(\frac{\partial u}{\partial y}\right)_p^2 \right) \right)$$

WLS model

$$\min_S \sum_p \left\{ (S_p - I_p)^2 + \lambda \cdot C(S) \right\}, C(S) = \#\left\{ p \,\big|\, |\partial_x S_p| + |\partial_y S_p| \neq 0 \right\}$$

L0 model

$$\arg\min_s \sum_p (S_p - I_p)^2 + \lambda \left(\frac{D_x(p)}{L_x(p)+\varepsilon} + \frac{D_y(p)}{L_y(p)+\varepsilon} \right)$$

RTV model

Calculus of Variations

- Classification based on the nature of solution
  - Variational Optimization Problems
    - An optimization problem is called a variational problem when its possible solution set (feasible set) S is an infinite dimensional subset of a space of functions (defined on a continuous domain).
    - In certain optimization problems, the unknown optimal solution might not be a number or a vector, but rather a continuous quantity, for example a function or the shape of a body.
    - Such a problem is also called an infinite-dimensional optimization problem, because a continuous quantity can not be determined by a finite number of certain degrees of freedom.
    - Calculus of Variation is used for solving such optimization problems.

Calculus of Variations

- A functional is a function of another function (or curve).
- A functional assigns a real number to each function (or curve) in some class.
- Functionals have an important role in many problems arising in analysis, optimization, mechanics, geometry, etc.
- Generally, we could write a functional:

$$F[y(x)] = \int_a^b f(x, y(x), y'(x))dx.$$

- The function f in the integral is to be viewed as an ordinary function of the variables x, y and y'.
- It is required that the function f in the integral have continuous partial derivatives of x,y, and y'.

Calculus of Variations

- Suppose the functional F obtains a minimum (or maximum) value. How do we determine the curve y(x) that produces such a minimum (or maximum) value for F?
- We will see that the minimizing curve y(x) must satisfy a differential equation known as the *Euler-Lagrange Equation*.
- Theorem. If y(x) is a curve in $C_{[a,b]}$ which minimizes the functional

$$F[y(x)] = \int_a^b f(x, y(x), y'(x))dx.$$

then the following differential equation must be satisfied:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

This equation is called the Euler-Lagrange Equation.

$$f_y - \frac{d}{dx} f_{y'} = 0. \qquad \left(f_y - \frac{d}{dx} f_{y'} + \frac{d^2}{dx^2} f_{y''} = 0 \right).$$

Tikhonov Regularization vs TV reg.

- Edge smoothing effect

$$\min_u \left\{ E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 d\Omega + \frac{\lambda}{2}\int_\Omega (f-u)^2 d\Omega \right\}$$

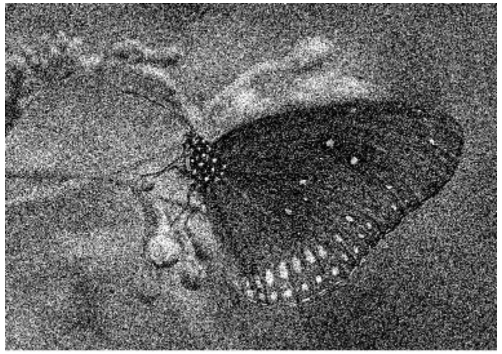
10	20	30	40

0	0	30	30

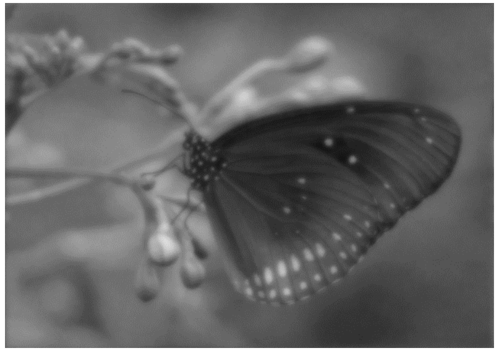
$$\min_u \left\{ \int_\Omega |\nabla u| \, d\Omega + \frac{\lambda}{2}\int_\Omega (u-f)^2 \, d\Omega \right\}$$

Tikhonov Regularization vs TV reg.

Original image



Noisy image



Tikhonov model

ROF model

Implementation of TV regularization models

Implementation of TV regularization models

- ROF model (1992)
- Linearization of Euler-Lagrange Equation (1996)
- Duality-based minimization (2004)
  - Primal-dual algorithm (2011)
- TV-L1 model (2006)
- Splitting techniques (2008~)
  - Variable splitting, split Bregman, ADMM (2009),,,

ROF model (1992)

$$\min_u \left\{ \int_\Omega |\nabla u| \, d\Omega + \frac{1}{2\lambda} \int_\Omega (u - f)^2 \, d\Omega \right\}$$

- Euler-Lagrange Equation of ROF model (1992)

$$-\nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) + \frac{1}{\lambda}(u - f) = 0.$$

$$\Big\downarrow \quad |\nabla u|_\varepsilon = \sqrt{|\nabla u|^2 + \varepsilon}$$

$$u - f - \lambda \nabla \cdot \left(\frac{\nabla u}{|\nabla u|_\varepsilon} \right) = 0 \longrightarrow \frac{\partial u}{\partial t} \approx \frac{u^{n+1} - u^n}{\Delta t} = u - f - \lambda \nabla \cdot \left(\frac{\nabla u}{|\nabla u|_\varepsilon} \right)$$

Time marching

Rudin, L. I.; Osher, S.; Fatemi, E. (1992). "Nonlinear total variation based noise removal algorithms". *Physica D*. **60**: 259–268.

Linearization for Euler-Lagrange Equation

- Vogel and Oman (1996)

$$\nabla \cdot \left(\frac{\nabla u^{k+1}}{\left| \nabla u^{k} \right|_{\varepsilon}} \right) - \lambda (u^{k+1} - f) = 0 \longrightarrow (1 + \alpha L(u^{k}))u^{k+1} = f$$

$$L(u)u = -\nabla \cdot \left(\frac{1}{\sqrt{\left|\nabla u\right|^{2} + \varepsilon}} \nabla u \right)$$

Using Jacobi algorithm to solve sparse large linear system

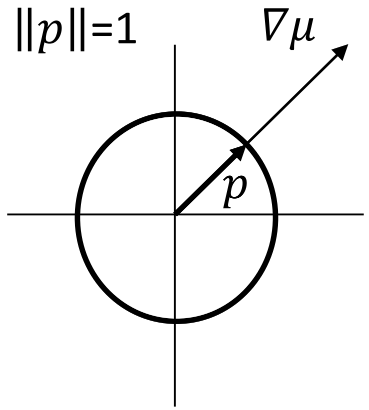
$$u_{i,j}^{n+1,l+1} = \frac{C_1^n u_{i+1,j}^{n+1,l} + C_2^n u_{i,j+1}^{n+1,l} + C_3^n u_{i-1,j}^{n+1,l} + C_4^n u_{i,j-1}^{n+1,l} + \frac{1}{\lambda} f_{i,j}}{C_1^n + C_2^n + C_3^n + C_4^n + \frac{1}{\lambda}}$$

Vogel and Oman, “Iterative methods for total vartiation denoising,”
SIAM J. Sci.Computing, 17 (1996)

Duality-based algorithm

- Chambolle (2004)

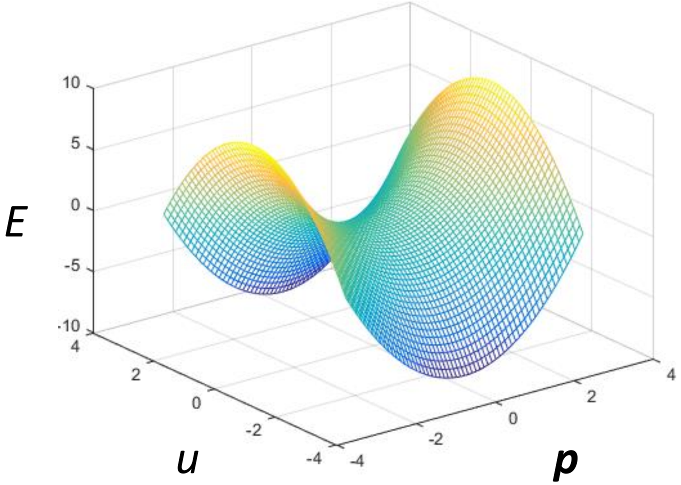
"An algorithm for total variation minimization and applications," JMIV, 2004



(1) $$|\boldsymbol{v}| = \max\{\boldsymbol{p}\cdot\boldsymbol{v} : \|\boldsymbol{p}\| \le 1\}$$

(2) $$\min E_{TV} = \min_{u} \max_{\|\boldsymbol{p}\|\le 1} \left\{ \int_\Omega \boldsymbol{p}\cdot\nabla u d\Omega + \frac{1}{2\lambda}\int_\Omega (u-f)^2 d\Omega \right\}$$

(3) $$\boldsymbol{p}^{n+1}(x,y) = \boldsymbol{p}^{n}(x,y) + \tau\cdot\nabla u^{n}(x,y)$$



(4) $$\boldsymbol{p}^{n+1}(x,y) = \frac{\boldsymbol{p}^{n}(x,y) + \tau\cdot\nabla u^{n}(x,y)}{\max\left(1, \left|\boldsymbol{p}^{n}(x,y) + \tau\cdot\nabla u^{n}(x,y)\right|\right)}$$

to guarantee $\left|\boldsymbol{p}^{n+1}\right| \le 1$

$$\mathbf{project}_P(p) = \frac{p}{max(\|p\|, 1)}$$

Duality-based algorithm

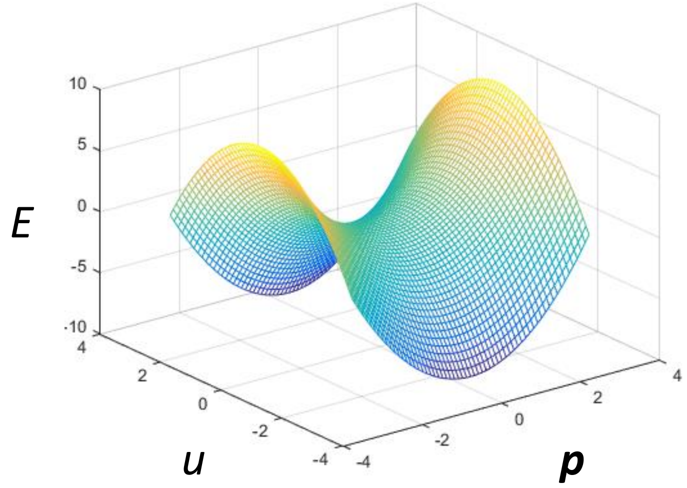
- Chambolle (2004)

Since $\langle \nabla u, \boldsymbol{p} \rangle = \langle -div[\boldsymbol{p}], u \rangle$

(5) $$\iint \langle \nabla u(x,y), \boldsymbol{p}(x,y) \rangle dxdy = -\iint u(x,y) \cdot div[\boldsymbol{p}(x,y)] dxdy$$

When $\partial E_{TV}(u, \boldsymbol{p}) / \partial u = 0$ with fixed ***p***,

(6) $$u^{n+1}(x,y) = f(x,y) + \lambda \cdot div[\boldsymbol{p}^{n+1}(x,y)]$$



Algorithm:
Step1:

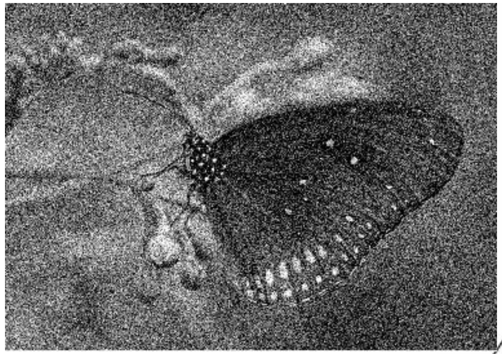
$$\boldsymbol{p}^{n+1}(x,y) = \frac{\boldsymbol{p}^{n}(x,y) + \tau \cdot \nabla u^{n}(x,y)}{\max\left(1, \left| \boldsymbol{p}^{n}(x,y) + \tau \cdot \nabla u^{n}(x,y) \right|\right)}$$

Step2:

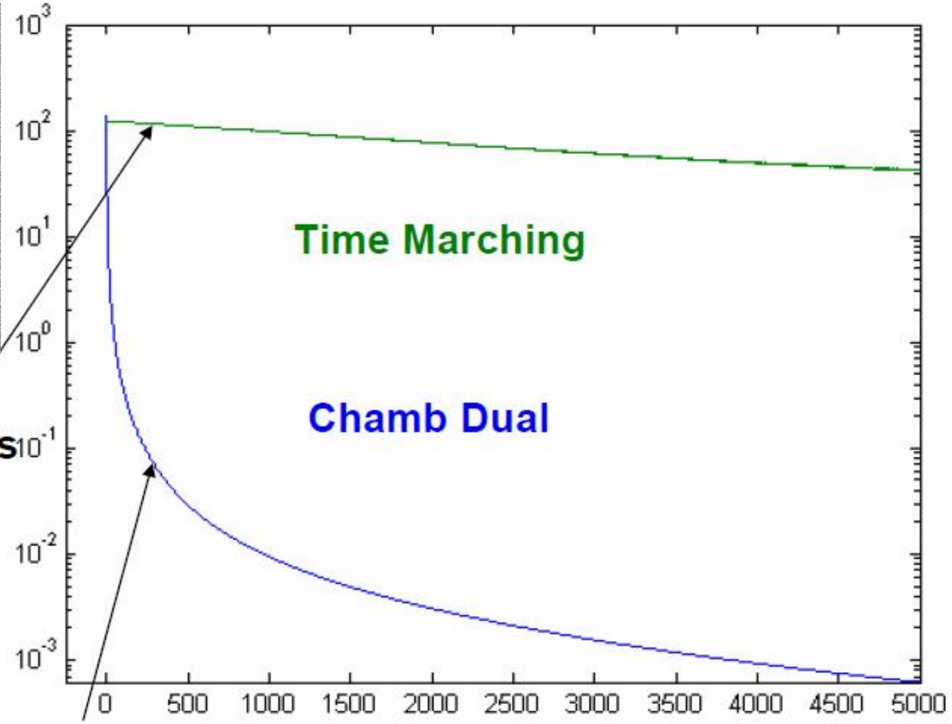
$$u^{n+1}(x,y) = f(x,y) + \lambda \cdot div[\boldsymbol{p}^{n+1}(x,y)]$$

Note $\langle div[\boldsymbol{p}(x,y)] \rangle = \frac{\partial p_1}{\partial x} + \frac{\partial p_2}{\partial y}$

ROF & Dual: Residual vs #Iterations



Observed image



TM 50 iters CD TM 100 iters CD




TV denoising (Chambolle)

- http://www.mathworks.com/matlabcentral/fileexchange/16201-toolbox-image/content/toolbox_image/toolbox/perform_tv_denoising.m
- Download toolbox_image.zip to perform_tv_denoising.m, which is the Chambolle's method.
- The Implementation method is described in appendix A

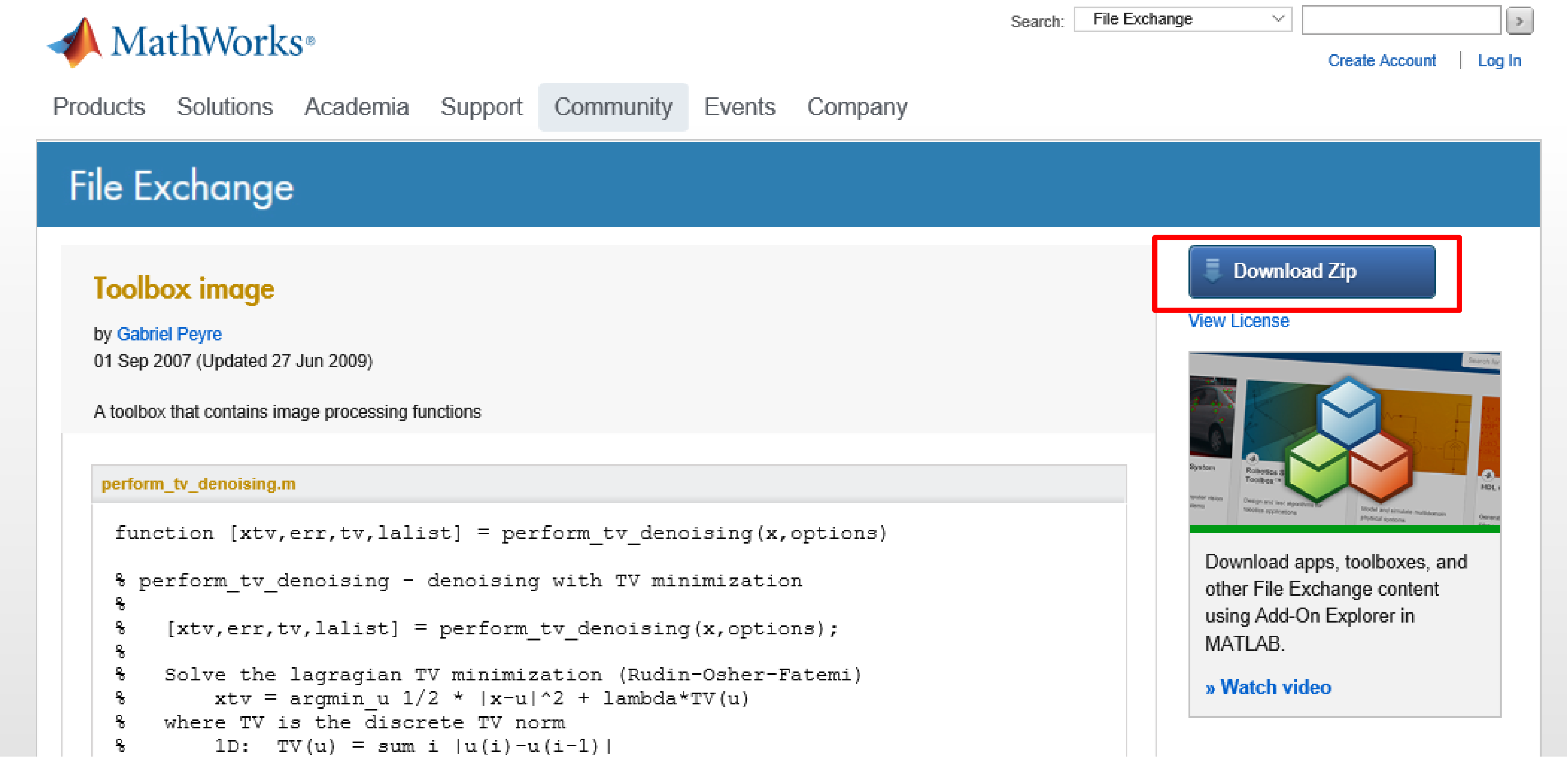
Original Image

Noisy Image

Denoising(TV(Chambolle))

Appendix A: Implementation method

- Go to the below web site and download the toolbox
  - http://www.mathworks.com/matlabcentral/fileexchange/16201-toolbox-image/content/toolbox_image/toolbox/perform_tv_denoising.m



Appendix A: Implementation method

- Go to the downloaded toolbox_image folder and implement the matlab
- Input the below codes in the command window

```
addpath('toolbox');
n = 128;
M0 = load_image('lena',256);
M0 = rescale(crop(M0,n));
sigma = .1;
M = M0 + randn(size(M0))*sigma;
% display
clf; imageplot({clamp(M0) clamp(M)},{'Original' 'Noisy'});
% some parameter for the algorithm
options.verb = 0;
options.display = 0;
options.niter = 300; % number of iterations
options.etgt = 1.1*sigma*n;
options.lambda = .3; % initial regularization
% now we perform the denoising
[Mtv,err,tv,lambda] = perform_tv_denoising(M,options);
clf; imageplot({clamp(M) clamp(Mtv)},{'Noisy' 'TV Denoised'});
```

감사합니다