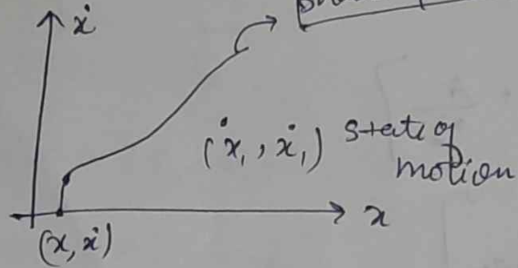
Phase space dynamics.

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$



phase space trajectory — dimension.

1D → 2	
2D → 4	
3D – 6	

phase space trajectory can't intersect. (autonomus) system.
but it should in periodic motion.

Tutorial : 06

1. $x = a\cos(\omega t + \alpha)$

$\Rightarrow x_0 = a\cos\alpha$

$v = -~~a\cos~~ - a\omega\sin(\omega t + \alpha)$

$v = 0 \Rightarrow \alpha = n\pi\ (0)$

$x_0 = a$

$x(t) = x_0 \cos\omega t$

$\Rightarrow v = \dot{x}(t) = -x_0 \omega \sin\omega t$

$v_{max} = x_0 \omega$ ——— (1)

Damped.

$$x(t) = c_1 \exp\left[\left(-\lambda + \sqrt{\lambda^2 - \omega^2}\right)t\right] + c_2 \exp\left[\left(-\lambda - \sqrt{\lambda^2 - \omega^2}\right)t\right]$$

$x(0) = x_0 = c_1 + c_2$

$\dot{x}(t) = v = c_1(-\lambda + \omega_d)\exp[\ \ ] + c_2(-\lambda - \omega t)\exp[$

$\dot{x}(0) = 0 = c_1(-\lambda + \omega_d) + c_2(-\lambda - \omega_d)$ —— (3)

$= -\lambda(c_1 + c_2) + \omega_d(c_1 - c_2)$.

$\Rightarrow -\lambda x_0 + \omega_d(c_1 - c_2)$

$\Rightarrow c_1 - c_2 = \frac{\lambda x_0}{\omega_d}$

$c_1 + c_2 = x_0$

$c_1 = \frac{x_0}{2}\left[1 + \frac{\lambda}{\omega_d}\right]$

$c_2 = \frac{x_0}{2}\left[1 - \lambda/\omega_d\right]$

then
$\dot{x}(t) = v$

$\frac{dv}{dt} = 0$

$\approx \dot{x}_1, \dot{x}_2$ --

$\frac{d^2x}{dt^2} < 0$

then then $\dot{x}_{max}$.

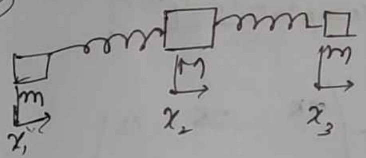
(i) $v_{max} = \omega x_0$

(ii) $v_{max} = -\omega x_0 \left[\frac{\lambda - \omega_d}{\lambda + \omega_d}\right]^{\frac{\lambda}{2\omega_d}}$

$\frac{V^2}{V'} = R = \left[\frac{\lambda - \omega_d}{\lambda + \omega_d}\right]^{\frac{+\lambda}{2\omega_d}}$

strong damp: $\lambda >> \omega$
critical damp:- $\lambda \sim \omega$ } find for these now

②



$m\ddot{x}_1 = -k(x_1 - x_2)$ $\quad -kx_2 + kx_1 - kx_2 + kx_3$

$m\ddot{x}_3 = -k(x_3 - x_2)$

$M\ddot{x}_2 =$

$= -k(x_2 - x_1) - k(x_2 - x_3)$

$= -k(2x_2 - x_1 - x_3)$

$V = \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2$

$\Rightarrow \frac{1}{2}k\left[2x_2^2 + x_1^2 + x_3^2 - x_1x_2 - x_2x_1 - x_3x_2 - x_2x_3\right]$

Since

$m\ddot{x}_1 + k(x_1 - x_2) = 0$

$m\ddot{x}_3 + k(x_3 - x_2) = 0$

$M\ddot{x}_2 + k(2x_2 - x_1 - x_3) = 0$

trail sol^n = $x_i = A_i e^{i\omega t}$

A lot of eq^n so we write them in form of matrix.

$\Rightarrow -m\omega^2 A_1 e^{i\omega t} + k[A_1 - A_2]e^{i\omega t} = 0$

$\Rightarrow (-m\omega^2)A_1 + k(A_1 - A_2) = 0$

$$\begin{pmatrix} k - m\omega^2 & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

Now All eqn in form of matrix without help of last equation which is complete sol^n.

$$(-\omega^2 e^{i\omega t})\begin{pmatrix} m & & \\ & M & \\ & & m \end{pmatrix} + e^{i\omega t}\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} = 0$$

1.

$$\Rightarrow \begin{pmatrix} k-m\omega^2 & -k & 0 \\ -k & 2k-M\omega^2 & -k \\ 0 & -k & k-m\omega^2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

Det A = 0.

$$\omega_1 = \sqrt{\frac{k}{m}}$$

$$(k-m\omega^2)\left[(2k-M\omega^2)(k-m\omega^2) - k^2\right] + k\left[-k(k-m\omega^2)\right] = 0$$

$$\omega_2 = 0$$

$$\omega_3 = \sqrt{k\left(\frac{1}{m}+\frac{2}{M}\right)}$$

2.

(i).

$$\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

$A_1 = A_2$

$A_2 = A_3$

$-A + 2A_2 - A_3 = 0$

$$|A_1|^2 = |A_2|^2 + |A_3|^2 = 1$$

(ii) $\omega = \sqrt{\frac{k}{m}}\left[\pm\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right]$

(iii) $\omega = \sqrt{\frac{2k}{M} + \frac{k}{m}}\begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}$

3.

$$\begin{pmatrix} 1 & 2 & -1 \\ -2 & 1 & 2 \\ 1 & 0 & 5 \end{pmatrix}$$

$$P^{-1} \overset{\text{original matrix.}}{S} P = \underset{\text{normalised matrix}}{M_D}$$

} diagonalisation.

already given.

$$M_D = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 6 \end{pmatrix}$$

$$P = \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{5} & -1/\sqrt{30} \\ -2/\sqrt{6} & 1/\sqrt{5} & 2/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \end{pmatrix}$$

$$\begin{pmatrix} x = au + bv + cw \\ y = du + ev + fw \\ z = gu + hv + kw \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

and if

$u \to \lambda u$
$v \to \lambda v$
$w \to \lambda w$. then $x \to \lambda x$, $y \to \lambda y$, $z \to \lambda z$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix as operator

$M \to$ is real symmetric matrix then it is diagonalisable

$$\begin{pmatrix} \cdot & & \\ & \cdot & \\ & & \cdot \end{pmatrix} \xrightarrow{\lambda_1, \lambda_2, \lambda_3} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

It is a ~~singular~~ similarity transformation matrix. M_D diagonalis

$M_D = P^{-1} M P$

P → Orthogonal matrix. Column will contain ~~eigen~~ eigenvector corresponding to λ_1 and λ_2 and λ_3.

Two $M_{3\times3}$ having same eigenvalue then $M_{3\times3}$ is similar matrix.

$PM_D = IMP$

$PM_D P^{-1} = IMI$

$$\boxed{PM_D P^{-1} = M}$$

Eigenvalues $\longrightarrow$ given eigenvector.

λ_1: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{normalise}} \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ λ_2: $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ λ_3: $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

let say $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$$(1 \quad 1 \quad 1)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0$$

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\boxed{P^{-1} = P^T}$$

$P \Rightarrow$ orthogonal

$$\begin{pmatrix} a & b & c \\ e & f & g \\ i & j & k \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \psi \begin{pmatrix} \ \\ \ \\ \ \end{pmatrix}$$

Eigenvectors of ( ) and then return this ψ ( )

this is Eigen value $\begin{pmatrix} a & b & c \\ 1 & 1 & 1 \end{pmatrix}$

$$\underset{3\times3}{M}\ \underset{3\times1}{X} = \psi\ \underset{3\times1}{X}.$$

$$MX = \psi I X$$

$$\boxed{(M - \psi I)X = 0}$$
$$X \neq 0$$

$$|M - \psi I| = 0$$

$$\Downarrow$$

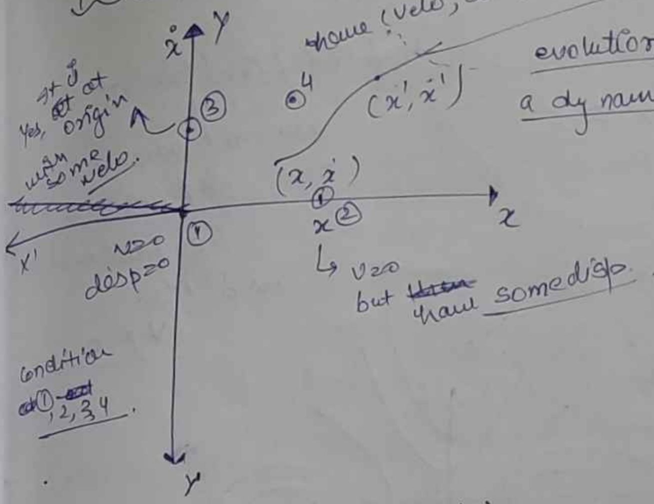
$$\begin{vmatrix} a-\psi & b & c \\ d & e-\psi & f \\ g & h & i-\psi \end{vmatrix} = 0$$

$\psi \to$ Eigenvalue ~~and root~~

- An Eigenvector v of a matrix M is a vector that gets taken into a multiple of itself when acted upon by M. That is $Mv = \lambda v$, where λ is some number (the eigenvalue). This can be rewritten as $(M-\lambda I)v = 0$, where I is the identity matrix. By ~~our~~ our usual reasoning about invertible matrices, a nonzero vector vector v exists only if λ satisfies $\det|M-\lambda I| = 0$

Date. 29/9.

phase space ~~disg~~ dynamics.



point of intersection have two possible future which is not possible

evolution of a dynamical system.

For autonomous system: $x(t)$

$$\dot{x} = v$$
$$\ddot{x} = f(x, \dot{x})$$
$$= f(x)$$

If we have $\boxed{\begin{matrix}\dot{x} = 0 \\ \ddot{x} = 0\end{matrix}}$

$\Rightarrow$ we have several points on x-axis.

Fixed points are the pt on phase space at which no further evolution take place if the particle is placed once. (End point of evolution)

$U(x) = -\frac{x^2}{2} + \frac{x^4}{4}$ $\quad (m=1)$

$F(x) = +x - x^3$

$x(1-x^2) = 0$

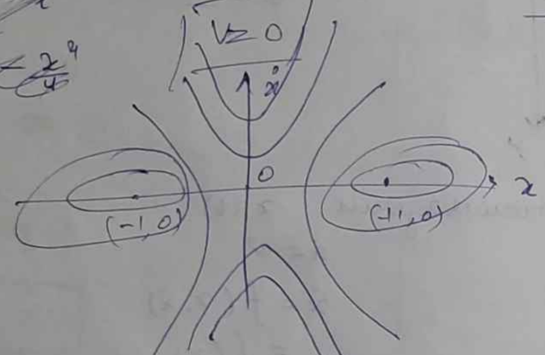
$x = 0$

$x = \pm 1$.

to find fixed point you need make $F(x) = 0$ i.e acceleration $=$ and Also $v = 0$

~~$\dot{x} = x - x^3$~~

~~$x = \frac{x^2}{2} - \frac{x^4}{4}$~~



$\dot{x} = v$

$\dot{v} = x - x^3$

$v \to 0$

$\dot{x} = v$

$v \simeq x$ $\quad \frac{\dot{v}}{x} = \frac{\frac{dv}{dt}}{\frac{dx}{dt}} = \frac{dv}{dx} = \frac{x}{v}$

$v^2 = x^2 + \text{const}$.

$\boxed{v^2 - x^2 = \text{const}}$

$x \simeq +1$

$x = 1 + \varepsilon$

$\dot{x} = 1 + \dot{\varepsilon}$

$\ddot{x} = x - x^3$

$= (1+\varepsilon) - (1+\varepsilon)^3$

$\simeq 1 + \varepsilon - 1 - 3\varepsilon$

$\simeq -2\varepsilon$.

$\dot{\varepsilon} = v$

$\dot{v} = -2\varepsilon$

orbit, stability point

Critical points and Stability.

let us take a dimensional flow (x, y) (equivalent to a phase space having p, q).

Let us consider the evolution of dynamics is given by

$$\left.\begin{aligned}\dot{x} &= ax + by\\ \dot{y} &= cx + dy\end{aligned}\right\} \quad \frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy.$$

Let us define. $X = \begin{pmatrix} x \\ y \end{pmatrix}$ then we have

$$\frac{dX}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \boxed{\frac{dX}{dt} = AX} \quad (1)$$

let (λ, V) be the eigenvalue of $A \Rightarrow$ $AV = \lambda V$, then $e^{\lambda t} V$ is a solⁿ of (1)

showing: If $X = e^{\lambda t} V \Rightarrow \frac{dX}{dt} = \lambda e^{\lambda t} V$

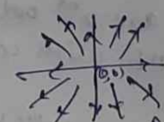
$\Rightarrow e^{\lambda t}(AV) = AX$.

now we start by considering some simple example:

$\dot{x} = x \Rightarrow \frac{dx}{dt} = x \Rightarrow x \sim e^t \Rightarrow x = x_0 e^t$

$\dot{y} = y \Rightarrow y \sim e^t$

$y = y_0 e^t$



If it starts from $(x_0, y_0) = (0, 0)$ it stays there.

But, if they start any other (x_0, y_0) where $x_0 \neq 0$, $y_0 \neq 0$ they move away from $(0,0) \Rightarrow (0,0)$ is a repeller

"unstable node".

Ans. In the neighbourhood of (0,0) we have straight lines.

Now let us write $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\lambda_1 + \lambda_2 = 2$

$\lambda_1 \lambda_2 = 0 \Rightarrow \lambda_1 + \frac{1}{\lambda_1} = 2 \Rightarrow \lambda_1^2 + 1 = 2\lambda$

$(\lambda_1 - 1)^2 = 0$

$\boxed{\lambda = 1,1}$

$\therefore$ solution $x = c_1 e^t v + c_2 t e^t v$ (unstable)

Now let us go back to the initial equation:

$\dot{x} = ax + by, \quad \dot{y} = cx + dy$] (i)

Now. $\ddot{x} = a\dot{x} + b\dot{y} \quad \ddot{y} = c\dot{x} + d\dot{y}$] (ii)

from (i) we have

$c\dot{x} = cax + bcy \quad a\dot{y} = acx + ady$

$\therefore c\dot{x} - a\dot{y} = (bc - ad)y$

$c\dot{x} = a\dot{y} + (bc - ad)y$

$\therefore \ddot{y} = a\dot{y} + (bc - ad)y + d\dot{y}$ (from (ii))

$\Rightarrow (a+d)\dot{y} + (bc - ad)y$

$\Rightarrow \ddot{y} - (a+d)\dot{y} + (ad - bc)y = 0$

& similarly $\ddot{x} - (a+d)\dot{x} + (ad - bc)x = 0$

$$\left.\begin{array}{l} \text{Trace } A = (a+d) = T \\ \text{Det } A = ad - bc = \Delta \end{array}\right\} \begin{array}{l} \ddot{y} - T\dot{y} + \Delta y = 0 \\ \ddot{x} - T\dot{x} + \Delta x = 0 \end{array}$$

Now if $x = x_0 e^{\lambda t}$ & $y = y_0 e^{\lambda t}$ then, (since they are 2nd order linear ODE with const. coeff.)

λ should satisfy,

$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$

It simply says that the two root of λ will be the eigenvalue of A.

(something which we obtained earlier).

Tutorial 7

1. $U(x) = \frac{x^2}{2} + \frac{x^3}{3}$

(a) $F(x) = -x - x^2$

(b) $(x, v) = (0,0)$

$= (-1, 0)$

(c) $x \to \epsilon$

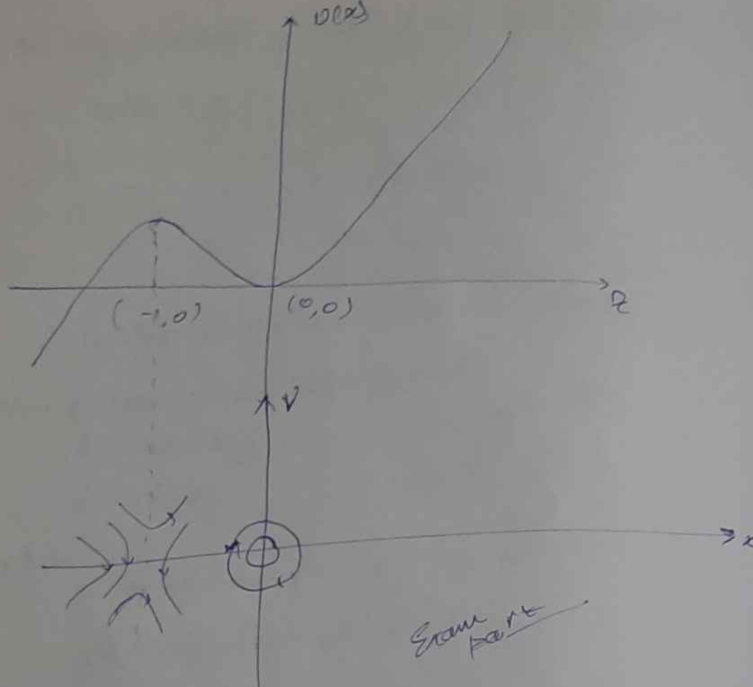
$v = \dot{x} = \dot{\epsilon}$

$\dot{v} = -x - x^2$

$= -\epsilon - \cancel{\epsilon^2}$

$= -\epsilon$

$\frac{dv}{dx} = -\frac{x}{v} \Rightarrow \frac{x^2}{2} + \frac{v^2}{2} = \text{const}$.



Exam part

$x = -1+\epsilon$

$\dot{x} = v = \dot{\epsilon}$

$\dot{v} = -x - x^2$

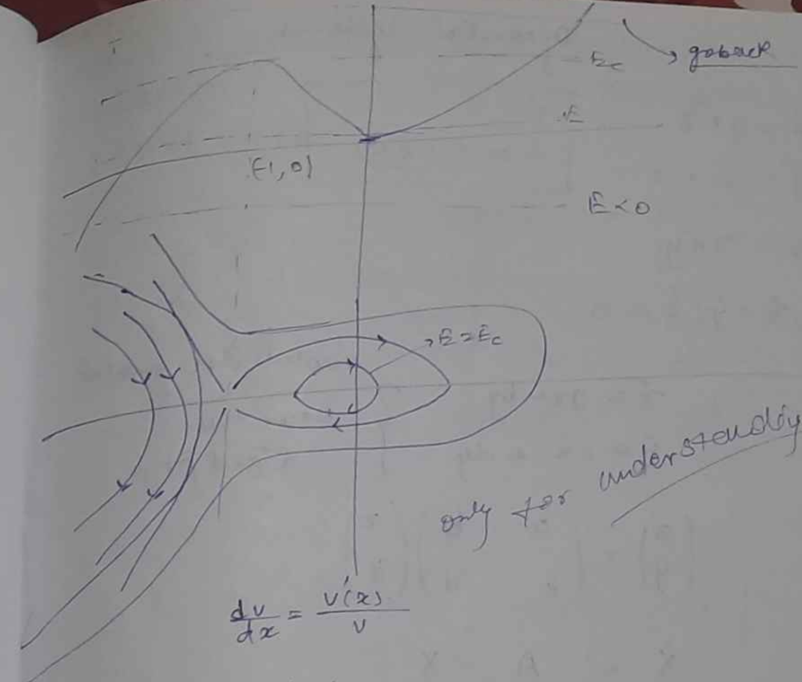
$= (1-\epsilon) - (-1+\epsilon)^2$

$= \cancel{1} - \epsilon - \cancel{1} - \epsilon^2 + 2\epsilon$

$\Rightarrow \epsilon$

$\Rightarrow (x+1)$ - ——— (4)

Eq $\Rightarrow$ $\boxed{\frac{(x+1)^2}{2} - \frac{v^2}{2} = k}$



only for understanding

$\frac{dv}{dx} = \frac{V'(x)}{v}$

By 2nd derevative test

(2) $\dot{x} = x[3 - x - 2y]$

$\dot{y} = y[2 - x - y]$

(i) $(0,0)$

(ii) $x=0$ $y \neq 0$ $\Rightarrow (0,2)$

(iii) $y=0$ $x \neq 0$ $\Rightarrow (3,0)$

(iv) $(1,1)$

(2) $x = x$ $(x \to 0)$

$y = 2 + y'$ $(\epsilon \to 0)$

$\dot{x} = x(3 - x - 2y)$

$\Rightarrow x(3 - x - 2(2+\epsilon))$

$\Rightarrow -x$

$\dot{y} = y(2 - x - (2+\epsilon))$

$\dot{y}' = -2x - 2y'$

$\begin{pmatrix} \dot{x} \\ \dot{y}' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Stable

$\dot{x} = 3x$

$\dot{y} = 2y$

$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Eigenvalue = 3, 2.

both λ are +ve, unstable equilibrium

Dynamical Systems

$\dot{x} = y + z$

$\dot{y} = z + x$

$\dot{z} = x + y$

$\dot{x} + \dot{y} + \dot{z} = 0$

$$\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= F(x, v) \end{aligned}}$$

evolution matrix

$$\left.\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}\right\} \text{near the fixed pt.}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{X} = A X$$

$A \longrightarrow (\lambda, v)$ $\qquad AV = \lambda V$

$\therefore \dot{X} = AX$

$X = e^{\lambda t} V$

$$\frac{d}{dt}(e^{\lambda t} V) = V \lambda e^{\lambda t} = e^{\lambda t} \lambda V$$

$$\Rightarrow e^{\lambda t} AV$$

$$\Rightarrow A(e^{\lambda t} V) \Rightarrow \underline{XA}$$

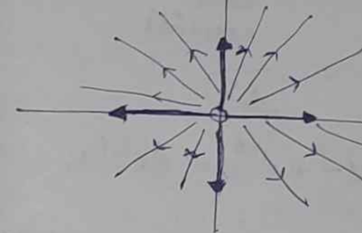
It doesn't correspond to phase space diagram.

$$\left.\begin{aligned} \dot{x} &= x \\ \dot{y} &= y \end{aligned}\right\} \qquad x = x_0 e^t, \quad y = y_0 e^t$$

+ Initial Condition $\qquad$ fixed pt $(0,0)$.

$$\dot{X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} X$$

where you have two eigenvalue which are similar and +ve then in phase space you have picture like



$$\boxed{\begin{aligned} \dot{x} &= x \\ \dot{y} &= y \end{aligned}}$$

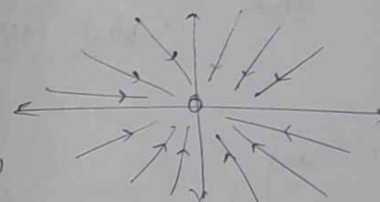
motion — if not thinking that it is going on the axes

Repeller.

$\dot{x} = -x$

$\dot{y} = -y$

E.v $\lambda = -1, -1$



$$\boxed{\begin{aligned} x &= x_0 e^{-t} \\ y &= y_0 e^{-t} \end{aligned}}$$

Asymptotically stable node.

$\dot{x} = x$

$\dot{y} = 2y$

F. pt $(0,0)$

$x = x_0 e^t$

$y = y_0 e^{2t}$

Phase space trajectory $x^2 = Ay$

$\dot{x} = -x$

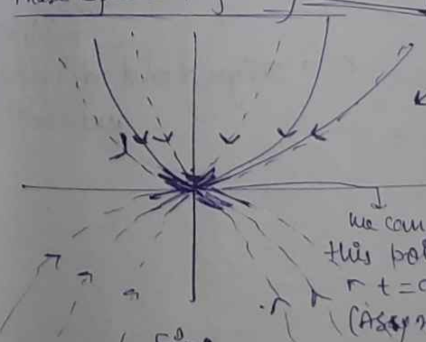
$\dot{y} = -2y$

F. pt $(0,0)$

$x = x_0 e^{-t}$

$y = y_0 e^{-2t}$

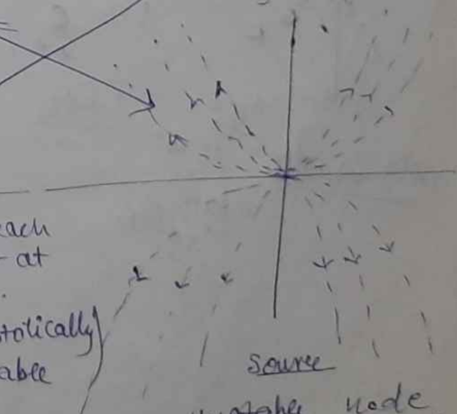
$x^2 = Ay$



we can reach this point at $t = \infty$. (Asymptotically stable

Sink

Stable node



Source

unstable node.

Fixed points of dynamical systems

Strogatz

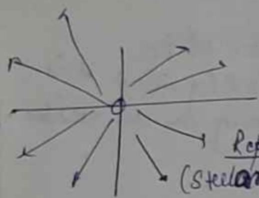
$X = \begin{pmatrix} x \\ y \end{pmatrix}$ 2d flow

phase space vector

phase space coordinate

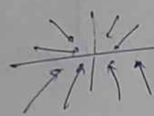
$\dot{X} = AX$ $\boxed{\lambda, V}$

$X = A e^{\lambda_1 t} V_1 + B e^{\lambda_2 t} V_2$



Repeller.
(Steller unstable node)

source



$t \to +\infty$
$\to$ stable

Steller stable node

sink

Simplest Attractor

$\dot{x} = -x$
$\dot{y} = 2y$

$A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

$\lambda_1, \lambda_2 = -1, 2$

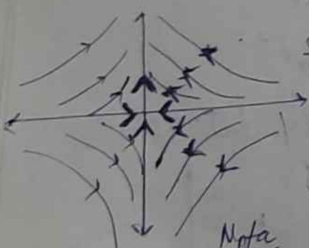
$x = x_0 e^{-t}$
$y = y_0 e^{2t}$

$\frac{dy}{dx} = \frac{2y}{-x}$

$\frac{dy}{-2y} = \frac{dx}{x}$

$y = \frac{c}{x^2}$

saddle



Not a saddle, it not a pure repeller.

not a saddle.

$c \to$ magnitude told they should vanish at fix pt. i.e $\dot{x}, \dot{y} = 0$

$c \to$ sign told different quadrant.

$\dot{x} = y$
$\dot{y} = -x$

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\frac{dy}{dx} = -\frac{x}{y}$

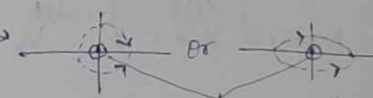
trace $\lambda_1 + \lambda_2 = 0$ $\lambda_1 = -\lambda_2$

det $\lambda_1 \lambda_2 = 1$ $-\lambda_2^2 = 1$

$X = A e^{it} V_1 + B e^{-it} V_2 \Rightarrow \lambda_1, \lambda_2 = i, -i$

eigen vector.

$V_1, V_2 \Rightarrow$ (2×1) column matrix.

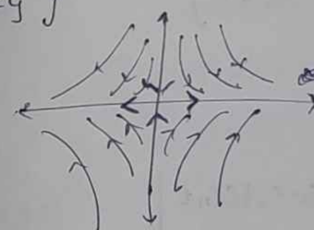


or

fixed point called center.

Fixed points in 2D flow.

$\left.\begin{aligned} \dot{x} &= -x \\ \dot{y} &= 2y \end{aligned}\right\}$ $\lambda_1, \lambda_2 = (-1, 2)$



also ...

* for A conservative system center can't be a attractor.
* for a dissipative system center is a attractor.

case: $\lambda_1, \lambda_2 \Rightarrow \lambda \pm i\mu$

$\begin{cases} \dot{x} = -x - y \\ \dot{y} = x - y \end{cases}$

$-x - y = 0$
$x - y = 0$ F.P $\Rightarrow (x, y) = (0, 0)$

$A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$

$\lambda_1 + \lambda_2 = -2$

$\lambda_1 \lambda_2 = 1 + 1 = 2$

$\lambda_1 = -2 - \lambda_2$

$-2\lambda_1 - \lambda_2^2 = 2$

$\lambda_2^2 + 2\lambda_1 + 2 = 0$

$\Rightarrow (\lambda_1 + 1)^2 + 1 = 0$

$\lambda_1 + 1 = \pm i$

$\lambda_1 = -1 \pm i$
$\lambda_2 = -1 \mp i$

$X = A e^{-1\pm it} v_1 + B e^{-1\mp it} v_2$

$\boxed{AV = \lambda V}$

$V_{-1+i} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

$V_{-1-i} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

$$\begin{pmatrix}-1+1+i & -1\\ 1 & -1+1-i\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$x = iy$

$y = -ix$

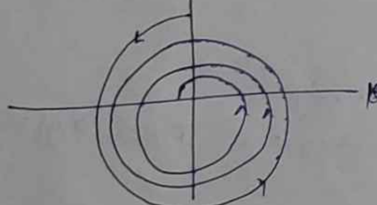
$X = c_1 e^{(-1+i)t} v_1 + c_2 e^{(-1-i)t} V_2$

$\Rightarrow \frac{e^{-t}}{\sqrt{2}}\left[c_1 e^{it}\begin{pmatrix}1\\-i\end{pmatrix} + c_2 e^{-it}\begin{pmatrix}1\\i\end{pmatrix}\right]$

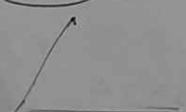
$x = \frac{1}{\sqrt{2}} e^{-t}\left[c_1 e^{it} + c_2 e^{-it}\right]$

$\Rightarrow \frac{1}{\sqrt{2}} e^{-t}\left[(c_1+c_2)\cos t + i(c_1-c_2)\sin t\right]$

$y \Rightarrow \frac{e^{-t}}{\sqrt{2}}\left[(c_1+c_2)\sin t - i(c_1-c_2)\cos t\right]$



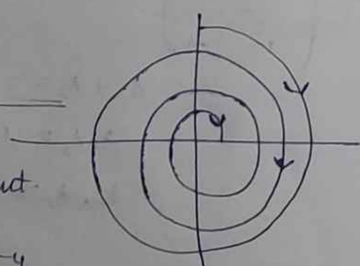
other possibility.



Check the ~~right~~ correct by

$\dot{x} = -x-y$

$\dot{y} = x-y$



Tutorial sheet-8.

1. $X = e^{\lambda t} v$ is a soln of $\dot{X} = AX$ then ~~$A\lambda = \lambda v$~~ $\boxed{AV = \lambda V}$

$\dot{x} = ax + by$ --- (i)

$\dot{y} = cx + dy$ --- (ii)

$A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $X = \begin{pmatrix}x\\y\end{pmatrix}$

(i) $\times c \Rightarrow c\dot{x} = acx + bcy$

(ii) $\times a \Rightarrow a\dot{y} \Rightarrow acx + ady$

$\Rightarrow c\dot{x} - a\dot{y} = (bc-ad)y$

Diff (ii) wrt t

$\ddot{y} = c\dot{x} + d\dot{y}$

$\ddot{y} - (d+a)\dot{y} + (ad-bc)y = 0$ --- (iii)

$\ddot{x} - (a+d)\dot{x} + (ad-bc)x = 0$ --- (iv)

Trial soln.

$x = x_0 e^{\lambda t}$

$y = y_0 e^{\lambda t}$

$\lambda^2 - \underbrace{(a+d)}_{Tr(A)}\lambda + \underbrace{(ad-bc)}_{\det A} = 0$

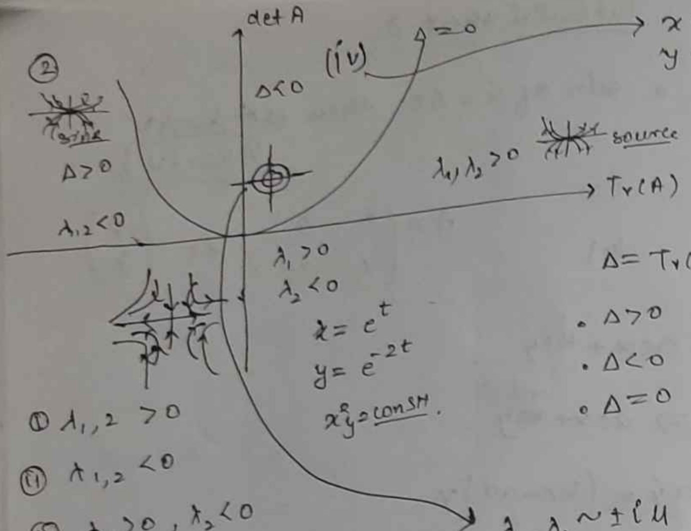
$\lambda = \frac{Tr A \pm \sqrt{(Tr(A))^2 - 4\det A}}{2}$

$\lambda x_0 = ax_0 + by_0$

$\lambda y_0 = cx_0 + dy_0$

$\Rightarrow \lambda V = AV$

②



$x \sim e^{\lambda t} e^{i\mu t}$
$y \sim e^{\lambda t} e^{-i\mu t}$

outward spiral due to +ve amplitude

$x = e^{t}$
$y = e^{-2t}$
$x^2 y = \text{constt}.$

$\Delta = Tr(A)^2 - 4 \det A$

- $\Delta > 0$
- $\Delta < 0$
- $\Delta = 0$

(i) $\lambda_{1,2} > 0$
(ii) $\lambda_{1,2} < 0$
(iii) $\lambda_1 > 0, \lambda_2 < 0$
(iv) $\lambda_{1,2} = \lambda \pm i\mu$
(v) $\lambda_{1,2} = -\lambda \pm i\mu$
(vi) $\lambda_{1,2} = \pm i\mu$

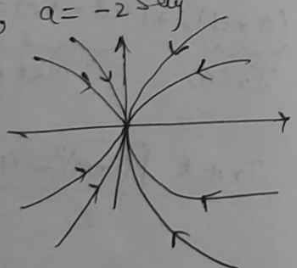
$\lambda_1, \lambda_2 \sim \pm i\mu$

$\mu \to 0$
$t \to \infty$
Centre.
(Collide)

③ $A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \dot{x} = ax \Rightarrow x = x_0 e^{at}$
$\dot{y} = -y \Rightarrow y = y_0 e^{-t}$

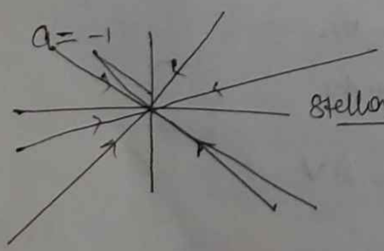
(i) $-\infty < a < -1$, $a = -2$ say

$x = x e^{-2t}$
$y = y_0 e^{-t}$
$x \sim y^2$



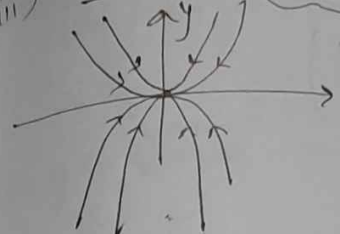
Assymtotically Stable.

(ii) $a = -1$

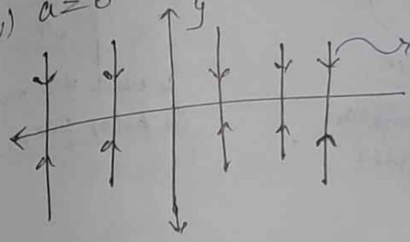


Stellon.

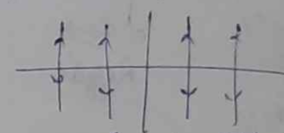
(iii) $-1 < a < 0$. ($a = -\frac{1}{2}$ say)



(iv) $a = 0$

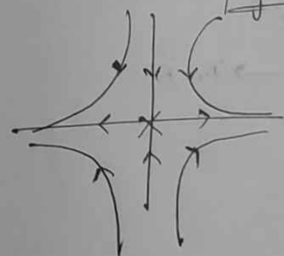


line of Attraction.



line of reppler

(v) $a > 0$ $\quad y = y_0 e^{-t}$
$x = x_0 e^{2t}$
$\boxed{y^2 x = C}$

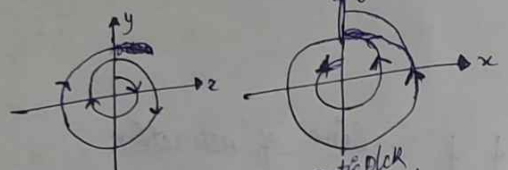


12/Oct

Fixed point.

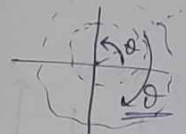
Spiral $\dot{x} = x - y$, $\dot{y} = x + y$ $\Big\} \to A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\lambda = 1 \pm i$



clockwise Repeller

anticlock. Repeller.

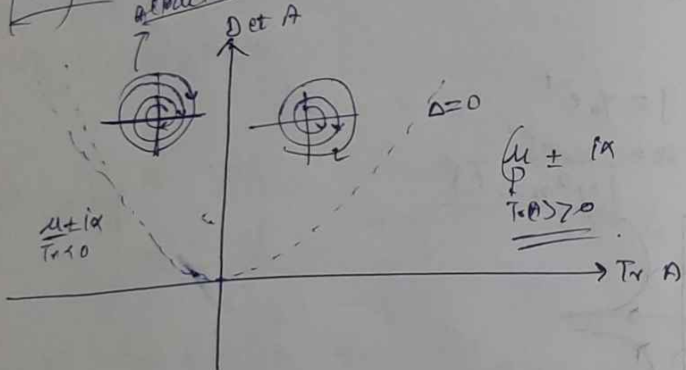


So write the eq^n in polar form.

H/W

$\dot{r}$
$\dot{\theta}$

attractor.



other fixed pt: temporal node (2 degenerate).

* Limit cycle is an isolated closed trajectory in phase space which can act as repeller or attractor but doesn't permit any other trajectory in neighbourhood

Limit cycle is not possible in linear system.

A cot trajectory to very very changes in Energy

case study

$\dot{x} = \mu x + y + x(x^2 + y^2)$

$\dot{y} = -x + \mu y + y(x^2 + y^2)$ → become nearly zero in neighbourhood of (0,0)

one of the fixed pt $(x, y) = (0, 0)$ and etc.

$\dot{x} = \mu x + y$

$\dot{y} = -x + \mu y$ $\quad A = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$

$Tr(A) = 2\mu$

$\det(A) = \mu^2 + 1$.

$|A - \lambda I| = 0.$

$\lambda_1 + \lambda_2 = 2\mu$

$\lambda_1 \lambda_2 = \mu^2 + 1$

$$\left| \begin{pmatrix} \mu - \lambda & 1 \\ -1 & \mu - \lambda \end{pmatrix} \right| = 0$$

$(\mu - \lambda)^2 = -1$

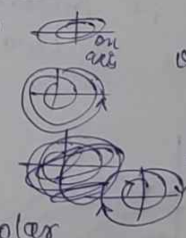
$\mu - \lambda = \pm i$

$\boxed{\lambda = \mu \mp i}$

$\mu = 0$ on axis

$\mu > 0$

$\mu < 0$



Parametric situation.

When a trajectory depends on the constant of Eigenvalue. like μ in our case.

now solving the thing in polar

$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}. \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow 2r\dot{r} = 2x\dot{x} + 2y\dot{y}$ → substitute the original eq^n

$r\dot{r} = \mu r^2 + r^4$

$\boxed{\dot{r} = (\mu + r^2) r}$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{x^2\left[1 + \left(\frac{y}{x}\right)^2\right]}$$

$$\dot{\theta} \Rightarrow \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}$$

$\boxed{r^2\dot{\theta} = x\dot{y} - \dot{y}x}$

$\Rightarrow \dot{\theta} = -1$. ~~cl~~ clockwise.

(i) $\mu > 0$, $\dot{r} > 0$. clockwise outward spiral
$\dot{\theta} = -1$

(ii) $\mu = 0$
$\dot{\theta} = -1$.

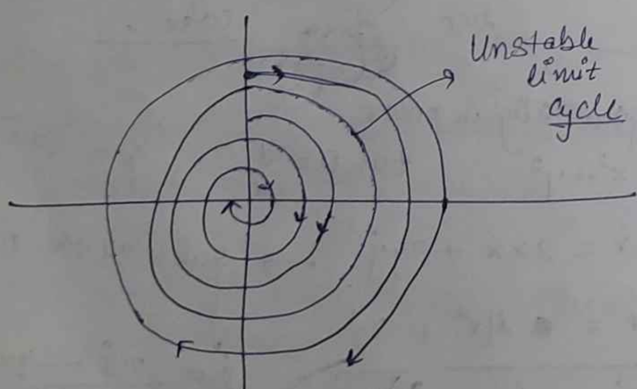
(iii) $\mu = -r^2$
$\dot{r} = 0$
circle.

(iv) $\mu < 0$ and $|\mu| > r^2$.
inward spiral.

$\mu > 0$ | $\dot{r} > 0$, $\dot{\theta} = -1$ } outward spiral
There will be no limit cycle.

$\mu < 0$ | $\dot{r} = (r^2 - \alpha) r$, $\alpha = -\mu$.
$\dot{\theta} = -1$

$\alpha = r^2$ circle.
$\alpha < r^2$ inward spiral
$\alpha > r^2$ outward

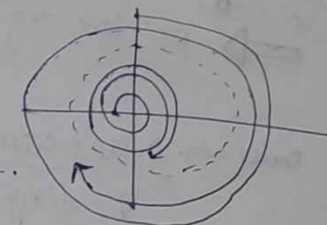


13/10

limit ~~Nahmetics~~.

$\dot{x} = \mu x + y + x(x^2 + y^2)$
$\dot{y} = -x + \mu y + y(x^2 + y^2)$ already discussed

$\Rightarrow$ $\dot{r} = r(\mu + r^2)$
$\dot{\theta} = -1$



~~Advance~~
advanced phase space trajectory.

Closed trajectory
Yes → may / may not be limit cycle.
No → No limit cycle

1) $\dot{x} = \frac{\partial V}{\partial x}$, $\dot{y} = \frac{\partial V}{\partial y}$ } → Gradient system

for such system we can never have a closed trajectory.

$\dot{x}dx + \dot{y}dy = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy = dV$

If the phase space region does not permit index

2) $\dot{x} = f(x,y)$ $V(x) \rightarrow$ Lyapunov f^n
$\dot{y} = g(x,y)$

If the sys consist of the real value f^n such that $V(x) > 0$ ~~except~~ $\forall x \mid x = x_c)$

$\boxed{\text{it } f^n \text{ of } x \text{ only}}$ $\dot{V} < 0 \quad \forall x \mid x = x_c$.

its not velo.

$\Longrightarrow$ No close trajectory and Hence no limit cycle

3) ~~Dulex~~ ~~or~~ Dulac's Criterion.

$\dot{x} = f(x,y)$ $f \in C'$
$\dot{y} = g(x,y)$ $g \in C'$ > continous & differential.

If $\exists\ h(x,y) \in C'$

$$\boxed{\frac{\partial(h\dot{x})}{\partial x} + \frac{\partial(h\dot{y})}{dh}}$$

$\Rightarrow$ No limit cycle possible.

4) Poincaré – Bendixon.

$\Downarrow$

2d Autonomous system in Euclidean plane

* Determinitically chaotic system No chaos.

body force:- whenever A force applied on a body uniformly.

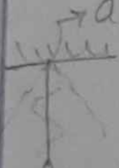
surface force:-  friction expirence by this surface only.

Constraint: $\xrightarrow[\text{of equality}]{\text{in the form}}$ Bilateral $\underline{r=a}$

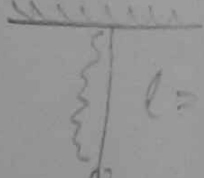
A can assosiated with inequality.

Unilateral

$\underline{r<a}$.



constraint $\xrightarrow[\text{appears explicity}]{\text{time}}$ Rheonomic



constraint $\xrightarrow[\text{time}]{\text{no explicts}}$ Scleronomic

⇒ They expressed in Algebraic eqn.
" only contain time & coordinates.

such a contraints are useful to reduce the degree of freedom.

↓

Holonomic constraints.

Langarangeous

Generalized coordinates

$\vec{r} = \vec{r}(x, y) = x\hat{x} + y\hat{j}$

$\vec{r} = \vec{r}(r, \theta) = r\hat{r}$

$\quad r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$

$\vec{r} = \vec{r}(\theta, \phi)$

normal coordinates

Look at ~~the~~ your system

↓

Find the holonomic constraints.

↓

choose a coordinate system where the holomonic constraint can effectively reduce degree of freedom.

$\vec{r} = r(\theta, \phi) \qquad x^2 + y^2 + z^2 = r^2$

↳ Generalized coordinates

↓

$\{q_i\}_{i=1,\dots N} = \{\theta, \phi\}$

$\{\dot{q}_i\} = \{\dot{\theta}, \dot{\phi}\}$ → set of Genralized velocity.

$\dot{\vec{r}} = \dot{\vec{r}}\{\theta, \phi, \dot{\theta}, \dot{\phi}\}$

$L = L(q_i, \dot{q}_i, t)$ Lagrangian

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \forall q_i$$ → effective degree of freedom

$E = T + V$.

$L = T - V$ → constant.

For a free particle, $L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$

for x_1

$$\frac{d}{dt}\left(\frac{dL}{d\dot{x}_1}\right) - \frac{dL}{dx_1} = 0$$

$m\ddot{x}_1 = 0$

similarly. $m\ddot{x}_2 = 0$

$m\ddot{x}_3 = 0$

$L = T - V$ → particle in plane where $V = V(r)$.

$\Rightarrow \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2\right] - V(r)$ $\quad(r, \theta)$ → generalized coordinates

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{r}}\right] - \frac{\partial L}{\partial r} = 0 \longrightarrow m\ddot{r} = mr\dot{\theta}^2 - V'(r)$$

$$\Rightarrow m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial V(r)}{dr}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

↳ Showing conservation of angular momentum.

Cyclic coordinates :- (r, θ, ϕ)
$\hookrightarrow$ constant

Properties of EL Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad 1, 2, \ldots n$$

$$\overset{L}{m\ddot{r}} = \vec{F}$$

$$\frac{\partial L}{\partial x_3} = 0 \implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) = 0$$

$\hookrightarrow$ a const. of motion

$\hookrightarrow p_3 \to$ generalise momentum.

x_3 = cyclic coordinate

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V(x_1, x_2, x_3)$$

$$\frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 \qquad \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$\boxed{\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}}$$

$$L' = L + c(t)$$

L is defined only upto an $f(t)$.

$L \to \alpha L$.

$$\frac{\partial L}{\partial t} = 0$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_i^n \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i^n \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

$$\implies \cancel{\frac{\partial L}{\partial t}}^{\,0} + \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\ddot{q}_i$$

$$\implies \frac{d}{dt}\left(\sum \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i}\dot{q} - L\right) = 0$$

$$\underbrace{\sum_i p_i\dot{q}_i - L}_{\text{Energy function}} \longrightarrow \text{is a constant of motion if } \frac{\partial L}{\partial t} = 0$$

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V(x_1, x_2, x_3)$$

$$\dot{q}_i p_i = \frac{\partial L}{\partial \dot{x}_i}\dot{x}_i = m\dot{x}_i^2$$

$\therefore$ Energy f^3

$$H = T + V \quad \left\{ \begin{array}{l} \to m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \\ -\frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \\ + V \end{array} \right.$$

$\mathcal{E}$

$$L \to L' = L + \frac{d}{dt}f(q_i, t)$$

Equivalent Lagrangians.

$$\frac{\partial}{\partial \dot q_i}\left(\frac{\partial f}{dt}\right) = \frac{\partial}{\partial \dot q_i}\left[\frac{\partial F}{\partial t} + \sum_j \frac{\partial F}{\partial q_j} \dot q_j\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\cancel{\left(\frac{\partial f}{\partial \dot q_i}\right)}^{0} + \sum_j \frac{\partial f}{\partial q_j} \delta_{ij}$$

coronical delta

$\delta_{ij} = 1$
$\because i = j$

$$\Rightarrow \left(\frac{\partial F}{\partial q_i}\right)_{i=j}$$

$$\Rightarrow \cancel{\frac{\partial F}{\partial t}} \frac{\partial}{\partial q_i}\left(\frac{dF}{dt}\right) = \frac{\partial}{\partial q_i}\left[\frac{\partial F}{\partial t} + \sum_j \frac{\partial f}{\partial q_j} \dot q_j\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\left[\frac{\partial F}{\partial q_i}\right] + \sum \frac{\partial}{\partial q_i}\left(\frac{\partial f}{\partial q_i}\right) \dot q_j$$

$$\Rightarrow \frac{d}{dt}\left[\frac{\partial f}{\partial q_i}\right]$$

Tutorial 10.

① $H = \sum_i p_i \dot q_i - L(q, \dot q, t)$

$$\Rightarrow \sum_i \left(\frac{\partial L}{\partial \dot q_i}\right)\dot q_i - L \qquad , L = T - V$$

$$\Rightarrow \sum_i \left(\frac{\partial T}{\partial \dot q_i}\right)\dot q_i - L$$

matrix $(m \times n)$

$$T = \sum_m \sum_n a_{mn} \dot x_m \dot x_n$$

$$\frac{\partial T}{\partial \dot x_i} = \sum_{m,n} a_{mn} \underbrace{\frac{\partial \dot x_m}{\partial \dot x_i}}_{\delta_{mi}} \dot x_n + \sum_{m,n} a_{mn} \dot x_m \underbrace{\frac{\partial \dot x_n}{\partial \dot x_i}}_{\delta_{ni}}$$

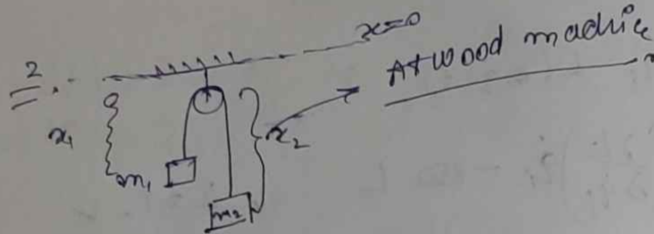
$$\Rightarrow \sum_n a_{in} \dot x_n + \sum_m a_{mi} \dot x_m \Rightarrow 2\sum_n a_{in} \dot x_n$$

$$\therefore \sum_i \left(\frac{\partial T}{\partial \dot q_i}\right) \dot q_i - L$$

$$\Rightarrow \sum_i 2 \sum_n a_{in} \dot x_n \dot x_i - L$$

$$\Rightarrow 2T - L \Rightarrow \underline{T + V}.$$

2.



$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$V = -m_1 g x_1 - m_2 g x_2$$

$$x_1 + x_2 = l$$

$$x_2 = l - x_1$$

$$\dot{x}_2 = -\dot{x}_1$$

$$L = T - V$$

$$\Rightarrow \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + (m_1 - m_2) g x_1 + m_2 g l$$

$$L = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\Rightarrow (m_1 + m_2)\ddot{x}_1 - (m_1 - m_2) g = 0$$

$$\ddot{x}_1 = \frac{(m_1 - m_2) g}{(m_1 + m_2)}$$

3. $L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$

(a) $$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}(m\dot{x}) + kx = 0$$

$$m\ddot{x} + kx = 0$$

(b) $E = p\dot{q} - L$ $\quad$ $(E = H)$

$$\Rightarrow \frac{\partial L}{\partial \dot{x}} \Rightarrow m\dot{x}$$

$$\left(\frac{\partial L}{\partial \dot{x}}\right)\dot{x} \Rightarrow m\dot{x}^2$$

$$H = p\dot{x} \cdot L$$

$$\Rightarrow m\dot{x}^2 - \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k x^2$$

$$E \Rightarrow \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k x^2$$

(c) $L_1 = L + \alpha t^2$

$$\Rightarrow L + \frac{d}{dt}\left(\frac{\alpha t^3}{3}\right) \qquad f = \frac{\alpha t^3}{3}$$

$$\Rightarrow L + \frac{df}{dt}$$

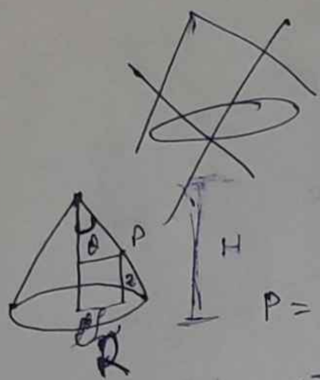
$\frac{\partial L_1}{\partial t} \neq 0$, but E is still conserved

$$\left(\frac{\partial L_1}{\partial x}\right)\dot{x} - L_1 \Rightarrow \left(\frac{\partial L}{\partial x}\right)\dot{x} - L - \alpha t^2$$

$$\Rightarrow E - \alpha t^2$$

~~Not Conserved~~

4.



$P = \rho\hat{\rho} + z\hat{z}$

$\cancel{\rho\cos\phi\hat{x}}$

$\Rightarrow \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} + z\hat{z}$

$\dfrac{\rho}{(h-z)} \Rightarrow \tan\theta$

$\vec{r} \Rightarrow (h-z)\tan\theta\cos\phi\,\hat{x} + (h-z)\tan\theta\sin\phi\,\hat{y} + z\hat{z}$

$L = \frac{1}{2}m\dot{\vec{r}}^2 - mgz$

$\dot{\vec{r}} \Rightarrow -(h-z)\,t_\theta\, s_\phi\,\dot{\phi}\,\hat{x} + (h-z)\,t_\theta\, c_\phi\,\dot{\phi}\,\hat{y} + \dot{z}\hat{z}$

$- \dot{z}\, t_\theta\, c_\phi\,\hat{x} - \dot{z}\, t_\theta \sin\phi\,\hat{y}$

$L = \frac{1}{2}m\left[\sec^2\theta\,\dot{z}^2 + (h-z)^2\tan^2\theta\,\dot{\phi}^2\right] - mgz$

$\dfrac{\partial L}{\partial \phi} =$ conserved.

$\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\phi}}\right) - \dfrac{\partial L}{\partial \phi} = 0 \rightarrow m\sec^2\theta\,\ddot{z} + g = 0 + (h-z)\,t_\theta^2\,\dot{\phi}^2$

$\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{z}}\right) - \dfrac{\partial L}{\partial z} = 0 \Rightarrow m(h-z)^2 t_\theta^2\,\dot{\phi} = \text{const}.$

$H = \left(\dfrac{\partial L}{\partial \dot{\phi}}\right)\dot{\phi} + \left(\dfrac{\partial L}{\partial \dot{z}}\right)\dot{z} - L$

properties of EL eq^{n}s.

$\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) - \dfrac{\partial L}{\partial q_i} = 0$

$L' = L + \dfrac{df}{dt}(q_i, t)$

$\dfrac{d}{dt}\left(\dfrac{\partial L'}{\partial \dot{q}_i}\right) - \dfrac{\partial L'}{\partial q_i} = 0 \quad \forall\, i = 1, 2 \,\ldots$

$\dfrac{d}{dt}\dfrac{\partial}{\partial \dot{q}_i}\left(\dfrac{\partial f}{\partial t}\right) - \dfrac{\partial}{\partial q_i}\left(\dfrac{df}{dt}\right) = 0$

$f(q_1, q_2, q \,\ldots\, q_n, t)$

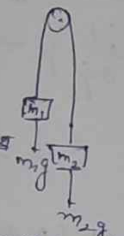
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$T - m_1 g = m_1 \ddot{x}_1$

$m_2 g - T = m_2 \ddot{x}_2$

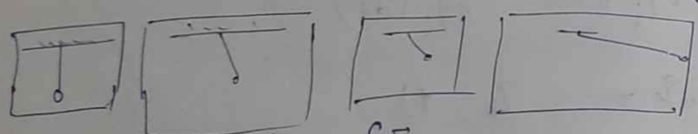
$x_1 + x_2 = \text{Const}$

$\dot{x}_1 = -\dot{x}_2$



total virtual ~~work~~ work

for usual mechanic sys the ~~virtual disp~~ done by the constraint force (non dissipative both time dependent on independent) always vanishes



virtual displacement $\rightarrow \delta \vec{r}$

$$\sum_i m_i \ddot{\vec{r}}_i = \sum (\vec{F}_e + \vec{F}_c)$$

e → external
c → Constraint

$$\sum_i \delta \vec{r}_i \cdot m_i \ddot{\vec{r}}_i = \sum_i (\vec{F}_{e_i} + \vec{F}_{c_i}) \cdot \delta \vec{r}_i$$

$$\Rightarrow \sum_i (\vec{F}_{ei} - m_i \ddot{\vec{r}}_i) \cdot \delta \vec{r}_i = 0$$

D'Alembert's Principle;

8/11/23

D'Alembert's principle.

$$\sum_i \vec{F}_{ci} \, d\vec{r}_i = 0 \qquad \Big| \qquad \sum_i \vec{F}_i \cdot d\vec{r}_i = 0$$

$$\sum_i (m_i \ddot{\vec{r}}_i - \vec{F}_{ci}) \cdot \delta \vec{r}_i = 0$$

for 1 particle :-

$$(m\ddot{\vec{r}} - \vec{F}_c) \cdot \delta \vec{r} = 0$$

$$x_i = x_i(q_1, q_2, q_3 - \dots q_n, t)$$

$$\dot{x}_i = \frac{dx_i}{dt} = \frac{\partial x_i}{\partial t} + \sum_k \frac{\partial x_i}{\partial q_k} \dot{q}_k$$

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_m} = \frac{\partial x_i}{\partial q_k} \delta_{km} \Rightarrow \frac{\partial x_i}{\partial q_m} \quad (m=k)$$

$$T = \frac{1}{2} m \sum_i \dot{x}_i^2 \Rightarrow \frac{1}{2} m \sum_i \left[\left(\frac{\partial x_i}{\partial t} + \sum_k \frac{\partial x}{\partial q_k} \dot{q}_k \right) \left(\frac{\partial x_i}{\partial t} + \sum_j \frac{\partial x_i}{\partial q_j} \dot{q}_j \right) \right]$$

$$\frac{\partial T}{\partial \dot{x}_k} = \frac{\partial}{\partial \dot{x}_k} \left(\frac{1}{2} m \sum_i v_i^2 \right)$$

$$\Rightarrow \frac{1}{2} m \left[\frac{\partial}{\partial \dot{x}_k} \sum \dot{x}_i^2 \right]$$

$$\Rightarrow \frac{1}{2} m [2\dot{x}_k] = m\dot{x}_k$$

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_i m v_i \frac{\partial v_i}{\partial \dot{q}_k} = \sum_i m v_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k} \Rightarrow \sum_i m v_i \frac{\partial x_i}{\partial q_k}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_i m v_i \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right) + \sum m \dot{v}_i \frac{\partial x_i}{\partial q_k}$$

$$\# \; \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right) \Rightarrow \left[\frac{\partial}{\partial t} + \sum_l \frac{\partial}{\partial q_l}\dot{q}_l\right]\left(\frac{\partial x_i}{\partial q_k}\right)$$

$$\Rightarrow \frac{\partial}{\partial q_k}\frac{\partial x_i}{\partial t} + \frac{\partial}{\partial q_k}\sum_l \frac{\partial x_i}{\partial q_k}\dot{q}_l \Rightarrow \frac{\partial}{\partial q_k}\left(\frac{\partial x_i}{\partial t}\right)$$

$$\Rightarrow \sum \frac{\partial}{\partial q_k}\left(\frac{1}{2} m v_i^2\right) + \sum m \dot{v}_i \frac{\partial x_i}{\partial q_k}$$

$$\Rightarrow \cancel{\Sigma} \frac{\partial}{\partial q_k}\sum \frac{1}{2} m v_i^2 + \sum m \dot{v}_i \frac{\partial x_i}{\partial q_k}$$

$$\Rightarrow \frac{\partial T}{\partial q_k} + \sum m \dot{v}_i \frac{\partial x_i}{\partial q_k}$$

$$dx_i = \frac{\partial x_i}{\partial t}dt + \frac{\sum \partial x_i}{\partial q_j}\partial q_j$$

$$\delta x_i = dx_i\Big|_{t = \text{frozen}} = \sum \frac{\partial x_i}{\partial q_j}\delta q_j$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \sum m \dot{v}_i \frac{\partial x_i}{\partial q_k}$$

$L = T - V$

$$\delta x_i = \sum_j \frac{\partial x_i}{\partial q_j}\delta q_j$$

$$\frac{\partial V}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_k}\right)$$

$$(m\ddot{\vec{r}} - \vec{F}_c)\cdot\delta\vec{r} = 0$$