Art Gallery Theorems and Algorithms

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Art Gallery

- **Problem**: determine the minimum number of guards sufficient to cover the interior of an n-wall art gallery
  - Victor Klee, 1973
  - Vasek Chvatal, 1975

Main reference for this material:

Art Gallery Theorems and Algorithms, Joseph O'Rourke, Oxford University Press, 1987

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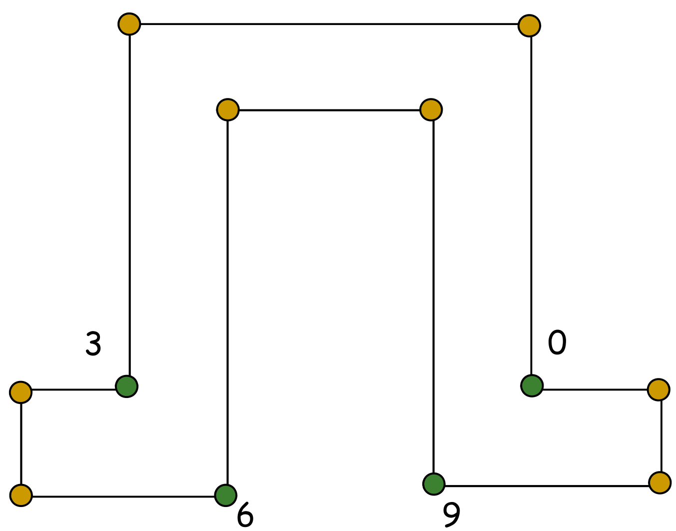
Contents

Definitions

- P is a simple polygon (i.e., does not cross over itself)
- Point $x \in P$ "covers" a point $y \in P$ if $xy \subseteq P$
- Let $G(P)$ be the minimum number k of points of P, such that for any $y \in P$, some $x=x_1 \ldots x_k$ covers y
- Let $g(n)$ be the $\max(G(P))$ over all polygons of n vertices
  - Thus, $g(n)$ guards are occasionally necessary and always sufficient

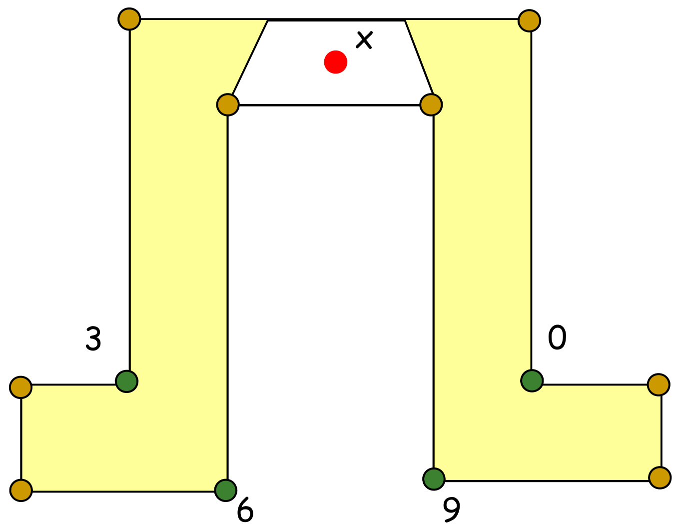
Guard Placement

- 1. Can we just place one guard on every 3rd vertex?



Guard Placement

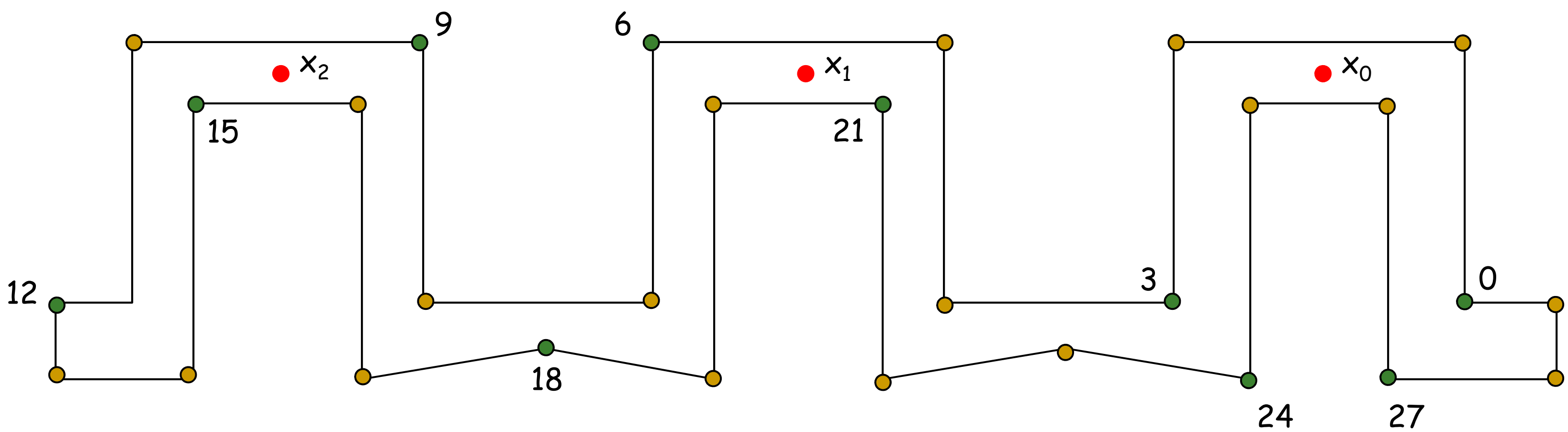
- 1. Can we just place one guard on every 3rd vertex? – No!



x is not visible

Guard Placement

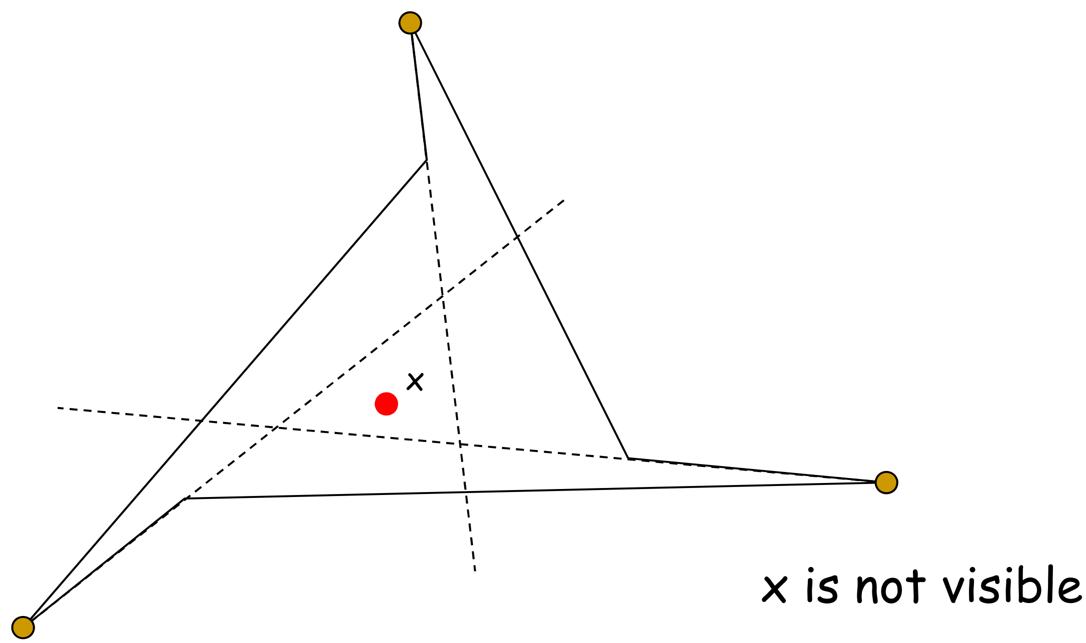
- 1. Can we just place one guard on every 3rd vertex? – No!



one of x_i is not visible

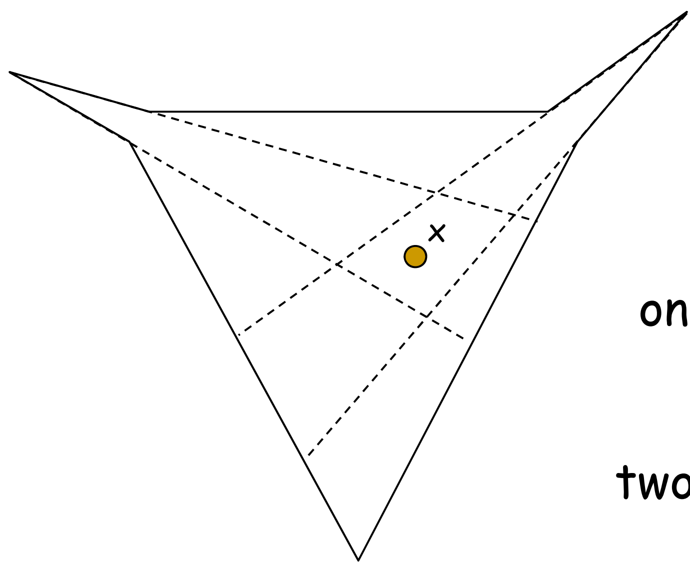
Guard Placement

- 2. If guards placed so they can see all the walls, does that imply they can see all the interior?
  - No!



Guard Placement

- 3. If we restrict guards to vertices, is $g_v(n) = g(n)$?
  - In general, yes, equal for $g(n) = \max(G(P))$



one point guard x

or

two vertex guards?

Art Gallery

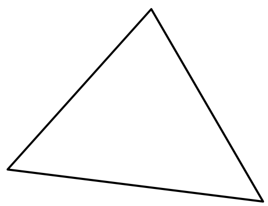
- **Theorem**: floor(n/3) guards are occasionally necessary and always sufficient to cover a polygon of n vertices
  - "Chvatal's Art Gallery Theorem"
  - "Watchman Theorem"

Fisk's Proof

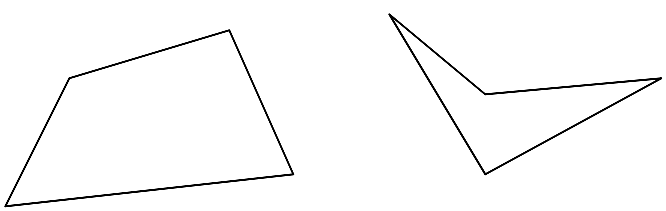
- $g(n) = floor(n/3)$
  - Published in 1978 (three years are Chvatal's original proof, but it is much more compact)
- Necessity
  - $g(n) \geq floor(n/3)$ are sometimes necessary
- Sufficiency
  - $g(n) \leq floor(n/3)$ are always sufficient

Necessity: Base Cases

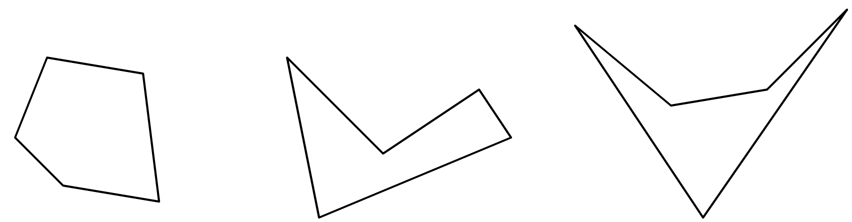
- n=3



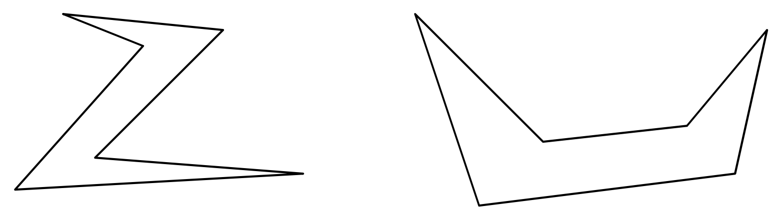
- n=4



- n=5

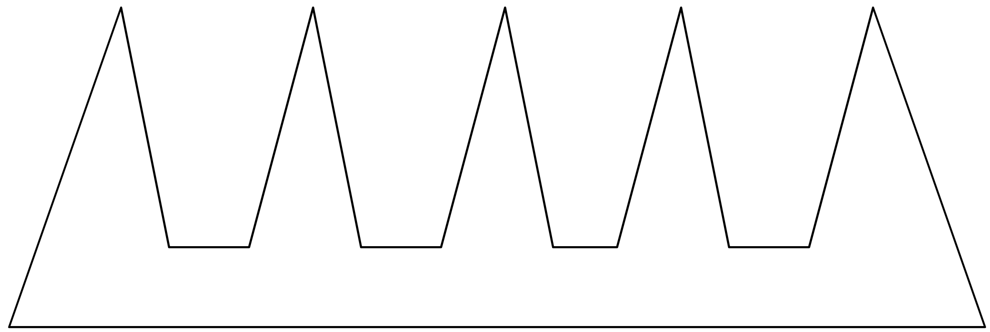


- n=6



Necessity: Base Cases

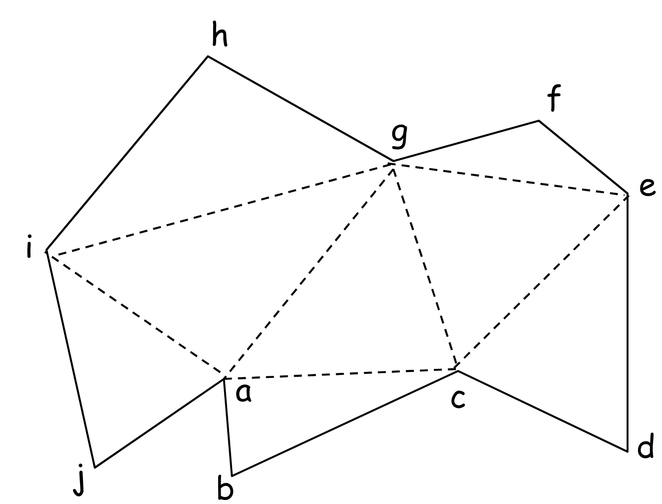
- $n \geq 6$



$g(n) \geq floor(n/3)$

Sufficiency: Fisk's Proof

- Step 1 of 3
  - Triangulate the polygon P by adding only internal diagonals

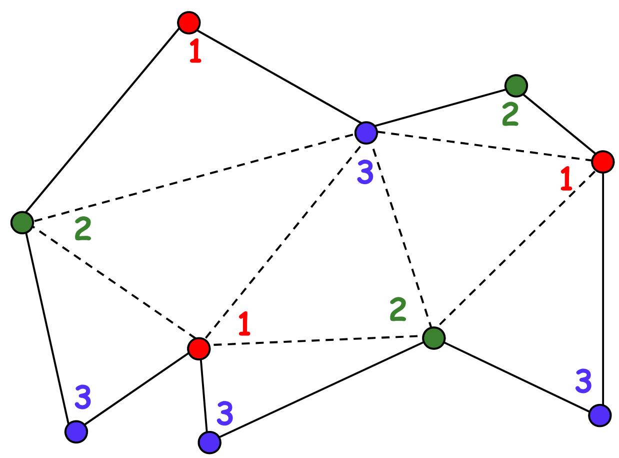


Triangulation Theorem

- A polygon of n-vertices may be partitioned into n-2 triangles by the addition of n-3 internal diagonals

Sufficiency: Fisk's Proof

- Step 2 of 3
  - Perform a 3-coloring of the triangulation graph
    - Using three colors, no two adjacent nodes have same color



Four Color Theorem

- Problem stated in 1852 by Francis Guthrie and Augustus De Morgan
  - "Given a map on a flat plane, what is the minimum number of colors needed to color the different regions of the map in such a way that no two adjacent regions have the same color."

Four Color Theorem

- Several attempted proofs and algorithms
  - Kempe (1879), Tait (1880), Birkhoff (1922), …
- Appel and Haken - first complete proof (1976)
- Robertson, Sanders, Seymour, and Thomas - second more compact proof (1994)

Four Color Theorem

- The proof creates a large number of cases (~1700 for Appel-Haken and ~600 for Robertson et al.)
- A computer is used to rigorously check the cases
- Solution is (still) controversial because of the use of a computer

Sufficiency: Fisk's Proof

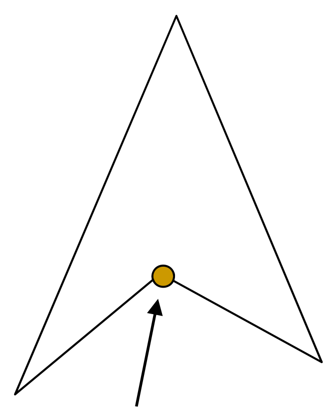
- Step 3 of 3
  - Note that one of three colors must be used no more than floor(1/3) of the time
    - Let a,b,c be # of nodes of each color
    - $a \leq b \leq c$ and $n = a + b + c$
    - If $a > n/3$, then $(a+b+c) \geq n$
    - Thus $a \leq \text{floor}(n/3)$
    - Since each triangle is a complete graph, each triangle has a node of color 'a'
    - Since each triangle is convex and the triangles partition all of P, at most 'a' guards are necessary!

Fisk's Proof

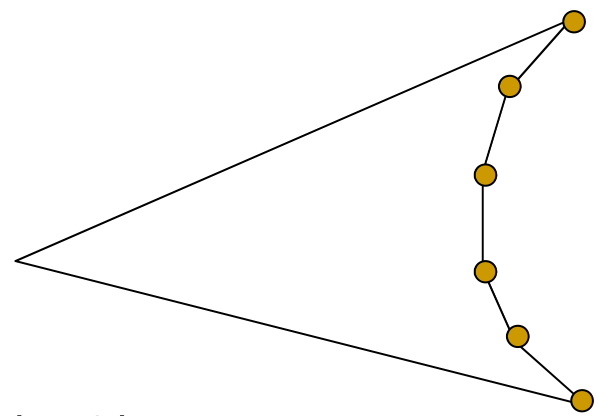
- Necessity
  - $g(n) \geq floor(n/3)$ are sometimes necessary
- Sufficiency
  - $g(n) \leq floor(n/3)$ are always sufficient
- Thus, $g(n) = floor(n/3)$
- $O(n\log n)$ overall algorithm

Reflex Vertices

- We wish to investigate the art gallery question as a function of r (the number of reflex vertices of a polygon)



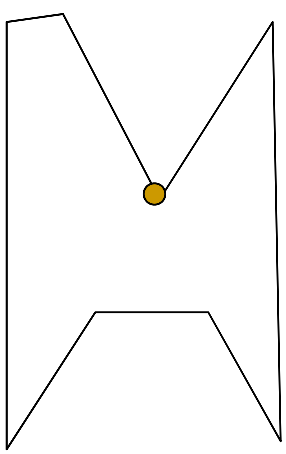
reflex vertex



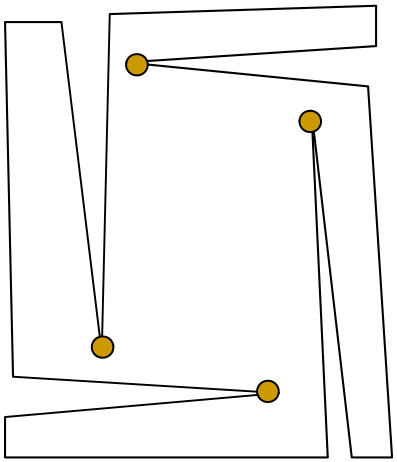
$r \le (n-3)$

Reflex Vertices

- Necessity
  - How many reflex-vertex guards are necessary?



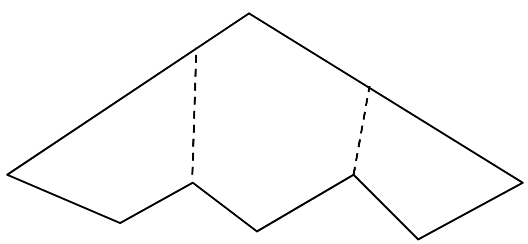
1 needed



r needed

Reflex Vertices

- Necessity
  - r guards are sometimes necessary
- Sufficiency
  - Place 1 guard at each reflex vertex
    - Proved via a convex partitioning of the polygon P
    - Any polygon P can be partitioned into at most r+1 convex pieces

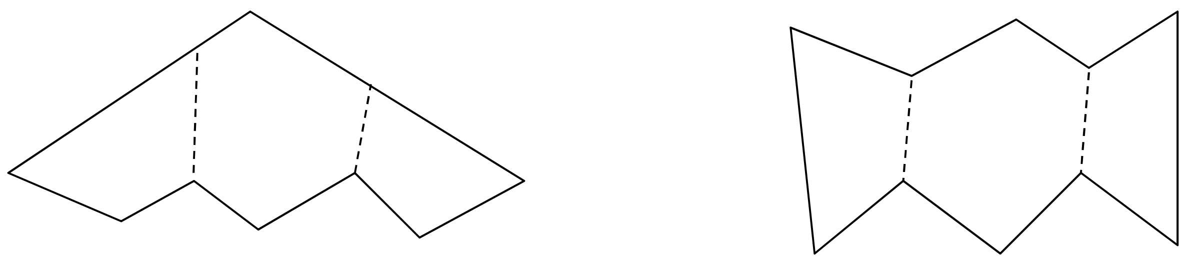


Reflex Vertices

- Necessity
  - r guards are sometimes necessary
- Sufficiency
  - Place 1 guard at each reflex vertex
- Theorem
  - r guards are occasionally necessary and always sufficient to see the interior of a n-gon of $r \geq 1$ reflex vertices

Convex Partitioning

- Naïve Algorithm (Chazelle 1980)



- Because at most two reflex vertices can be resolved by a single cut, the minimum number of pieces is m=ceil(r/2)+1
- This approach achieves no more than $r+1 \leq 2m$ in $O(rn)=O(n^2)$ time

Convex Partitioning

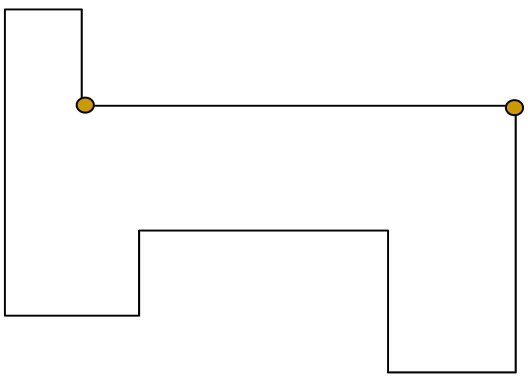
- A fast algorithm: $O(n \log \log n)$
  - Any triangulation can be divided into $2r+1$ convex pieces by removing diagonals

Convex Partitioning

- Chazelle 1980
  - $O(n^3)$ optimal algorithm using dynamic programming
  - (description is 97 pages long)

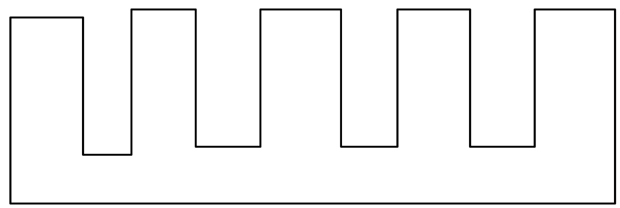
Orthogonal Polygons

- Kahn, Klawe, Kleitman 1980
  - Floor(n/4) guards are occasionally necessary and always sufficient
  - Based on convex quadrilateralization
    - Any orthogonal polygon P is convexly quadrilaterizable (theorem)



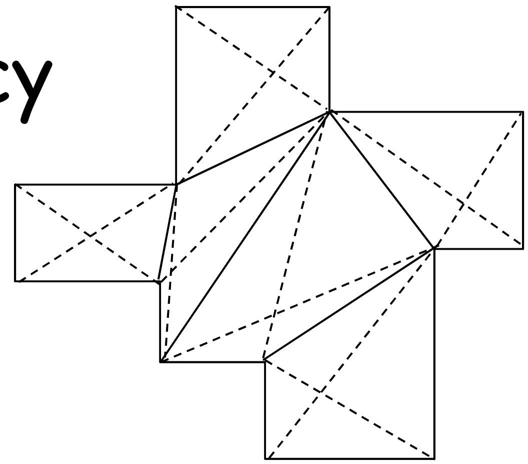
Orthogonal Polygons

- Necessity



$g(n) \geq floor(n/4)$

- Sufficiency



Four-colorable, and thus:

$g(n) \leq floor(n/4)$

- Theorem: $g(n)=floor(n/4)$

Orthogonal Polygons

- In an orthogonal polygon
  - n vertices
  - c interval vertices with $\pi/2$
  - r interval vertices with $3\pi/2$
  - $n = c + r$
  - sum of internal angles $(n-2)\pi$
  - yields $n=2r+4$
- Theorem restated as $g(n)=\text{floor}(r/2)+1$

Quadrilateralization

- Sacks's Algorithm
  - O(nlogn)
- Lubiw's Algorithm
  - O(nlogn)

Mobile Guards

- Theorem

Shape	Stationary	Mobile
General	floor(n/3)	floor(n/4)
Orthogonal	floor(n/4)	floor((3n+4)/16)

  - In general, only ¾ as many mobile guards are needed as stationary guards

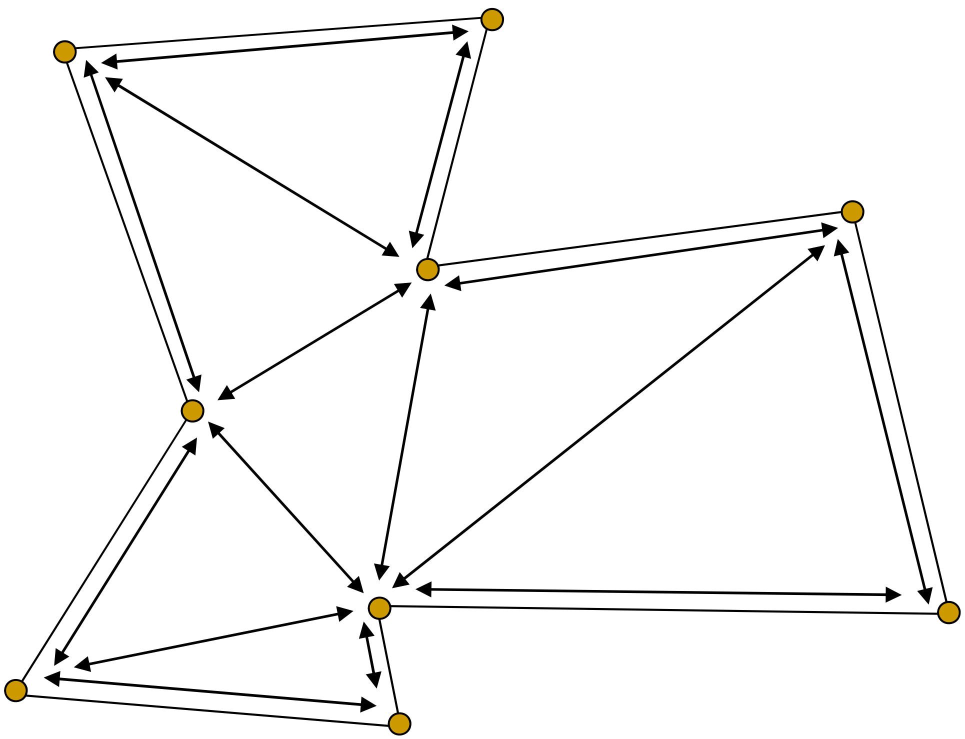
Mobile Guards

- General Polygons

Vertex guards

Edge guards

Diagonal guards

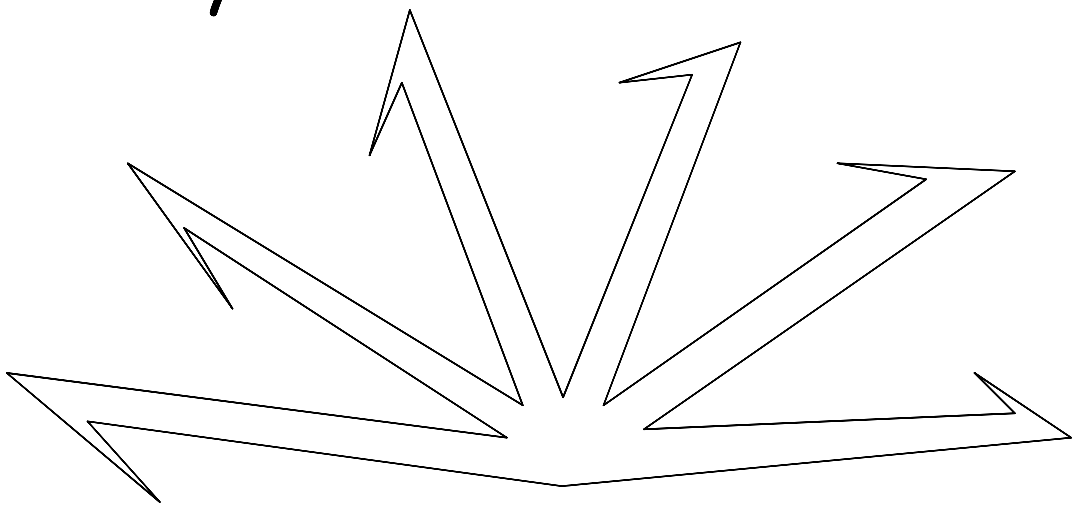


Mobile Guards

- Goal of the proof
  - Given a triangulation graph T
    - Vertex guard = node
    - Edge guard = adjacent arc
    - Diagonal guard = any arc
  - The analog of covering is domination
  - A collection of guards $C=\{g_1,\dots,g_k\}$ dominates triangulation graph T if every face has at least one of its three nodes in some $g_i \in C$.

Mobile Guards

- Necessity



Polygon that requires floor(n/4) edge, diagonal (or line) guards

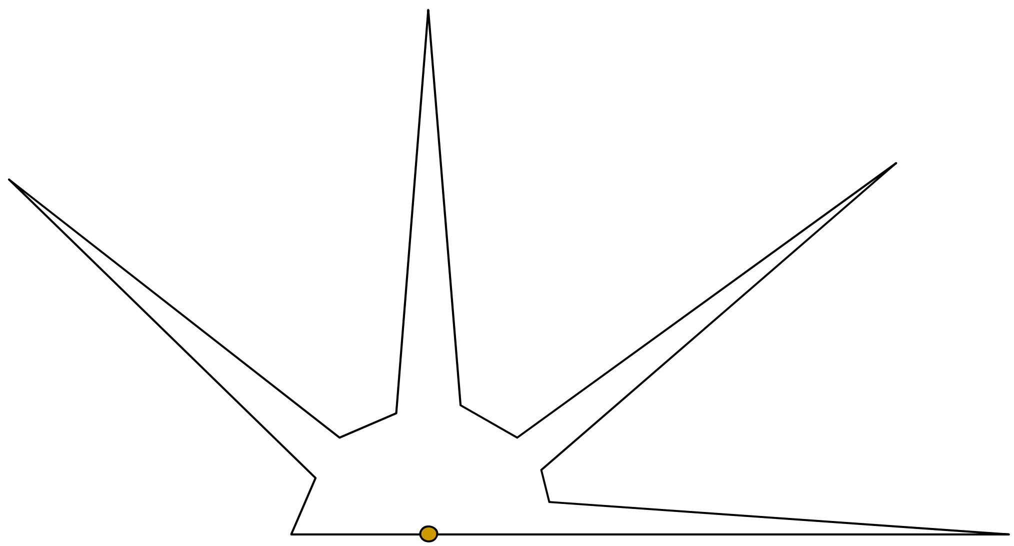
- Sufficiency: a little more complicated…

Miscellaneous Shapes

- (General polygon, convex, orthogonal)
- Star, spiral, monotone

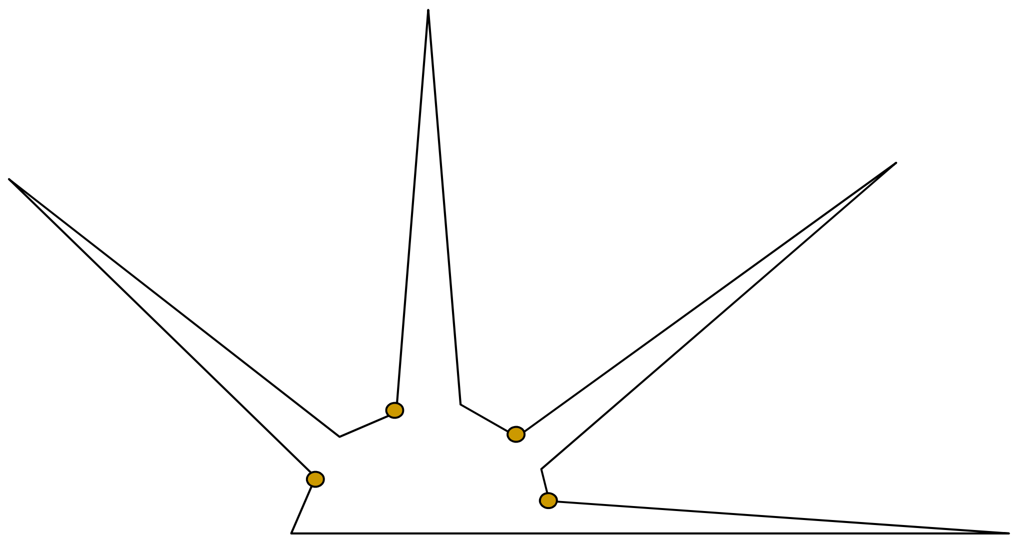
Star Shape

- A star polygon P is a polygon that may be covered by a single point guard



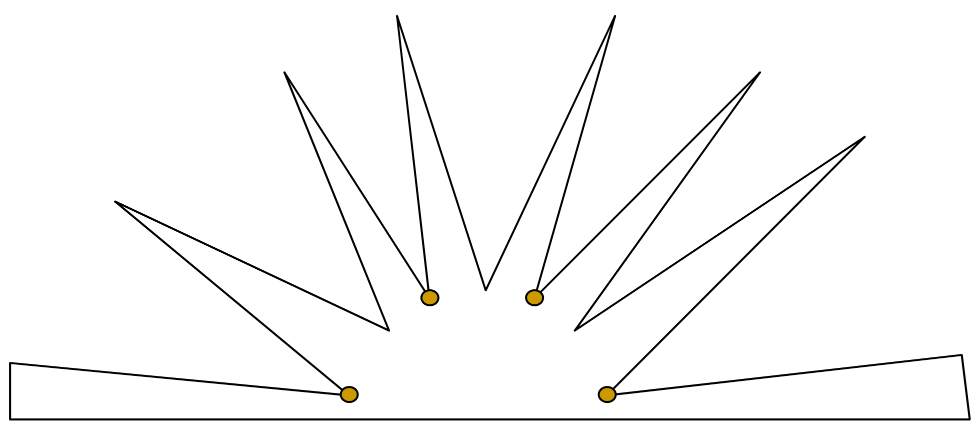
Star Shape

- Toussaint's Theorem
  - A star polygon P requires floor(n/3) vertex guards



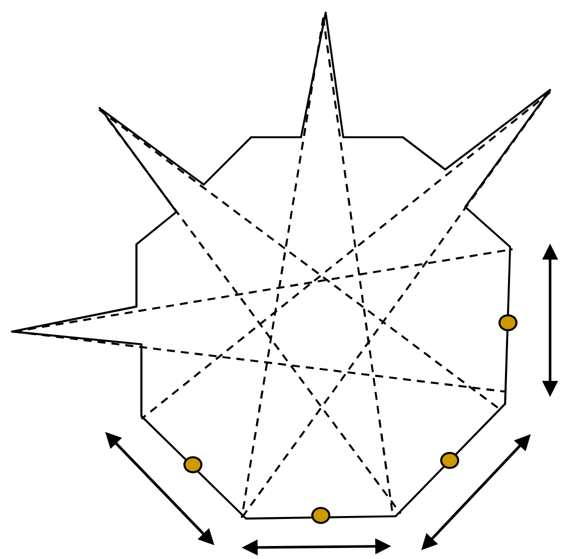
Star Shape

- Toussaint's Theorem
  - A star polygon P requires floor(r/2)+1 reflex guards



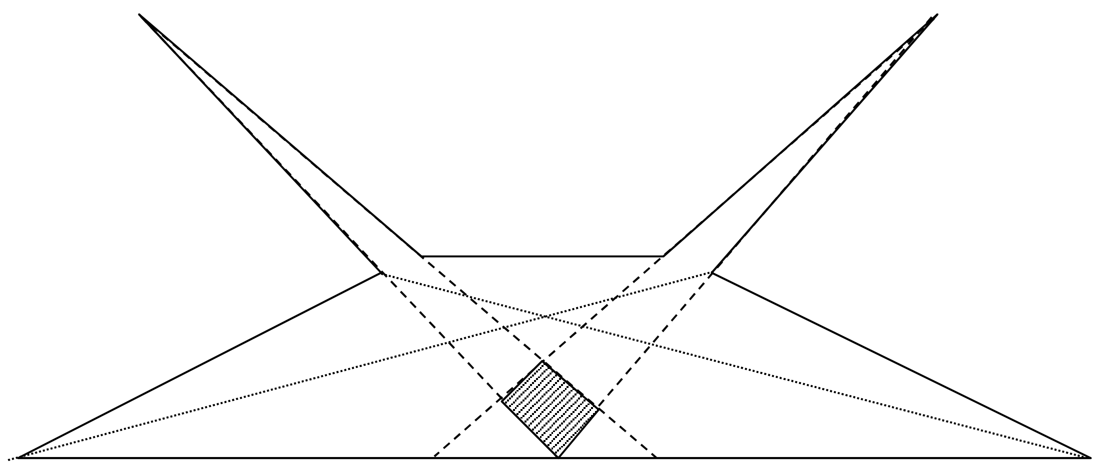
Star Shape

- Toussaint's Theorem
  - A star polygon P requires at least floor(n/5) edge guards



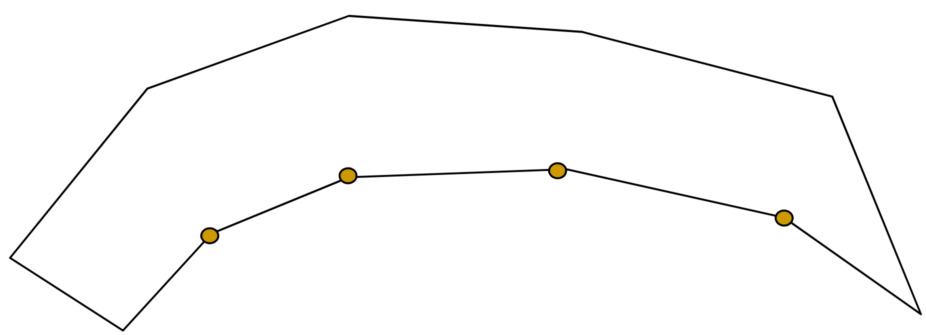
Star Shape

- Toussaint's Theorem
  - For a star polygon P
    - Unrestricted patrol, one line guard is needed
    - Restricted to diagonal lines, two are needed



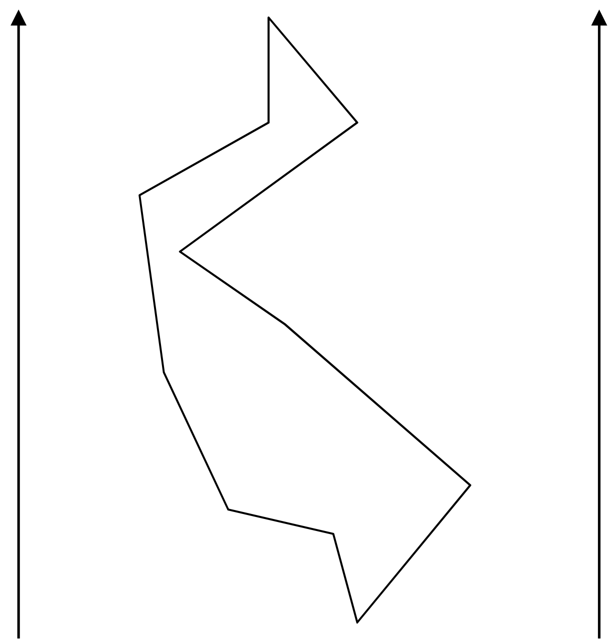
Spiral Polygon

- A spiral polygon is a polygon with at most one chain of reflex vertices



Monotone Polygon

- A polygon with no “doubling back” with respect to a line



Spiral and Monotone Polygons

- Aggarwal's Theorem
  - floor(n/3) vertex guards are needed
  - floor(r/2)+1 reflex-vertex guards are needed
  - floor((n+2)/5) diagonals guards are needed

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  - Prison Yard Problem
- Minimal Guards

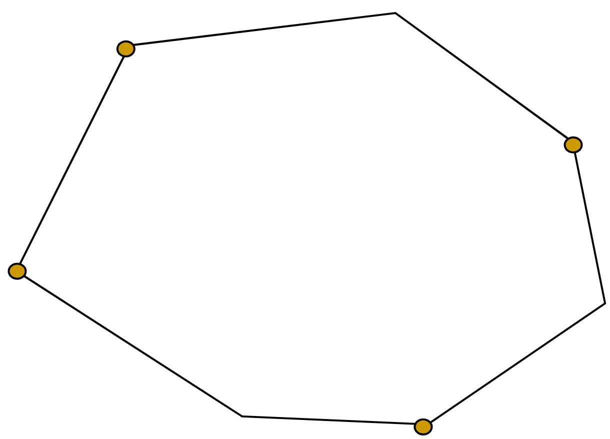
Exterior Visibility

- “Fortress Problem”
- “Prison Yard Problem

(independently stated by Derick Wood and Joseph Malkelvitch, early 1980s)

Fortress Problem

- How many vertex guards are needed to see the exterior of a polygon P?
- Simplex convex polygon

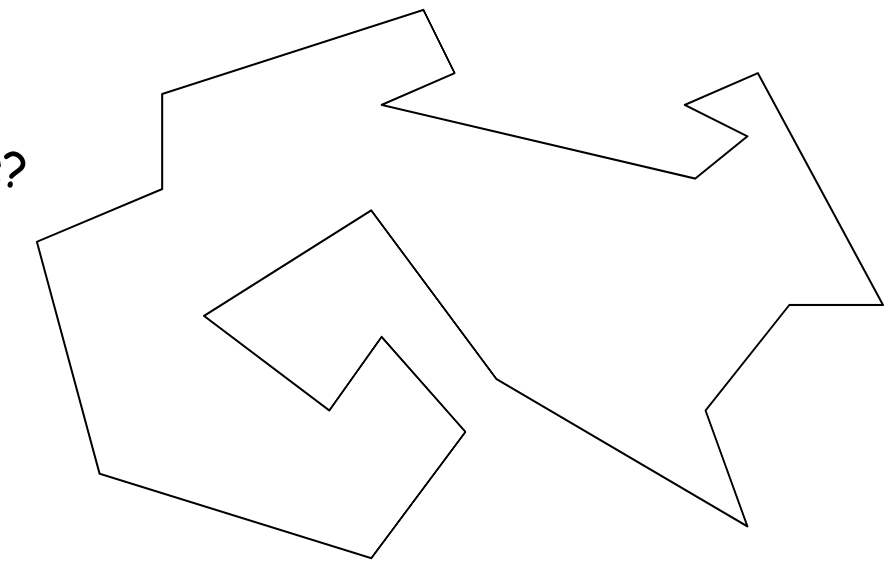


ceil(n/2) vertex guards

Fortress Problem

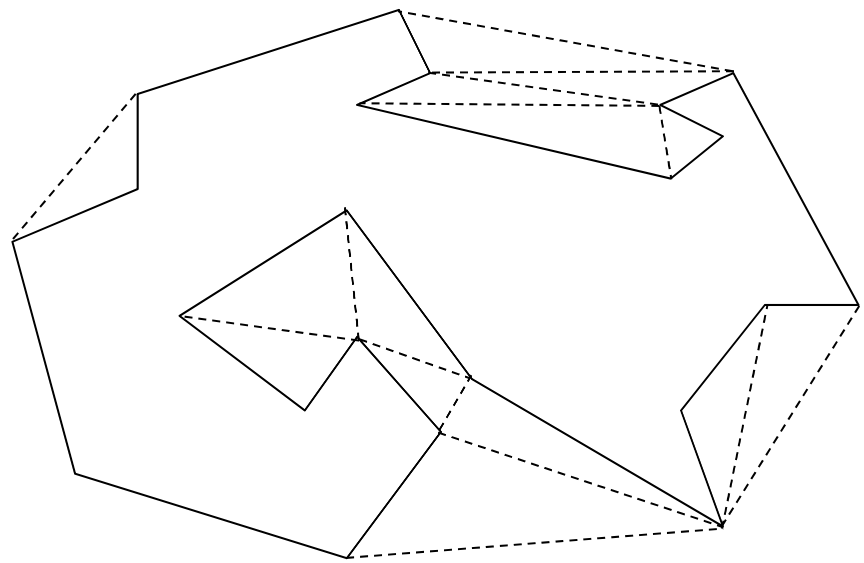
- Arbitrary polygon

ceil(n/2) vertex guards?



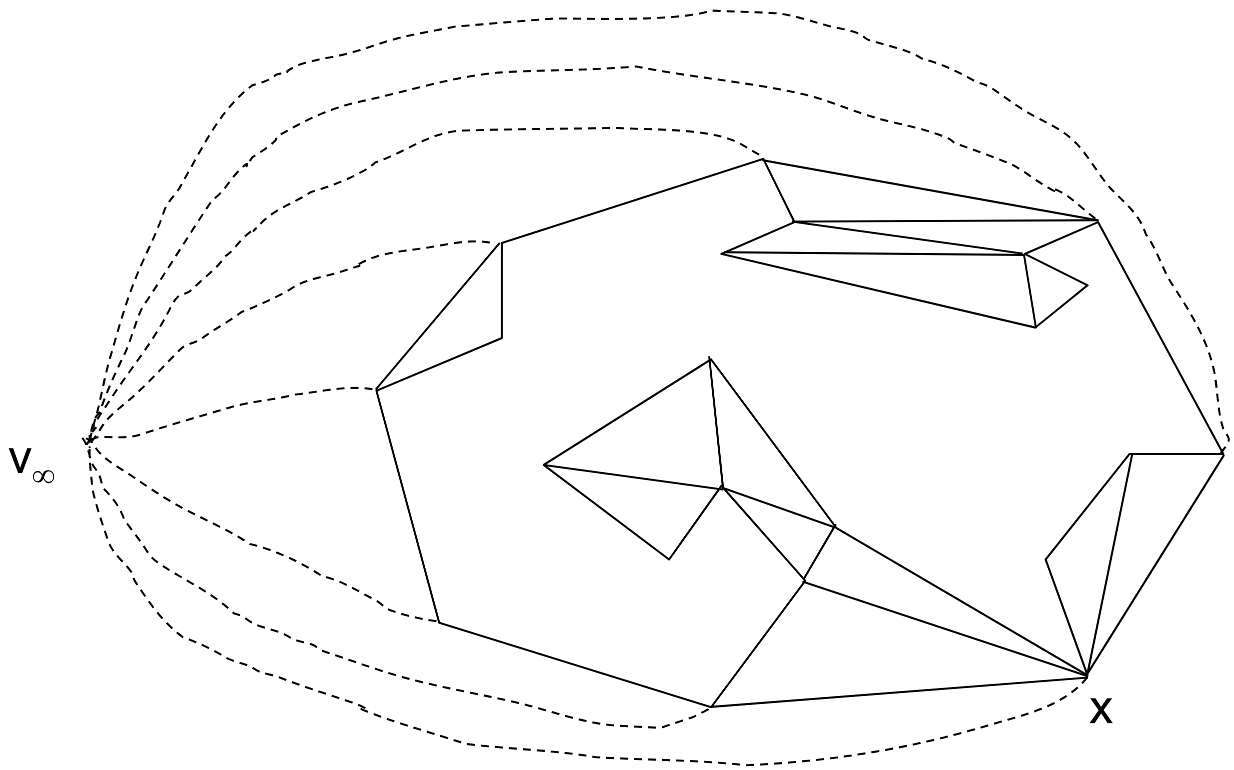
Fortress Problem

- Arbitrary polygon



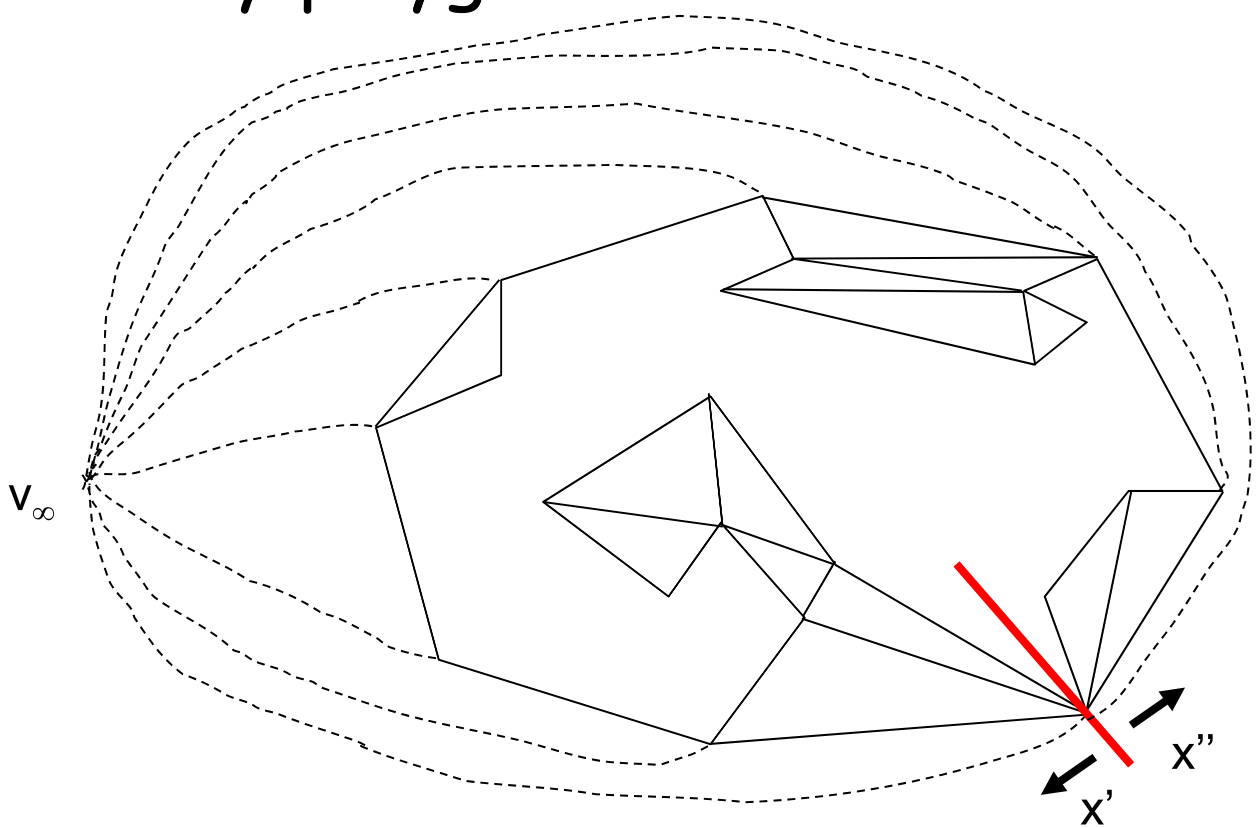
Fortress Problem

- Arbitrary polygon



Fortress Problem

- Arbitrary polygon



Fortress Problem

- Arbitrary polygon
  - Three-color the resulting triangulation graph T (of n+2 nodes)

Fortress Problem

- Arbitrary polygon
  - If least frequently used color is red and v_∞ is **not** red then,
    - floor((n+2)/3) vertex guards are needed

Fortress Problem

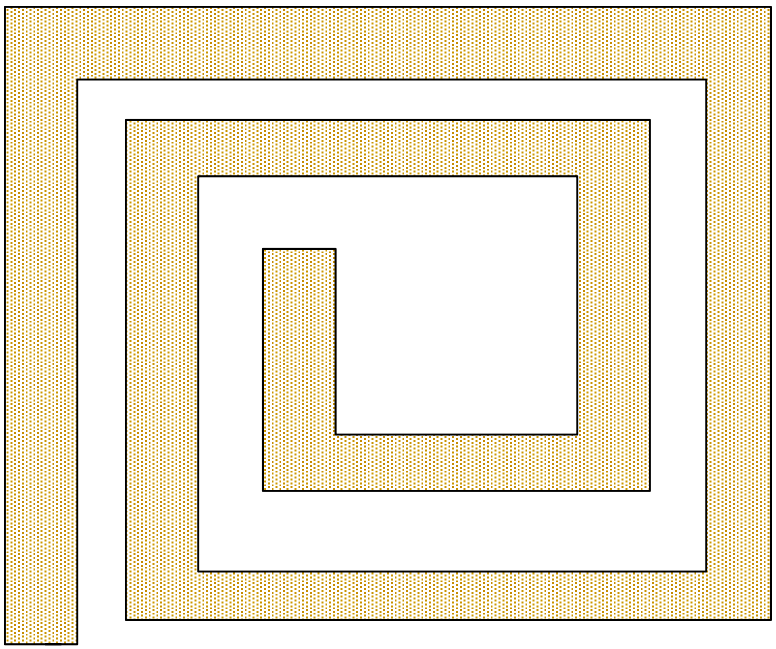
- Arbitrary polygon
  - If least frequently used color is red and v_∞ **is** red then,
    - No guard can be placed at v_∞ because it's not part of original polygon
    - Thus, place guards at second least frequently used color
    - $a \leq b \leq c$ and $a + b + c = n + 2$
    - $a \geq 1$ and $b + c \leq n + 1$
    - $b \leq$ floor((n+1)/2)=ceil(n/2) vertex guards are needed

Fortress Problem

- Arbitrary polygon (Summary)
  - 1. Triangulate the convex hull of the polygon P
  - 2. Add edges from all exterior vertices to new vertex v_∞
  - 3. Split a vertex x into x′ and x″
  - 4. Open-up the convex hull, straighten the lines to v_∞, and form a triangulation graph T of (n+2) nodes
  - 5. Three-color graph T
  - 6. Use least or second least frequently used color
- At most ceil(n/2) vertex guards are needed

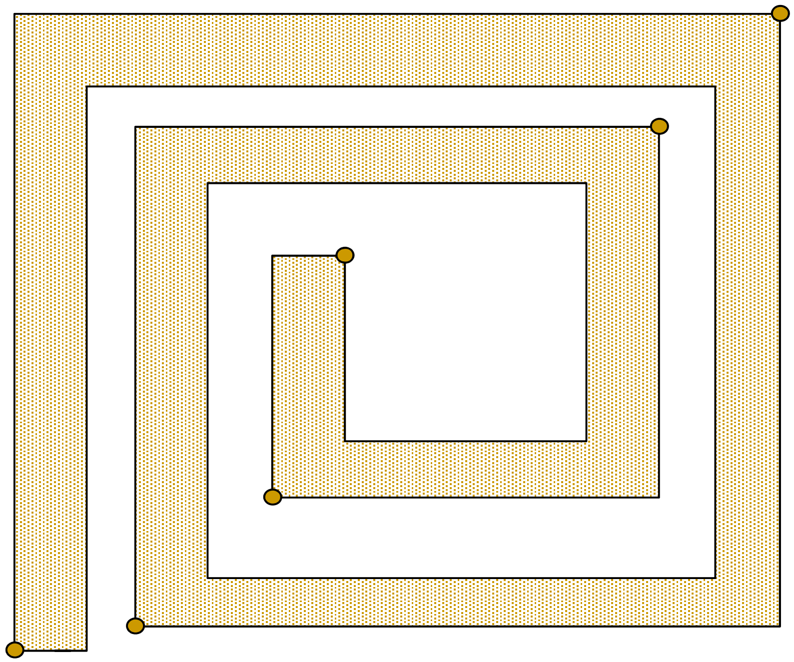
Fortress Problem

- Orthogonal polygon



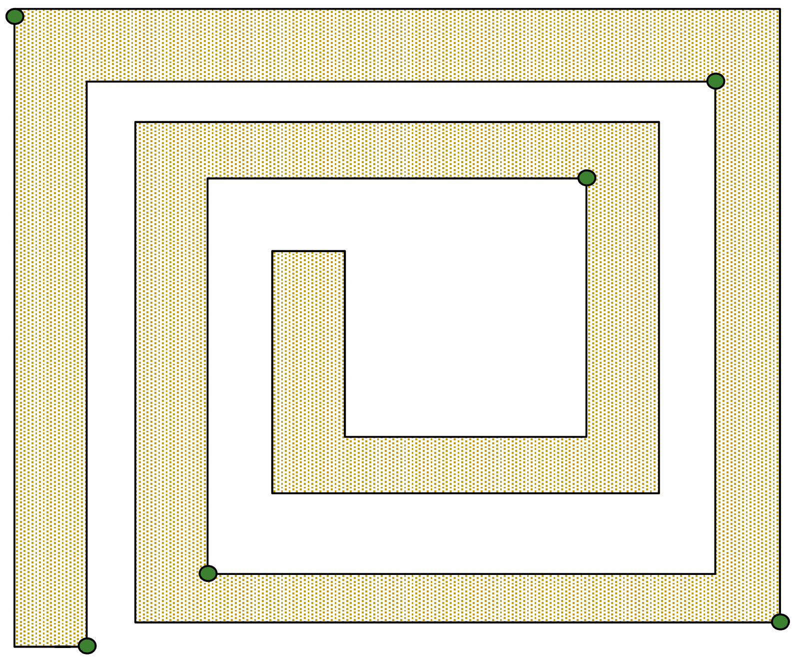
Fortress Problem

- Orthogonal polygon



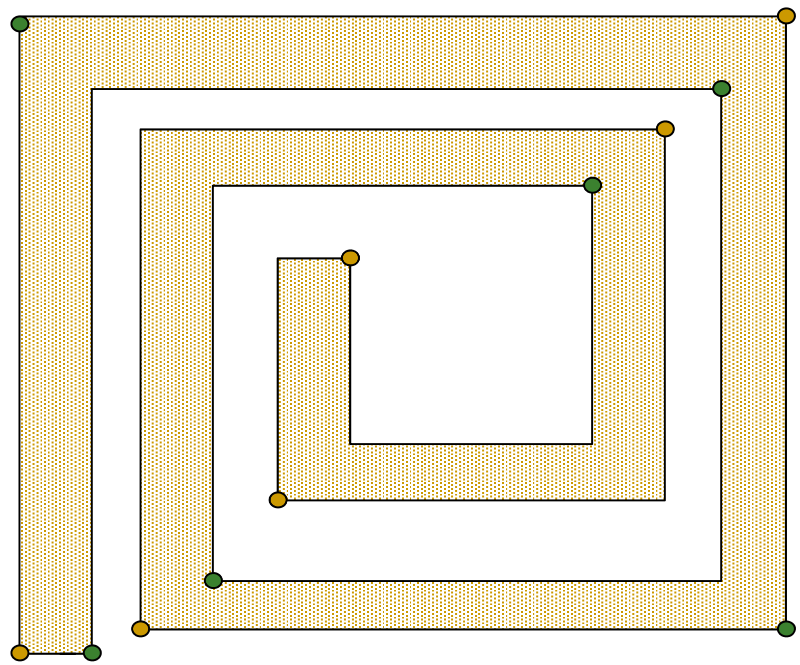
Fortress Problem

- Orthogonal polygon



Fortress Problem

- Orthogonal polygon

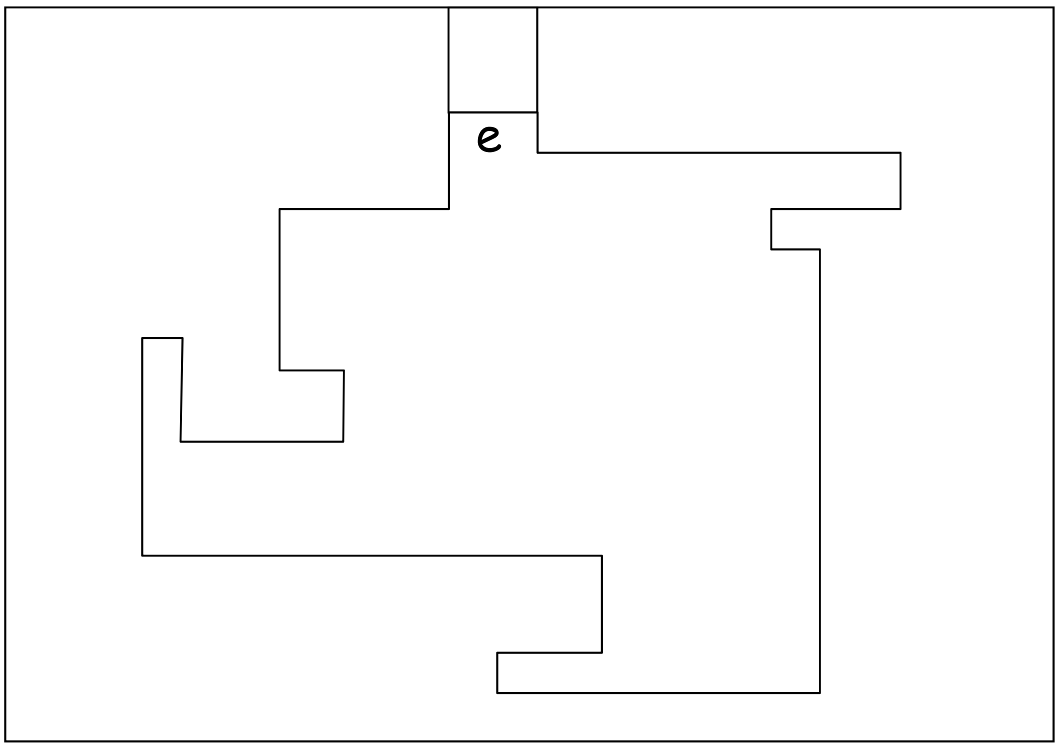


- Solution A
- Solution B

ceil(n/4)+1 vertex guards necessary

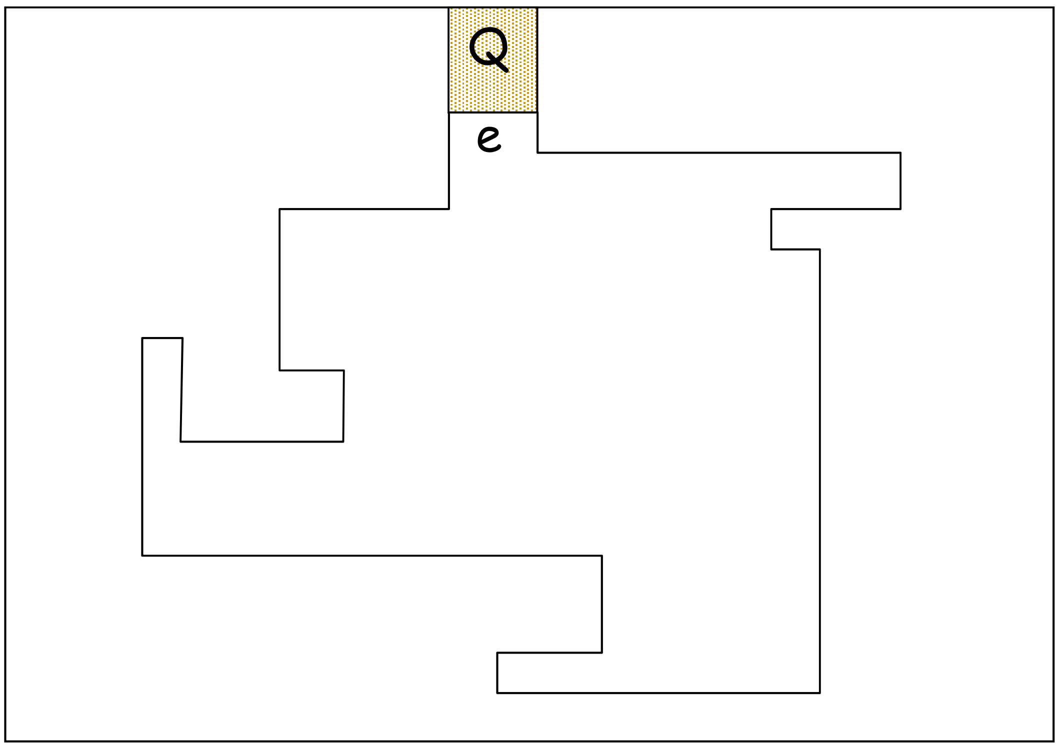
Fortress Problem

- Orthogonal polygon



Fortress Problem

- Orthogonal polygon



Fortress Problem

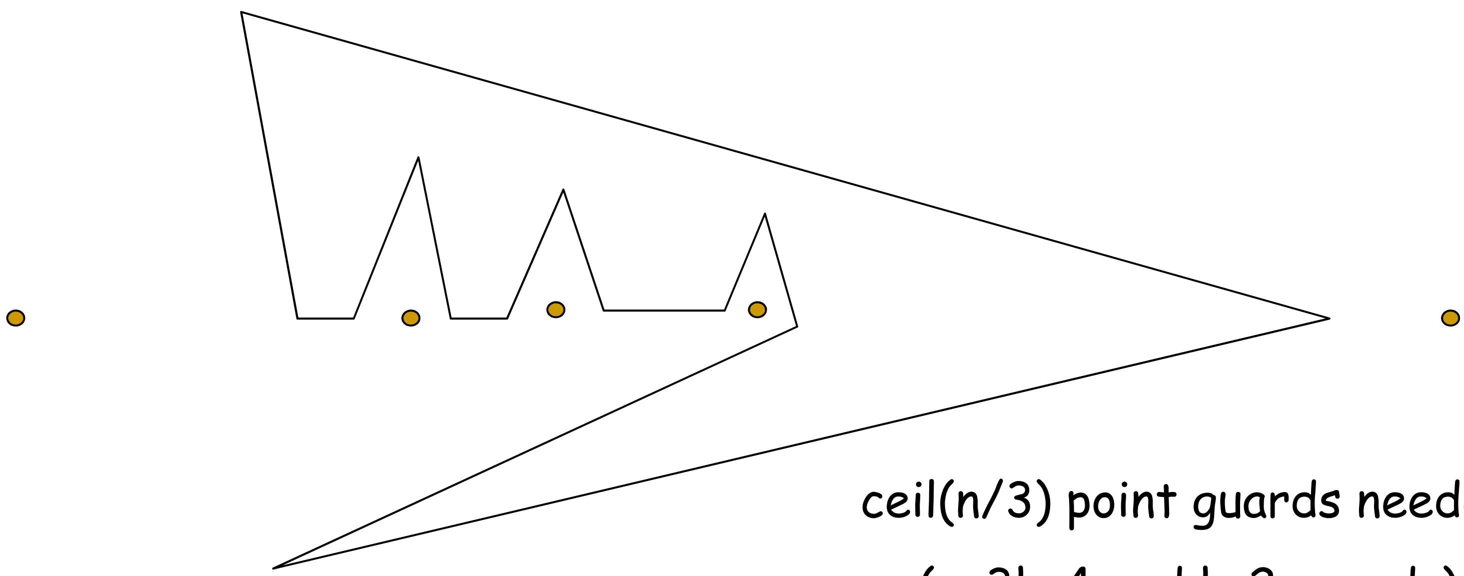
- Orthogonal polygon
  - Interior of new polygon P' coincides with the immediate exterior of P, except for Q which is exterior to both

Fortress Problem

- Orthogonal polygon
  - For P' of n+4 vertices, floor(r/2)+1 or floor((n+4)/4) vertex guards suffice to cover the interior
    - None of the new vertices of P' are reflex vertices
    - Need an additional one for Q
  - Thus, floor(n/4)+2 vertex guards are sufficient
    - For n mod 4=0, ceil(n/4)+1

Fortress Problem

- Guards in the plane
  - Necessity

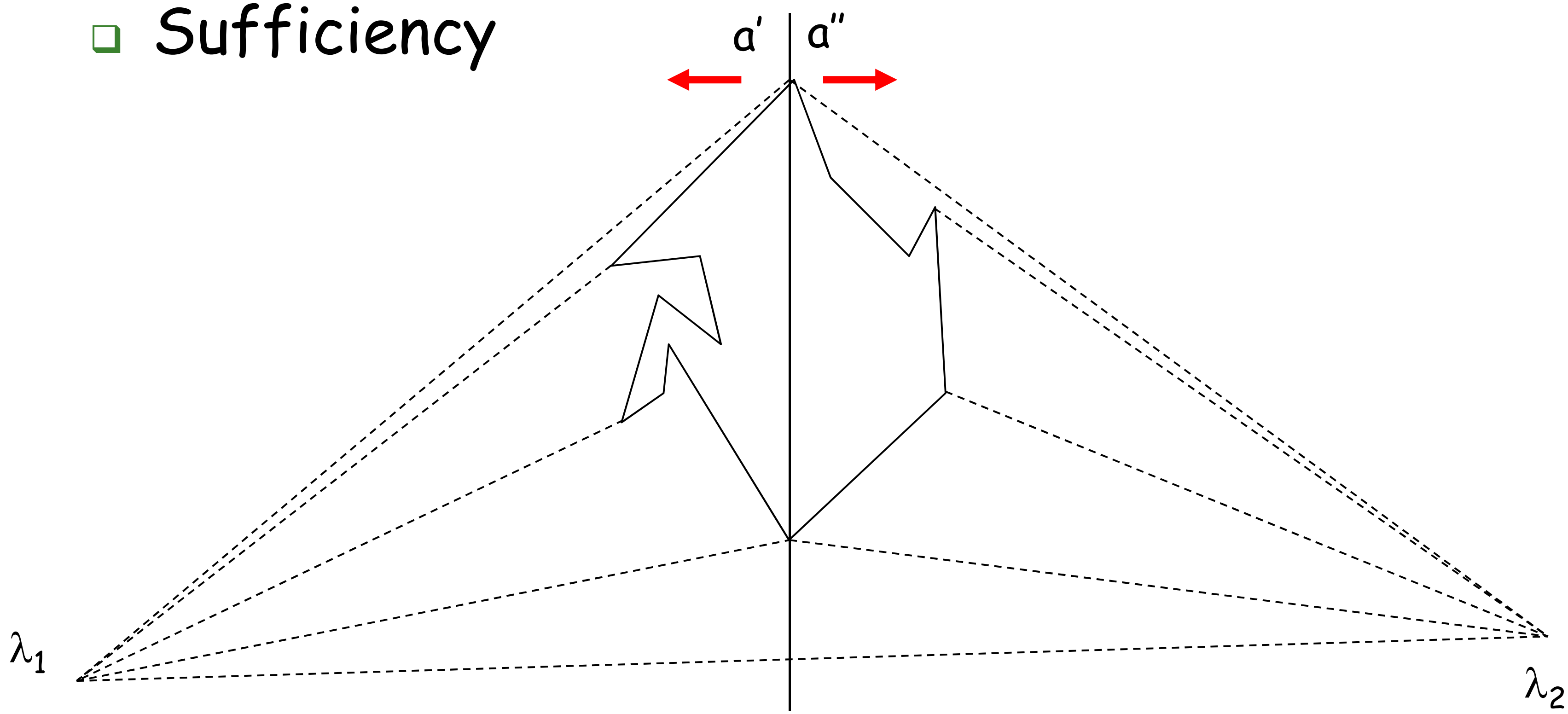


ceil(n/3) point guards needed

(n=3k+4 and k+2 guards)

Fortress Problem

- Guards in the plane
  - Sufficiency



Fortress Problem

- Guards in the plane
  - New triangulated polygon P′ of n+3 vertices
    - floor((n+3)/3) = ceil((n+1)/3) point guards

Fortress Problem

- Guards in the plane
  - More lengthy proof to remove "1/3 of a guard"
    - Add only 2 guards and 3-color triangulation
    - If even hull vertices, trivial
    - If odd hull vertices, need some extra work
  - Result: ceil(n/3) point guards are necessary to cover the exterior of a polygon P of n vertices
    - Nice duality with floor(n/3) for the interior

Prison Yard Problem

- How many vertex guards are needed to simultaneously see the exterior and interior of polygon P?

Prison Yard Problem

- General Polygons
  - Worst-case is a convex polygon
    - ceil(n/2) vertex guards needed
  - Multiply-connected polygons
    - min(ceil(n/2), floor((n+ceil(h/2))/2), floor(2n/3))

Prison Yard Problem

- Orthogonal Polygons
  - floor((7n/16)+5) vertex guards are needed

Fortress/Prison Yard Problem

Problem	Techniques	Guards	Time
Fortress			
General	Triangulation, 3-coloring	ceil(n/2)	O(T)
Orthogonal	L-shaped partition	ceil(n/4)+1	O(T)
Prison Yard			
General	Exterior	ceil(n/2)+r	O(T)
	Triangulation, 4-coloring	floor((n+ceil(h/2))/2)	$O(n^2)$
	Triangulation, 4-coloring	floor(2n/3)	$O(n^2)$
	Exterior, triangulation, 3-coloring	floor(2n/3+1)	O(T)
Orthogonal	Exterior, quad., 4-coloring	floor((7n/16)+5)	O(T)

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Minimal Guard Coverage

- Seek the placement of a minimal number of guards that cover a polygon P
  - In general, a NP-complete problem

Minimal Guard Coverage

Polygon	**Cover**		**Partition**	
	w. Steiner	*w/o Steiner*	*w. Steiner*	*w/o Steiner*
Simple Polygons	NP-hard	NP-complete	?	$O(n^7 \log n)$
Polygons with Holes	NP-hard	NP-complete	NP-hard	NP-complete