Lecture notes on risk management, public policy, and the financial system

Credit risk models

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Key metrics of credit risk

Probability of default π_t defined over a time horizon t, e.g. one year

Exposure at default: amount the lender can lose in default

- For a loan or bond, par value plus accrued interest
- For *OTC derivatives*, also driven by market value
  - Net present value (NPV) $\lesseqqgtr 0$ ($\rightarrow$counterparty risk)
  - But exposure at default ≥ 0

Recovery: creditor generally loses fraction of exposure $R < 100$ percent

Loss given default (LGD) equals exposure minus recovery (a fraction $1 - R$)

Expected loss (EL) equals default probability $\times$ LGD or fraction $\pi_t \times (1 - R)$

- Credit risk management focuses on **unexpected loss**

Credit Value-at-Risk related to a quantile of the credit return distribution

- Differs from market risk in *excluding* EL
- Credit VaR at confidence level of α defined as:

$$1 - \alpha\text{-quantile of credit loss distribution} - \text{EL}$$

Estimating default probabilities

Risk-neutral default probabilities based on market prices, esp. credit spreads

- Data sources include credit-risky securities and CDS
- Risk-neutral default probabilities may incorporate risk premiums
- Used primarily for market-consistent pricing

Physical default probabilities based on fundamental analysis

- Based on historical default frequencies, scenario analysis, or credit model
- Associated with credit ratings
- Used primarily for risk measurement

Types of credit models

Differ on inputs, on what is to be derived, and on assumptions:

Structural models or **fundamental models** model default, derive measures of credit risk from fundamental data

- Firm's balance sheet: volumes of assets and debt
- Standard is the **Merton default model**

Reduced-form models or **intensity models** take estimates of default probability or LGD as inputs

- Often used to simulate default times as one step in portfolio credit risk modeling
- Often risk-neutral
- Common example: **copula models**

Factor models: company, industry, economy-wide fundamentals, but highly schematized, lends itself to portfolio risk modeling.

Some models fall into several of these categories

What risks are we modeling?

Credit risk: models are said to operate in

- **Migration mode** taking into account credit migration a well as default, or
- **Default mode** taking into account default only

Spread risk: credit-risk related market risk

Rating migration rates, 1920–2016

From/To:	Aaa	Aa	A	Baa	Ba	B	Caa	Ca–C	WR	Default
Aaa	86.7	7.8	0.8	0.2	0.0	0.0	0.0	0.0	4.4	0.0
Aa	1.1	84.2	7.6	0.7	0.2	0.0	0.0	0.0	6.1	0.1
A	0.1	2.7	85.0	5.6	0.6	0.1	0.0	0.0	5.7	0.1
Baa	0.0	0.2	4.3	82.7	4.6	0.7	0.1	0.0	7.0	0.3
Ba	0.0	0.1	0.5	6.1	73.9	6.9	0.7	0.1	10.6	1.2
B	0.0	0.0	0.2	0.6	5.6	71.7	6.2	0.5	11.9	3.3
Caa	0.0	0.0	0.0	0.1	0.6	6.9	67.3	2.9	13.7	8.4
Ca–C	0.0	0.0	0.1	0.0	0.6	3.0	8.0	48.4	18.7	21.1

Average one-year letter rating migration rates, 1920-2016, percent. Each row shows the probability of starting the year with the rating in row heading and ending with the rating in the column heading. "WR" denotes withdrawn rating. *Source*: Moody's Investor Service.

Modeling default time

- Occurrence of default event for single company over discrete time horizon t can be modeled as Bernoulli distribution
  - Default *occurrence* the random variable
- Alternatively: **default intensity models**, model the time—specific instant τ—at which default occurs
  - Default *time* the random variable
  - Look out over horizon from now (time 0) to time t
  - Default probability over $[0, t)$: $\mathbf{P}[0 \leq \theta < t]$, firm defaults by time t
- Example of **jump** or **Poisson process** with exactly one jump possible
- Default probability increases as horizon grows longer
- Every firm defaults eventually: $\lim_{t \to \infty} \mathbf{P}[0 \leq \tau < t] = 1$
- **Survival probability** is $1 - \mathbf{P}[0 \leq \tau < t]$:
  - Obligor remains solvent until at least time t

Default time distributions

- Default probabilities can be expressed through **cumulative default time distribution function**
- Simple form: $\mathbf{P}[0 \leq \tau < t] = 1 - e^{-\lambda t}$
  - Survival probability is $e^{-\lambda t}$
  - If t expressed in years, 1-year default probability is $1 - e^{-\lambda}$
- Corresponding p.d.f. is derivative w.r.t. $t : \lambda e^{-\lambda t}$
  - For tiny time interval dt

$$\mathbf{P}[t \leq \tau < t + dt] \cong \lambda e^{-\lambda t}$$

Hazard rates

- Default can only occur once ⇒

$$\mathbf{P}\left[t \le \tau < t + dt\right] = \mathbf{P}\left[t \le \tau < t + dt \cup t < \tau\right]$$

- Conditional probability of default, given it has not occurred before t:

$$\begin{aligned}\mathbf{P}\left[t \le \tau < t + dt \mid t < \tau\right] &= \frac{\mathbf{P}\left[t \le \tau < t + dt \cup t < \tau\right]}{\mathbf{P}\left[t < \tau\right]} \\ &= \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} \\ &= \lambda\end{aligned}$$

- λ called **hazard rate** or **default intensity**
  - Viewed from time 0, probability of default over dt is λdt
- λ can be modeled as a constant or as changing over time
- In insurance, **force of mortality,** probability of death of a population member over next short time interval

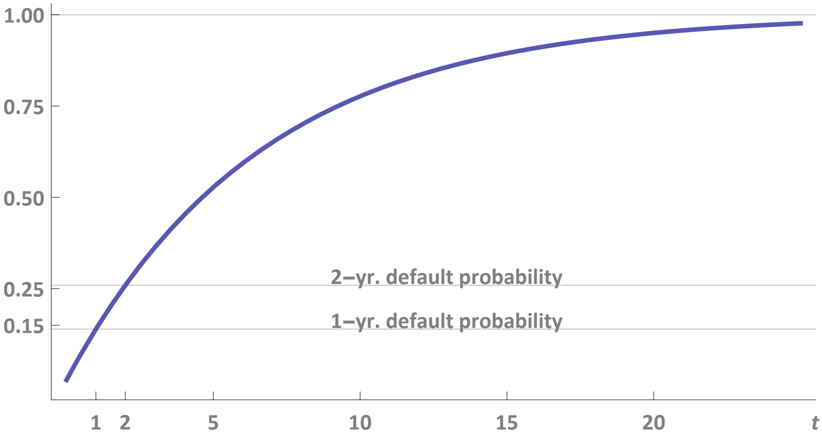
Conditional default probability

- **Conditional default probability**: probability of default over a future time horizon, *given* no default before then
- With constant hazard rate:
  - Unconditional one-year default probability lower for more remote years
  - But time to default *memoryless*: *if* no default occurs next year, probability of default over subsequent year is same as next year
- λ: *instantaneous* conditional default probability
  - Probability of default over next instant, given no prior default

Default probability analytics: example

Hazard rate	λ	0.15
1-yr. default probability	$1-e^{-\lambda}$	0.1393
2-yr. default probability	$1-e^{-2\lambda}$	0.2592
1-yr. survival probability	$e^{-\lambda}$	0.8607
1-yr. conditional default probability	$1-e^{-\lambda}$	0.1393

Default time distribution



Cumulative default time distribution function π_t, constant hazard rate $\lambda = 0.15$, t measured in years, π_t and λ at an annual rate.

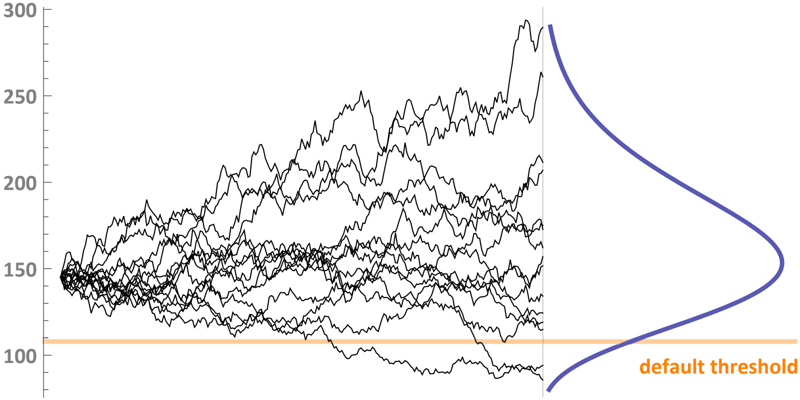
Merton model: overview

- Widely-used structural model based on fluctuations in debt-issuing firm's asset value
- Default occurs when asset value falls below **default threshold**, at which
    - Equity value extremely low or zero
    - Asset value close to par value of debt (plus accrued interest)
- Simplest version:
    - Default occurs when equity value hits zero
    - Default threshold equals par value of debt (plus accrued interest)

Equity and debt as options

- Assets assumed to display return volatility, so can apply option-pricing theory
- Equity can be viewed as a long call on the firm's assets, with a strike price equal to the par value of the debt
- Debt can be viewed as a portfolio:
    - A riskless bond with the same par value as the debt
    - Plus an implicit short put on the firm's assets, with a strike price equal to the par value of the debt
- If the lender bought back the short put, it would immunize itself against credit risk
    - $\Rightarrow$The value of the implicit short a measure of credit risk

Merton default model



Left: 15 daily-frequency sample paths of the geometric Brownian motion process of the firm's assets with a drift of 15 percent and an annual volatility of 25 percent, starting from a current value of 145. Right: probability density of the firm's asset value on the maturity date, one year hence, of the debt. The grid line represents the debt's par value (100) plus accrued interest at 8 percent.

Applying the Merton default model

- Immediate consequence: higher volatility (risk) benefits equity at expense of debtholders
- Model can be used to compute credit spread, expected recovery rate
- Two ways to frame model, depending on how mean of underlying return process interpreted

  Risk-neutral default probability: expected value equal to firm's dividend rate

  Physical default probability: expected value equal to asset rate of return
- Model timing of default, compute default probability
- KMV Moody's (and other practitioner applications):
  - Equity vol plus leverage→asset vol
  - Plus book value of liabilities→default threshold
  - Historical data+secret sauce to map into default frequency

Structure of single-factor model

- Basic similarity to Merton model
  - Default occurs when asset value falls below default threshold
- Asset returns depend on two random variables:

  Market risk factor m affects all firms, but not in equal measure
    - Expresses influence of general business conditions, state of economy on default risk
    - Latent factor: not directly observed, but influences results indirectly via model parameters

  Idiosyncratic risk factor ϵ affects just one firm
    - Expresses influence of individual firm's situation on default risk
- Fixed time horizon, e.g. one year
- Returns and shocks are measured as deviations from expectations or from a "neutral" state
- Most often used to model portfolio credit risk rather than single obligor

Parameters of single-factor model

Default probability π or, equivalently, default threshold k

- Combination of adverse market and idiosyncratic shocks sufficient to push borrower into default

Correlation β of asset return to market risk factor m

- High correlation implies strong influence of general business conditions on firm's default risk
- Correlations of individual firms' asset returns key driver of extent to which defaults of different firms coincide
- $\rightarrow$Portfolio credit models and **default correlation**

Single-factor model: asset return behavior

- Mertion model framework: default threshold is hit when firm's asset return r large and negative
- Asset return a function of market and idiosyncratic risk factors m, ϵ:

$$r = \beta m + \sqrt{1-\beta^2}\,\epsilon$$

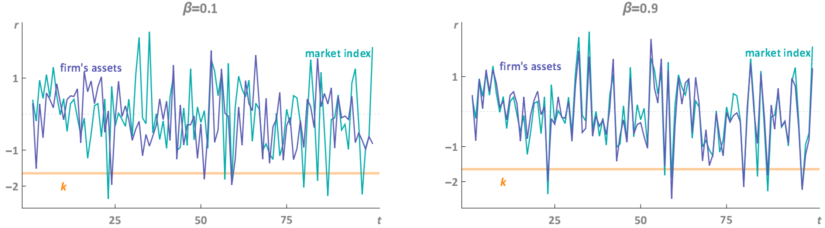
- β: correlation between firm's asset return and market factor m
- m and ϵ uncorrelated standard normal variates:

$$\begin{aligned} m &\sim \mathcal{N}(0,1) \\ \epsilon &\sim \mathcal{N}(0,1) \\ \mathrm{Cov}[m,\epsilon] &= 0 \end{aligned}$$

- Therefore r is a standard normal variate, expressed in "volatility units": $r \sim \mathcal{N}(0,1)$, with
  - r and m expressed as deviations from neutral state of business cycle

$$\begin{aligned} \mathbf{E}[r] &= 0 \\ \mathrm{Var}[r] &= \beta^2 + 1 - \beta^2 = 1 \end{aligned}$$

Asset and market returns in the single-factor model



Each panel shows a sequence of 100 simulations from the single-factor model. Cyan plot: returns on the market index m. Purple plot: associated returns $r = \beta m + \sqrt{1-\beta^2}\epsilon$ on firm's assets with the specified β to the market. Plots are generated by simulating m and ϵ as a pair of uncorrelated $\mathcal{N}(0,1)$ variates, using the same random seed for both panels.

Single-factor model: default probability

- Default probability an assigned parameter
  - Rather than an *output*, as in the Merton model, the default probability is an *input* in the single-factor model
- Expressed via default threshold k or **distance-to-default**
  - Default threshold a negative number, distance-to-default initially equal to $-k$
  - Default if r negative and large enough to wipe out equity:

$$\beta m + \sqrt{1 - \beta^2}\,\epsilon \leq k$$

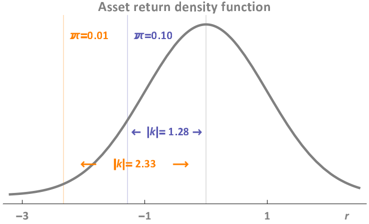
- Or, equivalently, **distance-to-default** $-k = |k|$
- Finding the initial default threshold: set k to match stipulated default probability π via

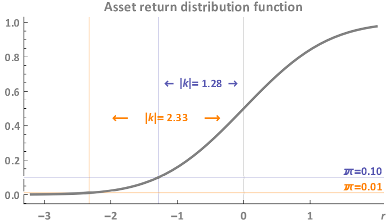
$$\pi = \mathbf{P}\left[r \leq k\right] \Leftrightarrow k = \Phi^{-1}(\pi),$$

  where $\Phi(\cdot)$ is the standard normal CDF
- **Example**:

	$\pi = 0.01$	$\pi = 0.10$
Distance-to-default ($-k$)	2.33	1.28

Single-factor model: default probability





Vertical grid lines mark the default threshold corresponding to default probabilities of 0.01 and 0.10.

Single-factor model and CAPM

- Single-factor model vs. CAPM beta
  - Since $\mathrm{Var}[r] = 1$, β analogous to the *correlation* of market and firm, rather than CAPM beta
  - Relationship of asset rather than equity values to market factor
- Systematic and idiosyncratic risk: fraction of asset return variance explained by variances of
  - Market risk factor: β^2
  - Idiosyncratic risk factors: $1 - \beta^2$
  - Example:

	$\beta = 0.40$	$\beta = 0.90$
Market factor β^2	0.16	0.81
Idiosyncratic factor $1 - \beta^2$	0.84	0.19

Market factor and conditional independence

- Suppose we know the "state of the economy,", i.e. the particular realization $\bar{m}$ of m
- Obligor i asset return r_i now has only one random driver: idiosyncratic factor ϵ_i

$$r_i = \beta_i \bar{m} + \sqrt{1 - \beta_i^2}\epsilon_i, \qquad i = 1, 2, \ldots$$

- Distance-to-default—the default-triggering return—becomes $-k_i + \beta_i \bar{m}$
- ϵ_i independent$\Rightarrow$conditional returns of two different obligors

$$\sqrt{1 - \beta_i^2}\epsilon_i, \sqrt{1 - \beta_j^2}\epsilon_j, i \neq j$$

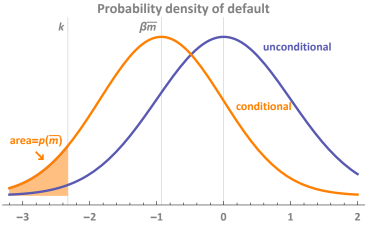
  are independent
- $\Rightarrow$**Conditional independence**: defaults of two firms independent
  - Conditioning is on realization of market risk factor
  - $\not\leftrightarrow$ discussion of hazard rates, in which "conditional" default probability refers to non-default in prior period

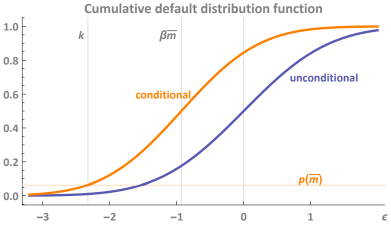
Conditional default distribution of a single obligor

- Conditional on $m = \bar{m}$:
  - **Mean** of the return distribution **changes**: $0 \rightarrow \beta_i \bar{m}$
  - **Variance** of the return distribution **reduced**: $1 \rightarrow 1 - \beta_i^2$
    - Because we have eliminated market factor as source of variation
    - And **standard deviation** from $1 \rightarrow \sqrt{1 - \beta_i^2}$
  - **Distance-to-default changes**: $-k_i \rightarrow -(k_i - \beta_i \bar{m})$
    - In standard units: $-k_i \rightarrow -\frac{k_i - \beta_i \bar{m}}{\sqrt{1-\beta_i^2}}$
  - **Default probability changes**: $\pi_i = \Phi(k_i) \rightarrow \Phi\left(\frac{k_i - \beta_i \bar{m}}{\sqrt{1-\beta_i^2}}\right)$
    - With $\Phi(x)$ the CDF of a standard normal variate x
- $\Rightarrow$**Conditional default probability distribution function**:

$$\boxed{p_i(m) = \mathbf{P}\left[r_i \leq k_i | m\right] = \Phi\left(\frac{k_i - \beta_i m}{\sqrt{1 - \beta_i^2}}\right), \qquad i = 1, 2, \ldots}$$

Conditional default probability: given market shock





Density and cumulative probability as a function of idiosyncratic shock. Graph assumes $\beta_i = 0.4, k_i = -2.33 \quad (\Leftrightarrow \pi_i = 0.01)$, and $\bar{m} = -2.33$. The unconditional default distribution is a standard normal, while the conditional default distribution is $\mathcal{N}(\beta_i \bar{m}, \sqrt{1-\beta_i^2}) = \mathcal{N}(-0.9305, 0.9165)$. The orange area in the density plot and horizontal grid line in the cumulative distribution plot identify $p(\bar{m})$, as in the example.

Conditional default distributions: example

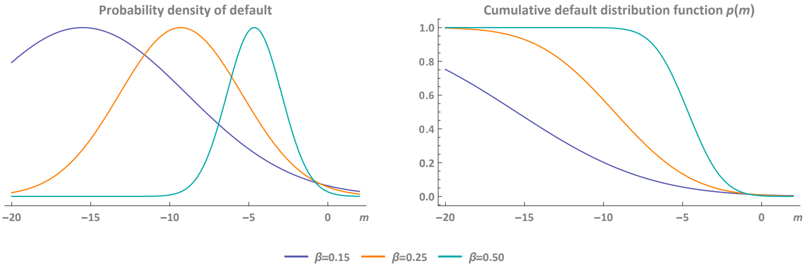
- Firm: $\beta_i = 0.4$, $k_i = -2.33$ (so $\pi_i = 0.01$)
- Market shock: $\bar{m} = -2.33$ (sharp downturn)

	Unconditional	Conditional	Change
Mean return	0	-0.9305	-0.9305
Return variance	1	0.8400	-0.1600
Return std. deviation	1	0.9165	-0.0835
Distance-to-default	2.33	1.3958	-0.9305
(standardized)	2.33	1.5230	-0.8034
Default probability	0.01	0.0639	0.0539

Properties of the conditional distribution

- Once the market factor is realized, the default distributions of individual loans/obligors are independent
- But the market factor continues to be a random variable—together with idiosyncratic risk—driving default
- Both parameters β_i and k_i continue to influence the shape of the distribution function

Conditional default distributions



Probability of default of a single obligor, conditional on the realization of m (x axis). Default probability 1 percent ($k = -2.33$). conditional cumulative distribution function of default $p(m)$. Values of the distribution function run from 1 to 0 because it is plotted against m rather than $\frac{k-\beta m}{\sqrt{1-\beta^2}}$.