

Aim

A useful source of information on the default risk of an obligor are the market prices of bonds and other defaultable securities that were issued by this obligor, and the prices of CDSs referencing this obligor's credit risk. To analyze the relationship between the prices of defaultable bonds and the default risk of the obligor, we will need to find a method to construct a term structure of credit spreads from observed market prices. Such a term structure of credit spreads has a direct interpretation in terms of a simple default risk model and is essential for pricing simple credit derivatives.

We will see that the prices of zero-recovery defaultable and default-free bonds contain all the information on the distribution of the time of default that we need. However, the prices of such a set of zero-recovery defaultable ZCB prices are not available in real market. Hence we would like to construct such a term structure from real traded assets. In order to do so, we will first construct a simple method to calculate a model price for real traded assets such as defaultable coupon bonds or CDSs for a given term structure of defaultable ZCBs. Then we invert this pricing relationship in order to find the term structure of defaultable ZCBs that yield model prices which equal the observed market prices. This term structure is then the implied term structure of default risk.

Bond Price Based Model

- **Notation:** $B(t, T)$ - price of default-free zero-coupon bond (ZCB) price
 $\bar{B}(t, T)$ - defaultable ZCB price
Clearly $0 < \bar{B}(t, T) \leq B(t, T) \leq 1$.

- **Assumptions:**
 - No recovery if default occurs
 - Risk-free interest rate dynamics independent of default time

Denote $I(t) = \begin{cases} 1, & \tau > t \\ 0, & \tau \leq t \end{cases}$ the default indicator.

Under the spot (bank account) martingale measure (risk-neutral measure),

$$B(t, T) = E \left[e^{-\int_t^T r(s) ds} \cdot 1 \right] \Rightarrow \bar{B}(t, T) = E \left[e^{-\int_t^T r(s) ds} I(T) \right] = E \left[e^{-\int_t^T r(s) ds} \right] E[I(T)] = B(t, T) P(t, T),$$

where $P(t, T)$ is the implied probability of survival between t and T .

The implied default probability over $[t, T]$ is $P^{def}(t, T) = 1 - P(t, T)$.

It is reasonable to assume that $P(t, t) = 1$ and $P(t, \infty) = 0$.

Then the conditional probability of survival given that there is no default before T_1 is

$$P(t, T_1, T_2) = \frac{P(t, T_2)}{P(t, T_1)}$$

where $t \leq T_1 < T_2$.

Hazard Rates

- Discrete implied hazard rate of default over $[T, T + \Delta t]$ as seen from time t :

$$H(t, T, T + \Delta t) = \frac{1}{\Delta t} \left(\frac{P(t, T)}{P(t, T + \Delta t)} - 1 \right) = \frac{1}{\Delta t} \frac{P^{def}(t, T, T + \Delta t)}{P(t, T, T + \Delta t)}$$

- Continuous implied hazard rate of default at time T :

$$h(t, T) = \lim_{\Delta t \rightarrow 0} H(t, T, T + \Delta t) = -\frac{1}{P(t, T)} \frac{\partial}{\partial T} P(t, T)$$

This is analogous to the definition of simply and continuously compounding forward rates

$$\begin{aligned} F(t, T, T + \Delta t) &= \frac{1}{\Delta t} \left(\frac{B(t, T)}{B(t, T + \Delta t)} - 1 \right) & f(t, T) &= -\frac{1}{B(t, T)} \frac{\partial}{\partial T} B(t, T) \\ \bar{F}(t, T, T + \Delta t) &= \frac{1}{\Delta t} \left(\frac{\bar{B}(t, T)}{\bar{B}(t, T + \Delta t)} - 1 \right) & \bar{f}(t, T) &= -\frac{1}{\bar{B}(t, T)} \frac{\partial}{\partial T} \bar{B}(t, T) \end{aligned}$$

It easily follows from the definitions above that

$$\begin{aligned} H(t, T_1, T_2) &= \frac{\bar{F}(t, T_1, T_2) - F(t, T_1, T_2)}{1 - (T_2 - T_1)F(t, T_1, T_2)} = \frac{B(t, T_2)}{B(t, T_1)} [\bar{F}(t, T_1, T_2) - F(t, T_1, T_2)] \\ h(t, T) &= \bar{f}(t, T) - f(t, T) \end{aligned}$$

Probability of default in $[T, T + \Delta t]$ is proportional to Δt with the coefficient $h(t, T) = \bar{f}(t, T) - f(t, T)$, or locally (if $t = T$)

$$p(t, t + \Delta t) \approx h(t, t) \Delta t = (\bar{r}(t) - r(t)) \Delta t = \lambda(t) \Delta t.$$

Recovery Modelling

Recovery is assumed to take place at the time of default.

If default occurs in $[T, T + \Delta t]$ then the value of recovery payment at t

$$\begin{aligned} e(t, T, T + \Delta t) &= E[B(t, T + \Delta t)(I(T) - I(T + \Delta t)) / \mathcal{F}_t] = B(t, T + \Delta t) P(t, T) - \bar{B}(t, T + \Delta t) \\ &= \bar{B}(t, T + \Delta t) \left[\frac{B(t, T + \Delta t) P(t, T)}{\bar{B}(t, T + \Delta t)} - 1 \right] = \bar{B}(t, T + \Delta t) \left[\frac{P(t, T)}{P(t, T + \Delta t)} - 1 \right] \\ &= \Delta t \bar{B}(t, T + \Delta t) H(t, T, T + \Delta t) \end{aligned}$$

Taking $\Delta t \rightarrow 0$ we obtain

$$e(t, T) = \lim_{\Delta t} \frac{1}{\Delta t} e(t, T, T + \Delta t) = \bar{B}(t, T) h(t, T)$$

So if a security pays $\pi(s)$ if defaulting at s its value at t will be

$$\int_t^T \pi(s) e(t, s) ds = \int_t^T \pi(s) \bar{B}(t, s) h(t, s) ds$$

If in addition the recovery amount is a random variable π' and

$$\pi^e(t, T) = E(\pi' / \mathcal{F}_t \cap [\tau = T])$$

is conditional (on default at T) distribution then

$$\int_t^T \pi(s) e(t, s) ds = \int_t^T \pi^e(t, s) B(t, s) P(t, s) h(t, s) ds$$

Pricing Credit Derivatives

For discrete version we assume the payments dates are time grid points $\{0 = T_0, T_1, \dots, T_n\}$.

The prices of default-free ZCBs:

$$B(0, T_k) = \prod_{i=1}^k \frac{1}{1 + (T_i - T_{i-1}) F(0, T_{i-1}, T_i)} \quad B(0, T) = \exp \left\{ - \int_0^T f(0, s) ds \right\}$$

The prices of defaultable ZCBs with zero recovery:

$$\begin{aligned} \bar{B}(0, T_k) &= B(0, T_k) P(0, T_k) = B(0, T_k) \prod_{i=1}^k \frac{1}{1 + (T_i - T_{i-1}) H(0, T_{i-1}, T_i)} = \prod_{i=1}^k \frac{1}{1 + (T_i - T_{i-1}) \bar{F}(0, T_{i-1}, T_i)} \\ \bar{B}(0, T) &= \exp \left\{ - \int_0^T (f(0, s) + h(0, s)) ds \right\} \end{aligned}$$

The value of \$1 at T_{k+1} if a default occurred in $[T_k, T_{k+1}]$:

$$\begin{aligned} e(0, T_k, T_{k+1}) &= (T_{k+1} - T_k) H(0, T_k, T_{k+1}) \bar{B}(0, T_{k+1}) \\ e(0, T) &= h(0, T) \bar{B}(0, T) \end{aligned}$$

- **Defaultable fixed-coupon bond**

If coupons are paid on $\{T_{k_i}, i = 1, \dots, N\}$, which is the subset of time grid points, then

$$\bar{C}(0) = \underbrace{\sum_{i=1}^N c \bar{B}(0, T_{k_i})}_{\text{coupons}} + \underbrace{\bar{B}(0, T_{k_N})}_{\text{principal}} + \underbrace{\pi \sum_{i=1}^{k_N} e(0, T_{k_{i-1}}, T_{k_i})}_{\text{recovery}}$$

- **Credit default swaps**

Fixed leg: payment of $s(T_{k_i} - T_{k_{i-1}})$ if no default until $T_{k_{i-1}}$.

The value of the fixed leg is: $s \sum_{i=1}^N (T_{k_i} - T_{k_{i-1}}) \bar{B}(0, T_{k_i})$.

Floating leg: payment of $(1-\pi)$ at T_k if default in $[T_{k-1}, T_k]$.

The value of the floating leg is: $(1-\pi) \sum_{i=1}^{k_N} e(0, T_{k-1}, T_k) = (1-\pi) \sum_{i=1}^{k_N} (T_k - T_{k-1}) H(0, T_{k-1}, T_k) \bar{B}(0, T_k)$

When the deal is done the spread s is chosen so that the value of the swap is 0.

$$s = (1-\pi) \frac{\sum_{i=1}^{k_N} (T_k - T_{k-1}) H(0, T_{k-1}, T_k) \bar{B}(0, T_k)}{\sum_{i=1}^{k_N} (T_k - T_{k-1}) \bar{B}(0, T_k)} = (1-\pi) \sum_{i=1}^{k_N} w_i H(0, T_{k-1}, T_k)$$

This is similar to the interest rate swap rate which is a weighted sum of forward rates.

Calibration.

From the previous section we know that the default-free forward rate $F = (F_1, F_2, \dots, F_K)$, the recovery and the credit forward spread H are input parameters in credit derivative pricing models. Unfortunately, these quantities are not directly observable in the market for a wide range of maturities. They must, therefore, be estimated from existing bond prices or yields.

Calibration procedures:

1. Default-free interest rate calibration: We want to find F such that $\hat{C}^{(i)} = \bar{C}^{(i)}(F)$ where $\hat{C}^{(i)}$ is the observed market price of the i^{th} security and $\bar{C}^{(i)}$ is the corresponding model price. There are a number of estimation methodologies for deriving forward curves from observed data. Each technique, however, can provide surprisingly different shapes for these curves. As a result, the selection of a specific estimation technique depends on its final use. For example, Bank of Canada utilizes the Super-Bell model for extracting the forward interest rates from Government of Canada bond yields.
2. Recovery rate π must be estimated separately to the calibration procedure. This is because recoveries and hazards rates have almost the same influence on the price of defaultable bonds, which result in a identification problem. Sometimes this problems can be solved by a joint calibration under inclusion of additional calibration securities. But this always produces unstable values. So we usually calibrate the recovery rate separately.
3. Spread (hazard rate) calibration: After obtaining F and π , our final task is to find H such that $\sum_{i \in I} w_i \left\| \hat{C}^{(i)} - \bar{C}^{(i)}(F, H, \pi) \right\| \rightarrow \min$, where I is the set of defaultable calibration securities. w_i are weights and can be chose to be $1/\hat{C}^{(i)}$ or $1/(\text{bid-ask spread})$ etc depending on our need.

Now we will illustrate the spread calibration.

Spread calibration methods:

- Parametric forms:

When calibrating the spreads, a functional form of $H(0,T)$ is assumed and some measure (e.g. mean-square) of deviation of the market prices from the calculated prices is minimized. The following are some popular choices of parametric forms:

Constant(one parameter): $H(0,T) = \beta_0$

Constant offset to a given function(one parameter):

Linear (two parameters): $H(0,T) = \beta_0 + \beta_1 T$.

Quadratic (three parameters): $H(0,T) = \beta_0 + \beta_1 T + \beta_2 T^2$

Nelson and Siegel (Four parameters): $H(0,T) = \beta_0 + (\beta_1 + \beta_2) \frac{1 - e^{-T/\gamma}}{T/\gamma} + \beta_2 e^{-T/\gamma}$

After choose a specific function form of the spread, we use some optimization method mentioned in calibration procedure 3 to calibrate the parameters of the form and hence obtain the spread.

The advantage of using parametric forms of spread curves is that the number of variables in the optimization problem is greatly reduced and one has full control over the possible shapes of the resulting spread curves. Regularity and smoothness of the resulting spread curve are ensured.

- Bootstrap

The bootstrap method is a popular method to extract the default-free forward rate curves from observed coupon bond prices which can also be applied to spread curve construction. It produces a piecewise constant spread curve.

We assume the spread curve is piecewise constant. By starting from the shortest maturity, the spread is ‘bootstrapped’ similarly to the bootstrapping procedure for the interest rate term structure:

1. Assume $H(0,T) = H_i$ iff $t_{i-1} < T < t_i$, where t_i is the maturity date of the i^{th} calibration security.

2. When $i = 1$, we find H_1 such that the first bond price is calibrated: $\bar{C}^{(1)}(H_1) = \hat{C}^{(1)}$.

3. Step: $i - 1 \rightarrow i$

Given H_1, H_2, \dots, H_{i-1} , we find H_i such that $\bar{C}^{(i)}(H_1, H_2, \dots, H_{i-1}; H_i) = \hat{C}^{(i)}$.

The bootstrapping technique does not require the solution of a minimization problem, and it imposes less structure on the shape of the spread curve than a parametric function.

In the following section we will give an example of calibrating spread using bootstrap technique under the intensity based model.

Bootstrapping spread curve under the Intensity Based Model

Assume the intensity follows inhomogeneous Poisson process, i.e. λ is a function of time $\lambda(t)$. It has the same properties as the homogenous Poisson process (λ is a constant) and

$$P[N_s - N_t = n] = \frac{1}{n!} \left(\int_t^s \lambda(u) du \right) \exp \left(- \int_t^s \lambda(u) du \right).$$

The default probability is given by (assume $s > t$)

$$P(\tau \leq s | \tau > t) = P(N_s - N_t \geq 1) = 1 - \exp\left(-\int_t^s \lambda(u) du\right).$$

We also assume the constant recovery rule, i.e. a fixed fraction R (recovery rate) of bond's face value is paid at maturity date T .

The equivalent value at T is equal to $\bar{B}(T, T) = 1_{\{\tau > T\}} + R1_{\{\tau \leq T\}}$. Then the zero-coupon bond price at time t is

$$\begin{aligned}\bar{B}(0, T) &= E\left(e^{-rT} \bar{B}(T, T)\right) = e^{-rT} E\left(1_{\{\tau > T\}} + R1_{\{\tau \leq T\}}\right) \\ &= B(0, T)[P(\tau > T) + R(1 - P(\tau > T))] \\ &= B(0, T)[R + (1 - R)P(\tau > T)],\end{aligned}$$

where $B(0, T)$ denotes the default-free ZCB price at time 0.

Then we obtain the credit spread:

$$S(0, T) = -\frac{1}{T} \ln \frac{\bar{B}(0, T)}{B(0, T)} = -\frac{1}{T} \ln[R + (1 - R)P(\tau > T)].$$

If we assume zero recovery i.e. $R = 0$, then we can get

$$S(0, T) = -\frac{1}{T} \ln[P(\tau > T)] = \frac{1}{T} \int_0^T \lambda(u) du.$$

The defaultable ZCB and the intensity has the relationship $\bar{B}(0, T_i) e^{rT_i} = P[\tau > T] = e^{-\int_0^{T_i} \lambda(u) du}$

Now we want to calibrate the spread curve using bootstrapping technique by assuming $\lambda(t)$ is a piecewise constant function:

$$\lambda(t) = a_1 + a_2 1_{\{t \geq T_1\}} + a_3 1_{\{t \geq T_3\}} + \dots + a_n 1_{\{t \geq T_{n-1}\}}.$$

We now determine the coefficient a_1 by $\bar{B}(0, T_1)$, a_2 by $\bar{B}(0, T_2)$, and so on. Here we assume r to be a constant. For instance,

$$\bar{B}(0, T_1) e^{rT_1} = e^{-\int_0^{T_1} a_1 du} = e^{-T_1 a_1}$$

$$\bar{B}(0, T_2) e^{rT_2} = e^{-\int_0^{T_2} a_1 + a_2 1_{\{u \geq T_1\}} du} = e^{-T_2 a_1 - (T_2 - T_1) a_2}$$

...

$$\bar{B}(0, T_n) e^{rT_n} = e^{-\int_0^{T_n} a_1 + a_2 1_{\{u \geq T_1\}} + \dots + a_n 1_{\{u \geq T_{n-1}\}} du} = e^{-T_n a_1 - (T_n - T_1) a_2 - \dots - (T_n - T_{n-1}) a_n}$$

allows us to compute first a_1 , then a_2, \dots, a_n . Hence our piecewise constant intensity model is fully calibrated to market bond prices.

In the implementation, since defaultable ZCB prices are not available for wide range of maturities, we will simulate those prices. We assume that the intensity follows CIR model, i.e.

$$d\lambda(t) = k(\theta - \lambda(t))dt + \sigma\sqrt{\lambda(t)}dW^Q(t),$$

where Q is the risk-neutral measure.

The defaultable bond price is given by $\bar{B}(0, T_i) = e^{-rT_i} e^{-\int_0^{T_i} \lambda(u) du}$. By the bond price formula under the CIR model, we can obtain