Credit Risk Models IV: Understanding and pricing CDOs

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December 2005†

Abstract

Some investors in the Collateralized Debt Obligations (CDOs) market have been publicly accused of not fully understanding the risks and dynamics of these products. They won't have an excuse any more. This report explains the mechanics of CDOs: their implied cash flows, the variables affecting those cash flows, their pricing, the sensitivity of CDO prices to those variables, the functioning of the markets where they are traded, their different types, the conventions used for trading CDOs, ... We built our description of CDOs pricing upon the Vasicek asymptotic single factor model because of its simplicity and the insights it provides regarding the pricing of CDOs. Additionally, we provide an extensive and updated review of the literature which extends the Vasicek model by relaxing its, somehow restrictive, assumptions in order to build more realistic and, as a consequence, more complicated CDO pricing models.

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†First version: July 2005. An updated version is available at `www.abelelizalde.com`.

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This paper is part of a series of surveys on credit risk modelling and pricing. The complete list of surveys is available at `www.abelelizalde.com`, and consists on the following:

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5. Credit Default Swap Valuation: An Application to Spanish Firms.

“**there is a minority of investors - perhaps 10 per cent - who do not fully understand what they are getting into**.”

Michael Gibson, head of trading risk analysis at the US Federal Reserve (Financial Times, 2005a).

“**Understanding the credit risk profile of CDO tranches poses challenges even to the most sophisticated participants.**”

Alan Greenspan, chairman of the US Federal Reserve (Financial Times, 2005b).

The “sharp increase in the complexity of credit derivative products being traded in the past couple of years ... may also mean that **investors do not fully understand what they are purchasing in areas such as collateralised debt obligations (CDOs)** - or pools of debt linked securities.”

(Financial Times, 2005c).

“Last month, Bank of America and Italian bank Banca Popolare di Intra (BPI) settled their 40 million euro lawsuit, in which BPI claims it was mis-sold several CDO investments by Bank of America. ... It would be naive to think that this is the last court case that will emerge. **A number of investors and regulators have already voiced concern about the level of complexity in some investment products.** With something as complicated as CDO-squared, it's not hard to imagine more investors claiming they were mis-sold investments if the credit cycle takes a turn for the worse.”

Nick Sawyer, Editor (Risk, 2005).

1 Introduction

Imagine a pool of defaultable instruments (bonds, loans, credit default swaps CDSs, ...) from different firms is put together. The losses on the initial portfolio value due to the default of the underlying firms depend on the default probability of each firm and the losses derived from each default (losses given default). Additionally, the degree of dependence between the firms' default probability, usually known as default correlation, plays an important role on the timing of the firms' defaults (whether they tend to cluster or they are independent) and, as a consequence, on the distribution of the portfolio losses.

Next, imagine we, the owners of the portfolio, decide to buy protection against the possible losses due to the defaults of the underlying firms, but we can not sell the portfolio. One way to do it is buying Credit Default Swaps (CDSs) of each firm, but that's not the way we are interested in here. We can *sell the portfolio in tranches*, i.e. we can buy protection for those losses in tranches. A Collateralized Debt Obligation (CDO) consists on *tranching* and selling the credit risk of the underlying portfolio. For example, a tranche with attachment points $[K_L, K_U]$ will bear the portfolio losses in excess of K_L percent of the initial value of the portfolio, up to a K_L percent. The tranche absorbing the first losses, called equity tranche, is characterized by $K_L = 0$ and $K_U > 0$. The holders of a tranche characterized by attachment points $[K_L, K_U]$ won't suffer any loss as long as the total portfolio loss is lower than K_L percent of its initial value. When the total portfolio loss goes above K_L percent, the tranche holders are responsible for the losses exceeding K_L percent, up to K_U percent. Losses above K_U percent of the initial portfolio value do not affect them. The lower attachment point K_L of each tranche corresponds to the upper attachment point K_U of the previous (more junior) tranche.

Obviously, the holders of each tranche (sellers of credit risk protection) have to be compensated for bearing those losses: they receive a periodic fee, called premium, until the maturity of the CDO (point in which they also stop being responsible for future losses in the portfolio.) The premium of the equity tranche will be the highest because its holders absorb the first losses of the portfolio. In order for the holders of more senior tranches to start suffering losses, the holders of more junior tranches would have already born all losses they were exposed to ($K_U - K_L$ percent of the initial portfolio value). As a consequence, the higher the seniority of the tranche the lower the premiums holders receive.

The whole problem lies in determining the tranches' premiums. They have to compensate tranche holders for the expected losses they will suffer and, therefore, they depend on the distribution of the portfolio losses which, as we argued above, depends on the underlying firms' default probabilities, default correlations, and losses given default.

Our review of CDO pricing models focus on a particular branch of this literature: the ones based on structural models. The main distinguishing characteristic of such models with respect to the other credit risk modelling alternative, reduced form models, is the link they provide between the probability of default and the firms' fundamental financial variables: assets and liabilities. The way structural models incorporate the dependence between the firms' default probabilities (which is a key ingredient for CDO pricing) is by making such fundamental variables depend on a set of, generally unobserved, common factors.

In contrast, reduced form models rely on market prices of the firms' defaultable instruments to extract both their default probabilities and their credit risk dependencies. These models rely on the market as the only source of information regarding the firms' credit structure and do not consider any information coming from their balance sheets. Although easier to calibrate, reduced form models lack the link between credit risk and the firms' financial situation incorporated in their assets and liabilities. Anyway, reduced form models provide an alternative way of pricing CDOs which one shouldn't forget, besides their lower popularity in this area.[1] In fact, these

[1]See, among others, Chava and Jarrow (2004), Driessen (2005), and Elizalde (2005d) for intensity models incorporating the correlation structure across firms, and Galiani (2003), Duffie and Garleanu

models provide the dynamics needed to price some of the recent exotic CDO products which we review in Section 4.5.4.

The paper starts from the theoretical foundations of the Vasicek model. It presents, in Section 2, a review of credit risk structural models, in order to understand the motivations behind such models. Section 3 describes in detail the Vasicek asymptotic single risk factor model, which has become the market standard for CDO pricing, and which is also referred to as the normal or Gaussian copula model, because that is the dependence structure it implies for the firms' default correlation.[2]

With all those tools in hand, Section 4 dives into CDOs: mechanics, types, pricing, premium sensitivity to the model parameters, trading issues (implied and base correlations), extensions of the Vasicek model and, finally, a few words on the calibration of the reviewed models.

The assumptions of the Vasicek asymptotic single risk factor model about the characteristics of the underlying portfolio (homogeneous infinitely large portfolio, ...) simplify the analytical derivation of CDO premiums but are not very realistic. The extensions we present relax these assumptions, making the model more suitable for CDO pricing. Investment banks and rating agencies devote a large amount of effort (and money) to fine tune and improve such models in order to benefit from the increase in pricing accuracy.

The text ends with an Appendix describing the application of the Vasicek factor model to bank capital regulation in Basel II, the brand new accord for banking supervision and regulation. Although this is not directly related with the main topic of the text, we include it because (i) it might interest some readers, (ii) it is the other most popular application of the Vasicek model, and (iii) it is straightforward using the material presented in Sections 2 and 3.

Throughout the text we provide an extensive list of references which the reader

(2004), and Willemann (2004) for applications of CDO pricing using intensity based methods via Monte Carlo simulation of default times.

[2] See Nelsen (1999), Embrechts, McNeil and Straumann (2002), and Cherubini and Luciano (2004) for an analysis of copula functions.

further interested in any of the covered topics might find useful.

2 Structural model for credit risk: Merton (1974)

There are two primary types of models that attempt to describe default processes in the credit risk literature: structural and reduced form models.[3]

Structural models use the evolution of firms' structural variables, such as asset and debt values, to determine the time of default. Merton's model (1974) was the first modern model of default and is considered the first structural model. In Merton's model a firm defaults if, at the time of servicing the debt, its assets are below its outstanding debt. A second approach, within the structural framework, was introduced by Black and Cox (1976). In this approach defaults occur as soon as firm's asset value falls below a certain threshold. In contrast to the Merton approach, default can occur at any time.

Reduced form models do not consider the relationship between default and firm financial situation in an explicit manner. In contrast to structural models, the time of default in intensity models is not determined via the value of the firm, but it is the first jump of an exogenously given jump process. The parameters governing the default hazard rate are inferred from market data.

Structural default models provide a link between the credit quality of a firm and the firm's economic and financial conditions. Thus, defaults are endogenously generated within the model instead of exogenously given as in the reduced approach.

Merton (1974) makes use of the Black and Scholes (1973) option pricing model to value corporate liabilities. As we shall see, this is an straightforward application only if we adapt the firm's capital structure and the default assumptions to the requirements of the Black-Scholes model.

Assume that the dynamics of firm n's asset value $A_{n,t}$ follow a continuous-time diffusion given, under the physical or real probability measure $\mathbf{P}$, by the following

[3] For a literature review of credit risk models see Elizalde (2005b and c).

geometric Brownian motion:

$$\frac{dA_{n,t}}{A_{n,t}} = \mu_n dt + \sigma_n dW_{n,t}, \tag{1}$$

where μ_n is the total expected return, σ_n is the asset's (relative) instantaneous volatility, and $W_{n,t}$ is a standard Brownian motion under $\mathbf{P}$.

Let us assume that the capital structure of firm n is comprised by equity and by a zero-coupon bond with maturity T and face value of D_n. The firm's asset value $A_{n,t}$ is simply the sum of equity and debt values. Under these assumptions, equity represents a call option on the firm's assets with maturity T and strike price of D_n. It is assumed that the firm defaults if, at maturity T the firm's asset value $A_{n,T}$ is not enough to pay back the face value of the debt D_n to bondholders. As a consequence, the probability at time $t < T$ of the firm defaulting at T is given by

$$p_{n,t,T} = \mathbf{P}\left[A_{n,T} < D_n \mid A_{n,t}\right]. \tag{2}$$

This approach assumes that default can only happen at the maturity of the zero-coupon bond.

It can be shown using Itô's lemma that the diffusion process (1) allows us to express the asset value at time T as a function of the current asset value $A_{n,t}$ as follows

$$A_{n,T} = A_{n,t} \exp\left\{\left(\mu_n - \frac{\sigma_n^2}{2}\right)(T-t) + \sigma_n \sqrt{T-t} X_{n,t,T}\right\}, \tag{3}$$

where $X_{n,t,T}$ is given by

$$X_{n,t,T} = \frac{W_{n,T} - W_{n,t}}{\sqrt{T-t}}, \tag{4}$$

and follows a standard normal distribution with zero mean and variance one.[4]

At time t, we can express the condition for firm n defaulting at time T in terms of the random variable $X_{n,t,T}$:

$$A_{n,T} < D_n \Leftrightarrow X_{n,t,T} < K_{n,t,T}, \tag{5}$$

[4] By definition of a Brownian motion, the difference $W_{n,T} - W_{n,t}$ follows a normal distribution with zero mean and standard deviation $\sqrt{T-t}$.

where

$$K_{n,t,T} = \frac{\ln D_n - \ln A_{n,t} - \left(\mu_n - \frac{\sigma_n^2}{2}\right)(T-t)}{\sigma_n \sqrt{T-t}}. \tag{6}$$

As a consequence we can rewrite (2) as

$$p_{n,t,T} = \Phi\left(K_{n,t,T}\right), \tag{7}$$

where $\Phi(\cdot)$ is the distribution function of a standard normal random variable.

Equivalently, if instead of considering the dynamics of the asset value $A_{n,t}$ under the physical probability measure $\mathbf{P}$, one considers its dynamics under the risk neutral probability measure $\mathbf{Q}$, firm n's risk neutral default probability is obtained.

In order to simplify notation hereafter we fix the actual time to $t = 0$, which allows us to eliminate the first time subindex of $p_{n,t,T}$, $X_{n,t,T}$, and $K_{n,t,T}$, which become $p_{n,T}$, $X_{n,T}$, and $K_{n,T}$.

3 Vasicek asymptotic single factor model

This Section builds on Vasicek (1987, 1991 and 2002).[5] We are interested in the default probabilities at time $t > 0$ of a group of $n = 1, ..., N$ firms with the asset and liabilities structure described in the preceding section.

The probability of default of each firm n at time t is denoted $p_{n,t}$ and given by

$$\begin{aligned} p_{n,t} &= \Phi\left(K_{n,t}\right), && (8) \\ K_{n,t} &= \frac{\ln D_n - \ln A_{n,0} - \left(\mu_n - \frac{\sigma_n^2}{2}\right)t}{\sigma_n \sqrt{t}}. && (9) \end{aligned}$$

Imagine we have a portfolio composed of loans to each one of the above firms (one loan per firm), and we are interested in the distribution function for the *portfolio default rate*, i.e. the fraction of defaulted credits in the portfolio at time t. Note that

[5] See also Finger (1999) and Schönbucher (2000).

the portfolio default rate is not the variable we will ultimately interested in, which is the loss in the initial portfolio value or *portfolio loss rate*.

The aim (and attractiveness) of the Vasicek single factor model we are about to present is to come up with a simple and closed-form formula for the distribution function of both the portfolio default and loss rates. Deriving a closed form solution requires making a set of simplifying assumptions. We will progressively introduce these assumptions and their implications for the model.

To derive the portfolio default rate, knowing the individual probabilities $p_{1,t}, ..., p_{N,t}$ of the firms is not enough; we also need to know their correlation structure. Since the only random variable affecting the status of each firm n at time t (default or not default) is $X_{n,t}$, the correlation structure between the firms' default probabilities have to be introduced through the normal random variables $X_{1,t}, ...X_{N,t}$. We assume that the correlation coefficient of each pair of random variables $X_{n,t}$ and $X_{m,t}$ is $\rho_{n,m,t}$.

Assumption 1. The correlation coefficient $\rho_{n,m,t}$ between each pair of random variables X_n and X_m is the same for any two firms:

$$corr\left(X_{n,t}, X_{m,t}\right) = \rho_{n,m,t} = \rho_t \quad \text{for any } n \neq m. \tag{10}$$

The random part driving all firms' asset values is characterized by a common correlation coefficient. We can think that there exists a random factor or source of uncertainty affecting all firms in exactly the same way. Moreover, we can write the random variables $X_{1,t}, ..., X_{N,t}$ as

$$X_{n,t} = \sqrt{\rho_t} Y_t + \sqrt{1-\rho_t} \varepsilon_{n,t}, \tag{11}$$

for all $n = 1, ..., N$, where Y_t, $\varepsilon_{1,t}$, ..., $\varepsilon_{N,t}$ are i.i.d. standard normal random variables. We can interpret (11) as follows: each random variable $X_{n,t}$, whose realization determines whether firm n defaults at t, can be expressed as the sum of two risk factors: one common or systematic risk factor Y_t affecting all firms in the same way, and an idiosyncratic risk factor $\varepsilon_{n,t}$ independent across firms.

Conditional on the realization of the common factor Y_t the default probability at time t of each firm n is denoted by $p_n(Y_t)$ and given by

$$\begin{aligned} p_n(Y_t) &= \mathbf{P}\left[X_{n,t} < K_{n,t} \mid Y_t\right] && (12)\\ &= \mathbf{P}\left[\sqrt{\rho_t}Y_t + \sqrt{1-\rho_t}\varepsilon_{n,t} < K_{n,t} \mid Y_t\right] && (13)\\ &= \mathbf{P}\left[\varepsilon_{n,t} < \frac{K_{n,t} - \sqrt{\rho_t}Y_t}{\sqrt{1-\rho_t}} \mid Y_t\right] && (14)\\ &= \Phi\left(\frac{K_{n,t} - \sqrt{\rho_t}Y_t}{\sqrt{1-\rho_t}}\right). && (15) \end{aligned}$$

Moreover, conditional on the value of the systematic factor Y_t, the random variables $X_{1,t}, ..., X_{N,t}$ (and the default probabilities $p_1(Y_t), ..., p_N(Y_t)$) are independent.

Assumption 2. We know the individual default probabilities of each firm defaulting at time t: $p_{1,t}, ...p_{N,t}$.

In that case, we can work out, from (8), the value of $K_{n,t}$ for each firm, $K_{n,t} = \Phi^{-1}(p_{n,t})$, and substitute it into the previous equation for the conditional default probability

$$p_n(Y_t) = \Phi\left(\frac{\Phi^{-1}(p_{n,t}) - \sqrt{\rho_t}Y_t}{\sqrt{1-\rho_t}}\right). \qquad (16)$$

Although the underlying theoretical model we are using is a structural one, in particular the Merton (1978) model, the previous assumption does not specify the way in which the default probabilities $p_{1,t}, ...p_{N,t}$ are computed. The underlying structural model presented in the previous section serves as an stylized theoretical foundation for the Vasicek single risk factor model. However, the default probabilities $p_{1,t}, ...p_{N,t}$ can be obtained in different ways (see Section 4.7).

Marginal default probabilities $p_{1,t}, ...p_{N,t}$ are taken as given; the Vasicek asymptotic single factor model is just a way of introducing dependence between them. Moreover, because it considers a single common factor and both common and idiosyncratic factors are normal, the Vasicek asymptotic single factor model is equivalent to a normal or Gaussian copula.[6]

[6]Alternative ways of linking the firms' default probability exist: using more factors, other dis-

Assumption 3. The default probability of all firms is the same and it is denoted by p_t:

$$p_{n,t} = p_t \quad \text{for all } n = 1, ..., N. \tag{17}$$

This assumption implies the same conditional default probability $p(Y_t)$ for all firms given the systematic risk factor Y_t

$$p(Y_t) = \Phi\left(\frac{\Phi^{-1}(p_t) - \sqrt{\rho_t}Y_t}{\sqrt{1-\rho_t}}\right). \tag{18}$$

Consider, for each firm n, the random variable $L_{n,t}$ which takes value 0 if the firm has not defaulted before (or at) t and 1 otherwise. Define the random variable L_t as the sum of the random variables $L_{1,t}, ...L_{N,t}$. L_t represents the number of defaults in our portfolio.

If we divide the number of defaults in the portfolio L_t by the total number of firms N in the portfolio, we obtain the fraction of defaults in the portfolio at time t, denoted by Ω_t. Ω_t can be interpreted as the portfolio default rate at time t. The unconditional cumulative distribution function of the default rate Ω_t of a portfolio characterized by a default probability p and a correlation coefficient ρ_t is given by

$$F(\omega; p_t, \rho_t) = \mathbf{P}[\Omega_t \leq \omega]. \tag{19}$$

Assumption 4. The number of credits (one to each firm) in our portfolio is very large, $N \to \infty$.

As Schönbucher (2000) and Vasicek (2002) explain, since defaults are independent when conditioned to the realization of the common factor Y_t, the assumption of an infinitely large equal-size portfolio of credits implies that, using the law of the large numbers, the fraction of defaulted credits in the portfolio Ω_t converges to the individual default probability of each individual credit $p(Y_t)$ (assumed to be equal

tribution functions different than the normal, other copulas, ... We briefly review them in Section 4.6.2.

across credits).[7] As a consequence

$$\begin{aligned}
F(\omega; p_t, \rho_t) &= \mathbf{P}[\Omega_t \leq \omega] && (20)\\
&= \mathbf{P}[p(Y_t) \leq \omega] && (21)\\
&= \mathbf{P}\left[\Phi\left(\frac{\Phi^{-1}(p_t) - \sqrt{\rho_t}Y_t}{\sqrt{1-\rho_t}}\right) \leq \omega\right] && (22)\\
&= \mathbf{P}\left[Y_t \geq \frac{\Phi^{-1}(p_t) - \sqrt{1-\rho_t}\Phi^{-1}(\omega)}{\sqrt{\rho_t}}\right] && (23)\\
&= 1 - \Phi\left(\frac{\Phi^{-1}(p_t) - \sqrt{1-\rho_t}\Phi^{-1}(\omega)}{\sqrt{\rho_t}}\right) && (24)\\
&= \Phi\left(\frac{\sqrt{1-\rho_t}\Phi^{-1}(\omega) - \Phi^{-1}(p_t)}{\sqrt{\rho_t}}\right). && (25)
\end{aligned}$$

When $\rho_t = 0$ defaults are statistically independent, so $\Omega_t = p_t$ with probability 1, while when $\rho_t = 1$ defaults are perfectly correlated, so $\Omega_t = 0$ with probability $1 - p_t$, and $\Omega_t = 1$ with probability p_t.

The distribution function $F(\omega; p_t, \rho_t)$ is increasing in ω, with $F(0; p_t, \rho_t) = \Phi(-\infty) = 0$ and $F(1; p_t, \rho_t) = \Phi(\infty) = 1$. Moreover, it can be shown that

$$E(\Omega_t) = p_t, \tag{26}$$

and

$$Var(\Omega_t) = \Phi_2(\Phi^{-1}(p_t), \Phi^{-1}(p_t); \rho_t) - p_t^2, \tag{27}$$

where $\Phi_2(\cdot, \cdot; \rho_t)$ is the distribution function of a zero mean bivariate normal random variable with standard deviation equal to one and correlation coefficient ρ_t; see Vasicek (2002, p.161). Therefore, the expected value of the default rate is precisely the probability of default p_t, while its variance is increasing with the correlation parameter ρ_t, with $Var(\Omega_t) = 0$ for $\rho_t = 0$ and $Var(\Omega_t) = p_t(1 - p_t)$ for $\rho_t = 1$.

Assumption 5. The loss given default on each credit, denoted by λ_t, is deterministic and the same for all firms.

[7] A more detailed proof can be found in Vasicek (1987, 1991) and Finger (1999).

λ_t is the loss on each credit due to default at t, expressed as a percentage of its size. $1-\lambda_t$ is the so-called recovery rate.

Assumption 6. The size of each credit in the portfolio is similar.

This assumption allows a one-to-one relationship between the default rate and the loss rate or percentage loss of the total initial value of the portfolio. If a fraction Ω_t of the portfolio has defaulted by t, the percentage loss of the total initial value of the portfolio, denoted by Z_t, is $\lambda_t \Omega_t$.

4 CDOs

4.1 Mechanics

CDOs are probably the most important type of multiname credit derivative. A CDO consists on a portfolio of defaultable instruments (loans, credits, bonds or default swaps) whose credit risk is sold to investors who, in return for an agreed payment (usually a periodic fee), will bear the losses in the portfolio derived from the default of the instruments in the portfolio.

The credit risk of the portfolio underlying the CDO is sold in *tranches*. A tranche is defined by a lower and an upper *attachment points*. The buyers of the tranche with lower attachment point K_L and higher attachment point K_U will bear all losses in the portfolio value in excess of K_L, and up to K_U, percent of the initial value of the portfolio. As an example, Table 1 represents the upper and lower tranches of a fictitious CDO.

		Attachment points (%)	
Tranche number	Tranche name	Lower K_L	Upper K_U
1	Equity	0	3
2	Mezzanine 1	3	7
3	Mezzanine 2	7	10
4	Mezzanine 3	10	15
5	Senior	15	30

Table 1. Example of a CDO tranche structure.

Imagine the CDO underlying portfolio experiences a loss of 9 percent of its initial value. In that case, the holders of the equity tranche would bear the first 3% of those losses, the holders of the first mezzanine tranche would bear the next 4% of them, and the holders of the second mezzanine tranche will bear just 1% of the portfolio losses. The holders of more senior tranches (mezzanine 3 and senior) would not suffer any losses.

CDO tranching allows the holders of each tranche to limit their loss exposure to $K_U - K_L$ percent of the initial portfolio value.

Let t denote the time (in years) passed since the CDO was originated, T the maturity (in years) of the CDO, M the initial value of the portfolio, and Z_t the percentage loss in the portfolio value at time t. At time t the total loss in the portfolio value is $Z_t M$. The loss suffered by the holders of tranche j from the origination (at time 0) of the CDO up to time t is a percentage $Z_{j,t}$ of the portfolio notional value M :

$$Z_{j,t} = \min \left\{ Z_t, K_{U_j} \right\} - \min \left\{ Z_t, K_{L_j} \right\}, \tag{28}$$

where K_{U_j} and K_{L_j} are the upper and lower attachment points of tranche j.

The losses are paid by the tranche holders during the life of the CDO, with a predetermined frequency. Let η denote such frequency in years. Usually $\eta = 0,25$, i.e. one quarter. At each payment date, tranche holders will pay the losses on the portfolio realized since the last payment date. If t was the last time in which losses were paid by tranche holders, the payment that the holders of tranche j will have to pay at time $t + \eta$ is the new loss suffered from time t : a fraction $Z_{j,t+\eta} - Z_{j,t}$ of the CDO notional M.

So far we have outlined the way in which the portfolio losses are shared among the holders of the different tranches. However, they have to be compensated for bearing the risk of such losses. The holders of tranche j receive a periodic payment, with frequency η years, equal to a premium s_j of the outstanding notional amount of tranche number j. At time t, the outstanding notional of tranche j, denoted by $\Gamma_{j,t}$, is its initial notional $\left(K_{U_j} - K_{L_j}\right) M$ minus the total losses suffered by its holders up to time t, given by $Z_{j,t}M$:

$$\begin{aligned} \Gamma_{j,t} &= \left(K_{U_j} - K_{L_j}\right) M - Z_{j,t}M && (29) \\ &= \left(K_{U_j} - K_{L_j} - Z_{j,t}\right) M && (30) \\ & && (31) \\ &= \begin{cases} \left(K_{U_j} - K_{L_j}\right) M, & \text{if } Z_t < K_{L_j}, \\ \left(K_{U_j} - Z_t\right) M, & \text{if } K_{L_j} \leq Z_t \leq K_{U_j}, \\ 0, & \text{if } Z_t > K_{U_j}. \end{cases} && (32) \end{aligned}$$

If $t = 0$ is the origination time, payment dates are: η, 2η, ..., T. The structure of the cash flows for the holders of tranche j is as follows. At each payment date during the life of the CDO:

- they receive an amount

$$s_j \eta \Gamma_{j,t}, \tag{33}$$

  and

- pay an amount

$$\left(Z_{j,t} - Z_{j,t-\eta}\right) M, \tag{34}$$

for $t = \eta$, 2η, ..., T.

The premium s_j does not vary during the life of the CDO. However, the notional of tranche j, $\Gamma_{j,t}$, is a decreasing function of the total portfolio losses $Z_t M$:

$$\frac{\partial \Gamma_{j,t}}{\partial \left(Z_t M\right)} = \begin{cases} 0, & \text{if } Z_t < K_{L_j}, \\ -1, & \text{if } K_{L_j} \leq Z_t \leq K_{U_j}, \\ 0, & \text{if } Z_t > K_{U_j}. \end{cases} \tag{35}$$

The outstanding notional (30) of tranche j becomes zero as soon as the percentage loss in the portfolio Z_t becomes higher than the tranche upper attachment point K_{U_j}, $Z_t \geq K_{U_j}$, which implies $Z_{j,t} = K_{U_j} - K_{L_j}$. When that happens, $\Gamma_{j,t+a\eta} = 0$ (and

$Z_{j,t+a\eta} = Z_{j,t}$) for all $a \geq 0$ and, as a consequence, the amount they have to receive (and pay) in future payment dates $t+\eta$, $t+2\eta$, ... is always zero.

The lower the seniority of the tranche, the higher are the expected losses suffered by its holders and, therefore, the higher will be the premium they receive.

A CDO whose underlying portfolio consists of credits, loans or debt instruments from different firms is called a *cash CDO*. However, the originator of a CDO, i.e. the one who buys protection (and pays the premium), does not need to physically own a portfolio of credits, loans or bonds. A portfolio of CDSs generates the same credit exposure than the portfolio of credits, loans or bonds. When the CDO is constructed using a portfolio of CDSs it receives the name of *synthetic CDO*.

Bluhm (2003) analyzes the different factors which have contributed to the success of CDO trading: spread arbitrage opportunities, regulatory capital relief, funding and economic risk transfer.

Gibson (2004) presents one of the most insightful works about the mechanics of CDOs. The author summarizes the development of CDO markets and presents a simple CDO pricing model which allows him to analyze, among other things, the risk and leverage inherent in each tranche as well as the sensitivity of each tranche to the business cycle.

Plantin (2003) presents a theoretical analysis of the rationale for tranching and securitization activities such as CDOs. The author shows that they arise as a natural profit maximizing strategy of investment banks, and predicts that investors with increasing sophistication acquire tranches with decreasing seniority. Mitchell (2004) reviews the financial literature which justifies the creation of structured assets such as CDOs.

4.2 Pricing

In order to simplify the presentation, we introduce two further assumptions:

Assumption 7. Complete market and absence of arbitrage opportunities.

For our purposes we shall use the class of equivalent probability measures $\mathbf{Q}$, where non-dividend paying asset processes discounted using the default-free short rate are martingales. Absence of arbitrage is a necessary requirement for the existence of (at least) one equivalent probability measure, and the assumption of market completeness guarantees its uniqueness. Such an equivalent measure is called a *risk neutral measure* and will be used to derive bonds and CDS pricing formulas.[8]

Assumption 8. Independence of the firms' credit risk and the default-free interest rates under the risk neutral probability measure.

Although the model could accommodate correlation between interest rates and survival probabilities, such an assumption would add a higher degree of complexity into the model because a process would have to be estimated for the default-free short rate as well.[9]

Pricing a CDO consists on finding the appropriate premium s_j for each tranche j. The premium s_j is fixed in such a way that the net present value of the cash flows received/paid by its holders is zero, which implies that, as in a swap or CDS, there is no payment up-front.[10]

Similar to a plain vanilla interest rate swap or a CDS, a CDO consists of two legs: a fixed and a floating leg.[11] The fixed leg represents the payments tranche holders receive (positive cash flows), whereas the floating leg represents the payments they pay (negative cash flows). Consider a CDO with payment dates $\{t_1, ..., t_K\}$, maturity t_K, and notional M, where $\eta = t_{k+1} - t_k$ for all $k = 0, ..., K$. The contract starts at time $t_0 = 0$, and the first premium is due at t_1.

At each payment date t_k the holders of tranche j receive (33), for $k = 1, ..., K$.

[8] See Elizalde (2005b, Appendix A) for an analysis of the different scenarios under which the transition from the physical to the equivalent (or risk neutral) probability measure can be accomplished.

[9] Elizalde (2005d) estimates a reduced form model in which default probabilities depend on default-free interest rates. The results show that the effect of default-free rates on default probabilities is small. Moreover, different empirical papers find different signs for that effect.

[10] Although, as we mention below, the equity tranche does not usually follow this convention, it can be priced using similar arguments.

[11] Elizalde (2005a) describes a CDS pricing model in a similar framework.

Thus, the value of the fixed leg at time t_0, denoted by $X_{F,j}$, is equal to

$$X_{F,j} = \sum_{k=1}^{K} \beta(t_0, t_k) s_j \eta E\left[\left(K_{U_j} - K_{L_j} - Z_{j,t_k}\right) M\right], \tag{36}$$

where $\beta(t_0, t_k)$ is the discount factor from t_0 to t_k.

At each payment date t_k the holders of tranche j pay (34), for $k = 1, ..., K$. Thus, the value of the floating leg at time t_0, denoted by $X_{V,j}$, is equal to

$$X_{V,j} = \sum_{k=1}^{K} \beta(t_0, t_k) E\left[\left(Z_{j,t_k} - Z_{j,t_{k-1}}\right) M\right]. \tag{37}$$

The premium s_j is chosen in such a way that

$$X_{F,j} = X_{V,j}, \tag{38}$$

which implies

$$s_j = \frac{\sum_{k=1}^{K} \beta(t_0, t_k)\left(E\left[Z_{j,t_k}\right] - E\left[Z_{j,t_{k-1}}\right]\right)}{\sum_{k=1}^{K} \beta(t_0, t_k)\eta\left(K_{U_j} - K_{L_j} - E\left[Z_{j,t_k}\right]\right)}. \tag{39}$$

Z_{j,t_k}, given by (28), is the accumulated loss suffered by the holders of tranche j from the origination of the CDO up to time t_k, expressed as a percentage of the portfolio notional value M.

Given the attachment points K_{U_j} and K_{L_j}, the payment dates $t_1, ..., t_k, ..., t_K$ and the discount factors $\beta(\cdot, \cdot)$, we need to evaluate the expectations appearing in (39) in order to compute the tranche premium s_j. In particular, we need to evaluate $E\left[Z_{j,t_k}\right]$:[12]

$$E\left[Z_{j,t_k}\right] = E\left[\min\left\{Z_{t_k}, K_{U_j}\right\} - \min\left\{Z_{t_k}, K_{L_j}\right\}\right], \tag{40}$$

for $k = 1, ..., K$.

The characteristics (size, number of firms, default probability of each firm, default correlations between firms, loss given default, ...) of the portfolio will determine the

[12] Note that $E\left[Z_{j,t_0}\right] = 0$ for all tranches j.