Name: Naman Goyal
Roll No: 180123029

Solⁿ ①: given $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$, $\lambda \to$ parameter

Compute dy/dx (y')

$$\frac{2x}{a^2+\lambda} + \frac{2yy'}{b^2+\lambda} = 0 \Rightarrow \lambda = \frac{-(b^2x + a^2yy')}{(x+yy')}$$

Now $(a^2+\lambda) = a^2 - \frac{b^2x + a^2yy'}{x+yy'} = \frac{(a^2-b^2)x}{(x+yy')} \to ①$

Similarly $(b^2+\lambda) = \frac{-(a^2-b^2)yy'}{(x+yy')} \to ②$

Putting ① and ② in original equatⁿ: $(a^2+\lambda), (b^2+\lambda)$ in the original eqⁿ.

we get $\frac{(x+yy')}{(a^2-b^2)}\left[x - \frac{y}{y'}\right] = 1. \to ③$

Hence above eqⁿ is the differential form of the original given family.

To find the orthogonal trajectory replace $y' \to -1/y'$ in ③.

which gives $\frac{(x - y/y')(x+yy')}{(a^2-b^2)} = 1. \to ④$

④ comes out to be same as ③. Hence, we can say that the family is self-orthogonal.

Name: Naman Goyal Roll No: 180123029

Sol^n ②: Let $x(t)$, $y(t)$ be the population of prey and predator at time t. The assumptions made are:

(a) If there is no prey, the predator species will decline at a rate proportional to the population of predator species.

(b) if there are no predators, the prey species will grow at a rate proportional to the population of prey species.

(c) the presence of both predators and prey is beneficial to the growth of predator species and is harmful to growth of prey species. More specifically the predator species ↑ and prey species ↓ at rates proportional to product of both the populations.

rate of prey's ~~deaths~~ births $= b_1 x(t)$ rate of prey natural deaths $= a_1 x(t)$

rate of predators death $= a_2 y(t)$ rate of prey deaths $= c_1 y(t)$

where b_1, a_1, a_2, c_1 are respective population densities.

rate at which prey are eaten $= c_1 y(t) x(t)$

hence, predator birth rate has a component that is proportional to this rate of prey eaten.

$$\left\{ \begin{array}{c} \text{rate of predators} \\ \text{births} \end{array} \right\} = b_2 y(t) + f c_1 y(t) x(t)$$

$f \rightarrow$ the constant of proportionality.

Name: Naman Goyal   Roll No: 160123029

so we obtain: using rates of change of prey, predators-

$$\frac{dx}{dt} = b_1 x - a_1 x + c_1 xy$$

$$\frac{dy}{dt} = b_2 y + fc_1 xy - a_2 y$$

consider $a = b_1 - a_1$, $b = +c_1$, $c = -b_2 + a_2$, $d = fc_1$

Hence: $\frac{dx}{dt} = ax - bxy$

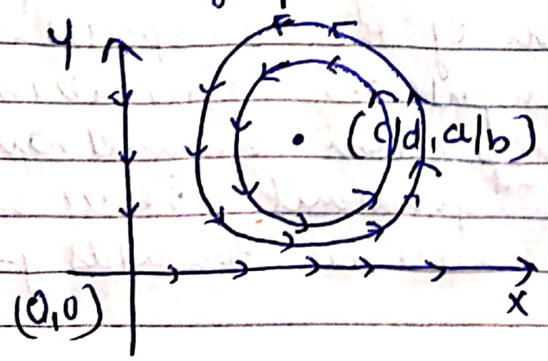
$\frac{dy}{dt} = dxy - cy = y(dx - c)$

we assume all the constants $a, b, c, d > 0$.

Equilibrium pts: $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$
→ pts where

giving us the sol^n $\{(x=0, y=0), (x = c/d, y = a/b)\}$

Consider the graph:



Thus on ~~distributing~~ disturbing from origin along x-axis it will increase indefinitely. Hence $(0,0) \to$ unstable.

↳ But on disturbing from $(c/d, a/b)$ it doesn't return back but stays in vicinity. Hence pt $(c/d, a/b)$ is stable but not asymptotically stable.

Name: Naman Goyal Roll No: 180123029

Sol^n 3:

$$\frac{dx}{dt} = x(12-4x-3y)$$

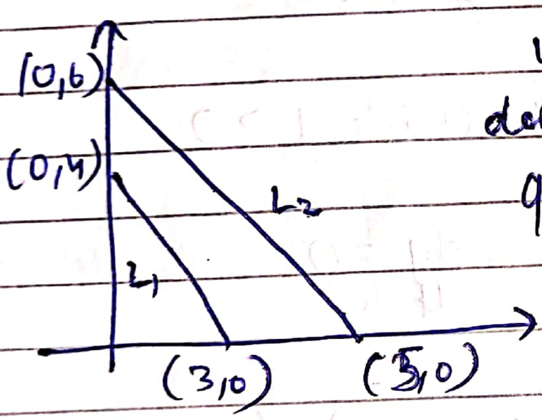
$$\frac{dy}{dt} = y(30-6x-5y) \quad x \geq 0, y \geq 0$$

for $\frac{dx}{dt} = 0$, equilibrium pt must satisfy

$$L_1: 12-4x-3y = 0 \rightarrow (1)$$

Similarly for $\frac{dy}{dt} = 0 \Rightarrow L_2: 30-6x-5y = 0 \rightarrow (2)$

plotting (1) and (2) on a graph.



we can see that L_1 and L_2 doesn't intersect in the first quadrant $(x \geq 0, y \geq 0)$, hence the system given above doesn't have nonzero equilibrium.

The directions show that the trajectories all move the equilibrium point on the y-axis. This corresponds to the extinction of the x-species, in a long term. If the compute the equilibrium pt using L_1, L_2 intersection we will have $(y' > 0, x' < 0)$ that's in 2^{nd} quadrant hence we can see that the direction will be towards y-axis only.

Scanned by CamScanner

Name: Naman Goyal   Roll No: 180123029

Sol^n (4): For dynamic equilibrium:

$$d(t) = s(t)$$

Hence, $\Rightarrow$ equating them

$$(b_2 - a_2)\frac{dp}{dt} + (b_1 - a_1)p = a_0 - b_0$$

Solving the above ODE:
we get

$$\boxed{p(t) = A + (p(0) - A)e^{\lambda t}}$$

where $A = \dfrac{a_0 - b_0}{(b_1 - a_1)}$, $\lambda = \dfrac{a_1 - b_1}{(b_2 - a_2)}$

Hence, behaviour of $p(t)$ depends on whether $p(0)$ or A is large and whether $\lambda > 0$ or $\lambda < 0$. hence $p(t)$ depends on both. Hence we can say that the model is highly unstable.

Sol^n (5): $x(n+1) = \dfrac{\alpha x(n)}{1 + \beta x(n)}$ $\alpha > 1, \beta > 0$

at equilibrium: we have $x(n) = x(n+1) = x^*$ (let's say)

$$x^* = \frac{\alpha x}{1+\beta x} \Rightarrow x^* + \beta x^{*2} = \alpha x$$

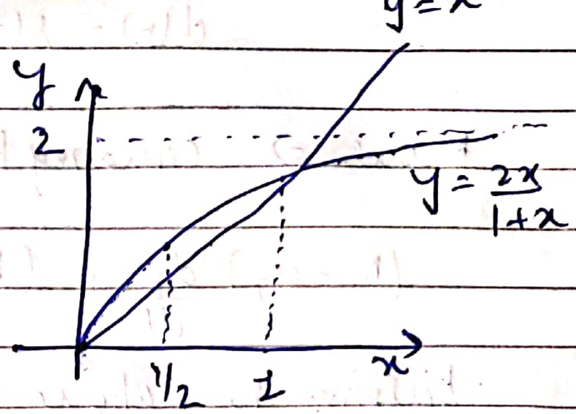
$$x^* = \frac{(\alpha - 1)}{\beta} \rightarrow$$

$\therefore \alpha > 1, \beta > 0 \Rightarrow \boxed{x^* > 0} \rightarrow$ +ve equilibrium point.

Taking $\alpha = 2, \beta = 1$: $x(n+1) = \frac{2x(n)}{1+x(n)} \Rightarrow x_{eqn} = 1$

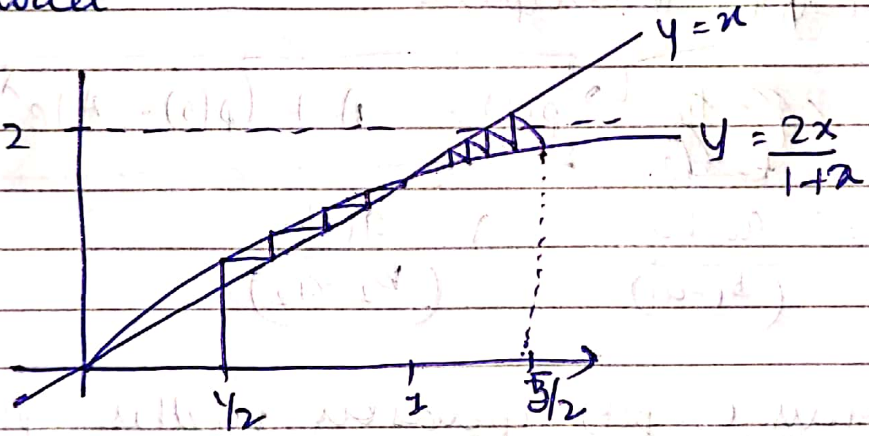
Taking x_n in place of $x(n)$

$\Rightarrow x_{n+1} = \frac{2x_n}{1+x_n}$.



Consider the curve $y = \frac{2x}{1+x}$

To find out if it's which equilibrium using stair step model:



If we consider $x_0 = 1/2 \Rightarrow x_1 = 2/3 \Rightarrow x_2 = 8/9 \Rightarrow x_3 = 16/17$

that is when $0 < x < 1$ it's approaching 1.

Consider $x_0 = 5/2 \Rightarrow x_1 = 10/7 \Rightarrow x_2 = 20/17 \Rightarrow x_3 = 40/37$.

that is when $x > 1$, it's again approaching 1.

Hence, for $x > 1$, $\frac{2x}{1+x} < x \Rightarrow$ it converges to 1.

for $0 < x < 1$, $x < \frac{2x}{1+x} \Rightarrow$ it's converges to 1.

Hence for any pt x_0 between 0 to ∞, x will converge to 1. Hence, x is a stable equilibrium point.

Name: Naman Goyal Roll No: 180123029

Sol^n 6:

$y' = -2y+1$

$y' = -2(y - 1/2)$

$\frac{dy}{dt} = -2(y - 1/2)$

$\int \frac{dy}{y - 1/2} = -\int 2dt$

$= \ln(y - 1/2) = -2t + c$

$\boxed{y = 1/2 + ke^{-2t}}$ $\Rightarrow \because y(0) = 1/2 \Rightarrow \underline{k=0}$

$\Rightarrow$ [$y = 1/2$ is the exact sol^n of given IVP.]

$\therefore \quad y_{n+1} = y_{n-1} + 2h f_n$

$y_{n+1} = y_{n-1} + 2h(-2y_n + 1) \quad \because f_n = -2y_n + 1$

$\Rightarrow y_{n+1} = y_{n-1} - 4hy_n + 2h$

Put $n = 1$ $y_2 = y_0 - 4hy_1 + 2h$

using $y(0) = 1/2 \Rightarrow y_1 = 1/2 \quad \therefore y \to y = 1/2$ is the exact sol^n.

$y_2 = 1/2 - 4h(1/2) + 2h = \boxed{1/2}$

Similarly $y_3 = 1/2,\ y_4 = 1/2 \ldots = y_n = 1/2 \ \forall n$.

↳ Hence we can say that difference scheme is stable. as also exact sol^n and difference scheme produces same result and hence stable.