

Individual Choice:

- Suppose there are three alternatives x, y, z . An individual has to choose from this set of three alternatives.
- Alternative we ask the individual to rank or order the alternatives.
- The choice or ranking is based on preference of the individual over the alternatives.
- We assume that the preferences are given. Taking a given type of preferences of the individuals whether the individual can rank or order all the alternatives or make choices.
- We do not study the way preferences are formed.

- Preferences over alternatives are based on binary relation.
- What is a binary relation? It is relation between the elements of a set. For example “is brother of” is a binary relation. Suppose the set is family members. $S = \{a, b, c, d\}$ is the set of family members. a= father, b= mother, c= elder son, d = younger son.
- In this example we have c is brother of d , and d is brother of c. a is not brother of d, a is not brother of b, a is not brother of c, b is not brother of a, b is not brother of d, b is not brother of c.
- We define it technically. We denote the binary relation by R . Let the set be $S = \{A, B, C, D\}$. We get cRd and dRc , $\sim aRb$ etc. In this example the binary relation can be represented as $R = \{(c, d), (d, c)\}$. A binary relation is a set of ordered pairs (x, y) such that x and y belongs to S where S is the set of alternatives.

Another example of binary relation “ is taller than ”. Suppose the set of alternative is $S = \{p1, p2, p3, p4\}$ these are the heights of four peaks, $p1= 1000\text{ft}$, $p2=2000\text{ ft}$, $p3= 3000\text{ ft}$, $p4= 1500\text{ ft}$. R is taller than so we have

$$R = \{(p2, p1), (p3, p2), (p3, p4), (p3, p1), (p4, p1), (p2, p4)\}.$$

We define the individual choice based on binary relation. The binary relation we take is R : atleast as good as .

$$S = \{x, y, z, w\},$$

$$xRy, yRx, xRz, xRw, xRx, yRy, yRw, zRz, wRy, zRz.$$

We can write it as

$$\begin{aligned} R = \{ & (x, y), (y, x), (x, z), (x, w), (x, x), \\ & (y, y), (y, w), (z, z), (w, y), (w, w) \} \end{aligned}$$

Properties of binary relation

$$S = \{x, y, z, w\}.$$

Reflexivity

A binary relation R is reflexive, for all $x, \in S$, xRx .

R : is taller than is not reflexive, p1 is taller than p1 is not possible.

R : is atleast as good as is reflexive when

$$(x, x), (y, y), (z, z), (w, w) \in R.$$

Completeness

A binary relation R is complete, for all $x, y \in S$, if $(x \neq y)$ then xRy or yRx .

R is taller than satisfies completeness,

$$R = \{(p2, p1), (p3, p2), (p3, p4), (p3, p1), (p4, p1), (p2, p4)\}$$

R : is atleast as good as satisfies completeness, if atleast these ordered pairs $(x, y), (y, z), (z, w), (x, z) \in R$.

Suppose $S' = \{p1, p2, p3, p4, p5\}$ and $p5=3000$. We cannot have $p3Rp5$ and $p3Rp5$. So R as is taller than is not complete when the set is S' .

Transitivity

For all $x, y, z \in S$, if xRy and yRz then xRz .

If x is atleast as good as y , and y is atleast as good as z , then x is atleast as good as z .

R atleast as good as satisfies transitivity for the set S , if $(x, y), (y, z) \in R$ then $(x, z) \in R$, if $(z, w), (x, z) \in R$ then $(x, w) \in R$.

Suppose $R = \{(x, y), (y, z), (x, z), (z, w), (x, w)\}$. This is transitive but not complete.

R : is taller than is transitive. When $(p_2, p_1), (p_3, p_2) \in R$ then $(p_3, p_1) \in R$. Check for all the combinations.

Suppose $X = \{x, y, z\}$ and $x = \text{grand father}$, $y = \text{father}$, $z = \text{son}$.
 R ; is son of. It is binary relation.

$$R = \{(z, y), (y, x)\}.$$

$(z, x) \notin R$. Therefore, this is not transitive.

Anti-symmetry:

For all $x, y \in S$, if xRy and yRx then $x = y$.

$X = \{2, 3, 4, 5, 2\}$.

R : is greater than equal to

$R = \{(3, 2), (2, 2), (2, 2), (5, 4), (4, 3), (4, 2), (4, 2)\}$.

It satisfies anti- symmetry.

R ; is taller than

It also satisfies anti-symmetry. It is trivially true.

R ; at as good as . This may not satisfy anti-symmetry.
We will see later than xRy and yRx implies $x \sim y$. If x is at least as good as y and y is atleast as good as x then x is indifferent to y .
They may not be same object.

Asymmetry:

For all $x, y \in S$, if xRY then $\sim yRx$ or yRx is not possible.

R ; is taller than . It satisfies asymmetry.

R : is brother of $X = \{x, y, z\}$ x = daughter, y = elder son, z = younger son.

$R = \{(y, x), (z, x), (y, z), (z, y)\}$. It violates asymmetry because $(y, z), (z, y) \in R$.

Symmetry:

For all $x, y \in S$, if xRy then yRx .

R : is atleast as good as

$R = \{(x, y), (y, x), (y, z), (z, x), (x, z), (z, x)\}$. The above R is symmetric.

$X = \{a, b, c\}$, $a =$ elder son, middle son, younger son. R : is brother of. It satisfies symmetry.

R : is taller than violates symmetry.

A binary relation satisfying reflexivity, completeness and transitivity is called ordering.

If a binary relation is ordering then based on that binary relation we can rank or order the alternatives.

$X = \{x, y, z\}$. R : is atleast as good as.

$R = \{(x, x), (y, y), (z, z), (x, y), (y, z), (x, z)\}$. This is an ordering, it satisfies all the three properties.

Now suppose $R = \{(x, x), (y, y), (z, z), (x, y), (x, z)\}$. This is not an ordering because it is not complete .

Now suppose $R = \{(x, x), (y, y), (z, z), (x, y), (y, z), (z, x)\}$. This is not an ordering.

A binary relation satisfying reflexivity and transitivity is called quasi ordering.

R : is atleast as good as is also weak preference. From this weak preference, we define strict preference (P) and indifference (I) relation.

xPy if and only if $[xRy \text{ and } \sim yRx]$.

x is preferred to y if and only if x is atleast as good y and y is not atleast as good as x .

$R = \{(x, x), (y, y), (z, z), (x, y), (y, z), (x, z)\}$. Here xPy , yPz and xPz .

$x \text{ly}$ if and only if $[xRy \text{ and } yRx]$.

x is indifferent to y if and only if x is atleast as good as y and y is atleast as good as x .

$$R = \{(x, x), (y, y), (z, z), (x, y), (y, x), (y, z), (z, y), (x, z), (z, x)\}.$$

Here we have $x \text{ly}$, $y \text{lz}$ and $x \text{lz}$.

Suppose $S = \{x, y, z\}$ and R : is atleast as good as.

An element x in S is a maximal element of S with respect to binary relation R if and only if there does not exist any y such that $y \in S$ and y is preferred to x .

In symbols

An element x in S is a maximal element of S with respect to binary relation R if and only if $\sim [\exists y : (y \in S \& yPx)]$.

An element of a set is maximal if it is not dominated or preferred by any other element of that set.

The set of maximal elements in S is called its maximal set and is denoted by $M(S, R)$.

When $R = \{(x, x), (y, y), (z, z), (x, y), (y, z), (x, z)\}$,
 $x \in M(S, R)$.

When

$R = \{(x, x), (y, y), (z, z), (x, y), (y, x), (y, z), (z, y), (x, z), (z, x)\}$,
 $x, y, z \in M(S, R)$.

An element x in S is a best element of S with respect to a binary relation R if and only if for all $y \in S$ x is atleast as good as y .

In symbols

An element x in S is a best element of S with respect to a binary relation R if and only if

$$\forall y : (y \in S \rightarrow xRy).$$

The set of best elements in S is called its choice set and is denoted $C(S, R)$.

When $R = \{(x, x), (y, y), (z, z), (x, y), (y, z), (x, z)\}$, $x \in C(S, R)$.

When

$R = \{(x, x), (y, y), (z, z), (x, y), (y, x), (y, z), (z, y), (x, z), (z, x)\}$,
 $x, y, z \in C(S, R)$.

From the definition of maximal element and best element, it is clear that if an element is a best element of a set, then it is also maximal element. So, the choice set is a subset of maximal set.
 $C(S, R) \subset M(S, R)$.

A maximal element may not be best element. Suppose $S = \{x, y, z\}$ and $R = \{(x, y), (z, y)\}$. In this example we have xPy and zPy . We do not have xRz and zRx . So The set of best element is empty. We have maximal elements $x, z \in M(S, R)$.

This implies that $M(S, R)$ is not a subset of $C(S, R)$.

We can have situations where both choice set and maximal set is empty.

Suppose $S = \{x, y, z\}$ and $R = \{(x, y), (y, z), (z, x)\}$. We have xPy , yPz and zPx . We dont have any $x \in S$ such that xRy for all $y \in S$. There is always an x such that xPz for all y . So maximal set is empty.

Indifference relation

for all $x, y \in S$, xRy and yRx then $x \sim y$

Preference relation

for all $x, y \in S$, xRy and $\sim yRx$ then xPy

Some results on strict preference and indifference

If R is an ordering, then for all $x, y, z \in S$

- 1) $xly \ \& \ ylz \rightarrow xIz.$
- 2) $xPy \ \& \ yIz \rightarrow xPz.$
- 3) $xly \ \& \ yPz \rightarrow xPz.$
- 4) $xPy \ \& \ yPz \rightarrow xPz.$

Proof of 1). $xly \ \& \ ylz \rightarrow (xRy \ \& \ yRx) \ \& \ (yRz \ \& \ zRy).$

$\rightarrow (xRy \ \& \ yRz) \ \& \ (zRy \ \& \ yRz)$

$\rightarrow xRz \ \& \ zRx$ from transitivity

$\rightarrow xIz.$

Proof of 2. $xPy \ \& \ yIz \rightarrow (xRy \ \& \ \sim yRx) \ \& \ (yRz \ \& \ zRy)$.

We have $xRy \ \& \ yRz$ implying xRz from transitivity.

Suppose zRx it implies xIz . We have yIz . Therefore using the first result we get xIy . We are given xPz . A contradiction. Therefore zRx is not true. We have xRz and $\sim zRx$, it implies xPz .

The proof of 3 is similar to the proof of 2.

Proof of 4. $xPy \ \& \ yPz \rightarrow (xRy \ \& \ \sim yRx) \ \& \ (yRz \ \& \ \sim zRy)$.

It implies $(xRy \ \& \ yRz)$. It implies xRz from transitivity.

Suppose zRx . We get zIx . Take yPz and zIx . Using the second result we have yPx . It is a contradiction because we have xPy .

Thus, we cannot have zRx . Therefore, we have xRz and $\sim zRx$. It implies xPz .

Result:

Any finite quasi ordered set has at least one maximal element.

Proof. Suppose the set of maximal set is empty. It implies that that for every $y \in S$ there exist one x such that xPy . Take for example $x_1, x_2, x_3 \in S$. From above we have x_1Px_2 , x_2Px_3 and x_3Px_1 . This implies that it violates transitivity. Therefore, R is not quasi ordered. This implies that set of maximal element is not empty.

We define choice function.

A choice function $C(S, R)$ defined over S is a functional relation such that the choice set $C(S, R)$ is non-empty for every non empty subset s of S .

It means that there is a non-empty choice set or there are best elements for every non-empty subset of S .

We have seen that whenever completeness is violated, choice set is empty. It implies that choice function does not exist.

If reflexivity is violated then also choice set is empty . For single element set, reflexivity is required for non-empty choice set.

If transitivity is violated, choice set can be empty. Example xPy , yPz and zPx . In this situation choice set is empty for $\{x, y, z\}$. We cannot have a choice function for this subset of S .

If R is an ordering defined over a finite set S , then a choice function $C(S, R)$ is defined over S .

Proof: Suppose the choice set over S is empty. This means that choice function is not defined. If choice set is empty it implies that it violates any one of the following property; completeness, reflexivity, or transitivity.

This implies that R is not an ordering if anyone of the above property is violated. This implies that choice set is not empty since R is an ordering. Thus, choice function defined for S .

We have proved in the last class

If R is an ordering, then for all $x, y, z \in S$

$$xPy \ \& \ yPz \rightarrow xPz$$

This is all the definition of quasi-transitivity.

If for all $x, y, z \in S$, xPy and $yPz \rightarrow xPz$, then R is quasi transitive.

We have shown that reflexivity, completeness and transitivity of R over S implies that there is a choice function $C(S, R)$.

We take a weaker condition quasi transitivity of R .

Result:

If R is reflexive, complete and quasi-transitive over a finite set X , then a choice function $C(S, R)$ is defined over X .

Proof: Suppose there are n elements in S and $S \subset X$. The elements are $x_1, x_2, x_3, \dots, x_n$. Take the first pair (x_1, x_2) , by reflexivity and completeness of R , there is a best element in this pair (x_1, x_2) .

Consider a set of j elements (x_1, x_2, \dots, x_j) . Suppose a_j is a best element of for the set (x_1, x_2, \dots, x_j) . This implies that $a_j Rx_k$ for $k = 1, 2, 3, \dots, j$.

Now consider the set $(x_1, x_2, \dots, x_j, x_{j+1})$, in this set we can have $a_j Rx_{j+1}$ or $x_{j+1} Ra_j$. If we have $a_j Rx_{j+1}$ then a_j is a best element for the set $(x_1, x_2, \dots, x_j, x_{j+1})$. So, there exist a best element for this set. It implies that choice function exist.

Suppose $x_{j+1}Pa_j$. In this x_{j+1} is not a best element only when x_kPx_{j+1} for some $k = 1, 2, 3\dots j$. Let this be x_k . We have x_kPx_{j+1} and $x_{j+1}Pa_j$. By quasi transitivity we have x_kPa_j . But we have a_jRx_k since a_j is the best element of the set (x_1, x_2, \dots, x_j) . A contradiction. So we cannot have x_kPx_{j+1} for some $k = 1, 2, 3\dots j$. Thus, x_{j+1} is a best element when $x_{j+1}Pa_j$.

Thus, we get that reflexivity, completeness and quasi transitivity implies that choice function can be defined.

We define Acyclicity

R is acyclical over S if and only the following holds:

for all $x_1, x_2 \dots, x_j \in S$, if $x_1Px_2 \ \& \ x_2Px_3 \ \& \ \dots \& x_{j-1}Px_j$ then x_1Rx_j .

It is much weaker than quasi transitivity condition.

Result:

If R is reflexive and complete, then a necessary and sufficient condition for $C(S, R)$ to be defined over finite S is that R be acyclical over S .

Proof: We first proof there necessary part, that is if $C(S, R)$ is defined over S then R must be acyclical over S given reflexivity and completeness are satisfied by R .

Suppose R is not acyclical. This implies that there is a subset of j alternatives in S such that $x_1Px_2, \dots, x_{j-1}Px_j$ and x_jPx_1 . This implies that there is no best element in this subset, so choice function is not defined over S .

Sufficiency part. Suppose all the alternatives are indifferent to each other. It implies all the elements are in the best element. Suppose there is only one pair satisfying the strict preference that is x_2Px_1 . We have x_2 is the best element. But x_2 cannot be best element of S if there is some element say x_3 in S such that x_3Px_2 . If x_1Px_3 then by acyclicity we have x_1Rx_2 . It contradicts x_2Px_1 . So x_3 is the best element of the set x_1, x_2, x_3 . If we continue in this way, we can exhaust all the elements and have best choice set being non-empty. We get the acyclicity is a sufficient condition for the choice function to be defined when reflexivity and completeness are given.

The existence of a choice function explains rational choice. By specifying certain properties on choice function we define rational choice.

For example suppose we choose x from the set x, y, z and choose y from the set x, y . We cannot rationalize this choice outcome. We define certain properties. They are consistency properties.

Property α : $x \in S_1 \subset S_2 \rightarrow [x \in C(S_2) \rightarrow x \in C(S_1)]$ for all x .
If some element of a subset S_1 of S_2 is best in the set S_2 then it must be best in S_1 .

It means if x is best in $\{x, y, z\}$ then x must be best in $\{x, z\}$ subset of $\{x, y, z\}$.

Property β :

$$[x, y \in C(S_1) \& S_1 \subset S_2] \rightarrow [x \in C(S_2) \text{ if and only if } y \in C(S_2)].$$

If x, Y are best element of a set S_1 which is a subset of S_2 , then if x is a best element of S_2 implies y is also a best element of S_2 and if y is a best element of S_2 it implies x is also a best element of S_2 .

Result:

Every choice function $C(S, R)$ generated by a binary relation R satisfies property α but not necessarily property β .

Proof: If x belongs to $C(S, R)$ then xRy for all $y \in S$. Therefore xRy for all y in all the subsets of S . This is property α .

Suppose we have xly , xPz and zPy for a triple x, y, z . Note that acyclicity is satisfied. $C([x, y]) = [x, y]$ and $C([x, y, z]) = [x]$. It violates property β .

We have seen in the last class that property α is always satisfied by choice function when it is defined. It means when that the binary relation satisfies reflexivity, completeness, and acyclicity.

Consider an example xly , xPz , zPy , then the choice sets are $C([x, y], R) = [x, y]$ and $C([x, y, z], R) = [x]$. It violates property β .

We have done that if R is an ordering then, for all $x, y, z \in S$,
 $xPy \ \& \ yIz \rightarrow xPz$.

When the above is satisfied by a binary relation R we call R to satisfy PI transitivity.

Relation R is PI transitive over S if and only if for all $x, y, z \in S$,
 $xPy \ \& \ yIz \rightarrow xPz$.

Result:

A choice function $C(S, R)$ generated by a binary relation R satisfies property β if and only if R is PI transitive.

Proof: R must satisfy reflexivity and completeness then only choice function can be defined. We have seen it. We need one more property that is acyclicity to be able to define choice function. We have seen that reflexivity, completeness and acyclicity does not satisfy property β .

Suppose PI transitivity is not satisfied. It implies that there is a triple $x, y, z \in S$ such that xRy , yRz and zRx . From this we $C([y, z], R) = [y, z]$. And $C([x, y, z], R) = [z]$ and $y \in C([x, y, z], R)$ is not true. It violated property β . If R is not PI transitive then property β is violated. We get that PI transitivity is a necessary condition.

We proof the sufficiency part. Suppose property β is violated. Then we have a pair $x, y \in S_1$ such that $x, y \in C(S_1, R)$ and $x \in C(S_2, R)$, $y \notin C(S_2, R)$ when $S_1 \subset S_2$. This implies that there exist a $z \in S_2$ such that zPy and xRz . Since $x, y \in C(S_1, R)$. It implies that xly . Suppose PI transitivity holds, so we have zPy and yIx implying zPx . We already have xRz . A contradiction. Therefore, PI transitivity cannot hold. Thus, we get PI implies that property β is satisfied.

We get that PI is a stronger condition then acyclicity.

We have when R satisfies reflexivity, completeness and PI transitivity, the choice function defined gives rational outcomes. It means that consistent outcomes.

If R_i is an ordering and it satisfies continuity and monotonicity then we can define a real valued function called utility function $U(x)$.

We move to social choice . We assume that X is the set of social states. There are n individuals. The preference relation of the i th individual is R_i , $i = 1, 2, 3, \dots, n$. R denotes the social preference relation.

We assume that each individual i has an ordering over the elements in X . Each R_i satisfies reflexivity, completeness and transitivity. We dont assume that R social preference relation has to satisfy these three properties. We will find when R is going to be an ordering.

A collective choice rule is a functional relation F such that for any set of n individual orderings R_1, R_2, \dots, R_n one and only social preference relation R is determined,

$$R = f(R_1, R_2, \dots, R_n).$$

Suppose x, y, z . There are three individual. The preference relation are:

1 2 3

x x x

y z yz

z y

The collective choice rule is $f \begin{pmatrix} x & x & x \\ y & z & yz \\ z & y & \end{pmatrix}$

A collective choice rule is decisive if and only if its range is restricted to complete preference relations R .

For all $x, y \in X$

$x\bar{R}y$ if and only if for all $i : xR_i y$.

$x\bar{P}y$ if and only if $x\bar{R}y$ & $\sim(y\bar{R}x)$. Pareto preference.

$x\bar{I}y$ if and only if $x\bar{R}y$ & $y\bar{R}x$. Indifference.

From these Pareto relations we define collective choice rule under certain condition.

Result:

Relation \bar{R} is a quasi-ordering for every logically possible combination of individual preferences.

Proof: \bar{R} is reflexive as for all $x \in X$: we have $xR_i x$. Since R_i are reflexive.

For all $x, y, z \in X$, $x\bar{R}y$ and $y\bar{R}z \rightarrow$ for all i $xR_i y$ and $yR_i z$.

Since R_i are transitive.

It implies for all i : $xR_i z$. This implies $x\bar{R}z$. Thus, \bar{R} is transitive.

Completeness can be violated. For example xP_1y and yP_2x then $x\bar{R}y$ and $y\bar{R}x$ are not defined.

Result:

A necessary and sufficient condition for \bar{R} to be an ordering and for $R = \bar{R}$ to be decisive collective choice rule is that for all $x, x, y \in X$ if there exist i such that xP_iy then xR_jy for all j .

Proof: For any pair x, y if xI_iy for all i , then the condition is trivially true. If for i xP_iy then for all j we must have xR_jy . It implies $x\bar{R}y$.

If condition is violated than there exist i such that xP_iy and there exist j such that yP_jx . It implies that $x\bar{R}y$ is not possible and $y\bar{R}x$ is not possible. So \bar{R} is not complete.

Logic:

We argue based on statements.

In mathematics, we are concern with two types of statements. — true or false. It can only take these two values.

- For example:

- 1) 2 is greater than 1 - true
- 2) There are finite numbers of integers - false
- 3) This sentence is false. (neither true or false.)

If the above sentence is true - then the sentence can't be false.
If the statement is false - the sentence is true. — \circlearrowleft Therefore, it can't be neither true or nor false.

- 4) $1+1=2$ (always true) - this statement is always true.
- 5) $x^2 \geq 1$, true when $x=1$ or $x=-1$.
It is false when $x=5$.

(2)

So the last sentence is true based on context. We need to specify the context.

Based on the simple sentences which are true or false we can form complex statements. This is through connectives.

logical connectives:

We do denote sentences through A and B.

① negation. , A can take two truth values: true, false.

| A | $\sim A$ |
|-------|----------|
| True | False |
| False | True |

~~If A is~~ when A is true, negation of A is false. And when A is false negation of A is true.

(3)

(4)

| | A | $\neg(\neg A)$ |
|-------|-------|----------------|
| True | True | True |
| False | False | False |

double negation
is same as the
original statement.

(2)

Conjunction of statements A, B is denoted as A and B

In symbol $A \wedge B$

Truth value of $A \wedge B$

| A | B | $A \wedge B$ |
|----------|-----------|--------------|
| True (T) | True | T |
| T | False (F) | F |
| F | T | F |
| F | F | F |

A statement -

A and B ($A \wedge B$)
is true when
both A and B
are true.

If when any one
of them is false
the statement is
false.

(3)

Disjunction of A, B is denoted as
 A or B , in symbol $A \vee B$

The truth table is:

(4)

| A | B | $A \vee B$ |
|---|---|------------|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

$A \vee B$ / A or B is a true statement when both A and B are true, any one of them (A, B) is true. It is false when both A, B are false.

3) Implication of statements A, B ,

If A then B , $A \rightarrow B$.

Truth table is :

| A | B | $A \rightarrow B$ |
|---|---|-------------------|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

The statement $A \rightarrow B$ is true when B is true and A is true. Whenever A is false, it is always true.

This statement is false when A is true and B is false.

4) Equivalence of statements A, B

A if and only if B , $\bullet \neg A \Leftrightarrow B$.

(5)

Truth table is:

| A | B | $A \leftrightarrow B$ |
|---|---|-----------------------|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

The statement

$A \leftrightarrow B$ (A if and only if B ,
is true when
both A and B ~~are~~ true,
and when both
 A and B are false.

The statement is false when one of them
 A or B is false.

$$(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$$

| A | B | $A \rightarrow B$ | $\neg A$ | $\neg B$ | $\neg B \rightarrow \neg A$ | $(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$ |
|---|---|-------------------|----------|----------|-----------------------------|---|
| T | T | T | F | F | T | T |
| T | F | F | F | T | F | T |
| F | T | T | T | F | T | T |
| F | F | T | T | T | T | T |

always true irrespective

of truth value of A & B

tautology.

6

$$\cancel{(A \wedge B)} \Leftrightarrow (\cancel{\sim A} \vee \cancel{\sim B}) \Leftrightarrow \cancel{(A \vee B)} \Leftrightarrow \cancel{\sim(A \wedge B)}$$

| A | B | $A \wedge B$ | $\sim A$ | $\sim B$ | $\sim A \vee \sim B$ | $\sim(\sim A \vee \sim B)$ | $(A \Rightarrow B) \Leftrightarrow \sim(\sim A \vee \sim B)$ |
|---|---|--------------|----------|----------|----------------------|----------------------------|--|
| T | T | T | F | F | F | T | T |
| T | F | F | F | T | T | F | T |
| F | T | F | T | F | T | F | T |
| F | F | F | T | T | T | F | T. |
| | | | | | | | |

$$(A \wedge B) \Leftrightarrow \sim(\sim A \vee \sim B) \quad / \text{Hau}\ddot{\text{o}}\text{ logy.}$$

=.

$$(A \Rightarrow B) \Leftrightarrow \sim(A \wedge \sim B)$$

| A | B | $A \Rightarrow B$ | $\sim B$ | $A \wedge \sim B$ | $\sim(A \wedge \sim B)$ | $(A \Rightarrow B) \Leftrightarrow \sim(A \wedge \sim B)$ |
|---|---|-------------------|----------|-------------------|-------------------------|---|
| T | T | T | F | F | T | T |
| T | F | F | T | T | F | T |
| F | T | T | F | F | T | T |
| F | F | T | T | F | T | T. |
| | | | | | | |

$$(A \Rightarrow B) \Leftrightarrow \sim(A \wedge \sim B) \text{ is a tautology.}$$

$$\sim(A \wedge B) \Leftrightarrow \sim A \vee \sim B$$

| A | B | $A \wedge B$ | $\sim(A \wedge B)$ | $\sim A$ | $\sim B$ | $\sim A \vee \sim B$ | $\sim(A \wedge B) \Leftrightarrow \sim A \vee \sim B$ |
|---|---|--------------|--------------------|----------|----------|----------------------|---|
| T | T | T | F | F | F | F | T |
| T | F | F | T | F | T | T | T |
| F | T | F | T | T | F | T | T |

$\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$ (7)
It is a tautology.

Try $\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$

Consider a set of statements like

~~A, B, C~~ A, B, C, D, E, F

- (1) $A \rightarrow B$
- (2) $\sim B \rightarrow C$
- (3) $D \rightarrow E$
- (4) $E \wedge G \rightarrow F$
- (5) $B \wedge \sim A \rightarrow \sim E$
- (6) D

$\sim A \wedge F$

The above set of statements are connected in the given ways. Based on these statements we get the implication that $\sim A \wedge F$.

(8)

He will argue in the following way:

Assume each of the ~~the~~ 6 statements
~~are~~ is true.

(6) D is true and (3) is true
 so E is true.

~~For~~ Statement (5) to be true, B ~~is~~ is
 true, ~~so~~ false.

so for statement ~~as~~ (1) to be true
 A is false..

~~Since~~ since B is false so $\sim B$ is
 true. so statement (2) is true
 when C is true.

C is true and E is true so F is
 true, since statement (4) is true

~~We can conclude that~~
 since A is false so $\sim A$ is true.
 so ~~A~~ and $\sim A \wedge F$ is true.

Another Example

$$① M \wedge A \rightarrow (S \rightarrow L)$$

$$② M \wedge \sim A \rightarrow \sim L$$

$$③ C \rightarrow (B \leftrightarrow L)$$

$$\overline{MAC \rightarrow (A \wedge B) \vee (\sim A \wedge \sim B)}$$

① MAC is true

and $(A \wedge B) \vee (\sim A \wedge \sim B)$ is true

- $MAC \rightarrow (A \wedge B) \vee (\sim A \wedge \sim B)$ is true.

C is true & ③ it implies
 $(B \leftrightarrow L)$ is true for $C \rightarrow (B \leftrightarrow L)$
to be true.

$B \leftrightarrow L$ is true when
B and L is true.

Or B and L both false

- If when L is true, then $\sim L$ is false
This implies. $M \wedge \sim A$ must be false

So for $M \wedge \sim A \rightarrow \sim L$.

If A is true, ^{it implies} _{then} $M \wedge \sim A$ is false.

—
M is true and suppose A then

$M \wedge A$ is true. It means implies

$S \rightarrow L$ must be true.

$S \rightarrow L$ is always true.
when L is true.

True.

—
When A, M, B, L, C are true all the
true statement are true.

—
Now $M \wedge C$ is true.

A and B is true so

$(A \wedge B) \vee (\sim A \wedge \sim B)$ is also true.

Quantifiers and Context

For example $x^2=1$, is either statement true for all integers.
we need to specify the set. (context).

\exists : for all values.

\exists : there exists, for some (existential quantifier);
 \forall : for all, for every (universal quantifier).

There exists an integer x such that
 $x^2=1$.

$x^2=1$ is true for $x=1$ and $x=-1$
 both of them are integers.

$$(\exists x)(x \in I)(x^2=1) \quad (\text{In symbols})$$

Examples of quantifiers:

For every integer x , there exists
 an integer y such that $x+y=0$.

$$(\forall x)(\exists y)(x+y=0).$$

Explanation: Take any integer x . 10,
 then, $\text{if } x=10, y=-10, x+y=0$

This is true statement!

(12)

Ex: There exists an integer y , such that-
 for every integer x , $\text{then } x+y=0$

$$(\exists y)(\forall x)(x+y=0)$$

Take any two integers $= 10, 20$, $x=10$, $x=20$.
and fix can we have a same y
 such that $10+y=0$, and $20+y=0$

- Not possible. So the above statement is false.
- The sequence of quantifiers ~~are~~ is very important. It changes the meaning.

Completeness can be violated. For example xP_1y and yP_2x then $x\bar{R}y$ and $y\bar{R}x$ are not defined.

Result:

A necessary and sufficient condition for \bar{R} to be an ordering and for $R = \bar{R}$ to be decisive collective choice rule is that for all $x, y \in X$ if there exist i such that xP_iy then xR_jy for all j .

Proof: For any pair x, y if xI_iy for all i , then the condition is trivially true. If for i xP_iy then for all j we must have xR_jy . It implies $x\bar{R}y$.

If condition is violated than there exist i such that xP_iy and there exist j such that yP_jx . It implies that $x\bar{R}y$ is not possible and $y\bar{R}x$ is not possible. So \bar{R} is not complete.

We define Pareto Optimality:

For any n tuple of individual preferences (R_1, R_2, \dots, R_n) , a state $x \in X$ is Pareto-optimal in X if and only if $\sim [\exists y \in X : y \bar{P} x]$.

A Pareto-optimal state is also called economically efficient.

It also implies that if an alternative or social state is preferred over other alternatives by all the individuals then society should also prefer it over other alternatives.

If $x P_i y$ for all i then society should prefer x over y .

If x is not Pareto optimal it means that there exist an alternative Y which is preferred to x by all individuals.

Result:

For every set of individual preferences (R_1, R_2, \dots, R_n) over any finite set of social states X , there is at least one Pareto optimal state.

Proof: The binary relation \bar{R} already defined is Pareto relation. We have already shown that \bar{R} is quasi ordering. It is due to R_i being orderings. And Pareto optimal set is the maximal set of X with respect to \bar{R} . So set of Pareto optimal set is $M(X, \bar{R})$. We have also proved that Maximal set is non-empty when X is finite and the binary relation is quasi ordering. Since \bar{R} is quasi ordering and X is finite, so $M(X, \bar{R})$ is non-empty.

Example: $X = \{x, y, z\}$.

Preference of the individuals:

| | 1 | 2 | 3 |
|--|---|---|---|
|--|---|---|---|

| | | |
|---|---|---|
| x | x | x |
|---|---|---|

| | | |
|---|----|---|
| z | zy | y |
|---|----|---|

| | | |
|---|--|---|
| y | | z |
|---|--|---|

Here in this example x is only Pareto optimal state. y is not Pareto optimal because xPy for all individuals. Similarly z is not Pareto optimal, xPz for all individuals.

Another example

$$X = \{x, y, z\}.$$

Individual preferences:

1 2 3

x y x

y x z

z z y

Here in this example x and y are Pareto optimal state. z is not Pareto optimal because xPz for all individuals.

Assumptions

A social welfare function (SWF) is a collective choice rule f ,, the range of which is restricted to the set of ordering over X .
We impose certain conditions on this social welfare function f .

Unrestricted domain (Condition U); The domain of the rule f must include all logically possible combinations of individual orderings.

f must have an image for all possible individual orderings over social states.

Example: $X = \{x, yz\}$. Examples of possible orderings are

| | | | | | | | | |
|---|---|---|---|-----|---|---|---|----|
| 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| x | y | z | z | xyz | y | y | x | zx |
| y | z | x | y | | z | z | z | y |
| z | x | y | x | | x | x | y | |

Take the example of Pareto principle, we have \bar{R} based on R_i . This \bar{R} is defined if it is valid for all individuals. If we have

| | | |
|---|---|---|
| 1 | 2 | 3 |
| x | y | z |
| y | z | x |
| z | x | y |

We dont have $x\bar{R}y$, $x\bar{R}y$, $z\bar{R}y$, $y\bar{R}z$, $x\bar{R}z$ $z\bar{R}x$.

\bar{R} is not complete, in this case.

Majority decision rule:

It is a collective choice rule where x is socially at least as good as y if and only if at least as many people prefer x to y as prefer y to x that $N(xPy) \geq N(yPx)$.

It violates unrestricted domain.

Take example , 6 individuals, xP_iy for 4 individuals, yP_iz for 4 individuals and zp_ix for 3 individuals. The social welfare violates transitivity.

Pareto Principle (Condition P): For any $x, y \in X$, for all i
 $xP_i y \rightarrow xPy$.

If everyone prefers x to y then society must also prefer x to y .
The social welfare function violates Pareto principle when
everybody prefer x to y and the social relation gives y is preferred
to x .

Independence of irrelevant alternatives (Condition I): let R and r' be the social binary relations determined by f corresponding respectively to two sets of individual preferences, (R_1, R_2, \dots, R_n) and R'_1, R'_2, \dots, R'_n . If for all pairs of alternatives x, y in a subset of S of X , $xR_iy \leftrightarrow xR'_iy$, for all i then $C(S, R)$ and $C(S, R')$ are the same.

If for any two alternative x and y , the two binary relations R_i and R'_i have same relations then social preference must be same over x and y .

Pareto Principle (Condition P): For any $x, y \in X$, for all i
 $xP_i y \rightarrow xPy$.

If everyone prefers x to y then society must also prefer x to y .
The social welfare function violates Pareto principle when
everybody prefer x to y and the social relation gives y is preferred
to x .

Example

$$\begin{matrix} 1 & 2 & 3 \\ x & y & x \\ y & z & y \\ z & x & z \end{matrix} \quad f \begin{pmatrix} x & y & y \\ y & z & y \\ z & x & z \end{pmatrix} = \begin{pmatrix} x \\ yz \end{pmatrix}$$

Independence of irrelevant alternatives (Condition I): let R and R' be the social binary relations determined by f corresponding respectively to two sets of individual preferences, (R_1, R_2, \dots, R_n) and R'_1, R'_2, \dots, R'_n . If for all pairs of alternatives x, y in a subset of S of X , $xR_iy \leftrightarrow xR'_iy$, for all i then $C(S, R)$ and $C(S, R')$ are the same.

If for any two alternatives x and y , the two binary relations R_i and R'_i have same relations then social preference must be same over x and y .

Borda Rule:

The preferences of 3 individuals over $\{x, y, z\}$ are;

1 2 3

x y x

y z z

z x y

In Borda rule rule assign number integers to each alternatives of individuals. The individuals have report their full preference ordering. Since there are three alternatives, so 2 points is given to the best alternatives, 1 is given to the second position and 0 is given to the last position. We aggregate the scores of each alternatives based on the individual preference ordering. The social preference relation is based on the aggregate score.

| | | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| x | 2 | y | 2 |
| y | 1 | z | 1 |
| z | 0 | x | 0 |

The aggregate score of x is $2 + 0 + 2 = 4$

$$y : 1 + 2 + 0 = 3$$

$$z : 0 + 1 + 1 = 2$$

The social preference relation based on these scores is xPy , yPz and xPz . This is Borda outcome.

$$f \begin{pmatrix} x & y & x \\ y & z & z \\ z & x & y \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Take another preference profile of three individuals, name is B and previous one is A

1 2 3

x y x

y z zy

z x

If we consider preference profile of A and B over the alternative x, y , we see that they are same.

xP_1y, yP_2x, xP_3y in A.

xP'_1y, yP'_2x, xP'_3y in B.

Only difference between A and B is between z , and y for individual 3.

According to independence of irrelevant alternatives, the social preference relation between x and y should not be different in case of A and B.

Borda count of B is

1 2 3

x 2 y 2 x 2

y 1 z 1 zy 1

z 0 x 0

So $x; 2 + 0 + 2 = 4$

$y : 1 + 2 + 1 = 4$

$z : 0 + 1 + 1 = 2.$

The social preference relation based on these scores is xly , xPz and yPz .

We see that the social preference relation between x, y is not same in A and B. Therefore Borda rule violates independence of irrelevant alternative condition.

Majority rule: it satisfies independence of irrelevant alternatives.

Individual 1: xyz , Individual 2: zxy , Individual 3: yxz

$n(xpy) = 2$ and $n(yPx) = 1$.

Now change the position of z .

Like In Individual 1 $x(yz)$, zxy , $(zx)y$.

Individual 2 $x(yz)$, $(zx)y$, xzy .

Individual 3: yzx , zyx , $(yz)x$ and $y(xz)$. With any of these changes in the preferences of z , $n(xpy) = 2$ and $n(yPx) = 1$ remain same.

Non-dictatorship (Condition D); There is no individual i such that for every element in the domain of rule f , for all $x, y \in X$,
 $xP_i y \rightarrow xPy$.

There should not be any individual such that the social preference is based on the preference of this person.

For example if $xP_1 y$ and $yP_i x$ for all i except 1 and social preference is xPy . It violates non dictatorship.

Further we assume that there are atleast two individuals and atleast three alternatives.

A set of individuals V is almost decisive for x against y if xPy whenever xP_iy for every $i \in V$. and yP_ix for every $i \notin V$.

For example

$$\begin{matrix} 1 & 2 & 3 \\ x & y & x \\ y & z & z \\ z & x & y \end{matrix} \quad f \begin{pmatrix} x & y & x \\ y & z & z \\ z & x & y \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

For x, y individual 1 and 3 forms a almost decisive set.

A set of individuals V is decisive for x against y if xPy when xP_iy for every $i \in V$.

$$\begin{matrix} 1 & 2 & 3 \\ x & z & x \\ y & yx & z \\ z & y & \end{matrix} \quad f \begin{pmatrix} x & z & x \\ y & yx & z \\ z & & y \end{pmatrix} = \begin{pmatrix} x \\ yz \end{pmatrix}$$

Individual 1 and 3 forms a decisive set for x, y . It is almost decisive for x, z .

Suppose there is an individual J who is almost decisive over x and y and denote it by $D(x, y)$. $D(\bar{x}, y)$ denotes that J is decisive over x, y .

A set of individuals V is almost decisive for x against y if xPy whenever xP_iy for every $i \in V$ and yP_ix for every $i \notin V$.

A set of individuals V is almost decision for a pair $x, y \in X$,
 $[xP_iy \forall i \in V \& yP_ix \forall i \notin V] \rightarrow xPy$.

A set of individuals V is decisive for x against y if xPy when xP_iy for every $i \in V$.

A set of individuals V is decisive for a pair $x, y \in X$
 $[xP_iy \forall i \in V] \rightarrow xPy$.

Notice that it is possible that a set V can be almost decisive but not decisive.

Suppose $xP_iy \forall i \in V$ is true and there exists $j \notin V$ such that xR_jy and the social preference is yRx . It satisfies almost decisive but not decisive.

Therefore, it is not always true that $D(x, y) \rightarrow \bar{D}(x, y)$.

But $\bar{D}(x, y) \rightarrow D(x, y)$.

If $xP_iy \forall i \in V$ and $j \notin V$, xR_jy , and xPy then if part of the definition of almost decisive set is false so the state is trivially true. It is decisive.

Lemma 1

If there is some individual J who is almost decisive for any ordered pair of alternative, then an Social welfare function satisfying conditions U , P , and I implies that J must be a dictator.

Proof: Suppose that person J is almost decisive for some x against some y , it means $\exists x, y \in X : D(x, y)$. Let z be another alternative and let i refers to all individuals other than J . Assume $xP_jy \& yP_jz$, and that $yP_ix \& yP_iz$. We have not specified the preferences of the persons other than J between x and z .

Now $[D(x, y) \& xP_jy \& yP_iz] \rightarrow xPy$. Further $[yP_jz \& yP_iz] \rightarrow yPz$ from the condition P Pareto principle. But $[xPy \& yPz] \rightarrow xPz$, by the transitivity of the strict social preference relation.

We have xPz without any assumption on the preference relation over x and z on part of the individuals other than J . We have only assumed yP_jz and yP_jx . Now, these ranking of x and y and the ranking of y and z has any effect on the ranking of x and z it violates condition I , independence of irrelevant alternatives.. Hence xPz must be independent of the assumptions on x and y and y and z . Also, xPz is the consequence of xP_jz alone without having any effect from the ordering of i s. This means that J is decisive for x against z .

We get $D(x, y) \rightarrow \bar{D}(x, z)$ (1).

Now suppose $zP_jx \& xP_jy$, and $zP_ix \& yP_ix$. We have zPx from Pareto condition. And $D(x, y) \& xP_jy \& yP_ix$, we have xPy . Using transitivity of strict social preference relation we have,
 $zPx \& xPy \rightarrow zPy$. Again, we have got the social relation over z and y without specifying the individual preference relation of i s.
Hence J is decisive for z against y .

We get $D(x, y) \rightarrow \bar{D}(z, y)$ (2).

We can also show $D(x, z) \rightarrow \bar{D}(y, z)$ (3)
by interchanging z with y in (2).

Again putting x in place of z , z in place of y , and y in place of x , we obtain from (1)

$$D(y, z) \rightarrow \bar{D}(y, x) \text{ (4).}$$

We have $D(x, y) \rightarrow \bar{D}(x, z)$ from (1)

$\rightarrow D(x, z)$ using the definition of decisive and almost decisive

$\rightarrow \bar{D}(y, z)$ from (3)

$\rightarrow D(y, z)$ from definition

$\rightarrow \bar{D}(y, x)$ from (4).

We get that $D(x, y) \rightarrow \bar{D}(y, x)$ (5).

By interchanging x and y in (1), (2) and (5), we get

$$D(x, y) \rightarrow [\bar{D}(y, z) \& \bar{D}(z, x) \& \bar{D}(x, y)] \text{ (6).}$$

Now, $D(x, y) \rightarrow \bar{D}(y, x)$ from (5)
 $\rightarrow D(y, x)$.

Hence from (6) we have

$D(x, y) \rightarrow [\bar{D}(y, z) \& \bar{D}(z, x) \& \bar{D}(x, y)]$ (7).

Combining (1), (2), (5) and (7), it is seen that $D(x, y)$ implies that individual J is decisive for every ordered pair of alternatives from the set of three alternative $\{x, y, z\}$ given the condition U, P and I . Thus, J is a dictator over any set of three alternatives containing x and y .

Now, consider a larger number of alternatives. Take any two alternatives u and v out of the entire set of alternatives. If u and v are so chosen that they are same as x and y , then $\bar{D}(u, v)$ holds, as can be shown by taking a triple consisting of u, v and any other alternative z . If one of u and v is same as one of x and y , say u and x are same but not v and y , then take the triple consisting of x (or u), y , and v . Since $D(x, y)$, holds it again follows that $\bar{D}(u, v)$ and also $\bar{D}(v, u)$.

Let both u and v be different from x and y . Now first take $\{x, y, u\}$ and we get $\bar{D}(x, u)$ which implies $D(x, u)$. Now take the triple $\{x, u, v\}$. Since $D(x, u)$, it follows from previous argument that $\bar{D}(u, v)$ and also $\bar{D}(v, u)$. Thus, $D(x, y)$ for some x and y , implies $\bar{D}(u, v)$ for all possible ordered pairs (u, v) . Therefore, individual J is a dictator.

Example

| | | |
|---|---|---|
| 1 | 2 | 3 |
| x | y | y |
| y | z | x |
| z | x | z |

Suppose individual 1 is the almost decisive individual over x, y . We have yP_2x and yP_3x . So social preference over x, y is xPy . We have $yP_iz, i = 1, 2, 3$. So from Pareto principle, we have yPz . Now using transitivity we have $xPy \& yPz \rightarrow xPz$.

| | 1 | 2 | 3 |
|---|---|---|---|
| x | | y | z |
| y | | z | y |
| z | | x | x |

Here in this example individual 1 is almost decisive over x, y . So we have xPy . For other pair (x, z) and (y, z) , we cannot say anything by simply using the properties U, P, I . Thus, in this case we do not have a social preference relation.

Here, the if part in the second part of the statement of lemma is false. So lemma is trivially true when there is an almost decisive individual.

Theorem: There is no social welfare function satisfying conditions U , P , I and D .

Proof: For any pair of alternatives, there is atleast one decisive set, the set of all individuals. It is due to Pareto principle. Thus, for every pair of alternatives there is also at least one almost decisive set, since a decisive set is also almost decisive. Lets compare all the sets of individuals that are almost decisive for some pair, it may not be same pair of alternatives. From these set of almost decisive sets chose the smallest decisive set. Let this set be called V and let it be almost decisive for x against y .

If V contains only one individual, then the lemma we get that individual is decisive over all pairs of alternative, thus dictator. Suppose V contains two or more individuals. We divide V into two parts, V_1 containing a single individual, and V_2 contains the rest of V . V_3 contains all individuals not contain in V .

Due to U we can take any logically possible combination of individual orderings. Suppose we have

For all $i \in V_1$, $xP_i y \& yP_i z$.

For all $j \in V_2$, $zP_j x \& xP_j y$.

For all $k \in V_3$, $yP_k x \& zP_k x$.

Since V is almost decisive for x against y and since X is preferred to y for all individuals in V and y is preferred to x for all in V_3 , so xPy .

Note that zP_jy for $j \in V_2$ and for the rest we have y is preferred to z . It implies that V_2 is almost decisive for y, z , if we have zPy . Then we get a contradiction because V_2 is smaller than V . We have assume that V is the smallest almost decisive set. Therefore, we cannot have zPy . It implies that for y and z pair we have yRz . Now $xPy \& yRz \rightarrow xPz$. Now only individual in V_1 prefers x to z and all others prefer z to x . Thus, we have V_1 as almost decisive set for the pair x and z . Thus, we get a contraction since we assume V is the smallest almost decisive set. Thus, our assumption that V contains two or more is not true. Thus, V contains only one individual. Using the lemma we get that this individual is decisive for all the pairs of alternatives. Thus, there is a dictator whenever a social welfare function satisfies U, P and I .

Example:

1 2 3

x z x

z x y

y y z

What can we say about the social preference?

From Pareto principle we can say xPy . We cannot say anything further based on the four conditions of Arrow's Theorem.

Suppose We have one more additional information that yPz . Then using transitivity we have xPz . Thus the social preference is

$$f \begin{pmatrix} x & z & x \\ z & x & y \\ y & y & z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Note here that individual 3 has become the dictator.
The additional information is giving that individual 3 is almost decisive for the pair y, z .

Instead if we had the additional information zPy . In this we have xPy and zPy . We cannot say anything about z and x .
We need further information, this can be either zPx or xPz . If zPx then we individual 2 as dictator. If xPz the we have individual 1 as dictator.

If we have xIz . We dont have any dictator. In this case also we need additional restriction. Here it is not in the form of almost decisive set.

Example:

| | | |
|---|---|---|
| 1 | 2 | 3 |
| x | z | x |
| y | x | z |
| z | y | y |
| t | t | t |

From the Pareto principle we get

xPy , yPt , zPt , and xPt .

We need additional information on y and z pair. If yPz , then by transitivity we have xPz . Thus, the social preference ordering is

$$f \begin{pmatrix} 1 & 2 & 3 \\ x & z & x \\ y & x & z \\ z & y & y \\ t & t & t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}.$$

Note Individual 1 is decisive, so the dictator.

Condorcet Paradox:

There exists profiles of individual preferences such that method of majority decision does not constitute a social welfare function.

Proof: Suppose there are three individual, The preference profile is

1 2 3

x y z

z x y

y z x

The method of majority decision gives social preference relation in the following way

$$xRy \leftrightarrow N(i \in N, xP_i y) \geq N(i \in N, yP_i x).$$

$$xRy \leftrightarrow N(i \in N, xR_i y) \geq N(i \in N, yR_i x).$$

Here, $N(yP_ix) = 2 > N(xP_iy) = 1$ it implies yPx .

$N(zP_iy) = 2 > N(yP_iz) = 1$, it implies zPy .

$N(xP_iz) = 2 > N(zP_ix) = 1$, it implies xPz .

We have $zPy \& yPx \rightarrow zPx$, from transitivity. But we have xPz . It violates transitivity. Therefore, it is not a social welfare function.

Voting Methods:

- Based on scoring method:
 - Plurality Method
 - Borda Count
- Pairwise comparison
 - Simple Majority
 - Absolute Majority
 - Two-third Majority
- Voting procedure
 - Instant run-off

Scoring Method

Based on preferences of the individual scores are given to the alternative. Based on aggregate score we derive the social preference.

Suppose individual preferences are

| | | |
|---|---|---|
| 1 | 2 | 3 |
| x | y | y |
| y | z | x |
| z | x | z |

Plurality Method: scoring rule is $(1, 0, 0)$. Give one point to the most preferred alternative and zero to all other alternatives. So, the scores in above preference profile is

| | 1 | 2 | 3 |
|-----|---|-----|---|
| x | 1 | y | 1 |
| y | 0 | z | 0 |
| z | 0 | x | 0 |

Aggregate scores are $x : 1$, $y : 2$, $z : 0$.

So the social preference is

$$\begin{pmatrix} y \\ x \\ z \end{pmatrix}$$

The winner in the election using plurality voting method is y .

Borda Count: scoring rule is $(2, 1, 0)$. If there are four alternatives, the scoring rule is $(3, 2, 1, 0)$. The scores in the given individual preference profile is

| | | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| x | 2 | y | 2 |
| y | 1 | z | 1 |
| z | 0 | x | 0 |

The aggregate scores are

$x : 3, y : 5, z : 1$.

The social preference ordering is

$$\begin{pmatrix} y \\ x \\ z \end{pmatrix}.$$

The winner is y , if this method is used.

The social ranking may not same in the case of Plurality and Borda count.

1 2 3

x y y

z x xz

y z

Plurality rule gives social preference as $\begin{pmatrix} y \\ x \\ z \end{pmatrix}$.

Borda Count method gives social preference as

$\begin{pmatrix} xy \\ z \end{pmatrix}$.

Pairwise comparison

Simple majority : $xRy \leftrightarrow N(xP_iy) \geq N(yP_ix)$.

1 2 3

x y y

y z x

z x z

Simple majority rule: $N(xP_iy) = 1 < N(yP_ix) = 2$,

$N(yP_iz) = 3 > N(zP_iy) = 0$, $N(xP_iz) = 2 < N(zP_ix) = 1$.

Social preference: yPx , yPz , xPz .

Absolute majority: $xRy \leftrightarrow N(xP_iy) \geq \frac{N}{2} + 1$. If we have xRy based on absolute majority , we will also have xPy since yRx is not possible.

1 2 3

x y y

y z x

z x z

Absolute majority: $N(xP_iy) = 1 < N(yP_ix) = 2$,
 $N(yP_iz) = 2 > N(zP_iy) = 1$, $N(xP_iz) = 2 < N(zP_ix) = 1$.
Social preference : yPx , yPz , xPz .

Two-third majority: $xRy \leftrightarrow N(xP_iy) \geq \frac{2N}{3}$.

1 2 3

x y y

y z x

z x z

Two-third majority: $N(xP_iy) = 1 < N(yP_ix) = 2$,

$N(yP_iz) = 3 > N(zP_iy) = 0$, $N(xP_iz) = 2 < N(zP_ix) = 1$.

Social preference : yPx , yPz , xPz .

Example:

| | | |
|---|---|----|
| 1 | 2 | 3 |
| x | x | yz |
| y | z | x |
| z | y | |

Simple majority: xPy , yIz and xPz .

Absolute majority: xPy and xPz , not possible to compare y and z .

It is not complete.

Two-third majority: xPy and xPz , not possible to compare y and z . It is not complete.

Possibility result:

Simple majority violates transitivity. If we move away from social welfare function which requires transitivity, can we generate a reflexive, complete and acyclic social ordering. So that we have social choice function .

A social decision function is a collective choice rule f , the range of which is restricted to those preference relations R , each of which generates a choice function $C(S, R)$ over the whole set of alternatives X .

Result: There is an Social decision function satisfying condition U, P, I and D .

Proof: We provide one example. Since we have to show existence, it proofs the statement.

Suppose $xRy \leftrightarrow \sim [(\forall i : yR_i x) \& (\exists i : yP_i x)]$.

This R is reflexive and complete. It is easy to see that P is satisfied. For pair x, y , the social relation is based on individual preferences over x and y , there is no role any other alternative , say z . So it satisfies I .

Suppose xP_1y and yP_2x and xP_3y , here we have both xRy and yRx , so social preference is xly . The condition D is satisfied.

We have to show that quasi-ordering is satisfied.

$$[xPy \& yPz] \rightarrow [\{\forall i : xR_i y \& \exists i : xP_i y\} \& \forall i : yR_i z]$$

$$\rightarrow [\forall i : xR_i z \& \exists i : xP_i z].$$

$$\rightarrow xPz.$$

Thus, R is quasi-transitive. We know that if R is quasi transitive, complete and reflexive, there is going to be best element for each non-empty subset S . Thus, there exist a social decision function.

A collective choice rule gives x is preferred to y if it is Pareto-superior to y . xRy if y is not Pareto superior to x .

1: xyz , 2: yzx , 3: zxy .

We have y is not Pareto superior to x , x is not Pareto superior to y , y is not Pareto superior to z , z is not Pareto superior to y . So We have xly and ylz . For x, z also, we have xlz .

Individual Choice:

- Suppose there are three alternatives x, y, z . An individual has to choose from this set of three alternatives.
- Alternative we ask the individual to rank or order the alternatives.
- The choice or ranking is based on preference of the individual over the alternatives.
- We assume that the preferences are given. Taking a given type of preferences of the individuals whether the individual can rank or order all the alternatives or make choices.
- We do not study the way preferences are formed.

- Preferences over alternatives are based on binary relation.
- What is a binary relation? It is relation between the elements of a set. For example “is brother of” is a binary relation. Suppose the set is family members. $S = \{a, b, c, d\}$ is the set of family members. a= father, b= mother, c= elder son, d = younger son.
- In this example we have c is brother of d , and d is brother of c. a is not brother of d, a is not brother of b, a is not brother of c, b is not brother of a, b is not brother of d, b is not brother of c.
- We define it technically. We denote the binary relation by R . Let the set be $S = \{A, B, C, D\}$. We get cRd and dRc , $\sim aRb$ etc. In this example the binary relation can be represented as $R = \{(c, d), (d, c)\}$. A binary relation is a set of ordered pairs (x, y) such that x and y belongs to S where S is the set of alternatives.

Another example of binary relation “ is taller than ”. Suppose the set of alternative is $S = \{p1, p2, p3, p4\}$ these are the heights of four peaks, $p1= 1000\text{ft}$, $p2=2000\text{ ft}$, $p3= 3000\text{ ft}$, $p4= 1500\text{ ft}$. R is taller than so we have

$$R = \{(p2, p1), (p3, p2), (p3, p4), (p3, p1), (p4, p1), (p2, p4)\}.$$

We define the individual choice based on binary relation. The binary relation we take is R : atleast as good as .

$$S = \{x, y, z, w\},$$

$$xRy, yRx, xRz, xRw, xRx, yRy, yRw, zRz, wRy, zRz.$$

We can write it as

$$\begin{aligned} R = \{ & (x, y), (y, x), (x, z), (x, w), (x, x), \\ & (y, y), (y, w), (z, z), (w, y), (w, w) \} \end{aligned}$$

Properties of binary relation

$$S = \{x, y, z, w\}.$$

Reflexivity

A binary relation R is reflexive, for all $x, \in S$, xRx .

R : is taller than is not reflexive, p1 is taller than p1 is not possible.

R : is atleast as good as is reflexive when

$$(x, x), (y, y), (z, z), (w, w) \in R.$$

Completeness

A binary relation R is complete, for all $x, y \in S$, if $(x \neq y)$ then xRy or yRx .

R is taller than satisfies completeness,

$$R = \{(p2, p1), (p3, p2), (p3, p4), (p3, p1), (p4, p1), (p2, p4)\}$$

R : is atleast as good as satisfies completeness, if atleast these ordered pairs $(x, y), (y, z), (z, w) \in R$.

Suppose $S' = \{p1, p2, p3, p4, p5\}$ and $p5=3000$. We cannot have $p3Rp5$ and $p3Rp5$. So R as is taller than is not complete when the set is S' .

Transitivity

For all $x, y, z \in S$, if xRy and yRz then xRz .

If x is atleast as good as y , and y is atleast as good as z , then x is atleast as good as z .

R atleast as good as satisfies transitivity for the set S , if $(x, y), (y, z) \in S$ then $(x, z) \in R$, if $(z, w), (x, z) \in R$ then $(x, w) \in R$.

Suppose $R = \{(x, y), (y, z), (x, z), (z, w), (x, w)\}$. This is transitive but not complete.

R : is taller than is transitive. When $(p_2, p_1), (p_3, p_2) \in R$ then $(p_3, p_1) \in R$. Check for all the combinations.

Suppose $X = \{x, y, z\}$ and $x = \text{grand father}$, $y = \text{father}$, $z = \text{son}$.
 R ; is son of. It is binary relation.

$$R = \{(z, y), (y, x)\}.$$

$(z, x) \notin R$. Therefore, this is not transitive.

Anti-symmetry:

For all $x, y \in S$, if xRy and yRx then $x = y$.

$X = \{2, 3, 4, 5, 2\}$.

R : is greater than equal to

$R = \{(3, 2), (2, 2), (2, 2), (5, 4), (4, 3), (4, 2), (4, 2)\}$.

It satisfies anti-symmetry.

R ; is taller than

It also satisfies anti-symmetry. It is trivially true.

R ; at as good as . This may not satisfy anti-symmetry.
We will see later than xRy and yRx implies $x \sim y$. If x is at least as good as y and y is atleast as good as x then x is indifferent to y .
They may not be same object.

Asymmetry:

For all $x, y \in S$, if xRy then $\sim yRx$ or yRx is not possible.

R ; is taller than . It satisfies asymmetry.

R : is brother of $X = \{x, y, z\}$ $x =$ daughter, $y =$ elder son, $z =$ younger son.

$R = \{(y, x), (z, x), (y, z), (z, y)\}$. It violates asymmetry because $(y, z), (z, y) \in R$.

Symmetry:

For all $x, y \in S$, if xRy then yRx .

R : is atleast as good as

$R = \{(x, y), (y, x), (y, z), (z, x), (x, z), (z, x)\}$. The above R is symmetric.

$X = \{a, b, c\}$, $a =$ elder son, middle son, younger son. R : is brother of. It satisfies symmetry.

R : is taller than violates symmetry.

Problem Set

1. Show whether the following logical sentences are tautologies.
 - (a) $\sim(P \rightarrow Q) \leftrightarrow (P \wedge \sim Q)$
 - (b) $\sim Q \wedge (P \rightarrow Q) \rightarrow \sim P.$
2. A set of statements are logically consistent if all the statement can be true together. Show whether the following set of statements are consistent.
 i) $[P \rightarrow (Q \rightarrow R)] \rightarrow [(P \rightarrow Q) \rightarrow (P \rightarrow R)];$ ii) $P \wedge \sim R;$ iii) $Q \vee \sim R.$
3. Is the argument given true?

$$\begin{array}{c} A \rightarrow B \\ B \rightarrow (C \wedge D) \\ \sim D \\ \hline \sim A \end{array}$$

4. Questions on binary relation
 - (a) Give an example of binary relation R satisfying reflexivity and completeness and not transitivity.
 - (b) Give an example of a binary relation satisfying completeness , transitivity and not reflexivity.
 - (c) Give an example of binary relation satisfying completeness and asymmetry.
 - (d) Give an example of binary relation satisfying completeness, transitivity and symmetry.
5. Find the maximal set and choice set for the following preference relations.
 - (a) $xIy, xPz, zIy.$
 - (b) $xPz, yPz, uPx, uPz.$
6. Find the Pareto optimal state for the following individual preference profiles:

| | 1 | 2 | 3 |
|-----|-----|-----|------|
| (a) | x | x | xy |
| | y | z | z |
| | z | y | |
| | 1 | 2 | 3 |
| (b) | x | y | x |
| | y | z | z |
| | z | x | y |

| | | | |
|-----|---|---|---|
| | 1 | 2 | 3 |
| | x | y | y |
| (c) | y | z | x |
| | z | t | t |
| | t | x | z |

7. Find the set of almost decisive individuals and set of decisive individuals in the following profiles of individual preferences

| | | | |
|-----|---|---|---|
| | 1 | 2 | 3 |
| | x | y | y |
| (a) | y | z | t |
| | z | x | x |
| | t | t | z |

| | | | |
|-----|---|---|---|
| | 1 | 2 | 3 |
| (b) | x | z | y |
| | y | y | x |
| | z | x | z |

8. Suppose the preference profile of three individual is

| | | | |
|--|---|---|---|
| | 1 | 2 | 3 |
| | x | t | y |
| | y | x | x |
| | z | z | t |
| | t | y | z |

- (a) What is the social preference relation when we use simple majority rule in the case of the above individual preference profile?
- (b) What is the social preference relation when we use Borda count in the case of the above individual preference profile?
- (c) What is the social preference relation when we use Plurality voting method in the case of the above individual preference profile?

9. Suppose there are four alternatives and number of individual is also four. Construct an example of individual preference profile such that simple majority rule generates cycle.

10. Suppose there are four alternatives and four individuals. Construct an example of individual preference profile which has a social preference ordering or ranking when absolute majority rule is used.

11. Give an example of social welfare function which satisfies condition U, I, D and not P when there are three alternatives.

12. Suppose the individual preference profile is

| | | | | | |
|--|---|---|----|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| | x | z | y | z | x |
| | y | x | x | y | z |
| | z | y | zx | y | |

Suppose the social preference relation is based on the following rule: if the majority that

prefer x to y consists of an even number of individuals then society prefers x to y ; if the majority that prefers x to y consist of an odd number of individuals then society prefers y to x . What is going to be the social preference relation when the above rule is used in the case of above individual preference profile ?

13. Suppose the social preference relation gives the following outcome for the given individual preference profile. What assumption is violated?

$$f \begin{pmatrix} x & t & z & y \\ y & y & x & t \\ z & z & t & x \\ t & x & y & z \end{pmatrix} = \begin{pmatrix} z \\ x \\ t \\ y \end{pmatrix}$$

14. Suppose the individual preference profile is

$$\begin{matrix} 1 & 2 & 3 \\ x & z & z \\ y & x & t \\ z & y & x \\ t & t & y \end{matrix}$$

Suppose condition U , p , I are satisfied. Individual 1 is almost decisive over y against z . What is the social preference relation in this case?

15. Suppose the individual preference profile is

$$\begin{matrix} 1 & 2 & 3 \\ x & y & y \\ y & z & x \\ z & x & t \\ t & t & z \end{matrix}$$

If individual 1 is almost decisive over x against y . What is additional information is required to have Individual 1 to be a dictator ?

16. Suppose individual 1 and 2 are decisive over the distinct pair (x, y) and (u, v) respectively. Give an example of individual preference profile satisfying Pareto principle such that the social welfare function (ranking)can be defined when there are four alternatives and three individuals.

17. Suppose individual 1 and 2 are decisive over the distinct pair (x, y) and (u, v) respectively. Give an example of individual preference profile satisfying Pareto principle such that we do not have a social decision function when there are four alternatives and three individuals.

Q1

①

a)

$$\sim(P \rightarrow Q) \Leftrightarrow (P \wedge \sim Q)$$

| P | Q | $P \rightarrow Q$ | $\sim(P \rightarrow Q)$ | $\sim Q$ | $P \wedge \sim Q$ | $\sim(P \rightarrow Q) \Leftrightarrow (P \wedge \sim Q)$ |
|---|---|-------------------|-------------------------|----------|-------------------|---|
| T | T | T | F | F | F | T |
| T | F | F | T | T | T | T |
| F | T | T | F | F | F | T |
| F | F | T | F | T | F | T |

It is a tautology: \Rightarrow

$$b) [\sim Q \wedge (P \rightarrow Q)] \rightarrow \sim P$$

| P | Q | $\sim P$ | $\sim Q$ | $P \rightarrow Q$ | $\sim Q \wedge (P \rightarrow Q)$ | $[\sim Q \wedge (P \rightarrow Q)] \rightarrow \sim P$ |
|---|---|----------|----------|-------------------|-----------------------------------|--|
| T | T | F | F | T | F | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | F | T |
| F | F | T | T | F | T | T |

It is a tautology.

\equiv

2).

$$\textcircled{O} \quad [P \rightarrow (Q \rightarrow R)] \rightarrow [(P \rightarrow Q) \rightarrow (P \rightarrow R)]$$

$$P \wedge \neg R.$$

$$Q \vee \neg R.$$

- we want the 3 statements
to be true together.

- $P \wedge \neg R$ is true when P is true
and R is false.

so $\neg R$ is true: $\therefore Q \vee \neg R$
it implies

is always:
when P is true and R is false
the last two statements

are true.

$P \rightarrow (Q \rightarrow R)$ is true when Q is false
since R is false. It implies
 $Q \rightarrow R$ is true when Q is false.

(27)

P can be true or false

when Q is false, $P \rightarrow [Q \rightarrow R]$ is always true.

~~$(P \rightarrow Q)$ is true when~~

$P \rightarrow R$ is true when P is false since R is false.

so Q is false.

It implies. $P \rightarrow R$ is true.

Now, we already have

P is false and Q is also false

so $P \rightarrow Q$ is true.

Thus. $(P \rightarrow Q) \rightarrow (P \rightarrow R)$ is true.

when P is false.

when P is

thus ~~$[P \rightarrow (Q \rightarrow R)] \rightarrow [(P \rightarrow Q) \rightarrow (P \rightarrow R)]$~~

we need $P \wedge \neg R$ to be true
so P is true.

So P is true.

This implies $P \rightarrow R$ is false

$P \rightarrow Q$ is also false since

P is true and Q is false.

Thus $(P \rightarrow Q) \rightarrow (P \rightarrow R)$ is true.

It implies

$$[P \rightarrow (Q \rightarrow R)] \rightarrow [(P \rightarrow Q) \rightarrow (P \rightarrow R)]$$

is true.

When P is true, Q is false
and R is false.

- We get that

when P is true, Q is false, R is false

The statements are consistent.

=====

3)

$$A \rightarrow B$$

$$B \rightarrow (C \wedge D)$$

$$\sim D$$

$$\sim A$$

(3)

$$\left[[A \rightarrow B] \wedge [B \rightarrow (C \wedge D)] \wedge [\neg D] \right] \rightarrow \neg A.$$

-

Suppose A is false then $\neg A$ is true.

and $A \rightarrow B$ is true.

$\neg D$ is true so D is false

so $C \wedge D$ is false. It implies

for $B \rightarrow (C \wedge D)$ to be true

B must be false.

- Thus when A is false,

B is false, D is false and

C can be both true or false

- The given deduction is

true.

=====

4) a) $\{x, y, z, u\}$

$$R_2 = \left\{ (x, x), (y, y), (z, z), (u, u), (x, y), (y, z), (z, x), (z, u), (u, z), (y, u), (x, u), (u, y), (u, x) \right\}.$$

It satisfies reflexivity, completeness
and not transitivity.

b). $\{x, y, z\}$.

$$R_2 = \left\{ (x, y), (y, z), (x, z) \right\}$$

c). $\{x, y, z\}$.

$$R_2 = \left\{ (x, y), (y, z), (x, z) \right\}$$

(4)

a) $\{x, y, z\}.$

$$\mathbb{R}^2 \left\{ (x, y), (y, x), (y, z), (z, y) \right. \\ \left. (x, z), (z, x) \right\}.$$

5).

(a) $\{x, y\}$ - maximal
nt.

$$\{x, y\} \rightarrow \text{choice nt.}$$

b)

$$\{u, y\} \rightarrow \text{maximal nt.}$$

- Choice nt is empty.

=

6)

a) $\{x, y\}$ - Pareto optimal
nt.

=

b). $\{x, y, z\}$ - Pareto optimal state.

c). $\{x, y\}$ - Pareto optimal state.

7. a) 1: $x y z t$

2: $y z x t$

3: $y t x z$.

Individual 1, can be almost decisive over x against y .

Individual 2 can be almost decisive over z against x .

Individual 3 can be almost decisive over t against 2. and also over t against 1.

(5)

Individual 1 can be decisive over

x against y , over y against z ,
 over z against t , over x against z
 over x against t , over y against z .

- Similarly Individual 2 and 3 can be
 decisive over all the ordered pairs
 based on their preference ordering
 or ranking.

The set of decisive individual is
 also the set of almost-decisive
 for Individual 1. almost-
 decisive over x against y
 is non-trivial. For
 other ordered pairs, it
 is trivially true

b) 1: $x y z$

2: $z y x$

3: $y z x$

Individual 1 can be decisive over
 x against y , over y against z , and over
 x against z .

Individual 1 can be decisive on.

(x,y) against y, y against z
, z against t, x against z,
x against t, y against t.

Similarly for individual 2 & 3 over
all the pairs as per their ranking.

Q)

1: n y z

2: z y x

3: y n z

— can be
Individual 1 almost decisive on.

x against y.

Individual 2

(6)

Individual 1 is almost decisive over x against y non-trivially. For other ordered pairs, it is trivially becoming almost decisive.

Individual 2 is almost decisive over z against y non-trivially. For other ordered pairs, it is trivially almost decisive.

Individual 3 is ~~not~~ trivially almost decisive over all ordered pairs.

- 8) a) 1: $xyzt$
- 2: $txzy$
- 3: $yntz$.

a) ~~Given~~ Simple majority rule:
 $n(xPy)=2 > n(yPz)=1 \Rightarrow \underline{xPy}$.

$$N(y P_2) = 2 > N(z P_2) = 1 \Rightarrow y P_2$$

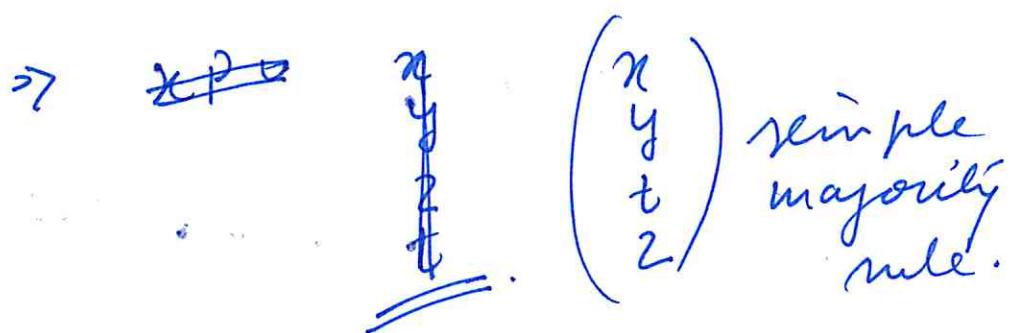
$$N(z P_i t) = 1 < N(t P_i z) = 2 \Rightarrow t P_2.$$

so

$$N(y P_2 t) = 2 > N(t P_i y) = 1 \Rightarrow \cancel{y P t}$$

$$N(x P_2 z) = 3 > N(z P_2 x) = 0 \Rightarrow x P_2.$$

$$N(x P_i t) = 2 > N(t P_i x) = 1 \Rightarrow x P t.$$



b). ~~x~~ Borda Count

$$\text{Score of } x: 3 + 2 + 2 = 7$$

$$\text{Score of } y: 3 + 2 + 0 = 5$$

$$\text{Score of } z: 1 + 1 + 0 = 2$$

$$\text{Score of } t: 3 + 1 + 0 = 4.$$

(7)

$$\begin{pmatrix} n \\ y \\ t \\ z \end{pmatrix}$$

→ Borda Count.

c). Score of Plurality method

Score of x : 1

Score of y : 1

Score of z : 0

Score of t : 1

→ $\begin{pmatrix} nyt \\ z \end{pmatrix}$ } plurality method.

Q. Individual 1: ~~x~~: ny 2 ~~x~~

~~y~~ Individual 2: ~~y~~ 2 ~~x~~

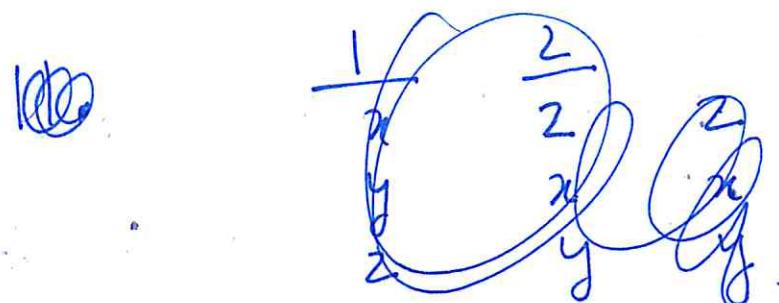
~~x~~ Individual 3: ~~z~~ 2 ~~ny~~

Individual 4: ny 2

In this example there is a cycle.
when we use simple majority rule.

10.

Indiv 1
3 individuals, ^{1, 2, 3}, have following
similar preference
 $y \succ z \succ u$. or $\begin{matrix} y \\ z \\ u \end{matrix}$
Individual 4 has preference
profile $y \succ z \succ u \succ n$. or $\begin{matrix} y \\ z \\ u \\ n \end{matrix}$.



(8)

11. Suppose social welfare function given the following or social ranking.
 xPy, yPz, xPz . always.

$$f\left(\begin{matrix} x & y & z \\ y & z & x \\ z & x & y \end{matrix}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

It violates Pareto.

$$f\left(\begin{matrix} y & y & y \\ x & z & 2x \\ z & x & n \end{matrix}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The image of this social function is same whatever may be the profile of individual preference profile.

12.

$$\begin{array}{ccccc} \frac{1}{y} & \frac{2}{x} & \frac{3}{2x} & \frac{4}{y} & \frac{5}{2y} \\ \hline y & x & 2x & y & 2y \\ z & y & & z & \end{array}$$

$$N(xPy) = 3 > N(yPx) = 2,$$

since $N(xPy) = 3$, is odd.

$w \bullet \underline{yPx}$.

$$N(yP_i z) = 2 = N(zP_i y) = 2.$$

$$\rightarrow \underline{yIz}.$$

$$N(xP_i z) = 2 = N(zP_i x) = 2.$$

$$\rightarrow \underline{xIz}.$$

$\bullet \underline{yPx, yIz, xIz.}$

13. Non dictatorialism condition
is violated.

14. $\underline{zP_1 y \rightarrow \cancel{zP_2 xPy}}$.

~~($\cancel{zP_2 x}$) from Pareto principle.~~

yPz ~~from Pareto principle.~~

yPz ~~from Pareto principle.~~

14)

$y P_1 z \rightarrow y P_2$ since $Individ$

is almost decisive
over y against z .

$2 Pt$, from Pareto principle.

$x Py$, from Pareto principle.

—

Social preference must be
transitive ~~too~~, no.

It implies xPy, yPz

then implies xPz .

yPz, zPt implies yPt .

xPy, yPz, zPt implies xPt .

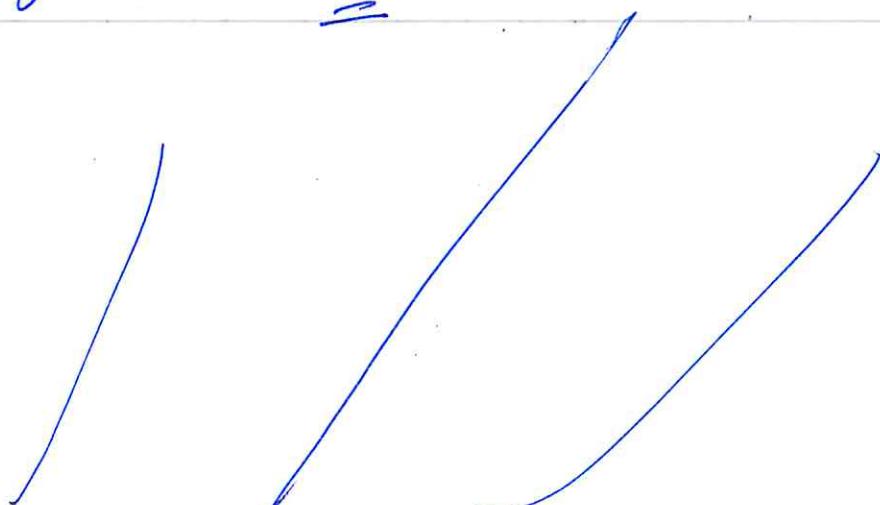
20.

$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ is the social
preference relation.

—

Q Additional information required
is 2 Pt. If individual 1 is
deciding over 2 against t.
Then using $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ is the social
preference relation.
It is same as the preference
relation of individual 1. So 1 is
deciding over all ~~all~~ of its
ordered pairs. Thus it is a
dictator.

100



(15)

$xP_1y \rightarrow xPy$, Individual
is almost decisive over
 x against y :

$yP_2 \rightarrow$ Pareto principle.

$yPt \rightarrow$ Pareto principle.

$xPt \rightarrow$ Pareto principle.

—

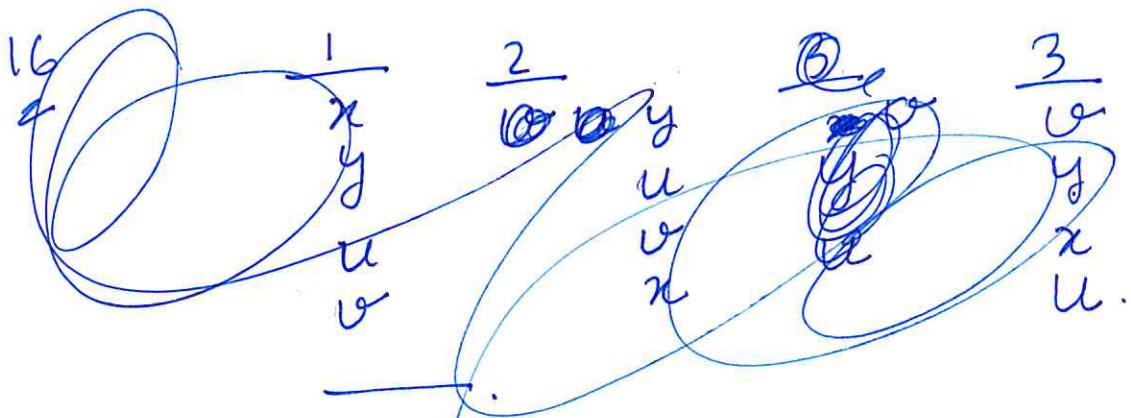
From transitivity we have:

$xPy, yPz \rightarrow xPz$.

Additional information required
is zPt , Individual 1 is
decisive over z against t .

so. $xg \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} =$

(16)



Individual 1 is deciding over.

$$(x, y) \text{ vs } xPy.$$

Individual 2 is deciding over

$$(u, v) \text{ vs } uPv.$$

- ~~xPy GPO yPu from~~

Pareto principle.

~~xPy, dyPu, uPv.~~

| <u>16</u> | <u>1</u> | <u>2</u> | <u>3</u> |
|-----------|----------|----------|----------|
| | x | y | y |
| | y | x | x |
| | u | u | u |
| | v | v | u |

=

$x \text{Py}$: Since individual 1 is
decisive over y against y .

$u \text{Pu}$: Since individual 2 is
decisive over u against u .

$y \text{Pu}$: Pareto principle.

$x \text{Pxu}$: Pareto principle.

$y \text{Pxu}$: Pareto principle.

$x \text{Pyu}$: Pareto principle.

—
 x
 y . — ~~social~~ social
 u choice set
 v contains x .
=.

17.

| | 1. | 2 | 3 |
|-----|-----|------------------|---|
| x | y | v | |
| y | u | v y | |
| u | v | x | |
| v | x | u | |

xPy - Individual 1 is decision over (x, y)

uPv - Individual 2 is decision over (u, v) .

yPu - By Pareto principle.

- we can't say anything on.

(x, u) pair, and (y, v) pair.

so, this individual preference profile generates an incomplete social preference relation.

so choice set is empty.