10Q7

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

$$\Rightarrow \cancel{\frac{2x}{a^2+\lambda} + \frac{2yy'}{b^2+\lambda}}$$

$$\Rightarrow (b^2+\lambda)x^2 + (a^2+\lambda)y^2 = (a^2+\lambda)(b^2+\lambda)$$

$\Rightarrow$ diff. w.r.t. x

$$2x(b^2+\lambda) + 2(a^2+\lambda)yy' = 0$$

$$\Rightarrow (xb^2 + a^2yy') + \lambda(x+yy') = 0$$

$$\Rightarrow \lambda = -\frac{(xb^2+a^2yy')}{x+yy'} \quad \text{---(i)}$$

So

$$a^2+\lambda = a^2 - \frac{(xb^2+a^2yy')}{x+yy'}$$

$$= \frac{(a^2-b^2)x}{x+yy'} \quad \text{---(ii)}$$

$$b^2+\lambda = b^2 - \frac{(xb^2+a^2yy')}{x+yy'} = \frac{(b^2-a^2)yy'}{x+yy'} \quad \text{---(iii)}$$

Thus, ~~so~~ diff. eq. we get

$$\frac{x^2(x+yy')}{(a^2-b^2)x} + \frac{y^2(x+yy')}{(b^2-a^2)yy'} = 1$$

$$\Rightarrow \frac{(x+yy')}{a^2-b^2}\left\{x - \frac{y}{y'}\right\} = 1$$

$$\Rightarrow \frac{(x+yy')(xy'-y)}{a^2-b^2} = y' \quad \text{——— (A)}$$

Now, for getting ortho. traject.

$$y' \to -\frac{1}{y'}$$

so

$$\frac{\left(x - \frac{y}{y'}\right)\left(-\frac{x}{y'} - y\right)}{a^2-b^2} = -\frac{1}{y'}$$

$$\Rightarrow \frac{(xy'-y)(x+yy')}{a^2-b^2} = y'$$

Hence, we get the same diff. equation so, same family has orthogonal ~~t~~rajectory to it.

~~Q3~~

$$\cancel{\begin{cases} \dfrac{dx}{dt} = x(12-4x-3y) \\[2ex] \dfrac{dy}{dt} = y(30-6x-5y), \quad x\geq 0,\ y\geq 0 \end{cases}}$$

$\boxed{\text{Q5}}$

$$x(n+1) = \frac{\alpha x(n)}{1+\beta x(n)}, \quad \alpha>1,\ \beta>0$$

For equilibrium point

$$x(n+1) = x(n) = \lambda$$

So

$$\lambda = \frac{\alpha\lambda}{1+\beta\lambda} \Rightarrow \lambda\left\{1-\frac{\alpha}{1+\beta\lambda}\right\} = 0$$

$$\Rightarrow \lambda(1+\beta\lambda-\alpha) = 0$$

$$\Rightarrow \lambda = 0 \quad \text{or} \quad \lambda = \frac{\alpha-1}{\beta}$$

Thus,

$$f(x) = \frac{\alpha x}{1+\beta x}$$

$$f'(x) = \frac{\alpha}{1+\beta x} - \frac{\alpha x\beta}{(1+\beta x)^2} = \frac{(1+\beta x)\alpha - \alpha x\beta}{(1+\beta x)^2}$$

$$= \frac{\alpha}{(1+\beta x)^2}$$

So for positive eqm pt., $\lambda = \frac{\alpha-1}{\beta}$

$$f'(x)\Big|_{x=\frac{\alpha-1}{\beta}} = \frac{\alpha}{(1+\alpha-1)^2} = \frac{\alpha}{\alpha^2} = \frac{1}{\alpha}$$

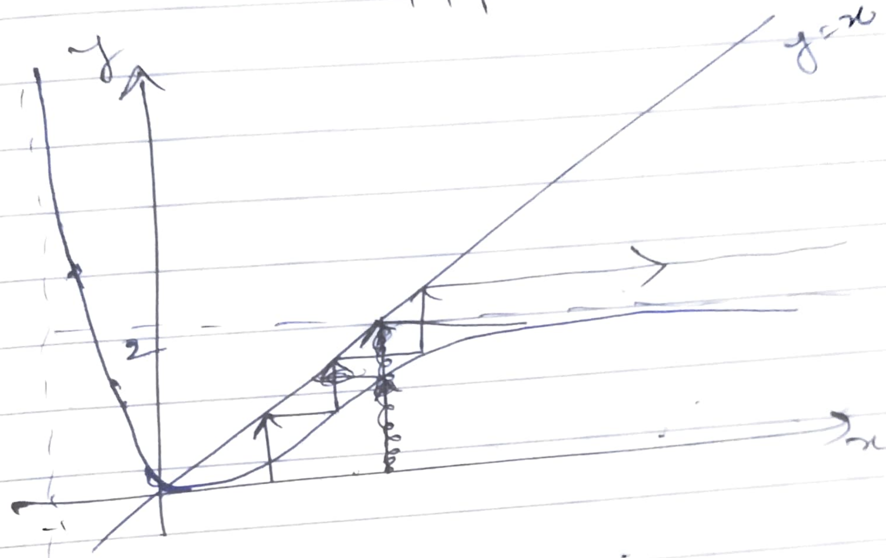
So, for stability $\left|\frac{1}{\alpha}\right| < 1 \Rightarrow \underline{\underline{\alpha>1}}$

Hence, $\lambda = \dfrac{\alpha - 1}{\beta}$ is stable

Now, for, $\alpha = 2, \beta = 1$

we have $\underline{\underline{\lambda = 1}}$ ~~which~~

$$f(x) = \frac{\alpha x}{1+\beta x} = \frac{2x}{1+x}$$



Q3

$$\frac{dx}{dt} = 2x(12-4x-3y) = f_1(x,y)$$

$$\frac{dy}{dt} = y(30-6x-5y), \quad x \geq 0, \; y \geq 0$$
$$= f_2(x,y)$$

for eq'm point

Jacobian

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 12-8x-3y & -3x \\ -6y & 30-10y-6x \end{bmatrix}$$

Also, for stable point

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 0$$

$$\begin{cases} x(12-4x-3y) = 0 \\ y(30-6x-5y) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x=0 \quad \text{or} \quad 12-4x-3y=0 \\ y=0 \quad \text{or} \quad 30-6x-5y=0 \end{cases}$$

Case I $\quad x=0$

then $\quad y=0$

or $\quad 30-6x-5y=0 \Rightarrow y=6$

Case II $\quad 12-4x-3y=0$

then, $x=3$ or $\begin{cases} 30-6x-5y=0 \\ 12-4x-3y=0 \end{cases}$

$(0,0)$, $(0,6)$, $(3,0)$, $(-15,24)$

So,

For ~~x~~. $(0,0)$ as eq'm point

$$J = \begin{bmatrix} 12 & 0 \\ 0 & 30 \end{bmatrix}$$

eigen values are $12 > 1$, $30 > 1$

So unstable

For $(0,6)$

$$J = \begin{bmatrix} \cancel{12} -6 & 0 \\ -36 & -30 \end{bmatrix}$$

eigen values $|-6| \leq 1$ $|-30| > 1$

similarly for ~~others~~ $(3,0)$, ~~(-15, 24)~~

~~By~~

$\boxed{Q4}$

$$\cancel{d(t) = a_0 + a_1 p(t) + a_2 p'(t)}$$

$$\cancel{S(t) = b_0 + b_1 p(t) + b_2 p'(t)}$$

$\boxed{Q2}$

In prey-predator mode following assumptions we make.

$b_1 x(t)$ * In absence of predator, prey population growth depends on its population

$a_2 Y(t)$ * In absence of prey, predator decline rate depends on its population

$b_2 Y + c_1 Y(t) X(t)$ * In presence of prey, predator growth depends on its population & also product of prey-predator population.

$c_1 Y(t) X(t)$ * In presence of ~~prey~~ predator, prey killed $\propto$ product of both population

$a_1 x(t)$ * ~~In presence~~ and natural deth of prey $\propto$ its population

So, combining all assumptions in equation we have

$$\begin{cases} \dfrac{dX}{dt} = (b_1 - a_1)X - c_1 XY \\[2ex] \dfrac{dY}{dt} = b_2 Y + c_1 XY - a_2 Y \end{cases}$$

We can denote $\beta_1 = b_1 - a_1$

$\beta_2 = b_2 - a_2$

so,

$$\begin{cases} \dfrac{dX}{dt} = \beta_1 X - c_1 XY = f_1(X,Y) \\[2ex] \dfrac{dY}{dt} = \beta_2 Y + c_1 XY = f_2(X,Y) \end{cases}$$

$$[J] = \begin{bmatrix} \beta_1 - c_1 Y & -c_1 X \\ c_1 Y & \beta_2 + c_1 X \end{bmatrix}$$

for steady state

$$\frac{dX}{dt} = 0, \quad \frac{dY}{dt} = 0$$

$$\begin{cases} \beta_1 X - c_1 XY = 0 \\ \beta_2 Y + c_1 XY = 0 \end{cases} \Rightarrow \begin{cases} X(\beta_1 - c_1 Y) = 0 \\ Y(\beta_2 + c_1 X) = 0 \end{cases}$$

possible pts $(0,0)$,

so $$[J]\big|_{(0,0)} = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}$$

thus eigen-values β_1, β_2

should be

$|b_1 - a_1| < 1$ $\leftarrow$ $|\beta_1| < 1$

$|b_2 - a_1| < 1$ $\quad$ $|\beta_2| < 1$

Q6

~~$y_{n+1} = y_{n-1} + 2h f_n$~~

$$y' = -2y + 1 \quad , \; y(0) = 1/2$$

~~$\Rightarrow \int \frac{dy}{1-2y} = \int dx$~~

~~$\Rightarrow \frac{1}{-2} \ln|1-2y| = x + c_1$~~

~~$\Rightarrow 1 - 2y = A e^{-2x}$~~

~~putting $y(0) = 1/2 \rightarrow 0 = A e^{-2x} \Rightarrow A = 0$~~

$$y' = -2y + 1$$

$$\Rightarrow \quad y' + 2y = 1 \qquad \cancel{(D+2)y = 1}$$

Homogeneous part

~~$\Rightarrow y = \frac{1}{D+2}(1)$~~

~~$\frac{y'}{y} = -2 + \frac{1}{y} \Rightarrow \frac{y'}{y} + \frac{1}{y} = -2$~~

$$\Rightarrow \quad y' + 2y = 1$$

$$\Rightarrow \quad \text{If} = e^{2x}$$

So

$$e^{2x} y' + 2y e^{2x} = e^{2x}$$

$$\Rightarrow \frac{d(e^{2x} y)}{dx} = e^{2x}$$

$$\Rightarrow e^{2x} y = \frac{e^{2x}}{2} + c_1$$

$$\Rightarrow \boxed{y = \frac{1}{2} + c_1 e^{-2x}}$$

So, $y(0) = 1/2 \rightarrow y = 1/2$ always for $y_{n+1} = y_{n-1} + 2hf_n$

$\Rightarrow y_{n+1} - y_n = -(y_n - y_{n-1}) + 2hf_n$

$\Rightarrow \Delta y_n = -\Delta y_{n-1} + 2hf_n$

$y_{n+1} = y_{n-1} + 2hf_n$

$\Rightarrow \dfrac{y_{n+1} - y_{n-1}}{2h} = f_n$

$\Rightarrow \dfrac{(y_{n+1} - y_n) - (y_n - y_{n-1})}{2h} = f_n$

$\Rightarrow$

Q4

$$\begin{cases} d(t) = a_0 + a_1 p(t) + a_2 p'(t) \\ s(t) = b_0 + b_1 p(t) + b_2 p'(t) \end{cases}$$

~~At~~ steady state,

demand = supply

so

$$(a_2 - b_2) p'(t) + (a_1 - b_1) p(t) + a_0 - b_0 = 0$$

$$\Rightarrow p'(t) + \frac{a_1 - b_1}{a_2 - b_2} p(t) + \frac{a_0 - b_0}{a_2 - b_2} = 0$$

$$\Rightarrow IF \left\{ p'(t) + \left(\frac{a_1 - b_1}{a_2 - b_2}\right) p(t) + \frac{a_0 - b_0}{a_2 - b_2} = 0 \right\}$$

$$IF = e^{\frac{a_1 - b_1}{a_2 - b_2} t}$$

So

$$\frac{d(IF \cdot p(t))}{dt} = -\frac{(a_0 - b_0)}{a_2 - b_2} IF$$

$$\Rightarrow IF \cdot p(t) = -\frac{(a_0 - b_0)}{(a_1 - b_1)} IF + c_1$$

$$\Rightarrow \boxed{p(t) = -\frac{(a_0 - b_0)}{a_1 - b_1} + c_1 e^{-\left(\frac{a_1 - b_1}{a_2 - b_2}\right) t}}$$

$$p(t) = \frac{a_0 - b_0}{b_1 - a_1} + c_1 e^{\left(\frac{b_1 - a_1}{a_2 - b_2} t\right)}$$

Stability occurs only if

$$p(0) = \frac{a_0 - b_0}{b_1 - a_1}$$

Because otherwise,

$b_1 - a_1 > 0$

$a_2 - b_2 > 0$

& otherwise term would explode exponential.