Weak Galerkin Finite Element Methods

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Second Order Elliptic Equation

Consider second order elliptic problem:

$$
\begin{aligned}
-\nabla\cdot a\nabla u &= f, \quad \text{in}\,\Omega && (1)\\
u &= 0, \quad \text{on}\,\partial\Omega. && (2)
\end{aligned}
$$

Testing (1) by $v \in H_0^1(\Omega)$ gives

$$-\int_\Omega \nabla\cdot a\nabla u v dx = \int_\Omega a\nabla u\cdot\nabla v dx - \int_{\partial\Omega} a\nabla u\cdot\mathbf{n} v ds = \int_\Omega f v dx.$$

$$(a\nabla u,\ \nabla v) = (f,\ v),$$

where $(f,g) = \int_\Omega fg dx$.

PDE and Its Weak Form

PDE: find u satisfies

$$\begin{aligned} -\nabla \cdot a\nabla u &= f, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

Its weak form: find $u \in H_0^1(\Omega)$ such that

$$(a\nabla u,\ \nabla v) = (f,\ v), \quad \forall v \in H_0^1(\Omega).$$

Continuous Finite Element Method

Find $u_h \in V_h \subset H_0^1(\Omega)$ such that

$$(a\nabla u_h,\ \nabla v) = (f,\ v), \quad \forall v \in V_h. \tag{3}$$

Let $V_h = \text{span}\{\phi_1, \cdots, \phi_n\}$ and $u_h = \sum_{j=1}^{n} c_j\phi_j$, the finite element formulation (3) becomes

$$\sum_{j=1}^{n}(a\nabla\phi_j, \nabla\phi_i)c_j = (f,\ \phi_i), \quad i = 1, \cdots, n.$$

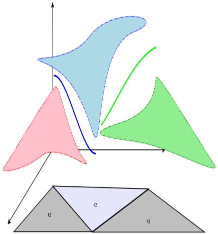
The system above is symmetric and positive definite. Solve it to obtain the continuous finite element solution u_h.

Weak Galerkin Finite Element Methods

Weak Galerkin method is such a method that uses discontinuous elements and has a formulation:

$$(a\nabla_w u_h, \nabla_w v_h) + s(u_h, v_h) = (f, v_h).$$

Weak Galerkin methods introduce **weak functions** and **weak derivatives** for discontinuous functions.



Weak function

Weak Functions and Weak Derivatives

- **Weak Function**: $v = \{v_0, v_b\}$ such that

$$v = \begin{cases} v_0, & \text{in } T^0 \\ v_b, & \text{on } \partial T \end{cases}$$

Weak Galerkin finite element space:

$$V_h = \{v = \{v_0, v_b\} : v_0|_T \in P_k(T), v_b \in P_{k-1}(e), e \subset \partial T, v_b = 0 \text{ on } \partial\Omega\}.$$

- **Weak Gradient**: $\nabla_w v \in [P_{k-1}(T)]^d$ for $v \in V_h$ on each element T:

$$(\nabla_w v, q)_T = -(v_0, \nabla \cdot q)_T + \langle v_b, q \cdot \mathbf{n} \rangle_{\partial T}, \qquad \forall q \in [P_{k-1}(T)]^d.$$

Weak Galerkin Finite Element Formulation

Find $u_h \in V_h \subset L^2(\Omega)$ such that for any $v = \{v_0, v_b\} \in V_h$

$$(a\nabla_w u_h, \nabla_w v) + \sum_T h_T^{-1} \langle Q_b u_0 - u_b, Q_b v_0 - v_b \rangle_{\partial T} = (f, v), \tag{4}$$

$$a(u_h, v) = (a\nabla_w u_h, \nabla_w v) + \sum_T h_T^{-1} \langle Q_b u_0 - u_b, Q_b v_0 - v_b \rangle_{\partial T},$$

where Q_b is the L^2 projection: $L^2(e) \to P_{k-1}(e)$ for $e \in \partial T$.

Let $V_h = \text{span}\{\phi_1, \cdots, \phi_n\}$ and $u_h = \sum_{j=1}^n c_j \phi_j$, then weak Galerkin formulation (4) becomes

$$\sum_{j=1}^n a(\phi_j, \phi_i) c_j = (f,\ \phi_i), \quad i = 1, \cdots, n.$$

The system above is symmetric and positive definite. Solve it to obtain the weak Galerkin finite element solution u_h.

WG Method vs Continuous Finite Element Method

General Finite Element Method:

Let $V_h = \text{span}\{\phi_1, \cdots, \phi_n\}$ and $u_h = \sum_{j=1}^{n} c_j \phi_j$, then

$$\sum_{j=1}^{n} a(\phi_j, \phi_i) c_j = (f,\ \phi_i), \quad i = 1, \cdots, n.$$

• Continuous Galerkin Finite Element Method:

$$a(\phi_j, \phi_i) = (a\nabla\phi_j, \nabla\phi_i).$$

• Weak Galerkin Finite Element Method:

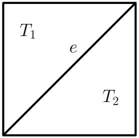
$$\begin{aligned} a(\phi_j, \phi_i) &= (a\nabla_w\phi_j, \nabla_w\phi_i) + \sum_T h_T^{-1} \langle Q_b\phi_{j,0} - \phi_{j,b}, Q_b\phi_{i,0} - \phi_{i,b} \rangle_{\partial T} \\ &= (a\nabla_w\phi_j, \nabla_w\phi_i) + s(\phi_j, \phi_i) \end{aligned}$$

where $\phi_i = \{\phi_{i,0}, \phi_{i,b}\}$.

Implementation of Weak Galerkin Method for $k = 1$ in 2D

$$V_h = \{v = \{v_0, v_b\} : v_0|_T \in P_1(T), v_b \in P_0(e), e \subset \partial T, v_b = 0 \text{ on } \partial\Omega\}.$$

Let $V_h = \text{span}\{\phi_1, \cdots, \phi_7\}$, $\mathscr{T}_h = T_1 \cup T_2$ and $\phi_i = \{\phi_{i,0}, \phi_{i,b}\}$. Sample basis functions:



$$\phi_1 = \begin{cases} 1 & \text{on } T_1^0, \\ 0 & \text{otherwise}, \end{cases} \quad \phi_2 = \begin{cases} x & \text{on } T_1^0, \\ 0 & \text{otherwise}, \end{cases} \quad \phi_3 = \begin{cases} y & \text{on } T_1^0, \\ 0 & \text{otherwis} \end{cases}$$

$$\phi_4 = \begin{cases} 1 & \text{on } T_2^0, \\ 0 & \text{otherwise}, \end{cases} \quad \phi_5 = \begin{cases} x & \text{on } T_2^0, \\ 0 & \text{otherwise}, \end{cases} \quad \phi_6 = \begin{cases} y & \text{on } T_2^0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\phi_7 = \begin{cases} 1 & \text{on } e, \\ 0 & \text{otherwise}, \end{cases}$$

Computing $(\nabla_w \phi_j, \nabla_w \phi_i)$

$\nabla_w v \in [P_0(T)]^2$ for $v \in V_h$ on each element T:

$$(\nabla_w v, q)_T = -(v_0, \nabla \cdot q)_T + \langle v_b, q \cdot \mathbf{n} \rangle_{\partial T}, \qquad \forall q \in [P_0(T)]^2. \tag{5}$$

$[P_0(T)]^2 = \text{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ and $\nabla_w \phi_i = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example 1. Find $\nabla_w \phi_1$.

Let $\nabla_w \phi_1 = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Letting $v = \phi_1$ and $q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in (5) give

$$c_1(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix})_{T_1} + c_2(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix})_{T_1} = (1, 0)_{T_1} + \langle 0, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \mathbf{n} \rangle_{\partial T_1} = 0$$

$$c_1(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix})_{T_1} + c_2(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix})_{T_1} = (1, 0)_{T_1} + \langle 0, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \mathbf{n} \rangle_{\partial T_1} = 0$$

Solving the two equations above yields $c_1 = 0$ and $c_2 = 0$, i.e. $\nabla_w \phi_1 = 0$ on T_1. Similarly, we have $\nabla_w \phi_1 = 0$ on T_2. It is easy to see $\nabla_w \phi_i = 0$ for $i = 1, \cdots, 6$.

Computing $(\nabla_w \phi_j, \nabla_w \phi_i)$

Example 2. Find $\nabla_w \phi_7$.

Let $\nabla_w \phi_7 = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Letting $v = \phi_7$ and $q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in (5) give

$$c_1 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)_{T_1} + c_2 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)_{T_1} = (0,0)_{T_1} + \left\langle \phi_{7,b}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \mathbf{n} \right\rangle_{\partial T_1} = |e| n_{1,T_1}.$$

$$c_1 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)_{T_1} + c_2 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)_{T_1} = (0,0)_{T_1} + \left\langle \phi_{7,b}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \mathbf{n} \right\rangle_{\partial T_1} = |e| n_{2,T_1}.$$

Thus $c_1 = \frac{|e|}{|T_1|} n_{1,T_1}$ and $c_2 = \frac{|e|}{|T_1|} n_{2,T_1}$. Then $\nabla_w \phi_7 = \frac{|e|}{|T_1|} \mathbf{n}_{T_1}$ on T_1. Similarly,

$\nabla_w \phi_7 = \frac{|e|}{|T_2|} \mathbf{n}_{T_2}$ on T_2.

Computing $(\nabla_w \phi_j, \nabla_w \phi_i)$

We have

$$\nabla_w \phi_j = 0, \quad j = 1, \cdots, 6,$$

and

$$\nabla_w \phi_7|_T = \frac{|e|}{|T|}\mathbf{n}_T, \quad T \in \mathscr{T}_h = T_1 \cup T_2.$$

We have

$$(a\nabla_w \phi_j, \nabla_w \phi_j) = 0$$

except ($a = I$ for simplicity)

$$(\nabla_w \phi_7, \nabla_w \phi_7) = (\nabla_w \phi_7, \nabla_w \phi_7)_{T_1} + (\nabla_w \phi_7, \nabla_w \phi_7)_{T_2} = \frac{|e|^2}{|T_1|} + \frac{|e|^2}{|T_2|}.$$

Computing $s(\phi_j, \phi_i)$

For any $v \in V_h$, $Q_b v|_e = v(m)$ where m is the midpoint of $e \subset \partial T$. Let $h_T = h_{T_1} = h_{T_2} = \sqrt{2}$.

Example 3. Find $s(\phi_1, \phi_5)$.

$$
\begin{aligned}
s(\phi_1, \phi_5) &= h_T^{-1}(\langle Q_b\phi_{1,0} - \phi_{1,b}, Q_b\phi_{5,0} - \phi_{5,b}\rangle_{\partial T_1} + \langle Q_b\phi_{1,0} - \phi_{1,b}, Q_b\phi_{5,0} - \phi_{5,b}\rangle_{\partial T_2} \\
&= h_T^{-1}(\langle Q_b\phi_{1,0}, Q_b\phi_{5,0}\rangle_{\partial T_1} + \langle Q_b\phi_{1,0}, Q_b\phi_{5,0}\rangle_{\partial T_2}) \\
&= h_T^{-1}(\langle 1, 0\rangle_{\partial T_1} + \langle 0, Q_b x\rangle_{\partial T_2}) = 0
\end{aligned}
$$

Example 4. Find $s(\phi_1, \phi_2)$.

$$
\begin{aligned}
s(\phi_1, \phi_2) &= h_T^{-1}(\langle Q_b\phi_{1,0} - \phi_{1,b}, Q_b\phi_{2,0} - \phi_{2,b}\rangle_{\partial T_1} + \langle Q_b\phi_{1,0} - \phi_{1,b}, Q_b\phi_{2,0} - \phi_{2,b}\rangle_{\partial T_2} \\
&= h_T^{-1}(\langle Q_b\phi_{1,0}, Q_b\phi_{2,0}\rangle_{\partial T_1} + \langle Q_b\phi_{1,0}, Q_b\phi_{2,0}\rangle_{\partial T_2}) \\
&= h_T^{-1}\langle Q_b\phi_{1,0}, Q_b\phi_{2,0}\rangle_{\partial T_1} \\
&= h_T^{-1}(\int_0^1 \frac{1}{2}dx + \int_e \frac{1}{2}ds) = \frac{\sqrt{2}+1}{2\sqrt{2}}.
\end{aligned}
$$

Example 5.

Find $s(\phi_1, \phi_7)$.

$$
\begin{aligned}
s(\phi_1, \phi_7) &= s(\phi_1, \phi_7)|_{T_1} = h^{-1}\langle Q_b\phi_{1,0} - \cancelto{0}{\phi_{1,b}}, \cancelto{0}{Q_b\phi_{7,0}} - \phi_{7,b}\rangle_{\partial T_1} \\
&= h^{-1}\langle Q_b\phi_{1,0}, -\phi_{7,b}\rangle_e = 1
\end{aligned}
$$

Error Analysis

Q_0: L^2 projection: $L^2(T) \to P_k(T)$ and Π_h: L^2 projection: $[L^2(T)]^d \to [P_{k-1}(T)]^d$.

Define $|||\cdot|||$ and $Q_h u$ as

$$|||v|||^2 = a(v, v), \quad Q_h u = \{Q_0 u, Q_b u\} \in V_h$$

Lemma 1. Let u be the solution of (1). We have

$$\nabla_w Q_h u = \Pi_h \nabla u.$$

Proof. For any $q \in [P_{k-1}]^d$, then

$$\begin{aligned}
(\nabla_w(Q_h u), q)_T &= -(Q_0 u, \nabla \cdot q)_T + \langle Q_b u, q \cdot \mathbf{n} \rangle_{\partial T} \\
&= -(u, \nabla \cdot q)_T + \langle u, q \cdot \mathbf{n} \rangle_{\partial T} \\
&= (\nabla u, q)_T \\
&= (\Pi_h(\nabla u), q)_T,
\end{aligned}$$

For simplicity, assume a being piecewise constant.

Lemma 2. Let u be the solution of (1) and $v=\{v_0,v_b\}\in V_h$. We have

$$-(\nabla\cdot a\nabla u, v_0)=(a\nabla_w Q_h u,\nabla_w v)-\ell(u,v),$$

where $\ell(u,v)=\sum_{T\in\mathscr{T}_h}\langle a(\nabla u-\Pi_h\nabla u)\cdot\mathbf{n}, v_0-v_b\rangle_{\partial T}$.

Proof. Using Lemma 1 and the definition of ∇_w, then

$$\begin{aligned}
-(\nabla\cdot a\nabla u, v_0) &= \sum_T ((a\nabla u,\nabla v_0)_T-\langle a\nabla u\cdot\mathbf{n}, v_0\rangle_{\partial T})\\
&= \sum_T ((a\nabla_w Q_h u,\nabla v_0)_T-\langle a\nabla u\cdot\mathbf{n}, v_0\rangle_{\partial T})\\
&= \sum_T ((-v_0,\nabla\cdot a\nabla_w Q_h u)_T+\langle v_0, a\nabla_w Q_h u\cdot\mathbf{n}\rangle_{\partial T}-\langle a\nabla u\cdot\mathbf{n}, v_0-v_b\rangle_{\partial T})\\
&= \sum_T (a\nabla_w Q_h u,\nabla_w v)_T+\langle v_0-v_b, a\nabla_w Q_h u\cdot\mathbf{n}\rangle_{\partial T}-\langle a\nabla u\cdot\mathbf{n}, v_0-v_b\rangle_{\partial T})\\
&= \sum_T (a\nabla_w Q_h u,\nabla_w v)_T-\ell(u,v).
\end{aligned}$$

Error Equation

Testing (1) by v_0 gives

$$-(\nabla\cdot a\nabla u, v_0) = (f, v_0).$$

Using Lemma 2, then

$$(a\nabla_w Q_h u, \nabla_w v) = (f, v_0) + \ell(u, v).$$

Adding $s(Q_h u, v)$ to the both sides of the above equation yields

$$a(Q_h u, v) = (a\nabla_w Q_h u, \nabla_w v) + s(Q_h u, v) = (f, v_0) + \ell(u, v) + s(Q_h u, v). \quad (6)$$

The difference of (6) and (4) implies

$$a(Q_h u - u_h, v) = \ell(u, v) + s(Q_h u, v). \quad (7)$$

Error Estimate

Lemma 3. For $u \in H^{k+1}(\Omega)$ and $v = \{v_0, v_b\} \in V_h$, then

$$\begin{aligned} |s(Q_h u, v)| &\leq Ch^k \|u\|_{k+1} |\!|\!| v |\!|\!|, && (8) \\ \left|\ell_(u, v)\right| &\leq Ch^k \|u\|_{k+1} |\!|\!| v |\!|\!|, && (9) \end{aligned}$$

Theorem. Let u be the true solution and u_h be the weak Galerkin finite element approximation of u. Then

$$|\!|\!| Q_h u - u_h |\!|\!| \leq Ch^k \|u\|_{k+1}.$$

Proof. By letting $v = Q_h u - u_h$ in (7), we have

$$|\!|\!| Q_h u - u_h |\!|\!|^2 = \ell(u, Q_h u - u_h) + s(Q_h u, \ Q_h u - u_h). \qquad (10)$$

It then follows from (8) and (9) that

$$|\!|\!| Q_h u - u_h |\!|\!|^2 \leq Ch^k \|u\|_{k+1} |\!|\!| Q_h u - u_h |\!|\!|.$$