Lecture-4: FEM for Elliptic PDE
MA 573: Finite Element Methods for PDEs

Dr. Bhupen Deka*

In the previous lecture, we have introduced basic steps in FEM for ODE. This lecture is devoted for finite element approximation of elliptic BVP. Over the last decades the finite element method, which was introduced by engineers in the 1960's, has become the perhaps most important numerical method for partial differential equations, particularly for equations of elliptic and parabolic types. This method is based on the variational form of the boundary value problem and approximates the exact solution by a piecewise polynomial functions. It is more easily adapted to geometry of underlying domain then the finite difference method, and for symmetric positive definite elliptic problems it reduces to a finite linear system with a symmetric positive definite matrix.

1 Model Problem

We consider following elliptic boundary value problem

$$-\nabla.(\nabla u(x,y)) = f(x,y) \quad \text{in } \Omega = (0,1)\times(0,1) \tag{1.1}$$

subject to the boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

We, now, illustrate basic steps in the finite element method for the above model problem.

Step-1 (Weak Formulation): As a first step towards the finite element approximation to (1.1)-(1.2), we first introduce the weak formulation. Recall following variational formulation (see, Lecture-2): Find $u \in H_0^1(\Omega)$ such that

$$A(u,v) = L(v), \tag{1.3}$$

where

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v dx \text{ and } L(v) = \int_\Omega f v dx. \tag{1.4}$$

It is easy to verify that $A(\cdot,\cdot)$ is bilinear, continuous and positive in $H_0^1(\Omega)$. Further, L is continuous in $H_0^1(\Omega)$. **(Homework)**

Step-2 (Discretization of the Domain): First, the domain is represented as a collection of a finite number n of subdomains, namely triangles. This is called discretization or triangulation of the domain. Each subdomain is called

*Department of Mathematics, IIT Guwahati, North Guwahati, India, Guwahati- 781039, India (bdeka@iitg.ac.in).

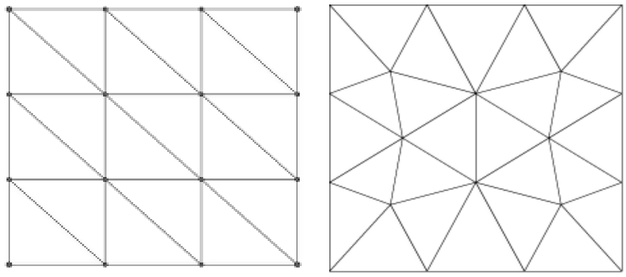


Figure 1.1: A uniform triangulation (left) and a nonuniform triangulation (right).

an element. The collection of elements is called the finite element mesh. The elements are connected to each other at points called nodes. The triangles can be of different sizes. When all elements are of the same size, the mesh is called uniform (see, Figure 1.1), otherwise, it is called a nonuniform (see, Figure 1.1).

In general, for $\Omega \subset \mathbb{R}^2$, we now define a triangulation of a computational domain Ω.

Definition 1.1. Triangulation. A triangulation $\mathcal{T}_h$ is a partition of $\bar{\Omega}$ into a finite number of subsets K such that it satisfies the following conditions:

($\mathcal{A}1$) $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$.

($\mathcal{A}2$) $K = \bar{K}$, $int(K) \neq \Phi$, $K \in T_h$,

($\mathcal{A}3$) *If K_1, $K_2 \in \mathcal{T}_h$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is a common vertex or edge of both triangles. That is no vertex of any triangle lies in the interior of an edge of another triangle.*

($\mathcal{A}4$) *For each triangle $K \in \mathcal{T}_h$, let r_K, $\overline{r}_K$ be the radii of its inscribed and circumscribed circles, respectively. Let $h = \max\{\overline{r}_K : K \in \mathcal{T}_h\}$. We assume that, for some fixed $h_0 > 0$, there exists two positive constants C_0 and C_1 independent of h such that*

$$C_0 h \leq diam(K) \leq C_1 h \quad \forall K \in \mathcal{T}_h, \quad \forall h \in (0, h_0).$$

Step-3 (Construction of V_h) Based on a triangulation $\mathcal{T}_h$ of our computational domain Ω, we construct a finite dimensional piecewise polynomial space V_h in $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$, known as finite element space. This is done by constructing polynomial spaces over elements K.

Definition 1.2. The triple $(K, P_r, \sum_K)$ is called a finite elements. Simply $\sum_k$ is set of points in K such that any $p \in P_r$ can be determined uniquely.

Choice of $(K, P_r, \sum_K)$ (Lagrange finite elements): As a choice of P_r, we consider polynomial space of degree $\leq r$. Clearly, in $\mathbb{R}^2$, we have

$$\dim P_r = \frac{(r+1)(r+2)}{2} = n_r.$$

Then any $p \in P_r$ is uniquely determined on the set

$$L_r(K) = \left\{ x = \sum_{i=1}^{3} \lambda_i a_i : \ \lambda_i \in \left\{ 0, \ \frac{1}{r}, \ \frac{2}{r}, ..., \frac{r-1}{r}, \ 1 \right\} \right\}.$$

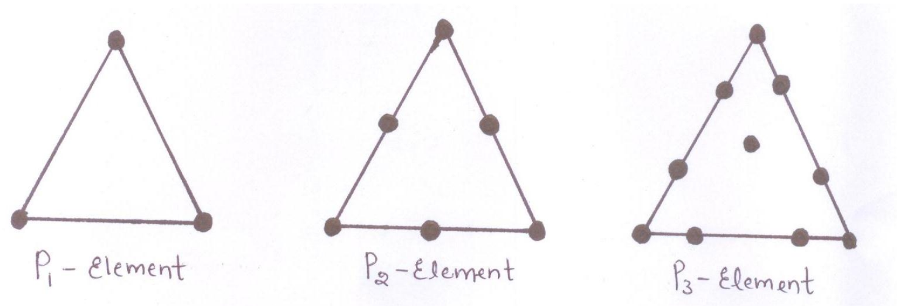


Figure 1.2: Different Lagrange elements.

As $\mathrm{card}(L_r(K)) = \frac{(r+1)(r+2)}{2} = n_r$. As an example, we will discuss linear and quadratic Lagrange elements.

Definition 1.3. ***Nodal basis function:*** *Let $N_h(\bar{\Omega}) = \{x_i : 1 \leq i \leq n_h\}$ be the collection of all nodes of a triangulation. The function $\phi_i \in V_h$ $(1 \leq i \leq n_h)$ satisfying $\phi_i(x_j) = \delta_{ij}$ is called nodal basis function.*

Construction of $(K, P_1, \sum_K)$**:** It is done by introducing area coordinate. Suppose K is a triangle with vertices $a_1 = A$, $a_2 = B$ and $a_3 = C$. Suppose P is an arbitrary point in K. Corresponding to $a_1 = A$, we define

$$\lambda_1(x) = \frac{\mathrm{area}(\triangle PBC)}{\mathrm{area}(\triangle K)} = \frac{A_1}{A}.$$

Similarly, corresponding to $a_2 = B$ and $a_3 = C$, we define

$$\begin{aligned} \lambda_2(x) &= \frac{\mathrm{area}(\triangle PAC)}{\mathrm{area}(\triangle K)} = \frac{A_2}{A}, \\ \lambda_3(x) &= \frac{A_3}{A} = \frac{1 - A_1 - A_2}{A} = 1 - \lambda_1(x) - \lambda_2(x). \end{aligned}$$

For P_1 element, $N_h(K) = \{a_1, a_2, a_3\}$ and basis functions corresponding to node a_i is defined as

$$\phi_i(x) = \lambda_i(x) \text{ so that } \phi_i(a_j) = \delta_{ij}.$$

Construction of $(K, P_2, \sum_K)$**:** For P_2 element note that

$$N_h(K) = \{a_1, a_2, a_3, a_{12}, a_{23}, a_{31}\}, \quad a_{ij} = \frac{a_i + a_j}{2}.$$

Basis function corresponding to a_i is defined as

$$\phi_i(x) = \lambda_i(x)(2\lambda_i(x) - 1), \ 1 \leq i \leq 3.$$

Basis function corresponding to a_{ij} is given by

$$\phi_{ij} = 4\lambda_i(x)\lambda_j(x), \ 1 \leq i < j \leq 3.$$

Then the finite element space is defined as

$$V_h = \left\{ v_h : \bar{\Omega} \to \mathbb{R} : v_h|_K \in P_r(K), \ K \in \mathcal{T}_h \right\}.$$

Based on finite element space V_h, finite element method can be grouped into two categories, namely conforming FEM if $V_h \subseteq H$ and non-conforming FEM if $V_h \not\subseteq H$. Here, H is the solution space for the given boundary value problem.

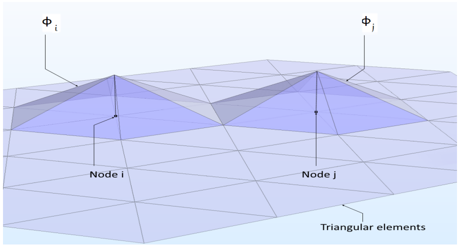


Figure 1.3: Lagrange basis functions corresponding to nodes i and j.

In this lecture, we will concentrate only on linear elements $(K, P_1, \sum_K)$ and our finite element space is defined as

$$V_h = \left\{ v_h \in C(\bar{\Omega}) : \ v_h|_K \in P_1(K), \ K \in \mathcal{T}_h \ \& \ v_h|_{\partial\Omega} = 0 \right\}. \tag{1.5}$$

Since V_h is the collection of continuous and piecewise linear functions, and vanishing on the boundary $\partial\Omega$, therefore $V_h \subset H_0^1(\Omega)$.

Let us consider a triangulation of $\Omega = (0,1) \times (0,1)$ with mesh parameter $h = \frac{1}{3}$ (see, Figure 1.4). Corresponding to the triangulation $\mathcal{T}_h$, we mark the elements as $1, \ 2, \ 3, \ \ldots$ (see, Figure 1.4 **with green color**). Then we fix the unknown grid points and then assign them node numbers $1, \ 2, \ 3, \ \ldots$ **by red color**, as could be seen in Figure 1.4. Clearly, our unknown nodes are **1**, **2**, **3** and **4** with respect to grid points $X_1 = (x_1, y_1)$, $X_2 = (x_2, y_1)$, $X_3 = (x_1, y_2)$ and $X_4 = (x_2, y_2)$, respectively. Note that

$$x_0 = 0, \ x_1 = x_0 + h, \ x_2 = x_1 + h, \ x_3 = x_2 + h \ \text{ and}$$
$$y_0 = 0, \ y_1 = y_0 + h, \ y_2 = y_1 + h, \ y_3 = y_2 + h, \ \text{ with } h = \frac{1}{3}.$$

Then, corresponding to unknown grid points **1**, **2**, **3** and **4**, we proceed to find Lagrange basis functions $\phi_1, \ \phi_2, \ \phi_3$ and ϕ_4, where ϕ_i's are piecewise linear function as shown in the Figure 1.3. Clearly, the set $\{\phi_1, \ \phi_2, \ \phi_3, \ \phi_4\}$ is linearly independent.

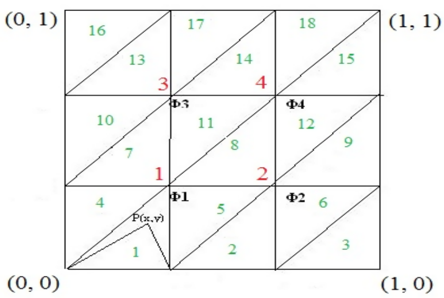


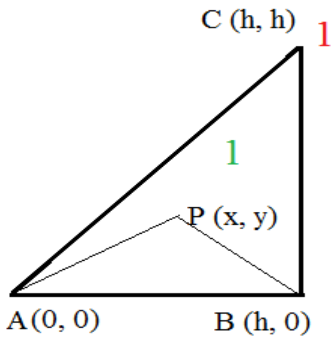
Figure 1.4: A triangulation $\mathcal{T}_h$ of our domain $\Omega = (0,1) \times (0,1)$ with $h = \frac{1}{3}$.

Remark 1.1. *Following Table illustrates the relation between unknown grid points and associated triangles*

Unknown grid points	*Coordinate*	*Associated Triangles*
1	$X_1 = (x_1, y_1)$	**1, 5, 8, 11, 7, 4**
2	$X_2 = (x_2, y_1)$	**2, 6, 9, 12, 8, 5**
3	$X_3 = (x_1, y_2)$	**7, 11, 14, 17, 13, 10**
4	$X_4 = (x_2, y_2)$	**8, 12, 15, 18, 14, 11**

Consider the unknown grid point 1. Using the construction of $(K, P_1, \sum_K)$ in the triangle **1**, we find that

$$\begin{aligned}\phi_1(x,y) &= \frac{\text{area}\Delta PAB}{\text{area}\Delta ABC}\\ &= \frac{1}{h^2}\begin{vmatrix} 1 & x & y\\ 1 & 0 & 0\\ 1 & h & 0\end{vmatrix} = \frac{yh}{h^2}\end{aligned}$$



Therefore, in triangle **1**, we have

$$\frac{\partial \phi_1}{\partial x} = 0 \;\;\&\;\; \frac{\partial \phi_1}{\partial y} = \frac{1}{h}.$$

Similarly, we have following calculations

Triangles	$\frac{\partial\phi_1}{\partial x}$	$\frac{\partial\phi_1}{\partial y}$
1	$\frac{\partial\phi_1}{\partial x}=0$	$\frac{\partial\phi_1}{\partial y}=\frac{1}{h}$
5	$\frac{\partial\phi_1}{\partial x}=\frac{-1}{h}$	$\frac{\partial\phi_1}{\partial y}=\frac{1}{h}$
8	$\frac{\partial\phi_1}{\partial x}=\frac{-1}{h}$	$\frac{\partial\phi_1}{\partial y}=0$
11	$\frac{\partial\phi_1}{\partial x}=0$	$\frac{\partial\phi_1}{\partial y}=\frac{-1}{h}$
7	$\frac{\partial\phi_1}{\partial x}=\frac{1}{h}$	$\frac{\partial\phi_1}{\partial y}=\frac{-1}{h}$
4	$\frac{\partial\phi_1}{\partial x}=\frac{1}{h}$	$\frac{\partial\phi_1}{\partial y}=0$

Triangles	$\frac{\partial\phi_2}{\partial x}$	$\frac{\partial\phi_2}{\partial y}$
2	$\frac{\partial\phi_2}{\partial x}=0$	$\frac{\partial\phi_2}{\partial y}=\frac{1}{h}$
6	$\frac{\partial\phi_2}{\partial x}=\frac{-1}{h}$	$\frac{\partial\phi_2}{\partial y}=\frac{1}{h}$
9	$\frac{\partial\phi_2}{\partial x}=\frac{-1}{h}$	$\frac{\partial\phi_2}{\partial y}=0$
12	$\frac{\partial\phi_2}{\partial x}=0$	$\frac{\partial\phi_2}{\partial y}=\frac{-1}{h}$
8	$\frac{\partial\phi_2}{\partial x}=\frac{1}{h}$	$\frac{\partial\phi_2}{\partial y}=\frac{-1}{h}$
5	$\frac{\partial\phi_2}{\partial x}=\frac{1}{h}$	$\frac{\partial\phi_2}{\partial y}=0$

Triangles	$\frac{\partial\phi_3}{\partial x}$	$\frac{\partial\phi_3}{\partial y}$
7	$\frac{\partial\phi_3}{\partial x}=0$	$\frac{\partial\phi_3}{\partial y}=\frac{1}{h}$
11	$\frac{\partial\phi_3}{\partial x}=\frac{-1}{h}$	$\frac{\partial\phi_3}{\partial y}=\frac{1}{h}$
14	$\frac{\partial\phi_3}{\partial x}=\frac{-1}{h}$	$\frac{\partial\phi_3}{\partial y}=0$
17	$\frac{\partial\phi_3}{\partial x}=0$	$\frac{\partial\phi_3}{\partial y}=\frac{-1}{h}$
13	$\frac{\partial\phi_3}{\partial x}=\frac{1}{h}$	$\frac{\partial\phi_3}{\partial y}=\frac{-1}{h}$
10	$\frac{\partial\phi_3}{\partial x}=\frac{1}{h}$	$\frac{\partial\phi_3}{\partial y}=0$

Triangles	$\frac{\partial\phi_4}{\partial x}$	$\frac{\partial\phi_4}{\partial y}$
8	$\frac{\partial\phi_4}{\partial x}=0$	$\frac{\partial\phi_4}{\partial y}=\frac{1}{h}$
12	$\frac{\partial\phi_4}{\partial x}=\frac{-1}{h}$	$\frac{\partial\phi_4}{\partial y}=\frac{1}{h}$
15	$\frac{\partial\phi_4}{\partial x}=\frac{-1}{h}$	$\frac{\partial\phi_4}{\partial y}=0$
18	$\frac{\partial\phi_4}{\partial x}=0$	$\frac{\partial\phi_4}{\partial y}=\frac{-1}{h}$
14	$\frac{\partial\phi_4}{\partial x}=\frac{1}{h}$	$\frac{\partial\phi_4}{\partial y}=\frac{-1}{h}$
11	$\frac{\partial\phi_4}{\partial x}=\frac{1}{h}$	$\frac{\partial\phi_4}{\partial y}=0$

Remark 1.2. *In the above calculation, we have used similarity and the fact that triangulation is uniform. For example,*

- ϕ_1 *in triangle* **1** *is same with* ϕ_2 *in triangle* **2**. ϕ_1 *in triangle* **1** *is same with* ϕ_3 *in triangle* **7**. ϕ_1 *in triangle* **1** *is same with* ϕ_4 *in triangle* **8**.

Similar remarks holds for other triangles.

Once the construction of basis functions corresponding to unknown grids are completed, we formally construct our finite element space V_h as

$$V_h = \text{span}\{\phi_1,\ \phi_2,\ \phi_3,\ \phi_4\}.$$

Step-4 (Finite Element Approximation) The finite element approximation to u satisfying weak formulation (1.3) is defined as: Find $u_h \in V_h$ such that

$$A(u_h, v_h) = L(v_h)\ \forall v_h \in V_h, \tag{1.6}$$

where bilinear map $A(\cdot,\cdot)$ and functional L are as defined in (1.4).

Since $u_h \in V_h$, there exists unique real constants $d_1,\ d_2,\ d_3$ and d_4 such that

$$u_h = d_1\phi_1 + d_2\phi_2 + d_3\phi_3 + d_4\phi_4.$$

Now, substitute u_h in the equation (1.6) to have

$$A\left(\sum_{i=1}^{4} d_i\phi_i \; v_h\right) = L(v_h).$$

Using the linearity of $A(\cdot,\cdot)$, we have

$$d_1A(\phi_1, v_h) + d_2A(\phi_2, v_h) + d_3A(\phi_3, v_h) + d_4A(\phi_4, v_h) = L(v_h) \;\; \forall v_h \in V_h.$$

Setting $v_h = \phi_1,\ \phi_2,\ \phi_3,\ \phi_4$ in the above equation, we have following system of equations

$$\begin{aligned}
d_1A(\phi_1,\phi_1) + d_2A(\phi_2,\phi_1) + d_3A(\phi_3,\phi_1) + d_4A(\phi_4,\phi_1) &= L(\phi_1)\\
d_1A(\phi_1,\phi_2) + d_2A(\phi_2,\phi_2) + d_3A(\phi_3,\phi_2) + d_4A(\phi_4,\phi_2) &= L(\phi_2)\\
d_1A(\phi_1,\phi_3) + d_2A(\phi_2,\phi_3) + d_3A(\phi_3,\phi_3) + d_4A(\phi_4,\phi_3) &= L(\phi_3)\\
d_1A(\phi_1,\phi_4) + d_2A(\phi_2,\phi_4) + d_3A(\phi_3,\phi_4) + d_4A(\phi_4,\phi_4) &= L(\phi_4)
\end{aligned}$$

In matrix notation

$$AD = F, \tag{1.7}$$

where $A = (a_{i,j})_{4\times 4}$ and $F = (f_{i,1})_{4\times 1}$ with

$$\begin{aligned}
a_{i,j} &= A(\phi_j,\phi_i) = \int_\Omega \nabla\phi_j \cdot \nabla\phi_i dx\\
&= \int_\Omega \nabla\phi_i \cdot \nabla\phi_j dx\\
&= a_{j,i} && (1.8)\\
f_{i,1} &= L(\phi_1) = \int_\Omega f\phi_1. && (1.9)
\end{aligned}$$

Using positivity of bilinear map $A(\cdot,\cdot)$, it is easy to verify that matrix A is positive definite and so matrix A is invertible.

Step-5 (Calculation) As the integration may involve arbitrary functions, for example $f_{i,1}$ involves integration of arbitrary function f, it may be difficult to evaluate the integration exactly. In that case, we use some well known quadrature rule (numerical integration). Suppose $K = \Delta ABC$ is a triangle with vertices A, B and C. Then, we use following numerical integration rule which is exact for linear polynomials

$$\int_K PdX \approx \frac{\text{area of K}}{3}\Big[P(A) + P(B) + P(C)\Big].$$

Using this quadrature rule, we obtain

$$\begin{aligned}
f_{1,1} = \int_\Omega f\phi_1 dX &= \int_{\mathbf{1}} f\phi_1 dX + \int_{\mathbf{5}} f\phi_1 dX + \int_{\mathbf{8}} f\phi_1 dX\\
&\quad + \int_{\mathbf{11}} f\phi_1 dX + \int_{\mathbf{7}} f\phi_1 dX + \int_{\mathbf{4}} f\phi_1 dX\\
&\approx 6 \times \frac{\text{area of K}}{3} \times f(x_1,y_1), \;\; \text{since } \phi_i(X_j) = \delta_{ij}\\
&= 6 \times \frac{\frac{1}{2}h^2}{3} \times f(x_1,y_1)\\
&= h^2 f(x_1,y_1) = h^2 f(X_1).
\end{aligned}$$

In general, we have

$$f_{i,1} \approx h^2 f(X_i),\ i = 1,\ 2,\ 3,\ 4.$$

Next, note that

- Basis function $\phi_1 \neq 0$ only on the triangles **1, 5, 8, 11, 7, 4** and similar type observations can be made for other basis functions $\phi_2,\ \phi_3,\ \phi_4$.

So that, using the Tables in page 5 and the fact that

$$\int_K dx = \text{area of K},$$

we obtain

$$\begin{aligned}
a_{1,1} &= \int_\Omega \nabla\phi_1 \cdot \nabla\phi_1 dx \\
&= \int_{\mathbf{1}} (\nabla\phi_1)^2 dx + \int_{\mathbf{5}} (\nabla\phi_1)^2 dx + \int_{\mathbf{8}} (\nabla\phi_1)^2 dx \\
&\quad + \int_{\mathbf{11}} (\nabla\phi_1)^2 dx + \int_{\mathbf{7}} (\nabla\phi_1)^2 dx + \int_{\mathbf{4}} (\nabla\phi_1)^2 dx \\
&= 4 \times \frac{1}{h^2} \times \frac{1}{2} h^2 + 2 \times \left(\frac{1}{h^2} + \frac{1}{h^2} \right) \times \frac{1}{2} h^2 = 2 + 2 = 4.
\end{aligned}$$

Similarly, in general, we have following diagonal entries for coefficient matrix A

$$a_{i,i} = 4,\quad i = 1,\ 2,\ 3,\ 4. \tag{1.10}$$

For other entries of the matrix A, we use following grid movement

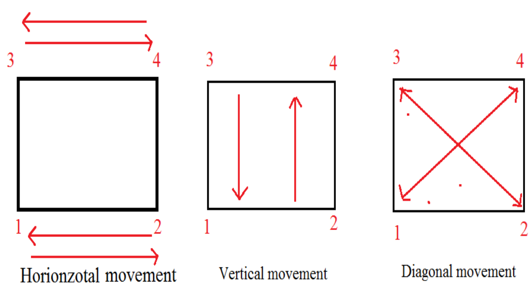


Figure 1.5: Grid movement for the entries of matrix A

Due to horizontal movement, we have

$$\begin{aligned}
a_{1,2} = A(\phi_2, \phi_1) &= \int_\Omega \nabla\phi_2 \cdot \nabla\phi_1 dx \\
&= \int_{\mathbf{5}} \nabla\phi_2 \cdot \nabla\phi_1 dx + \int_{\mathbf{8}} \nabla\phi_2 \cdot \nabla\phi_1 dx \\
&= -\frac{1}{2} - \frac{1}{2} = -1 = a_{2,1}.
\end{aligned} \tag{1.11}$$

Similarly, we have

$$a_{3,4} = -1 = a_{4,3},\quad a_{1,4} = 0 = a_{4,1}\ \ \&\ \ a_{2,3} = 0 = a_{3,2}. \tag{1.12}$$

Due to vertical movement, we have following entries

$$\begin{aligned} a_{1,3} = A(\phi_3, \phi_1) &= \int_\Omega \nabla\phi_3 \cdot \nabla\phi_1 dx \\ &= \int_{\mathbf{7}} \nabla\phi_2 \cdot \nabla\phi_1 dx + \int_{\mathbf{11}} \nabla\phi_2 \cdot \nabla\phi_1 dx \\ &= -\frac{1}{2} - \frac{1}{2} = -1 = a_{3,1}. \end{aligned} \tag{1.13}$$

Similarly

$$a_{2,4} = -1 = a_{4,2}. \tag{1.14}$$

Due to diagonal movement, we have following entries

$$\begin{aligned} a_{1,4} = A(\phi_4, \phi_1) &= \int_\Omega \nabla\phi_4 \cdot \nabla\phi_1 dx \\ &= \int_{\mathbf{8}} \nabla\phi_4 \cdot \nabla\phi_1 dx + \int_{\mathbf{11}} \nabla\phi_4 \cdot \nabla\phi_1 dx \\ &= 0 + 0 = 0 = a_{4,1}. \end{aligned} \tag{1.15}$$

Similarly

$$a_{2,3} = 0 = a_{3,2}. \tag{1.16}$$

Finally, using above calculations in (1.7), we obtain

$$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} h^2 f(X_1) \\ h^2 f(X_2) \\ h^2 f(X_3) \\ h^2 f(X_4) \end{pmatrix}. \tag{1.17}$$

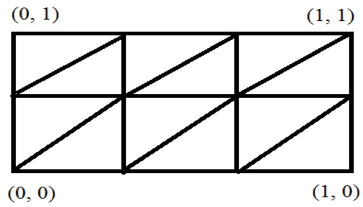
For given f, we can solve above system for unknown matrix D so that finite element approximation is given by

$$u_h(x) = d_1\phi_1(x) + d_2\phi_2(x) + d_3\phi_3(x) + d_4\phi_4(x) \quad \forall x \in \Omega. \tag{1.18}$$

Example 1.1. (Homework) Consider following boundary value problem

$$-\Delta u = 1 \ \text{ in } \Omega = (0,1) \times (0,1) \ \ \& \ \ u = 0 \ \text{ on } \partial\Omega.$$

For the given uniform triangulation, find finite element solution at $(\frac{1}{5}, \frac{1}{7})$.



Example 1.2. (Homework) Consider following boundary value problem

$$-\Delta u = 2(x+y) - 4 \ \text{ in } \Omega = (0,1) \times (0,1) \ \ \& \ \ u = 0 \ \text{ on } \partial\Omega.$$

For the given nonuniform triangulation, with $X_1 = (\frac{1}{3}, \frac{2}{3})$, find finite element solution at $(\frac{2}{3}, \frac{1}{3})$.