# A SIMPLE NUMERICAL METHOD FOR PRICING AMERICAN POWER PUT OPTIONS

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# Introduction

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- **American option**
  - An American option is a style of options contract that allows holders to exercise their rights at any time before and including the expiration date.
- **Put option**
  - A put option is a contract giving the option buyer the right, but not the obligation, to sell specified amount of an underlying security at a predetermined price within a specified time frame.
  - This predetermined price at which the buyer of the put option can sell the underlying security is called the strike price.
- **Power option**
  - An option whose payoff is based on the price of an underlying asset raised to a power.
- **Payoff of power put option**
  - Payoff of power put option is given as
$$(K - S^n(T))^+ = \max\{K - S^n(T), 0\}.$$

# Optimal exercise boundary

- **Definition**
  - Optimal exercise boundary separates the area where one should continue to hold the option and the area where one should exercise it.
- **Importance**
  - The price problem requires complex analytical calculations. Due to the fact that the option holder has an early exercise right, the problem becomes a free boundary problem. As a result, determining the best exercise border is a major task when it comes to valuing American options.

# Finite Difference method

- **Definition**
  - In numerical analysis, finite-difference methods (FDM) are a class of numerical techniques for solving differential equations by approximating derivatives with finite differences.
  - Both the spatial domain and time interval (if applicable) are discretized, or broken into a finite number of steps, and the value of the solution at these discrete points is approximated by solving algebraic equations containing finite differences and values from nearby points.
- **Importance**
  - After determining the optimal exercise boundary, the American power put option values are calculated by applying the finite difference method (FDM).
  - The proposed method provides fast and accurate results with respect to the calculation of the optimal exercise boundary and pricing of the American power put options.

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# Formulation

- PDE
  - Black-Scholes pde
  - Boundary Condition
  - Initial Condition
- Optimal exercise boundary (Beta)

# PDE

- The Black–Scholes PDE for the price of an American power put option with a non-dividend yield can be given as

$$\frac{\partial P}{\partial \tau} - n\left\{\frac{1}{2}\sigma^2(n-1) + r\right\} S^n \frac{\partial P}{\partial S^n} - \frac{1}{2}\left(\sigma n S^n\right)^2 \frac{\partial^2 P}{\partial \left(S^n\right)^2} + rP = 0 \quad (1)$$

- Then (1) is subjected to the following boundary conditions.

$$\lim_{S^n \to \infty} P\left(\tau, S^n\right) = 0 \quad (2)$$

$$P(\tau, \beta(\tau)) = K - \beta(\tau) \quad (3)$$

$$P_{\mathrm{S}^n}(\tau, \beta(\tau)) = -1 \quad (4)$$

and the initial condition

$$P\left(0, S^n\right) = \max\left\{K - S^n, 0\right\} \quad (5)$$

# Optimal Exercise Boundary

- The best time to exercise an American option and its price are determined by the optimal exercise boundary denoted as $\beta = \{\beta(\tau) : \tau \in [0, \mathrm{T}]\}$.
- Here, we suppose that the optimal exercise boundary $\beta(\tau)$ is continuously nonincreasing with $\beta(0) = K$. For each time $\tau \in [0, \mathrm{T}]$, there exists an optimal exercise boundary $\beta(\tau)$, below which the American power put option should be exercised early, i.e., if

$$S^n(\tau) \leq \beta(\tau), \text{ then } P(\tau, S^n) = \max\{K - S^n(\tau), 0\} \tag{6}$$

and if

$$S^n(\tau) > \beta(\tau), \text{ then } P(\tau, S^n) > \max\{K - S^n(\tau), 0\} \tag{7}$$

# Optimal Exercise Boundary - ctd.

- The time and asset price space are divided into two regions. A continuation region is the one in which it is optimal to hold, commonly known as $\Omega_C = [0, \mathrm{T}] \times(\beta(\tau), \infty)$, and the region in which it is optimal to exercise, generally called the exercise (or stopping) region, is defined as $\Omega_E = [0, \mathrm{T}] \times[0, \beta(\tau)]$.
- Following are the asymptotically optimal exercise boundaries for perpetual American power puts $\beta(\infty)$:

$$\beta(\infty) = \frac{\gamma}{\gamma + 1} K \tag{8}$$

where

$$\gamma = \frac{2r}{n^2\sigma^2}$$

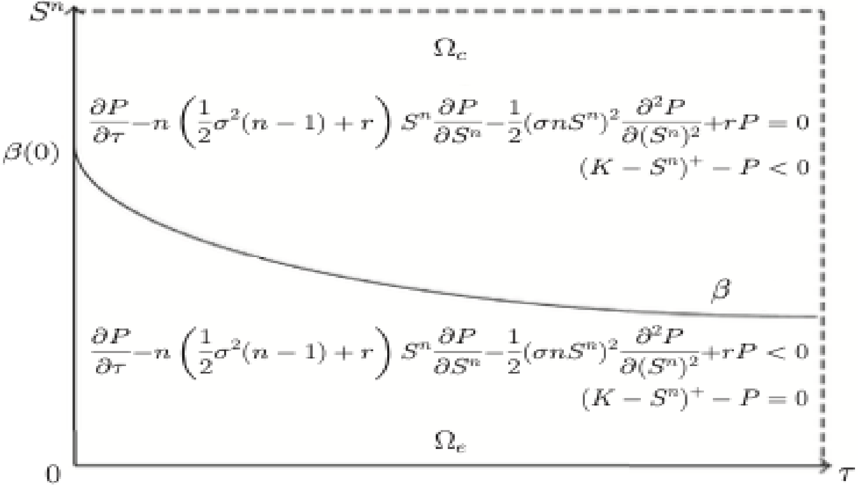


Figure: Optimal exercise boundary of an American power put option

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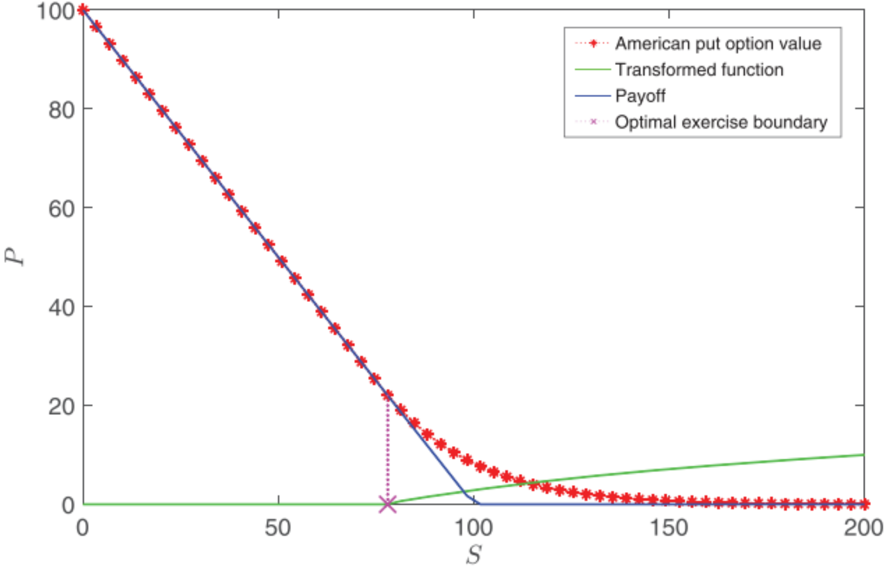
# Transformed Function

- Before computing the PDE solution, we shall determine the ideal exercise border ($\beta(\tau)$) in this step. We give the following transformed function

$$Q\left(\tau, S^{n}\right)=\sqrt{P\left(\tau, S^{n}\right)-\left(K-S^{n}\right)} \tag{9}$$

- Since the solution is a horizontal line in the exercise zone and an inclined line in the continuation region, the aforementioned function is employed. As a result, $Q(\tau, S_n)$ equals 0 in the workout region and $Q(\tau, S_n) > 0$ in the continuation region. The boundary may be clearly detected since the transformed function create a sufficiently large angle with the horizontal line.

# Angle b/w Q and Exercise region



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# Numerical Method

- A numerical method developed based on the transformed function. The independent variables in the Black–Scholes PDE are time ($\tau$) and the underlying stock price ($S^n$).
- To solve this problem using the FDM, we divide the time interval $[0, T]$ into $L$ subintervals, that is $\tau_l = l\Delta\tau, l = 0, 1, 2, .., L-1, L \; where \; \Delta\tau = \frac{T}{L}$ and the stock price interval $[0, S^n_M]$ into M subintervals, that is $S^n_i = i\Delta S^n, i = 0, 1, 2, ..., M-1, M, \;\Delta S^n = \frac{S^n_M}{M}$.
- We intend to define a numerical method for computing the grid values $P^l_i \approx P(\tau_l, S^n_i)$ and the optimal exercise boundary values $\beta_l = \beta(\tau_l), for \; i = 0, 1, 2, ..., M-1, M, \; and \; l = 0, 1, 2, ..., L-1, L$. Herein, i denotes the horizontal node index, l denotes the time step index, and $\tau_{l-1}$ represents the $(l1)th$ previous time step and $\tau_l$ represents the $lth$ current time step.
- For the proposed problem, $\beta(\tau_{l-1})$ and $P^{l-1}_i$ are given and the objective is to compute $\beta(\tau_l)$ and $P^l_i$. We proceed step by step now:

# Step 1

- We begin with the initial value at time T , which provides the value of the payoff function $(K - S_i^n)^+ = max\{K - S_i^n, 0\}$ and the optimal exercise boundary $\beta(0) = K$.
- Because the optimal exercise boundary $\beta(\tau_{l-1})$ is dependent on the future boundary value $\beta(\tau_l)$ , it must be determined by setting the initial boundary value to the strike price $\beta(\tau_0) = K$ and working backward through time from $\mathrm{T} = \tau_0$ to $\mathrm{T} = \tau_L$.

# Step 2

- We obtain the current optimal exercise boundary $\beta(\tau_l)$ based on the previous optimal exercise boundary $\beta(\tau_{l-1})$ . The current optimal exercise boundary can be obtained using a transformed function $Q(\tau_l, S^n) = \sqrt{P(\tau_l, S^n) - (K - S^n)}$.
- Herein, to obtain Q at the current time step, the option price $P(\tau_l, S^n)$ should be obtained. Further, to find $\beta(\tau_l)$, we derive the relation between $Q(\tau_l, S^n)$ and $\beta(\tau_l)$.
- Now We find $P(\tau_l, S^n)$ with respect to the current time step using a three-point FDM based on a uniform mesh.
- If the optimal exercise boundary does not depend on the grid point, then we cannot use the FDM based on a uniform mesh. Therefore, we use natural cubic spline interpolation. Based on the mesh point information, we can obtain value between the mesh points.

- Define:
$$S_1 = \beta(\tau_{l-1}) + \Delta S^n$$
$$S_2 = \beta(\tau_{l-1}) + 2\Delta S^n$$
then, we find $P(\tau_l, S_1)$.
- Because the previous optimal exercise boundary $\beta(\tau_{l-1})$ is dependent on the current optimal exercise boundary $\beta(\tau_l)$ , we calculate the option price $P(\tau_l, S_1)$ on $S_1$ in the current time step using the explicit method.

- The Black Sholes equation can be approximated as:

$$P\left(\tau_{l}, S_{1}\right)=(1-r\Delta\tau)P\left(\tau_{l-1}, S_{1}\right)+n\left\{\tfrac{1}{2}\sigma^{2}(n-1)+r\right\} S_{1}\Delta\tau \frac{\partial P(\tau_{l-1},S_{1})}{\partial S}+\frac{1}{2}\left(\sigma n S_{1}\right)^{2}\frac{\partial^{2} P(\tau_{l-1},S_{1})}{\partial S^{2}}\Delta\tau \tag{10}$$

- The derivatives in above equations could be formulated using:

$$\frac{\partial P\left(\tau_{l-1}, S_{1}\right)}{\partial S} \simeq \frac{P\left(\tau_{l-1}, S_{2}\right)-P\left(\tau_{l-1}, S_{1}\right)}{2\left(S_{2}-S_{1}\right)} \tag{11}$$

$$\frac{\partial^{2} P\left(\tau_{l-1}, S_{1}\right)}{\partial S^{2}} \simeq \frac{P\left(\tau_{l-1}, \beta\left(\tau_{l-1}\right)\right)-2P\left(\tau_{l-1}, S_{1}\right)+P\left(\tau_{l-1}, S_{2}\right)}{\left(S_{2}-S_{1}\right)^{2}} \tag{12}$$

- We now apply the cubic spline interpolation in above 2 equations to find $P(\tau_{l-1}, S_1)$ and $P(\tau_{l-1}, S_2)$ ; thus, we obtain $P(\tau_{l-1}, S_1) \approx f(\tau_{l-1}, S_1)$ , $P(\tau_{l-1}, S_2) \approx f(\tau_{l-1}, S_2)$ , where f is the cubic spline function.
- Because $P(\tau, S^n) = Q^2(\tau, S^n) + K - S^n$, we obtain $P(\tau_{l-1}, \beta(\tau_{l-1})) = K - \beta(\tau_{l-1})$ near the optimal exercise boundary (Q=0). Thus, we obtain $P(\tau_l, S_1)$. As we can see that

$$Q\left(\tau_l, S_1\right) = \sqrt{P\left(\tau_l, S_1\right) - \left(K - S_1\right)} \tag{13}$$

- We find the relation between $\beta(\tau_{l-1}$ and $Q(\tau_l, S_1)$ using second-order Taylor expansion of Q:

$$\begin{aligned} Q\left(\tau, S^n\right) = & Q(\tau, \beta) + Q_{S^1}(\tau, \beta)\left(S^n - \beta\right) \\ & + \frac{1}{2} Q_{S^n\ S1}\left(\tau, \beta\right)\left(S^n - \beta\right)^2 + \mathcal{O}\left(S^n - \beta\right)^3 \end{aligned} \tag{14}$$

- After some mathematical calculations we obtain:

$$\begin{aligned}6(\sigma n\beta)^2 Q_{S^n} Q =& 6\left(\sigma n\beta Q_{S^n}\right)^2 \left(S^n - \beta\right) \\ & - \left\{2Q_{S^n}^2\left(\beta' + A\beta + \sigma^2 n^2 \beta\right) - (A - r)\right\}\left(S^n - \beta\right)^2\end{aligned} \tag{15}$$

  where

$$\frac{\beta'}{\beta} \approx \frac{\ln\left(\frac{S^n}{\beta}\right) - \left(\frac{S^n}{\beta} - 1\right)}{\Delta\tau} \tag{16}$$

  and

$$A = n\left\{\frac{1}{2}\sigma^2(n-1) + r\right\} \tag{17}$$

  Combining the above 2 equations we can rewrite is as:

$$a\beta^5 + b\beta^4 + c\beta^3 + d\beta^2 + e\beta + f = 0 \tag{18}$$

  where $a, b, c, d, e, f$ are constants.
- This equation could be solved using Newton–Raphson method. Hence, the solution $\beta$ could be found at the current optimal exercise boundary.

# Step 3

- Now task is to obtain the option price using $\beta(\tau_l)$. let j be the smallest spatial grid index over $\beta(\tau_l)$ . If $\beta(\tau_l)$ is too close to $S_j^n$ , we select the index j+1 instead of j.
  Using central difference for the spatial discretization, then:

$$\frac{\partial P_j^l}{\partial S_j^n} = \frac{P_{j+1}^l - P_{j-1}^l}{2\Delta S^n}, \frac{\partial^2 P_j^l}{\partial \left(S_j^n\right)^2} = \frac{P_{j+1}^l - 2P_j^l + P_{j-1}^l}{(\Delta S^n)^2} \tag{19}$$

- Now using the implicit scheme for time discretization, equation (1.5) is discretized as follows:

$$(1 + r\Delta\tau)P_j^l - AS_j^n \frac{P_{j+1}^l - P_{j-1}^l}{2\Delta S^n}\Delta\tau - \frac{1}{2}\left(\sigma n S_j^n\right)^2 \frac{P_{j+1}^l - 2P_j^l + P_{j-1}^l}{(\Delta S^n)^2}\Delta\tau = P_j^{l-1}. \tag{20}$$

- Generally, the optimal exercise boundary may not be located at the grid points. Therefore, we cannot use FDM directly. Because we know the values of P at $\beta$, we consider using linear interpolation between adjacent data points.
- The value of P at $\beta$ can be written as

$$P\left(\tau_l, \beta\left(\tau_l\right)\right) = \frac{\left(S_j^n - \beta\left(\tau_l\right)\right) P_{j-1}^l + \left(\Delta S - S_j^n + \beta\left(\tau_l\right)\right) P_j^l}{\Delta S^n} = K - \beta\left(\tau_l\right) \tag{21}$$

- Hence we obtain:

$$P_{j-1}^l = \frac{\left(K - \beta\left(\tau_l\right)\right) \Delta S^n - \left(\Delta S^n - S_j^n + \beta\left(\tau_l\right)\right) P_j^l}{S_j^n - \beta\left(\tau_l\right)} \tag{22}$$

- After mathematical calculations:

$$a_1(K - \beta(\tau_l)) + b_1 P_j^l + c_1 P_{j+1}^l = P_j^{l-1} \tag{23}$$

where

$$
\begin{aligned}
a_1 =& -\frac{1}{2}\left(\sigma n S_j^n\right)^2 \Delta\tau \frac{1}{\left(S_j^n-\beta\left(\tau_l\right)\right)\Delta S^n} + A S_j^n \Delta\tau \frac{1}{2\left(S_j^n-\beta\left(\tau_l\right)\right)} \\
b_1 =& 1 + r\Delta\tau - A S_j^n \Delta\tau \frac{\Delta S^n - S_j^n + \beta\left(\tau_l\right)}{2\left(S_j^n-\beta\left(\tau_l\right)\right)\Delta S^n} \\
&+ \frac{1}{2}\left(\sigma n S_j^n\right)^2 \Delta\tau \left\{\frac{S_j^n - \beta\left(\tau_l\right) + \Delta S^n}{\left(S_j^n-\beta\left(\tau_l\right)\right)\left(\Delta S^n\right)^2}\right\} \\
c_1 =& - A S_j^n \Delta\tau \frac{1}{2\Delta S^n} - \frac{1}{2}\left(\sigma n S_j^n\right)^2 \Delta\tau \frac{1}{\left(\Delta S^n\right)^2}
\end{aligned}
\tag{24}
$$

- We can rewrite a recursive formula like:

$$
a_i P_{j-1}^l + b_i P_j^l + c_i P_{j+1}^l = P_j^{l-1} \tag{25}
$$

where

$$a_i = \mu_i - \lambda_i, b_i = (1 + r\Delta\tau) + 2\lambda_i, c_i = -(\mu_i + \lambda_i)$$

$$\mu_i = A\frac{\Delta\tau}{2\Delta S^n} S^n_{j+i-1}, \lambda_i = \frac{(\sigma n)^2}{2(\Delta S^n)^2}\Delta\tau\left(S^n_{j+i-1}\right)^2, 2 \le i \le M - j + 1 \tag{26}$$

- The Dirichlet boundary condition at $\beta(\tau_l)$ gives $K - \beta(\tau_l)$. Therefore we can observe a set of simultaneous equations which could be solved using matrix of from $\mathrm{W}\mathbf{x} = \mathbf{y}$.

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & b_1 & c_1 & \cdots & 0 & 0 \\ 0 & a_2 & b_2 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{M-j} & b_{M-j} & c_{M-j} \\ 0 & 0 & \cdots & 0 & a_{M-j+1} & b_{M-j+1} \end{pmatrix} \begin{pmatrix} P^l_\beta \\ P^l_j \\ P^l_{j+1} \\ \vdots \\ P^l_{M-1} \\ P^l_M \end{pmatrix} = \begin{pmatrix} K - \beta(\tau_l) \\ P^{l-1}_j \\ P^{l-1}_{j+1} \\ \vdots \\ P^{l-1}_{M-1} \\ P^{l-1}_M \end{pmatrix} \tag{27}$$

where

$$a_{M-j+1} = 2\mu_{M-j+1}, b_{M-j+1} = 1 + r\Delta\tau - 2\mu_{M-j+1} \tag{28}$$

# Solution for W

- To solve this system, we verify if the coefficient matrix W is invertible. By following Proposition given below, we verify the existence and uniqueness of the system of equations.
- **Definition:** An n $\times$ n matrix is said to be strictly diagonally dominant when

$$|a_{ss}| > \sum_{t=1, s\neq t}^{n} |a_{st}| \tag{29}$$

  holds for s = 1,2,...,n, where $a_{st}$ denotes the entry in the sth row and tth column.
- **Proposition:** If

$$\frac{1}{\Delta\tau} + r > A\tilde{C}(M+1) \tag{30}$$

  then W is strictly diagonally dominant, where $\tilde{C}$ is a constant. Therefore, the linear system has a unique solution.

- By some complex mathematical formulations we can show that W in equation is strictly diagonally dominant and it can be shown that every strictly diagonally dominant matrix is invertible.
- **Lemma:** Following conditions are equivalent on the n $\times$ n square matrix W.
  (1) *The matrix $W$ is invertible.*
  (2) *The linear system $W\mathbf{x} = \mathbf{y}$ is consistent for every $\mathbf{y}$.*
  (3) *The linear system $W\mathbf{x} = \mathbf{y}$ has a unique solution for every $\mathbf{y}$.*
- Hence, by above lemma, we prove the existence and uniqueness of the solution to system of equations. The system can be solved by finding the inverse matrix $W^{-1}$. Therefore, we can obtain the option value for the next time level.

# Why is every strict diagonally dominant matrix is invertible

For an elementary proof, assume there exists a vector $x \neq 0$ such that $Ax = 0$. This implies $\sum_{j=1}^{n} a_{ij}x_j = 0, \forall i \in \{1, \dots, n\}$. Let $x_k = \|x\|_\infty \neq 0$, i.e. $x_k$ is the the largest entry of $x$ by absolute value. We have:

$$0 = \sum_{j=1}^{n} a_{kj}x_j \implies a_{kk}x_k = -\sum_{j\neq k} a_{kj}x_j \implies a_{kk} = -\sum_{j\neq k} a_{kj}\frac{x_j}{x_k}$$

By taking the absolute value we get:

$$|a_{kk}| = \left|\sum_{j\neq k} a_{kj}\frac{x_j}{x_k}\right| \leq \sum_{j\neq k} |a_{kj}| \underbrace{\left|\frac{x_j}{x_k}\right|}_{\leq 1} \leq \sum_{j\neq k} |a_{kj}|$$

This is a contradiction since $A$ is strictly diagonally dominant. This means that $0 \notin \sigma(A)$, hence $A$ is invertible.

# Step 4

- We repeat the previously mentioned process until $\tau_L$ and obtain the optimal exercise boundary in a time-recursive manner.

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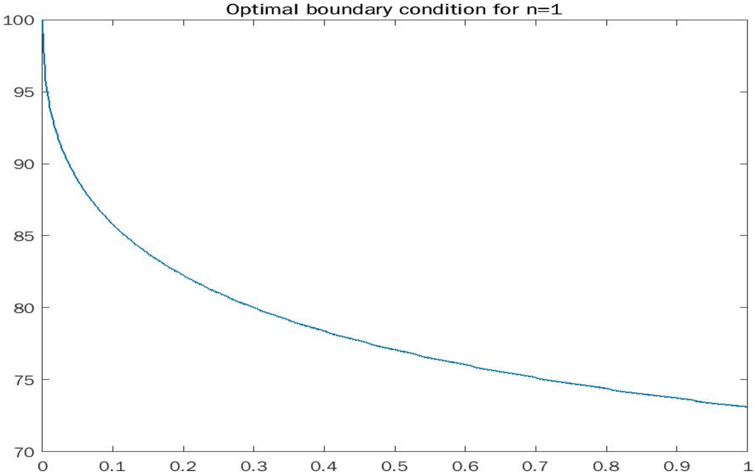
# Numerical Experiments

- Optimal boundary condition for n=1
- American power put price vs asset price for n = 1
- Price of option vs asset price and Payoff vs asset price
- Beta for different values of n

# Optimal boundary condition for n=1

- We consider the strike price K = 100, the interest rate r = 0.1, thevolatility= 0.3, and the time to maturity T = 1 (year). Further, we construct the computational domain with 250 spatial steps and 1000 timesteps.
- At the maturity time, the optimal exercise boundary theoretically is $\beta(T) = 76.0964$, whereas it is $\beta(T) = 73.1258$ in the proposed method.
- The asymptotically optimal exercise boundary with respect to the perpetual American power put option is $\beta(\infty) = 68.9655$.

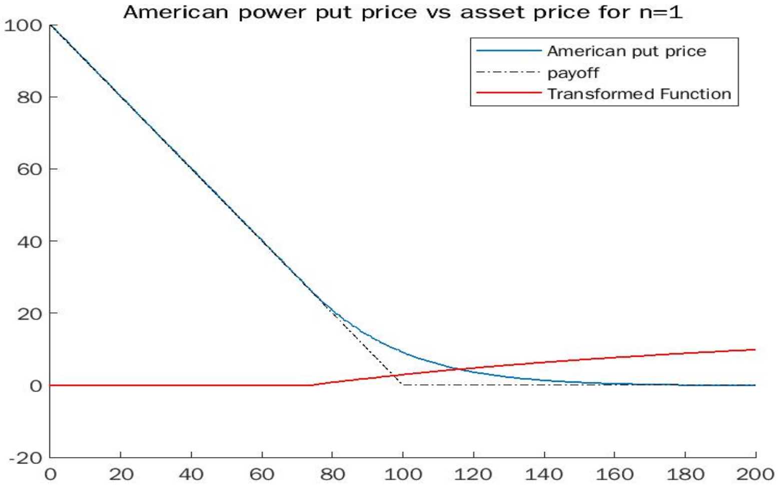
# Optimal boundary condition for n=1 ctd.



# American power put price vs asset price for n = 1

- The parameter values used to calculate the optimal exercise boundary and values of the American put options are r = 0.1, $\sigma = 0.3$, K = 100, T = 1 and n = 1 and the computational domain can be constructed with 250 spatial steps and 1000 time steps.

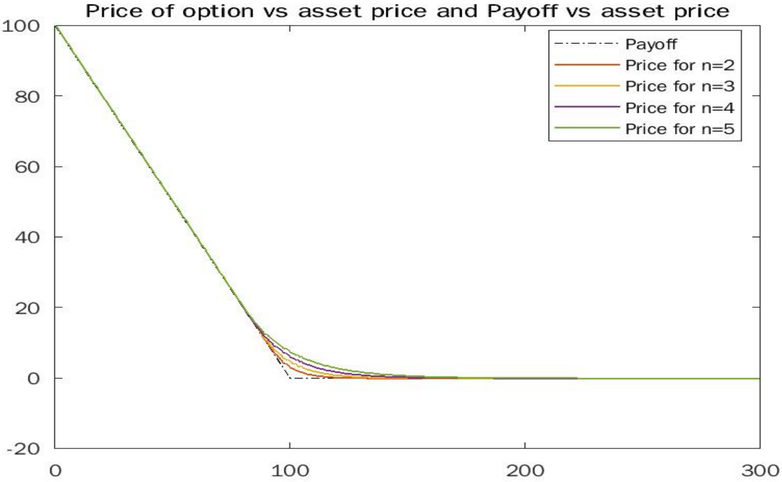
# American power put price vs asset price for n = 1 ctd.



# Price of option vs asset price and Payoff vs asset price

- The below figure shows that the American power put option values obtained when power n = 2, 3, 4, 5. The American power put option values increase with the increasing value of power n.
- The feature of nonlinear payoffs of power options provides the buyer with a potential to receive a considerably higher payoff than that received from a vanilla option.

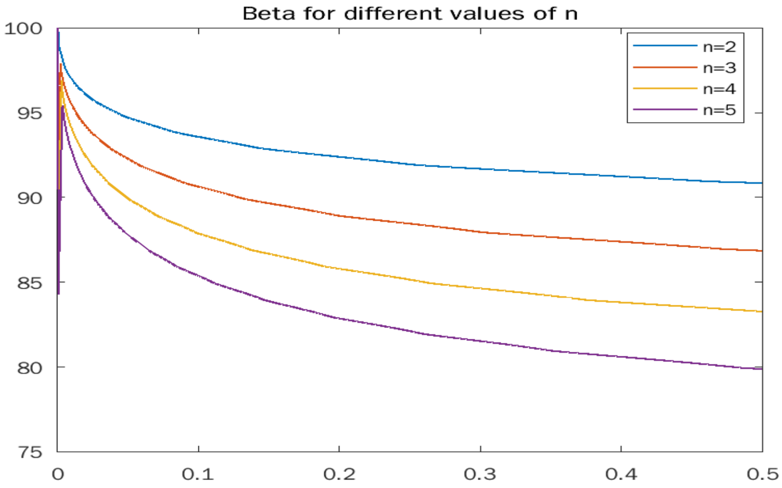
# Price of option vs asset price and Payoff vs asset price ctd.



# Beta for different values of n

- We use $\sigma = 0.1$, r = 0.08, K = 100,T = 0.5, and L $\times$ M = 2000 $\times$ 300. The below figure optimal exercise boundary for different values of power n = 2, 3, 4, 5.
- Further, the optimal exercise boundaries are plotted as a function of time $\tau$. The optimal exercise boundaries decrease with the increasing power.

# Beta for different values of n ctd.



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# Conclusions

- We provided a numerical method and examples for pricing the American power put option for non-dividend yields.
- we determined the optimal exercise boundary using the transformed function.
- We have compared the other results with the results obtained using the proposed method for the valuation of the American power put option when $n = 1$ to provide sufficient numerical analysis.
- We provided the optimal exercise boundary and American power put option values for different values of power n.
- The numerical experiment denotes that the proposed method is accurate, flexible, and efficient and provides accurate prices with respect to the critical stock price for various parameter combinations.