WEAK GALERKIN FINITE ELEMENT SOLVER WITH POLYGONAL MESHES

A Project Report Submitted

for the Course

MA498 Project I

by

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to the

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CERTIFICATE

This is to certify that the work contained in this project report entitled "Weak galerkin finite element solver using polygonal meshes" submitted by Ansh Bhatt (Roll No.: 180123005) and Naman Goyal (Roll No.: 180123029) to the Department of Mathematics, Indian Institute of Technology Guwahati towards partial requirement of Bachelor of Technology in Mathematics and Computing has been carried out by them under my supervision.

It is also certified that, along with literature survey, a few new results are established/computational implementations have been carried out/simulation studies have been carried out/empirical analysis has been done by the student under the project.

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ABSTRACT

In mathematics and various associated fields Finite Element Method is used to numerically approximate the solutions to partial differential equations which are not possible to solve analytically. In this project we try to develop an efficient solver for Finite Element approximations using polygonal meshes. We start with an introduction to the method, followed by its computational techniques and then proceed to look at the various aspects of an FEM solver.

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Chapter 1

Introduction

Differential equations are used in a variety of sectors of science and engineering to represent mathematical models. The majority of the time, solving these differential equations analytically is extremely difficult. Numerical approaches that approximate the answer are required to obtain the solutions to these mathematical models. We are transitioning towards numerical methods that efficiently and accurately approximate the solutions to complex mathematical models as high-speed computers develop.

One of the numerical methods is the Finite Element Method. Instead of computing the solution of the model on the entire domain, FEM is utilised to do it in a finite number of sub-domains. The computational domain is broken down into sub-domains, each of which is called an element. The mesh is made up of all of these pieces combined together. At every location in the domain, FEM produces a numerical estimate of the model's solution.

After a general introduction to the method in which we look at the issue formulation and attempt to describe the mathematics behind the method, we will go on to using FEM to solve a boundary value problem. We'll learn about the many aspects of creating a FEM solver in this section.

Let $\Omega \in \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. We consider following elliptic boundary value problem

$$\left.\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= 0 \text{ in } \partial\Omega. \end{aligned}\right\} \tag{1.1}$$

We need to study the convergence of finite element solution to the exact solution of (1.1) with respect to H^1-norm and L^2-norm.

1.1 Basic Notations

We shall now introduce the standard notation for Sobolev spaces and norms to be used.

Definition 1.1.1. Let $\Omega \in R^2$ and p is a real number with the property $1 \leq p \leq \infty$, *then* $L^p(\Omega)$ denotes the following space

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}.$$

Further, L^p (Ω) is a normed linear space with respect to the following norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}$$

Definition 1.1.2. Let m be a positive integer and $1 \leq p < \infty$. Then the Sobolev spaces $W^{m,p}(\Omega)$ is defined by

$$W^{m,p} = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), 0 \leq \alpha \leq m\},$$

where $$D^\alpha u = \frac{\partial^\alpha u}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2}, \ \alpha = \alpha_1 + \alpha_2$$

is the weak derivative of u at the αth order. To put it another way, $W^{m,p}(\Omega)$ is a collection of all functions in L^p (Ω) that have all of their weak derivatives up to order m in L^p (Ω). In addition, $W^{m,p}(\Omega)$ is a normed linear space in terms of the norm.

$$\|v\|_{m,p,\Omega} = \sum_{0\leq\alpha\leq m} \|D^\alpha v\|_{L^2(\Omega)}$$

For our later use, we introduce following semi-norm

$$|v|_{m,p,\Omega} = \sum_{\alpha=m} \|D^\alpha v\|_{L^2(\Omega)}$$

Now in order to introduce the weak derivative, we consider the following equation

$$\frac{dy}{dx} = g(x) \ in \ \Omega \in \mathbb{R}$$

It is well understood that if $g \in C(\Omega)$, then $y \in C^1(\Omega)$. If $g \in L^2(\Omega)$, the difficulty emerges, and we cannot expect a solution in $C^1(\Omega)$. Of course, we suppose that $y \in C(\Omega) \supset C^1(\Omega)$. C($\Omega$) functions, on the other hand, do not have to be differentiable. As a result, we can't find $\frac{dy}{dx}$ in the traditional sense. As a result, even if y is neither differentiable or at all continuous, we must give the derivative of y a meaning. The weak derivative of such functions is used to do this.

Let us try to define the weak derivative of a differentiable function *f*, and then generalise that definition to a bigger class of functions to encourage us to look for weak derivatives of such functions. Let $C_0^\infty(\Omega)$ be the collection of all C^∞ functions defined over Ω that vanish at the boundary $\partial\Omega$ of Ω. Then we obtain

$$\int_\Omega \frac{df}{dx} v dx = fv|_{\partial\Omega} - \int_\Omega f\frac{dv}{dx}dx = -\int_\Omega f\frac{dv}{dx}dx$$

for $v \in C_0^\infty(\Omega)$. As a result, under the sign of integration, the derivative of f decreases to the derivative of v. We define non-smooth functions' weak derivatives in this way. Before we go into the formal definition of weak derivatives, let's speak about function support.

Definition 1.1.3. Let ϕ be a real(or complex) valued continuous functions on an open set $\Omega \in \mathbb{R}^n$. The support of ϕ is denoted by supp(ϕ) and defined as

$$supp(\phi) = \overline{\{x \in \Omega : \phi(x) \neq 0\}}$$

If this closed set is compact as well, then supp(ϕ) is said to be of compact support.

The set of all infinitely differentiable function on an open set $\Omega \in \mathbb{R}^n$ with compact support is a vector space which will henceforth be denoted by $D(\Omega)$. Therefore, first order weak derivative of f is given by

$$D_f : D(\Omega) \to \mathbb{R} \textit{ such that } D_f(v) = -\int_\Omega f v' dx \ \forall v \in D(\Omega)$$

We can construct the weak derivative for those functions f that are neither differentiable or continuous because the right hand side does not require the derivative of f. The *mth* order weak derivative of f is generally defined as

$$D_f^m(v) = (-1)^m \int_\Omega \frac{d^m v}{dx^m} f dx \ \forall v \in D(\Omega)$$

Definition 1.1.4. For p = 2, the Sobolev space $W^{m,p}(\Omega)$ is a Hilbert space and it is denoted by $H^m(\Omega)$. In particular, $H^1(\Omega) = \{v \in L^2(\Omega) : Dv \in L^2(\Omega)\}$ and the corresponding norm is defined by

$$\|v\|_{H^1(\Omega)} = \|v\|_{L^2(\Omega)} + \|Dv\|_{L^2(\Omega)}$$

Definition 1.1.5. It can be proved that $C_0^\infty(\bar{\Omega})$ is dense in $H^1(\Omega)$. If $f \in C^\infty(\bar{\Omega})$ we define the trace of f, namely γf, by $\gamma f = f|_\Gamma, \Gamma = \partial\Omega$. Further the map $\gamma : C^\infty(\bar{\Omega}) \to L^2(\Gamma)$ is continuous and linear satisfying

$$\|\gamma u\|_{L^2(\Gamma)} \leq C\|u\|_{H^1(\Omega)}$$

Hence this can be extended as continuous linear map from $H^1(\Omega)$ to $L^2(\Gamma)$. This map γ is called trace map.

Definition 1.1.6. The collection of all H^1 functions vanishing on the boundary is a closed subspace of H^1 and it is denoted by

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : \gamma v = 0\}$$

. For $H_0^1(\Omega)$ functions, following important inequality, due to Poincar´e, holds true

$$\int_\Omega |v|^2 dx \leq C \int_\Omega |Dv|^2 dx \ \forall v \in H_0^1(\Omega)$$

We'll wrap up this part by talking about Sobolev quotient space. Let $P_k(\Omega)$ be the set of all polynomials of degree less than or equal to k ≥ 0, defined over Ω . We define

$$\begin{gathered} V = W^{k+1,p}(\Omega)/P_k(\Omega) \\ = \{[v] | [v] = \{v + q | q \in P_k(\Omega)\}, v \in W^{k+1,p}(\Omega)\} \end{gathered}$$

and its corresponding norm

$$\|[v]\|_V = \inf_{p \in P_k(\Omega)} \|v + q\|_{k+1,p,\Omega}$$

Lemma 1.1.7. *For any Lipschitz domain $\Omega \subset \mathbb{R}^d$, there is a constant $C >$ 0, depending only on Ω, such that*

$$\inf_{p\in P_k(\Omega)} \|v+p\|_{k+1,\Omega} \le c|v|_{k+1,\Omega} \ \forall v \in H^{k+1}(\Omega) \tag{1.2}$$

1.2 Variation Formulation of Elliptic BVP

We present a weak formulation for an elliptic boundary value problem in this section. Furthermore, the Lax-Milgram finding is used to investigate the variational problem's well-posedness.

Multiply (1.1) by a smooth function $v \in D(\Omega)$, where $D(\Omega)$ is the set of all infinitely differentiable function on Ω with compact support, and then integrate over Ω to have

$$\int_\Omega -\triangle u.v dx \ = \int fv dx \tag{1.3}$$

Now, we may recall classical Green's theorem

Theorem 1.2.1. ***Green's Theorem.*** *Let Ω be a connected domain in R^2. Then, we have*

$$-\int_\Omega \nabla.(\nabla G) w dxdy \ = \int_\Omega \nabla G.\nabla w dxdy - \oint_\Gamma \frac{\partial G}{\partial \eta}$$

where η is the outward normal to the boundary $\partial\Omega = \Gamma$.

Now, apply Green's theorem and the fact v = 0 on $\partial\Omega$ in (1.3) to have

$$\int_\Omega \nabla u.\nabla v dx \ = \int_\Omega fv dx$$

Then variational problem to (1.1) is defined as: Find $u \in H_0^1(\Omega)$ such that

$$A(u,v) = L(v) \ \forall v \in H_0^1(\Omega) \tag{1.4}$$

Here, A : $H_0^1(\Omega) \times H_0^1(\Omega) \to$ R and L : $H_0^1(\Omega) \to \mathbb{R}$ are defined as

$$A(w, v) = \int_\Omega \nabla u.\nabla v dx \; \forall w, \; v \in H_0^1(\Omega)$$

$$L(v) = \int_\Omega f v dx \; \forall v \in H_0^1(\Omega)$$

The variational issue is approximated using the finite element approach (1.4). As a result, the existence and uniqueness of the variational problem become extremely crucial. In this regard, we must comprehend the Lax-Milgram Lemma, which is a well-known finding. Before we go any further, let's define a few key words.

Definition 1.2.2. Billinear Form: If X and Y are vector spaces, a billinear form A : X × Y → R is defined to be an operator with following properties

$$A(\alpha u + \beta w, v) = \alpha A(u, v) + \beta A(w, v) \; \forall u, w \in X, v \in Y, \alpha, \beta \in \mathbb{R}$$

$$A(u, \alpha v + \beta w) = \alpha A(u, v) + \beta A(u, w) \; \forall u \in X, v, w \in Y$$

Definition 1.2.3. Continuous Billinear Map: Let A : X×Y → ($\mathbb{R}$) be a billinear map, where X and Y are normed linear spaces equipped with norms $\|.\|_X$ and $\|.\|_Y$, respectively. Then A(., .) is said to be continuous/bounded if there is a positive number C such that

$$|A(u, v)| \leq C\|u\|_X \|v\|_Y \; \forall u \in X, v \in Y$$

Definition 1.2.4. H-Elliptic Billinear Map/ Positive Billinear Map:

Given a billinear form A : H × H$\mathbb{R}$, where H is an inner product space, we say that A(., .) is H-elliptic/positive if there exist a constant C > 0 such that

$$A(v, v) \geq C\|v\|_H^2 \; \forall v \in H$$

where $\|.\|$ is the norm associated with inner product.

We can now move to discussing the key finding in this chapter, which offers conditions that are necessary for the existence and uniqueness of the variational problem, now that we have presented all of the terminology.

Theorem 1.2.5. *Let H be a Hilbert space and let $A : H \times H \to \mathbb{R}$ be a continuous, positive billinear map defined on $H \times H$. Then, for a given continuous linear functional L on H, there exist a unique element u in H such that*

$$A(u, v) = L(v) \ \forall v \in H$$

1.3 Regularity Result

Depending on the smoothness of the available data, this section describes the existence, uniqueness, and smoothness of the issue (1.4) solutions. Consider the following equation for clarity.

$$\frac{dy}{dx} = f, x \in (a, b)$$

If $f \in L^2(a, b)$ then $\frac{dy}{dx} \in L^2(a, b)$ so y $\in H^1(a, b)$. Similarly, for $f \in H^1(a, b)$, $y' \in H^1(a, b)$ and $y'' \in L^2(a, b)$ so that y $\in H^2(a, b)$. Any result which describe the smoothness of solution depending on the smoothness of given data is called regularity result.

Theorem 1.3.1. *For $f \in L^2(\Omega)$, the problem (1.4) has unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying following a priori estimate*

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

1.4 Embedding Result

The following result connects Sobolev space and classical function spaces, and thus concludes this chapter. For our convenience, we will now state the Sobolev embedding theorem and Rellich's theorem. It's reasonable to wonder if members of $H^m(\Omega)$ are just functions that are continuous as well as their derivatives of order $\leq$ m $-$ 1. After all, it's not easy to picture a non-continuous function in $H^1(\Omega)$, for example. Sobolev's famous theorem states that, as expected, all members of $H^1(a,b)$ are continuous functions, but that this is not true for higher dimensional domains. Such results are called Embedding results.

Let X, Y are Banach Spaces with X $\subseteq$ Y . We say that X is continuously embedded in Y i.e. X $\to$ Y if the identity map I : X $\to$ Y is continuous that is

$$\|I(x)\|_Y \leq C\|x\|_X \ for\ some\ constant\ C > 0$$

For Sobolev spaces to be embedded in the space of continuous functions certain conditions need to be met. The following theorem describes those conditions.

Theorem 1.4.1. The Sobolev Embedding Theorem. *Let Ω be a bounded domain in R^n with Lipschitz boundary Γ. If $m - k > n/2$, then every function in $H^m(\Omega)$ belongs to $C^k(\bar{\Omega})$. Furthermore, the embedding*

$$H^m(\Omega) \subset C^k(\bar{\Omega}) \tag{1.5}$$

is continuous.

The aforementioned finding must be interpreted with caution. Remember that $H^m(\Omega)$ members are function equivalence classes since they are L^2 members, whereas continuous functions are clearly specified. As a result, the embedding (1.5) must be interpreted in the sense that each member of $H^m(\Omega)$ can be linked to a $C^k(\bar{\Omega})$ function, possibly after changing the function's values on a set of measure zero.

According to the Sobolev Embedding Theorem, the functions in $H^1(\Omega)$ are continuous if n = 1 and Ω is a subset of the real line. However, we require that a function be a member of $H^2(\Omega)$ in order to assure the continuity of functions with domains that are subsets of the plane.

Chapter 2

Computational Aspects of FEM

The discretization of the domain into elements is discussed in this chapter, followed by the construction of a finite element space. Finally, we look at evaluating the model using reference elements on a simplified domain.

2.1 Finite Element Discretization

The first step towards FEM is to discretize the computational domain $\Omega \in R^2$ into finitely many elements.

Definition 2.1.1. Triangulation A triangulation T_h is a partition of $\bar{\Omega}$ into a finite number of triangular subsets K for splitting the domain into discrete pieces, such that:

1. $\bar{\Omega} = \cup_{K \in T_h} K$
2. $K = \overline{K}$ and $int(K) \neq \phi$ where $K \in T_h$
3. If $K_1, K_2 \in T_h$ and $K_1 \neq K_2$ then either $K_1 \cap K_2 = \phi$ or $K_1 \cap K_2$ is a common vertex or edge of K_1 and K_2. This implies that no vertex of any triangle can lie on the interior of an edge of another triangle.

4. For any triangle $K \in T_h$, let $r_k, \overline{r_k}$ be the radii of the inscribed and the circumscribed circles of K respectively. Let $h = max\{\overline{r_k} : K \in T_h\}$. We assume that for some fixed $h_0 > 0$, there exist two positive constants C_0 and C_1 such that

$$C_0 h \leq diam(K) \leq C_1 h \quad \forall K \in T_h, \ \forall h \in (0, h_0)$$

We construct our finite element space V_h, which is finite dimensional and piece-wise polynomial, based on the triangulation T_h. Over the elements K, we create polynomial spaces.

Definition 2.1.2. The triple $(K,\ P_r,\ \Sigma_K)$ is called a finite elements. Σ_K represents the set of conditions such that unique determination of any $p \in P_r$ is possible.

Lagrange Finite Elements:

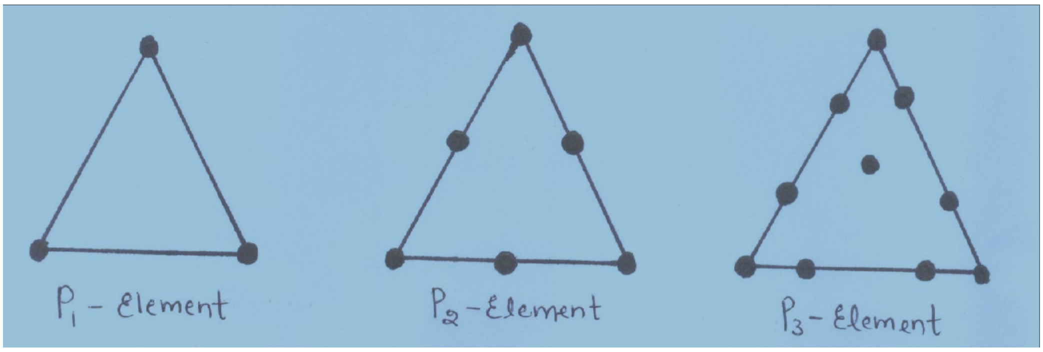

Figure 2.1: Different Lagrange Elements

For given P_r we consider polynomial space of degree $\leq r$. In $\mathbb{R}^2$,

$$dim(P_r) = \frac{(r+1)(r+2)}{2} = n_r$$

.

We will explore linear P_1 and quadratic P_2 elements as examples of lagrange elements, as well as the building of finite element space for the elements. Let us begin by introducing the nodal basis function.

Definition 2.1.3. Nodal Basis Function: Firstly, let $N_h(\bar{\Omega}) = \{x_i : 1 \leq i \leq n_h\}$ denote set of all nodes generated by triangulation. Then, ϕ_i $\forall(1 \leq i \leq n_h)$ such that it satisfies $\phi_i(x_j) = \delta_{ij}$ is called the nodal basis function.

Construction of (K, P_1, Σ_K)

This utilizes area of triangles as the factor for the basis functions. Consider, element K is a triangle with vertices (nodes) $a_1 = A$, $a_2 = B$, $a_3 = C$. Further assume, P to be arbitrarily located in K. Now, corresponding to $a_1 = A$, let us define:

$$\phi_1(P) = \frac{Area(\Delta PBC)}{Area(\Delta ABC)}$$

The same can be done corresponding to points $a_2 = B$ and $a_3 = C$:

$$\phi_2(P) = \frac{Area(\Delta PCA)}{Area(\Delta ABC)}$$

$$\phi_3(P) = \frac{Area(\Delta PAB)}{Area(\Delta ABC)}$$

We can observe that $\phi_1(x) + \phi_2(x) + \phi_3(x) = 1$.

More importantly, for the P_1 element and $N_h(K) = \{a_1, a_2, a_3\}$, we have ϕ_i as the nodal basis function, since $\phi_i(a_j) = \delta_{ij}$.

Construction of (K, P_2, Σ_K)

The basis functions calculated for the P_1 element can be extended as below. We have, $N_h = \{a_1, a_2, a_3, a_{12}, a_{23}, a_{31}\}$. The basis function ϕ_i corresponding to node a_i is defined as:

$$\phi_i(x) = \lambda_i(x)(2\lambda_i(x) - 1) \qquad \forall\, 1 \leq i \leq 3$$

The basis function corresponding to a_{ij} can be defined as below:

$$\phi_{ij}(x) = 4\lambda_i(x)\lambda_j(x) \qquad \forall\, 1 \leq i, j \leq 3$$

Here, λ_i represents the nodal basis function for the P_1 element.

Now, the finite element space is defined as

$$V_h = \{v_h : \bar{\Omega} \to \mathbb{R} : v_{h|K} \in P_r(K), K \in T_h\}.$$

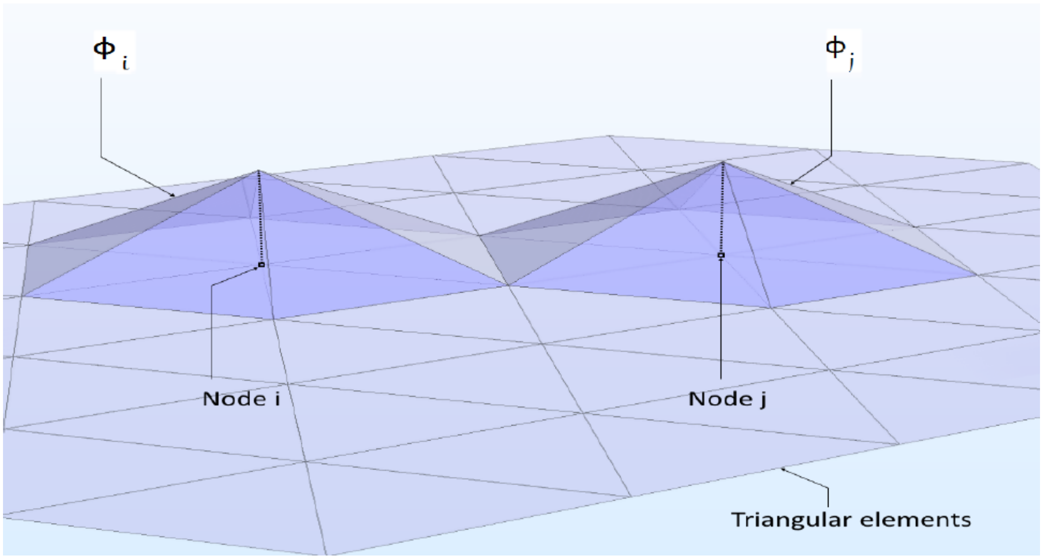

Figure 2.2: Nodal Basis Function for Lagrange Element

Based on the finite element space V_h, two categories of FEM can be defined. If $V_h \in H$ it is known as conforming FEM, whereas if $V_h \notin H$ it is known as non-conforming FEM. Here, H represents the solution space for the given BVP. This project only deals with conforming FEM.

2.2 Reference Element

Using the weak formulation of the BVP and the finite elements we generated above, we can proceed to find the solution $u_h \in V_h$ such that

$$A(u_h, v_h) = L(v_h) \quad \forall\ v_h \in V_h$$

Many calculations need to be done separately for each triangular element K. To avoid repeating similar calculations for different elements and to make the algorithm more efficient, the Reference Element Technique is used.

Definition 2.2.1. Reference Triangle: Any arbitrary triangle $K \in \mathbb{R}^2$ can be represented as an image of the unit triangle $\hat{K}$ under a map:

$$F_k : \mathbb{R}^2 \to \mathbb{R}^2 \ i.e.\ \hat{x} \to F_K(\hat{x}) = B_K(\hat{x}) + b_K$$

where $B_K \in \mathbb{R}^2 \times \mathbb{R}^2$ represents a non-singular matrix and $b_K \in \mathbb{R}^2$. The unit triangle $\hat{K}$ is referred to as the reference triangle.

Let $\hat{K}$ be the triangle ΔABC, where $a_1 = (0,0) = A,\ a_2 = (1,0) = B,\ a_3 = (0,1) = C$. The nodal basis functions over $\hat{K}$ can be given by:

$$\phi_1(x,y) = 1 - x - y,\ \phi_2(x,y) = x,\ \phi_3(x,y) = y$$

Suppose $K \in T_h$ is element with vertices $P_1^K = (x_1, y_1),\ P_2^K = (x_2, y_2),\ P_3^K = (x_3, y_3)$. Here, we can define $F_K(\hat{X}) = P_1^K \phi_1(\hat{X}) + P_2^K \phi_2(\hat{X}) + P_3^K \phi_3(\hat{X})$.

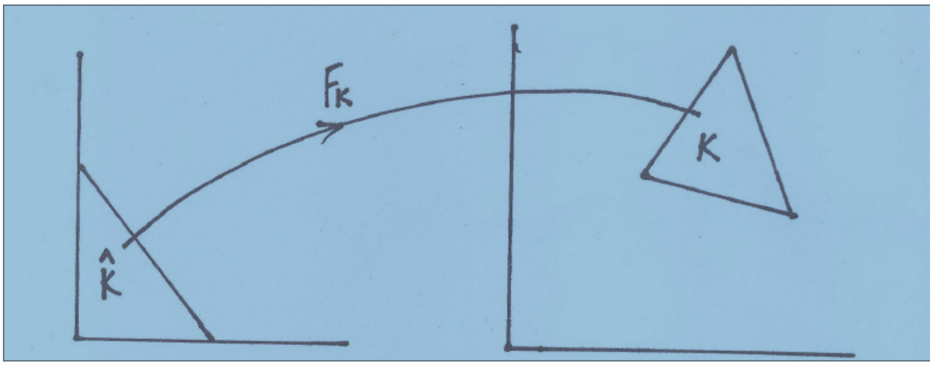

Figure 2.3: Reference Element

We obtain that the nodes of the reference triangle $\hat{K}$ map to the nodes of our given triangle K. Also, the map F_K is well defined so that each $\hat{X} \in \hat{K}$ maps to a unique $X \in K$. We obtain

$$F_K(\hat{X}) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Here, $B_K = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}$ which we can verify easily to be non-singular. Also, $b_K = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$.

We get $\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = B_K^{-1} \begin{bmatrix} x - x_1 \\ y - y_1 \end{bmatrix}$. So we conclude that F_K^{-1} exists.

Further let $N_h(K) = \{x_1, x_2, x_3, \cdots, x_{nr}\}$ denote the set of nodes for K, and let ϕ_i denote the nodal basis functions. We can now observe,

$$x_i = F_K(\hat{x}_i)$$

$$\phi_i(x_j) \;=\; \delta_{ij} \;=\; \hat{\phi}_i(\hat{x}_j) \;=\; \hat{\phi}_i \cdot F_K^{-1}(x_j)$$

$$\therefore\;\phi_i = \hat{\phi}_i \cdot F_K^{-1}$$

Making use of the reference element technique, computation becomes much simpler since all the computations need to be made only on the reference element, which can then be transformed to the required element.

Chapter 3

Two Dimensional Mesh Generation and Future Work

The first step in FEM is generating a mesh of elements to work upon. We first look at uniform triangulation of a rectangular domain. Then, we shall discuss other mesh types and their implementation.

3.1 Triangulation

The following algorithm generates triangular elements of a given rectangular domain with given step size. After obtaining the array of nodes and the list of elements, we can obtain more information regarding the elements including all the edges, the boundary nodes and elements, and neighbouring elements. Using all of this information, we can proceed to the finite element approximation.

Algorithm 1 Triangulation of Rectangular Domain

Require: $[x_0, x_1, y_0, y_1]$ denoting the domain and *step size* denoting the step size

Ensure: node contains array of all nodes. elem contains list of all elements.

$square \leftarrow [x_0, x_1, y_0, y_1]$
$h \leftarrow step\ size$
▷ Creating Array of Nodes
$x_0 \leftarrow square(1), x_1 \leftarrow square(2), y_0 \leftarrow square(3), y_1 \leftarrow square(4)$
$h_x \leftarrow h(1), h_y \leftarrow h(2)$
Generate meshgrid from $[x_0 : h_x : x_1] \times [y_0 : h_y : y_1]$
node $\leftarrow$ array of vertices from meshgrid
▷ Listing Elements
$ni \leftarrow$ number of rows in mesh
$k \leftarrow 1$
while $k \leq$ Number of nodes $- ni$ **do**
  $elem \leftarrow [elem, (k, k + ni, k + ni + 1)]$
  $elem \leftarrow [elem, (k, k + 1, k + ni + 1)]$
  $k \leftarrow k + 1$
end while

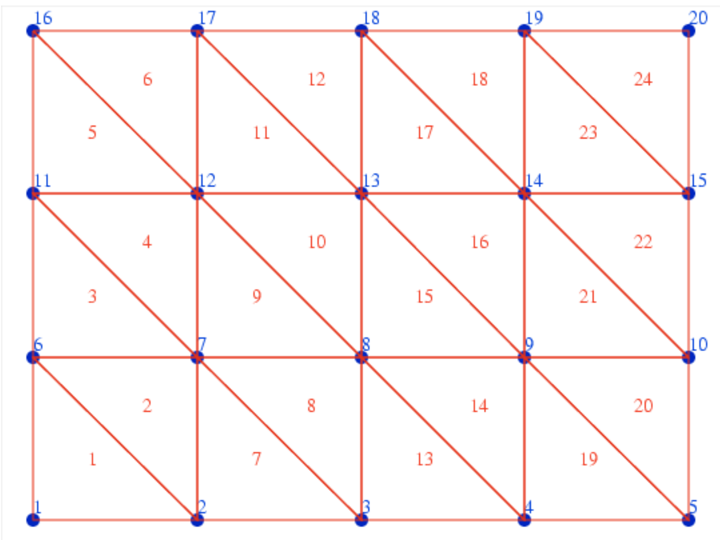

Figure 3.1: Creation of nodes and elements in a Rectangular Domain

3.2 Future Work

Triangulation on a rectangular domain is a very specific case of mesh generation. The domain can take any arbitrary shape on a two dimensional plane. We will consider that next. Further, we will look at mesh generation in a three dimensional domain. There too, we shall start with a general case of cuboidal domain and then consider an arbitrary three dimensional domain.

Finally, we shall work towards finite element approximation of a model boundary value problem.

Bibliography

[1] B. Daya Reddy. *Introductory Functional Analysis.* Springer Science+Business Media, LLC), 1998.

[2] FJ Sayas. *A Gentle Introduction to the Finite Element Method.* 2008.