WEAK GALERKIN FINITE ELEMENT SOLVER WITH POLYGONAL MESHES

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Outline

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Objective

- To understand finite element approximation and its computational techniques.
- To implement 3D mesh generation and develop an efficient FEM solver.

Outline

Abstract

- In mathematics and various associated fields Finite Element Method is used to numerically approximate the solutions to partial differential equations which are not possible to solve analytically. In this project we try to develop an efficient solver for Finite Element approximations using polygonal meshes. We start with an introduction to the method, followed by its computational techniques and then proceed to look at the various aspects of an FEM solver.

Outline

$L^p(\Omega)$ Space and Norm

- For a function f, the norm $\|f_{L^p(\Omega)}\|$ is defined as:

$$\|f_{L^p(\Omega)}\| = \left(\int_{\Omega} |f(x)|^p dx\right)^{\frac{1}{p}}$$

- We can now define space of functions f,

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : \left(\int_{\Omega} |f(x)|^p dx\right)^{\frac{1}{p}} < \infty \right\}$$

L^2 Space

- By definition, L^2 space represents square-integrable functions.
- Square-integrable functions need not be continuous. Example,

$$f(x) = \begin{cases} x, & x \in (0,1/2) \\ -x, & x \in (1/2,1) \end{cases}$$

- Continuous functions need not be square-integrable. $f(x) = \dfrac{1}{x}$
- $f \in L^2$ might not have classical derivative.

Weak Derivatives

- To begin with, let f be a differentiable function and $v \in C_0^\infty(\Omega)$
- $\int_\Omega \frac{df}{dx} v dx = fv|_{\partial\Omega} - \int_\Omega f \frac{dv}{dx} dx = -\int_\Omega f \frac{dv}{dx} dx$
- Weak derivative of f is defined as function g which satisfies,

$$\int_\Omega g v dx = -\int_\Omega f \frac{dv}{dx} dx$$

- This can be extended to define weak derivative for all functions.
- In general, $\int_\Omega f^{(m)} v dx = (-1)^m \int_\Omega f v^{(m)} dx$

Sobolev Space

- Let $D^\alpha u = \dfrac{\partial^\alpha u}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2}$, $\alpha = \alpha_1 + \alpha_2$ denote the weak derivative of u at the α^{th} order.
- $W^{m,p} = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), 0 \leq \alpha \leq m\}$ represents the Sobolev space $W^{m,p}$.
- Sobolev space $W^{m,p}$ includes all functions $u \in L^p(\Omega)$ such that all the weak derivatives of u upto order m belong to $L^p(\Omega)$.
- $\|v\|_{m,p,\Omega} = \sum_{0 \leq \alpha \leq m} \|D^\alpha v\|_{L^2(\Omega)}$
- $|v|_{m,p,\Omega} = \sum_{\alpha = m} \|D^\alpha v\|_{L^2(\Omega)}$
- For $p = 2$, the Sobolev space $W^{m,p}(\Omega)$ is a Hilbert space and it is denoted by $H^m(\Omega)$

Outline

Elliptic BVP

Consider the elliptic boundary value problem as follows

- $-\Delta u = f$ in Ω
- $\displaystyle\int_\Omega -\triangle u.v dx \;=\int f v dx$ where $v \in D(\Omega)$
- $\displaystyle\int_\Omega \nabla u.\nabla v dx \;=\int_\Omega f v dx$ using Green's Theorem and taking $v = 0$ on $\partial\Omega$
- The elliptic BVP reduces to the variational problem: Find $u \in H_0^1(\Omega)$ such that $A(u, v) = L(v) \quad \forall v \in H_0^1(\Omega)$

Existence and Uniqueness

Let H be a Hilbert space and let $A : H \times H \to \mathbb{R}$ be a **continuous, positive bilinear map** defined on $H \times H$. Then, for a given continuous linear functional L on H, there exist a unique element $u \in H$ such that

$$A(u, v) = L(v)\ \forall v \in H$$

Bilinear Form

If X and Y are vector spaces, a bilinear form $A : X \times Y \to \mathbb{R}$ is defined to be an operator with following properties

$$A(\alpha u + \beta w, v) = \alpha A(u, v) + \beta A(w, v) \ \forall u, w \in X, v \in Y, \alpha, \beta \in \mathbb{R}$$

$$A(u, \alpha v + \beta w) = \alpha A(u, v) + \beta A(u, w) \ \forall u \in X, v, w \in Y, \alpha, \beta \in \mathbb{R}$$

Continuous Bilinear Map

Let $A : X \times Y \to \mathbb{R}$ be a bilinear map, where X and Y are normed linear spaces equipped with norms $\|.\|_X$ and $\|.\|_Y$, respectively. Then $A(.,.)$ is said to be continuous if there is a positive number C such that

$$|A(u,v)| \leq C\|u\|_X\|v\|_Y \; \forall u \in X, v \in Y$$

Positive Bilinear Map

Given a billinear form $A : H \times H \to \mathbb{R}$, where H is an inner product space, we say that $A(.,.)$ is positive if there exist a constant $C > 0$ such that

$$A(v,v) \geq C\|v\|_H^2 \ \forall v \in H$$

Outline

Mesh Generation

- Dividing the domain into finitely many sub-domains
- Each sub-domain is called an element.
- Set of all elements is called a mesh.

Triangulation

A triangulation T_h is a partition of $\bar{\Omega}$ into a finite number of triangular subsets K for splitting the domain into discrete pieces, such that:

1. $\bar{\Omega} = \cup_{K \in T_h} K$
2. $K = \overline{K}$ and $int(K) \neq \phi$
3. If $K_1, K_2 \in T_h$ and $K_1 \neq K_2$ then either $K_1 \cap K_2 = \phi$ or $K_1 \cap K_2$ is a common vertex or edge of K_1 and K_2.
4. Let $r_k, \overline{r_k}$ be the radii of the inscribed and the circumscribed circles of K.
   Let $h = max\{\overline{r_k} : K \in T_h\}$.
   For some fixed $h_0 > 0$, there exist two positive constants C_0 and C_1 such that

$$C_0 h \leq diam(K) \leq C_1 h \quad \forall K \in T_h, \; \forall h \in (0, h_0)$$

Lagrange Elements

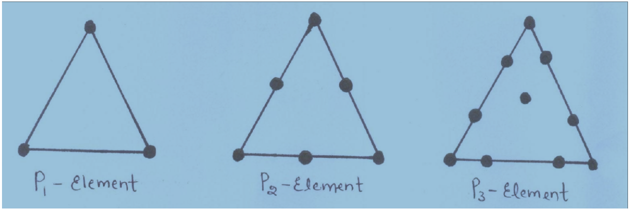

Figure: Lagrange Finite Elements

Nodal Basis Function Let $N_h(\bar{\Omega}) = \{x_i \ : \ 1 \leq i \leq n_h\}$ denote set of all nodes generated by the triangulation.
$\phi_i \ \forall \ 1 \leq i \leq n_h$ such that it satisfies $\phi_i(x_j) = \delta_{ij}$ is called the nodal basis function.

Lagrange Elements

- P_1 Element K is a triangle with vertices (nodes) $a_1 = A$, $a_2 = B$, $a_3 = C$
- P is arbitrarily located in $\triangle ABC$
- $$\phi_1(P) = \frac{Area(\Delta PBC)}{Area(\Delta ABC)}$$
  $$\phi_2(P) = \frac{Area(\Delta PCA)}{Area(\Delta ABC)}$$
  $$\phi_3(P) = \frac{Area(\Delta PAB)}{Area(\Delta ABC)}$$
- u_h in the element can therefore be represented as linear combination of ϕ_1, ϕ_2, ϕ_3.

Reference Element

- Many similar calculations have to be done for all the elements
- To reduce redundancy, reference element technique is used
- Each triangular sub-domain is mapped to the unit triangle
- Each point in reference element maps to a unique point in required element
- Each node of reference element maps to a node of the required element
- Calculations are carried out only on the reference element
  Results for other elements are obtained via the mapping

Reference Element

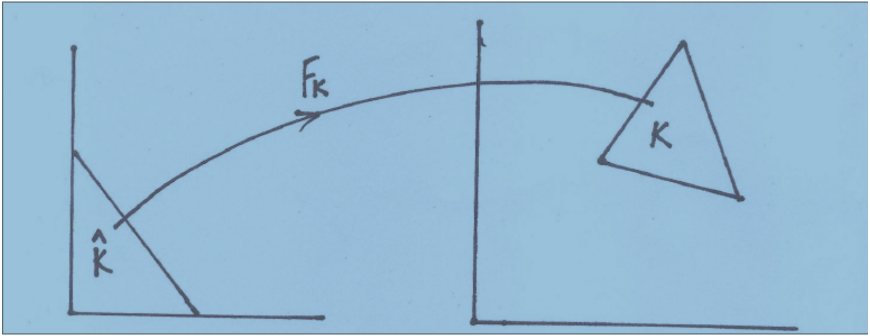

Figure: Reference Element Mapping

2D Triangulation on Rectangular Domain

Algorithm 1 Triangulation of Rectangular Domain

Require: $[x_0, x_1, y_0, y_1]$ denoting the domain and *step size* denoting the step size

Ensure: node contains array of all nodes. elem contains list of all elements.

$square \leftarrow [x_0, x_1, y_0, y_1]$
$h \leftarrow step\ size$

▷ Creating Array of Nodes

$x_0 \leftarrow square(1), x_1 \leftarrow square(2), y_0 \leftarrow square(3), y_1 \leftarrow square(4)$
$h_x \leftarrow h(1), h_y \leftarrow h(2)$
Generate meshgrid from $[x_0 : h_x : x_1] \times [y_0 : h_y : y_1]$
node $\leftarrow$ array of vertices from meshgrid

▷ Listing Elements

$ni \leftarrow$ number of rows in mesh
$k \leftarrow 1$
while $k \leq$ Number of nodes $- ni$ **do**
    $elem \leftarrow [elem, (k, k + ni, k + ni + 1)]$
    $elem \leftarrow [elem, (k, k + 1, k + ni + 1)]$
    $k \leftarrow k + 1$
end while

Outline

Future Work

- Understanding two dimensional mesh generation, labelling of elements and finding edges, neighbors, boundary nodes and elements.
- Implementing mesh generation on a three dimensional domain.
- Integrating mesh generation to obtain FEM solver for a three dimensional domain.