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Sol^n (3): using FTCS:

$$w_{i,v+1} = \lambda w_{i-1,v} + (1-2\lambda) w_{i,v} + \lambda w_{i+1,v}$$

$$\lambda = \frac{\Delta c}{\Delta x^2}$$

So coeff matrix =

$$A = \begin{bmatrix} 1-2\lambda & \lambda & 0 & \cdots & 0 \\ \lambda & 1-2\lambda & \lambda & \cdots & 0 \\ 0 & \vdots & & & \\ 0 & \cdots & 0 & \lambda & 1-2\lambda \end{bmatrix}$$

& $w^{v+1} = A w^v$ for $v = 0, 1, 2, \ldots, v_{max}-1$

Error analysis: $w^{v+1} = A w^v + d^v$

$w^{(v)}$ exact vectors of $w^{v+1} = A w^v + d^v$

$w^v \rightarrow$ subject to rounding errors

Lesser errors $e^v = \bar{w}^{(v)} - w^v$

and result is

$$\bar{w}^{(v+1)} = A\bar{w}^v + \cancel{\text{ }} r^{v+1}$$

↳ rounding err.

Effect of initial rounding error $e^{(0)}$ have on the iteration is (let $r^v = 0$ for $v > 1$)

$$Ae^v = A\bar{w}^v - Aw^v = \bar{w}^{(v+1)} - w^{v+1} = e^{v+1}$$

$$\Rightarrow Ae^v = e^{v+1} \Rightarrow \boxed{e^v = A^v e^{(0)}}$$

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for stability we require $A^{v} e^{(0)} \to 0$ for $v \to \infty$

$\lim_{v\to\infty} A^{v} e^{(0)} = 0$

which is true iff spectral radius of $A < 1$

$$\rho(A) = \max|\mu_i| < 1$$

where $\mu_1 \dots \mu_{m-1} \to$ eigen values of A

for $G = \begin{bmatrix} \alpha & \beta & \cdots & 0 \\ \gamma & \cdots & \cdots & \cdots \\ \vdots & \cdots & \cdots & \beta \\ \vdots & \cdots & \gamma & \alpha \end{bmatrix} \in R^{N\times N}$

the eigen values are $\mu_k^{G} = \alpha + 2\beta\sqrt{\gamma/\beta}\cos\left(\frac{N\pi}{N+1}\right)$

$\therefore$ $A = 1 - \lambda \begin{bmatrix} 2 & -1 & \cdots & 0 \\ -1 & 2 & -1 & -1 \\ \cdots & \cdots & -1 & 2 \end{bmatrix}$

$= I - \lambda G$.

$\mu_k^{A} = 1 - \mu_k^{G}$ $\quad \alpha = 2, \beta = \gamma = -1, N = m-1$

$\mu_k^{G} = 2 - 2\cos\left(\frac{k\pi}{m}\right) = 4\sin^2\left(\frac{k\pi}{2m}\right)$

$\mu_k^{A} = \left|1 - 4\lambda\sin^2\left(\frac{k\pi}{2m}\right)\right|$.

$\Rightarrow$ for stability $\mu_k^{A} < 1 \Rightarrow \left|1 - 4\lambda\sin^2\left(\frac{k\pi}{2m}\right)\right| < 1$.

$k = 1, 2, \dots m-1$.

$\Rightarrow \lambda > 0$ and $-1 < 1 - 4\lambda\sin^2\left(\frac{k\pi}{2m}\right)$.

$\Rightarrow \lambda\sin^2\left(\frac{k\pi}{2m}\right) < 1/2. \Rightarrow \boxed{\lambda \le \frac{1}{2}}$

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Hence $0 < \lambda \leq 1/2$ is sufficient.

Hence for above λ, the explicit method FTCS

$w^{v+1} = w^v$ is stable. $\lambda = \frac{\Delta t}{\Delta x^2}$.

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Q.4): To show:
Cryer problem:

for $\hat{b} = b - Ay$
$Ax - y = \hat{b}$ $\quad y \geq 0, \; x^T y = 0$

is equivalent to

$\left(\min_{x \geq 0} G(x) \right)$ $\quad G(x) = \frac{1}{2}(x^T A x) - b^T x$.

is strictly convex.

Sol^n (4)

↳ Since the derivatives of G are $G_x = Ax - \hat{b}$ and $G_{xx} = A$. we can use Lemma and can see that A has the eigen values. Hence the hessian matrix G_{xx} is symmetric and positive definite. So G is strictly convex. and has a unique minimum on each convex set in R^n. for $x \geq 0$.

The Kuhn and Tucker theorem minimizes $G(x)$ under $H_i(x) \leq 0$ $\; i = 1, \ldots m$.
Acc to the theorem, a vector x_0 to be a minimum is equivalent to existence of a langrange multiplier $y \geq 0$ with.

$$\text{grad } G(x_0) + \left(\frac{\partial H(x_0)}{\partial x} \right)^T y = 0 \qquad y^T H(x_0) = 0$$

The set $\{x \geq 0\}$ leads to define $H(x) = -x$.
Hence putting this in above eq^n leads to cond^n $Ax - \hat{b} + (-I)^T y = 0$, $y^T x = 0$
which is the initial cryer problem.

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Q5): early exercise curve $(S_f(t))$ $(t<T)$

$$S_f(t) < \lim_{t \to T} S_f(t) = \min(K, r/\delta K)$$

for $t<T$: the value of $V_p^{American}$ = payoff

sol^n (5) $\Rightarrow V_p^{Amer.}(S,T) = K-S$ for $S<K$.

Put this in Black Scholes eq^n:

$$\frac{\partial V(S,T)}{\partial t} + 0 - (r-\delta) S - rV = 0 \Rightarrow \frac{\partial V}{\partial t} = r(S+V) - \delta S.$$

$$\Rightarrow \frac{\partial V(S,T)}{\partial t} = (rK - \delta S).$$

Claim $\Rightarrow rK - \delta S \leq 0 \Rightarrow \frac{\partial V(S,T)}{\partial t} \leq 0$ — (1)

otherwise for $t \to T \Rightarrow V \geq$ payoff which is not possible. Hence for $t = T$ and $S<K$.

$$rK - \delta S \leq 0 \quad , \Rightarrow \boxed{S \geq r/\delta K}$$

~~Hence~~, we can also deduce that $\delta > r$ otherwise $S > K$ which is not good.

↳ There could exist multiple possibilities like either

$S_f(T) = \lim_{t \to T} S_f(t)$ $(t<T)$ satisfies $S_f(T) = r/\delta K$.

or other 2 cases possible maybe
(i) $S_f(T) < r/\delta K$ (ii) $S_f(T) > r/\delta K$.

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(i) There is S such that $S_f(T) < S < \frac{\gamma}{\delta} K$.

Then $\frac{\partial v(s,T)}{\partial t} = rK - \delta S > 0$

but claim (1) this is not possible.

(ii) There is S such that $\gamma/_\delta K < S < S_f(T)$.

Then $rK < \delta S$ and $K(e^{rdt} - 1) < S(e^{\delta dt} - 1)$ which implies that dividend earns more than the interest on K, early exercise ~~should be~~ shouldn't be executed. Thus it contradicts the fact that $S < S_f(T)$.

Last case $\rightarrow$ $\delta \leq r$.

$\therefore$ $S_f(T) > K$ is not possible. $\Rightarrow$

Assume $S_f(T) < K$ then for $S_f(T) < S < K$ and $t \approx T$.

$\underbrace{\frac{dv}{dt}}_{\text{has to be } \leq 0} = rK - S\delta$ but $rK - S\delta > 0$

which can't be possible as LHS ≤ 0 and RHS > 0.

Hence, $S_f(T) = K$ for $\delta \leq r$.

Hence we can ~~conclude~~ conclude from above discussion that:

$S_f(t) K \lim_{t \to T} S_f(t) = \min(K, \gamma/_\delta K)$.

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