Chapter 6 Pricing of Exotic Options

In Chapter 4 we discussed the pricing of vanilla options (standard options) by means of finite differences. The methods were based on the simple partial differential equation (4.2),

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2} \, ,$$

which was obtained from the Black–Scholes equation (4.1) for $V(S,t)$ via the transformations (4.3). These transformations could be applied because $\frac{\partial V}{\partial t}$ in the Black–Scholes equation is a linear combination of terms of the type

$$c_j S^j \frac{\partial^j V}{\partial S^j}$$

with constants c_j, $j = 0, 1, 2$.

Exotic options lead to partial differential equations that are not of the simple structure of the basic Black–Scholes equation (4.1). In the general case, the transformations (4.3) are no longer useful and the PDEs must be solved directly. Thereby numerical instabilities or spurious solutions may occur, which do not play any role for the methods of Chapter 4. To cope with the "new" difficulties, Chapter 6 introduces ideas and tools not needed in Chapter 4. Exotic options often involve higher-dimensional problems. This significantly adds to the complexity. The aim of this chapter will not be to formulate algorithms, but to give an outlook and lead the reader to the edge of several aspects of recent research. Some of the many possible methods will be exemplified on Asian options.

Sections 6.1 and 6.2 give a brief overview on important types of exotic options. An exhaustive discussion of the wide field of exotic options is far beyond the scope of this book. Section 6.3 introduces approaches for path-dependent options, with the focus on Asian options. Then numerical aspects of convection-diffusion problems are discussed (in Section 6.4), and upwind schemes are analyzed (in Section 6.5). After these preparations the Section 6.6 arrives at a state of the art high-resolution method.

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6.1 Exotic Options

So far, this book has mainly concentrated on standard options. These are the American or European call or put options with payoff functions (1.1C) or (1.1P) as discussed in Section 1.1, based on a single underlying asset. The options traded on official exchanges are mainly standard options; there are market prices quoted in relevant newspapers.

All nonstandard options are called exotic options. That is, at least one of the features of a standard option is violated. One of the main possible differences between standard and exotic options lies in the payoff; examples are given in this section. Another extension from standard to exotic is an increase in the dimension, from single-factor to multifactor options; this will be discussed in Section 6.2. The distinctions between put und call, and between European and American options remain valid for exotic options. Financial institutions have been imaginative in designing exotic options to meet the needs of clients. Many of the products have a highly complex structure. Exotic options are traded outside the exchanges (OTC), and often there are no market prices. Exotic options must be priced based on models. In general, their parameters are taken from the results obtained when standard options with comparable terms are calibrated to market prices. The simplest models extend the Black–Merton–Scholes model summarized by Assumption 1.2.

Next we list a selection of some important types of exotic options. For more explanation we refer to [Hull00], [Wi98].

Compound Option: Compound options are options on options. Depending on whether the options are put or call, there are four main types of compound options. For example, the option may be a call on a call.

Chooser Option: After a specified period of time the holder of a chooser option can choose whether the option is a call or a put. The value of a chooser option at this time is

$$\max\{V_C, V_P\}$$

Binary Option: Binary options have a discontinuous payoff, for example

$$V_T = \Psi(S_T) := C \cdot \begin{cases} 1 & \text{if } S_T < K \\ 0 & \text{if } S_T \geq K \end{cases}$$

for a fixed amount C. See Section 3.5.5 for a two-dimensional example.

Path-Dependent Options

Options where the payoff depends not only on S_T but also on the path of S_t for previous times $t < T$ are called *path dependent*. Important path-dependent options are the *barrier option*, the *lookback option*, and the *Asian option*.

Barrier Option: For a barrier option the payoff is contingent on the underlying asset's price S_t reaching a certain threshold value B, which is called barrier.

Barrier options can be classified depending on whether S_t reaches B from above *(down)* or from below *(up)*. Another feature of a barrier option is whether it ceases to exist when B is reached *(knock out)* or conversely comes into existence *(knock in)*. Obviously, for a down option, $S_0 > B$ and for an up option $S_0 < B$. Depending on whether the barrier option is a put or a call, a number of different types are possible. For example, the payoff of a European *down-and-out* call is

$$V_T = \begin{cases} (S_T - K)^+ & \text{in case } S_t > B \text{ for all } t \\ 0 & \text{in case } S_t \leq B \text{ for some } t \end{cases}$$

In the Black–Merton–Scholes framework, the value of the option before the barrier has been triggered still satisfies the Black–Scholes equation. The details of the barrier feature come in through the specification of the boundary conditions, see [Wi98].

Lookback Option: The payoff of a lookback option depends on the maximum or minimum value the asset price S_t reaches during the life of the option. For example, the payoff of a lookback option is

$$\max_t S_t - S_T \ .$$

Average Option/Asian Option: The payoff from an Asian option depends on the average price of the underlying asset. This will be discussed in more detail in Section 6.3.

The exotic options of the above short list gain complexity when they are multifactor options.

Pricing of Exotic Options

Several types of exotic options can be reduced to the Black–Scholes equation. In these cases the methods of Chapter 4 are adequate. For a number of options of the European type the Black–Scholes evaluation formula (A4.10) can be applied. For related reductions of exotic options we refer to [Hull00], [WDH96], [Kwok98]. Approximations are possible with binomial methods or with Monte Carlo simulation. The Algorithm 3.6 applies, only the calculation of the payoff (step 2) must be adapted to the exotic option.

6.2 Options Depending on Several Assets

The options listed in Section 6.1 depend on one underlying asset. Options depending on several assets are discussed next. Two large groups of multifactor options are the *rainbow options* and the *baskets*. The subdivision into the groups is by their payoff. Assume n assets are underlying, with prices $S_1, \ldots, S_n$. Different from the notation in previous chapters, the index refers to the number of the asset. Recall that two examples of exotic options with two underlyings occurred earlier in this text: Example 3.8 of a binary put, and Section 5.4 with a basket-barrier call.

Rainbow options compare the value of individual assets [Smi97]. Examples of payoffs are

$\max(S_1, \ldots, S_n)$	"n-color better-of option"
$\min(S_1, S_2)$	"two-color worse-of option"
$\max(S_2 - S_1, 0)$	"outperformance option"
$\max(\min(S_1 - K, \ldots, S_n - K), 0)$	"min call option"

A basket is an option with payoff depending on a portfolio of assets. An example is the payoff of a basket call,

$$\left(\sum_{i=1}^{n} c_i S_i - K \right)^+ ,$$

where the weights c_i are given by the portfolio. It is recommendable to sketch the above payoffs for $n = 2$.

For the pricing of multifactor options the instruments introduced in the previous chapters apply. This holds for the four large classes of methods discussed before, namely, the PDE methods, the tree methods, the evaluation of integrals by quadrature, and the Monte Carlo methods. Each class subdivides into further methods. For the choice of an appropriate method, the dimension n is crucial. For large values of n, in particular PDE methods suffer from the *curse of dimension*. When in any one dimension m nodes are required, then in $\mathbb{R}^n$ already m^n nodes are involved —at least for standard finite difference methods. At present state it is not possible to decide at what level the threshold of n might be, above which PDE standard discretizations are too expensive. At least for $n = 2$ and $n = 3$, such elementary PDE approaches are competitive. Otherwise sparse-grid technology or multigrid are better choices, see the references in Section 3.5.1 and at the end of Chapter 4. Generally in a multidimensional situation, finite elements are recommendable. But FE methods suffer from the curse of dimension too.

PDE methods require relevant PDEs *and* boundary conditions. Often a Black–Merton–Scholes scenario is assumed. To extend the one-factor model, an appropriate generalization of geometric Brownian motion is needed. We begin with the two-factor model, with the prices of the two assets S_1 and S_2. The assumption of a constant-coefficient GBM is then expressed as

$$\begin{aligned} dS_1 &= \mu_1 S_1 \, dt + \sigma_1 S_1 \, dW^{(1)} \\ dS_2 &= \mu_2 S_2 \, dt + \sigma_2 S_2 \, dW^{(2)} \\ \mathsf{E}(\, dW^{(1)} \, dW^{(2)}) &= \rho \, dt \,, \end{aligned} \tag{6.1a}$$

where ρ is the correlation between the two assets, $-1 \leq \rho \leq 1$. Note that the third equation in (6.1a) is equivalent to $\mathsf{Cov}(\, dW^{(1)}, \, dW^{(2)}) = \rho \, dt$, because $\mathsf{E}(\, dW^{(1)}) = \mathsf{E}(\, dW^{(2)}) = 0$. Compared to more general systems as in (1.41), the version (6.1a) with correlated Wiener processes has the advantage that each asset price has its own growth factor μ and volatility σ, which can be estimated from data. The correlation ρ is given by the correlation of the returns $\frac{dS}{S}$, since

$$\mathsf{Cov}\left(\frac{\mathrm{d}S_1}{S_1}, \frac{\mathrm{d}S_2}{S_2}\right) = \mathsf{E}(\sigma_1 \,\mathrm{d}W^{(1)} \sigma_2 \,\mathrm{d}W^{(2)}) = \rho\sigma_1\sigma_2 \,\mathrm{d}t \; . \tag{6.1b}$$

Note that following Section 2.3.3 and Exercise 2.9, the correlated Wiener processes are given by

$$\begin{aligned} \mathrm{d}W^{(1)} &= \mathrm{d}Z_1 \\ \mathrm{d}W^{(2)} &= \rho \,\mathrm{d}Z_1 + \sqrt{1-\rho^2}\,\mathrm{d}Z_2 \; , \end{aligned} \tag{6.1c}$$

where Z_1 and Z_2 are independent standard normally distributed processes. This was used already in (3.28). The resulting two-dimensional Black–Scholes equation was applied in Section 5.4, see equation (5.26). This is derived by the two-dimensional version of the Itô-Lemma ($\longrightarrow$ Appendix B2) and a no-arbitrage argument. The resulting PDE (5.26) has independent variables (S_1, S_2, t). Usually, the time variable is not counted when the dimension is discussed. In this sense, the PDE (5.26) is two-dimensional, whereas the classic Black–Scholes PDE (1.2) is considered as one-dimensional.

The general n-factor model is analogous. The appropriate model is a straightforward generalization of (6.1a),

$$\begin{aligned} \mathrm{d}S_i &= (\mu_i - \delta_i)S_i \,\mathrm{d}t + \sigma_i S_i \,\mathrm{d}W^{(i)} \; , \quad i = 1, \ldots, n \\ \mathsf{E}(\mathrm{d}W^{(i)}\mathrm{d}W^{(j)}) &= \rho_{ij} \,\mathrm{d}t \; , \quad i, j = 1, \ldots, n \end{aligned} \tag{6.2a}$$

where ρ_{ij} is the correlation between asset i and asset j, and δ_i denotes the dividend flow paid by the ith asset. For a simulation of such a stochastic vector process see Section 2.3.3. The Black–Scholes-type PDE of the model (6.2a) is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r-\delta_i)S_i\frac{\partial V}{\partial S_i} - rV = 0 \; . \tag{6.2b}$$

Boundary conditions depend on the specific type of option. For example in the two-dimensional situation, one boundary can be defined by the plane $S_1 = 0$ and the other by the plane $S_2 = 0$. It may be appropriate to apply the Black–Scholes vanilla formula (A4.10) along these planes, or to define one-dimensional sub-PDEs only for the purpose to calculate the values of $V(S_1, 0, t)$ and $V(0, S_2, t)$ along the boundary planes.

For **tree methods**, the binomial method can be generalized canonically [BoEG89]. But already for $n = 2$ the recombining standard tree with M time levels requires $\frac{1}{3}M^3 + O(M^2)$ nodes, and for $n = 3$ the number of nodes is of the order $O(M^4)$. Tree methods also suffer from the curse of dimension. But obviously not all of the nodes of the canonical binomial approach are needed. The ultimate aim is to approximate the lognormal distribution, and this can be done with fewer nodes. Nodes in $\mathbb{R}^n$ should be constructed in such a way that the number of nodes grows comparably slower than the quality of the approximation of the distribution function. An example of a two-dimensional approach is presented in [Lyuu02]. Generalizing the trinomial approach to higher dimensions is not recommendable because of storage requirements, but other geometrical structures as icosahedral volumes can be applied. For different tree approaches, see [McW01]. For a convergence analysis of tree methods, and for an extension to Lévy processes, see [FoVZ02], [MaSS06].

An advantage of tree methods and of **Monte Carlo methods** is that no boundary conditions are needed. The essential advantage of MC methods is that they are much less affected by high dimensions, see the notes on Section 3.6. An example of a five-dimensional American-style option is calculated in [BrG04], [LonS01]. It is most inspiring to perform Monte Carlo experiments on exotic options. For European-style options, this amounts to a straightforward application of Section 3.5 ($\longrightarrow$ Exercise 6.1).

6.3 Asian Options

The price of an Asian option[1] depends on the average price of the underlying and hence on the history of S_t. We choose this type of option to discuss some strategies of how to handle path-dependent options. Let us first define different types of Asian options via their payoff.

6.3.1 The Payoff

There are several ways how an average of past values of S_t can be formed. If the price S_t is observed at discrete time instances t_i, say equidistantly with time interval $h := T/n$, one obtains a times series $S_{t_1}, S_{t_2}, \ldots, S_{t_n}$. An obvious choice of average is the arithmetic mean

$$\frac{1}{n}\sum_{i=1}^{n} S_{t_i} = \frac{1}{T}\; h\sum_{i=1}^{n} S_{t_i}\ .$$

If we imagine the observation as continuously sampled in the time period $0 \le t \le T$, the above mean corresponds to the integral

$$\widehat{S} := \frac{1}{T}\int_0^T S_t \,\mathrm{d}t \tag{6.3}$$

[1] Again, the name has no geographical relevance.

The arithmetic average is used mostly. Sometimes the geometric average is applied, which can be expressed as

$$\left(\prod_{i=1}^{n} S_{t_i}\right)^{1/n} = \exp\left(\frac{1}{n}\log\prod_{i=1}^{n} S_{t_i}\right) = \exp\left(\frac{1}{n}\sum_{i=1}^{n}\log S_{t_i}\right) .$$

Hence the continuously sampled geometric average of the price S_t is the integral

$$\widehat{S}_{\mathrm{g}} := \exp\left(\frac{1}{T}\int_0^T \log S_t \, \mathrm{d}t\right) .$$

The averages $\widehat{S}$ and $\widehat{S}_{\mathrm{g}}$ are formulated for the time period $0 \le t \le T$, which corresponds to a European option. To allow for early exercise at time $t < T$, $\widehat{S}$ and $\widehat{S}_{\mathrm{g}}$ are modified appropriately, for instance to

$$\widehat{S} := \frac{1}{t}\int_0^t S_\theta \, \mathrm{d}\theta .$$

With an average value $\widehat{S}$ like the arithmetic average of (6.3) the payoff of Asian options can be written conveniently:

Definition 6.1 (Asian option)

With an average $\widehat{S}$ of the price evolution S_t the payoff functions of Asian options are defined as

$$\begin{array}{ll} (\widehat{S} - K)^+ & \textit{average price call} \\ (K - \widehat{S})^+ & \textit{average price put} \\ (S_T - \widehat{S})^+ & \textit{average strike call} \\ (\widehat{S} - S_T)^+ & \textit{average strike put} \end{array}$$

The price options are also called *rate options*, or *fixed strike options*; the strike options are also called *floating strike options*. Compared to the vanilla payoffs of (1.1P), (1.1C), for an Asian price option the average $\widehat{S}$ replaces S whereas for the Asian strike option $\widehat{S}$ replaces K. The payoffs of Definition 6.1 form surfaces on the quadrant $S > 0$, $\widehat{S} > 0$. The reader may visualize these payoff surfaces.

6.3.2 Modeling in the Black–Scholes Framework

The above averages can be expressed by means of the integral

$$A_t := \int_0^t f(S_\theta, \theta) \, \mathrm{d}\theta , \tag{6.4}$$

where the function $f(S,t)$ corresponds to the type of chosen average. In particular $f(S,t) = S$ corresponds to the continuous arithmetic average (6.3), up to scaling by the length of interval. For Asian options the price V is a function of S, A and t, which we write $V(S, A, t)$. To derive a partial differential equation for V using a generalization of Itô's Lemma we require a differential equation for A. But this is given by (6.4), it lacks a stochastic $\mathrm{d}W_t$-term,[2]

$$\begin{aligned} &\mathrm{d}A = a_A(t)\,\mathrm{d}t + b_A\,\mathrm{d}W_t \ , \\ &\text{with} \quad a_A(t) := f(S_t, t) \quad , \quad b_A := 0 \ . \end{aligned}$$

For S_t the standard GBM of (1.33) is assumed. By the multidimensional version (B2.1) of Itô's Lemma adapted to $Y_t := V(S_t, A_t, t)$, the two terms in (1.44) or (1.45) that involve b_A as factors to $\frac{\partial V}{\partial A}, \frac{\partial^2 V}{\partial A^2}$ vanish. Accordingly,

$$\mathrm{d}V_t = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f(S,t)\frac{\partial V}{\partial A} \right) \mathrm{d}t + \sigma S \frac{\partial V}{\partial S}\,\mathrm{d}W_t \ .$$

The derivation of the Black–Scholes-type PDE goes analogously as outlined in Appendix A4 for standard options and results in

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + f(S,t)\frac{\partial V}{\partial A} - rV = 0 \ . \tag{6.5}$$

Compared to the original vanilla version (1.2), only one term in (6.5) is new, namely,

$$f(S,t)\frac{\partial V}{\partial A} \ .$$

As we will see below, the lack of a second-order derivative with respect to A may cause numerical difficulties. The transformations (4.3) cannot be applied advantageously to (6.5). — As an alternative to the definition of A_t in (6.4), one can scale by t. This leads to a different "new term" ($\longrightarrow$ Exercise 6.2).

6.3.3 Reduction to a One-Dimensional Equation

Solutions to (6.5) are defined on the domain

$$S > 0 \ , \ A > 0 \ , \ 0 \le t \le T$$

of the three-dimensional (S, A, t)-space. The extra A-dimension leads to significantly higher costs when (6.5) is solved numerically. This is the general situation. But in some cases it is possible to reduce the dimension. Let us discuss an example, concentrating on the case $f(S, t) = S$ of the arithmetic average.

We consider a European arithmetic average strike call with payoff

[2] The ordinary integral A_t is random but has zero quadratic variation [Shr04].

$$\left(S_T - \frac{1}{T}A_T\right)^+ = S_T\left(1 - \frac{1}{TS_T}\int_0^T S_\theta \,\mathrm{d}\theta\right)^+ .$$

An auxiliary variable R_t is defined by

$$R_t := \frac{1}{S_t}\int_0^t S_\theta \,\mathrm{d}\theta ,$$

and the payoff is rewritten

$$S_T\left(1 - \frac{1}{T}R_T\right)^+ = S_T \cdot \text{function}(R_T, T) .$$

This motivates trying a separation of the solution in the form

$$V(S, A, t) = S \cdot H(R, t) \tag{6.6}$$

for some function $H(R, t)$. In this role, R is an independent variable. But note that the integral R_t satisfies an SDE. From

$$\begin{aligned} R_{t+\mathrm{d}t} &= R_t + dR_t \\ \mathrm{d}S_t &= \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t \end{aligned}$$

the SDE

$$\mathrm{d}R_t = (1 + (\sigma^2 - \mu)R_t)\,\mathrm{d}t - \sigma R_t \,\mathrm{d}W_t \tag{6.7}$$

follows.

Substituting the separation ansatz (6.6) into the PDE (6.5) leads to a PDE for H,

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR)\frac{\partial H}{\partial R} = 0 \tag{6.8}$$

($\longrightarrow$ Exercise 6.2). To solve this PDE, boundary conditions are required. Their choice in general is not unique. The following considerations suggest boundary conditions.

A right-hand boundary condition for $R \to \infty$ follows from the payoff

$$H(R_T, T) = (1 - \tfrac{1}{T}R_T)^+ ,$$

which implies $H(R_T, T) = 0$ for $R_T \to \infty$. The integral R_t is bounded, hence $S \to 0$ for $R \to \infty$. For $S \to 0$ a European call option is not exercised, which suggests

$$H(R, t) = 0 \quad \text{for} \quad R \to \infty . \tag{6.9}$$

At the left-hand boundary $R = 0$ we encounter more difficulties. Even if $R_0 = 0$ holds, the equation (6.7) shows that $\mathrm{d}R_0 = \mathrm{d}t$ and R_t will not stay at 0. So there is no reason to expect $R_T = 0$, and the value of the payoff cannot be predicted. Another kind of boundary condition is required.

To this end, we start from the PDE (6.8), which for $R \to 0$ is equivalent to

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + \frac{\partial H}{\partial R} = 0 \; .$$

Assuming that H is bounded, one can prove that the term

$$R^2 \frac{\partial^2 H}{\partial R^2}$$

vanishes for $R \to 0$. The resulting boundary condition is

$$\frac{\partial H}{\partial t} + \frac{\partial H}{\partial R} = 0 \quad \text{for} \quad R \to 0 \; . \tag{6.10}$$

The vanishing of the second-order derivative term is shown by contradiction: Assuming a nonzero value of $R^2 \frac{\partial^2 H}{\partial R^2}$ leads to

$$\frac{\partial^2 H}{\partial R^2} = O\left(\frac{1}{R^2}\right) \; ,$$

which can be integrated twice to

$$H = O(\log R) + c_1 R + c_2 \; .$$

This contradicts the boundedness of H for $R \to 0$.

For a numerical realization of the boundary condition (6.10) in the finite-difference framework of Chapter 4, we may use the second-order formula

$$\left.\frac{\partial H}{\partial R}\right|_{0\nu} = \frac{-3H_{0\nu} + 4H_{1\nu} - H_{2\nu}}{2\Delta R} + O(\Delta R^2) \; . \tag{6.11}$$

The indices have the same meaning as in Chapter 4. We summarize the boundary-value problem of PDEs to

$$\boxed{\begin{aligned} &\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR)\frac{\partial H}{\partial R} = 0 \\ &H = 0 \quad \text{for} \quad R \to \infty \\ &\frac{\partial H}{\partial t} + \frac{\partial H}{\partial R} = 0 \quad \text{for} \quad R = 0 \\ &H(R_T, T) = \left(1 - \tfrac{R_T}{T}\right)^+ \end{aligned}} \tag{6.12}$$

Solving this problem numerically for $0 \le t \le T$, $R \ge 0$, gives $H(R,t)$, and via (6.6) the required values of V.

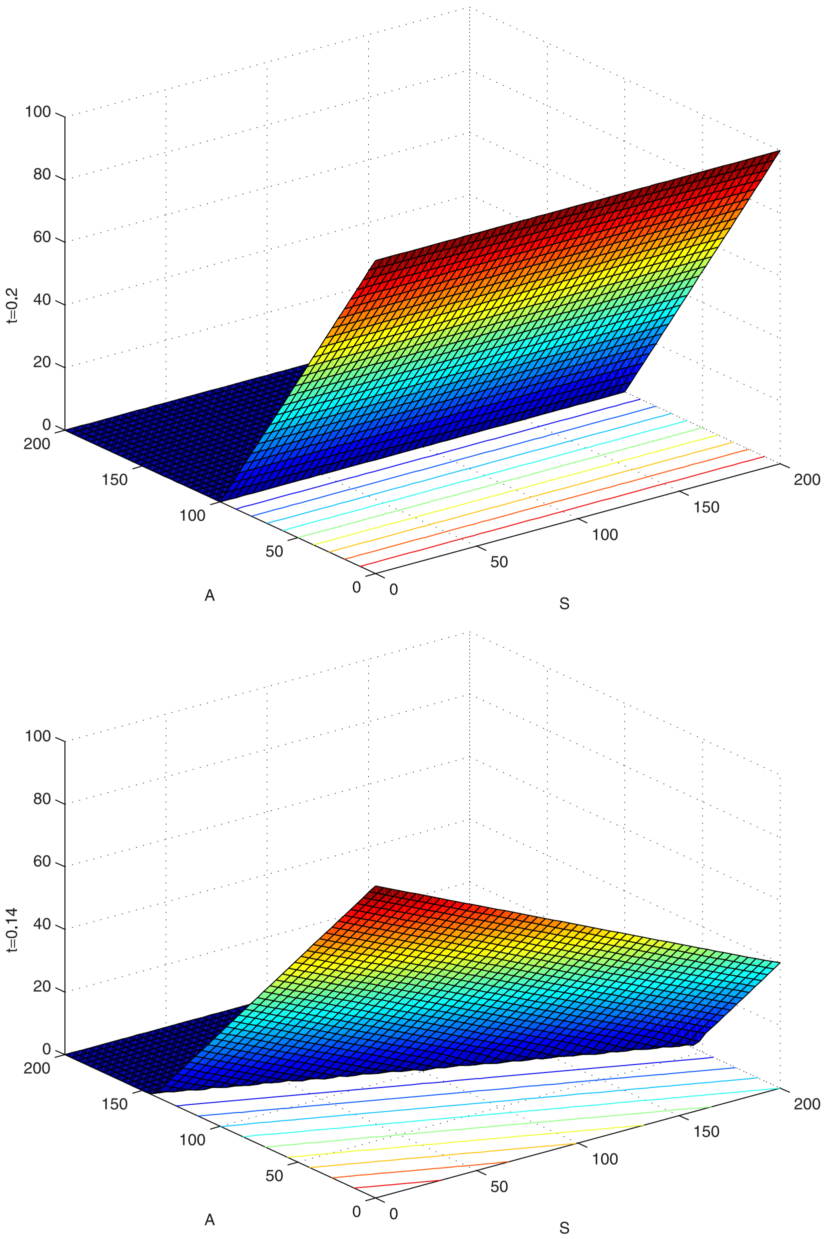


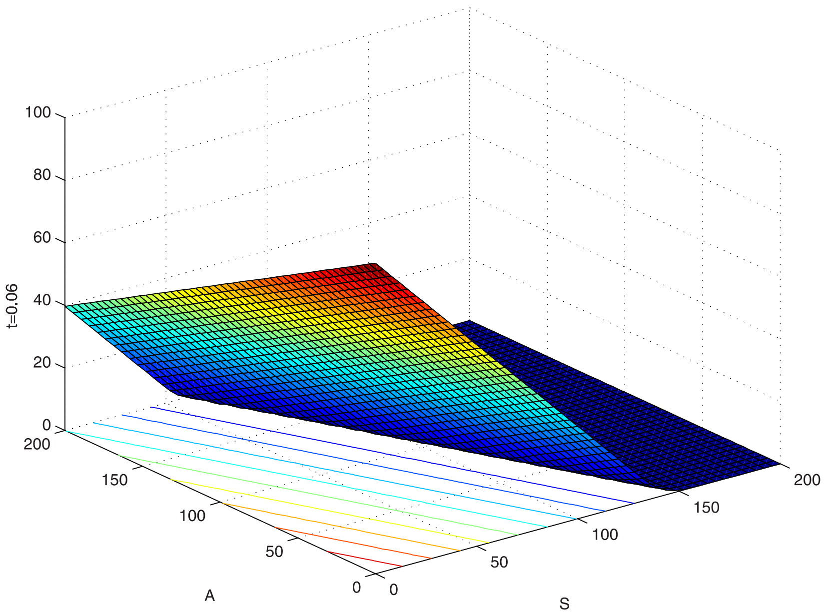
Fig. 6.1. Asian European fixed strike put, $K = 100$, $T = 0.2$, $r = 0.05$, $\sigma = 0.25$, payoff ($t = 0.2$) and three solution surfaces for $t = 0.14$, $t = 0.06$, and $t = 0$. (Figure continued on facing page)

6.3.4 Discrete Monitoring

Instead of defining a continuous averaging as in (6.3), a realistic scenario is to assume that the average is monitored only at discrete time instances

$$t_1, t_2, \ldots, t_M \ .$$

These time instances are not to be confused with the grid times of the numerical discretization. The discretely sampled arithmetic average at t_k is given by



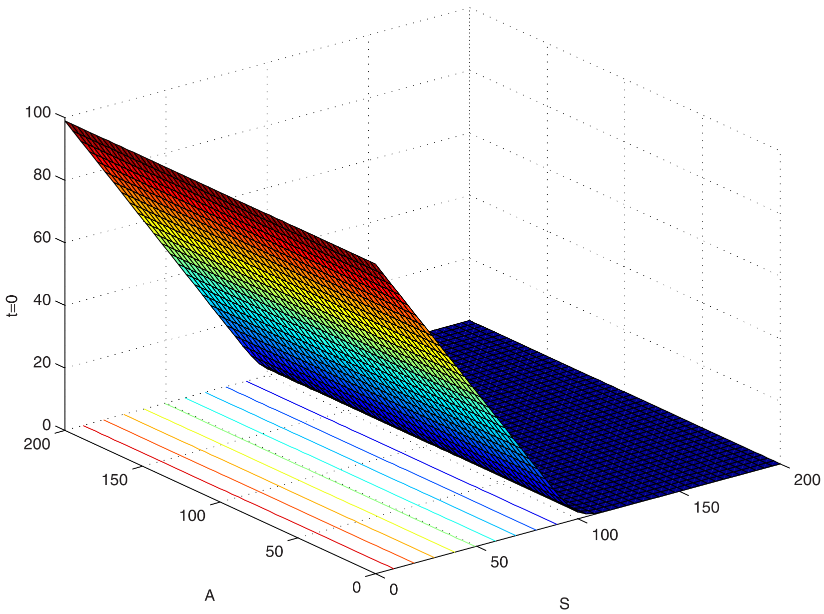


Fig. 6.1. continued

$$A_{t_k} := \frac{1}{k} \sum_{i=1}^{k} S_{t_i}, \ k = 1, \ldots, M \ . \tag{6.13}$$

A new average is updated from a previous one by

$$A_{t_k} = A_{t_{k-1}} + \frac{1}{k}(S_{t_k} - A_{t_{k-1}})$$

or

$$A_{t_{k-1}} = A_{t_k} + \frac{1}{k-1}(A_{t_k} - S_{t_k}) \ .$$

The latter of these update formulas is relevant to us, because we integrate backwards in time. The discretely sampled A_t is constant between sampling times, and it jumps at t_k with the step

$$\frac{1}{k-1}(A_{t_k} - S_{t_k}) \,.$$

For each k this jump can be written

$$A^-(S) = A^+(S) + \frac{1}{k-1}(A^+(S) - S), \text{ where } S = S_{t_k} \,. \tag{6.14a}$$

A^- and A^+ denote the values of A immediately before and immediately after sampling at t_k. The no-arbitrage principle implies continuity of V at the sampling instances t_k in the sense of continuity of $V(S_t, A_t, t)$ for any realization of a random walk. In our setting, this continuity is written

$$V(S, A^+, t_k) = V(S, A^-, t_k) \,. \tag{6.14b}$$

But for a *fixed* (S, A) this equation defines a **jump** of V at t_k.

The numerical application of the jump condition (6.14) is as follows: The A-axis is discretized into discrete values A_j, $j = 1, \ldots, J$. For each time period between two consecutive sampling instances, say for $t_{k+1} \to t_k$, the option's value is independent of A because in our discretized setting A_t is piecewise constant; accordingly $\frac{\partial V}{\partial A} = 0$. So J one-dimensional Black–Scholes equations are integrated separately and independently from t_{k+1} to t_k, one for each j. Each of the one-dimensional Black–Scholes problems has its own terminal condition. For each A_j, the "first" terminal condition is taken from the payoff surface for $t_M = T$. Proceeding backwards in time, at each sampling time t_k the J parallel one-dimensional Black–Scholes problems are halted because new terminal conditions must be derived from the jump condition (6.14). The new values for $V(S, A_j, t_k)$ that serve as terminal values (starting values for the backward integration) for the next time period $t_k \to t_{k-1}$, are defined by the jump condition, and are obtained by interpolation. Only at these sampling times the J standard one-dimensional Black–Scholes problems are coupled; the coupling is provided by the interpolation. In this way, a sequence of surfaces $V(S, A, t_k)$ is calculated for $t_M = T, \ldots, t_1 = 0$. Figure 6.1 shows[3] the payoff and three surfaces calculated for an Asian European fixed strike put. As this illustration indicates, there is a kind of rotation of this surface as t varies from T to 0.

[3] After interpolation; MATLAB graphics; courtesy of S. Göbel; similar [ZFV99].