A SIMPLE NUMERICAL METHOD FOR PRICING AMERICAN POWER PUT OPTIONS

A Term Paper Report Submitted

for the Course

MA473

by

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ABSTRACT

In this paper, we present numerical methods to determine the optimal exercise boundary in case of an American power put option with non-dividend yields. The payoff of a power option is typified by its underlying share price raised to a constant power. The nonlinear payoffs of power options offer considerable flexibility to investors and can be applied in various applications. Herein, we exploit a transformed function to obtain the optimal exercise boundary of the American power put option. Employing it, we can easily determine the optimal exercise boundary. After determining the optimal exercise boundary, we calculate the American power put option values using the finite difference method. Generally, the optimal exercise boundary may not be observed at the grid points. Therefore, the interpolation method is used to determine the value of the American power put option.

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1.1 Introduction

Using a transformed function, this report provides a method for rapidly and accurately pricing American power put options. The most common derivative transaction in today's financial markets is the option. American power put options are the put options whose payoff depends on underlying asset price raised to a constant power and can be exercised on or before Maturity Due to the vast majority of American options traded on regulated markets, finding a way to price them appropriately is essential.

The price problem requires complex analytical calculations. Due to the fact that the option holder has an early exercise right, the problem becomes a free-boundary problem. The optimal exercise boundary is usually referred to as the free boundary. Ideal exercise boundaries (also known as early exercise boundaries) are determined by time and are part of the solution before the option expires. As a result, determining the best exercise border is a major task when it comes to valuing American options.

Finite Difference Methods are used to address free boundary problems

(FDM). Once the model is discretized using the finite difference approach, the pricing of American options may be addressed as a linear complementarity problem (LCP) or variational inequality. LCP is a key problem in mathematical programming because of two properties: linearity and complementarity. The projected sequential overrelaxation approach is one of the most prevalent and efficient procedures (PSOR).

We provide a numerical technique for pricing American power put options using a transformed function. To avoid degeneracy of the solution surface near the optimal exercise boundary, we use a transformed function with a Lipschitz-like free boundary. The primary purpose of this paper is to develop a numerical approach for calculating the appropriate exercise boundary for the American power put option. We also demonstrate this numerically, employing a transformed function to determine the ideal training bounds. At the grid spots, the suitable workout border is not always apparent.

As a result, the interpolation method is used to calculate the value of the American power put option. After determining the optimal exercise boundary, the American power put option values are derived using the finite difference technique (FDM).

1.2 Formulation

The purpose of this section is to introduce mathematical models for the American power put option. Let $S^n(t)$ be the price of an underlying asset as a function of time t and let n be the option's power. We assume that $S^n(t)$ follows

$$dS^n(t) = n\left\{\frac{1}{2}\sigma^2(n-1)+r\right\}S^n(t)dt + \sigma n S^n(t)dW(t), \forall t \in [0,T] \quad (1.1)$$

where $r > 0$ is the constant risk interest rate, $\sigma > 0$ is the underlying asset price volatility, and W(t) is the typical Brownian motion.

In the instance of an American power put option with the underlying asset price $S^n(t)$, exercise price K, and time to expiration T, the payoffs are as follows:

$$(K - S^n(T))^+ = \max\{K - S^n(T), 0\} \quad (1.2)$$

The value of an American power put option is denoted as $P(\tau, S^n)$, where τ(:= T-t) is the time to expiration for $\tau \in$ [0,T]. We intend to determine the value $P(\tau, S^n)$ for an American power put option, which can be exercised by the holder at any time to obtain a payoff $(K - S^n(T))^+$.

The best time to exercise an American option and its price are determined by the optimal exercise boundary denoted as $\beta = \{\beta(\tau) : \tau \in [0, T]\}$. Here, we suppose that the optimal exercise boundary $\beta(\tau)$ is continuously nonincreasing with $\beta(0) = K$. For each time $\tau \in$ [0, T], there exists an optimal exercise boundary $\beta(\tau)$, below which the American power put option should be exercised early, i.e., if

$$S^n(\tau) \leq \beta(\tau), \text{ then } P(\tau, S^n) = \max\{K - S^n(\tau), 0\} \quad (1.3)$$

and if

$$S^n(\tau) > \beta(\tau), \text{ then } P\left(\tau, S^n\right) > \max\left\{K - S^n(\tau), 0\right\} \tag{1.4}$$

The time and asset price space are divided into two regions. A continuation region is the one in which it is optimal to hold, commonly known as Ω_C = [0, T] $\times(\beta(\tau), \infty)$, and the region in which it is optimal to exercise, generally called the exercise (or stopping) region, is defined as Ω_E = [0, T] $\times$[0, $\beta(\tau)$].

The Black–Scholes PDE for the price of an American power put option with a non-dividend yield can be given as

$$\frac{\partial P}{\partial \tau} - n\left\{\frac{1}{2}\sigma^2(n-1) + r\right\} S^n \frac{\partial P}{\partial S^n} - \frac{1}{2}\left(\sigma n S^n\right)^2 \frac{\partial^2 P}{\partial \left(S^n\right)^2} + rP = 0 \text{ in } \Omega_C \tag{1.5}$$

Then (1.5) is subjected to the following boundary conditions.

$$\lim_{S^n \to \infty} P\left(\tau, S^n\right) = 0 \tag{1.6}$$

$$P(\tau, \beta(\tau)) = K - \beta(\tau) \tag{1.7}$$

$$P_{\mathrm{S}^n}(\tau, \beta(\tau)) = -1 \tag{1.8}$$

and the initial condition

$$P\left(0, S^n\right) = \max\left\{K - S^n, 0\right\} \tag{1.9}$$

With the use of the time variable τ = T-t, we are faced with an initial-value problem. The existence and uniqueness of P and beta as solutions to PDE.

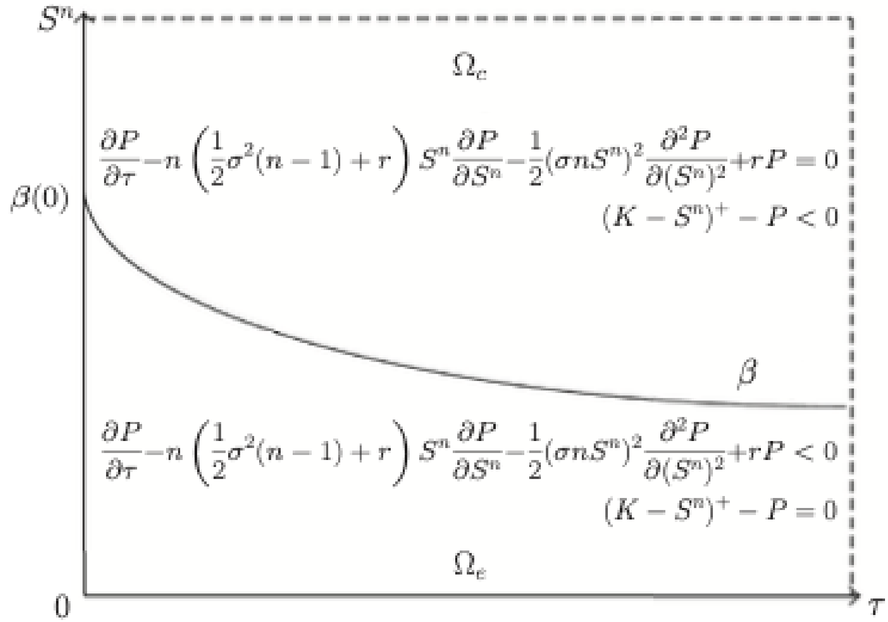

Figure 1.1: Optimal exercise boundary of an American power put option

The above figure shows an illustration of an optimal exercise boundary β with an American power put option price function P. P must satisfy

$$\max_{\tau, S^n}\left\{\frac{\partial P}{\partial \tau} - n\left(\frac{1}{2}\sigma^2(n-1)+r\right)S^n\frac{\partial P}{\partial S^n} - \frac{1}{2}\left(\sigma n S^n\right)^2\frac{\partial^2 P}{\partial \left(S^n\right)^2} + rP\right\} = 0 \tag{1.10}$$

Following are the asymptotically optimal exercise boundaries for perpetual American power puts $\beta(\infty)$:

$$\beta(\infty) = \frac{\gamma}{\gamma+1}K \tag{1.11}$$

where

$$\gamma = \frac{2r}{n^2\sigma^2}$$

1.3 Transformed Function

Before computing the PDE solution, we shall determine the ideal exercise border ($\beta(\tau)$) in this step. We give the following transformed function with Lipschitz character at the optimal exercise border:

$$Q\left(\tau, S^n\right) = \sqrt{P\left(\tau, S^n\right) - \left(K - S^n\right)} \tag{1.12}$$

Because the solution is a horizontal line in the exercise zone and an inclined line in the continuation region, the aforementioned function is employed. As a result, $Q(\tau, S_n)$ equals 0 in the workout region and $Q(\tau, S_n) > 0$ in the continuation region. The boundary may be clearly detected since the transformed function create a sufficiently large angle with the horizontal line.

Theorem 1.3.1. *Let*

$$Q\left(\tau, S^n\right) = \sqrt{P\left(\tau, S^n\right) - \left(K - S^n\right)} \tag{1.13}$$

Q possesses Lipschitz properties, such as nonsingularity and nondegeneracy around the optimal workout boundary. Then,

$$0 < \xi < Q_{S^n} < \eta, \tag{1.14}$$

where

$$\xi = \sqrt{\frac{1}{K}\left(1 - \frac{1}{n}\right)\left(\frac{1}{2} + \frac{r}{\sigma^2 n}\right) + \frac{rK}{(\sigma n K)^2}} \tag{1.15}$$

$$\eta = \sqrt{\frac{1}{\beta(\infty)}\left(1 - \frac{1}{n}\right)\left(\frac{1}{2} + \frac{r}{\sigma^2 n}\right) + \frac{rK}{(\sigma n \beta(\infty))^2}} \tag{1.16}$$

We obtain the angle between the exercise region and the transformed function curve such that $0 < \xi < Q_{S^n} < \eta$, for some constants ξ and η.

1.4 Numerical Methods

In this section, we describe the numerical method developed based on the transformed function. The independent variables in the Black–Scholes PDE are time (τ) and the underlying stock price (S^n). To solve this problem using the FDM, we divide the time interval $[0, T]$ into L subintervals, that is $\tau_l = l\Delta\tau, l = 0, 1, 2, .., L-1, L \ where \ \Delta\tau = \frac{T}{L}$ and the stock price interval $[0, S^n_M]$ into M subintervals, that is $S^n_i = i\Delta S^n, i = 0, 1, 2, ..., M-1, M, \ \Delta S^n = \frac{S^n_M}{M}$. We intend to define a numerical method for computing the grid values $P^l_i \approx P(\tau_l, S^n_i)$ and the optimal exercise boundary values $\beta_l = \beta(\tau_l), for \ i = 0, 1, 2, ..., M-1, M, \ and \ l = 0, 1, 2, ..., L-1, L$. Herein, i denotes the horizontal node index, l denotes the time step index, and τ_{l-1} represents the $(l1)th$ previous time step and τ_l represents the lth current time step. For the proposed problem, $\beta(\tau_{l-1})$ and P^{l-1}_i are given and the objective is to compute $\beta(\tau_l)$ and P^l_i.

We define step by step method for solving this problem:

Step 1 : We begin with the initial value at time T , which provides the value of the payoff function $(K - S^n_i)^+ = max\{K - S^n_i, 0\}$ and the optimal exercise boundary $\beta(0) = K$. Because the optimal exercise boundary $\beta(\tau_{l-1})$ is dependent on the future boundary value $\beta(\tau_l)$, it must be determined by setting the initial boundary value to the strike price $\beta(\tau_0) = K$ and working backward through time from T $= \tau_0$ to T $= \tau_L$.

Step 2 : We obtain the current optimal exercise boundary $\beta(\tau_l)$ based on the previous optimal exercise boundary $\beta(\tau_{l-1})$. The current optimal exercise boundary can be obtained using a transformed function $Q(\tau_l, S^n) = \sqrt{P(\tau_l, S^n) - (K - S^n)}$. Herein, to obtain Q at the current time step, the option price $P(\tau_l, S^n)$ should be obtained. Further, to find $\beta(\tau_l)$, we derive the relation between $Q(\tau_l, S^n)$ and $\beta(\tau_l)$.

Now We find $P(\tau_l, S^n)$ with respect to the current time step using a three-point FDM based on a uniform mesh. If the optimal exercise boundary does not depend on the grid point, then we cannot use the FDM based on a uniform mesh. Therefore, we use natural cubic spline interpolation. Based on the mesh point information, we can obtain value between the mesh points. Define:

$$S_1 = \beta(\tau_{l-1}) + \Delta S^n$$
$$S_2 = \beta(\tau_{l-1}) + 2\Delta S^n$$

then, we find $P(\tau_l, S_1)$. Because the previous optimal exercise boundary $\beta(\tau_{l-1})$ is dependent on the current optimal exercise boundary $\beta(\tau_l)$, we calculate the option price $P(\tau_l, S_1)$ on S_1 in the current time step using the explicit method.

The Black Sholes equation can be approximated as:

$$\begin{aligned} P\left(\tau_l, S_1\right) = (1 - r\Delta\tau) P\left(\tau_{l-1}, S_1\right) + n\left\{\tfrac{1}{2}\sigma^2(n-1) + r\right\} S_1 \Delta\tau \\ \tfrac{\partial P(\tau_{l-1}, S_1)}{\partial S} + \tfrac{1}{2}\left(\sigma n S_1\right)^2 \tfrac{\partial^2 P(\tau_{l-1}, S_1)}{\partial S^2} \Delta\tau \end{aligned} \tag{1.17}$$

The derivatives in above equations could be formulated using:

$$\frac{\partial P\left(\tau_{l-1}, S_1\right)}{\partial S} \simeq \frac{P\left(\tau_{l-1}, S_2\right) - P\left(\tau_{l-1}, S_1\right)}{2\left(S_2 - S_1\right)} \tag{1.18}$$

$$\frac{\partial^2 P\left(\tau_{l-1}, S_1\right)}{\partial S^2} \simeq \frac{P\left(\tau_{l-1}, \beta\left(\tau_{l-1}\right)\right) - 2P\left(\tau_{l-1}, S_1\right) + P\left(\tau_{l-1}, S_2\right)}{\left(S_2 - S_1\right)^2} \tag{1.19}$$

We now apply the cubic spline interpolation in above 2 equations to find $P(\tau_{l-1}, S_1)$ and $P(\tau_{l-1}, S_2)$; thus, we obtain $P(\tau_{l-1}, S_1) \approx f(\tau_{l-1}, S_1)$, $P(\tau_{l-1}, S_2) \approx f(\tau_{l-1}, S_2)$, where f is the cubic spline function. Because

$P(\tau, S^n) = Q^2(\tau, S^n) + K - S^n$, we obtain $P(\tau_{l-1}, \beta(\tau_{l-1})) = K - \beta(\tau_{l-1})$ near the optimal exercise boundary (Q=0). Thus, we obtain $P(\tau_l, S_1)$.

As we can see that

$$Q\left(\tau_l, S_1\right) = \sqrt{P\left(\tau_l, S_1\right) - \left(K - S_1\right)} \tag{1.20}$$

We find the relation between $\beta(\tau_{l-1}$ and $Q(\tau_l, S_1)$ using second-order Taylor expansion of Q:

$$\begin{aligned} Q\left(\tau, S^n\right) =& Q(\tau, \beta) + Q_{S^1}(\tau, \beta)\left(S^n - \beta\right) \\ &+ \frac{1}{2} Q_{S^n \text{ S1}}\left(\tau, \beta\right)\left(S^n - \beta\right)^2 + \mathcal{O}\left(S^n - \beta\right)^3 \end{aligned} \tag{1.21}$$

After some mathematical calculations we obtain:

$$\begin{aligned} 6(\sigma n \beta)^2 Q_{S^n} Q =& 6\left(\sigma n \beta Q_{S^n}\right)^2\left(S^n - \beta\right) \\ &- \left\{2 Q_{S^n}^2\left(\beta' + A\beta + \sigma^2 n^2 \beta\right) - (A - r)\right\}\left(S^n - \beta\right)^2 \end{aligned} \tag{1.22}$$

where

$$\frac{\beta'}{\beta} \approx \frac{\ln\left(\frac{5^n}{\beta}\right) - \left(\frac{5^n}{\beta} - 1\right)}{\Delta\tau} \tag{1.23}$$

Combining the above 2 equations we can rewrite is as:

$$a\beta^5 + b\beta^4 + c\beta^3 + d\beta^2 + e\beta + f = 0 \tag{1.24}$$

where a, b, c, d, e, f are constants that are:

$$\begin{aligned}
a &= -C\frac{(A-r)^2}{4B\sqrt{B}} \\
b &= C\frac{A-r}{\sqrt{B}} + C\frac{(A-r)^2\tilde{\beta}}{2B\sqrt{B}} + 5\Delta\tau(A-r) \\
&+ \frac{2(A-r)}{(\sigma n)^2}\left\{\ln\left(\frac{S^n}{\tilde{\beta}} + 1 + \Delta\tau\left(A+(\sigma n)^2\right)\right)\right\} \\
c &= 2C\sqrt{B} - C\frac{(A-r)\tilde{\beta}}{\sqrt{B}} - C\frac{(A-r)^2\tilde{\beta}^2}{4B\sqrt{B}} - \Delta\tau\left\{4(A-r)S^n - 6rK\right\} \\
&+ \frac{2rK}{(\sigma n)^2}\left\{\ln\left(\frac{S^n}{\tilde{\beta}}\right) + 1 + \Delta\tau\left(A+(\sigma n)^2\right)\right\} \\
&- \frac{2(A-r)S^n}{(\sigma n)^2}\left\{3 + 2\ln\left(\frac{S^n}{\tilde{\beta}}\right) + 2\Delta\tau\left(A+(\sigma n)^2\right)\right\} \\
d &= -6rKS^n\Delta\tau - (A-r)\Delta\tau\left(S^n\right)^2 \\
&- \frac{2rKS^n}{(\sigma n)^2}\left\{3 + 2\ln\left(\frac{S^n}{\tilde{\beta}}\right) + 2\Delta\tau\left(A+(\sigma n)^2\right)\right\} \\
&+ \frac{2(A-r)\left(S^n\right)^2}{(\sigma n)^2}\left\{3 + \ln\left(\frac{S^n}{\tilde{\beta}}\right) + \Delta\tau\left(A+(\sigma n)^2\right)\right\} \\
e &= \frac{2rK\left(S^n\right)^2}{(\sigma n)^2}\left\{3 + \ln\left(\frac{S^n}{\tilde{\beta}}\right) + \Delta\tau\left(A+(\sigma n)^2\right)\right\} - \frac{2(A-r)}{(\sigma n)^2}\left(S^n\right)^3 \\
f &= -\frac{2rK}{(\sigma n)^2}\left(S^n\right)^3
\end{aligned} \tag{1.25}$$

and

$$A = n\left\{\frac{1}{2}\sigma^2(n-1) + r\right\} \tag{1.26}$$

These equations could be solved using Newton–Raphson method. Hence, the solution β could be found at the current optimal exercise boundary.

Step 3 : Now task is to obtain the option price using $\beta(\tau_l)$. let j be the smallest spatial grid index over $\beta(\tau_l)$. If $\beta(\tau_l)$ is too close to S_j^n , we select the index j+1 instead of j.

Using central difference for the spatial discretization, then:

$$\frac{\partial P_j^l}{\partial S_j^n} = \frac{P_{j+1}^l - P_{j-1}^l}{2\Delta S^n}, \frac{\partial^2 P_j^l}{\partial \left(S_j^n\right)^2} = \frac{P_{j+1}^l - 2P_j^l + P_{j-1}^l}{\left(\Delta S^n\right)^2} \tag{1.27}$$

Now using the implicit scheme for time discretization, equation (1.5) is discretized as follows:

$$(1 + r\Delta\tau)P_j^l - AS_j^n \tfrac{P_{j+1}^l - P_{j-1}^l}{2\Delta S^n}\Delta\tau - \tfrac{1}{2}\left(\sigma n S_j^n\right)^2 \tfrac{P_{j+1}^l - 2P_j^l + P_{j-1}^l}{\left(\Delta S^n\right)^2}\Delta\tau = P_j^{l-1}. \tag{1.28}$$

Generally, the optimal exercise boundary may not be located at the grid points. Therefore, we cannot use FDM directly. Because we know the values of P at β, we consider using linear interpolation between adjacent data points.

The value of P at β can be written as

$$P\left(\tau_l, \beta\left(\tau_l\right)\right) = \frac{\left(S_j^n - \beta\left(\tau_l\right)\right)P_{j-1}^l + \left(\Delta S - S_j^n + \beta\left(\tau_l\right)\right)P_j^l}{\Delta S^n} = K - \beta\left(\tau_l\right) \tag{1.29}$$

Hence we obtain:

$$P_{j-1}^l = \frac{\left(K - \beta\left(\tau_l\right)\right)\Delta S^n - \left(\Delta S^n - S_j^n + \beta\left(\tau_l\right)\right)P_j^l}{S_j^n - \beta\left(\tau_l\right)} \tag{1.30}$$

Using equation 1.27 in 1.24 and after mathematical calculations:

$$a_1 K + b_1 P_j^l + c_1 P_{j+1}^l = P_j^{l-1} \tag{1.31}$$

where

$$
\begin{aligned}
a_1 =& -\frac{1}{2}\left(\sigma n S_j^n\right)^2 \Delta\tau \frac{1}{\left(S_j^n - \beta\left(\tau_l\right)\right)\Delta S^n} + A S_j^n \Delta\tau \frac{1}{2\left(S_j^n - \beta\left(\tau_l\right)\right)} \\
b_1 =& 1 + r\Delta\tau - A S_j^n \Delta\tau \frac{\Delta S^n - S_j^n + \beta\left(\tau_l\right)}{2\left(S_j^n - \beta\left(\tau_l\right)\right)\Delta S^n} \\
&+ \frac{1}{2}\left(\sigma n S_j^n\right)^2 \Delta\tau \left\{\frac{S_j^n - \beta\left(\tau_l\right) + \Delta S^n}{\left(S_j^n - \beta\left(\tau_l\right)\right)\left(\Delta S^n\right)^2}\right\} \\
c_1 =& -A S_j^n \Delta\tau \frac{1}{2\Delta S^n} - \frac{1}{2}\left(\sigma n S_j^n\right)^2 \Delta\tau \frac{1}{\left(\Delta S^n\right)^2}
\end{aligned}
\tag{1.32}
$$

from equation 1.25 we can find a recurrence relation like this:

$$
a_i P_{j-1}^l + b_i P_j^l + c_i P_{j+1}^l = P_j^{l-1} \tag{1.33}
$$

where

$$
\begin{aligned}
&a_i = \mu_i - \lambda_i, b_i = (1 + r\Delta\tau) + 2\lambda_i, c_i = -\left(\mu_i + \lambda_i\right) \\
&\mu_i = A\tfrac{\Delta\tau}{2\Delta S^n} S_{j+i-1}^n, \lambda_i = \tfrac{(\sigma n)^2}{2(\Delta S^n)^2}\Delta\tau\left(S_{j+i-1}^n\right)^2, 2 \le i \le M - j + 1
\end{aligned}
\tag{1.34}
$$

Two boundary conditions have to be implemented. The Dirichlet boundary condition at $\beta(\tau_l)$ gives $K - \beta(\tau_l)$. $\beta(\tau_l)$ and we apply the following linear boundary condition at S_M^n.

$$
\frac{\partial^2 P}{\partial\left(S^n\right)^2}\left(\tau, S_M^n\right) = 0, 0 \le \tau \le T \tag{1.35}
$$

It assumes that the second derivative of the option value with respect to the underlying asset price S^n vanishes to zero for a large value of the asset price. Thus,

$$
\frac{P_{M-2} - 2P_{M-1} + P_M}{\left(\Delta S^n\right)^2} = 0 \tag{1.36}
$$

Therefore we can observe a set of simultaneous equations which could be solved using matrix of from W$\mathbf{x} = \mathbf{y}$.

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & b_1 & c_1 & \cdots & 0 & 0 \\ 0 & a_2 & b_2 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{M-j} & b_{M-j} & c_{M-j} \\ 0 & 0 & \cdots & 0 & a_{M-j+1} & b_{M-j+1} \end{pmatrix} \begin{pmatrix} P_\beta^l \\ P_j^l \\ P_{j+1}^l \\ \vdots \\ P_{M-1}^l \\ P_M^l \end{pmatrix} = \begin{pmatrix} K-\beta(\tau_l) \\ P_j^{l-1} \\ P_{j+1}^{l-1} \\ \vdots \\ P_{M-1}^{l-1} \\ P_M^{l-1} \end{pmatrix} \tag{1.37}$$

where

$$a_{M-j+1} = 2\mu_{M-j+1}, b_{M-j+1} = 1 + r\Delta\tau - 2\mu_{M-j+1} \tag{1.38}$$

To solve this system, we verify if the coefficient matrix W is invertible. In (1.34), for the sake of simplicity we divide both sides by τ. By following Proposition given below, we verify the existence and uniqueness of the system of equations in (1.34).

Definition: An n $\times$ n matrix is said to be strictly diagonally dominant when

$$|a_{ss}| > \sum_{t=1,s\neq t}^{n} |a_{st}| \tag{1.39}$$

holds for s = 1,2,...,n, where a_{st} denotes the entry in the sth row and tth column.

Proposition: If

$$\frac{1}{\Delta\tau} + r > A\tilde{C}(M+1) \tag{1.40}$$

then W is strictly diagonally dominant, where $\tilde{C}$ is a constant. Therefore, the linear system (1.34) has a unique solution.

By some complex mathematical formulations we can show that W in equation (1.34) is strictly diagonally dominant and it can be shown that every strictly diagonally dominant matrix is invertible.

Lemma: Following conditions are equivalent on the n $\times$ n square matrix W.

(1) *The matrix W is invertible.*

(2) *The linear system $W\boldsymbol{x} = \boldsymbol{y}$ is consistent for every $\boldsymbol{y}$.*

(3) *The linear system $W\boldsymbol{x} = \boldsymbol{y}$ has a unique solution for every $\boldsymbol{y}$.*

Hence, by above lemma, we prove the existence and uniqueness of the solution to (1.34). The system can be solved by finding the inverse matrix W^{-1} . Therefore, we can obtain the option value for the next time level.

Step 4 : If there are spatial grids, where the option value is not calculated among them over the optimal exercise boundary, then we obtain the put option value by extrapolation.

Step 5 : We repeat the previously mentioned process (from Steps 1 to 4) until τ_L and obtain the optimal exercise boundary in a time-recursive manner.

1.5 Numerical experiments

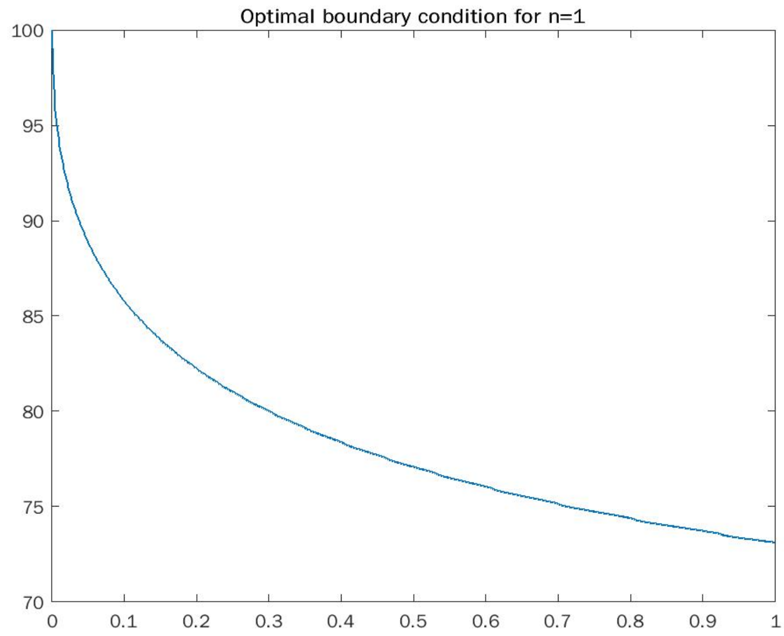

Figure 1.2: Optimal boundary condition for n = 1

We consider the strike price K = 100, the interest rate r = 0.1, the volatility $\sigma = 0.3$, and the time to maturity T = 1 (year). Further, we construct the computational domain with 250 spatial steps and 1000 time steps. At the maturity time, the optimal exercise boundary theoretically is $\beta(\mathrm{T}) = 76.0964$, whereas it is $\beta(\mathrm{T}) = 73.1258$ in the proposed method. The asymptotically optimal exercise boundary with respect to the perpetual American power put option is $\beta(\infty) = 68.9655$.

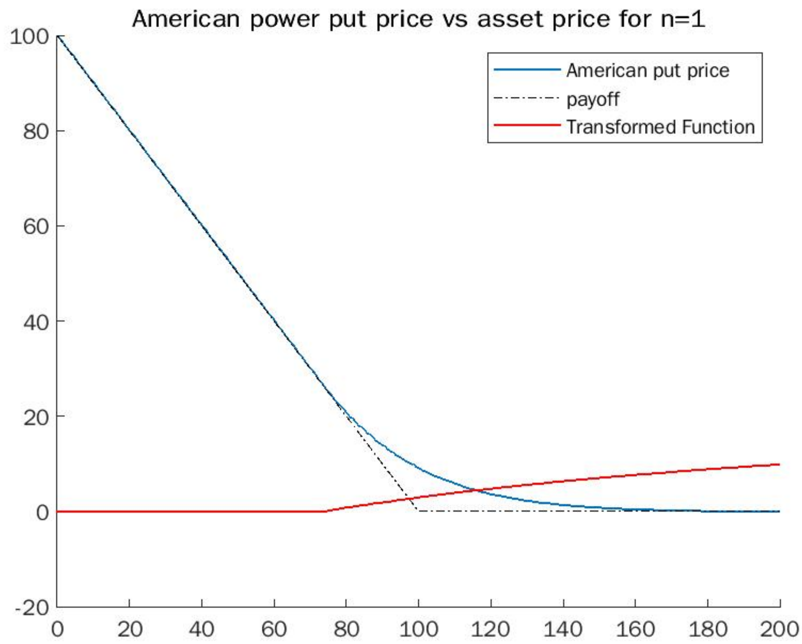

Figure 1.3: American power put price vs asset price for n = 1

Fig. 1.3 show the option values. The parameter values used to calculate the optimal exercise boundary and values of the American put options are r = 0.1, σ = 0.3, K = 100, T = 1 and n = 1 and the computational domain can be constructed with 250 spatial steps and 1000 time steps.

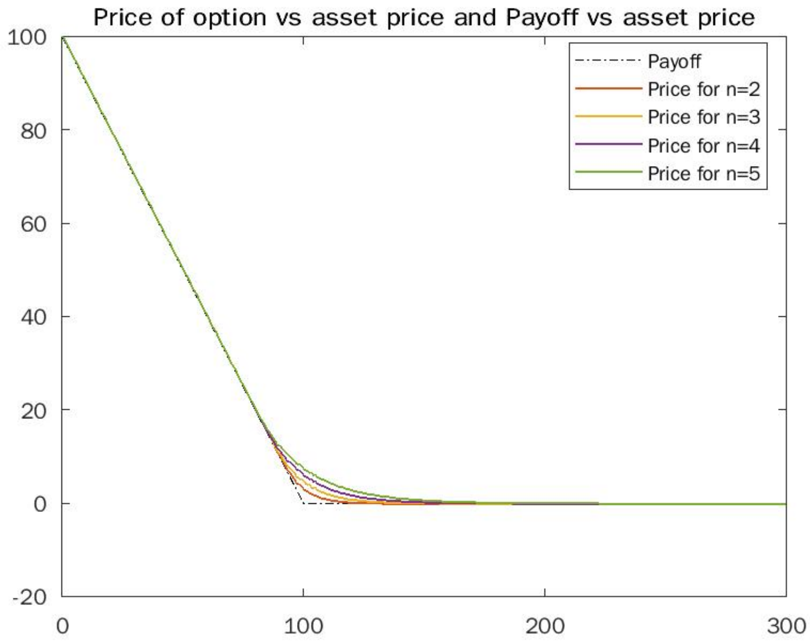

Figure 1.4: Price of option vs asset price and Payoff vs asset price

Fig. 1.4 demonstrates the American power put option values obtained when power n = 2, 3, 4, 5. The American power put option values increase with the increasing value of power n. The feature of nonlinear payoffs of power options provides the buyer with a potential to receive a considerably higher payoff than that received from a vanilla option.

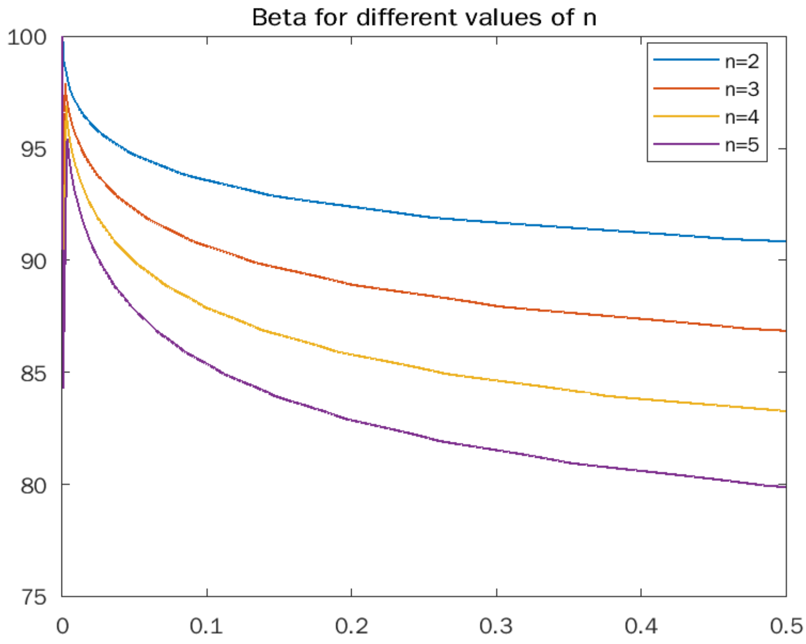

Figure 1.5: Beta for different values of n

We use $\sigma = 0.1$, r = 0.08, K = 100,T = 0.5, and L $\times$ M = 2000 $\times$ 300. Fig. 5 reports the optimal exercise boundary for different values of power n = 2, 3, 4, 5. Further, the optimal exercise boundaries are plotted as a function of time τ. The optimal exercise boundaries decrease with the increasing power.

1.6 Conclusion

We present a numerical technique and examples for pricing the American power put option for non-dividend yields in this research. Using the transformed function, we can also immediately find the optimal exercise boundary. Financial markets may be adequately comprehended in such a quickly changing environment by adopting a fast and efficient numerical technique. To offer appropriate numerical analysis, we compared the other findings to the results produced using the suggested approach for valuing the American power put option when n = 1. Furthermore, the numerical experiments carried out with the suggested technique show stability and good convergence with the linear rate.

In addition, for different values of power n, we present the best exercise boundary and American power put option pricing. This is due to the leverage characteristic of the power option, which allows for a slight change in the underlying of the power option to significantly impact the price of the power option. The numerical experiment shows that the suggested technique is accurate, adaptable, and efficient, and that it offers accurate prices in relation to the crucial stock price for different parameter combinations.