Lecture-2: Variational Formulation
MA 573: Finite Element Methods for PDEs

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In the previous lecture, we have introduced notion for weak derivatives. Then, we have used weak derivative to introduce Hilbertian Sobolev spaces H^m satisfying following inclusion relation

$$L^2(\Omega) \supseteq H^1(\Omega) \supseteq H^2(\Omega) \supseteq \ldots \supseteq H^m(\Omega).$$

In this lecture, we will introduce variational formulation for boundary value problem, which is the first step towards the finite element approximation.

1 Introduction

We first introduce variational formulation for a simple ODE

$$-y'' + y = f, \quad x \in \Omega = (a, b) \tag{1.1}$$

with homogeneous Dirichlet boundary condition

$$y(a) = 0 \;\;\&\;\; y(b) = 0. \tag{1.2}$$

For $f \in L^2(\Omega)$, we assume that $y \in H$ is the solution for the BVP (1.1)-(1.2). Here, H is a suitable function space, which need to be introduced.

For the variational formulation, we multiply the equation (1.1) by $v \in V$ and then integrate from a to b to have

$$\int_a^b \big\{ -y'' + y \big\} v dx = \int_a^b f v dx.$$

Integration by parts, we obtain

$$-\big[y'v\big]_a^b + \int_a^b \big\{y'v' + yv\big\} dx = \int_a^b f v dx. \tag{1.3}$$

Note that we don't have the information of y' at a and b, therefore, we set $v(a) = 0 = v(b)$ so that (1.4) yields

$$\int_a^b \big\{y'v' + yv\big\} dx = \int_a^b f v dx \quad \text{OR} \quad A(y, v) = (f, v), \tag{1.4}$$

where $A(y, v) = \int_a^b \big\{y'v' + yv\big\} dx \;\;\&\;\; (f, v) = \int_a^b f v dx$.

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Remark 1.1 (**Multiplier Space**) Observe, due to Chaucy-Schwarz inequality, that

$$\begin{aligned}
|A(y,v)| &\leq \left[\int_a^b |y'|^2 dx\right]^{\frac{1}{2}} \left[\int_a^b |v'|^2 dx\right]^{\frac{1}{2}} + \left[\int_a^b |y|^2 dx\right]^{\frac{1}{2}} \left[\int_a^b |v|^2 dx\right]^{\frac{1}{2}} \\
&= \|y'\|_{L^2(\Omega)} \|v'|_{L^2(\Omega)} + \|y|_{L^2(\Omega)} \|v|_{L^2(\Omega)} \\
&\leq \big(\|y'\|_{L^2(\Omega)} + \|y\|_{L^2(\Omega)}\big)\big(\|v'|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\big) \\
&= \|y\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}. \qquad (1.5)
\end{aligned}$$

Clearly, $|A(y,v)| < \infty$ for y and v belongs to $H^1(\Omega)$. Again $y(a) = 0 = y(b)$ and $v(a) = 0 = v(b)$ that is y and v vanishes on the boundary of domain $\Omega = (a,b)$. Therefore, we assume $v \in H_0^1(\Omega)$ and hence, multiplier space $V = H_0^1(\Omega)$. Next question arises whether $y \in H_0^1(\Omega)$ and satisfies (1.4). More precisely, we need to discuss the existence and uniqueness of the problem: *Find* $y \in H_0^1(\Omega)$ *such that*

$$A(y,v) = L(v) \;\; \forall v \in H_0^1(\Omega), \qquad (1.6)$$

where $A : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ *and* $L : H_0^1(\Omega) \to \mathbb{R}$ *are functions defined by*

$$A(y,v) = \int_a^b \big\{y'v' + yv\big\} dx \;\; \& \;\; L(v) = (f,v) = \int_a^b fv dx.$$

Remark 1.2 *Problem defined by (1.6) is known as variational formulation (problem) or weak formulation or integral formulation of the original BVP (1.1)-(1.2). Original equation (1.1) demands solution y to be twice differentiable, but variational formulation (1.6) requires first derivative of solution y. Due to this weaker requirement of the derivative, solution y satisfying (1.6) is called* **weak solution** *and the formulation is called weak formulation.*

Remark 1.3 *Clearly, a solution to the BVP (1.1)-(1.2) is also a solution to weak formulation (1.6). What about the converse? Converse is not always true that is a weak solution need not be a solution to the BVP (1.1)-(1.2) because original equation requires higher derivative of the solution. Therefore, a solution to the equation (1.1) is called* **strong solution.**

Remark 1.4 *For a continuous function* f*, we know that there exists an unique* $y \in C^2[a,b]$ *satisfying BVP (1.1)-(1.2). For* $f \in L^2(\Omega)$*, we do not know the existence theory for the BVP (1.1)-(1.2).* **So, we first try to ensure existence of a weak solution and then through that weak solution we try to prove existence of a strong solution.**

Existence of a weak solution requires a celebrated result known as **Lax-Milgram Lemma**, which is an extension of Riesz representation lemma. Before, we proceed further, let us introduce few basic definitions.

Definition 1.1 *Let* H *be a normed linear space with norm* $\|\cdot\|_H$*. A map* $A : H \times H \to \mathbb{R}$ *is said to be:*

(a) **Bi-linear** *if* A *is linear in both the arguments, that is, for any scalars* c_1 *and* c_2*, we have*

$$\begin{aligned}
&T(c_1 w_1 + c_2 w_2, v) = c_1 T(w_1, v) + c_2 T(w_2, v) \;\; \forall w_1,\; w_2,\; v \in H \, and \\
&T(w, c_1 v_1 + c_2 v_2) = c_1 T(w, v_1) + c_2 T(w, v_2) \;\; \forall w,\; v_1,\; v_2 \in H.
\end{aligned}$$

(b) **Continuous/Bounded** *if there exists a constant $C > 0$ such that*

$$|A(u,v)| \leq C\|u\|_H \|v\|_H \ \forall u,\ v \in H.$$

(c) **Positive/Coercive** *if there exists a constant $M > 0$ such that*

$$A(u,u) \geq M\|u\|_H^2 \ \ \forall u \in H.$$

Definition 1.2 *Let H be a normed linear space with norm $\|\cdot\|_H$. A map $L : H \to \mathbb{R}$ is said to be continuous/bounded if there exists a positive constant m such that*

$$|L(v)| \leq m\|v\|_H \ \ \forall v \in H.$$

Now, it is a good time to recall the following Lax-Milgram theorem which tells about the existence of a unique weak solution.

Theorem 1.1 *For a given Hilbert space H, a continuous, coercive bilinear map $A(.,.) : H \times H \to \mathbb{R}$ and a continuous linear functional $L : H \to \mathbb{R}$, there exists an unique $y \in H$ such that*

$$A(y,v) = L(v) \ \ \forall v \in H.$$

It is easy to verify that map $A(\cdot,\cdot)$ defined by (1.6) is bi-linear. Again, from (1.5), it is clear that $A(\cdot,\cdot)$ is continuous in $H^1(\Omega)$ and so it is continuous in $H_0^1(\Omega) \subset H^1(\Omega)$. Further, we observe that

$$\begin{aligned} A(v,v) &= \int_a^b |v'|^2 dx + \int_a^b |v|^2 dx \\ &= \|v'\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 \\ &= \|v\|_{H^1(\Omega)}^2 \geq m\|v\|_{H^1(\Omega)}^2, \ \ m = \frac{1}{2}. \end{aligned} \tag{1.7}$$

Thus, $A(\cdot,\cdot)$ is positive in $H^1(\Omega)$ and so it is positive in $H_0^1(\Omega) \subset H^1(\Omega)$. For the operator L defined in (1.6), we find that

$$\begin{aligned} |L(v)| = \Big| \int_a^b fv dx \Big| &\leq \Big(\int_a^b f^2 dx \Big)^{\frac{1}{2}} \Big(\int_a^b v^2 dx \Big)^{\frac{1}{2}} \\ &= \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq C\|v\|_{H^1(\Omega)}, \ C = \|f\|_{L^2(\Omega)} < \infty. \end{aligned} \tag{1.8}$$

Here, we have used the facts that $f \in L^2(\Omega)$ and $\|v\|_{L^2(\Omega)} \leq \|v\|_{H^1(\Omega)}$.

Remark 1.5 (**Weak Solution Space**) All assumptions in Lax-Milgram Lemma are satisfied in $H_0^1(\Omega)$ by the bilinear map $A(\cdot,\cdot)$ and operator L defined as in (1.6). Therefore, there exists unique $y \in H_0^1(\Omega)$ such that

$$A(y,v) = L(v) \ \ \forall v \in H_0^1(\Omega).$$

Hence, weak solution corresponding to the original BVP (1.1)-(1.2) belongs to $H_0^1(\Omega)$. **If we somehow know that strong solution to the BVP (1.1)-(1.2) exists.** Let y be the strong solution. Then the strong solution will be a weak solution and it will belong to function space $H_0^1(\Omega)$. For the strong solution y, from (1.1), we have

$$y'' = f - y' \in L^2(\Omega) \ \text{ since } f \in L^2(\Omega) \ \ \& \ \ y \in H_0^1(\Omega).$$

Thus strong solution, if it exists, belongs to the space $H^2(\Omega) \cap H_0^1(\Omega)$. **Have we proved that strong solution to the BVP (1.1)-(1.2) exists? Not yet!** *We proceed as discussed in Remark 1.4.*

For $f \in L^2(\Omega)$, any function $y \in H_0^1(\Omega)$ satisfying variational problem (1.6) belongs to function space $H^2(\Omega)$. This result is known as **Elliptic Regularity** theorem. That is a weak solution becomes twice differentiable. Then from (1.6) and integration by parts, we obtain

$$\Big[y'v\Big]_a^b - \int_a^b \Big\{y'' - y\Big\} v dx = \int_a^b f v dx \ \ \forall v \in H_0^1(\Omega).$$

Using the fact that $v(a) = 0 = v(b)$, we have

$$\int_a^b \Big\{ -y'' + y\Big\} v dx = \int_a^b f v dx \ \ \forall v \in H_0^1(\Omega).$$

Equivalently

$$\int_a^b \Big\{ -y'' + y - f\Big\} v dx = 0 \ \ \forall v \in H_0^1(\Omega).$$

Due to arbitrariness of v, we find that

$$-y'' + y - f = 0.$$

Hence, a weak solution is also a strong solution.

Remark 1.6 *In general, a weak solution need not be a strong solution. But, in the present example we have seen that a weak solution is also a strong solution.* **Now onwards, we will assume that original BVP and its weak formulation are equivalent.**

Example 1.1 For $f \in L^2(\Omega)$, consider following BVP

$$-y''(x) + P(x)y'(x) + Q(x)y(x) = f(x), \ \ x \in \Omega$$

with boundary condition

$$y(a) = \alpha \ \ \& \ \ y(b) = \beta.$$

Here, coefficient functions P and Q are sufficiently smooth function. For suitable choice of P and Q, justify that above BVP has an unique weak solution.

Solution. Consider the given BVP

$$-y''(x) + P(x)y'(x) + Q(x)y(x) = f(x), \ \ x \in \Omega \tag{1.9}$$

with boundary condition

$$y(a) = \alpha \ \ \& \ \ y(b) = \beta. \tag{1.10}$$

For weak formulation, we multiply equation (1.9) by suitable v and then integrate in the interval $[a, b]$ to obtain

$$\int_a^b \Big\{ -y''(x) + P(x)y'(x) + Q(x)y(x)\Big\} v dx = \int_a^b f v dx.$$

Integration by parts yields

$$-\Big[y'v\Big]_a^b + \int_a^b y'v'dx + \int_a^b Py'vdx + \int_a^b Qyvdx = \int_a^b fvdx. \tag{1.11}$$

Here, $y'(a)$ and $y'(b)$ are not given, so set $v(a) = 0 = v(b)$ in (1.11) to have

$$A(y, v) = L(v), \tag{1.12}$$

where

$$A(y, v) = \int_a^b \Big\{y'v' + Py'v + Qyv\Big\}dx \;\; \& \;\; L(v) = \int_a^b fvdx. \tag{1.13}$$

Further, it is easy exercise to verify that $A(\cdot, \cdot)$ is bilinear and $L(\cdot)$ is linear.

Next, we try to fix the multiplier and weak solution space. Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|A(w, v)| &\leq \|y'\|_{L^2(\Omega)}\|v'\|_{L^2(\Omega)} + \|p\|_\infty\|y'\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\quad + \|Q\|_\infty\|y\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\leq \|y'\|_{L^2(\Omega)}\|v'\|_{L^2(\Omega)} + m_1\|y'\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + m_2\|y\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\leq \max\{1, m_1, m_2\}\Big(\|y'\|_{L^2(\Omega)} + \|y\|_{L^2(\Omega)}\Big)\Big(\|v'\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\Big) \\
&\leq C\|y\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.
\end{aligned} \tag{1.14}$$

For the operator $L(\cdot)$, we have

$$|L(v)| \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq C\|v\|_{H^1(\Omega)}, \;\; C = \|f\|_{L^2(\Omega)}. \tag{1.15}$$

From (1.14) and (1.15), we find that $A(\cdot, \cdot)$ and $L(\cdot)$ are well defined in $H^1(\Omega) \times H^1(\Omega)$ and $H^1(\Omega)$, respectively. More precisely, $A : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ and $L : H^1(\Omega) \to \mathbb{R}$, defined by (1.13), are continuous in $H^1(\Omega)$. Thus, multiplier $v \in H_0^1(\Omega)$ as $v(a) = 0 = v(b)$. But, $y \in H^1(\Omega)$ not $y \in H_0^1(\Omega)$ as $y(a) \neq 0$ and $y(b) \neq 0$. Hence, multiplier space is $H_0^1(\Omega)$ and weak solution space is $H^1(\Omega)$.

In order to use Lax-Milgram theorem, we try to convert the weak solution space to $H_0^1(\Omega)$ by a meaningful way. For given α and β, we find a linear polynomial g such that $g(a) = \alpha$ and $g(b) = \beta$. Then the function

$$w = y - g \in H_0^1(\Omega)$$

and satisfies following equation

$$A(w + g, v) = L(v), \tag{1.16}$$

which gives

$$A(w, v) = L(v) - A(g, v) = \tilde{L}(v) \;\; \forall v \in H_0^1(\Omega), \tag{1.17}$$

where $\tilde{L}(v) = L(v) - A(g, v)$. It is worth to note that existence of $w \in H_0^1(\Omega)$ satisfying (1.17) gives existence of weak solution $y = w + g$ satisfying (1.12). **So, we concentrate on the existence of w satisfying (1.17).**

From (1.14), we observe that bilinear map $A(\cdot, \cdot)$ is continuous in $H_0^1(\Omega)$. Now, it remains to verify positivity of bilinear map $A(\cdot, \cdot)$. Integration by parts, we obtain

$$\int_a^b Pv'vdx = \int_a^b P\frac{1}{2}\frac{d}{dx}\Big(v^2\Big)dx \;\; = \;\; \Big[\frac{P}{2}v^2\Big]_a^b - \int_a^b \Big(\frac{P}{2}\Big)' v^2dx.$$

For $v \in H_0^1(\Omega)$, we obtain

$$\int_a^b Pv'vdx = -\int_a^b \Big(\frac{P}{2}\Big)' v^2 dx,$$

which together with definition of $A(\cdot,\cdot)$, for all $v \in H_0^1(\Omega)$, leads to

$$\begin{aligned} A(v,v) &= \int_a^b \Big\{\Big(v'\Big)^2 + \Big(Q - \frac{P'}{2}\Big)v^2\Big\}dx \\ &= \int_a^b \Big\{\Big(v'\Big)^2 + Hv^2\Big\}dx, \;\; H = Q - \frac{P'}{2} \\ &\geq \int_a^b \Big(v'\Big)^2 dx, \quad \text{assuming } H = Q - \frac{P'}{2} \geq 0 \\ &\geq C\|v\|^2_{H^1(\Omega)}, \quad \text{due to Poincaré inequality.} \end{aligned} \tag{1.18}$$

Hence, $A(\cdot,\cdot)$ is positive in $H_0^1(\Omega)$. For **Poincaré inequality**, see equation (2.7).

For the continuity of $\tilde{L}$, we use (1.14)-(1.15) to have

$$\begin{aligned} |\tilde{L}(v)| \leq |L(v)| + |A(g,v)| &\leq C\|v\|_{H^1(\Omega)} + C\|g\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \\ &\leq \tilde{C}\|v\|_{H^1(\Omega)}. \end{aligned} \tag{1.19}$$

Thus, $\tilde{L}$ is linear and continuous in $H_0^1(\Omega)$, which together with the fact that bilinear map $A(\cdot,\cdot)$ is continuous and positive in $H_0^1(\Omega)$ yields that there exists unique $w \in H_0^1(\Omega)$ such that

$$A(w,v) = \tilde{L}(v) \;\; \forall v \in H_0^1(\Omega).$$

Finally, existence of w gives existence of $y = w + g \in H^1(\Omega)$ satisfying (1.12) and boundary condition $y(a) = \alpha$ and $y(b) = \beta$.

2 Variation Formulation for Poisson's Equation

In this section, we consider a linear elliptic problem of the form

$$-\nabla.(\nabla u(x,y)) = f(x,y) \quad \text{in } \Omega \tag{2.1}$$

subject to the boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

The above problem is also called Dirichlet boundary value problem. The equations of the form (2.1)-(2.2) are often encountered in stationary heat conduction problems, material sciences and fluid dynamics. As a model problem, we consider the stationary heat conduction problem in Ω. The smoothness of the solution u, depends on the smoothness of the given data f which is known as **elliptic regularity** of the solution. Concerning the problem (2.1)-(2.2), we have the following regularity result.

Theorem 2.1 Regularity Result *: For any $f \in L^2(\Omega)$, the Dirichlet problem has an unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and satisfies the following stability estimate*

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

As a first step towards the finite element approximation to (2.1)-(2.2), we first introduce the weak formulation. The following Green's formula is borrowed from Lecture-1 for our convenience.

$$\int_\Omega \Delta uvdx = -\int_\Omega \nabla u \cdot \nabla udx + \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} vds, \tag{2.3}$$

where $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$ is the outward normal derivative of u on $\partial\Omega$. Let $v \in {H_0}^1(\Omega)$, then multiplying the given elliptic problem by v and further using Green's theorem, we have

$$\int_\Omega \nabla u.\nabla vdx - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} vds = \int_\Omega fvdx. \tag{2.4}$$

Since the information of $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$ is not given on the boundary $\partial\Omega$, we select $v = 0$ on $\partial\Omega$, so that equation (2.4) reduces to

$$A(u, v) = L(v), \tag{2.5}$$

where

$$A(u, v) = \int_\Omega \nabla u \nabla vdx \text{ and } L(v) = \int_\Omega fvdx.$$

Remark 2.1 *The equation (2.5) involve only the first order partial derivative of u where as the original equation demands second order partial derivative of u. Therefore, (2.5) is called weak form of the original equation. Since the BVP (2.1)-(2.2) has a strong solution u, so it is also a weak solution. Now, we confirm that such weak solution satisfying (2.5) is unique.*

Applying Cauchy-Schwartz inequlity, we have

$$\begin{aligned}
|A(u,v)| &\leq \int_\Omega |\nabla u||\nabla v|dx \\
&\leq \Big(\int_\Omega |\nabla u|^2 dx\Big)^{\frac{1}{2}} \Big(\int_\Omega |\nabla v|^2 dx\Big)^{\frac{1}{2}} \\
&\leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\
&\leq (\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)})(\|\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}) \\
&\leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}.
\end{aligned} \tag{2.6}$$

Thus, $|A(u, v)| < \infty$ provided $u,\ v \ \in H^1(\Omega)$. Therefore $A(.,.) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by

$$A(u, v) = \int_\Omega \nabla u.\nabla vdxdy$$

is well defined. Further, it is easy to verify that $L(\cdot)$ is linear in $H^1(\Omega)$ and satisfies the following estimate

$$\begin{aligned}
|L(v)| \leq \Big(\int_\Omega |f|^2 dx\Big)^{\frac{1}{2}} \Big(\int_\Omega |v|^2 dx\Big)^{\frac{1}{2}} &\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\leq \|f\|_{L^2(\Omega)} (\|\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}) \\
&\leq C\|v\|_{H^1(\Omega)}, \ \ C = \|f\|_{L^2(\Omega)} < \infty.
\end{aligned}$$

Clearly, $L(\cdot)$ is well defined linear map in $H^1(\Omega)$. In fact, L is continuous linear map in $H^1(\Omega)$.

Regarding $A(\cdot,\cdot)$, we have the following observations:

Observation 1: For any scalars α and β, we have

$$A(\alpha w_1 + \beta w_2, v) = \alpha A(w_1, v) + \beta A(w_2, v)$$

that is $A(\cdot,\cdot)$ is bilinear map.

Proof.: For any scalars α and β, we have

$$\begin{aligned} A(\alpha w_1 + \beta w_2, v) &= \int_\Omega \nabla(\alpha w_1 + \beta w_2).\nabla v dx \\ &= \int_\Omega (\alpha \nabla w_1 + \beta \nabla w_2).\nabla v dx \\ &= \int_\Omega \alpha \nabla w_1.\nabla v dx + \int_\Omega \beta \nabla w_2.\nabla v dx \\ &= \alpha \int_\Omega \nabla w_1.\nabla v dx + \beta \int_\Omega \nabla w_2.\nabla v dx = \alpha A(w_1, v) + \beta A(w_2, v). \end{aligned}$$

Therefore $A(\cdot,\cdot)$ is a linear map in first argument w. Similarly, it can be shown that $A(\cdot,\cdot)$ is also linear in second argument v. This completes the rest of the proof. □

Observation 2: The bilinear map $A(\cdot,\cdot)$ is continuous in $H^1(\Omega)$.

Proof.: The bilinear map $A(\cdot,\cdot)$ is said to be continuous in $H^1(\Omega)$ if there exists a constant C such that

$$|A(u, v)| \leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)},$$

which is true due to (2.6). Therefore, $A(\cdot,\cdot)$ is continuous in $H^1(\Omega)$. □

Remark 2.2 *We know that* $\int_\Omega \left(\frac{dw}{dx}\right)^2 dx$ *can not leads to norm, due to the fact that* $\int_\Omega \left(\frac{dw}{dx}\right)^2 dx = 0$ *does not imply that* $w = 0$. *For instance, set* $w = 1 \neq 0$, *but,* $\int_\Omega \left(\frac{dw}{dx}\right)^2 dx = 0$. *In this regards, the Sobolev space* $H_0^1(\Omega) = \Big\{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\Big\}$ *is special due one of the most important inequality, know as* **Poincaré inequality.**

Lemma 2.1 (**Poincaré Inequality**) For any $w \in H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^d$, following inequality

$$\|w\|_{H^1(\Omega)} \leq C \|\nabla w\|_{L^2(\Omega)}, \tag{2.7}$$

holds for some constant $C > 0$.

Observation 3: Bilinear map $A(\cdot,\cdot)$ is positive in $H_0^1(\Omega)$. More precisely, for any $w \in H_0^1(\Omega)$, we have

$$A(w, w) \geq C \|w\|^2_{H^1(\Omega)},$$

for some cosntant $C > 0$.

Proof.: Using Poincare's inequality, we have

$$\begin{aligned} A(w, w) &= \int_\Omega |(\nabla w)(\nabla w)| dx = \int_\Omega |\nabla w|^2 dx = \|\nabla w\|^2_{L^2(\Omega)} \\ &\geq C \|w\|^2_{H^1(\Omega)}. \quad \square \end{aligned}$$

From the above discussion, we find that $A : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ and $L : H_0^1(\Omega) \to \mathbb{R}$ satisfies all assumptions of Lax-Milgram lemma. Therefore, **there exists unique** $u \in H_0^1(\Omega)$ **such that**

$$A(u, v) = L(v) \;\; \forall v \in H_0^1(\Omega).$$

Remark 2.3 *Suppose* u_1 *and* u_2 *are strong solutions for the BVP (2.1)-(2.2). Hence, they are also weak solutions and satisfies equation (2.5). But, weak solution to the BVP (2.1)-(2.2) is unique. So, we must have* $u_1 = u_2$.

Example 2.1 (Homework) Suppose u is a strong solution for the BVP

$$-\Delta u + u = f \;\text{ in } \Omega \;\;\&\;\; u = 0 \;\text{ on } \partial\Omega,$$

where $f \in L^2(\Omega)$ and $\Omega \subset \mathbb{R}^2$. Determine the weak formulation and justify that weak solution is unique. Further, show that the strong solution is unique and belongs to the function space $H^2(\Omega) \cap H_0^1(\Omega)$.