Theory Assignment for Module 1 & 2

Theory - Team 8

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1 Question 1

Proof. □

2 Question 2

Proof. Let V be any n×n invertible matrix. A is any $n \times n$ matrix and a scaler γ

$$\delta A = V^{-1}(fl(VA) - VA)$$

$$\implies \delta A_j = V^{-1}\Big((fl(VA) - VA)_j\Big) \tag{1}$$

$$\implies |\delta A_j| = |V^{-1}\Big((fl(VA) - VA)_j\Big)|$$

(Using basic matrix inequalities)

$$\implies |\delta A_j| \leq |V^{-1}||\Big((fl(VA) - VA)_j\Big)|$$

$$(fl(VA) - VA)ij = fl(\sum k = 1^n v_{ik}a_{kj}) - \sum_{k=1}^{n} v_{ik}a_{kj}$$

$fl(\sum_{k=1}^{n} v_{ik}a_{kj}) = \sum_{k=1}^{n} v_{ik}a_{kj}(1 + \gamma_{ijk})$ where $|\gamma_{ijk}| \leq nu + O(u^2)$. Hence

$$(fl(VA) - VA)ij = \sum k = 1^n v_{ik}a_{kj}(1 + \gamma_{ijk}) - \sum_{k=1}^{n} v_{ik}a_{kj}$$

$$\implies (fl(VA) - VA)ij = \sum k = 1^n \gamma_{ijk}v_{ik}a_{kj}$$

$$\implies (fl(VA)-VA)_j = \begin{bmatrix} \sum_{k=1}^{n} \gamma_{1k} v_{1k} a_{kj} \\ \sum_{k=1}^{n} \gamma_{2k} v_{2k} a_{kj} \\ \vdots \\ \sum_{k=1}^{n} \gamma_{nk} v_{nk} a_{kj} \end{bmatrix}$$

Define $\gamma = \max_{1 \leq i,j,k \leq n} \gamma_{ijk}$

$$\implies |(fl(VA)-VA)e_j| = |\begin{bmatrix} \sum_{k=1}^{n} \gamma_{1k} v_{1k} a_{kj} \\ \sum_{k=1}^{n} \gamma_{2k} v_{2k} a_{kj} \\ \vdots \\ \sum_{k=1}^{n} \gamma_{nk} v_{nk} a_{kj} \end{bmatrix}| \leq \gamma |\begin{bmatrix} \sum_{k=1}^{n} v_{1k} a_{kj} \\ \sum_{k=1}^{n} v_{2k} a_{kj} \\ \vdots \\ \sum_{k=1}^{n} v_{nk} a_{kj} \end{bmatrix}|$$

$$\begin{bmatrix} \sum_{k=1}^{n} v_{1k} a_{kj} \\ \sum_{k=1}^{n} v_{2k} a_{kj} \\ \vdots \\ \sum_{k=1}^{n} v_{nk} a_{kj} \end{bmatrix} = VA_j \implies |(fl(VA)-VA)e_j| \leq \gamma |VA_j| \leq \gamma |V||A_j|$$

$$\implies |\delta A_j| \leq \gamma |V^{-1}||V||A_j|$$

Hence, proved

$\square$

3 Question 3

Proof. For Gaussian elimination with partial pivoting, the permutation matrix P will not change $\max_{i,j}|a_{ij}|$.

Hence, we can denote PA $= (a_{ij}^{(0)})_{mxm}$ where $\max_{i,j}|a_{ij}^{(0}| = \max_{i,j}|a_{ij}|$

Apply 1 step of Gaussian elimination to P A :

$$\begin{pmatrix} a_{11}^{(0)} & a_{12}^{(0)} \dots a_{1m}^{(0)} \\ a_{21}^{(0)} & a_{22}^{(0)} \dots a_{2m}^{(0)} \\ \vdots & \vdots \ddots \vdots \\ a_{m1}^{(0)} & a_{m2}^{(0)} \dots a_{mm}^{(0)} \end{pmatrix} \implies \begin{pmatrix} a_{11}^{(0)} & a_{12}^{(0)} \dots a_{1m}^{(0)} \\ 0 & a_{22}^{(0)} \dots a_{2m}^{(0)} \\ \vdots & \vdots \ddots \vdots \\ 0 & a_{m2}^{(0)} \dots a_{mm}^{(0)} \end{pmatrix}$$

where the entries $a_{ij}^{(1)} = a_{ij}^{(0)} - \dfrac{a_{i1}^{(0)}}{a_{11}^{(0)}} a_{1j}^{(0)}$

and the Gaussian elimination with partial pivoting, $|\dfrac{a_{i1}^{(0)}}{a_{11}^{(0)}}| \leq 1$

Hence, we can obtain,

$$|a_{ij}^{(1)}| \leq |a_{ij}^{(0)}| + |a_{1j}^{(0)}| \leq 2\max_{i,j}|a_{ij}^{(0)}| = 2\max_{i,j}|a_{ij}|$$

Repeat the above process, we can obtain after k steps of Gaussian elimination,

$|a_{ij}^{(k)}| \leq |a_{ij}^{(k-1)}| + |a_{kj}^{(k-1)}| \leq 2\max_{i,j}|a_{ij}^{(k-1)}|$

We need to do m - 1 steps to form U. Hence,

$|u_{ij}| = |a_{ij}^{(m-1)}| \leq |a_{ij}^{(m-2)}| + |a_{kj}^{(m-2)}| \leq 2\text{max}_{i,j}|a_{ij}^{(m-2)}| \leq 2^2\text{max}_{i,j}|a_{ij}^{(m-3)}| \leq .....$

$\leq 2^{m-1}max|a_{ij}^{(0)}| = 2^{m-1}max|a_{ij}|$

Hence, the growth factor

$\rho = \frac{\max_{i,j}|u_{ij}|}{\max_{i,j}|a_{ij}|} \leq 2^{m-1}$

For LHS

Given $A = [a_{ij}]nxn$, let $A(k) = [a_{ij}^{(k)}]_{nxn}$ is the matrix obtained after k^{th} step of Gaussian Elimination.

$$\rho(A) = \frac{\max\limits_{1\leq i,j\leq n, 1\leq k\leq n-1}|a_{ij}^{(k)}|}{\max\limits_{1\leq i,j\leq n}|a_{ij}|} \tag{1}$$

Now the total no. of steps in Gaussian elimination is n-1, let p be the step in which $\max\limits_{1\leq i,j\leq n, 1\leq k\leq n-1}|a_{ij}^{(k)}|$ is achieved i.e.

$$\max_{1\leq i,j\leq n, 1\leq k\leq n-1}|a_{ij}^{(k)}| = \max_{1\leq i,j\leq n}|a_{ij}^{(p)}| = M^{(p)}$$

$\because p \leq n-1, U$ will have entries which are less than $M^{(p)}$ and $||U||_\infty =$ max of row sums of U

$$||U||_\infty \leq nM^{(p)} \tag{2}$$

Now, $||A_\infty||$ =max of row sums of A thie implies

$$||A||\infty \geq \max_{1\leq i,j\leq n}|aij| \tag{3}$$

From (1), (2) and (3) we get

$$\frac{||U||\infty}{||A||\infty} \leq n\rho(A)$$

Thus,

$$\frac{||U||\infty}{||A||\infty} \leq n\rho(A) \leq n2^{n-1}$$

$\square$