- In order for the coefficient of $\dfrac{\partial^2 u}{\partial x^2}$ to be one, we let $\bar{\tau} = \sigma^2\tau/2$. If σ depends on t, then letting $\bar{\tau} = \frac{1}{2}\int_0^{\tau} \sigma^2(T-s)ds = \frac{1}{2}\int_t^T \sigma^2(s)ds$ can still make the coefficient of $\dfrac{\partial^2 u}{\partial x^2}$ be one.

3. From the explanation on the function of each single transformation given above, we can see that if r, D_0, and σ are not constant, but depend on t only, then the Black–Scholes equation can still be converted into a heat equation by the following transformation

$$\begin{cases} x = \ln S + \int_t^T \left[r(s) - D_0(s) - \sigma^2(s)/2\right] ds, \\ \bar{\tau} = \dfrac{1}{2}\int_t^T \sigma^2(s)ds, \\ V(S,t) = \mathrm{e}^{-\int_t^T r(s)ds} u(x, \bar{\tau}) \end{cases} \tag{2.79}$$

and the solution $V(S,t)$ possesses the following form:

$$\mathrm{e}^{-\int_t^T r(s)ds} u\left(\ln \frac{S\mathrm{e}^{-\int_t^T D_0(s)ds}}{\mathrm{e}^{-\int_t^T r(s)ds}} - \frac{1}{2}\int_t^T \sigma^2(s)ds,\ \frac{1}{2}\int_t^T \sigma^2(s)ds\right), \tag{2.80}$$

where $u(x, \bar{\tau})$ is a solution of the heat equation (see [84]). This is left for readers as an exercise (Problem 36). There, in order to see the function of each part of the transformation, readers are asked to reduce the Black–Scholes equation with time-dependent parameters to a heat equation through two steps.

4. The transformation to convert the Black–Scholes equation into a heat equation is not unique. In fact, we can let $x = \ln S$, $\bar{\tau} = \dfrac{1}{2}\sigma^2(T-t)$, $V(S,t) = \mathrm{e}^{\alpha x + \beta\bar{\tau}} u(x, \bar{\tau})$, and choose constants α and β such that $u(x, \bar{\tau})$ satisfies the heat equation (see [84]).

2.6.2 The Solutions of Parabolic Equations

In order for a parabolic differential equation to have a unique solution, one has to specify some conditions. For example, the initial value problem for a heat equation

$$\frac{\partial u}{\partial \bar{\tau}} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad \bar{\tau} \geq 0 \tag{2.81}$$

with

$$u(x, 0) = u_0(x) \tag{2.82}$$

has a unique solution under certain conditions that usually hold for cases considered in this book.

Let us find the solution of Eq. (2.81) with initial condition (2.82). The way to find the solution is not unique. Here, we use the following method (see [52]). We first try to find a special solution of Eq. (2.81) in the form

$$u(x, \bar{\tau}) = \bar{\tau}^{-1/2} U(\eta),$$

where

$$\eta = \frac{x - \xi}{\sqrt{\bar{\tau}}}, \qquad \xi \text{ being a parameter.}$$

Because

$$\begin{aligned}
\frac{\partial u}{\partial \bar{\tau}} &= -\frac{\bar{\tau}^{-3/2}}{2}\left(U + \eta \frac{dU}{d\eta}\right) = -\frac{\bar{\tau}^{-3/2}}{2}\frac{d}{d\eta}[\eta U(\eta)],\\
\frac{\partial u}{\partial x} &= \bar{\tau}^{-1/2}\frac{dU}{d\eta}\frac{1}{\sqrt{\bar{\tau}}} = \bar{\tau}^{-1}\frac{dU}{d\eta},\\
\frac{\partial^2 u}{\partial x^2} &= \bar{\tau}^{-3/2}\frac{d^2U}{d\eta^2},
\end{aligned}$$

from Eq. (2.81) we have

$$-\frac{\bar{\tau}^{-3/2}}{2}\frac{d}{d\eta}(\eta U) = \bar{\tau}^{-3/2}\frac{d^2U}{d\eta^2},$$

that is,

$$\frac{d^2U}{d\eta^2} + \frac{1}{2}\frac{d}{d\eta}(\eta U) = 0.$$

Integrating this equation, we have

$$\frac{dU}{d\eta} + \frac{\eta}{2}U = c_1,$$

where c_1 is a constant. Let us choose $c_1 = 0$, so now we have a linear homogeneous equation. The solution of this equation is

$$U(\eta) = c\mathrm{e}^{-\eta^2/4},$$

where c is a constant. Thus, for the diffusion equation we have a special solution in the form

$$c\bar{\tau}^{-1/2}\mathrm{e}^{-(x-\xi)^2/4\bar{\tau}}.$$

If we further require

$$\int_{-\infty}^{\infty} c\bar{\tau}^{-1/2}\mathrm{e}^{-(x-\xi)^2/4\bar{\tau}}d\xi = 1,$$

then

$$c = \frac{1}{\displaystyle\int_{-\infty}^{\infty} \bar{\tau}^{-1/2} \mathrm{e}^{-(x-\xi)^2/4\bar{\tau}} d\xi} = \frac{1}{\sqrt{2}\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-\eta^2/2} d\eta} = \frac{1}{2\sqrt{\pi}}$$

and the special solution is

$$\frac{1}{2\sqrt{\pi\bar{\tau}}}\mathrm{e}^{-(x-\xi)^2/4\bar{\tau}}.$$

This solution is called the fundamental solution, or Green's function, for the heat equation (2.81). Let $g(\xi; x, \bar{\tau})$ represent this class of functions with ξ as parameters. It is clear that the relation

$$\frac{\partial g(\xi; x, \bar{\tau})}{\partial \bar{\tau}} = \frac{\partial^2 g(\xi; x, \bar{\tau})}{\partial x^2}$$

holds for any ξ. Thus, for any $u_0(\xi)$ we have

$$\int_{-\infty}^{\infty} u_0(\xi) \frac{\partial g(\xi; x, \bar{\tau})}{\partial \bar{\tau}} d\xi = \int_{-\infty}^{\infty} u_0(\xi) \frac{\partial^2 g(\xi; x, \bar{\tau})}{\partial x^2} d\xi,$$

that is,

$$\frac{\partial \left[\displaystyle\int_{-\infty}^{\infty} u_0(\xi) g(\xi; x, \bar{\tau}) d\xi\right]}{\partial \bar{\tau}} = \frac{\partial^2 \left[\displaystyle\int_{-\infty}^{\infty} u_0(\xi) g(\xi; x, \bar{\tau}) d\xi\right]}{\partial x^2}.$$

Consequently,

$$u(x, \bar{\tau}) = \int_{-\infty}^{\infty} u_0(\xi) \times \frac{1}{2\sqrt{\pi\bar{\tau}}}\mathrm{e}^{-(x-\xi)^2/4\bar{\tau}} d\xi \tag{2.83}$$

is also a solution of Eq. (2.81). Because

$$\lim_{\bar{\tau}\to 0} \frac{1}{2\sqrt{\pi\bar{\tau}}}\mathrm{e}^{-(x-\xi)^2/4\bar{\tau}} = \begin{cases} 0, & x - \xi \neq 0, \\ \infty, & x - \xi = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\bar{\tau}}}\mathrm{e}^{-(x-\xi)^2/4\bar{\tau}} d\xi = 1$$

is true for any $\bar{\tau}$, we have

$$\lim_{\bar{\tau}\to 0} \frac{1}{2\sqrt{\pi\bar{\tau}}}\mathrm{e}^{-(x-\xi)^2/4\bar{\tau}} = \delta(x - \xi)$$

and

$$\lim_{\bar{\tau}\to 0} \int_{-\infty}^{\infty} u_0(\xi) \times \frac{1}{2\sqrt{\pi\bar{\tau}}}\mathrm{e}^{-(x-\xi)^2/4\bar{\tau}} d\xi = u_0(x).$$

Consequently, Eq. (2.83) is the solution of the initial-value problem

$$\begin{cases} \dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad \bar{\tau} > 0, \\ u(x, 0) = u_0(x), \ -\infty < x < \infty. \end{cases}$$

2.6.3 Solutions of the Black–Scholes Equation

Because the solution of the problem (2.77) is the expression (2.83), from the relation (2.78) we know that the solution of the final value problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0) S \dfrac{\partial V}{\partial S} - rV = 0, & 0 \le S, \quad 0 \le t \le T, \\ V(S,T) = V_T(S), & 0 \le S \end{cases}$$

is

$$\begin{aligned} V(S,t) &= \mathrm{e}^{-r(T-t)} \int_{-\infty}^{\infty} u_0(\xi) \frac{1}{2\sqrt{\pi\bar{\tau}}} \mathrm{e}^{-(x-\xi)^2/4\bar{\tau}} d\xi \\ &= \mathrm{e}^{-r(T-t)} \int_{-\infty}^{\infty} V_T\left(\mathrm{e}^{\xi}\right) \frac{1}{2\sqrt{\pi\bar{\tau}}} \mathrm{e}^{-(\xi-x)^2/4\bar{\tau}} d\xi \\ &= \mathrm{e}^{-r(T-t)} \frac{1}{\sigma\sqrt{2\pi(T-t)}} \\ &\quad \times \int_0^{\infty} V_T(S') \mathrm{e}^{-\left\{\ln S' - \left[\ln S + (r - D_0 - \sigma^2/2)(T-t)\right]\right\}^2 / 2\sigma^2 (T-t)} \frac{dS'}{S'}. \end{aligned}$$

This result can be written as

$$V(S,t) = \mathrm{e}^{-r(T-t)} \int_0^{\infty} V_T(S') G(S', T; S, t) dS', \tag{2.84}$$

where

$$\begin{aligned} &G(S', T; S, t) \\ &= \frac{1}{\sigma\sqrt{2\pi(T-t)}S'} \mathrm{e}^{-\left\{\ln S' - \left[\ln S + (r - D_0 - \sigma^2/2)(T-t)\right]\right\}^2 / 2\sigma^2 (T-t)}. \end{aligned} \tag{2.85}$$

Equations (2.84) and (2.85) are usually referred to as the general solution and Green's function of the Black–Scholes equation, respectively. From Sect. 2.1.3, we know that this function is also the probability density function for a lognormal distribution, that is, we can say that S' is a lognormal random variable and according to the result (2.6) its expectation is

$$\mathrm{E}\left[S'\right] = S\mathrm{e}^{(r-D_0)(T-t)}. \tag{2.86}$$

In order to make the expression of this function short, we rewrite it as

$$G(S', T; S, t) = \frac{1}{\sqrt{2\pi} b S'} \mathrm{e}^{-\left[\ln(S'/a) + b^2/2\right]^2 / 2b^2},$$

where

$$a = S\mathrm{e}^{(r-D_0)(T-t)} \quad \text{and} \quad b = \sigma\sqrt{T-t}.$$