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Soln (4): 1) $\frac{\partial V}{\partial t} = S\frac{\partial H}{\partial t}(R,t)$

$$\frac{\partial V}{\partial S} = H(R,t) + S\frac{\partial}{\partial S}H(R,t) = H(R,t) + S\frac{\partial H(R,t)}{\partial R}\frac{\partial R}{\partial S}$$

$$= H(R,t) + S\frac{\partial H}{\partial R}\left(-A/S^2\right)$$

$$= H(R,t) - \frac{R\partial H}{S\cdot\partial R}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial H}{\partial R}\times\frac{\partial R}{\partial S} - \frac{\partial R}{\partial S}\frac{\partial H}{\partial R} - R\frac{\partial^2 H}{\partial S\partial R}$$

$$= -A/S^2\frac{\partial H}{\partial R} + A/S^2\frac{\partial H}{\partial R} + A^2/S^3\frac{\partial^2 H}{\partial R^2}$$

$$= A^2/S^3\,\partial^2 H/\partial R^2$$

$$\Rightarrow \frac{\partial V}{\partial A} = \frac{\partial H}{\partial R}$$

Substituting all derivatives in the original eqn.

$$S\cdot\frac{\partial H(R,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 A^2/S^3 \frac{\partial^2 H}{\partial R^2} + rS\left(H(R,t) - R\frac{\partial H}{\partial R}\right)$$

$$+ S\frac{\partial H}{\partial R} - rV = 0$$

$\Rightarrow$ taking $S$ common and putting $R = A/S$.

$$\frac{\partial H}{\partial t} + \frac{\sigma^2 R^2}{2}\frac{\partial^2 H}{\partial R^2} + rV - rR\frac{\partial H}{\partial R} + \frac{\partial H}{\partial R} - rV = 0$$

$$\Rightarrow \frac{\partial H}{\partial t} + \frac{\sigma^2 R^2}{2}\frac{\partial^2 H}{\partial R^2} + (1 - rR)\frac{\partial H}{\partial R} = 0$$

Hence eqn is obtained.

Now Obtaining boundary cond$^n$s:

→ from the payoff we can say that for $R \to \infty$

$$H(R, T) = \left(1 - \frac{R_T}{T}\right)^+$$

$$\Rightarrow H(R_T, T) = 0 \text{ for } R_T \to \infty$$

∵ $R_t$ is bounded, hence $S \to 0$ for $R \to \infty$

for $S \to 0$, option won't be exercised

$$\Rightarrow H(R, t) = 0 \text{ for } R \to \infty$$

Now for $R \to 0$ we can prove that $R^2 \frac{\partial^2 H}{\partial R^2} \to 0$ if $H \to$ bounded.

for proof assume $H \to$ bounded.
and $R^2 \frac{\partial^2 H}{\partial R^2} \neq 0$

$$\Rightarrow \frac{\partial^2 H}{\partial R^2} = O\left(\frac{1}{R^2}\right) \Rightarrow H = O(\log R) + k_1 R + k_2$$

$k_1, k_2 \Rightarrow$ some constants.

But if $R \to 0$ we will have $\log R \to -\infty$ which means $H$ is not bounded. Hence contradiction.

Hence, if $H$ is bounded then $R^2 \frac{\partial^2 H}{\partial R^2} \to 0$ as $R \to 0$.

$\Rightarrow$ Boundary cond$^n$ is : $\frac{\partial H}{\partial t} + \frac{\partial H}{\partial R} = 0$ for $R \to 0$.

Summarizing all boundary and terminal cond^ns for

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR)\frac{\partial H}{\partial R} = 0$$

$H = 0$ for $R \to \infty$

$$\frac{\partial H}{\partial t} + \frac{\partial H}{\partial R} = 0 \quad \text{for } R = 0$$

$$H(R_T, T) = \left(1 - \frac{R_T}{T}\right)^+$$

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Sol^n (5): Strong order of convergence of a numerical scheme in SDE:

We say that a discretization $\hat{X}$ has strong order of convergence $\beta > 0$ if

$$E\left[\| \hat{X}(nh) - X(T) \|\right] \leq ch^{\beta}$$

for some constant $c$ and all sufficiently small $h$.

Weak order of convergence.

We say a discretization $\hat{X}$ has weak order of convergence ~~...~~ $\beta$ if

$$\left| E[f(\hat{X}(nh))] - E[f(X(T))] \right| \leq ch^{\beta}.$$

for some constant $c$ and all sufficiently small $h$.

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Sol$^n$ (6) : given DDE

$$\begin{cases} dX(t) = a(X(t))\,dt + b(X(t))\,dW(t) \\ X(0) = X_0 \end{cases}$$

To solve:

we associate some operators:-

$$L^0 = a\frac{d}{dx} + \frac{1}{2} b^2 \frac{d^2}{dx^2}, \quad L^1 = b\frac{d}{dx}$$

for any $f(x)$

$$L^0 f(x) = a(x) f'(x) + \frac{1}{2} b^2(x) f''(x)$$

$$L^1 f(x) = b(x) f'(x)$$

Applying Ito's

$$df(X(t)) = L^0 f(X(t))\,dt + L^1 f(X(t))\,dW(t)$$

Hence $\Rightarrow L^0 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2}{\partial x^2}$

take interval $[t, t+h]$, using the original given eq$^n$.

$$X(t+h) = X(t) + \int_t^{t+h} a(X(u))\,du + \int_t^{t+h} b(X(u))\,dW(u) \longrightarrow (1)$$

using approximation:-

$$a(X(u)) \simeq a(X(t)) \quad \text{for } u \in [t, t+h]$$

apply ito's to $a(X(u))$:

$$a(X(u)) = a(X(t)) + \int_t^u L^0 a(X(s))\,ds + \int_t^u L^1 a(X(s))\,dW(s)$$

Apply Euler approximation to both integrals as well.

$$L^0(a(X(s)) \simeq L^0(a(X|t)), \quad L^1(a(X(s)) \simeq L^1(a(X|t))$$

$$a(X(u)) \simeq a(X|t)) + L^0 a(X|t)) \int_t^u ds + L^1 a(X|t)) \int_t^u dW(s)$$

So now from (1): putting $a(X(u))$ in 1st integral:

$$\int_t^{t+h} a(X(s))ds \simeq a(X|t))h + L^0 a(X|t)) \int_t^{t+h}\int_t^u ds\,du$$

$$+ L^1 a(X|t)) \int_t^{t+h}\int_t^u dW(s)\,du$$

$$= a(X|t))h + L^0 a(X|t)) I_{0,0} + L^1 a(X|t)) I_{(1,0)}$$

where $I_{0,0}, I_{(1,0)}$ are double integrals. Hence we get approx value for the 1st integral for (1)

Now approximating $b(X|u))$
Similarly like $a(X|t))$ we have

$$b(X|u)) \simeq b(X|t)) + \int_t^u L^0 b(X(s))ds + \int_t^u L^1 b(X(s))dW(s)$$

$$\simeq b(X|t)) + L^0 b(X|t)) \int_t^u ds + L^1 b(X|t)) \int_t^u dW(s)$$

Now integral approximation:-

$$\int_t^{t+h} b(X(u))dW(u) \approx b(X|t))[W(t+h) - W(t)]$$

$$\simeq \cancel{b(X|t))} + L^0 b(X|t)) \int_t^{t+h}\int_t^u ds\,dW(u) +$$

$$L^1(b(X|t)) \int_t^{t+h}\int_t^u dW(s)\,dW(u)$$

$$= b(X(t))\left[W(t+h) - W(t)\right] + L^0 b(X(t))\, I_{(0,1)} + L^1(b(X(t)))\, I_{(1,1)}$$

where again $I_{(0,1)}, I_{(1,1)}$ are double integrals.

Hence putting both the obtained approximate integral in the original equation :-

$$X(t+h) \approx X(t) + ah + b\Delta W + \left(aa' + \tfrac{1}{2}b'a''\right) I_{(0,0)}$$

$$+ \left(ab' + \tfrac{1}{2}b^2 b''\right) I_{(0,1)} + ba' I_{(1,0)} + bb' I_{(1,1)}.$$

where $\Delta W = W(t+h) - W(t)$

Hence req^r portion is to compute the double integrals :

$$I_{(0,0)} = \int_t^{t+h}\int_t^{u} ds\, du = \tfrac{1}{2}h^2.$$

$$I_{(1,1)} = \int_t^{t+h}\left(W(u) - W(t)\right) dW(u) = \frac{1}{2}\left[\left(W(t+h) - W(t)\right)^2 - h\right]$$

Apply Ito's formula to $tW(t)$ :-

$$I_{(0,1)} = hW(t+h) - \int_t^{t+h} W(u)\, du$$

$$= h\,W(t+h) - hW(t) - \int_t^{t+h} \left(W(u) - W(t)\right) du$$

$$= h\Delta W - I_{(1,0)}$$

Now we need to compute $I_{(1,0)}$ in order to compute $I_{(0,1)}$.

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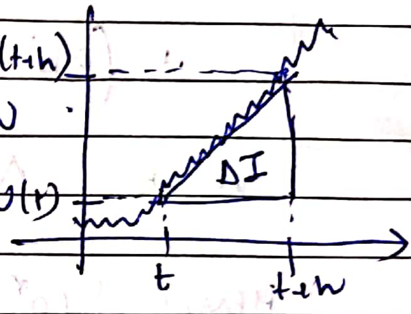
where $I_{(1,0)} = \int_t^{t+h} [W(u) - W(t)]\, du$

We can observe that given $W(t)$ the area $I_{(1,0)}$ and $\Delta W = W(t+h) - W(t)$ are jointly normal with mean $= 0$ and conditional variance of $\Delta W = h$ and that of $I_{(1,0)} = h^3/3$. We need to find their covariance.

firstly:




$$E\left[I_{(1,0)} \mid W(t), \Delta W\right] = \frac{1}{2} h \Delta W$$

using the figure we can have

$$\Rightarrow E\left[I_{(1,0)} \Delta W\right] = \frac{1}{2} h^2.$$

So we can ~~[illegible]~~ write the joint fxn as:-

$$\begin{pmatrix} \Delta W \\ \Delta I \end{pmatrix} \sim N(0, M) \quad \text{where } M = \begin{pmatrix} h & \frac{1}{2}h^2 \\ \frac{1}{2}h^2 & h^3/3 \end{pmatrix}$$

Hence we can write the 2nd order scheme as:

~~$\hat{X}((i+1)h)$~~

$$\hat{X}((i+1)h) = \hat{X}(ih) + ah + b\Delta W + \left(ab' + \frac{1}{2}b^2 b''\right)(h\Delta W - \Delta I)$$

$$+ a'b\,\Delta I + \frac{1}{2} bb'\left[\Delta W^2 - h\right] + \left(aa' + \frac{1}{2}b^2 a''\right)\frac{1}{2}h^2.$$

with the functions $a, b$ and their derivatives all evaluated at

$\hookrightarrow \hat{X}(ih)$.

Hence, the 2nd order scheme could be derived.