Error Analysis in Finite Element Method for Elliptic Boundary Value Problems

A Project Report Submitted
in Partial Fulfilment of the Requirements
for the Degree of

MASTER OF SCIENCE

in

Mathematics and Computing

by

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to the

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April 2017

CERTIFICATE

This is to certify that the work contained in this report entitled **"Error Analysis in Finite Element Method for Elliptic Boundary Value Problems"** submitted by **Kunal Verma** (**Roll No: 152123019**) to Department of Mathematics, Indian Institute of Technology Guwahati towards the requirement of the course **MA699 Project** has been carried out by him under my supervision.

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ABSTRACT

The main aim of this project is to study the convergence of finite element approximations to the actual solution of elliptic boundary value problems. The error analysis is based on interpolation approximation in Sobolev spaces. We first studied the interpolation error estimates in H^1-norm and L^2-norm. Then the interpolation approximation is extended to finite element error analysis. A numerical example is presented for the completeness of this work.

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Chapter 1

Introduction

1.1 problem Description

Partial Differential Equations (PDE's) are widely used to describe and model physical phenomena in different engineering fields and science. Only for simple and geometrically well-defined problems analytical solutions can be found, but for the most problems it is impossible. For these problems, also often with several boundary conditions, the solution of the PDE's can only be found with numerical methods. The most universal numerical method is based on finite elements. This method has a general mathematical fundament and clear structure. Thereby, it can be relative easily applied for all kinds of PDE's with various boundary conditions in nearly the same way. The finite element method (FEM) has its origin in the mechanics and since than it has been widely used in various problems (cf. [3]).

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. We consider following elliptic boundary value problem

$$\left.\begin{array}{rcl} -\Delta u & = & f \quad \text{in} \quad \Omega, \\ u & = & 0 \quad \text{in} \quad \partial\Omega. \end{array}\right\} \tag{1.1}$$

The objective of this work is to study the convergence of finite element solution to the exact solution of (1.1) with respect to H^1-norm and L^2-norm. In finite element method, we split our error into two components using an intermediate operator. The first component is the difference between the exact solution and a properly defined local projection of the exact solution, and the second component is the difference between the properly defined local projection of the exact solution and the finite element solution. Most of the cases, we use interpolation operator as an intermediate operator.

1.2 Basic Notation

We shall now introduce the standard notation for Sobolev spaces and norms to be used in this paper.

Definition 1.2.1. Let $\Omega \subset \mathbb{R}^2$ and p is a real number with the property $1 \leq p < \infty$, then $L^p(\Omega)$ denotes the following space

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : \left(\int_\Omega \mid f(x) \mid^p dx \right)^{\frac{1}{p}} < \infty \right\}.$$

Further, $L^p(\Omega)$ is a normed linear space with respect to the following norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega \mid f(x) \mid^p dx \right)^{\frac{1}{p}}.$$

Definition 1.2.2. Let $m > 0$ be an integer and let $1 \leq p < \infty$. Then the **Sobolev spaces** $W^{m,p}(\Omega)$ is defined by

$$W^{m,p} = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), 0 \leq \alpha \leq m\},$$

where

$$D^\alpha u = \frac{\partial^\alpha u}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2}, \alpha = \alpha_1 + \alpha_2,$$

is the αth order weak derivative of u. In other words, $W^{m,p}(\Omega)$ is the collection of all functions in $L^p(\Omega)$ such that all its weak derivatives upto order m are also in $L^p(\Omega)$. Further, $W^{m,p}(\Omega)$ is a normed linear space with respect to the norm

$$\|v\|_{m,p,\Omega} = \sum_{0 \leq \alpha \leq m} \|D^\alpha v\|_{L^2(\Omega)}.$$

For our later use, we introduce following semi-norm

$$|v|_{m,p,\Omega} = \sum_{\alpha = m} \|D^\alpha v\|_{L^2(\Omega)}.$$

Now in order to introduce the weak derivative, we consider the following equation

$$\frac{dy}{dx} = g(x) \text{ in } \Omega \subset \mathbb{R}.$$

It is known that if $g \in C(\Omega)$ then $y \in C^1(\Omega)$. The problem arises if $g \in L^2(\Omega)$, then we cannot expect the solution in $C^1(\Omega)$. Naturally, we assume that $y \in C(\Omega) \supset C^1(\Omega)$. But, functions of $C(\Omega)$ need not be differentiable. Thus, we cannot find $\frac{dy}{dx}$ in classical sense. Therefore, we need to attach a meaning to the derivative of y even y is not diffrentiable or not at all continuous. This is done by introducing weak derivative of such functions.

To motivate towards weak derivative of such functions, let us try to define the weak derivative of a diffrentiable function f and then we will generalize the same definition for larger class of functions. Let $C_0^\infty(\Omega)$ be the collection of all C^∞ functions define overΩ which vanishes on the boundary $\partial\Omega$ of Ω. Then, for $v \in C_0^\infty(\Omega)$, we have

$$\int_\Omega \frac{df}{dx} v dx = fv]_{\partial\Omega} - \int_\Omega f \frac{dv}{dx} dx = - \int_\Omega f \frac{dv}{dx} dx.$$

Thus, under the sign of integration, derivative of f reduces to the derivative of v. This fact motivate us to define weak derivatives for non smooth functions.

Before going to the formal definition of weak derivatives, we would like to discuss about the support of a function.

Definition 1.2.3. Let ϕ be a real(or complex) valued continuous functions on an open set $\Omega \subset \mathbb{R}^n$. The support of ϕ is denoted by supp(ϕ) and defined as

$$\text{supp}(\phi) = \overline{\{x \in \Omega : \phi(x) \neq 0\}}.$$

If this closed set is compact as well, then supp(ϕ) is said to be of compact support.

The set of all infinitely differentiable function on an open set $\Omega \subset \mathbb{R}^n$ with compact support is a vector space which will henceforth be denoted by

$D(\Omega)$.

Therefore, first order weak derivative of f is given by

$$D_f : D(\Omega) \to \mathbb{R} \quad \text{such that} \quad D_f(v) = -\int_\Omega f v' dx \; \forall v \in D(\Omega).$$

Since, right hand side does not involve the derivative of f, we can define the weak derivative for those functions f, which not at all differentiable or continuous. In general, the mth order weak derivative of f is defined as

$$D_f^m(v) = (-1)^m \int_\Omega \frac{d^m v}{dx^m} f dx \;\; \forall v \in D(\Omega).$$

Definition 1.2.4. For $p = 2$, the Sobolev Space $W^{m,p}(\Omega)$ is a Hilbert space and it is denoted by $H^m(\Omega)$. In particular, $H^1(\Omega) = \{v \in L^2(\Omega) : Dv \in L^2(\Omega)\}$ and the corresponding norm is defined by

$$\|v\|_{H^1(\Omega)} = \|v\|_{L^2(\Omega)} + \|Dv\|_{L^2(\Omega)}.$$

Definition 1.2.5. It can be proved that $C_0^\infty(\bar{\Omega})$ is dense in $H^1(\Omega)$. If $f \in C^\infty(\bar{\Omega})$ we define the trace of f, namely γf, by $\gamma f = f|_\Gamma$, $\Gamma = \partial\Omega$. Further the map $\gamma : C^\infty(\bar{\Omega}) \to L^2(\Gamma)$ is continuous and linear satisfying

$$\|\gamma u\|_{L^2(\Gamma)} \leq C\|u\|_{H^1(\Omega)}.$$

Hence this can be extended as continuous linear map from $H^1(\Omega)$ to $L^2(\Gamma)$. This map γ is called trace map.

Definition 1.2.6. The collection of all H^1 functions vanishing on the boundary is a closed subspace of H^1 and it is denoted by

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : \gamma v = 0\}.$$

For $H_0^1(\Omega)$ functions, following important inequality, due to Poincaré, holds true

$$\int_\Omega |v|^2 dx \leq C \int_\Omega |Dv|^2 dx \ \ \forall v \in H_0^1(\Omega).$$

We now wind-up this section by introducing Sobolev quotient space. Let $\mathbb{P}_k(\Omega)$ be the collection of all polynomials defined over Ω with degree less than or equal to $k \geq 0$. We define

$$\begin{aligned} V &= W^{k+1,p}(\Omega)/\mathbb{P}_k(\Omega) \\ &= \{[v] | [v] = \{v + q | q \in \mathbb{P}_k(\Omega)\}, \ v \in W^{k+1,p}(\Omega)\} \end{aligned}$$

and its corresponding norm

$$\|[v]\|_V = \inf_{q \in \mathbb{P}_k(\Omega)} \|v + q\|_{k+1,p,\Omega}.$$

Later in error analysis, we need following inequality involving the norm of the Sobolev quotient space V.

Lemma 1.2.7. *For any Lipschitz domain $\Omega \subset \mathbb{R}^d$, there is a constant $C > 0$, depending only on Ω, such that*

$$\inf_{p \in \mathbb{P}_k(\Omega)} \|v + p\|_{k+1,\Omega} \leq c|v|_{k+1,\Omega} \ \forall v \in H^{k+1}(\Omega). \tag{1.2}$$

1.3 Variation Formulation of Elliptic BVP

In this section, we introduce weak formulation for a elliptic boundary value problem. Further, Lax-Milgram result is cited to study the well possedness of the variational problem.

Multiply (1.1) by a smooth function $v \in D(\Omega)$, where $D(\Omega)$ is the set of all infinitely differentiable function on Ω with compact support, and then integrate over Ω to have

$$\int_{\Omega} -\Delta u.v dx = \int f v dx. \tag{1.3}$$

Now, we may recall classical Green's theorem

Theorem 1.3.1. ***Green's Theorem.*** *Let Ω be a connected domain in $\mathbb{R}^2$. Then, we have*

$$-\int_{\Omega} \nabla.(\nabla G) w dx dy = \int_{\Omega} \nabla G.\nabla w dx dy - \oint_{\Gamma} \frac{\partial G}{\partial \eta},$$

where η is the outward normal to the boundary $\partial\Omega = \Gamma$. $\square$

Now, apply Green's theorem and the fact $v = 0$ on $\partial\Omega$ in (1.3) to have

$$\int_{\Omega} \nabla u.\nabla v dx = \int_{\Omega} f v dx.$$

Then variational problem to (1.1) is defined as: Find $u \in H_0^1(\Omega)$ such that

$$A(u, v) = L(v) \;\; \forall v \in H_0^1(\Omega). \tag{1.4}$$

Here, $A : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ and $L : H_0^1(\Omega) \to \mathbb{R}$ are defined as

$$\begin{aligned} A(w, v) &= \int_\Omega \nabla u.\nabla v dx \ \ \forall w,\ v \in H_0^1(\Omega), \\ L(v) &= \int_\Omega f v dx \ \ \forall v \in H_0^1(\Omega). \end{aligned}$$

In finite element algorithm, we approximate the variational problem (1.4). Thus, existence and uniqueness of a variational problem is important. In this connection, we need a important result popularly known as Lax-Milgram Lemma (cf. [2]). Before proceeding further, we need to understand some important terms.

Definition 1.3.2. Billinear Form: If X and Y are vector spaces, a billinear form $A : X \times Y \to \mathbb{R}$ is defined to be an operator with following properties

$$A(\alpha u + \beta w, v) = \alpha A(u, v) + \beta A(w, v) \forall u, w \in X, v \in Y, \alpha, \beta \in \mathbb{R}$$

$$A(u, \alpha v + \beta w) = \alpha A(u, v) + \beta A(u, w) \forall u \in X, v, w \in Y.$$

Definition 1.3.3. Continuous Billinear Map: Let $A : X \times Y \to (R)$ be a billinear map, where X and Y are normed linear spaces equipped with norms $\|.\|_X$ and $\|.\|_Y$, respectively. Then $A(.,.)$ is said to be continuous/bounded if there is a positive number C such that

$$|A(u, v) \leq C\|u\|_X \|v\|_Y \forall u \in X, v \in Y.$$

Definition 1.3.4. H-Elliptic Billinear Map/ Positive Billinear Map:

Given a billinear form $A : H \times H\mathbb{R}$, where H is an inner product space, we say that $A(.,.)$ is H-elliptic/positive if there exist a constant $C > 0$ such that

$$A(v, v) \geq C\|v\|_H^2 \forall v \in H,$$

where $\|.\|$ is the norm associated with inner product.

As we have introduced all the terms now we are in the position to to move towards the main result of this chapter which illustrate sufficient conditions for the existence and uniqueness of the variational problem.

Theorem 1.3.5. *Let H be a Hilbert space and let $A : H \times H \to \mathbb{R}$ be a continuous, positive billinear map defined on $H \times H$. Then, for a given continuous linear functional L on H, there exist a unique element u in H such that*

$$A(u, v) = L(v) \;\; \forall v \in H. \quad \square$$

1.4 Regularity Result

This section describe the existence, uniqueness and smoothness of the solutions to the problem (1.4) depending on the smoothness of the given data. For simplicity consider the equation

$$\frac{dy}{dx} = f, x \in (a, b)$$

If $f \in L^2(a, b)$ then $\frac{dy}{dx} \in L^2(a, b)$ so $y \in H^1(a, b)$. Similarly, for $f \in H^1(a, b)$, $y' \in H^1(a, b)$ and $y'' \in L^2(a, b)$ so that $y \in H^2(a, b)$. Any result which describe the smoothness of solution depending on the smoothness of given data is called regularity result.

Regarding regularity of the solution to the problem (1.4), we have following result (see, [2])

Theorem 1.4.1. *For $f \in L^2(\Omega)$, the problem (1.4) has unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega))$ satisfying following a priori estimate*

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

1.5 Embedding Result

We conclude this chapter with the following result connecting Sobolev space and classical function spaces. We now state Sobolev embedding theorem and Rellich's theorem for our convenience. It is natural to seek whether it is true that members of $H^m(\Omega)$ are simply functions that, together with their derivatives of order $\leq m-1$, are continuous. After all it is not easy, for example, to conceive of a function in $H^1(\Omega)$ that is not continuous. A famous theorem due to Sobolev asserts that, as we would expect, all members of $H^1(a,b)$ are indeed continuous functions, but that this does not hold for higher dimensional domains. Such results are called Embedding results.

Let X, Y are Banach Spaces with $X \subseteq Y$. We say that X is continuously embedded in Y i.e. $X \hookrightarrow Y$ if the identity map $I : X \to Y$ is continuous that is

$$\|I(x)\|_Y \leq C\|x\|_X, \text{ for some constant } C > 0.$$

The following theorem gives conditions under which Sobolev spaces are embedded in spaces of continuous functions (see, [2]).

Theorem 1.5.1. The Sobolev Embedding Theorem. *Let Ω be a bounded domain in $\mathbf{R}^n$ with Lipschitz boundary Γ. If $m-k > n/2$, then every function in $H^m(\Omega)$ belongs to $C^k(\bar{\Omega})$. Furthermore, the embedding*

$$H^m(\Omega) \subset C^k(\bar{\Omega}) \tag{1.5}$$

is continuous.

Remark 1.5.2. Some care has to be exercised in the interpretation of above result. Recall that members of $H^m(\Omega)$ are equivalence classes of functions, given that they are members of L^2, whereas continuous functions, by contrast, are defined unambiguously. The embedding (1.5) has therefore to be interpreted in the sense that each member of $H^m(\Omega)$ may be identified with a function in $C^k(\bar{\Omega})$, possibly after changing its values on a set of measure zero.

According to the Sobolev Embedding Theorem, if $n = 1$ so that Ω is a subset of the real line, then the functions in $H^1(\Omega)$ are continuous. For domains that are subsets of the plane, though, $n = 2$ and we require that a function be a member of $H^2(\Omega)$ in order to guarantee its continuity.

Chapter 2

Interpolation Theory in Sobolev Spaces

In this Chapter, we have discussed finite element discretization of a computational domain. Then we construct a finite element space based on the discretization. Finally, we focus on the interpolation error estimates based on Theorem 1.2.7.

2.1 Finite Element Discretization

To proceed towards finite element algorithm, we now describe finite element discretization of a computational domain $\Omega \subset \mathbb{R}^2$.

Definition 2.1.1. Triangulation. A triangulation $\mathcal{T}_h$ is a partition of $\bar{\Omega}$ into a finite number of subsets K such that it satisfies the following conditions:

($\mathcal{A}1$) $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$.

($\mathcal{A}2$) $K = \bar{K},\ \text{int}(K) \neq \Phi,\ K \in T_h,$

($\mathcal{A}3$) If $K_1,\ K_2 \in \mathcal{T}_h$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is a common vertex or edge of both triangles. That is no vertex of any triangle lies in the interior of an edge of another triangle.

($\mathcal{A}4$) For each triangle $K \in \mathcal{T}_h$, let r_K, $\overline{r}_K$ be the radii of its inscribed and circumscribed circles, respectively. Let $h = \max\{\overline{r}_K : K \in \mathcal{T}_h\}$. We assume that, for some fixed $h_0 > 0$, there exists two positive constants C_0 and C_1 independent of h such that

$$C_0 h \leq \text{diam}(K) \leq C_1 h \quad \forall K \in \mathcal{T}_h, \quad \forall h \in (0, h_0).$$

Based on a triangulation $\mathcal{T}_h$, we construct finite dimensional piece-wise polynomial space V_h, known as finite element space. We first describe element wise construction of V_h. For $K \in \mathcal{T}_h$, $P_r,\ r \geq 1$ is a linear space of functions $p : K \to \mathbb{R}$ with $P_k \subset H^1(K)$ and $\text{dim}P_r = n_r$. Let $l_i : P_r \to \mathbb{R}$ $(1 \leq i \leq n_r)$ be bounded linear functionals i. e. $l_i \in P_r'$(dualspace of P_r). Define,

$$\sum_K = \{l_i(p) : p \in P_r,\ 1 \leq i \leq n_r\}$$

The member of $\sum_K$ are called degree of freedom.

Definition 2.1.2. The triple $(K, P_r, \sum_K)$ is called a finite elements. Simply $\sum_k$ is set of conditions such that any $p \in P_r$ can be determined uniquely. Then it is called unisolvent.

Choice of $(K, P_r, \sum_K)$ (Lagrange finite elements): As a choice of P_r,

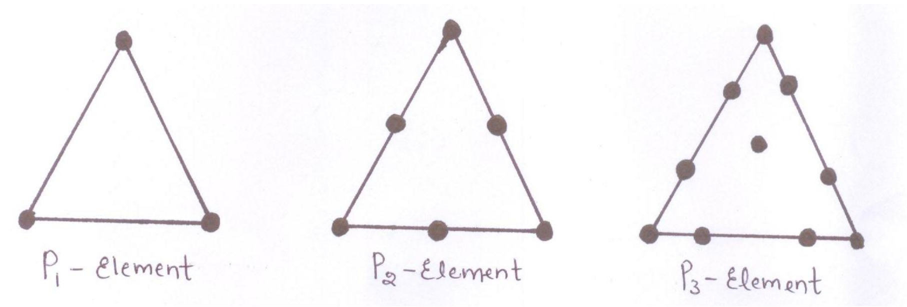

Figure 2.1: Different Lagrange elements

we consider polynomial space of degree $\leq r$. Clearly, in $\mathbb{R}^2$

$$\dim P_r = \frac{(r+1)(r+2)}{2} = n_r.$$

Then any $p \in P_r$ is uniquely determined on the set

$$L_r(K) = \left\{ x = \sum_{i=1}^{3} \lambda_i a_i : \ \lambda_i \in \left\{ 0,\ \frac{1}{r},\ \frac{2}{r}, ..., \frac{r-1}{r},\ 1 \right\} \right\}.$$

As $\text{card}(L_r(K)) = \frac{(r+1)(r+2)}{2} = n_r$. As an example, we will discuss linear and quadratic Lagrange elements.

Definition 2.1.3. Nodal basis function: Let $N_h(\bar{\Omega}) = \{x_i : 1 \leq i \leq n_h\}$ be the collection of all nodes of a triangulation. The function $\phi_i \in V_h$ $(1 \leq i \leq n_h)$ satisfying $\phi_i(x_j) = \delta_{ij}$ is called nodal basis function.

Construction of $(K, P_1, \sum_K)$**:** It is done by introducing area coordinate. Suppose K is a triangle with vertices $a_1 = A$, $a_2 = B$ and $a_3 = C$. Suppose P is an arbitrary point in K. Corresponding to $a_1 = A$, we define

$$\lambda_1(x) = \frac{\text{area}(\triangle PBC)}{\text{area}(\triangle K)} = \frac{A_1}{A}.$$

Similarly, corresponding to $a_2 = B$ and $a_3 = C$, we define

$$\begin{aligned}
\lambda_2(x) &= \frac{\text{area}(\triangle PAC)}{\text{area}(\triangle K)} = \frac{A_2}{A}, \\
\lambda_3(x) &= \frac{A_3}{A} = \frac{1 - A_1 - A_2}{A} = 1 - \lambda_1(x) - \lambda_2(x).
\end{aligned}$$

For P_1 element, $N_h(K) = \{a_1, a_2, a_3\}$ and basis functions corresponding to node a_i is defined as

$$\phi_i(x) = \lambda_i(x) \text{ so that } \phi_i(a_j) = \delta_{ij}.$$

Construction of $(K, P_2, \sum_K)$**:** For P_2 element note that $N_h(K) = \{a_1, a_2, a_3, a_{12}, a_{23}, a_{31}\}$. Basis function corresponding to a_i is defined as

$$\phi_i(x) = \lambda_i(x)(2\lambda_i(x) - 1), \ 1 \leq i \leq 3.$$

Basis function corresponding to a_{ij} is given by

$$\phi_{ij} = 4\lambda_i(x)\lambda_j(x), \ 1 \leq i < j \leq 3.$$

Then the finite element space is defined as

$$V_h = \left\{ v_h : \bar{\Omega} \to \mathbb{R} : v_h|_K \in P_r(K), \ K \in \mathcal{T}_h \right\}.$$

Based on finite element space V_h, finite element method can be grouped into two categories, namely conforming FEM if $V_h \subseteq H$ and non-conforming FEM if $V_h \nsubseteq H$. Here, H is the solution space for the given boundary value problem.

Characterization of V_h: Suppose $\mathcal{T}_h$ is a triangulation of Ω. For $r \geq 1$, $P_r(K) \subset H(K)$. For simplicity, we take $r = 1$. Then Lagrange finite element space $V_h \subset C(\bar{\Omega})$.

Consider any two adjacent elements $K_1, K_2 \in \mathcal{T}_h$. Let $E = E_1 \cap E_2$ be a common edge with nodes $a_1,\ a_2 \in N_h$. For $v_h \in V_h$, we define

$$v_h|_{K_1} = v_h^1,\ \ v_h|_{K_2} = v_h^2 \text{ along } E.$$

Due to continuity of v_h along edges, we obtain $v_h^1 = v_h^2$ along E. Then integration by parts, we have

$$\int_\Omega v_h \phi' dx = -\sum_{K \in T_h} \int_K v_h' \phi dx + \sum_{K \in T_h} \int_{\partial K} [v_h] \bar{n}_K \phi dx,$$

$[v_h]$ is the jump along common edges. Due to continuity $[v_h] = 0$ along edges. Hence weak Derivative of v_h exists in Ω and

$$\|Dv_h\|_{L^2(\Omega)} = \sum_{K \in T_h} \|v_h'\|_{L^2(K)} < \infty$$

Thus, any member $v_h \in V_h$ belongs to $H^1(\Omega)$.

2.2 Reference Element

Suppose $\mathcal{T}_h$ is a triangulation of a domain Ω. For $K \in \mathcal{T}_h$, we wish to evaluate $(K, P_r, \sum_K)$ with the help of a simplified domain which is called reference element. For higher dimension, the use of reference element, in our

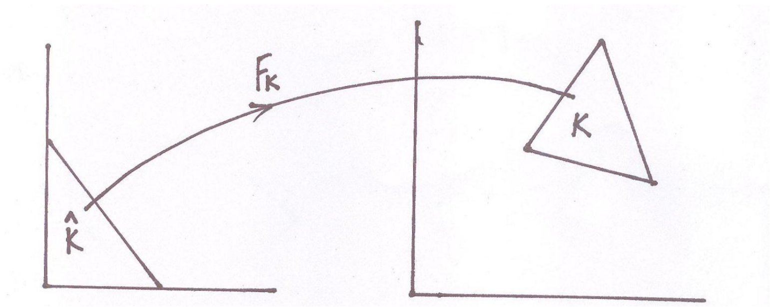

Figure 2.2: Affine map from reference element

case it is triangle, is essential for both theoretical error analysis and practical implementation of the finite element method.

Definition 2.2.1. Reference Triangle. Any non-degenerate triangle $K \subset \mathbb{R}^2$ is the image of the unit triangle $\hat{K}$ under a map

$$F_k : \mathbb{R}^2 \to \mathbb{R}^2 \;\; i.e. \;\; \hat{x} \to F_k(\hat{x}) = B_k(\hat{x}) + b_k$$

with a non-singular matrix $B_k \in \mathbb{R}^2 \times \mathbb{R}^2$ and $b_k \in \mathbb{R}^2$.

Suppose $\hat{K}$ be a triangle with vertices $\hat{a}_1(0,0)$, $\hat{a}_2(1,0)$, and $\hat{a}_3(0,1)$. Then $\hat{P}_1$ over $\hat{K}$ is generated by following nodal basis functions

$$\hat{\phi}_1(x,y) = (1 - \hat{x} - \hat{y}), \;\; \hat{\phi}_2(x,y) = \hat{x} \;\; \text{and} \;\; \hat{\phi}_3(x,y) = \hat{y}.$$

Suppose $K \in \mathcal{T}_h$ with vertices $P_1^K = (x_1, y_1)$, $P_2^K = (x_2, y_2)$ and $P_3^K = (x_3, y_3)$. Define $F_K : \hat{K} \to K$ by

$$F_K(\hat{X}) \;\; = \;\; P_1^K \hat{\phi}_1(\hat{X}) + P_2^K \hat{\phi}_2(\hat{X}) + P_3^K \hat{\phi}_3(\hat{X}).$$

Since $\hat{\phi}_i(\hat{a}_j) = \delta_{ij}$, we obtain

$$F_K(\hat{a}_i) = P_i^K,\ 1 \leq i \leq 3.$$

Again

$$K = \left\{ X = \sum_{i=1}^{3} \lambda_i P_i^K : 0 \leq \lambda_i \leq 1 \text{ and} \sum \lambda_i = 1 \right\}.$$

Thus, the map F_K is well defined in the sense that for each $\hat{X} \in \hat{K}$ there exists unique $X \in K$ such that $X = F_K(\hat{X}) \in K$. Expanding $F_K(\hat{x})$, we obtain

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

More precisely,

$$X = F_K(\hat{X}) = B_K \hat{X} + b_K,$$

with

$$B_K = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}, \quad b_K = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

It is easy to verify that $|B_K| \neq 0$ and hence

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = B_K^{-1} \begin{bmatrix} x - x_1 \\ y - y_1 \end{bmatrix}.$$

Therefore, F_K^{-1} exists.

Suppose $(\hat{K}, \hat{P}_r, \hat{\sum}_{\hat{K}})$ is a Lagrange finite element space over reference element $\hat{K}$ with nodes $\hat{N}_h(\hat{K}) = \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{n_r}\}$. Suppose $\hat{\phi}_i$ is a nodal basis corresponding to the node $\hat{x}_i$. Then the nodal points on K are given by

$$x_i = F_K(\hat{x}_i), \quad 1 \leq i \leq n_r$$

and the corresponding basis functions are defined as

$$\phi_i = \hat{\phi}_i \cdot F_K^{-1}$$

having the property $\phi_i(x_j) = \hat{\phi}_i(F_K^{-1}(x_j)) = \hat{\phi}_i(\hat{x}_j) = \delta_{ij}$. Then the polynomial space P_r over K is defined as

$$P_r = \text{span}\{\phi_1, \phi_2, \ldots, \phi_{n_r}\}.$$

More precisely

$$(K, P_r) = (F_K(\hat{K}), \hat{P}_r \cdot F_K^{-1}).$$

Let $\hat{X}$ be a polynomial space over $\hat{K}$ with $\dim(\hat{X}) = I$. We choose a set of nodal points $\{\hat{x}_i\}_{i=1}^I$ in $\hat{K}$ and any function $\hat{v} \in \hat{X}$ is uniquely determined by its values at the nodes $\{\hat{x}_i\}_{i=1}^I$, and we have the following formula

$$\hat{v}(\hat{x}) = \sum_{i=1}^{I} \hat{v}(\hat{x}_i)\hat{\phi}_i(\hat{x}).$$

The set $\{\hat{\phi}_i\}_{i=1}^I$ is a nodal basis for the set $\hat{X}$. Suppose $F_K : \hat{K} \to K$ is an invertible map with

$$F_K(\hat{x}) = T_K\hat{x} + b_K, \tag{2.1}$$

where T_K is an invertible 2×2 matrix and b_K is a translation vector.

For any function v defined on K, $\hat{v}$ denotes the corresponding function defined on $\hat{K}$ through $\hat{v} = v \cdot F_K$. Conversely, for any function $\hat{v}$ on $\hat{K}$, let v be the function on K defined by $v = \hat{v} \cdot F_K^{-1}$. Thus, we have the relation

$$v(x) = \hat{v}(\hat{x}) \ \forall x \in K, \ \hat{x} \in \hat{K}, \text{with } x = F_K(\hat{x}).$$

We then define a finite dimensional function space X_K formally by the following formula

$$X_K = \hat{X} \cdot F_K^{-1}.$$

Thus, both X_K and $\hat{X}$ are polynomial space with same degree. Then the nodal points on K are given by $x_i^K = F_K(\hat{x}_i), \ i = 1, \ 2, \ldots, I$, and nodal basis functions are defined by $\phi_i^K = \hat{\phi}_i \cdot F_K^{-1}$ having the property $\phi_i^K(x_j^K) = \delta_{ij}$.

Now, we present an important lemma which will be used in estimating finite element interpolation errors.

Lemma 2.2.2. *For the affine map $F_K : \hat{K} \to K$ defined by* 2.1*, we have the bounds,*

$$\|T_K\| \leq \frac{h_k}{\hat{\rho}} \ \textit{and} \ \|T_K^{-1}\| \leq \frac{\hat{h}_k}{\rho_K} \tag{2.2}$$

where $h_K = max\{\|x - y\| : x, y \in K\}$ and ρ_K is the diameter of the largest sphere S_k inscribed in K.

2.3 Interpolation Theory

We first introduce an interpolation operator $\hat{\Pi}$ for continuous functions on $\hat{K}$. We define

$$\hat{\Pi} : H^2(\hat{K}) \to \hat{X} \text{ with } \hat{\Pi}\hat{v}(\hat{x}) = \sum_{i=1}^{I} \hat{v}(\hat{x}_i)\hat{\phi}_i(\hat{x}). \tag{2.3}$$

On any element K, we define

$$\Pi_K : H^2(K) \to X_K, \text{ with } \Pi_K v = \sum_{i=1}^{I} v(x_i^K)\phi_i^K. \tag{2.4}$$

Theorem 2.3.1. *For the two interpolation operators $\hat{\Pi}$ and Π_K introduced above, we have $\hat{\Pi}(\hat{v}) = (\Pi_K v) \cdot F_K$.*

Now we first derive the interpolation error estimates on the reference element.

Theorem 2.3.2. *Let k and m be nonnegative integers with $k > 0$ and $k+1 \geq m$. Let $\hat{\Pi}$ be the operators defined in 2.3.Then there exists a constants $C > 0$ such that*

$$|\hat{v} - \hat{\Pi}\hat{v}|_{m,\hat{K}} \leq c|\hat{v}|_{k+1,K} \; \forall \hat{v} \in H^{k+1}(\hat{K}) \tag{2.5}$$

Proof. Notice that $k > 0$ implies $H^{k+1}(\hat{K}) \hookrightarrow C(\bar{}K)$, so $\hat{v} \in H^{k+1}(\hat{K})$ is continuous and $\hat{\Pi}\hat{v}$ is well defined. From (2.3), we obtain

$$\|\hat{\Pi}\hat{v}\|_{m,\hat{K}} \leq \sum_{i=1}^{I} |\hat{v}(\hat{x}_i)| \|\hat{\phi}_i\|_{m,\hat{K}} \leq C\|\hat{v}\|_{k+1,\hat{K}}.$$

By the assumption on the space $\hat{X}$, we have $\hat{\Pi}\hat{v} \in P_k(\hat{K}) \; \forall \hat{v} \in H^2(\hat{K})$. Thus, for $\hat{p} \in \mathbb{P}_k(\hat{K})$, we have

$$\begin{aligned} |\hat{v} - \hat{\Pi}\hat{v}|_{m,\hat{K}} \leq \|\hat{v} - \hat{\Pi}\hat{v}\|_{m,\hat{K}} &= \|\hat{v} - \hat{\Pi}\hat{v} + \hat{p} - \hat{\Pi}\hat{p}\|_{m,\hat{K}} \\ &\leq c\|\hat{v} + \hat{p}\|_{k+1,\hat{K}} \\ &\leq c \inf_{\hat{p}\in\mathbb{P}_k(\hat{K})} \|\hat{v} + \hat{p}\|_{k+1,\hat{K}} \\ &\leq c\|[\hat{v}]\|_{m,\hat{K}}. \end{aligned}$$

By an application of Corollary 1.2.7, we get the following estimate

$$|\hat{v} - \hat{\Pi}\hat{v}|_{m,\hat{K}} \leq c|v|_{k+1,\Omega}. \quad \square$$

Now, we consider the finite element interpolation error over each element $K \in \mathcal{T}_h$. So, to translate the result of Theorem 2.3.2 from the reference element $\hat{K}$ to K, we need to discuss the relations between Sobolev norms over the reference element and a general element.

Theorem 2.3.3. *Assume $x = T_K(\hat{x}) + b_K$ is a bijection from $\hat{K}$ to K. Then $v \in H^m K$ if and only if $\hat{v} \in H^m(\hat{K})$. Furthermore, for some constant C independent of K and $\hat{K}$, the estimates*

$$|\hat{v}|_{m,\hat{K}} \leq c\|T_K\|^m |detT_K|^{-1/2} |v|_{m,K} \tag{2.6}$$

and

$$|v|_{m,\hat{K}} \leq c\|T_K^{-1}\|^m |detT_K|^{1/2} |v|_{m,\hat{K}} \tag{2.7}$$

holds true.

We now combine preceding theorems to obtain an estimate for the interpolation error in the semi-norm $|v - \Pi_K v|_{m,K}$.

Theorem 2.3.4. *Let k and m be non negative integers with $k > 0$, $k+1 \geq m$ and $\mathbb{P}_k(\hat{K}) \subset \hat{X}$. Let Π_K be the operators defined in (2.4). Then there is a constant $C > 0$ depending only on $\hat{K}$ and $\hat{\Pi}$ such that*

$$|v - \Pi_K v|_{m,K} \leq C \frac{h_K^{k+1}}{\rho_K^m} |v|_{k+1,K} \ \forall v \in H^{k+1}(k). \tag{2.8}$$

Proof. From Theorem 2.3.1, we have $\hat{v} - \hat{\Pi}\hat{v} = (v - \Pi_K v) \cdot F_K$. Consequently, using (2.7), we have

$$|v - \Pi_K v|_{m,K} \leq C \|T_K^{-1}\|^m |\text{det} T_K^{1/2} |\hat{v} - \hat{\Pi}\hat{v}|_{m,\hat{K}}. \tag{2.9}$$

Using Theorem 2.3.2, we have

$$|v - \Pi_K v|_{m,K} \leq c \|T_K^{-1}\|^m |\text{det} T_K^{1/2} |\hat{v}|_{k+1,\hat{K}} \tag{2.10}$$

The inequality (2.6) with $m = k+1$ is

$$|\hat{v}|_{k+1,\hat{K}} \leq c \|T_K\|^{k+1} |\text{det} T_K|^{-1/2} |v|_{k+1,K}. \tag{2.11}$$

From (2.10), we obtain

$$|v - \Pi_K v|_{m,K} \leq c \|T_K^{-1}\|^m \|T_K\|^{k+1} |\hat{v}|_{k+1,\hat{K}}$$

With the help of Lemma 2.2.2, the estimate (2.8) follows immediately. $\square$

The error bounds in (2.8) depends on two parameters h_k and ρ_K. It will be convenient to use the parameter h_K only in an interpolation error bound. For this purpose, we introduce the notion of a *regular family* of finite elements.

Definition 2.3.5. A family $\{T_h\}_h$ of finite elements partitions is said to be regular if there exists a constant σ such that $h_K/\rho_K \leq \sigma$ for all elements $K \in T_h$ and for any h.

As an immediate consequence of Theorem 2.3.4, we obtain

Corollary 2.3.6. *We keep the assumptions stated in Theorem 2.3.4. Furthermore, assume that $\mathcal{T}_h$ is a regular family of finite elements. Then there is a constant $C > 0$ such that for any $K \in \mathcal{T}_h$ in the family, we have*

$$\|v - \Pi_K v\|_{m,K} \leq c h_K^{k+1-m} |v|_{k+1,K} \; \forall v \in H^{k+1}(K) \;\; \forall K \in \mathcal{T}_h. \tag{2.12}$$

Chapter 3

Finite Element Error Analysis for Elliptic BVP

In this chapter, we focuss on the error analysis of finite element solution and exact solution with respect to L^2-norm and H^1-norm. This has been done using Aubin-Nitsche lemma.

3.1 Cea's Lemma

We recall our variational formulation: Find $u \in H_0^1(\Omega)$ such that

$$A(u, v) = L(v) \;\; \forall v \in H_0^1(\Omega). \tag{3.1}$$

Bilinear map $A : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ and functional $L : H_0^1(\Omega) \to \mathbb{R}$ are as defined in Chapter 1.

For a triangulation $\mathcal{T}_h$ of the domain Ω and finite element space V_h based on $\mathcal{T}_h$, we define the finite element approximation to (3.1) as: Find $u_h \in V_h$ such that

$$A(u_h, v_h) = L(v_h) \;\; \forall v_h \in V_h. \tag{3.2}$$

In the context of our finite element approximation of a linear second order elliptic BVP, there holds the Cea's inequality, stated as

Lemma 3.1.1. Cea's Lemma. *let u and u_h be the solutions of (3.1) and (3.2), respectively. Then then there exists a constant $C > 0$ independent of h such that*

$$\|u - u_h\|_{H^1(\Omega)} \leq C \inf_{v \in V_h} \|u - v\|_{H^1(\Omega)}. \tag{3.3}$$

Proof. For simplicity, we write $V = H_0^1(\Omega)$. Since $V_h \subset V$, we have for all $v_h \in V_h$

$$a(u - u_h, v_h) = 0 \;\; \forall v_h \in V_h. \tag{3.4}$$

This is known as Galerkin orthogonality. Now, using positivity of the bilinear map $A(.,.)$, we obtain

$$\begin{aligned} \alpha_0 \|u - u_h\|_V^2 &\leq a(u - u_h, u - u_h) \\ &= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\ &= a(u - u_h, u - v_h). \end{aligned} \tag{3.5}$$

Further, using continuity of the bilinear map $A(.,.)$, we obtain

$$\alpha_0 \|u - u_h\|_V^2 \leq M \|u - u_h\|_V \|u - v_h\|_V. \tag{3.6}$$

Taking infimum over $v_h \in V_h$, we obtain

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \|u - v_h\|_V. \quad \square$$

For finite element error analysis, we relate above result with our interpolation theory discussed in Chapter 2. For a function $v \in C(\bar{\Omega})$, we construct its global interpolant $\Pi_h v|_K$ in the finite element space V_h by the formula

$$\Pi_h v|_K = \Pi_K v \;{}_K v \;\forall K \in T_h.$$

We have the representation formula

$$\Pi_h v = \sum_{i=1}^{N_h} v(x_i)\phi_i. \tag{3.7}$$

Regarding the error estimates for our global interpolant, we have following result

Theorem 3.1.2. *Assume that all the conditions of Corollary 2.3.6 hold. Then there exist a constant $C > 0$ independent of h such that*

$$\|v - \Pi_h v\|_{m,\Omega} \leq C h^{k+1-m} |v|_{k+1,\Omega} \;\forall v \in H^{k+1}(\Omega), \;\; m = 0,\ 1. \tag{3.8}$$

Proof. Apply corollary 2.3.6 with $m = 0$ and 1 to find

$$\begin{aligned}
\|u - \Pi_h u\|_{m,\Omega}^2 &= \sum_{K \in T_h} \|u - \Pi_K u\|_{m,K}^2 && (3.9)\\
&\leq \sum_{K \in T_h} c h_K^{2(k+1-m)} |u|_{k+1,K}^2 && (3.10)\\
&\leq C h_K^{2(k+1-m)} |u|_{k+1,\Omega}^2. && (3.11)
\end{aligned}$$

Taking the square root of above relation, we obtain the error estimates. □

Then from Cea's inequality, we have

$$\|u - u_h\|_{1,\Omega} \leq C \inf{}_{v_h \in V_h} \|u - v_h\|_{1,\Omega}. \tag{3.12}$$

and hence,

$$\|u - u_h\|_{1,\Omega} \leq c \|u - \Pi_h u\|_{1,\Omega}. \tag{3.13}$$

Finally, Theorem 3.1.2 leads to following convergence result.

Theorem 3.1.3. *Let $k > 0$ be an integer and let $V_h \subset V$ be affine-equivalent finite element spaces of piecewise polynomials of degree less than or equal to k, corresponding to a regular family of triangulations of $\bar{\Omega}$. Then the finite element method converges, i. e.*

$$\|u - u_h\|_V \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

Assume $u \in H^{k+1}(\Omega)$. Then there exists a constant $C > 0$ such that the *following error estimate holds*

$$\|u - u_h\|_{1,\Omega} \leq ch^k |u|_{k+1,\Omega}. \quad \square$$

Next, we derive error estimates for the finite element solutions in $L^2(\Omega)$ norm.

Theorem 3.1.4. *AUBIN-NITSCHE LEMMA*
In the context of our problems (3.1)-(3.2), we have

$$\|u - u_h\|_{L^2(\Omega)} \leq M\|u - u_h\|_{1,\Omega} \sup_{g \in L^2(\Omega)} \left(\frac{1}{\|g\|_{L^2(\Omega)}} \inf_{v_h \in V_h} \|\psi_g - v_h\|_{1,\Omega} \right)$$

where for each $g \in L^2(\Omega)$, $\psi_g \in V$ is the solution for $g \in L^2(\Omega)$ satisfying

$$A(v, \psi_g) = (g, v) \ \forall \ v \in V. \quad \square$$

For $g = e = u - u_h$, we consider following problem: Find $\psi_e \in V$ such that

$$A(v, \psi_e) = (e, v) \ \forall \ v \in V$$

and its finite element approximation: find $\psi_{eh} \in V_h$ such that

$$A(v_h, \psi_{eh}) = (e, v_h) \ \forall \ v_h \in V_h.$$

Then, Theorem 3.1.3 and regularity result leads to

$$\|\psi_e - \psi_{eh}\|_{1,\Omega} \leq Ch\|\psi_e\|_{H^2(\Omega)} \leq Ch\|e\|_{L^2(\Omega)}.$$

This together with Theorem 3.1.4, we have

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch\|u - u_h\|_{1,\Omega}$$

Finally, we have following L^2-norm error estimates

Theorem 3.1.5. *let u and u_h be the solutions of (3.1) and (3.2), respectively. Then then there exists a constant $C > 0$ independent of h such that*

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{k+1}|u|_{k+1,\Omega}, \quad k \geq 1. \quad \square$$

3.2 Numerical Result

Now we turn our attention to a numerical experiment illustrating the efficiency and accuracy of the proposed method for our scheme.
Consider the following elliptic boundry value Problem

$$-\Delta u = 2\pi^2 \sin \pi x \sin \pi y \text{ on } \quad \Omega = (0,1) \times (0,1), \tag{3.14}$$

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{3.15}$$

Exact solution of to the above problem is given by

$$u = \sin \pi x \sin \pi y. \tag{3.16}$$

The finite element approximation to our problem is defined as: find $u_h \in V_h$ such that

$$A(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h. \tag{3.17}$$

For numerical result, globally continuous piecewise linear finite element functions based on the triangulation of Ω as stated in chapter 2 were used. The L^2-norm errors for various step size h are presented in table below.

h	$\|u - u_h\|_{L^2(\Omega)}$	Ratio of error
$\frac{1}{4}$	0.0265	
$\frac{1}{8}$	0.0065	4.07
$\frac{1}{16}$	0.0016	4.06
$\frac{1}{32}$	0.00041	3.90

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