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Sol^n 1: given $(AB)^T = AB$, $(BA)^T = BA$, $ABA = A$, $BAB = B$

$A \in R^{n \times m}$, $B \in R^{m \times n}$.

To prove $B = A^+$

$A = U\Sigma V^T$. $\Sigma = diag(\sigma_1, \sigma_2 \cdots \sigma_p, 0, 0 \cdots) \in R^{n \times m}$

$\therefore A^+ = V\Sigma^+ U^T$. $\Sigma^+ = diag(\sigma_1^{-1}, \sigma_2^{-1}, \cdots \sigma_p^{-1}, 0, \cdots 0) \in R^{m \times n}$.

Let's say $rank(A) = p$.

$\Rightarrow \Sigma^+\Sigma = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \in R^{m \times m}$. $\rightarrow$ 1 Similarly $\Sigma\Sigma^+ \in R^{n \times n}$ $\rightarrow$ 2

Now solving the forward part that all eq^ns are satisfied if $B = A^+$ and then prove ~~that~~ ~~eqn~~ uniqueness to show ~~that~~ for B.

Hence $B = V\Sigma^+ U^T$ and $A = U\Sigma V^T$.

Hence, $BAB = V\Sigma^+ U^T = B$.

Similarly we can have $ABA = U\Sigma V^T = A$.

Now for $(AB)^T = (U\Sigma\Sigma^+ U^T)^T = U(\Sigma\Sigma^+)^T U^T$

$= U\Sigma\Sigma^+ U^T$. $(\Sigma\Sigma^+)^T = \Sigma\Sigma^+$

But as we can see from 1 and 2, $(\Sigma^+\Sigma)^T = \Sigma^+\Sigma$ and

Hence we can say that $\Sigma^+\Sigma, \Sigma\Sigma^+$ is ~~also~~ symmetric.

Hence we can insert a nxn Identity matrix $V^T V$ in middle: where V is orthogonal nxn:

$V^T V = I$.

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Hence:
$(AB)^T = U\Sigma V^T V \Sigma^+ U^T = AB$
→ Hence proved.

Now for $(BA)^T$ = we get
$(BA)^T = V(\Sigma^+\Sigma)^T V^T$
using the prev. arguement $\Sigma^+\Sigma$ is symmetric
$(BA)^T = V\Sigma^+\Sigma V^T$
Now insert U^TU in the middle where U is a $n \times n$ orthogonal matrix as $U^TU = I$.

Hence $(BA)^T = \underbrace{V\Sigma^+U^T}_{B}\ \underbrace{U\Sigma V^T}_{A} = BA$

Hence, we can see that $B = A^+$ satisfies the eq^n.

Now we have just have to prove uniqueness of it. Hence let take C which also satisfies the above eq.
then we have $(CA)^T = CA$, $(AC)^T = AC$,
$ACA = A$, $CAC = C$

Now we consider AB
$AB = (A)B = (ACA)B = ACAB = (AC)^T(AB)^T$
∴ B also satisfies the above eq^{n}s.

$AB = (AC)^T(AB)^T = C^TA^TB^TA^T$ (using property of matrices)
$= C^T(ABA)^T = C^TA^T$ ∵ $A = ABA$

$AB = C^TA^T = (AC)^T = \boxed{AC}$ → (1)

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~~Hence~~ similarly, we can prove for premultiply B

$\Rightarrow \boxed{BA = CA} \longrightarrow$ (2)

Hence: $B = BAB$ (given already)

$$B = B(AB) = B(AC) = (BA)C = CA(C) = CAC = C$$

using all the eqⁿs got ; and $C = CAC$ (given)

Hence we got $C = B$ which we wanted to prove

$\boxed{B = A^{+}}$

Hence we can say that only one matrix satisfies all the given equations

that is $\boxed{B = A^{+}}$.

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Sol^n 3 : since $UV^T \to$ matrix ~~of size~~, $u, v \in R^n \setminus \{0\}$
consider
$u = [u_1 \; u_2 \; \cdots \; u_n]^T$
$v = [v_1 \; v_2 \; \cdots \; v_n]^T$
$v^T = [v_1 \; v_2 \; \cdots \; v_n]$

$$uv^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} [v_1 \; v_2 \; \cdots \; v_n]$$

$$= \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & & & \\ u_n v_1 & & \cdots & u_n v_n \end{bmatrix}$$

~~Hence~~ since ~~$u_i v_j$~~ $u, v \in R^n \setminus \{0\}$
Hence uv^T is a rank 1 matrix unless and until all value of u, v are non-zero

$uv^T u = u(v^T u) = (v^T u)\, u$
Hence $u \to$ eigen vector of uv^T.
Hence rank $(uv^T) \leq 1$
$\Rightarrow$ nullity $\geq n-1$
$\Rightarrow (n-1)$ eigenvectors ~~of~~ of eigen value $= 0$.

$G_n = I_n + uv^T$
(1) has eigen vector u with eigen value $1 + v^T u$.
(2) ~~eigen vector 0 here~~

Sol^n 4. $A \rightarrow 2\times 2$ real orthogonal matrix with complex eigenvalues.

TPT: Real Schur form: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\because$ A is orthogonal

$|A| \Rightarrow (ad-bc) = \pm 1 \longrightarrow (1)$

and since it's having complex eigen value.

Let $\lambda \rightarrow$ Eigen value

Should satisfy: $|A - \lambda I| = 0$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - (a+d)\lambda + ad-bc = 0$$

Since λ's are complex.

$D < 0$

$$\Rightarrow (a+d)^2 - 4(ad-bc) < 0$$

$$(a-d)^2 + 4bc < 0 \longrightarrow (2)$$

from (1) and (2) we can see that

$ad-bc \neq -1$ as if it were to be

then $(a-d)^2 + 4ad + 4 < 0$

$(a+d)^2 + 4 < 0$ which is not possible

Hence $\boxed{ad-bc=1}$

$$\Rightarrow (a-d)^2 - 4 < 0$$

$$\boxed{-2 < a-d < 2} \longrightarrow (3)$$

Schur form: UAU^T

Sol^n 5.

$A - pI = Q_p R_p$ ①, $\hat{A} = R_p Q_p + pI$ ②

$\hat{A} - \tau I = Q_\tau R_\tau$ ③, $\tilde{A} = R_\tau Q_\tau + \tau I$ ④

Let $Q = Q_p Q_\tau$, $R = R_\tau R_p$.

To prove $(A - pI)(A - \tau I) = QR$.

G: $(A - pI)(A - \tau I)$ Let say given expression $= G$.

$A^2 - (p+\tau)I + p\tau I$

where $A = pI + Q_p R_p$ (from ①)

$= (pI + Q_p R_p)(pI + Q_p R_p) - (p+\tau)I + p\tau I$

$= Q_p R_p Q_p R_p + 2p Q_p R_p + \cancel{p^2 I} - (p+\tau) Q_p R_p$
$\cancel{-p^2 I} + \cancel{p\tau I} - \cancel{p\tau I}$

$G = Q_p R_p Q_p R_p + (p - \tau) Q_p R_p$

given $\hat{A} = R_p Q_p + pI = Q_\tau R_\tau + \tau I$ (from ② & ③)

$\Rightarrow R_p Q_p = Q_\tau R_\tau + (\tau - p) I$.

$= Q_p (Q_\tau R_\tau + (\tau - p) I) R_p + (p - \tau) Q_\tau R_\tau$

$= Q_p (Q_\tau R_\tau) R_p + \underbrace{(\tau - p) Q_p R_p}_{} + \underbrace{\cancel{(p - \tau)} Q_p R_p}_{}$

$G = Q_p (Q_\tau R_\tau) R_p$

$= (Q_p Q_\tau)(R_\tau R_p)$

But since given $Q_p Q_\tau = Q$, $R_\tau R_p = R$.

~~then~~ we have $G = QR$.

Hence proved

$(A - pI)(A - \tau I) = QR$.

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