Lecture-3: FEM for ODE
MA 573: Finite Element Methods for PDEs

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In the previous lectures on FEMs, we have introduced weak derivatives, Sobolev spaces and variational formulation for boundary value problems. In this lecture, we will introduce the finite element methods for ordinary differential equations with Dirichlet boundary conditions.

1 Basic Steps in FEM

In this section, we will introduce basic steps involve in FEMs through a model problem. We begin with following two point boundary value problem

$$-\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=f(x) \quad \text{for } 0<x<1 \tag{1.1}$$
$$y(0)=y(1)=0. \tag{1.2}$$

Equations of the form (1.1)-(1.2) can be viewed as a model of heat conduction in a finite rod $\Omega=(0,1)$ with source intensity f. Here, $H=Q-\frac{P'}{2}=1>0$. Therefore, from Lecture-2, we find that there exists unique strong solution $y \in H^2(\Omega)\cap H_0^1(\Omega)$.

FEMs for the above problem follows the following basic steps:

Step1 Weak formulation: From Lecture-2, we find that strong solution $y \in H^2(\Omega)\cap H_0^1(\Omega)$ satisfies following weak formulation

$$A(y,v)=L(v) \ \ \forall v \in H_0^1(\Omega), \tag{1.3}$$

where

$$A(y,v)=\int_0^1 \Big\{y'v'+y'v+yv\Big\}dx \ \ \& \ \ L(v)=\int_a^b fv dx. \tag{1.4}$$

Further, it is easy exercise to verify that $A(\cdot,\cdot)$ is bilinear and $L(\cdot)$ is linear.

Step 2 Discretization: We divide the interval $\Omega=(0,1)$ into N equal parts by the points $x_0=0,\ x_1=x_0+h,\ x_2=x_1+h,\dots,\ x_{N-1}=x_{N-2}+h,\ x_N=x_{N-1}+h=1$ with $h=1/N$ (see, Figure 1.1). This procedure is known as the discretization of the domain. The points $x_0,\ x_1,\ x_2,\dots,\ x_{N-1},\ x_N$ are called grid points. Each of the small interval are known as elements. The collection of all elements are called mesh of discretization. If the elements are same size then it is called uniform mesh otherwise it is called nonuniform mesh. The parameter h is known as mesh parameter.

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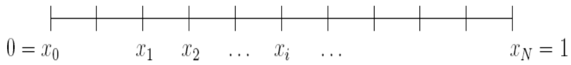

Figure 1.1: Discretization of domain $\Omega = (0, 1)$.

Step 3 Construction of finite element space: In this step, we construct a piecewise linear finite element space V_h with respect to the mesh parameter h. To do so, we construct basis functions $\Phi_1,\ \Phi_2,\ \ldots,\ \Phi_{N-1}$ corresponding to the unknown grid points $x_1, x_2, \ldots, x_{N-1}$, such that

$$\begin{aligned} \Phi_i(x_j) &= 1 \ \text{ if } i = j, \\ &= 0 \ \text{ if } i \neq j, \end{aligned}$$

with the property that each Φ_i is piecewise linear function and the set

$$\{\Phi_1,\ \Phi_2,\ \ldots,\ \Phi_{N-1}\}$$

is linearly independent. One obvious choice for Φ_i is the Lagrange basis function

$$\begin{aligned} \Phi_i(x) &= \frac{(x - x_{i-1})}{h} \quad \text{if } x_{i-1} \leq x \leq x_i, \\ &= \frac{(x_{i+1} - x)}{h} \quad \text{if } x_i \leq x \leq x_{i+1}, \\ &= 0 \quad \text{else.} \end{aligned}$$

Clearly, Φ_i is piecewise linear continuous function in $(0, 1)$ and vanishing on the boundary. Further, weak derivative of Φ_i is given by

$$\begin{aligned} \frac{d\Phi_i}{dx} &= \frac{1}{h} \ \text{ if } \ x_{i-1} \leq x < x_i, \\ &= -\frac{1}{h} \ \text{ if } \ x_i \leq x \leq x_{i+1}, \\ &= 0 \ \text{ else} \end{aligned} \tag{1.5}$$

and hence Φ_i is piecewise differentiable. Then the finite element space V_h is defined to be

$$V_h = \text{span}\{\Phi_1, \ldots, \Phi_{N-1}\}.$$

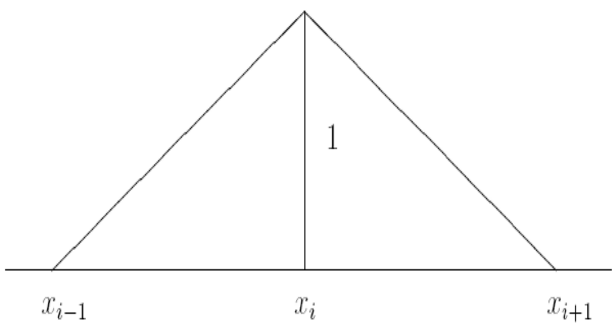

Figure 1.2: The piecewise linear finite element basis function Φ_i in $\Omega = (0, 1)$.

Few Important Observations:

Observation 1: First order derivatives for $\Phi_i,\ i = 1, 2, \ldots, N-1$, are exists but not continuous. This fact is clear from the expression (1.5).

Observation 2: For $|i-j| \geq 2$, any integration involving Φ_i and Φ_j are zero.
Proof. For $|i-j| \geq 2$, the result follows from the fact

$$\text{support of } \ \Phi_i \cap \text{support of } \ \Phi_j = \phi.$$

The support of any arbitrary function f is defined to be

$$\text{supp}(f) = \overline{\{x : \ f(x) \neq 0\}}.$$

Observation 3: The finite element space V_h is a subspace of $H_0^1(0,1)$.
Proof. It follows from the definition of Φ_i that

$$\begin{aligned}
\|\Phi_i\|^2_{L^2(\Omega)} &= \int_{x_i-1}^{x_i+1} {\Phi_i}^2 dx \\
&= \int_{x_i-1}^{x_i} {\Phi_i}^2 dx + \int_{x_i}^{x_i+1} {\Phi_i}^2 dx \\
&= \int_{x_i-1}^{x_i} \left(\frac{x - x_{i-1}}{h}\right)^2 dx + \int_{x_i}^{x_i+1} \left(\frac{x_{i+1} - x}{h}\right)^2 dx \\
&= \frac{\left(x - x_{i-1}\right)^3}{3h^2}\Bigg]_{x_i-1}^{x_i} + \frac{\left(x_{i+1} - x\right)^3}{3h^2}\Bigg]_{x_i}^{x_i+1} \\
&= \frac{2h}{3} < \infty \Rightarrow \Phi_i \in L^2(\Omega).
\end{aligned}$$

Similarly, it can be proved that

$$\left\|\frac{d\Phi_i}{dx}\right\|_{L^2(\Omega)} < \infty \Rightarrow \frac{d\Phi_i}{dx} \in L^2(\Omega).$$

Again, $\Phi_i(0) = \Phi_i(x_0) = 0$ and $\Phi_i(1) = \Phi_i(x_N) = 0$. Since each $\Phi_i \in H_0^1(0,1)$, we can conclude that $V_h \subset H_0^1(0,1)$.

In fact, we have following general result.

Theorem 1.1 *Suppose v is continuous and piece-wise linear function in $\Omega \subset \mathbb{R}^d,\ d \geq 1$, then $v \in H^1(\Omega)$.*

Step4 : Finite Element Approximation:
The finite element approximation to y is defined as: find $y_h \in V_h$ such that

$$A(y_h, v_h) = L(v_h) \ \ \forall\, v_h \in V_h. \tag{1.6}$$

Since $y_h \in V_h$, therefore $\exists$ unique real constants $d_1, d_2, \ldots, d_{N-1}$ such that $y_h = \sum_{i=1}^{N-1} d_i \Phi_i$ and $y_h(0) = 0,\ y_h(1) = 0$. Now, substituting y_h in the equation (1.6), we have

$$A\left(\sum_{i=1}^{N-1} d_i \Phi_i, v_h\right) = L(v_h).$$

Further, using linearity of A, we have

$$\sum_{i=1}^{N-1} d_i A\Big(\Phi_i, v_h\Big) \quad = \quad L(v_h) \ \ \forall\, v_h \in V_h. \tag{1.7}$$

Setting, $v_h = \Phi_j \in V_h,\ 1 \le j \le N-1$, in (1.7), we have

$$A(\Phi_1, \Phi_j)d_1 + A(\Phi_2, \Phi_j)d_1 + \cdots + A(\Phi_{N-1}, \Phi_j)d_1 = L(\Phi_j). \tag{1.8}$$

Which leads to a $(N-1) \times (N-1)$ system involving $(N-1)$ unknowns $d_1, \ldots, d_{N-1}$. In usual matrix notation this fact can be written as

$$AD = F, \tag{1.9}$$

where $A = (a_{i,j})_{(N-1)\times(N-1)}$ and $F = (L(\Phi_1)\ \ldots\ L(\Phi_{N-1}))^t$ with

$$a_{i,j} = A(\Phi_j, \Phi_i) = \int_0^1 \Big\{\Phi_j'\Phi_i' + \Phi_j'\Phi_i + \Phi_j\Phi_i\Big\}dx \ \ \& \tag{1.10}$$

$$L(\Phi_j) = \int_a^b f\Phi_j dx. \tag{1.11}$$

Further, $D = (d1\ d_2\ \ldots\ d_{n-1})^t$ is the unknown matrix.

Regarding the coefficient matrix A, we have the following result which is crucial for the unique solvability of the system (1.9).

Lemma 1.1 *The coefficient matrix A is non-singular.*

Proof. The proof follows from the fact that the matrix A is positive definite that is, for any non-zero column matrix $\eta \in \mathbb{R}^{N-1}$, we have

$$\begin{aligned} \eta^t A\eta = \sum_{i,j=1}^{N-1} \eta_j a_{ij}\eta_i &= \sum_{i,j=1}^{N-1} \eta_j A(\Phi_j, \Phi_i)\eta_i \\ &= A\Big(\sum_{j=1}^{N-1} \Phi_j\eta_j, \sum_{i=1}^{N-1} \Phi_i\eta_i\Big) \\ &= A(v, v) \ge C\|v\|^2_{H^1(\Omega)} > 0 \ \text{ since } v \ne 0. \end{aligned}$$

Therefore A is positive definite and hence non-singular. This completes the rest of the proof. □

As an immediate consequence of the previous lemma, we can now conclude that the system (1.9) has an unique solution $D = (d_1\ d_2\ \ldots\ d_{n-1})^t$ and this leads to unique finite element solution

$$u_h = d_1\Phi_1 + d_2\Phi_2 + \ldots + d_{N-1}\Phi_{N-1}.$$

More precisely, for all $x \in [0, 1]$, we have

$$u_h(x) = d_1\Phi_1(x) + d_2\Phi_2(x) + \ldots + d_{N-1}\Phi_{N-1}(x). \tag{1.12}$$

Remark 1.1 (Basic difference between FEM and FDM): We know that finite difference methods yield approximate solution only at the grid points. But, finite element methods give approximate solution at all points in the computational domain. *At grid point x_i, we obtain*

$$\begin{aligned} y_h(x_i) &= d_1\Phi_1(x_i) + d_2\Phi_2(x_i) + \ldots + d_i\Phi_i(x_i) + \ldots + d_{N-1}\Phi_{N-1}(x_i) \\ &= d_i\Phi_i(x_i) = d_i, \ \text{ since } \Phi_i(x_j) = \delta_{ij}. \end{aligned}$$

Remark 1.2 The coefficient matrix A in (1.9) is a tridiagonal. From **Observation 2**, we find that

$$\begin{aligned} a_{i,j} = A(\Phi_j, \Phi_i) &= \int_0^1 \Big\{ \Phi_j' \Phi_i' + \Phi_j' \Phi_i + \Phi_j \Phi_i \Big\} dx \\ &= 0, \ \text{ when } \ |i-j| \geq 2. \end{aligned}$$

Hence,

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & \dots & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & \dots & 0 \\ 0 & a_{3,2} & a_{3,3} & a_{3,4} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{N-1,N-2} & a_{N-1,N-1} \end{pmatrix}$$

Remark 1.3 (Evaluation of diagonal entries.) For $i = j$, use the fact $\Phi_i \neq 0$ in $[x_{i-1}, x_i] \cup [x_i, x_{i+1}]$ to have

$$\begin{aligned} a_{i,i} &= A(\Phi_i, \Phi_i) \\ &= \int_0^1 \Big\{ \Phi_i' \Phi_i' + \Phi_i' \Phi_i \Big\} dx + \|\Phi_i\|^2_{L^2(\Omega)} \\ &= \int_{x_{i-1}}^{x_{i+1}} \Big\{ \Phi_i' \Phi_i' + \Phi_i' \Phi_i \Big\} dx + \frac{2h}{3} \\ &= \int_{x_{i-1}}^{x_i} \Big\{ \Phi_i' \Phi_i' + \Phi_i' \Phi_i \Big\} dx + \int_{x_i}^{x_{i+1}} \Big\{ \Phi_i' \Phi_i' + \Phi_i' \Phi_i \Big\} dx + \frac{2h}{3} \\ &= \int_{x_{i-1}}^{x_i} \Big\{ \Big(\frac{1}{h}\Big)^2 + \frac{1}{h}\Phi_i \Big\} dx + \int_{x_i}^{x_{i+1}} \Big\{ \Big(\frac{1}{h}\Big)^2 - \frac{1}{h}\Phi_i \Big\} dx + \frac{2h}{3} \\ &= \Big(\frac{1}{h}\Big)^2 \times h + \frac{1}{h} \int_{x_{i-1}}^{x_i} \Phi_i dx + \Big(\frac{1}{h}\Big)^2 \times h - \frac{1}{h} \int_{x_i}^{x_{i+1}} \Phi_i dx + \frac{2h}{3} \\ &= \frac{2}{h} + \frac{2h}{3} + \frac{1}{h}\Big\{ \frac{h}{2}\big[\Phi_i(x_{i-1}) + \Phi_i(x_i)\big] \Big\} - \frac{1}{h}\Big\{ \frac{h}{2}\big[\Phi_i(x_i) + \Phi_i(x_{i+1})\big] \Big\}, \\ &\quad \text{(using Trapezoidal rule)} \\ &= \frac{2}{h} + \frac{2h}{3} + \frac{1}{2}\big[0+1\big] - \frac{1}{2}\big[1+0\big] = \frac{6+2h^2}{3h}. \end{aligned}$$

Remark 1.4 (Evaluation of lower diagonal entries.) Use the fact that $\text{supp}(\Phi_i) \cap \text{supp}(\Phi_{i+1}) = [x_i, x_{i+1}]$ to have

$$\begin{aligned} a_{i+1,i} &= A(\Phi_i, \Phi_{i+1}) \\ &= \int_0^1 \Big\{ \Phi_i' \Phi_{i+1}' + \Phi_i' \Phi_{i+1} + \Phi_i \Phi_{i+1} \Big\} dx = \int_{x_i}^{x_{i+1}} \Big\{ \Phi_i' \Phi_{i+1}' + \Phi_i' \Phi_{i+1} + \Phi_i \Phi_{i+1} \Big\} dx \\ &= \int_{x_i}^{x_{i+1}} \Big\{ -\frac{1}{h} \times \frac{1}{h} - \frac{1}{h}\Phi_{i+1} + \Phi_i \Phi_{i+1} \Big\} dx \\ &= -\frac{1}{h^2} \times h - \frac{1}{h} \int_{x_i}^{x_{i+1}} \Phi_{i+1} dx + \int_{x_i}^{x_{i+1}} \Phi_i \Phi_{i+1} dx \\ &\approx -\frac{1}{h} - \frac{1}{h} \times \frac{h}{2}\Big[\Phi_{i+1}(x_i) + \Phi_{i+1}(x_{i+1})\Big] + \frac{h}{2}\Big[(\Phi_i\Phi_{i+1})(x_i) + (\Phi_i\Phi_{i+1})(x_{i+1})\Big] \\ &= -\frac{1}{h} - \frac{1}{2} = -\frac{2+h}{2h}. \end{aligned}$$

Remark 1.5 (Upper diagonal entries.) Check that

$$a_{i,i+1} = \frac{h-2}{2h}.$$

Remark 1.6 Evaluation of matrix F: Consider the i-th row of column matrix F in (1.9) to have

$$\begin{aligned} L(\Phi_i) &= \int_0^1 f\Phi_i dx \\ &= \int_{x_{i-1}}^{x_i} f\Phi_i dx + \int_{x_i}^{x_{i+1}} f\Phi_i dx \\ &\approx \frac{h}{2}\Big[(f\Phi_i)(x_{i-1}) + (f\Phi_i)(x_i)\Big] + \frac{h}{2}\Big[(f\Phi_i)(x_i) + (f\Phi_i)(x_{i+1})\Big] \\ &= \frac{h}{2}f(x_i) + \frac{h}{2}f(x_i) = hf(x_i). \end{aligned} \tag{1.13}$$

Therefore,

$$F = \begin{pmatrix} hf(x_1) \\ hf(x_2) \\ \vdots \\ hf(x_{N-1}) \end{pmatrix}.$$

Example 1.1 (Homework) Consider following BVP

$$-y'' + Py' + Qy = f \text{ in } (a,b), \quad y(a) = 0 = y(b),$$

where P, Q and f are real valued functions such that strong solution exists. Then

(a) Find the weak formulation.

(b) Define finite element approximation and derive following matrix equation

$$AD = F.$$

Further, identify all entries in Matrix A as

$$a_{i,i} \approx \frac{2}{h} + hQ(x_i), \quad a_{i,i+1} \approx -\frac{1}{h} + \frac{1}{2}P(x_i) \;\; \& \;\; a_{i,i-1} = -\frac{1}{h} - \frac{1}{2}P(x_i).$$

(c) Set a condition on P and Q such that matrix A is invertible.

Example 1.2 Find the finite element solution for the BVP

$$-y'' - y = -x^2, \quad y(0) = 0 = y(1)$$

with three elements.

Solution. Comparing with equation

$$-y'' + Py' + Qy = f,$$

we find $P = 0,\ Q = -1$ and $f = -x^2$. For finite element solution, we need to solve following system of equations

$$AD = F,$$

where where $A = (a_{i,j})_{(N-1)\times(N-1)}$ and $F = (L(\Phi_1)\ \ldots\ L(\Phi_{N-1}))^t$ with

$$a_{i,j} = A(\Phi_j, \Phi_i) = \int_0^1 \Big\{\Phi_j'\Phi_i' - \Phi_j\Phi_i\Big\}dx = a_{j,i}\ \ \& \tag{1.14}$$

$$L(\Phi_j) = \int_a^b f\Phi_j dx. \tag{1.15}$$

Here, $N = 3$ is the number of elements so that

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\ \ \&\ \ F = \begin{pmatrix} L(\Phi_1) \\ L(\Phi_2) \end{pmatrix}.$$

From Example 1.3, we obtain

$$a_{1,1} = \frac{2}{h} + hQ(x_1) = \frac{2}{h} - h, \quad a_{2,2} = \frac{2}{h} - h,$$
$$a_{1,2} = -\frac{1}{h} + \frac{1}{2}P(x_1) = -\frac{1}{h} = a_{2,1}.$$

From Remark 1.6, we have

$$L(\Phi_1) = hf(x_1) = h \times \Big(-\frac{1}{h^2}\Big) = -\frac{1}{h}\ \ \&\ \ L(\Phi_2) = h \times \Big(-\frac{2}{h^2}\Big) = -\frac{4}{h}.$$

Therefore, we have following system of equations

$$\begin{pmatrix} \frac{2}{h} - h & -\frac{1}{h} \\ -\frac{1}{h} & \frac{2}{h} - h \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{h} \\ -\frac{4}{h} \end{pmatrix}.$$

Finally, solve above systems **(Homework)** for d_1 and d_2 so that finite element solution is given by

$$y_h(x) = d_1\Phi_1(x) + d_2\Phi_2(x), \quad x \in [0,1].$$

Let us consider a BVP with non-homogeneous Dirichlet boundary condition. For this purpose, **we need few facts from Example 1.1 in Lecture-2.**

Example 1.3 Find finite element solution at $x = \frac{\pi}{12}$ for the BVP

$$-y'' - \sin xy' + e^x y = 0 \ \text{ in } \Big(0, \frac{\pi}{2}\Big)$$

with boundary condition $y(0) = 0$ and $y\big(\frac{\pi}{2}\big) = 1$.

Solution. Comparing with equation

$$-y'' + Py' + Qy = f, \ \text{ in } \ \Omega = \Big(0, \frac{\pi}{2}\Big)$$

we find $P = -\sin x,\ Q = e^x$ and $f = 0$. Hence,

$$H = Q - \frac{P'}{2} = e^x + \frac{\cos x}{2} \geq 0 \ \text{ in } \Big(0, \frac{\pi}{2}\Big).$$

For the finite element approximation, we now definite weak formulation: Find $y \in H^1(\Omega)$ satisfying following equation

$$A(y, v) = L(v) \ \forall v \in H_0^1(\Omega), \tag{1.16}$$

where

$$A(y, v) = \int_0^{\frac{\pi}{2}} \Big\{y'v' + Py'v + Qyv\Big\} dx \ \ \& \ \ L(v) = \int_0^{\frac{\pi}{2}} fvdx = 0.$$

We know that Problem (1.16) is equivalent to the Problem: Find $w \in H_0^1(\Omega) = y - g$ such that

$$A(w + g, v) = 0 \ \forall v \in H_0^1(\Omega), \tag{1.17}$$

where g is a linear function such that $g(0) = 0$ and $g\big(\frac{\pi}{2}\big) = 1$. More precisely, $g(x) = \frac{2}{\pi}x$.

From equation (1.17), we find

$$A(w, v) = -A(g, v) = \tilde{L}(v) \ \forall v \in H_0^1(\Omega). \tag{1.18}$$

We, now, proceed to find finite element approximation for w.

As a second step, we discretize the domain Ω by the following points

$$x_0 = 0,\ x_1 = \frac{\pi}{4} \ \ \& \ \ x_2 = \frac{\pi}{2}.$$

Here, x_1 is the only unknown grid point. Let Φ_1 be the basis function corresponding to grid point x_1 such that Φ_1 is piecewise linear and

$$\Phi_1(x_0) = 0 = \Phi_1(x_2) \ \text{ and } \ \Phi_1(x_1) = 1.$$

Let

$$V_h = \text{span}\{\Phi_1\}.$$

Here, $V_h \subset H_0^1(\Omega)$ is the collection of all continuous and piecewise linear functions vanishing on the boundary. Let $w_h \in V_h$ be the finite element approximation of w such that

$$A(w_h, v_h) = \tilde{L}(v_h) \ \forall v_h \in V_h. \tag{1.19}$$

Since $w_h \in V_h \in \text{span}\{\Phi_1\}$, we have $w_h = d_1\Phi_1$ for some $d_1 \in \mathbb{R}$. Then (1.19) yields

$$A(d_1\Phi_1, v_h) = \tilde{L}(v_h) \ \forall v_h \in V_h. \tag{1.20}$$

Setting $v_h = \Phi_1$ in (1.20), we obtain

$$d_1 A(\Phi_1, \Phi_1) = \tilde{L}(\Phi_1). \tag{1.21}$$

Using the formula $a_{i,i} \approx \frac{2}{h} + hQ(x_i)$, we obtain

$$A(\Phi_1, \Phi_1) = a_{1,1} \approx \frac{2}{h} + hQ(x_1) = \frac{2}{h} + he^{x_1}. \tag{1.22}$$

Further, we have

$$
\begin{aligned}
\tilde{L}(\Phi_1) = -A(g,\Phi_1) &= -\int_0^{\frac{\pi}{2}} \Big\{ g'\Phi_1' + Pg'\Phi_1 + Qg\Phi_1 \Big\} dx \\
&= -\int_{x_0}^{x_1} \Big\{ \frac{2}{\pi}\Phi_1' + P\frac{2}{\pi}\Phi_1 + Qg\Phi_1 \Big\} dx \\
&\quad -\int_{x_1}^{x_2} \Big\{ \frac{2}{\pi}\Phi_1' + P\frac{2}{\pi}\Phi_1 + Qg\Phi_1 \Big\} dx \\
&= -\int_{x_0}^{x_1} \Big\{ \frac{2}{\pi}\Big(-\frac{1}{h}\Big) + P\frac{2}{\pi}\Phi_1 + Qg\Phi_1 \Big\} dx \\
&\quad -\int_{x_1}^{x_2} \Big\{ \frac{2}{\pi}\Big(\frac{1}{h}\Big) + P\frac{2}{\pi}\Phi_1 + Qg\Phi_1 \Big\} dx \\
&= \frac{2}{\pi} \times \frac{1}{h} \times h - \frac{2}{\pi}\int_{x_0}^{x_1} P\Phi_1 dx - \int_{x_0}^{x_1} Qg\Phi_1 dx \\
&\quad -\frac{2}{\pi} \times \frac{1}{h} \times h - \frac{2}{\pi}\int_{x_1}^{x_2} P\Phi_1 dx - \int_{x_1}^{x_2} Qg\Phi_1 dx \\
&\approx -\frac{2}{\pi} hP(x_1) - h(Qg)(x_1) \\
&= -\frac{2}{\pi} \times \frac{\pi}{4}\big(-\sin x_1\big) - \frac{\pi}{4} Q(x_1) g(x_1) \\
&= \frac{1}{2}\sin x_1 - \frac{\pi}{4}\frac{2}{\pi}\frac{\pi}{4} e^{x_1} = \frac{1}{2}\sin x_1 - \frac{\pi}{8} e^{x_1}. \qquad (1.23)
\end{aligned}
$$

Using (1.22)-(1.23) in (1.21), we obtain

$$
d_1 = \frac{\frac{2}{h} + he^{x_1}}{\frac{1}{2}\sin x_1 - \frac{\pi}{8}e^{x_1}}, \qquad (1.24)
$$

which gives

$$
w_h(x) = d_1 \Phi_1(x), \quad x \in \Big[0, \frac{\pi}{2}\Big]
$$

so that $y \approx w_h + g$ and hence,

$$
y\Big(\frac{\pi}{12}\Big) \approx w_h\Big(\frac{\pi}{12}\Big) + g\Big(\frac{\pi}{12}\Big) = d_1 \Phi_1\Big(\frac{\pi}{12}\Big) + g\Big(\frac{\pi}{12}\Big).
$$