

It is an approximation
for variational prob-

$$H \rightarrow \mathbb{R}^r \text{ space.}$$

$$V \rightarrow \mathbb{R}^r \text{ space.}$$

Variational prob - Find

$$u \in H \text{ s.t.}$$

$$(A(u, v) = L(v) \forall v \in H)$$

$$A: H \times H \rightarrow \mathbb{R}$$

$$L: H \rightarrow \mathbb{R}$$

$$u \approx u_h \in F.D.$$

$$\begin{aligned}
 & -y'' + y = f \quad (a, b) \\
 & \int_a^b y'' + y = \int_a^b f \\
 & \Rightarrow \left[-y' \right]_a^b + \int_a^b y' + \int_a^b y = \int_a^b f \\
 & \Rightarrow \left[-y' \right]_a^b + \int_a^b y' + \int_a^b y = \int_a^b f
 \end{aligned}$$

$$\begin{aligned}
 & V(a) = 0 = V(b) \\
 & \int_a^b y' + \int_a^b y = \int_a^b f \\
 & \Rightarrow A(y, y) = L(y) \\
 & A(y, y) = A(y, y) \rightarrow \text{Symm.} \\
 & A(y, y) = \int_a^b (y')^2 + \int_a^b y^2 \\
 & \geq 0
 \end{aligned}$$

$$A(c_1 y, v) = c_1 A(y, v), \quad A(y, cv) = c A(y, v)$$

$$A(y_1 + y_2, v) = A(y_1, v) + A(y_2, v)$$

$$A(v, v) \geq 0, \quad \text{if } v \neq 0$$

$$A(y, v) = A(v, y) \quad \text{symmetric}$$

$$A(cy, v) = c A(y, v), \quad A(y, cv) = c A(y, v)$$

bilinear
 linearity
 in both args.

Find: $u \in H$ $\xrightarrow{\quad}$ infinite dimensional
 $A(u, v) = L(v) \quad \forall v \in H$

Find: $u \in \underbrace{H}_{\text{s.f.}}$ $\xrightarrow{\quad}$ finite dimension

$$\underline{A(u_n, v_n) = L(v_n) \quad \forall v_n \in V_n}$$

$$A(u_h, v_h) = L(v_h). \quad \text{--- (1)}$$

$$u_h \in \tilde{H} = \text{span} \{ \underline{p}_1, q_1, \dots, q_n \}$$

$$u_h = \dots$$

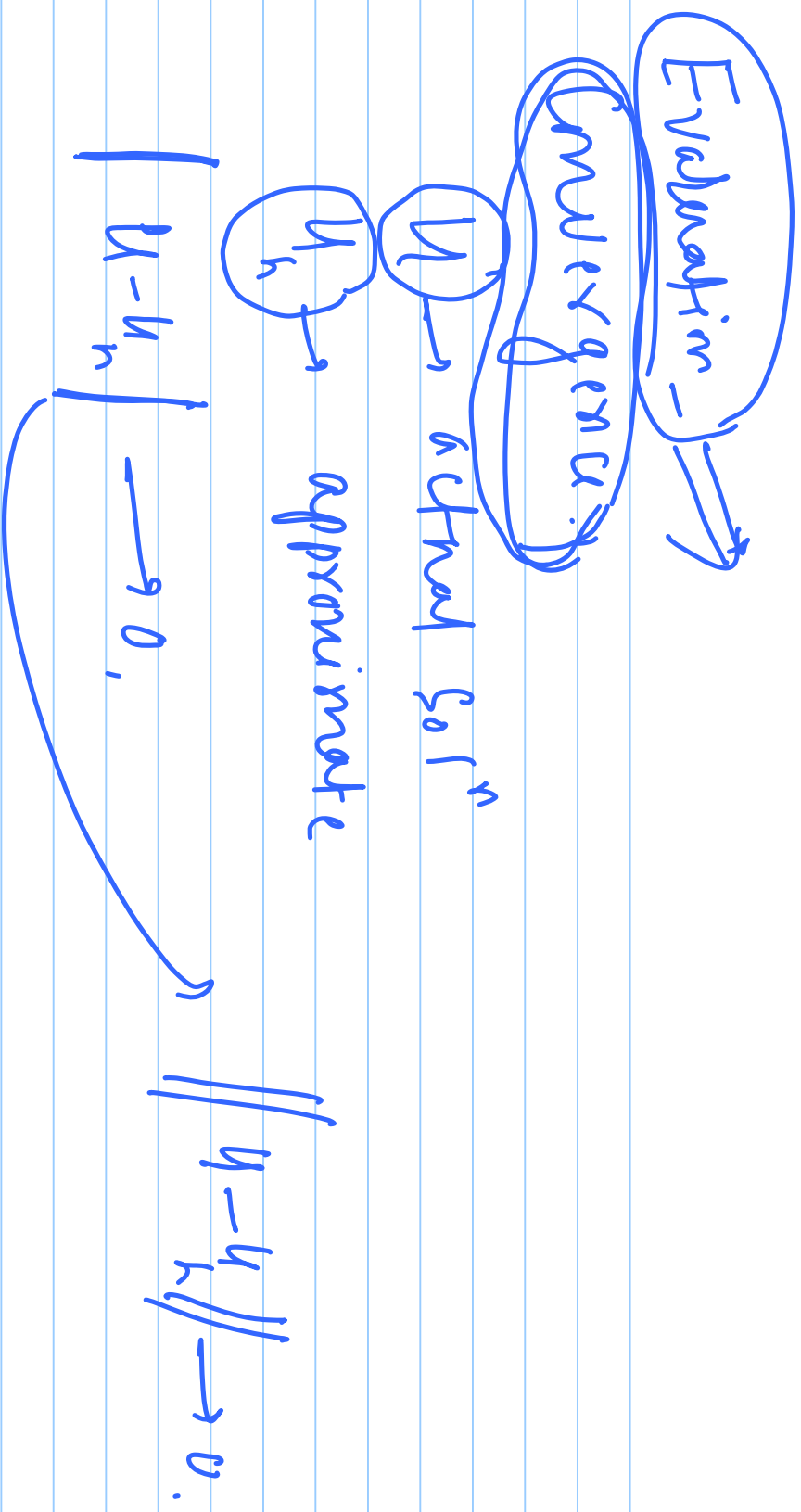
$$\sum_{i=1}^n c_i \cdot q_i$$

, $q_i \rightarrow$ basis fⁿ, known

$$\sum_{i=1}^n c_i \cdot A(q_i, v_h) = L(v_h), \Rightarrow$$

$$v_h = q_j, 1 \leq j \leq n$$

$$\left| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right| = \left| \begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_n \end{array} \right|$$



$$\|u - u_h\| = \left(\int_a^b |u - u_h|^2 dx \right)^{1/2}, \quad \underbrace{L^2_{norm}}_{\|u - u_h\| \rightarrow 0}$$

$$\|\Delta(u - u_h)\| = \left(\int_a^b |\Delta(u - u_h)|^2 dx \right)^{1/2}, \quad \underbrace{H^1_{norm}}$$

H^1 = collection of all L^2 -fct. s.t. its weak derivative is

Convergence analysis

Interpolation error:

$$f, \{ (x_0, f(x_0)), \dots, (x_n, f(x_n)) \}$$

$$\|f - p_n\| \leq C h^{k+1} \quad \xrightarrow{\text{Lagrange}} \quad C_{k+1} \quad \checkmark$$

$$\tilde{H} \subseteq \mathcal{P}_n$$

$$U - u_n$$

$$U - \frac{p_n(h)}{h}$$

interpolant

$$p_n(h) - u_n$$

$$|u - p_n|,$$

$$p_n - u_n \in \mathcal{P}_n$$

$$A(u, v) = L(v) \quad \forall v \in H$$

$$A(u_h, v_h) = L(v_h) \quad \forall v_h \in \tilde{H} \subseteq H$$

$$A(u - u_h, v_h) = 0 \quad \forall \underline{v_h \in \tilde{H}}$$

$$\| (u - u_h, u - u_h) \|_H \geq 0 \quad , \quad A(v, v) = \|v\|_H^2$$

$$\|u - u_h\|_H^2 = A(u - u_h, u - u_h)$$

$$= A(u - u_h + v_h - u_h, u - u_h)$$

$$= A(u - v_h, u - u_h) + A(v_h - u_h, u - u_h)$$

$$= A(u - v_h, u - u_h) + A(u - u_h, \underbrace{v_h - u_h}_{=0})$$

$$\stackrel{\sim}{\hookrightarrow} \in H$$

$$\|u - u_h\|^2 = A(u - v_h, u - u_h) \quad A \text{ inner product}$$

$$\leq \|u - u_h\|_H \|u - v_h\|_H$$

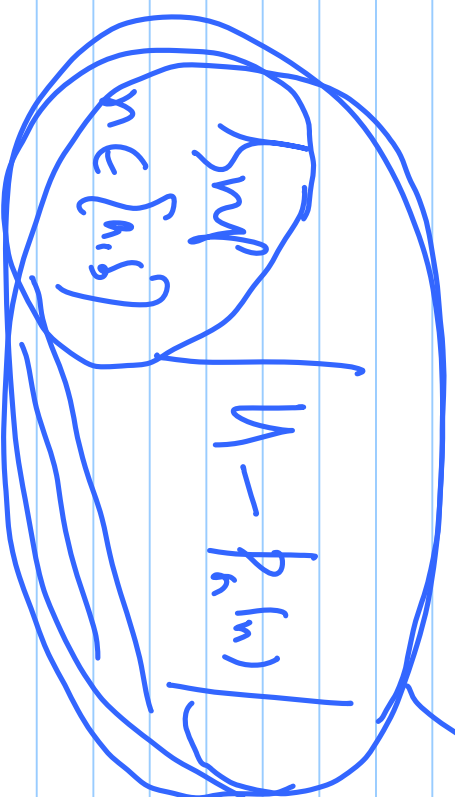
$$\therefore \|u - u_h\|_H \leq \|u - v_h\|_H \quad v_h = p_h(u_h) \quad v_h \in H \subseteq \mathcal{P}_h$$

$$\|u - u_n\|_{(H)} \leq \left(\|u - p_n(u)\|_{(H)} \right)$$

$p_n(u) \in H_n$

interpolating

n



Polynomial approx.
max in
Sobolev space.