Chaos, Solitons and Fractals 139 (2020) 110254

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

A simple numerical method for pricing American power put options

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ARTICLE INFO

Article history:
Received 5 February 2020
Revised 25 July 2020
Accepted 25 August 2020
Available online 13 September 2020

MSC:
Primary 60h07
Secondary 60g51

Keywords:
American power put option
Optimal exercise boundary
Transformed function
Cubic spline interpolation

ABSTRACT

In this paper, we present numerical methods to determine the optimal exercise boundary in case of an American power put option with non-dividend yields. The payoff of a power option is typified by its underlying share price raised to a constant power. The nonlinear payoffs of power options offer considerable flexibility to investors and can be applied in various applications. Herein, we exploit a transformed function to obtain the optimal exercise boundary of the American power put option. Employing it, we can easily determine the optimal exercise boundary. After determining the optimal exercise boundary, we calculate the American power put option values using the finite difference method. Generally, the optimal exercise boundary may not be observed at the grid points. Therefore, the interpolation method is used to determine the value of the American power put option. Furthermore, we present several numerical results obtained by comparing the proposed method and the existing methods.

1. Introduction

In this paper, a fast and accurate method that can price an American power put option based on a transformed function is presented. Options are the most commonly used derivative contracts in modern financial markets. Options are tradable financial contracts, the value of which is dependent on that of the underlying asset. A derivative contract that provides the holder the right to sell (buy) an underlying asset at a certain price (exercise price) is called the put (call) option. The option is called the European option when the right can be exercised only on the prespecified expiry date; however, it is called the American option when it can be exercised at any time until the prespecified expiry date. A method to accurately price the American options is extremely important because majority of the options traded on organized exchanges are American options.

Even though closed-form solutions have been derived for the European options by Black and Scholes [3] and Merton [30], no analogous result can be observed with respect to the American options. In case of American options, the major difficulty is associated with the fact that they can be exercised at any time before the expiration day. Thus, the pricing problem results in complicated analytic calculations. Mathematically, the fact that the holder of the option has an early exercise right will change the problem into a so-called free-boundary problem. McKean [29] and Merton [30] proved that the valuation of American options constitutes a free-boundary problem in which a change in boundary must be identified prior to maturity. The free boundary is commonly referred as the optimal exercise boundary. Moerbeke [31] extended this analysis and studied the properties of the optimal exercise boundary.

The optimal exercise boundary (or early exercise boundary) is dependent on time and is part of the solution before the expiry of the option. Thus, the major task associated with the valuation of American options is how to determine the optimal exercise boundary efficiently and accurately. Therefore, many researchers have conducted theoretical studies about the valuation of American options using numerical or approximation methods.

Brennan and Schwartz [7] have developed finite difference methods to solve the free-boundary problem. After the model is discretized using the finite difference method, the pricing of American option can be solved as a linear complementarity problem (LCP) or variational inequality. LCP is a fundamental problem associated with mathematical programming because of two characteristics: linearity and complementarity. Many methods have been proposed to solve the discretized LCP in case of the pricing of American options. Borici and Lüthi [4] introduced an algorithm to solve the discrete LCP that linearly converges according to the number of spatial grid points. The projected successive overrelaxation method (PSOR) is considered to be one of the popular and effective methods [37]. The usage of variational inequalities, which were developed by Bensoussan and Lions [2], to characterize

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https://doi.org/10.1016/j.chaos.2020.110254
0960-0779/

the price function is closely related to the free-boundary problem. Friedman [13] has discussed the relations of the free-boundary formulation with respect to the variational inequality formulation.

The front-fixing methods, such as those devised by Landau [19] and Wu and Kwok [40], utilize the change in variables to transform the free-boundary problem into a nonlinear problem in a fixed domain. Further, the penalty methods, including those proposed by Nielsen et al. [34], eliminate the free boundary by adding a nonlinear penalty term to the partial differential equation (PDE). Both the aforementioned methods solve a set of nonlinear equations, the computational speed and accuracy of which are considerably dependent on the initial guess, problem size, and underlying nonlinear solver. However, these methods are not considerably efficient for pricing American options and exhibit considerably generalized applicability. Moreover, Muthuraman [32] used a moving-boundary approach to convert the free-boundary problem into a sequence of fixed-boundary problems.

The homotopy analysis method (HAM) [21–24], which was initially proposed and widely applied in the field of mechanics [25,26,33], is an analytical method used in case of strongly nonlinear problems. This method can be combined with other mathematical methods, including the Padé method, series expansion method, integral transform methods, and numerical methods. Zhu [41] initially applied the HAM to an American put option problem. Cheng et al. [9] used HAM to obtain an explicit series approximation solution for the optimal exercise boundary of an American put option.

There are several methods that exploit the representation of the price as the expected pay-off under the risk-neutral measure. Binomial methods are common examples of such methods. In binomial methods, the pricing process of the underlying asset is performed using a binomial lattice. The binomial tree model proposed by Cox, Ross, and Rubinstein [10] is one of the earliest approximation methods and is widely used even now. Another example is the analytical approximation method. MacMillan [28], Barone–Adesi and Whaley [1] developed an approximate analytical formula for application to American options. Geske and Johnson [14] derived a valuation formula for American put options in terms of a series of compound option functions. The price of an American put option can also be evaluated using Monte Carlo simulations. The Monte Carlo simulation techniques have gained popularity because of their ability to price options on several underlying assets more quickly than other numerical methods. Boyle [5] initially proposed the application of simulation to option pricing, and Tilley [38] developed the first computational scheme that could implement the simulation techniques. As additional examples, Broadie and Glasserman [6], Carriere [8], and Longstaff and Schwartz [20] developed option pricing methods using Monte Carlo simulations.

Generally, the standard call or put options are referred to as vanilla options. All the non-standard options are referred to as exotic options. In finance, an exotic option is an option having that has various features, resulting in an increased complexity when compared with that associated with commonly traded vanilla options. The power options, also known as leveraged option, are one example. Power options are the options in which payoffs are based on the underlying asset raised to some power [27,39]. Thus, a power option is defined as a contingent claim on the product of powers of several underlying assets. The feature of nonlinear payoffs of power options provides the buyer with a potential to receive a considerably higher payoff than that received from a vanilla option. Heynen and Kat [15] and Tompkins [39] show that a package of vanilla options can be used to hedge a single power option when the dynamics of the underlying share price is a geometric Brownian motion. Esser [11] derived semi-closed-form solutions of the power options when the mean-reverting Ornstein–Uhlenbeck process is considered to be the volatility process. Macovschi and Quittard–Pinon [27] also considered the power options under Heston's stochastic volatility models without calculating the characteristic functions in detail. They introduced polynomial options under some special assumptions based on the power option formulas.

In this paper, we develop a numerical method in which a transformed function is used for pricing the American power put options. We exploit a transformed function with a free boundary exhibiting a Lipschitz character, thereby avoiding the degeneracy of the solution surface near the optimal exercise boundary, as stated by Kim et al. [18]. This paper mainly intends to develop a numerical method to obtain the optimal exercise boundary associated with the American power put option. The solution surface should be carefully examined near the optimal exercise boundary. Further, we prove this mathematically and obtain the optimal exercise boundary using a transformed function. Generally, the optimal exercise boundary may not be observed at the grid points. Therefore, the interpolation method is used to determine the value of the American power put option. After determining the optimal exercise boundary, the American power put option values are calculated by applying the finite difference method (FDM). The proposed method provides fast and accurate results with respect to the calculation of the optimal exercise boundary and pricing of the American power put options. In addition, we present several numerical results obtained by comparing the proposed method and the existing methods.

This paper is structured as follows. Section 2 presents the model formulation. Section 3 shows that using a transformed function to find the optimal exercise boundary is reasonable. The numerical methods are presented in Section 4, and the numerical results and comparative studies are presented in Section 5. Section 6 summarizes this paper.

2. Formulation

In this section, we introduce mathematical models with respect to the American power put option. Let $S^n(t)$ denote the value of an underlying asset price as a function of the current time t and n be the power of the option. We assume that $S^n(t)$ follows

$$dS^n(t) = n\left\{\frac{1}{2}\sigma^2(n-1)+r\right\}S^n(t)dt+\sigma nS^n(t)dW(t),\ \ \forall t\in[0,T], \tag{2.1}$$

where $r > 0$ represents the constant risk interest rate, $\sigma > 0$ represents the constant volatility of the underlying asset price, and $W(t)$ is the standard Brownian motion [35].

In case of an American power put option with the underlying price of the asset $S^n(t)$, exercise price K and time to expiry T, we have the payoffs for the American power put options as

$$(K - S^n(T))^+ = \max\{K - S^n(T), 0\}. \tag{2.2}$$

The valuation of an American power put option is denoted as $P(\tau, S^n)$, where $\tau (:= T - t)$ is the time to expiration for $\tau \in [0, T]$. We intend to determine the value $P(\tau, S^n)$ for an American power put option, which can be exercised by the holder at any time to obtain a payoff $(K - S^n(T))^+$.

The time at which the option should be exercised and the price of the American option depend on the optimal exercise boundary, which is denoted as $\beta = \{\beta(\tau) : \tau \in [0, T]\}$. Here, we suppose that the optimal exercise boundary $\beta(\tau)$ is continuously nonincreasing with $\beta(0) = K$. For each time $\tau \in [0, T]$, there exists an optimal exercise boundary $\beta(\tau)$, below which the American power put option should be exercised early, i.e.,

$$\text{if } S^n(\tau) \le \beta(\tau), \text{then } P(\tau, S^n) = \max\{K - S^n(\tau), 0\}, \tag{2.3}$$

$$\text{and if } S^n(\tau) > \beta(\tau), \text{then } P(\tau, S^n) > \max\{K - S^n(\tau), 0\}. \tag{2.4}$$

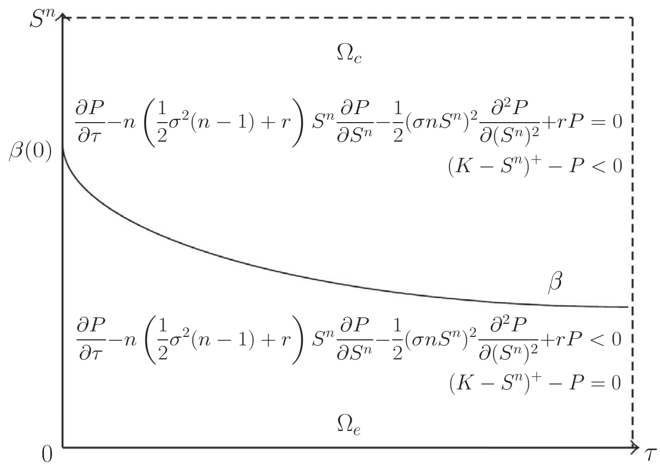


Fig. 1. Optimal exercise boundary of an American power put option.

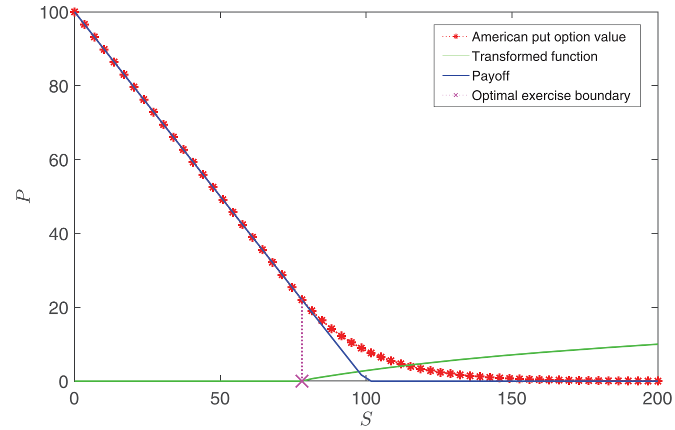


Fig. 2. Transformed function.

It divides the time and asset price space into two regions. The region in which it is optimal to hold, generally called the continuation region, is defined as $\Omega_C = [0, T] \times (\beta(\tau), \infty)$, and the region in which it is optimal to exercise, generally called the exercise (or stopping) region, is defined as $\Omega_E = [0, T] \times [0, \beta(\tau)]$. (See [36].)

The Black–Scholes PDE for the price of an American power put option with a non-dividend yield can be given as

$$\frac{\partial P}{\partial \tau} - n\left\{\frac{1}{2}\sigma^2(n-1)+r\right\}S^n\frac{\partial P}{\partial S^n} - \frac{1}{2}(\sigma n S^n)^2\frac{\partial^2 P}{\partial (S^n)^2} + rP = 0 \text{ in } \Omega_C, \tag{2.5}$$

where $\frac{\partial P}{\partial \tau}$, $\frac{\partial P}{\partial S^n}$, and $\frac{\partial^2 P}{\partial (S^n)^2}$ denote the partial derivatives. The analysis of McKean [29] implies that the American put option value $P(\tau, S^n)$ and the exercise boundary $\beta(\tau)$ can be used to jointly solve the free-boundary problem, when (2.5) is subjected to the following boundary conditions.

$$\lim_{S^n \to \infty} P(\tau, S^n) = 0, \tag{2.6}$$

$$P(\tau, \beta(\tau)) = K - \beta(\tau), \tag{2.7}$$

$$P_{S^n}(\tau, \beta(\tau)) = -1, \tag{2.8}$$

and the initial condition

$$P(0, S^n) = \max\{K - S^n, 0\}. \tag{2.9}$$

The use of the time variable $\tau = T - t$ provides us with an initial-value problem. For more detailed information about equations (2.5) to (2.9), please refer to [36]. The existence and uniqueness of P and β as solutions to PDE have been verified by Karatzas and Shreve [16].

Fig. 1 shows an illustration of an optimal exercise boundary β with an American power put option price function P. Based on Fig. 1, P must satisfy

$$\max_{\tau, S^n}\left\{\frac{\partial P}{\partial \tau} - n\left(\frac{1}{2}\sigma^2(n-1)+r\right)S^n\frac{\partial P}{\partial S^n} - \frac{1}{2}(\sigma n S^n)^2\frac{\partial^2 P}{\partial (S^n)^2} + rP\right\} = 0.$$

The asymptotically optimal exercise boundary for perpetual American power puts $\beta(\infty)$ can be obtained as follows:

$$\beta(\infty) = \frac{\gamma}{\gamma+1}K, \tag{2.10}$$

where $\gamma = \frac{2r}{n^2\sigma^2}$ [12].

3. Transformed function

Initially, we determine the optimal exercise boundary $\beta(\tau)$ before calculating the solution of the PDE (2.5). To obtain the optimal exercise boundary, we present a transformed function exhibiting Lipschitz character near the optimal exercise boundary as follows:

$$Q(\tau, S^n) = \sqrt{P(\tau, S^n) - (K - S^n)}. \tag{3.1}$$

Furthermore, we obtain a Lipschitz curve that avoids the degeneracy and singularity of the solution line near the optimal exercise boundary. In addition, the solution line near the optimal exercise boundary should be carefully determined. The aforementioned function is used because the solution line is a horizontal line in the exercise region and an inclined line in the continuation region. Hence, $Q(\tau, S^n) = 0$ in the exercise region, and $Q(\tau, S^n) > 0$ in the continuation region (Fig. 2). The transformed function $Q(\tau, S^n)$ forms a sufficiently large angle with the horizontal line corresponding to the exercise region; thus, the borderline can be easily distinguished.

Based on the theorem 3.1, we obtain the angle between the exercise region and the Q line such that $0 < \xi < Q_{S^n} < \eta$ for some constants ξ and η.

Theorem 3.1. *Let*

$$Q(\tau, S^n) = \sqrt{P(\tau, S^n) - (K - S^n)},$$

Q has Lipschitz character with nonsingularity and nondegeneracy near the optimal exercise boundary. Then,

$$0 < \xi < Q_{S^n} < \eta, \tag{3.2}$$

where

$$\xi = \sqrt{\frac{1}{K}\left(1 - \frac{1}{n}\right)\left(\frac{1}{2} + \frac{r}{\sigma^2 n}\right) + \frac{rK}{(\sigma n K)^2}},$$

$$\eta = \sqrt{\frac{1}{\beta(\infty)}\left(1 - \frac{1}{n}\right)\left(\frac{1}{2} + \frac{r}{\sigma^2 n}\right) + \frac{rK}{(\sigma n \beta(\infty))^2}}.$$

Proof. Let $Q = \sqrt{P(\tau, S^n) - (K - S^n)}$. Then, $P(\tau, S^n) = Q^2 + (K - S^n)$. From this, we obtain the following relations

$$P_{S^n} = 2QQ_{S^n} - 1, \; P_{S^nS^n} = 2Q_{S^n}Q_{S^n} + 2QQ_{S^nS^n}, \; P_\tau = 2QQ_\tau. \tag{3.3}$$

By substituting (3.3) into (2.5), we obtain

$$2QQ_\tau + n\left\{\frac{1}{2}\sigma^2(n-1) + r\right\}S^n - \frac{1}{2}(\sigma n S^n)^2 2(Q_{S^n})^2 + r\left(Q^2 + K - S^n\right) = 0. \tag{3.4}$$

Further, we obtain the following equation near the optimal exercise boundary ($Q = 0$).

$$n\left\{\frac{1}{2}\sigma^2(n-1)+r\right\}S^n-(\sigma n S^n Q_{S^n})^2+r(K-S^n)=0. \tag{3.5}$$

In addition, we obtain

$$(Q_{S^n})^2=\frac{1}{S^n}\left(1-\frac{1}{n}\right)\left(\frac{1}{2}+\frac{r}{\sigma^2 n}\right)+\frac{rK}{(\sigma n S^n)^2}. \tag{3.6}$$

Therefore,

$$Q_{S^n}=\sqrt{\frac{1}{S^n}\left(1-\frac{1}{n}\right)\left(\frac{1}{2}+\frac{r}{\sigma^2 n}\right)+\frac{rK}{(\sigma n S^n)^2}}. \tag{3.7}$$

We can obtain the following near the optimal exercise boundary ($Q = 0, S^n \to \beta$).

$$Q_{S^n}=\sqrt{\frac{1}{\beta}\left(1-\frac{1}{n}\right)\left(\frac{1}{2}+\frac{r}{\sigma^2 n}\right)+\frac{rK}{(\sigma n \beta)^2}}. \tag{3.8}$$

Because $0 < \beta(\infty) < \beta(\tau) < K$ for every τ, we obtain $\frac{1}{K} < \frac{1}{\beta} < \frac{1}{\beta(\infty)}$. Furthermore,

$$\begin{aligned}
&\frac{1}{K}\left(1-\frac{1}{n}\right)\left(\frac{1}{2}+\frac{r}{\sigma^2 n}\right) < \frac{1}{\beta}\left(1-\frac{1}{n}\right)\left(\frac{1}{2}+\frac{r}{\sigma^2 n}\right)\\
&< \frac{1}{\beta(\infty)}\left(1-\frac{1}{n}\right)\left(\frac{1}{2}+\frac{r}{\sigma^2 n}\right),
\end{aligned}$$

$$\frac{rK}{(\sigma n K)^2} < \frac{rK}{(\sigma n \beta)^2} < \frac{rK}{(\sigma n \beta(\infty))^2}.$$

Therefore,

$$\begin{aligned}
&\frac{1}{K}\left(1-\frac{1}{n}\right)\left(\frac{1}{2}+\frac{r}{\sigma^2 n}\right)+\frac{rK}{(\sigma n K)^2}\\
&< \frac{1}{\beta}\left(1-\frac{1}{n}\right)\left(\frac{1}{2}+\frac{r}{\sigma^2 n}\right)+\frac{rK}{(\sigma n \beta)^2}\\
&< \frac{1}{\beta(\infty)}\left(1-\frac{1}{n}\right)\left(\frac{1}{2}+\frac{r}{\sigma^2 n}\right)+\frac{rK}{(\sigma n \beta(\infty))^2}.
\end{aligned}$$

Here, $0 < \xi < Q_{S^n} < \eta$ is satisfied, where

$$\xi=\sqrt{\frac{1}{K}\left(1-\frac{1}{n}\right)\left(\frac{1}{2}+\frac{r}{\sigma^2 n}\right)+\frac{rK}{(\sigma n K)^2}},$$

$$\eta=\sqrt{\frac{1}{\beta(\infty)}\left(1-\frac{1}{n}\right)\left(\frac{1}{2}+\frac{r}{\sigma^2 n}\right)+\frac{rK}{(\sigma n \beta(\infty))^2}}.$$

□

This theorem gives the result $\frac{\sqrt{rK}}{\sigma K} < Q_S < \frac{\sqrt{rK}}{\sigma\beta(\infty)}$ for the case in which $n = 1$ [18].

4. Numerical method

In this section, we describe the numerical method developed based on the transformed function. The independent variables in the Black–Scholes PDE are time (τ) and the underlying stock price (S^n). To solve this problem using the FDM, we divide the time interval $[0, T]$ into L subintervals, i.e., $\tau_l = l\Delta\tau, l = 0, 1, 2, \ldots, L-1, L, \Delta\tau = T/L$, and the stock price interval $[0, S^n_M]$ into M subintervals, i.e., $S^n_i = i\Delta S^n, i = 0, 1, 2, \ldots, M-1, M, \Delta S^n = \frac{S^n_M}{M}$. We intend to define a numerical method for computing the grid values $P^l_i \approx P(\tau_l, S^n_i)$ and the optimal exercise boundary values $\beta_l = \beta(\tau_l)$, for $i = 0, 1, 2, \ldots, M-1, M$, and $l = 0, 1, 2, \ldots, L-1, L$. Herein, i denotes the horizontal node index, l denotes the time step index, and τ_{l-1} represents the $(l-1)$th previous time step and τ_l represents the lth current time step. For the proposed problem, $\beta(\tau_{l-1})$ and P^{l-1}_i are given and the objective is to compute $\beta(\tau_l)$ and P^l_i.

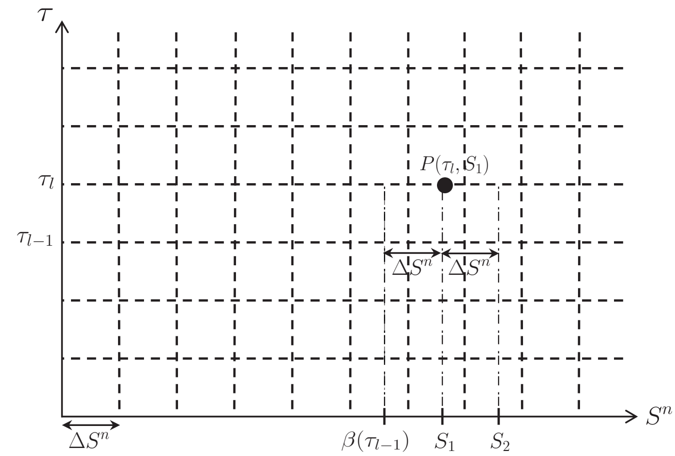


Fig. 3. Graphical representation of Step 2-1.

To solve this problem using the proposed method, there are five steps.

Step 1: We begin with the initial value at time T, which provides the value of the payoff function $(K - S^n_i)^+ = \max\{K - S^n_i, 0\}$ and the optimal exercise boundary $\beta(0) = K$. Because the optimal exercise boundary $\beta(\tau_{l-1})$ is dependent on the future boundary value $\beta(\tau_l)$, it must be determined by setting the initial boundary value to the strike price $\beta(\tau_0) = K$ and working backward through time from $T = \tau_0$ to $0 = \tau_L$.

Step 2: The second step is to obtain the current optimal exercise boundary $\beta(\tau_l)$ based on the previous optimal exercise boundary $\beta(\tau_{l-1})$. The current optimal exercise boundary can be obtained using a transformed function $Q(\tau_l, S^n) = \sqrt{P(\tau_l, S^n) - (K - S^n)}$. Herein, to obtain Q at the current time step, the option price $P(\tau_l, S^n)$ should be obtained. Further, to find $\beta(\tau_l)$, we derive the relation between $Q(\tau_l, S^n)$ and $\beta(\tau_l)$.

Step 2-1: We find $P(\tau_l, S^n)$ with respect to the current time step using a three-point FDM based on a uniform mesh. If the optimal exercise boundary does not depend on the grid point, then we cannot use the FDM based on a uniform mesh. Therefore, we use natural cubic spline interpolation. Based on the mesh point information, we can obtain value between the mesh points. We let $S_1 = \beta(\tau_{l-1}) + \Delta S^n$ and $S_2 = \beta(\tau_{l-1}) + 2\Delta S^n$; then, we find $P(\tau_l, S_1)$, as shown in Fig. 3. Because the previous optimal exercise boundary $\beta(\tau_{l-1})$ is dependent on the current optimal exercise boundary $\beta(\tau_l)$, we calculate the option price $P(\tau_l, S_1)$ on S_1 in the current time step using the explicit method.

The Black–Scholes equation (2.5) can be approximated as follows:

$$\begin{aligned}
P(\tau_l, S_1) &= (1 - r\Delta\tau)P(\tau_{l-1}, S_1) + n\left\{\frac{1}{2}\sigma^2(n-1)+r\right\}S_1\Delta\tau\\
&\frac{\partial P(\tau_{l-1}, S_1)}{\partial S}+\frac{1}{2}(\sigma n S_1)^2\frac{\partial^2 P(\tau_{l-1}, S_1)}{\partial S^2}\Delta\tau.
\end{aligned} \tag{4.1}$$

The derivatives in equation (4.1) can be represented as follows:

$$\frac{\partial P(\tau_{l-1}, S_1)}{\partial S} \simeq \frac{P(\tau_{l-1}, S_2) - P(\tau_{l-1}, S_1)}{2(S_2 - S_1)}, \tag{4.2}$$

$$\frac{\partial^2 P(\tau_{l-1}, S_1)}{\partial S^2} \simeq \frac{P(\tau_{l-1}, \beta(\tau_{l-1})) - 2P(\tau_{l-1}, S_1) + P(\tau_{l-1}, S_2)}{(S_2 - S_1)^2}. \tag{4.3}$$

At each time level, we apply the cubic spline interpolation in (4.2)–(4.3) to find $P(\tau_{l-1}, S_1)$ and $P(\tau_{l-1}, S_2)$; thus, we obtain $P(\tau_{l-1}, S_1) \approx f(\tau_{l-1}, S_1)$, $P(\tau_{l-1}, S_2) \approx f(\tau_{l-1}, S_2)$, where f is the cubic spline function. Because $P(\tau, S^n) = Q^2(\tau, S^n) + K - S^n$, we obtain $P(\tau_{l-1}, \beta(\tau_{l-1})) = K - \beta(\tau_{l-1})$ near the optimal exercise boundary ($Q = 0$). Thus, we obtain $P(\tau_l, S_1)$.

Step 2-2: By substituting (4.1) into

$$Q(\tau_l, S_1) = \sqrt{P(\tau_l, S_1) - (K - S_1)},$$

we obtain $Q(\tau_l, S_1)$.

Step 2-3: We find the relation between $\beta(\tau_l)$ and $Q(\tau_l, S_1)$. The second-order Taylor expansion of Q can be written as

$$\begin{aligned} Q(\tau, S^n) &= Q(\tau, \beta) + Q_{S^n}(\tau, \beta)(S^n - \beta) \\ &\quad + \frac{1}{2} Q_{S^n S^n}(\tau, \beta)(S^n - \beta)^2 + \mathcal{O}(S^n - \beta)^3. \end{aligned} \tag{4.4}$$

We rewrite (4.4) as follows:

$$a\beta^5 + b\beta^4 + c\beta^3 + d\beta^2 + e\beta + f = 0, \tag{4.5}$$

where a, b, c, d, e and f are constants. Subsequently, β can be calculated as the solution of (4.14)–(4.15).

From (3.5), we obtain

$$(Q_{S^n})^2 = \frac{AS^n + r(K - S^n)}{(\sigma n S^n)^2}, \tag{4.6}$$

$$Q_{S^n} = \frac{\sqrt{(A - r)\beta + rK}}{\sigma n \beta}, \tag{4.7}$$

where $A = n\left\{\frac{1}{2}\sigma^2(n-1) + r\right\}$. To obtain (4.14)–(4.15), we use the approximation of Q_{S^n}. In (4.7), we rewrite $\sqrt{(A-r)\beta + rK}$ as the form $\sum_{i=0}^{m} c_i \beta^i$, where m is the integer and c_i is a constant because the optimal exercise boundary is included in the square root representation.

Because $\beta(\tau_{l-1})$ is considerably close to $\beta(\tau_l)$, if we let $f(\beta) = \sqrt{(A-r)\beta + rK}$, the approximation of $f(\beta)$ at $\beta(\tau_{l-1})$ as follows:

$$\begin{aligned} f(\beta) &\approx f(\tilde{\beta}) + f'(\tilde{\beta})(\beta - \tilde{\beta}) + \frac{f''(\tilde{\beta})}{2!}(\beta - \tilde{\beta})^2 \\ &\approx \sqrt{(A-r)\tilde{\beta} + rK} + \frac{A - r}{2\sqrt{(A-r)\tilde{\beta} + rK}}(\beta - \tilde{\beta}) \\ &\quad - \frac{(A-r)^2}{8\left\{(A-r)\tilde{\beta} + rK\right\}\sqrt{(A-r)\tilde{\beta} + rK}}(\beta - \tilde{\beta})^2, \end{aligned} \tag{4.8}$$

where $\tilde{\beta}$ is the previous optimal exercise boundary $\beta(\tau_{l-1})$. In (4.8), we let $B = (A-r)\tilde{\beta} + rK$; thus, we obtain

$$Q_{S^n} \approx \frac{1}{\sigma n \beta}\left\{\sqrt{B} + \frac{A - r}{2\sqrt{B}}(\beta - \tilde{\beta}) - \frac{(A-r)^2}{8B\sqrt{B}}(\beta - \tilde{\beta})^2\right\}. \tag{4.9}$$

To find $Q_{S^n S^n}$, we calculate the partial derivative with respect to S^n in (2.5) as follows:

$$P_{\tau S^n} - AP_{S^n} - AS^n P_{S^n S^n} - \sigma^2 n^2 S^n P_{S^n S^n} - \frac{1}{2}(\sigma n S^n)^2 P_{S^n S^n S^n} + rP_{S^n} = 0. \tag{4.10}$$

From $P_{S^n S^n}$, we obtain $P_{S^n S^n S^n} = 6Q_{S^n} Q_{S^n S^n}$. From $P_{S^n} = -1$, we obtain $P_{S^n \tau} = -P_{S^n S^n}\beta'$, where β' is the rate of change of β over time. Hence, we obtain the following equation near the optimal exercise boundary ($Q = 0$).

$$Q_{S^n S^n} = -\frac{2Q_{S^n}^2(\beta' + A\beta + \sigma^2 n^2 \beta) - (A - r)}{3(\sigma n \beta)^2 Q_{S^n}}. \tag{4.11}$$

Further, we rewrite (4.4) as

$$\begin{aligned} Q &\approx \frac{1}{\sigma n \beta}\left\{\sqrt{B} + \frac{A - r}{2\sqrt{B}}(\beta - \tilde{\beta}) - \frac{(A-r)^2}{8B\sqrt{B}}(\beta - \tilde{\beta})^2\right\}(S^n - \beta) \\ &\quad - \frac{2Q_{S^n}^2\left(\beta' + A\beta + \sigma^2 n^2 \beta\right) - (A - r)}{6(\sigma n \beta)^2 Q_{S^n}}(S^n - \beta)^2, \end{aligned} \tag{4.12}$$

and multiply both sides with $6(\sigma n \beta)^2 Q_{S^n}$. Then, (4.12) can be written as

$$\begin{aligned} 6(\sigma n \beta)^2 Q_{S^n} Q &= 6(\sigma n \beta Q_{S^n})^2 (S^n - \beta) \\ &\quad - \left\{2Q_{S^n}^2\left(\beta' + A\beta + \sigma^2 n^2 \beta\right) - (A - r)\right\}(S^n - \beta)^2. \end{aligned} \tag{4.13}$$

By combining (4.13) with $\frac{\beta'}{\beta} \approx \frac{\ln(\frac{S^n}{\beta}) - (\frac{S^n}{\beta} - 1)}{\Delta\tau}$ in [18] and we let $C = 3\sigma n Q \Delta\tau$, then we obtain

$$a\beta^5 + b\beta^4 + c\beta^3 + d\beta^2 + e\beta + f = 0, \tag{4.14}$$

where

$$\begin{aligned} a &= -C\frac{(A-r)^2}{4B\sqrt{B}}, \\ b &= C\frac{A - r}{\sqrt{B}} + C\frac{(A-r)^2\tilde{\beta}}{2B\sqrt{B}} + 5\Delta\tau(A - r) \\ &\quad + \frac{2(A - r)}{(\sigma n)^2}\left\{\ln\left(\frac{S^n}{\tilde{\beta}} + 1 + \Delta\tau\left(A + (\sigma n)^2\right)\right)\right\}, \\ c &= 2C\sqrt{B} - C\frac{(A-r)\tilde{\beta}}{\sqrt{B}} - C\frac{(A-r)^2\tilde{\beta}^2}{4B\sqrt{B}} - \Delta\tau\{4(A-r)S^n - 6rK\} \\ &\quad + \frac{2rK}{(\sigma n)^2}\left\{\ln\left(\frac{S^n}{\tilde{\beta}}\right) + 1 + \Delta\tau\left(A + (\sigma n)^2\right)\right\} \\ &\quad - \frac{2(A-r)S^n}{(\sigma n)^2}\left\{3 + 2\ln\left(\frac{S^n}{\tilde{\beta}}\right) + 2\Delta\tau\left(A + (\sigma n)^2\right)\right\}, \\ d &= -6rKS^n\Delta\tau - (A - r)\Delta\tau(S^n)^2 \\ &\quad - \frac{2rKS^n}{(\sigma n)^2}\left\{3 + 2\ln\left(\frac{S^n}{\tilde{\beta}}\right) + 2\Delta\tau\left(A + (\sigma n)^2\right)\right\} \\ &\quad + \frac{2(A-r)(S^n)^2}{(\sigma n)^2}\left\{3 + \ln\left(\frac{S^n}{\tilde{\beta}}\right) + \Delta\tau\left(A + (\sigma n)^2\right)\right\}, \\ e &= \frac{2rK(S^n)^2}{(\sigma n)^2}\left\{3 + \ln\left(\frac{S^n}{\tilde{\beta}}\right) + \Delta\tau\left(A + (\sigma n)^2\right)\right\} - \frac{2(A-r)}{(\sigma n)^2}(S^n)^3, \\ f &= -\frac{2rK}{(\sigma n)^2}(S^n)^3. \end{aligned} \tag{4.15}$$

To solve for equations (4.14)–(4.15), we use the Newton–Raphson method. Then, we find the solution β at the current optimal exercise boundary.

Step 3: The next step is to obtain the option price over $\beta(\tau_l)$ of Step 2. As shown in Fig. 4, let j be the smallest spatial grid index over $\beta(\tau_l)$. If $\beta(\tau_l)$ is too close to S_j^n, we select the index $j+1$ instead of j.

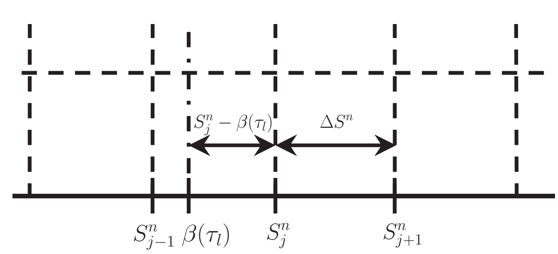


Fig. 4. Graphical representation of Step 3.

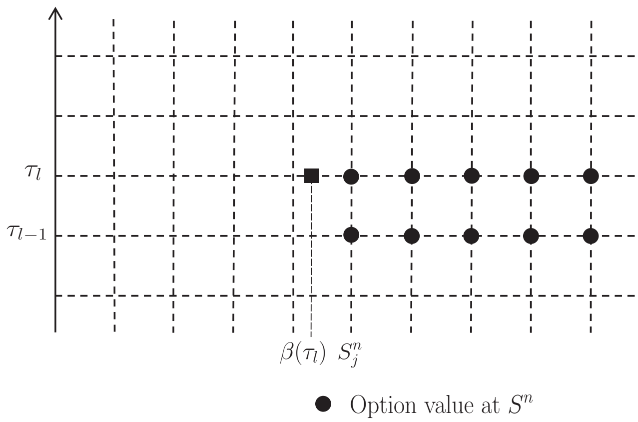


Fig. 5. Graphical representation of Step 3.

If we use the central difference for the spatial discretization, then

$$\frac{\partial P_j^l}{\partial S_j^n} = \frac{P_{j+1}^l - P_{j-1}^l}{2\Delta S^n}, \ \frac{\partial^2 P_j^l}{\partial \left(S_j^n\right)^2} = \frac{P_{j+1}^l - 2P_j^l + P_{j-1}^l}{(\Delta S^n)^2}. \tag{4.16}$$

Further, if we use the implicit scheme for time discretization, (2.5) is discretized as follows:

$$\frac{P_j^l - P_j^{l-1}}{\Delta\tau} - AS_j^n \frac{P_{j+1}^l - P_{j-1}^l}{2\Delta S^n} - \frac{1}{2}\left(\sigma n S_j^n\right)^2 \frac{P_{j+1}^l - 2P_j^l + P_{i-1}^l}{(\Delta S^n)^2} + rP_i^l = 0. \tag{4.17}$$

Thus, we obtain

$$(1 + r\Delta\tau)P_j^l - AS_j^n \frac{P_{j+1}^l - P_{j-1}^l}{2\Delta S^n}\Delta\tau - \frac{1}{2}\left(\sigma n S_j^n\right)^2 \frac{P_{j+1}^l - 2P_j^l + P_{j-1}^l}{(\Delta S^n)^2}\Delta\tau = P_j^{l-1}. \tag{4.18}$$

Generally, the optimal exercise boundary may not be located at the grid points. Therefore, we cannot use FDM directly. Because we know the values of P at β, we consider using linear interpolation between adjacent data points, as shown in Fig. 5.

The value of P at β can be written as

$$P(\tau_l, \beta(\tau_l)) = \frac{(S_j^n - \beta(\tau_l))P_{j-1}^l + (\Delta S - S_j^n + \beta(\tau_l))P_j^l}{\Delta S^n} = K - \beta(\tau_l). \tag{4.19}$$

From (4.19), we obtain

$$P_{j-1}^l = \frac{(K - \beta(\tau_l))\Delta S^n - (\Delta S^n - S_j^n + \beta(\tau_l))P_j^l}{S_j^n - \beta(\tau_l)}. \tag{4.20}$$

Hence, by substituting (4.20) into (4.16), we obtain

$$\frac{\partial P_j^l}{\partial S_j^n} = \frac{P_{j+1}^l - P_{j-1}^l}{2\Delta S} = \frac{P_{j+1}^l(S_j^n - \beta(\tau_l)) - \{(K - \beta(\tau_l))\Delta S - (\Delta S - S_j^n + \beta(\tau_l))P_j^l\}}{2(S_j^n - \beta(\tau_l))\Delta S^n}, \tag{4.21}$$

$$\frac{\partial^2 P_j^l}{\partial \left(S_j^n\right)^2} = \frac{P_{j+1}^l - 2P_j^l + P_{j-1}^l}{(\Delta S^n)^2} = \frac{\left(P_{j+1}^l - 2P_j^l\right)\left(S_j^n - \beta(\tau_l)\right)}{\left(S_j^n - \beta(\tau_l)\right)(\Delta S^n)^2} + \frac{(K - \beta(\tau_l))\Delta S^n - P_j^l\left(\Delta S^n - S_j^n + \beta(\tau_l)\right)}{\left(S_j^n - \beta(\tau_l)\right)(\Delta S^n)^2}. \tag{4.22}$$

Further, by substituting (4.21) and (4.22) into (4.18), we obtain

$$\begin{aligned}
&(1 + r\Delta\tau)P_j^l \\
&- AS_j^n \Delta\tau \frac{P_{j+1}^l\left(S_j^n - \beta(\tau_l)\right) + P_j^l\left(\Delta S^n - S_j^n + \beta(\tau_l)\right) - \Delta S^n(K - \beta(\tau_l))}{2\left(S_j^n - \beta(\tau_l)\right)\Delta S^n} \\
&- \frac{1}{2}\left(\sigma n S_j^n\right)^2 \Delta\tau \frac{\left(P_{j+1}^l - 2P_j^l\right)\left(S_j^n - \beta(\tau_l)\right)}{\left(S_j^n - \beta(\tau_l)\right)(\Delta S^n)^2} \\
&+ \frac{1}{2}\left(\sigma n S_j^n\right)^2 \Delta\tau \frac{P_j^l\left(\Delta S^n - S_j^n + \beta(\tau_l)\right) - (K - \beta(\tau_l))\Delta S^n}{\left(S_j^n - \beta(\tau_l)\right)(\Delta S^n)^2} \\
&= P_j^{l-1}.
\end{aligned} \tag{4.23}$$

We rewrite (4.23) as follows:

$$a_1 P_\beta^l + b_1 P_j^l + c_1 P_{j+1}^l = P_j^{l-1}, \tag{4.24}$$

where

$$\begin{aligned}
a_1 &= -\frac{1}{2}\left(\sigma n S_j^n\right)^2 \Delta\tau \frac{1}{\left(S_j^n - \beta(\tau_l)\right)\Delta S^n} + AS_j^n \Delta\tau \frac{1}{2\left(S_j^n - \beta(\tau_l)\right)}, \\
b_1 &= 1 + r\Delta\tau - AS_j^n \Delta\tau \frac{\Delta S^n - S_j^n + \beta(\tau_l)}{2\left(S_j^n - \beta(\tau_l)\right)\Delta S^n} \\
&\quad + \frac{1}{2}\left(\sigma n S_j^n\right)^2 \Delta\tau \left\{ \frac{S_j^n - \beta(\tau_l) + \Delta S^n}{\left(S_j^n - \beta(\tau_l)\right)(\Delta S^n)^2} \right\}, \\
c_1 &= -AS_j^n \Delta\tau \frac{1}{2\Delta S^n} - \frac{1}{2}\left(\sigma n S_j^n\right)^2 \Delta\tau \frac{1}{(\Delta S^n)^2}.
\end{aligned} \tag{4.25}$$

Except for the above case, we rewrite (4.18):

$$a_i P_{j-1}^l + b_i P_j^l + c_i P_{j+1}^l = P_j^{l-1}, \tag{4.26}$$

where

$$\begin{aligned}
&a_i = \mu_i - \lambda_i, b_i = (1 + r\Delta\tau) + 2\lambda_i, c_i = -(\mu_i + \lambda_i), \\
&\mu_i = A\frac{\Delta\tau}{2\Delta S^n} S_{j+i-1}^n, \lambda_i = \frac{(\sigma n)^2}{2(\Delta S^n)^2}\Delta\tau\left(S_{j+i-1}^n\right)^2, \ 2 \le i \le M - j + 1.
\end{aligned} \tag{4.27}$$

Two boundary conditions have to be implemented. By the condition (2.7), the Dirichlet boundary condition at $\beta(\tau_l)$ gives $K - \beta(\tau_l)$ and we apply the following linear boundary condition at S_M^n.

$$\frac{\partial^2 P}{\partial (S^n)^2}(\tau, S_M^n) = 0, 0 \le \tau \le T. \tag{4.28}$$

It assumes that the second derivative of the option value with respect to the underlying asset price S^n vanishes to zero for a large value of the asset price. Thus,

$$\frac{P_{M-2} - 2P_{M-1} + P_M}{(\Delta S^n)^2} = 0.$$

To obtain the option prices at time step l, a set of simultaneous linear equations (4.24)–(4.28) has to be solved, which can be cast in the matrix form $W\mathbf{x} = \mathbf{y}$.

$$\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
a_1 & b_1 & c_1 & \cdots & 0 & 0 \\
0 & a_2 & b_2 & c_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & a_{M-j} & b_{M-j} & c_{M-j} \\
0 & 0 & \cdots & 0 & a_{M-j+1} & b_{M-j+1}
\end{pmatrix}
\begin{pmatrix}
P_\beta^l \\ P_j^l \\ P_{j+1}^l \\ \vdots \\ P_{M-1}^l \\ P_M^l
\end{pmatrix}
=
\begin{pmatrix}
K - \beta(\tau_l) \\ P_j^{l-1} \\ P_{j+1}^{l-1} \\ \vdots \\ P_{M-1}^{l-1} \\ P_M^{l-1}
\end{pmatrix}, \tag{4.29}$$

where $a_{M-j+1} = 2\mu_{M-j+1}, b_{M-j+1} = 1 + r\Delta\tau - 2\mu_{M-j+1}$. To solve this system, we verify if the coefficient matrix W is invertible. In (4.29), for the sake of simplicity we divide both sides by $\Delta\tau$. By following Proposition 4.3, we verify the existence and uniqueness of the system (4.29).

Definition 4.1. An $n \times n$ matrix is said to be strictly diagonally dominant when

$$|a_{ss}| > \sum_{t=1, s\neq t}^{n} |a_{st}|$$

holds for $s = 1, 2, \ldots, n$, where a_{st} denotes the entry in the sth row and tth column.

Lemma 4.2. *The following are equivalent conditions on the $n \times n$ square matrix W.*

(1) *The matrix W is invertible.*
(2) *The linear system $W\mathbf{x} = \mathbf{y}$ is consistent for every $\mathbf{y}$.*
(3) *The linear system $W\mathbf{x} = \mathbf{y}$ has a unique solution for every $\mathbf{y}$.*

Proposition 4.3. *If*

$$\frac{1}{\Delta\tau} + r > A\tilde{C}(M+1), \tag{4.30}$$

then W is strictly diagonally dominant, where $\tilde{C}$ is a constant. Therefore, the linear system (4.29) has a unique solution.

Proof. In (4.29), it suffices to verify that $|b_i| - (|a_i| + |c_i|) > 0$ for all $i = 2, \ldots, M-j$.

Since

$$|a_i| + |c_i| \leq A\frac{S^n_{j+i-1}}{\Delta S^n} + \frac{(\sigma n)^2}{(\Delta S^n)^2}(S^n_{j+i-1})^2, \quad 2 \leq i \leq M-j.$$

We have the following

$$\begin{aligned} |b_i| - (|a_i| + |c_i|) &\geq \frac{1}{\Delta\tau} + r - A\frac{S^n_{j+i-1}}{\Delta S^n} = \frac{1}{\Delta\tau} + r - A(j+i-1) \\ &\geq \frac{1}{\Delta\tau} + r - AM, \quad 2 \leq i \leq M-j. \end{aligned}$$

When $i = 1$, it suffices to verify that $|b_1| - (|a_1| + |c_1|) > 0$.

Since

$$\begin{aligned} |a_1| + |c_1| \leq{}& \frac{(\sigma n S^n_j)^2}{2\Delta S^n}\left\{\frac{\Delta S^n + S^n_j - \beta(\tau_l)}{(S^n_j - \beta(\tau_l))\Delta S^n}\right\} \\ &+ \frac{A}{2}S^n_j\left\{\frac{\Delta S^n + S^n_j - \beta(\tau_l)}{(S^n_j - \beta(\tau_l))\Delta S^n}\right\}. \end{aligned}$$

We have that

$$\begin{aligned} |b_1| - (|a_1| + |c_1|) &\geq \frac{1}{\Delta\tau} + r - A\frac{S^n_j}{\Delta S^n}\left\{\frac{\Delta S^n + S^n_j - \beta(\tau_l)}{S^n_j - \beta(\tau_l)}\right\} \\ &\geq \frac{1}{\Delta\tau} + r - AM(1+L), \end{aligned}$$

where $L = \frac{\Delta S^n}{S^n_j - \beta(\tau_l)}$.

When $i = M-j+1$, it suffices to verify that $|b_{M-j+1}| - |a_{M-j+1}| > 0$.

Since

$$\left|a_{M-j+1}\right| \leq A\frac{S^n_M}{\Delta S^n} = AM.$$

We have that

$$\left|b_{M-j+1}\right| - \left|a_{M-j+1}\right| \geq \frac{1}{\Delta\tau} + r - 2AM.$$

Hence, if

$$\frac{1}{\Delta\tau} + r > A\tilde{C}M, \tag{4.31}$$

then W is strictly diagonally dominant, where $\tilde{C}$ is a constant. Clearly, (4.30) implies (4.31). Every strictly diagonally dominant matrix is invertible. By the Lemma 4.2, we prove the existence and uniqueness of the solution to (4.29). □

The system can be solved by finding the inverse matrix W^{-1}. Therefore, we can obtain the option value for the next time level.

Step 4: If there are spatial grids, where the option value is not calculated among them over the optimal exercise boundary, then we obtain the put option value by extrapolation.

Step 5: We repeat the previously mentioned process (from Steps 1 to 4) until τ_L and obtain the optimal exercise boundary in a time-recursive manner.

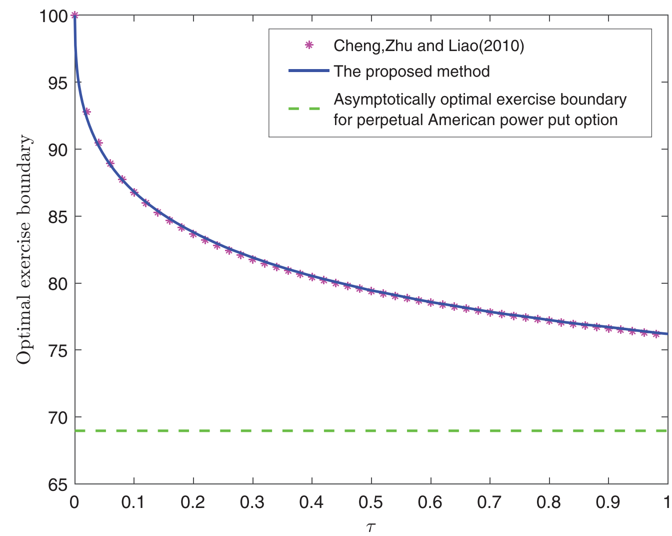


Fig. 6. Comparison of the optimal exercise boundaries.

5. Numerical experiments

In this section, we present numerical examples to illustrate the proposed method. All the implementations are performed using MATLAB with a 2.3 GHz Intel Core i5 CPU with 8GB RAM. Initially, we show the result obtained using the proposed method when $n = 1$ for the standard American option. An FDM with the backward Euler method is used as the proposed method. As shown in Fig. 6, we compare the results obtained using the proposed method with those obtained using the method proposed by Cheng, Zhu, and Liao [9] and the asymptotically optimal exercise boundary for the perpetual American power put option. In a previous study [9], we reference the 19th order $o(\tau^6)$ HAM approximation. We consider the strike price $K = 100$, the interest rate $r = 0.1$, the volatility $\sigma = 0.3$, and the time to maturity $T = 1$ (year). Further, we construct the computational domain with 250 spatial steps and 1000 time steps. At the maturity time, the optimal exercise boundary in [9] a numerical solution is $\beta(T) = 76.0964$, whereas it is $\beta(T) = 76.1994$ in the proposed method. The asymptotically optimal exercise boundary with respect to the perpetual American power put option is $\beta(\infty) = 68.9655$.

Fig. 7 show the option values. The parameter values used to calculate the optimal exercise boundary and values of the American put options are $r = 0.05, \sigma = 0.2, K = 100, T = 1$ and $n = 1$ and the computational domain can be constructed with 300 spatial steps and 2000 time steps.

Table 1 presents the values of the American put options for a specific parameter set. We also compare the results obtained using other numerical methods, including the binomial method (Binomial) developed by Cox et al. [10], the front-fixing method (Front-

Table 1
Comparison of the values of the American put options and RMSE.

(S, T, r, σ)	Binomial	Front-fixing	MBM-FDM	Simple method	The proposed method
(80,0.5,0.05,0.20)	20.0000	20.0000	20.0000	20.0000	20.0000
(90,0.5,0.05,0.20)	10.6661	10.6643	10.6680	10.6661	10.6646
(100,0.5,0.05,0.20)	4.6556	4.6501	4.6504	4.6549	4.6537
(110,0.5,0.05,0.20)	1.6681	1.6629	1.6631	1.6686	1.6669
(120,0.5,0.05,0.20)	0.4976	0.4961	0.4993	0.4985	0.4974
RMSE		0.0035	0.0034	0.0006	0.0012
(80,1.0,0.05,0.20)	20.0000	20.0000	20.0000	20.0000	20.0002
(90,1.0,0.05,0.20)	11.4928	11.4924	11.4857	11.4929	11.4907
(100,1.0,0.05,0.20)	6.0903	6.0893	6.0829	6.0905	6.0880
(110,1.0,0.05,0.20)	2.9866	2.9856	2.9854	2.9868	2.9847
(120,1.0,0.05,0.20)	1.3672	1.3654	1.3643	1.3674	1.3660
RMSE		0.0010	0.0048	0.0002	0.0017
(80,3.0,0.08,0.20)	20.0000	20.0000	20.0000	20.0000	20.0000
(90,3.0,0.08,0.20)	11.6974	11.9029	11.6892	11.6977	11.6954
(100,3.0,0.08,0.20)	6.9320	7.2527	6.9221	6.9321	6.9300
(110,3.0,0.08,0.20)	4.1550	4.4841	4.1443	4.1548	4.1527
(120,3.0,0.08,0.20)	2.5102	2.7760	2.4997	2.5102	2.5077
RMSE		0.2546	0.0088	0.0002	0.0020

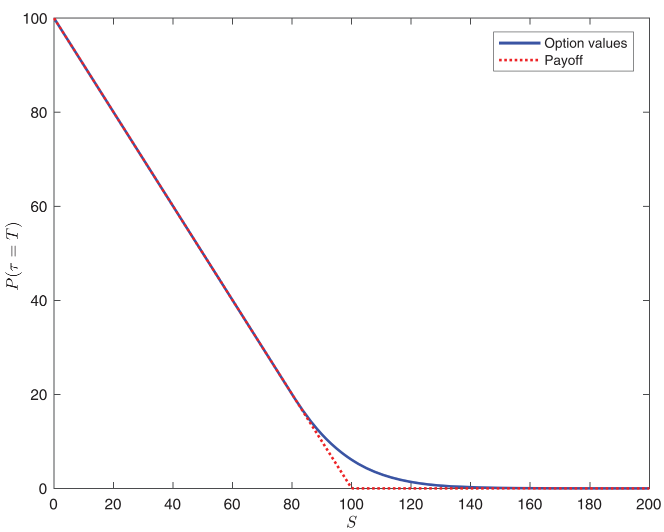
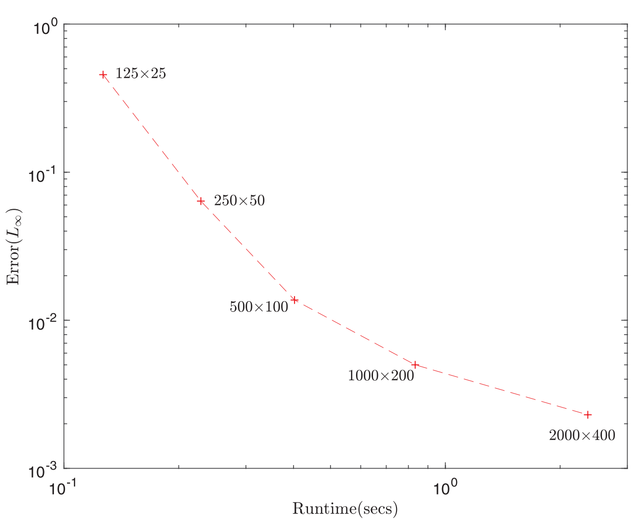


Fig. 7. American power put option value for $n = 1$.

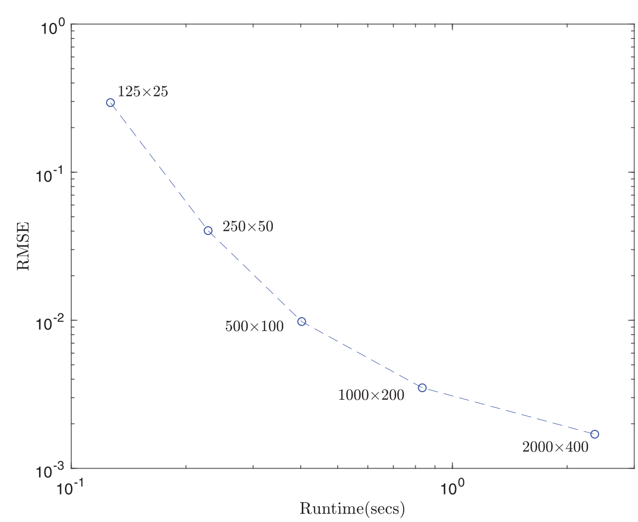
fixing) developed by Wu and Kwok [40], and the finite difference implementation of the moving-boundary method (MBM-FDM) developed by Muthuraman [32] and the simple numerical method (Simple method) developed by Kim, Ma, and Choe [17]. The benchmark results are obtained using the binomial method with 10000 time steps, and these results are considered to be the exact values associated with the American put options. In the following results, the root mean square error (RMSE) and maximum error (L_∞) are calculated based on the values obtained using the binomial method. In Fig. 8 and Table 2, we consider the parameter values used in Fig. 7 except for the discrete mesh. Further, we calculate the L_∞ and RMSE associated with the runtime (s). Thus, the

Table 2
Comparison of the values of the runtime and errors

Mesh size($L \times M$)	(runtime, L_∞)	(runtime, RMSE)
125 × 25	(0.1267,0.4556)	(0.1267,0.2951)
250 × 50	(0.2286,0.0638)	(0.2286,0.0403)
500 × 100	(0.4024,0.0137)	(0.4024,0.0098)
1000 × 200	(0.8337,0.005)	(0.8337,0.0035)
2000 × 400	(2.3630,0.0023)	(2.3630,0.0017)



(a) Comparison of L_∞ and runtimes



(b) Comparison of RMSE and runtimes

Fig. 8. Comparison of the values of the runtimes and errors.

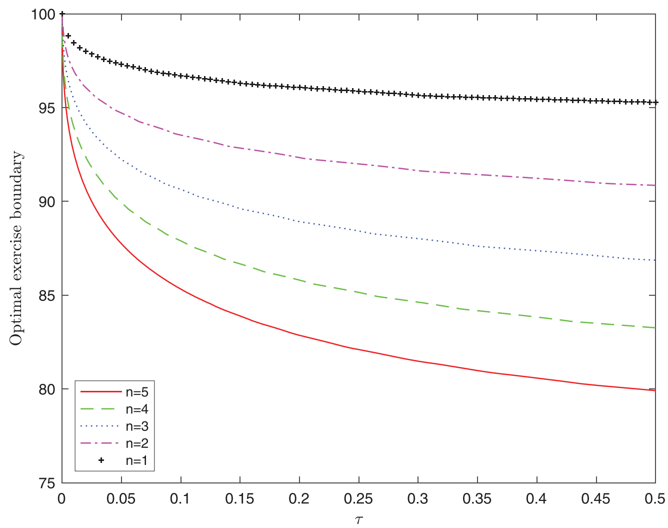


Fig. 9. Optimal exercise boundary for different values of power n.

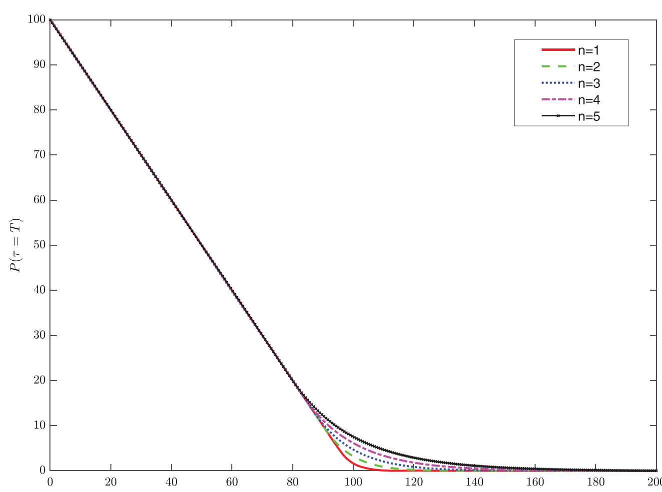


Fig. 10. American power put option values for different values of power n.

numerical convergence of the proposed method can be demonstrated. Note that the discrete meshes of $125 \times 25, 250 \times 50, 500 \times 100, 1000 \times 200$, and 2000×400 nodes are plotted in Fig. 8 and Table 2.

In the following experiments, we use $\sigma = 0.1, r = 0.08, K = 100, T = 0.5$, and $L \times M = 2000 \times 300$. Fig. 9 reports the optimal exercise boundary for different values of power $n = 1, 2, 3, 4, 5$. Further, the optimal exercise boundaries are plotted as a function of time τ. The optimal exercise boundaries decrease with the increasing power.

Fig. 10 demonstrates the American power put option values obtained when power $n = 1, 2, 3, 4, 5$. The American power put option values increase with the increasing value of power n. The feature of nonlinear payoffs of power options provides the buyer with a potential to receive a considerably higher payoff than that received from a vanilla option.

From Fig. 11, we observe the American power put option values for different values of power $n = 1.9, 1.95, 2.0, 2.05, 2.10$ and $\sigma = 0.1, 0.15, 0.2, 0.25, 0.3, 0.35$. This result is calculated at $S = 95$. Based on Fig. 11, the American power put option value increase with the increasing volatility.

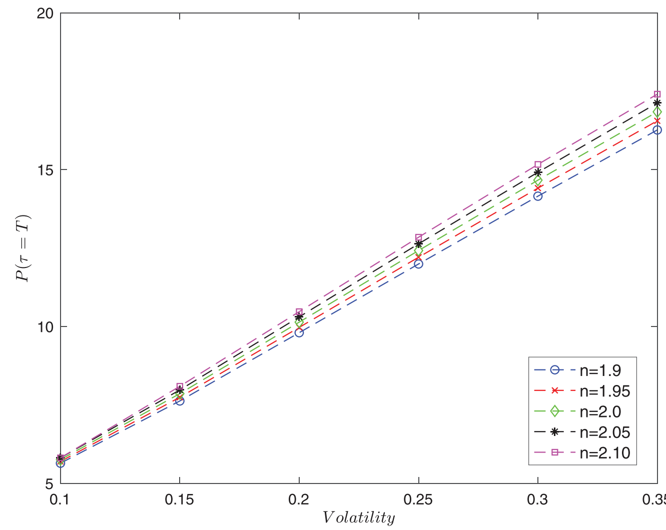


Fig. 11. American power put option values for different values of power n and σ.

6. Final remarks

In this paper, we provide a numerical method and examples for pricing the American power put option for non-dividend yields. Further, we can directly determine the optimal exercise boundary using the transformed function. In such a rapidly changing environment, the financial markets can be appropriately understood using a fast and efficient numerical method. We have compared the other results with the results obtained using the proposed method for the valuation of the American power put option when $n = 1$ to provide sufficient numerical analysis. In addition, the numerical experiments conducted using the proposed method exhibit stability and good convergence with the linear rate. In addition, we provide the optimal exercise boundary and American power put option values for different values of power n. This is expected because of the leverage feature of the power option, where a small change in the underlying of the power option may considerably change the price of the power option. The numerical experiment denotes that the proposed method is accurate, flexible, and efficient and provides accurate prices with respect to the critical stock price for various parameter combinations.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Jung-Kyung Lee: Conceptualization, Methodology, Software, Data curation, Formal analysis, Writing - original draft, Visualization, Investigation, Validation, Writing - review & editing.

Acknowledgements

The author is supported by the National Research Foundation of Korea (NRF) under Grant No. 2019R1G1A1099148.

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