and

$$\begin{aligned}\mathrm{Var}\,[S'] &= S^2 \mathrm{e}^{2\mu dt+\sigma^2 dt} - S^2 \mathrm{e}^{2\mu dt} \\ &= S^2 \mathrm{e}^{2\mu dt}\left(\mathrm{e}^{\sigma^2 dt} - 1\right),\end{aligned}$$

where we have used the relation (2.5).

If m and σ in the expression (2.4) are constants, then for a large time period $T-t$, we can have

$$\ln S' - \ln S = \int_t^T d\ln S = m\int_t^T dt + \sigma\int_t^T dX(t) = m(T-t) + \sigma\phi\sqrt{T-t},$$

where S' is the stock price at time T, S is the stock price at time t, and ϕ is a standardized normal random variable. Here we used the relation $\int_t^T dX(t) = \phi\sqrt{T-t}$, which can be obtained from the relation (2.2). Therefore, in this case, the probability density function for S' is

$$G(S') = \frac{1}{S'\sigma\sqrt{2\pi(T-t)}}\mathrm{e}^{-[\ln S' - \ln S - m(T-t)]^2/2\sigma^2(T-t)}$$

and

$$\begin{cases}\mathrm{E}\,[S'] = S\mathrm{e}^{\mu(T-t)}, \\ \mathrm{Var}\,[S'] = S^2\mathrm{e}^{2\mu(T-t)}\left[\mathrm{e}^{\sigma^2(T-t)} - 1\right],\end{cases} \tag{2.6}$$

where μ is given by the relation (2.5):

$$\mu = m + \frac{\sigma^2}{2}.$$

2.2 Derivation of the Black–Scholes Equation

2.2.1 Arbitrage Arguments

In the modern world, financial transactions may be done simultaneously in more than one market. Suppose the price of gold is \$324 per ounce in New York and 37,275 Japanese Yen in Tokyo, while the exchange rate is 1 U.S. dollar for 115 Japanese Yen. An arbitrageur, who is always looking for any arbitrage opportunities to make money, could simultaneously buy 1,000 ounces in New York, sell them in Tokyo to obtain a risk-free profit of

$$37{,}275 \times 1{,}000/115 - 324 \times 1{,}000 = \$130.43$$

if the transaction costs can be ignored. Small investors may not profit from such opportunity due to the transaction costs. However, the transaction costs for large investors might be a small portion of the profit, which makes the arbitrage opportunity very attractive.

Arbitrage opportunities usually cannot last long. As arbitrageurs buy the gold in New York, the price of the gold will rise. Similarly, as they sell the gold in Tokyo, the price of the gold will be driven down. Very quickly, the ratio between the two prices will become closer to the current exchange rate. In practice, due to the existence of arbitrageurs, very few arbitrage opportunities can be observed. Therefore, throughout this book we will assume that there are no arbitrage opportunities for financial transactions.

Let us make the following assumptions: both the borrowing short-term interest rate and the lending short-term interest rate are equal to r, short selling is permitted, the assets and options are divisible, and there is no transaction cost. Then, we can conclude that the absence of arbitrage opportunities is equivalent to all risk-free portfolios having the same return rate r.

Let us show this point. Suppose that Π is the value of a portfolio and that during a time step dt the return of the portfolio $d\Pi$ is risk-free. If

$$d\Pi > r\Pi dt,$$

then an arbitrageur could make a risk-free profit $d\Pi - r\Pi dt$ during the time step dt by borrowing an amount Π from a bank to invest in the portfolio. Conversely, if

$$d\Pi < r\Pi dt,$$

then the arbitrageur would short the portfolio and invest Π in a bank and get a net income $r\Pi dt - d\Pi$ during the time step dt without taking any risk. Only when

$$d\Pi = r\Pi dt$$

holds, is it guaranteed that there are no arbitrage opportunities.

In the next subsection, we will derive the equation the prices of derivative securities should satisfy by using the conclusion that all risk-free portfolios have the same return rate r.

2.2.2 The Black–Scholes Equation

Let V denote the value of an option that depends on the value of the underlying asset S and time t, i.e., $V = V(S,t)$. It is not necessary at this stage to specify whether V is a call or a put; indeed, V can even be the value of a whole portfolio of various options. For simplicity, readers may think of a simple call or put.

Assume that in a time step dt, the underlying asset pays out a dividend SD_0dt, where D_0 is a constant known as the dividend yield.[6] Suppose S satisfies Eq. (2.1):

[6] This dividend structure is a good model for an index. In this case, many discrete dividend payments are paid at many different times, and the dividend payment can be approximated by a continuous yield without serious error. Also, if the asset is a foreign currency, then the interest rate for the foreign currency plays the role of D_0.

$$\frac{dS}{S} = \mu(S,t)dt + \sigma(S,t)dX.$$

According to Itô's lemma (2.3), the random walk followed by V is given by

$$dV = \frac{\partial V}{\partial S}dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt. \tag{2.7}$$

Here we require V to have at least one t derivative and two S derivatives.

Now construct a portfolio consisting of one option and a number $-\Delta$ of the underlying asset. This number is as yet unspecified. The value of this portfolio is

$$\Pi = V - \Delta S. \tag{2.8}$$

Because the portfolio contains one option and a number $-\Delta$ of the underlying asset, and the owner of the portfolio receives SD_0dt for every asset held, the earnings for the owner of the portfolio during the time step dt is

$$d\Pi = dV - \Delta\left(dS + SD_0dt\right).$$

Using the relation (2.7), we find that Π follows the random walk

$$d\Pi = \left(\frac{\partial V}{\partial S} - \Delta\right)dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta SD_0\right)dt.$$

The random component in this random walk can be eliminated by choosing

$$\Delta = \frac{\partial V}{\partial S}. \tag{2.9}$$

This results in a portfolio whose increment is wholly deterministic:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta SD_0\right)dt. \tag{2.10}$$

Because the return for any risk-free portfolio should be r, we have

$$r\Pi dt = d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta SD_0\right)dt. \tag{2.11}$$

Substituting the relations (2.8) and (2.9) into Eq. (2.11) and dividing by dt, we arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} - rV = 0. \tag{2.12}$$

When we take different Π for different S and t, we can conclude that Eq. (2.12) holds on a domain. In this book, Eq. (2.12) is called the Black–Scholes partial differential equation, or the Black–Scholes equation,[7] even though $D_0 = 0$ in the equation originally given by Black and Scholes (see [11]). With its extensions and variants, it plays the major role in the rest of the book.

About the derivation of this equation and the equation itself, we give more explanation here.

[7] It is also called Black–Scholes–Merton differential equation (see [43]).

- The key idea of deriving this equation is to eliminate the uncertainty or the risk. $d\Pi$ is not a differential in the usual sense. It is the earning of the holder of the portfolio during the time step dt. Therefore, $\Delta S D_0 dt$ appear. In the derivation, in order to eliminate any small risk, Δ is chosen before an uncertainty appears and does not depend on the coming risk. Therefore, no differential of Δ is needed.
- The linear differential operator given by

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r$$

has a financial interpretation as a measure of the difference between the return on a hedged option portfolio

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} - D_0 S\frac{\partial}{\partial S}$$

and the return on a bank deposit

$$r\left(1 - S\frac{\partial}{\partial S}\right).$$

Although the difference between the two returns is identically zero for European options, we will later see that the difference between the two returns may be nonzero for American options.
- From the Black–Scholes equation (2.12), we know that the parameter μ in Eq. (2.1) does not affect the option price, i.e., the option price determined by this equation is independent of the average return rate of an asset price per unit time.
- From the derivation procedure of the Black-Scholes equation we know that the Black-Scholes equation still holds if r and D_0 are functions of S and t.
- If dividends are paid only on certain dates, then the money the owner of the portfolio will get during the time period $[t, t+dt]$ is

$$dV - \Delta dS - \Delta D(S,t)dt,$$

where $D(S,t)$ is a sum of several Dirac delta functions. Suppose that a stock pays dividend $D_1(S)$ at time t_1 and $D_2(S)$ at time t_2 for a share, where $D_1(S) \le S$ and $D_2(S) \le S$. Then

$$D(S,t) = D_1(S)\delta(t - t_1) + D_2(S)\delta(t - t_2),$$

where the Dirac delta function[8] $\delta(t)$ is defined as follows:

[8] It is the limit as $\varepsilon \to 0$ of the one-parameter family of functions:

$$\delta_\varepsilon(x) = \begin{cases} \dfrac{1}{2\varepsilon}, & -\varepsilon \le x \le \varepsilon, \\ 0, & |x| > \varepsilon. \end{cases}$$

$$\delta(t) = \begin{cases} 0, & \text{if } t \neq 0, \\ \infty, & \text{if } t = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) = 1.$$

In this case, the modified Black–Scholes equation is in the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + [rS - D(S,t)] \frac{\partial V}{\partial S} - rV = 0. \tag{2.13}$$

2.2.3 Final Conditions for the Black–Scholes Equation

From the derivation of the Black–Scholes equation (2.12), we know that this partial differential equation holds for any option (or portfolio of options) whose value depends only on S and t. In order to determine a unique solution of the Black–Scholes equation, the solution at the expiry, $t = T$, needs to be given. This condition is called the final condition for the partial differential equation. Different options satisfy the same partial differential equation, but different final conditions. Therefore, in order to determine the price of an option, we need to know the value of the option at time T. In what follows, we will derive the final conditions for call and put options.

Final Condition for Call Options. Let us examine what a holder of a call option will do just at the moment of expiry. If $S > E$ at expiry, it makes financial sense for the holder to exercise the call option, handing over an amount E for an asset worth S. The money earned by the holder from such a transaction is then $S - E$. On the other hand, if $S < E$ at expiry, the holder should not exercise the option because the holder would lose an amount of $E - S$. In this case, the option expires valueless. Thus, the value of the call option at expiry can be written as

$$C(S,T) = \max(S - E, 0). \tag{2.14}$$

This function giving the value of a call option at expiry is usually called the payoff function of a call option. In Fig. 1.9, we plot $\max(S-E,0)$ as a function of S, which is usually known as a payoff diagram. A call option with such a payoff is the simplest call option and is known as a vanilla call option.

Final Condition for Put Options. Each option or each portfolio of options has its own payoff at expiry. An argument similar to that given above for the value of a call at expiry leads to the payoff for a put option. At expiry, the put option is worthless if $S > E$ but has the value $E - S$ for $S < E$. Thus, the payoff function of a put option is

$$P(S,T) = \max(E - S, 0). \tag{2.15}$$

The payoff diagram for a put is shown in Fig. 1.10 where the line shows the payoff function $\max(E-S,0)$. In order to distinguish this put option from other more complicated put options, sometimes it is referred to as the vanilla put option.

2.2.4 Hedging and Greeks

The way to reduce the sensitivity of a portfolio to the movement of something by taking opposite positions in different financial instruments is called hedging. Hedging is a basic concept in finance. When we derived the Black–Scholes equation in Sect. 2.2.2, we chose the delta to be $\dfrac{\partial V}{\partial S}$, so that the portfolio Π became risk-free. This gives an important example on how hedging is applied. Let us see another example of hedging that is similar to what we have used in deriving the Black–Scholes equation.

Consider a call option on a stock. Figure 2.4 shows the relation between the call price and the underlying stock price. When the stock price corresponds to point A, the option price corresponds to point B and the Δ of the call is the slope of the line indicated. As an approximation

$$\Delta = \frac{\delta c}{\delta S},$$

where δS is a small change in the stock price and δc is the corresponding change in the call price.

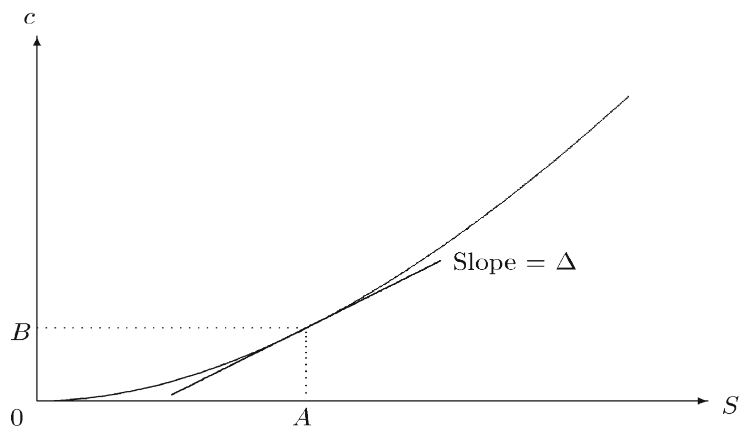

Fig. 2.4. Δ = the slope of a curve

Assume that the delta of the call option is 0.7 and a writer sold 10,000 units of call options. Then, the writer's position could be hedged by buying $0.7 \times 10{,}000 = 7{,}000$ shares of stocks. If the stock price goes up by \$0.50, the writer will earn \$3,500 from the 7,000 shares of stocks held. At the same time, the price of call options will go up approximately $0.7 \times 0.5 = \$0.35$, and he will lose $10{,}000 \times \$0.35 = \$3{,}500$ from 10,000 shares of option he sold. Therefore, the net profit or loss is about zero. If the price falls down by a small amount, the situation is similar. Consequently, the writer's position has been hedged quite well as long as the movement of the price is small. This is called delta hedging.

In the example above, we have considered only a call option. Actually, any portfolio can be hedged in this way. If Π denotes the price of option, then the slope is

$$\Delta = \frac{\partial \Pi}{\partial S}.$$

If the movement of the price is not very small, then it might be necessary to use the value of the second derivative of the portfolio with respect to S in order to eliminate most of the risk. The second derivative is known as gamma

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}.$$

When hedging in practice, some other values, for example, $\dfrac{\partial \Pi}{\partial t}$, $\dfrac{\partial \Pi}{\partial \sigma}$, $\dfrac{\partial \Pi}{\partial r}$, $\dfrac{\partial \Pi}{\partial D_0}$, may need to be known. Usually, $\dfrac{\partial \Pi}{\partial t}$, $\dfrac{\partial \Pi}{\partial \sigma}$, and $\dfrac{\partial \Pi}{\partial r}$ are called theta, vega, and rho, respectively; namely, the following notation is used:

$$\Theta = \frac{\partial \Pi}{\partial t}, \quad \mathcal{V} = \frac{\partial \Pi}{\partial \sigma},$$

and

$$\rho = \frac{\partial \Pi}{\partial r}.$$

In currency options, D_0 is the interest rate in the foreign country. Thus, $\dfrac{\partial \Pi}{\partial D_0}$ is also known as rho. In order to distinguish $\dfrac{\partial \Pi}{\partial r}$ and $\dfrac{\partial \Pi}{\partial D_0}$, here we define

$$\rho_d = \frac{\partial \Pi}{\partial D_0}.$$

These quantities are usually referred to as Greeks.

When σ depends on S, or the coefficient of $\dfrac{\partial V}{\partial S}$ is more complicated, analytic expressions of option prices may not exist. In this case, we have to use numerical methods. Also sometimes (for example, for a call option), the solution is unbounded. It is not convenient to solve a problem numerically on an infinite domain with an unbounded solution. Therefore in Sect. 2.2.5, we also provide a transformation under which the Black–Scholes equation on $[0, \infty)$ becomes an equation on $[0, 1)$ with a bounded solution.

2.2.5 Transforming the Black–Scholes Equation into an Equation Defined on a Finite Domain

Let us consider the following option problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le S < \infty, \quad t \le T, & \\ V(S,T) = V_T(S), \qquad\qquad\qquad\qquad\quad 0 \le S < \infty. & \end{cases} \tag{2.16}$$

The transformation to be described in this subsection works even when σ, r, or D_0 depends on S and t. For simplicity, we assume in the derivation that σ depends on S and that r, D_0 are constant. In this case, an analytic expression of the solution $V(S,t)$ may not exist, and numerical methods may be necessary. Also for a call option, the solution $V(S,t)$ is not bounded. Therefore, we introduce new independent variables and dependent variable through the following transformation:

$$\begin{cases} \xi = \dfrac{S}{S+P_m}, \\ \tau = T - t, \\ V(S,t) = (S+P_m)\overline{V}(\xi,\tau). \end{cases} \tag{2.17}$$

From Eq. (2.17) we have

$$S = \frac{P_m \xi}{1-\xi}, \quad S + P_m = \frac{P_m}{1-\xi}$$

and

$$\frac{d\xi}{dS} = \frac{P_m}{(S+P_m)^2} = \frac{(1-\xi)^2}{P_m}.$$

Because

$$\begin{aligned} \frac{\partial V}{\partial t} &= \frac{\partial}{\partial t}\left[(S+P_m)\overline{V}(\xi,\tau)\right] = -(S+P_m)\frac{\partial \overline{V}}{\partial \tau} = -\frac{P_m}{1-\xi}\frac{\partial \overline{V}}{\partial \tau}, \\ \frac{\partial V}{\partial S} &= \frac{\partial}{\partial S}\left[(S+P_m)\overline{V}(\xi,\tau)\right] = (S+P_m)\frac{\partial \overline{V}}{\partial \xi}\frac{d\xi}{dS} + \overline{V} = (1-\xi)\frac{\partial \overline{V}}{\partial \xi} + \overline{V}, \\ \frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial \xi}\left[(1-\xi)\frac{\partial \overline{V}}{\partial \xi} + \overline{V}\right]\frac{d\xi}{dS} = \frac{(1-\xi)^3}{P_m}\frac{\partial^2 \overline{V}}{\partial \xi^2}, \end{aligned}$$

and let

$$\bar{\sigma}(\xi) = \sigma(S(\xi)) = \sigma\left(\frac{P_m \xi}{1-\xi}\right),$$

the original equation becomes[9]

$$\frac{P_m}{1-\xi}\frac{\partial \overline{V}}{\partial \tau} = \frac{\bar{\sigma}^2(\xi) P_m \xi^2 (1-\xi)}{2}\frac{\partial^2 \overline{V}}{\partial \xi^2} + (r-D_0)P_m \xi \frac{\partial \overline{V}}{\partial \xi} + \frac{(r-D_0)\xi - r}{1-\xi}P_m \overline{V}$$

or

$$\frac{\partial \overline{V}}{\partial \tau} = \frac{\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2}{2}\frac{\partial^2 \overline{V}}{\partial \xi^2} + (r-D_0)\xi(1-\xi)\frac{\partial \overline{V}}{\partial \xi} - [r(1-\xi) + D_0 \xi]\overline{V},$$
$$0 \le \xi < 1, \quad 0 \le \tau.$$

[9] Actually, the same equation can be directly obtained by constructing a portfolio and using Itô lemma (see Problem 23).

Assume that $\overline{V}$ is a smooth function of ξ, then the equation also holds at $\xi = 1$. Because $V(S,T) = (S+P_m)\overline{V}(\xi,0) = \overline{V}(\xi,0)\dfrac{P_m}{1-\xi}$, the condition $V(S,T) = V_T(S)$ can be rewritten as $\overline{V}(\xi,0) = V_T\left(\dfrac{P_m\xi}{1-\xi}\right)\dfrac{1-\xi}{P_m}$. Consequently, the problem (2.16) becomes

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + (r-D_0)\xi(1-\xi)\dfrac{\partial \overline{V}}{\partial \xi} - [r(1-\xi)+D_0\xi]\overline{V}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le \xi \le 1, \quad 0 \le \tau, \\ \overline{V}(\xi,0) = \dfrac{1-\xi}{P_m}V_T\left(\dfrac{P_m\xi}{1-\xi}\right), \qquad\qquad\qquad 0 \le \xi \le 1. \end{cases} \tag{2.18}$$

Thus, the transformation (2.17) converts a problem on an infinite domain into a problem on a finite domain. For a parabolic equation defined on a finite domain to have a unique solution, besides an initial condition, boundary conditions are usually needed. However, in this equation the coefficients of $\dfrac{\partial^2 \overline{V}}{\partial \xi^2}$ and $\dfrac{\partial \overline{V}}{\partial \xi}$ at $\xi = 0$ and at $\xi = 1$ are equal to zero, i.e., the equation degenerates to ordinary differential equations at the boundaries. Actually, at $\xi = 0$ the equation becomes

$$\frac{\partial \overline{V}(0,\tau)}{\partial \tau} = -r\overline{V}(0,\tau)$$

and the solution is

$$\overline{V}(0,\tau) = \overline{V}(0,0)\mathrm{e}^{-r\tau}. \tag{2.19}$$

Similarly, at $\xi = 1$ the equation reduces to

$$\frac{\partial \overline{V}(1,\tau)}{\partial \tau} = -D_0\overline{V}(1,\tau),$$

from which we have

$$\overline{V}(1,\tau) = \overline{V}(1,0)\mathrm{e}^{-D_0\tau}. \tag{2.20}$$

Therefore for this equation, the two solutions of the ordinary differential equations provide the values at the boundaries, and no boundary conditions are needed in order for the problem (2.18) to have a unique solution.

Consequently, in order to price an option, we need to solve a problem on a finite domain if this formulation is adopted. From the point view of numerical methods, such a formulation is better. This is its first advantage. Actually, the uniqueness of solution for problem (2.18) can easily be proved (see Sect. 2.4). Indeed, not only the uniqueness can be proved, but the stability of the solution with respect to the initial value can also be shown easily. That is, this formulation makes proof of some theoretical problems easy. This is its other advantage.

For a call option, the payoff is

$$V(S,T) = \max(S-E,0),$$

so the initial condition in the problem (2.18) for a call is

$$\begin{aligned}\overline{V}(\xi,0) &= \max(S-E,0)(1-\xi)/P_m \\ &= \max\left(\frac{P_m\xi}{1-\xi}-E,0\right)(1-\xi)/P_m \\ &= \max\left(\xi-\frac{E}{P_m}(1-\xi),0\right).\end{aligned}$$

For a put option

$$V(S,T) = \max(E-S,0).$$

Therefore

$$\overline{V}(\xi,0) = \max\left(\frac{E}{P_m}(1-\xi)-\xi,0\right).$$

Let $P_m = E$, the two initial conditions become

$$\overline{V}(\xi,0) = \max(2\xi-1,0) \quad \text{and} \quad \overline{V}(\xi,0) = \max(1-2\xi,0),$$

respectively. Therefore, a European call option is the solution of the following problem:

$$\begin{cases}\dfrac{\partial\overline{V}}{\partial\tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2\overline{V}}{\partial\xi^2}+(r-D_0)\xi(1-\xi)\dfrac{\partial\overline{V}}{\partial\xi}-[r(1-\xi)+D_0\xi]\overline{V}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0\le\xi\le1,\quad 0\le\tau, \\ \overline{V}(\xi,0) = \max(2\xi-1,0), \qquad\qquad\qquad\qquad 0\le\xi\le1\end{cases} \tag{2.21}$$

and the solution of the problem

$$\begin{cases}\dfrac{\partial\overline{V}}{\partial\tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2\overline{V}}{\partial\xi^2}+(r-D_0)\xi(1-\xi)\dfrac{\partial\overline{V}}{\partial\xi}-[r(1-\xi)+D_0\xi]\overline{V}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0\le\xi\le1,\quad 0\le\tau, \\ \overline{V}(\xi,0) = \max(1-2\xi,0), \qquad\qquad\qquad\qquad 0\le\xi\le1\end{cases} \tag{2.22}$$

gives the price of a European put option. In the problem (2.21) the initial condition is bounded, so $\overline{V}(\xi,\tau)$, as a solution of a linear parabolic equation, is also bounded. Therefore in this case, the solution that needs to be found numerically is bounded.

So far, we assumed that σ depends only on S and that r and D_0 are constant. However, the result will be the same if σ depends on both S and t, and r and D_0 also depend on S and t.

Finally, we would like to point out that from the expression (2.20) we can have an asymptotic expression of the solution of the Black–Scholes equation at infinity. Because at $\xi = 1$ there is an analytic solution (2.20), noticing

$$V(S,t) = (S + P_m)\overline{V}(\xi, \tau),$$

for $S \approx \infty$ we have

$$\begin{aligned} V(S,t) &= (S + P_m)\overline{V}(\xi, \tau) \approx (S + P_m)\overline{V}(1, \tau) \\ &= (S + P_m)\overline{V}(1, 0)\mathrm{e}^{-D_0\tau} \\ &\approx V(S,T)\mathrm{e}^{-D_0\tau} = V(S,T)\mathrm{e}^{-D_0(T-t)}. \end{aligned} \tag{2.23}$$

This is an asymptotic expression of the solution of the Black–Scholes equation at infinity.

2.2.6 Derivation of the Equation for Options on Futures

As we know, a futures contract in finance is a standardized contract between two parties to exchange a specified asset of a standardized quantity and quality for a price K (the delivery price) agreed today with delivery occurring at a specified future date, while a forward contract in finance is a nonstandardized contract between two parties to buy or sell an asset at a specified future time at a price K agreed today. There are some differences between a futures contract and a forward contract, but both are a contract in which two parties agree to exchange a specified asset for a specified amount of cash at a specified future date. Here we derive the PDE for options on such a contract.

Suppose that the price of the underlying asset satisfies

$$dS = \mu S dt + \sigma S dX, \tag{2.24}$$

and it pays dividends continuously with a constant dividend yield D_0. We also assume that the interest rate r is a constant. Let T be the expiration date of the contract and t be the time today.

Before deriving the PDE, we point out that the value of a forward/futures contract at time t is

$$f = S\mathrm{e}^{-D_0(T-t)} - K\mathrm{e}^{-r(T-t)}, \tag{2.25}$$

from which we can have

$$S = \mathrm{e}^{D_0(T-t)}\left(f + K\mathrm{e}^{-r(T-t)}\right). \tag{2.26}$$

The reason is as follows. At time t, the seller of this contract, who gets f when the contract is sold, can borrow $K\mathrm{e}^{-r(T-t)}$ from a bank with an interest rate r and buy $\mathrm{e}^{-D_0(T-t)}$ units of the asset by spending $S\mathrm{e}^{-D_0(T-t)}$. At time T,