Note Title 25-08-2021

Variational prob - Find

$u \in H$ s.t.

$A(u, v) = L(v) \ \forall v \in H$

$A : H \times H \to \mathbb{R}$

$L : H \to \mathbb{R}$

$u \approx u_h \in$ F.D.

It is an approximation for variational prob.

$H \to f^n$ space.

$V \to f^n$ space.

$$-y'' + y = f, \quad (a,b)$$

$$-\int_a^b y''v + \int_a^b yv = \int_a^b fv$$

$$\Rightarrow \left[-y'v\right]_a^b + \int_a^b y'v' + \int_a^b yv = \int_a^b fv$$

$$v(a) = 0 = v(b)$$

$$\int_a^b y'v' + \int_a^b yv = \int_a^b fv$$

$$\Rightarrow A(y,v) = L(v).$$

$A(y,v) = A(v,y) \rightarrow$ symmt.

$$A(y,y) = \int_a^b (y')^2 + \int_a^b y^2 \geq 0$$

$A(c_1 y, v) = c_1 A(y, v)$, $A(y, cv) = c\,A(y, v)$

$A(y_1 + y_2, v) = A(y_1, v) + A(y_2, v)$

$A(v, v) \geq 0$, if n +ve

$A(y, v) = A(v, y)$ symmetric

$A(cy, v) = c\,A(y, v)$, $A(y, cv) = c\,A(y, v)$ bilinear

linearity in both argu.

A

Find: $u \in H$

$A(u, v) = L(v) \quad \forall v \in H$

infinite dimensional

Find: $u_h \in \tilde{H} \subset H$

finite dimension

s.t.

$$A(u_h, v_h) = L(v_h) \quad \forall v_h \in V_h$$

$A(u_h, v_h) = L(v_h)$. —— (1)

$u_h \in \tilde{H} = \text{span}\{\phi_1, \phi_2, \dots, \phi_n\}$

$u_h = \sum_{i=1}^{n} c_i \phi_i$, $\phi_i \to$ basis f^n, known

$\sum_{i=1}^{n} c_i A(\phi_i, v_h) = L(v_h)$, $\Rightarrow$ $\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [\;]$

$v_h = \phi_j$, $1 \le j \le n$

Evaluation_

Convergence.

$u \to$ actual sol^n

$u_h \to$ approximate

$|u-u_h| \to 0.$ $\qquad \|u-u_h\| \to 0.$

$$\|u-u_n\| = \left(\int_a^b |u-u_n|^2 \, dx\right)^{1/2}, \quad L^2\text{-norm}$$

$$\|u-u_n\| \to 0$$

$$\|\nabla(u-u_n)\| = \left(\int_a^b |\nabla(u-u_n)|^2 \, dx\right)^{1/2}, \quad H^1\text{-norm}$$

H^1 = collection of all L^2-fn s.t. its weak derivative in

Convergence analysis

Interpolation error.

$f, \{(x_0, f(x_0)), \ldots, (x_n, f(x_n))\}$

$p_n(x) \rightarrow$ Lagrange.

$\|f - p_n\| \leq C h^{n+1} \|u\|_{C^{k+1}}$ ✓

$\tilde{H} \subseteq P_n$

$$u - u_n = \underbrace{u - p_n(u)}_{\text{interpolant}} + p_n(u) - u_n$$

$p_n - u_n \in P_n$

$|u - p_n|$,

$$A(u, v) = L(v) \quad \forall v \in H$$

$$A(u_h, v_h) = L(v_h) \quad \forall v_h \in \tilde{H} \subseteq H$$

$$A(u - u_h, v_h) = 0 \quad \forall \underline{v_h \in \tilde{H}}$$

$$A(u - u_h, u - u_h) \geq 0 \quad , \quad A(v, v) = \|v\|_H$$

$$\|u-u_h\|_H^2 = A(u-u_h, u-u_h)$$

$$= A(u-v_h+v_h-u_h, u-u_h)$$

$$= A(u-v_h, u-u_h) + A(v_h-u_h, u-u_h)$$

$$= A(u-v_h, u-u_h) + \underbrace{A(u-u_h, v_h-u_h)}_{=0}$$

$v_h-u_h \in \hat{H}$

$$\|u - u_h\|^2 = A(u - v_h,\ u - u_h)$$

A induces an inner product

$$\leq \|u - u_h\|_H \|u - v_h\|_H$$

$$\therefore \quad \|u - u_h\|_H \leq \|u - v_h\|_H \quad \forall\ v_h \in \tilde{H} \subseteq P_h$$

$$v_h = P_h(u)$$

$$\|u - u_h\|_H \le \|u - p_h(u)\|_E$$

$$\sup_{x \in [a,b]} |u - p_h(u)|$$

$p_h(u)$ is the interpolant of u

Polynomial approximation in Sobolev space.