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Sol^n (1):

$$\mathcal{E} = S/_{S+P_m}.$$

$$\tau = T - t$$

$$V(s,t) = (S+P_m)\bar{V}(\mathcal{E},\tau)$$

$$S = \frac{P_m \mathcal{E}}{1-\mathcal{E}}, \quad S+P_m = \frac{P_m}{1-\mathcal{E}}$$

$$\Rightarrow \frac{d\mathcal{E}}{dS} = \frac{P_m}{(S+P_m)^2} = \frac{(1-\mathcal{E})^2}{P_m}.$$

because

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial t}\left[(S+P_m)\bar{V}(\mathcal{E},\tau)\right] = -(S+P_m)\frac{\partial \bar{V}}{\partial \tau} = -\frac{P_m}{1-\mathcal{E}}\frac{\partial \bar{V}}{\partial \tau}$$

$$\frac{\partial V}{\partial S} = \frac{\partial}{\partial S}\left[(S+P_m)\bar{V}(\mathcal{E},\tau)\right] = (S+P_m)\frac{\partial \bar{V}}{\partial \mathcal{E}}\frac{d\mathcal{E}}{dS} + \bar{V}$$

$$= (1-\mathcal{E})\frac{\partial \bar{V}}{\partial \mathcal{E}} + \bar{V}$$

and $$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial \mathcal{E}}\left((1-\mathcal{E})\frac{\partial \bar{V}}{\partial \mathcal{E}} + \bar{V}\right)\frac{\partial \mathcal{E}}{\partial S} = \frac{(1-\mathcal{E})^3}{P_m}\frac{\partial^2 \bar{V}}{\partial \mathcal{E}^2}$$

and assume $\bar{\sigma}(\mathcal{E}) = \sigma(S(\mathcal{E})) = \sigma\left(\frac{P_m \mathcal{E}}{1-\mathcal{E}}\right).$

Hence, the given PDE reduces to

$$\frac{P_m}{1-\mathcal{E}}\frac{\partial \bar{V}}{\partial \tau} = \frac{\bar{\sigma}^2(\mathcal{E})\,P_m \mathcal{E}^2(1-\mathcal{E})}{2}\frac{\partial^2 \bar{V}}{\partial \mathcal{E}^2} + (r-D_0)P_m \mathcal{E}\frac{\partial \bar{V}}{\partial \mathcal{E}} + \frac{((r-D_0)\mathcal{E} - r)}{1-\mathcal{E}} P_m \bar{V}$$

$$\Rightarrow \frac{\partial \bar{V}}{\partial \tau} = \frac{\bar{\sigma}^2(\varepsilon)\,\varepsilon^2(1-\varepsilon)^2}{2}\frac{\partial^2 \bar{V}}{\partial \varepsilon^2} + (r-D_0)\,\varepsilon(1-\varepsilon)\frac{\partial \bar{V}}{\partial \varepsilon} - \left[r(1-\varepsilon)+D_0\varepsilon\right]\bar{V}$$

where $\varepsilon \in [0,1)$. $(0 \le \varepsilon < 1)$

$0 \le \tau$

Assuming $\bar{V}$ is a smooth fn of ε, then the above eq^n holds for $\varepsilon = 1$ also.

$$\therefore V(S,\tau) = (S+P_m)\,\bar{V}(\varepsilon,0) = \bar{V}(\varepsilon,0)\frac{P_m}{1-\varepsilon}$$

$V(S,T) = V_T(S) \Rightarrow$ can be rewritten as ~~$\bar{V}(\varepsilon,0)$~~ =

$$\bar{V}(\varepsilon,0) = V_T\left[\frac{P_m\varepsilon}{1-\varepsilon}\right]\left(\frac{1-\varepsilon}{P_m}\right).$$

Hence the original eq^n could be written as:-

$$\begin{cases} \dfrac{\partial \bar{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\varepsilon)\,\varepsilon^2(1-\varepsilon)^2\dfrac{\partial^2 \bar{V}}{\partial \varepsilon^2} + (r-D_0)\,\varepsilon(1-\varepsilon)\dfrac{\partial \bar{V}}{\partial \varepsilon} \\ \quad -\left[r(1-\varepsilon)+D_0\varepsilon\right]\bar{V}. \qquad 0 \le \varepsilon \le 1,\ 0 \le \tau \\ \bar{V}(\varepsilon,0) = \dfrac{(1-\varepsilon)}{P_m}\,V_T\left(\dfrac{P_m\varepsilon}{1-\varepsilon}\right) \end{cases}$$

Hence we can obtain finite domain eq^n from infinite domain. ~~For parabolic eq^n defined on finite domain to have a unique sol^n besides an initial cond^n. needs a boundary co~~

For parabolic eq^n's boundary cond^n also needed with the initial cond^n's.

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However coefficients of $\frac{\partial^2 \bar{V}}{\partial \xi^2}$, $\frac{\partial \bar{V}}{\partial \xi}$ at $\xi = 0$ & $\xi = 1$ are 0.

Actually at $\xi = 0 \Rightarrow$

$$\frac{\partial \bar{V}(0,\tau)}{\partial \tau} = -\gamma \bar{V}(0,\tau)$$

$$\Rightarrow \bar{V}(0,\tau) = \bar{V}(0,0)\, e^{-\gamma\tau}$$

Similarly at $\xi = 1$ we have

$$\frac{\partial \bar{V}(1,\tau)}{\partial \tau} = -D_0 \bar{V}(1,\tau)$$

$$\Rightarrow \bar{V}(1,\tau) = \bar{V}(1,0)\, e^{-D_0\tau}.$$

Hence, we obtain two solutions of the ordinary differential equation and providing value at boundaries.

Hence eq^n is reduced to finite domain and boundary cond^n are obtained at $\xi = 0$, $\xi = 1$.

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Sol^n ②: K → ~~set of all~~ family of admissible fxns

for $v \in K$, let y be the exact sol^n of linear complementarity problem for american put options. Since y is the sol^n for this partial differential inequality ⇒ $y \to C^2$ smooth on the continuation region and $y \in K$.

Hence we can write: $K = \{ v \in C^0 : \frac{\partial v}{\partial x}$ piecewise C^0

$v(x,t) \geq g(x,t)$ for x, t

$v(x,0) = g(x,0)$

$v(x_{max}, t) = g(x_{max}, t)$

and

$v(x_{min}, t) = g(x_{min}, t)$

⇒ given $v \geq g$, $\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0$ the problem

$$\Rightarrow \int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \right) (v-g)\, dx \geq 0 \quad \to ①$$

Invoking the complimentarity,

$$\int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \right) (y-g)\, dx = 0 \quad \to ②$$

and subtraction given ⇒ ① − ② ⇒

$$\int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \right) (v-y)\, dx \geq 0$$

Applying integration by parts: we have

$$\int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau}(v-y) + \frac{\partial y}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial y}{\partial x} \right) \right) dx - \frac{\partial y}{\partial x}(v-y) \Big|_{x_{min}}^{x_{max}} \geq 0$$

* $\frac{\partial y}{\partial x}(v-y)$ vanishes because at x_{min}, x_{max}

In view of $v=g$, $y=g$ ~~then~~ $\Rightarrow v=y$.

Hence

Eq^n reduces to

$$\int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau}(v-y) + \frac{\partial y}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial y}{\partial x} \right) \right) dx \geq 0 \quad \forall \, v \in K$$

$\Uparrow$

~~let's~~ which is $I(y; v)$.

$I(y;v)$ holds for all $v \in K$. Considering the case when $v=y \Rightarrow I(y:v)$ takes its minimum value.

$$\min_{v \in K,\, v=y} I(y; v) = I(y;y) = 0.$$

(a) To obtain the fully discrete version of above minimizat^n problem:

we assume approximations for $\hat{y}$ and v in similar forms:

$$\sum_i w_i(\tau)\, \phi_i(x) \quad \text{for } \hat{y}$$

$$\sum_i u_i(\tau)\, \varphi_i(x) \quad \text{for } v.$$

The reduced smoothness match the requirements for K. we assumed that variables τ, x ~~are independent~~ ~~and to~~ which are independent are to be separated. Hence same x-grid is applied for all τ $\Rightarrow$ rectangular (x, τ) grid. The time ~~gradient~~ is introduced in fxn w_i + v_i. Since the basis function ϕ_i represent x_i grid. Putting values in the eq^n formed in previous part:

$$\int \left\{ \left(\sum_i \frac{dw_i}{d\tau} \phi_i \right) \left(\sum_j (v_j - w_j) \phi_j \right) + \left(\sum_i w_i \phi_i^1 \right)_x \left(\sum_j (v_j - w_j) \phi_j' \right) \right\} dx$$

$$= \sum_i \sum_j \frac{dw_i}{d\tau} (v_j - w_j) \int \phi_i \phi_j \, dx + \sum_i \sum_j w_i (v_j - w_j)_x \int \phi_i' \phi_j' \geq 0.$$

We can write above in vector notation like

$$\left(\frac{dw}{d\tau} \right)^T B(v - w) + w^T A(v - w) \geq 0.$$

$$\Rightarrow (v - w)^T \left(B \frac{dw}{d\tau} + Aw \right) \geq 0 \quad \rightarrow (3)$$

where A, B are corresponding matrices $\underbrace{\text{tridiagonal}}$

As time is discretized, define ~~$w^{(n)} := w(\tau_n)$~~

$$w^{(n)} := w(\tau_n) \quad , \quad v^{(n)} := v(\tau_n)$$

Using substitution and θ-averaging Aw term in (3):

$$\left(v^{(n+1)} - w^{(n+1)} \right)^T \left(B \frac{1}{\Delta\tau} \left(w^{n+1} - w^n \right) + \theta \left(A w^{(n+1)} \right) + (1 - \theta) A w^n \right) \geq 0. \quad \forall \, n.$$

$$\Rightarrow \left(v^{(n+1)} - w^{n+1} \right)^T \left((B + \Delta\tau\theta A) w^{n+1} + \left(\Delta\tau (1 - \theta) A - B \right) w^{(n)} \right) \geq 0$$

Using the variables given in ques:

$$r := \left(B - \Delta\tau(1-\theta)A\right) w^{(n)}$$

$$C := B + \Delta\tau\theta A.$$

we get

$\Rightarrow$

$$\left(v^{(n+1)} - w^{(n+1)}\right)^T \left(Cw^{(n+1)} - r\right) \geq 0.$$

Hence fully discrete version of above minimised problem.

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Sol^n (3) : To show that following problems are equivalent $\Rightarrow$

Consider the 1st case:

a) (FDM) sol^n implies (FEM) sol^n.

Let y ~~solve~~ be the sol^n of FDM. $\Rightarrow y \geq g$.

$$I = \cancel{y} \; (v-y)^T(Cy-r) = (v-g)^T(Cy-r) - (y-g)^T(Cy-r)$$

$\because (v-g)^T(Cy-r) \geq 0 \; ; \; (y-g)^T(Cy-r) = 0$

Hence $\boxed{I \geq 0}$ $\forall$ ~~$y\in$~~ $v \geq g$.

Hence sol^n of FDM also satisfies that of FEM.

(b) (FEM) sol^n $\Rightarrow$ (FDM) sol^n

Let y ~~sol~~ be sol^n of FEM eq^n $\Rightarrow$ $y \geq g$.

$$v^T(Cy-r) \geq y^T(Cy-r) \; \forall v \in R.$$

~~Suppose~~

Let's say ~~one~~ p^{th} component of $Cy-r < 0$

and make v_p very large.

Hence $v^T(Cy-r)$ becomes very small as compared which is not possible as LHS $\geq$ RHS.

Hence $\Rightarrow Cy-r \geq 0$

Since we have $y \geq g \Rightarrow (y-g)^T(Cy-r) \geq 0$.

Put in FEM eq^n: $v = g$ ~~$\Rightarrow$~~ $(y-g)^T(Cy-r) \leq 0$

$\Rightarrow (y-g)^T(Cy-r) = 0$ hence sol^n of FEM also satisfies FDM. Hence we could say both problems are equivalent.

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