MA 224 (Real Analysis)

Summary of Lectures - 3

Pointwise convergence of sequence of functions: Let $E(\neq \emptyset) \subset \mathbb{R}$. A sequence (f_n) of real-valued functions defined on E is said to be pointwise convergent on E if for each $x \in E$, the sequence $(f_n(x))$ converges in $\mathbb{R}$.
Thus (f_n) is pointwise convergent on E iff there exists a function $f : E \to \mathbb{R}$ such that $f_n(x) \to f(x)$ for each $x \in E$, *i.e.* for each $x \in E$ and for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq n_0$.
In this case, f is called the pointwise limit (function) of (f_n) on E and we write $f_n \to f$ pointwise on E.

Uniform convergence of sequence of functions: Let $E(\neq \emptyset) \subset \mathbb{R}$. A sequence (f_n) of real-valued functions defined on E is said to be uniformly convergent on E if there exists a function $f : E \to \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ satisfying $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq n_0$ and for all $x \in E$.
In this case, f is called the uniform limit (function) of (f_n) on E and we write $f_n \to f$ uniformly on E.

Remarks: Let $E(\neq \emptyset) \subset \mathbb{R}$. If $f : E \to \mathbb{R}$ and for each $n \in \mathbb{N}$, $f_n : E \to \mathbb{R}$ such that $f_n \to f$ uniformly on E, then $f_n \to f$ pointwise on E. Hence if the sequence (f_n) is not pointwise convergent on E, then (f_n) cannot be uniformly convergent on E. Also, if $f_n \to f$ pointwise on E and $f_n \not\to f$ uniformly on E, then the sequence (f_n) cannot converge uniformly on E.

Examples: If $f_n(x) = \frac{x}{n}$ for all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$, then the sequence (f_n) converges (a) pointwise on $\mathbb{R}$, (b) converges uniformly on $[a, b]$ for all $a, b \in \mathbb{R}$ with $a < b$, (c) does not converge uniformly on $\mathbb{R}$.

Cauchy's criterion for uniform convergence: Let $E(\neq \emptyset) \subset \mathbb{R}$. A sequence (f_n) of real-valued functions defined on E is uniformly convergent on E iff for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|f_m(x) - f_n(x)| < \varepsilon$ for all $m, n \geq n_0$ and for all $x \in E$.

An useful criterion for uniform convergence: Let $E(\neq \emptyset) \subset \mathbb{R}$. Let (f_n) be a sequence of real-valued functions defined on E and let $f_n \to f$ pointwise on E, where $f : E \to \mathbb{R}$. If $M_n = \sup\limits_{x \in E} |f_n(x) - f(x)|$ for all $n \in \mathbb{N}$ (with the convention that $M_n = +\infty$ if $\{|f_n(x) - f(x)| : x \in E\}$ is not bounded above), then $f_n \to f$ uniformly on E iff $\lim\limits_{n \to \infty} M_n = 0$.

Examples: Let $f_n(x) = \frac{x}{1+nx}$, $g_n(x) = \frac{x}{1+nx^2}$ and $\phi_n(x) = \frac{nx}{1+n^2x^2}$ for all $n \in \mathbb{N}$ and for all $x \in [0, 1]$. Then each of the sequences (f_n) and (g_n) is uniformly convergent on $[0, 1]$ and the sequence (ϕ_n) is pointwise convergent but not uniformly convergent on $[0.1]$.

Pointwise and uniform convergence of series of functions: Let $E(\neq \emptyset) \subset \mathbb{R}$. Let (f_n) be a sequence of real-valued functions defined on E and let $s_n(x) = \sum\limits_{i=1}^{n} f_i(x)$ for all $n \in \mathbb{N}$ and for all $x \in E$. The series $\sum\limits_{n=1}^{\infty} f_n$ is said to be pointwise (respectively, uniformly) convergent on E if the sequence (s_n) is pointwise (respectively, uniformly) convergent on E and we define $\sum\limits_{n=1}^{\infty} f_n = \lim\limits_{n \to \infty} s_n$ pointwise (respectively, uniformly) on E.

Examples: If $f_n(x) = x^n$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then the series $\sum\limits_{n=1}^{\infty} f_n$ is (a) not pointwise convergent on any nonempty subset of $\mathbb{R} \setminus (-1, 1)$ (b) pointwise convergent but not uniformly convergent on $(-1,1)$ (c) uniformly convergent on $[-r,r]$ if $r \in (0,1)$.

Weierstrass M-test: Let $E(\neq \emptyset) \subset \mathbb{R}$. Let (f_n) be a sequence of real-valued functions defined on E and let (M_n) be a sequence of non-negative real numbers such that $|f_n(x)| \leq M_n$ for all $n \in \mathbb{N}$ and for all $x \in X$. If the series $\sum\limits_{n=1}^{\infty} M_n$ converges (in $\mathbb{R}$), then the series $\sum\limits_{n=1}^{\infty} f_n$ is uniformly convergent on E.

Example: If $p > 1$, then the series $\sum\limits_{n=1}^{\infty} \frac{\sin nx}{n^p}$ converges uniformly on $\mathbb{R}$.

Pointwise and uniform convergence of power series: Let $R \in (0,\infty]$ be the radius of convergence of a power series $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$. Then $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ is pointwise convergent on $(x_0 - R, x_0 + R)$.
If $0 < r < R$, then $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ converges uniformly on $[x_0 - r, x_0 + r]$ but need not converge uniformly on $(x_0 - R, x_0 + R)$.

Uniform convergence and limit: Let $E(\neq \emptyset) \subset \mathbb{R}$ and let $f : E \to \mathbb{R}$. Let $x_0 \in \mathbb{R}$ be a limit point of E and for each $n \in \mathbb{N}$, let $f_n : E \to \mathbb{R}$ such that $\lim\limits_{x \to x_0} f_n(x)$ exists (in $\mathbb{R}$).

(a) If $f_n \to f$ uniformly on E, then the sequence $\left(\lim\limits_{x \to x_0} f_n(x)\right)$ converges in $\mathbb{R}$ and $\lim\limits_{x \to x_0} f(x) = \lim\limits_{n \to \infty} \left(\lim\limits_{x \to x_0} f_n(x)\right)$.

(b) If $\sum\limits_{n=1}^{\infty} f_n = f$ uniformly on E, then the series $\sum\limits_{n=1}^{\infty} \lim\limits_{x \to x_0} f_n(x)$ converges in $\mathbb{R}$ and $\lim\limits_{x \to x_0} f(x) = \sum\limits_{n=1}^{\infty} \lim\limits_{x \to x_0} f_n(x)$.

Remarks: The above result need not be true if we only assume that $f_n \to f$ pointwise on E or $\sum\limits_{n=1}^{\infty} f_n = f$ pointwise on E. For example, $\lim\limits_{x \to 1-} \left(\lim\limits_{n \to \infty} x^n\right) \neq \lim\limits_{n \to \infty} \left(\lim\limits_{x \to 1-} x^n\right)$ and $\lim\limits_{x \to 0} \sum\limits_{n=1}^{\infty} \frac{x^2}{(1+x^2)^{n-1}} \neq \sum\limits_{n=1}^{\infty} \lim\limits_{x \to 0} \frac{x^2}{(1+x^2)^{n-1}}$.

Uniform convergence and continuity: Let $E(\neq \emptyset) \subset \mathbb{R}$ and let $f : E \to \mathbb{R}$. For each $n \in \mathbb{N}$, let $f_n : E \to \mathbb{R}$ be continuous at $x_0 \in E$.

(a) If $f_n \to f$ uniformly on E, then f is continuous at x_0.

(b) If $\sum\limits_{n=1}^{\infty} f_n = f$ uniformly on E, then f is continuous at x_0.

Remarks: Let $E(\neq \emptyset) \subset \mathbb{R}$ and for each $n \in \mathbb{N}$, let $f_n : E \to \mathbb{R}$ be continuous at $x_0 \in E$. If $f : E \to \mathbb{R}$ such that (a) $f_n \to f$ pointwise on E or (b) $\sum\limits_{n=1}^{\infty} f_n = f$ pointwise on E, then f need not be continuous at x_0. For example, consider $f_n(x) = x^n$ for all $x \in [0,1]$ and for all $n \in \mathbb{N}$ in case of (a) and consider $f_n(x) = \frac{x^2}{(1+x^2)^{n-1}}$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$ in case of (b).

Uniform convergence and integration: For each $n \in \mathbb{N}$, let $f_n : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$ and let $f : [a,b] \to \mathbb{R}$.

(a) If $f_n \to f$ uniformly on $[a,b]$, then f is Riemann integrable on $[a,b]$ and $\int\limits_a^b f(x)\,dx = \lim\limits_{n \to \infty} \int\limits_a^b f_n(x)\,dx$.

(b) If $\sum\limits_{n=1}^{\infty} f_n = f$ uniformly on $[a,b]$, then f is Riemann integrable on $[a,b]$ and $\int\limits_a^b f(x)\,dx = \sum\limits_{n=1}^{\infty} \int\limits_a^b f_n(x)\,dx$.

Remarks: If for each $n \in \mathbb{N}$, $f_n : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and if $f : [a,b] \to \mathbb{R}$ such that $f_n \to f$ pointwise on $[a,b]$, then two cases can arise:

(a) f is not Riemann integrable on $[a,b]$.
For example, let $\mathbb{Q} \cap [0,1] = \{r_1, r_2, ...\}$ and for each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, ..., r_n\}, \\ 0 & \text{if } x \in [0,1] \setminus \{r_1, ..., r_n\}. \end{cases}$

(b) f is Riemann integrable on $[a,b]$ but $\int\limits_a^b f(x)\,dx \neq \lim\limits_{n\to\infty} \int\limits_a^b f_n(x)\,dx$.
For example, consider $f_n(x) = nx(1-x^2)^n$ for all $n \in \mathbb{N}$ and for all $x \in [0,1]$.

Uniform convergence and differentiation: For each $n \in \mathbb{N}$, let $f_n : [a,b] \to \mathbb{R}$ be differentiable.

(a) If the sequence $(f_n(x_0))$ converges for some $x_0 \in [a,b]$ and if the sequence (f_n') converges uniformly on $[a,b]$, then the sequence (f_n) converges uniformly on $[a,b]$ to a differentiable function $f : [a,b] \to \mathbb{R}$ and $f'(x) = \lim\limits_{n\to\infty} f_n'(x)$ for all $x \in [a,b]$.

(b) If the series $\sum\limits_{n=1}^{\infty} f_n(x_0)$ converges for some $x_0 \in [a,b]$ and if the series $\sum\limits_{n=1}^{\infty} f_n'$ converges uniformly on $[a,b]$, then the series $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly on $[a,b]$ to a differentiable function $f : [a,b] \to \mathbb{R}$ and $f'(x) = \sum\limits_{n=1}^{\infty} f_n'(x)$ for all $x \in [a,b]$.

Example: If $f(x) = \sum\limits_{n=1}^{\infty} \frac{\sin(nx^2)}{n^3+1}$ for all $x \in \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

Remarks: For each $n \in \mathbb{N}$, let $f_n : [a,b] \to \mathbb{R}$ be differentiable and let $f : [a,b] \to \mathbb{R}$ such that $f_n \to f$ uniformly on $[a,b]$. Then

(a) f need not be differentiable on $[a,b]$.
For example, consider $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ and $f(x) = |x|$ for all $n \in \mathbb{N}$ and for all $x \in [-1,1]$.

(b) assuming that $f : [a,b] \to \mathbb{R}$ is differentiable, the sequence (f_n') need not converge, even pointwise, to f' on $[a,b]$.
For example, consider $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ and $f(x) = 0$ for all $x \in [0,1]$ and for all $n \in \mathbb{N}$.