DEPARTMENT OF MATHEMATICS

Indian Institute of Technology Guwahati

MA 224 (Real Analysis)

Time : 10 hours 20th June, 2020 Maximum marks: $\boxed{50}$

End-Semester Examination

Answers without proper justification will fetch zero marks

1. Prove or disprove the following statements:

   (a) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable then for $a, b \in \mathbb{R}^n$, there is a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(b) - f(a) = T(b - a)$.

   (b) If (X, d) is a metric space such that any closed and bounded subset of X is compact then X complete.

   (c) The normed linear space $(\ell^1, \|\cdot\|_2)$ is complete.

   (d) Uniform limit of a sequence of differentiable functions is differentiable.

   (e) For $1 < p < \infty$ the set $G = \{x = (x_1, x_2, \cdots,) \in \ell^p : \sum_{n=1}^{\infty} x_n = 0\}$ is closed in ℓ^p.

   (f) For $k = 1, 2, \cdots$, let $f : A_k \subset \mathbb{R} \to [-\infty, \infty]$ be a measurable function. Then $f : \cup_{k=1}^{\infty} A_k \to [-\infty, \infty]$ is measurable.

   (g) Let $F : [0, 1] \to \mathbb{R}$ be Lebesgue integrable. Then for every $\epsilon > 0$, $\int_0^1 f^2\, dx \leq \epsilon^2 m\{x \in [0, 1] : |f(x)| > \epsilon\}$.

   (h) Let $F : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable. Then $\displaystyle\lim_{k\to\infty} \int_k^{k+1} f(x)\, dx = 0$. $\boxed{2\times 8}$

2. Let $S = \{f \in C[0, 1] : f(x) = f(1 - x),\ \forall x \in [0, 1]\}$. Show that S is complete with respect to supremum norm. $\boxed{4}$

3. Let $\{f_n\}$ be a sequence of continuous real valued functions defined on $[a, b]$. Let $\{a_n\}, \{b_n\}$ be two sequences in $[a, b]$ such that $\displaystyle\lim_{n\to\infty} a_n = a$ and $\displaystyle\lim_{n\to\infty} b_n = b$. If $f_n \to f$ uniformly as $n \to \infty$ then show that $\lim_{n\to\infty} \int_{a_n}^{b_n} f_n(x)dx = \int_a^b f(x)dx$. $\boxed{4}$

4. Let $F_n : \mathbb{R} \to [0, 1], n \geq 0$, be continuous functions satisfying
   (i) $F_n(x) \leq F_n(y)$ for all $x \leq y$,
   (ii) $\displaystyle\lim_{x\to-\infty} F_n(x) = 0$, and
   (iii) $\displaystyle\lim_{x\to\infty} F_n(x) = 1$.
   If F_n converges pointwise to F_0 on $\mathbb{R}$, then show that F_n converges uniformly to F_0 on $\mathbb{R}$. $\boxed{6}$

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5. Let A and B be two nonempty disjoint closed sets in a metric space (X, d). Show that there are disjoint open sets U, V with $A \subset U$ and $B \subset V$. $\boxed{4}$

6. For $A, B \subset \mathbb{R}$, define the distance $d(A, B) := \inf\{|x - y| : x \in A, x \in B\}$. Let C, D be two nonempty disjoint closed subsets of $\mathbb{R}$. If $d(C, D) = 0$ then both C and D are unbounded. $\boxed{4}$

7. If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, then there exists a Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $g = f$ a.e. $\boxed{4}$

8. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable. Show that the function $F(x) = \displaystyle\int_{-\infty}^{x} f(x)\, dx$ is continuous on $\mathbb{R}$. Further define $\psi(x) = \displaystyle\sum_{n=1}^{\infty} f(2^n x + \frac{1}{n})$. Is ψ measurable? Is ψ integrable? If yes, calculate $\int_{\mathbb{R}} \psi(x)\, dx$. $\boxed{4+4}$

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