1(a) $f: R^n \to R^m$ is differentiable.
$T: R^n \to R^m$
$\therefore$ for $a, b \in R^n \to f(b) - f(a) = T(b-a)$.

Sol^n for each $y \in R^m$

$\langle f(b) - f(a), y \rangle = \langle f'(c)(b-a), y \rangle$
for some $c \in R^n$.
$= \langle L(b-a), y \rangle \to (1)$
where L is a linear fxn. $L: R^n \to R^m$.

Let $= L = L_1 e_1 + L_2 e_2 + \cdots + L_m e_m$.
where $\{e_1, e_2 \ldots e_m\}$ are directional vectors.
Put $y = e_1$ in (1):

$\langle f(b) - f(a), e_1 \rangle = \langle L(b-a), e_1 \rangle$
$= f(b) e_1 - f(a) e_1 = L_1 (b-a) e_1^2 = L_1 (b-a)$
$= f_1(b) - f_1(a) = L_1(b-a)$

Similarly for $y = e_2, e_3 \ldots e_m$.

we have $f_i(b) - f_i(a) = L_i(b-a)$
$\forall i \in \{1, 2, \ldots m\}$.

Thus, we can say each component (independent) of $(f(b) - f(a))$ is a linear fxn of $(b-a)$ in R^m.

Hence, $\Rightarrow f(b) - f(a) = T(b-a)$
where T is a linear map.

(1b) Suppose X is not complete
$\Rightarrow \exists\ (x_n)$ which is cauchy but not convergent.

Now consider $A = \{(x_n), n \in \mathbb{N}\}$.
$\because$ Every cauchy is bounded $\Rightarrow$ A is bounded.

$\because$ A is bounded $\Rightarrow \exists\ B[x, r]$ such that ~~[illegible]~~, $A \in B[x, r]$ (closed ball of radius r around x)

$\because$ B is closed ball and bounded by radius r $\Rightarrow$ B is compact.
$\Rightarrow (x_n) \in A$ must have a convergent subsequence.
$\Downarrow$
But (x_n) is cauchy $\Rightarrow$ and it has a convergent subsequence $\Rightarrow$ converge itself.
$\Downarrow$

$\nexists\ (x_n)$ such that it's cauchy but not convergent. $\Rightarrow$ X is complete.

classmate

Date ______

Page ______

(1c) Consider $X_n = (1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, 0, 0, \dots)$

Consider $m, n \in N$ such that $m > n$.

$$\|X_m - X_n\|_2 = \|(0,0,0,\dots,\frac{1}{n+1},\dots \frac{1}{m},0\dots 0)\|$$

$$= \left(\sum_{k=n+1}^{m} 1/k^2\right)^{1/2} < \left(\sum_{k=n+1}^{\infty} 1/k^2\right)^{1/2}.$$

$\because \sum_{k=n+1}^{\infty} 1/k^2 \to$ converges $\left(\because \sum_{k=1}^{\infty} 1/k^2 \to \text{converges}\right)$

Hence (X_n) is cauchy.

Let (X_n) be convergent.

Let $\Rightarrow \lim_{n\to\infty} X_n = X$

Let $X = (x_1, x_2 \dots \dots) \Rightarrow x_k = 0$ for some k.

$\Rightarrow$ for large n, we always have the k^{th} value of $(X_n - X)$ ~~as~~ ~~$\frac{1}{k}$~~.

$\hookrightarrow (1/k - 0) = (1/k)$

$$\Rightarrow \|X_n - X\|_2 = \|(\dots, \frac{1}{k}, \dots)\|$$

$$= \cancel{\left(\frac{1}{k}\right)} = \left(\dots + \frac{1}{k^2} \dots\right)^{1/2} \geq 1/k.$$

Then $\|X_n - X\|_2 \not\to 0$ thus X is not the limit of (X_n)

$\Rightarrow$ Contradiction. Hence, it's not convergent

$\Downarrow$

Normed linear space $(\ell^1, \|.\|_2)$ is not complete.

classmate

Date

Page

1 (a) Let $f_n(x) = \sqrt{x^2+\frac{1}{n}}$ and $f(x) = |x|$.
$\forall n \in N$ $\forall x \in [-1,1]$

$\therefore$ f_n's are differentiable $\forall$ $n \geq 1$, $n \in N$.

$\therefore$ $f_n(x) \longrightarrow f(x)$
as $\lim_{n\to\infty} \sqrt{x^2+\frac{1}{n}} \longrightarrow \sqrt{x^2} \longrightarrow |x|$.
$= f(x)$.
$\forall x \in [-1,1]$

also $f_n(x) = \sqrt{x^2+\left(\frac{1}{\sqrt{n}}\right)^2} \implies |x| \leq f_n(x) \leq |x| + 1/\sqrt{n}$. $\rightarrow$ (1)

$= \sqrt{\underbrace{\left(x+\frac{1}{\sqrt{n}}\right)^2 - \frac{2x}{\sqrt{n}}}}$ $\forall x \in [-1,1]$.

$\leq \sqrt{\left(x+\frac{1}{n}\right)^2}$.

from (1) $f_n(x) \longrightarrow |x|$ as $n \longrightarrow \infty$,
but $f(x) = |x|$ is not differentiable
in $x \in [-1,1]$
that at $x = 0$.

Hence, (f_n) seq limit need not be differentiable.

(1e) $1 < p < \infty$

$$G = \{ x = (x_1, x_2 \dots x_n \dots) \in \ell^p : \sum_{n=1}^{\infty} x_n = 0 \}$$

Consider the sequence

$X_1 = (1, -1, 0, 0, \dots 0 \dots) \in \ell^p$ $\because |1|^p + |1|^p$
$X_2 = (1, 0, -1, 0, \dots)$ $= 2 < \infty$
$X_3 = (1, 0, 0, -1, \dots)$
$X_4 = (1, 0, 0, 0, -1 \dots)$
$\vdots$

Like we can consider more $\Rightarrow$
Let $X_n \to X$
then $X = (1, 0, 0, \dots 0, \dots)$
because first term of all X_i is 1.
and after that we can find
$k = \max\{k_1, k_2 \dots k_\infty\}$ such that
$\cancel{x_k}$: $x_k = 0$ $\forall$ X_i's.

Hence, they converge to 0.

$X \in \ell^p$ because $1^p < \infty$.
but $\sum x_n = 1 + 0 + 0 \dots = 1$.
which is not possible.

Hence X_n be a convergent sequence in G does not converge to a pt in G itself.

(1f) Given $f : A_K \subset R \to [-\infty, \infty]$ be a measurable fxn.
for $K = 1, 2, \ldots$

$\Rightarrow L_K = \{x \in A_K \mid f(x) > c\}$ is LM for each $c \in R$

Then $f : \bigcup_{K=1}^{\infty} A_K \to [-\infty, \infty]$

$$L = \{x \in \bigcup_{K=1}^{\infty} A_K \mid f(x) > c\}.$$

$$= \bigcup_{K=1}^{\infty} \{x \mid x \in A_K, f(x) > c\}.$$

$$= \bigcup_{K=1}^{\infty} A_K.$$

* $\because$ since we know that $(\bigcup)$ LM sets = LM
and L is ~~set of~~ union of LM sets
$\Rightarrow L = LM$.

Hence, f is said to be measurable fxn.
Hence, Proved.

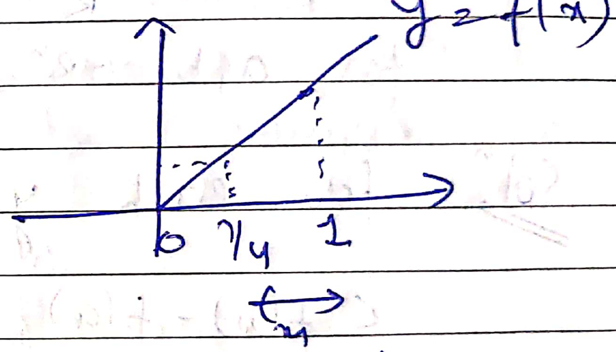
classmate

Date ______
Page ______

1(g) $f:[0,1]\to\mathbb{R}$ be Lebesgue Integral

for every $\varepsilon>0$, $\int_0^1 f^2\,dx \le \varepsilon^2 m\{x\in[0,1] : |f(x)|>\varepsilon\}$

Sol^n^ Let $f(x)=x$.



then $\int_0^1 x^2\,dx = 1/3$.

take $\varepsilon = 1/4$.

$m\{x\in[0,1], |f(x)|>\varepsilon\} = \underline{\underline{3/4}} \because (1-\frac{1}{4})$

Then LHS $= 1/3$

RHS $= \frac{1}{16}\times\frac{3}{4} = \frac{3}{64}$.

but $\frac{1}{3} \le \frac{3}{64}$ $\propto$ (contradiction)

Hence, wrong. (disproved).

(2) $S = \{ f \in C[0,1] : f(x) = f(1-x), \forall x \in [0,1] \}$.

To show: S is complete wrt supremum norm.

Soln.

Let there be a convergent sequence in S.
Let $f_n \longrightarrow f$ in S.

Convergence criteria states that
$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that
$d_\infty(f_n, f) < \varepsilon \quad \forall n \geq n_0$.

$\Rightarrow d_\infty(f_n, f) < \infty \quad \forall n \geq n_0 \ \forall x \in [0,1]$.
$= \sup |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq n_0 \ \forall x \in [0,1]$

$\Downarrow$

$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq n_0 \ \forall x \in [0,1]$.

$\Rightarrow f_n \xrightarrow{p} f$
(pointwise convergence)

$\Rightarrow f_n(x) \longrightarrow f(x) \longrightarrow (1)$
Put $x \rightarrow 1-x \Rightarrow f_n(1-x) \longrightarrow f(1-x) \rightarrow (2)$

but $f_n(1-x) = f_n(x) \Rightarrow f_n(x) \longrightarrow f(1-x)$
from (2)

$\therefore f_n$ is real we have $f(x) = f(1-x)$
$\Rightarrow f \in S \Rightarrow S$ is closed.

$\therefore C[0,1]$ is complete and $S \subset C[0,1]$ is closed. $\Rightarrow$ By theorem we have that S is complete.

Hence, proved.

classmate
Date
Page

Q. (3): $\lim a_n \Rightarrow a$
$\lim b_n = b$ If $f_n \to f$ uniformally as $n \to \infty$.

To show
$$\lim_{n\to\infty} \int_{a_n}^{b_n} f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Consider $\int_a^b f_n(x)\,dx$ $\because f_n \to$ continous and real valued fxn.

$$\int_a^b f_n(x)\,dx = \int_a^{a_n} f_n(x)\,dx + \int_{a_n}^{b_n} f_n(x)\,dx + \int_{b_n}^{b} f_n(x)\,dx.$$

Apply limit:

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \lim_{n\to\infty}\left(\int_a^{a_n} f_n(x)\,dx + \int_{a_n}^{b_n} f_n(x)\,dx + \int_{b_n}^{b} f_n(x)\,dx\right)$$

$$= \lim_{n\to\infty} \int_a^{a_n} f_n(x)\,dx + \lim_{n\to\infty} \int_{a_n}^{b_n} f_n(x)\,dx + \lim_{n\to\infty} \int_{b_n}^{b} f_n(x)\,dx$$

$\because$ all are continous fxns.

$$= \because a_n \to a \Rightarrow \lim_{n\to\infty} \int_a^{a_n} f_n(x)\,dx = 0$$

and $b_n \to b \Rightarrow \lim_{n\to\infty} \int_{b_n}^{b} f_n(x)\,dx = 0$

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n(x)\,dx = \lim_{n\to\infty} \int_{a_n}^{b_n} f_n(x)\,dx. \rightarrow (1)$$

$\because f_n \to f$ (uniform convergence)

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx \quad \text{(Theorem)} \rightarrow (2)$$

From (1) & (2):

$$= \int_a^b \underbrace{\left(\lim_{n\to\infty} f_n(x)\right)}_{= f(x)} dx = \lim_{n\to\infty} \int_{a_n}^{b_n} f_n(x)\,dx$$

$$\Rightarrow \int_a^b f(x)\,dx = \lim_{n\to\infty} \int_{a_n}^{b_n} f_n(x)\,dx.$$ Hence proved.

classmate
Date
Page

(5) Since B is closed $\Rightarrow A \cap B = \phi$.
and disjoint with A.

$\Rightarrow \forall a \in A$ we have $d(a, B) > 0$.
$(\because A, B$ are disjoint$)$

Define $f: X \to R$
$f(x) = d(x, B) \quad \forall x \in X$.
$\to f$ is continous.
$\forall a \in A, \; f(a) = d(a, B) > 0$.

$\forall a \in A$ choose a real no $(r_a \in R)$
such that $0 < r_a < f(a)$,
then $\{f^{-1}(r_a, \infty) : a \in A\}$ is an open set of A.
$\because f \to$ cont^s and (r_a, ∞) is open in R.
since A is compact, $\Rightarrow \exists$ finitely many pts $a_1, a_2 \ldots a_n \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} f^{-1}(r_{a_i}, \infty).$$

Let $r_{a_k} = \min(r_{a_i}, 1 \le i \le n)$.
Then $r_{a_k} > 0$ and $A \subseteq f^{-1}(r_{a_k}, \infty)$
ie: $d(a, B) > r_{a_k}$
$\forall a \in A$.

hence $\inf_{a \in A} d(a, B) \ge r_{a_k}$.

$\Downarrow$

$$d(A, B) \ge r_{a_k} > 0. \longrightarrow (1)$$

from (1) we can see that we can choose a $r \in R$
such that $0 < r < d(A, B)$.
again Consider $f(x) : X \to R$
$f(x) = \{ d(x, B) \quad \forall x \in X$.

Consider 2 set P, Q such that

$$P = f^{-1}(-\infty, \lambda)$$
$$Q = f^{-1}(\lambda, \infty)$$

we can easily see that P, Q are disjoint set

and $A \subseteq Q$ and $B \subseteq P$.

classmate
Date
Page

given:

(6) $\therefore d(A,B) = \inf\{|x-y| : x\in A, y\in B\}$.

C, D are 2 disjoint subsets of R that are closed.

$\Rightarrow C\cap D = \phi$

To show: if $d(C,D)=0 \Rightarrow C,D$ are unbounded.

Sol^n:

$\rightarrow$ We here prove by contradiction.

Suppose any of C or D be bounded in R.

Let's say C be bounded in R.

$\therefore$ C is closed and bounded in R

$\Rightarrow$ C is compact.

$\because d(C,D)=0 \Rightarrow \exists$ seq $(x_n)\in C$ and $(y_n)\in D$ such that

$|x_n - y_n| \to 0$ as $n\to\infty$. $\longrightarrow$ (1)

$\because$ C is compact, thus there exists a subsequence (x_{n_k}) which is convergent to let's say $x\in C$.

$\Rightarrow |x_{n_k} - x| \to 0$ as $k\to\infty$, $x\in C$. $\rightarrow$ (2)

Consider a sequence (y_{n_k}).

Applying Triangular Inequality:

$|y_{n_k} - x| \le |y_{n_k} - x_{n_k}| + |x_{n_k} - x|$.

But $|x_{n_k} - x| \to 0$ as $k\to\infty$ from (2)

and $|x_{n_k} - y_{n_k}| \to 0$ as $n\to\infty$ from (1).

$\Rightarrow |y_{n_k} - x| \to 0$ as $n\to\infty$

$\Rightarrow$ since D is closed $\Rightarrow x\in D$.

$\Rightarrow x\in C\cap D$ which is a contradiction as $C\cap D = \phi$.

· Hence, proved that C can't be bounded and similarly D can't be bounded.

(8) (a) given f is integrable.
$f: R \to R$

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

$\because$ f is integrable hence $F(x)$ is properly defined. for $\forall x \in R$. $\Rightarrow f_{\chi(-\infty, x]}$ is also integrable.
for continuity: consider the sequential criteria

$\Rightarrow$ consider a seq^n $(x_n) \to x$
and consider $f_n = f_{\chi(-\infty, x_n]}$.

$\Rightarrow$ we have $f_n \to f_{\chi(-\infty, x]}$. $\Rightarrow |f_n| \le f$. $\forall x \in R$.

$$\text{So } \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \int_{-\infty}^{x} f_n = \int_{-\infty}^{x} f = F(x).$$

↳ since f is Lebesgue Integrable.

Hence $F(x_n) \to F(x)$
Hence, by sequential criteria of continuity we can say that F is continuous on R.