MA 224 (Real Analysis)

Summary of Lectures - 5

Drawbacks of Riemann integral:

(a) The Riemann integral of a real-valued function f is not defined if either the domain of f is not a closed (bounded) interval or f is not bounded in that domain.
(However, this is not a very serious drawback as extensions of Riemann integrals to improper integrals can be useful.)
Also, the class of Riemann integrable functions on an interval $[a, b]$ is 'quite small' since a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff $m^*(\{x \in [a, b] : f \text{ is discontinuous at } x\}) = 0$.

(b) If $\{f_n\}$ is a sequence of Riemann integrable functions on $[a, b]$ and if $f : [a, b] \to \mathbb{R}$ such that $f_n \to f$ pointwise on $[a, b]$, then f need not be Riemann integrable on $[a, b]$ and even if f is Riemann integrable on $[a, b]$, it is not necessary that $\lim\limits_{n\to\infty} \int\limits_a^b f_n(x)\,dx = \int\limits_a^b f(x)\,dx$.

(c) The differentiation and the integration as inverse process of each other, as expressed by the fundamental theorem of calculus, no longer remains completely true with Riemann integral.

(d) The integral formula using Riemann integral for the length of a continuously differentiable path in the plane $\mathbb{R}^2$ no longer remains valid for rectifiable paths in $\mathbb{R}^2$.

Length of an interval: The length of an interval I in $\mathbb{R}$ is defined by
$\ell(I) = \begin{cases} b - a & \text{if } I \text{ is bounded with endpoints } a, b \text{ satisfying } a \le b, \\ +\infty & \text{if } I \text{ is unbounded.} \end{cases}$

Lebesgue outer measure: The Lebesgue outer measure of a subset A of $\mathbb{R}$ is defined by
$m^*(A) = \inf\Big\{ \sum\limits_{n=1}^{\infty} \ell(I_n) : A \subset \bigcup\limits_{n=1}^{\infty} I_n,\ I_n \in \mathcal{F} \text{ for all } n \in \mathbb{N} \Big\}$,
where $\mathcal{F}$ is the class of all bounded (nonempty) open intervals in $\mathbb{R}$.
Thus the Lebesgue outer measure on $\mathbb{R}$ is a set function $m^* : \mathcal{P}(\mathbb{R}) \to [0, +\infty]$.

Properties of m^*:

(a) For every countable subset A of $\mathbb{R}$, $m^*(A) = 0$.
In particular, $m^*(\emptyset) = 0$, $m^*(\mathbb{N}) = 0$, $m^*(\mathbb{Z}) = 0$, $m^*(\mathbb{Q}) = 0$ and $m^*(F) = 0$, where F is a nonempty finite subset of $\mathbb{R}$.
However, $m^*(A)$ can be equal to 0 for some uncountable subset A of $\mathbb{R}$. An example can be found below.

(b) For all $A, B \subset \mathbb{R}$ with $A \subset B$, $m^*(A) \le m^*(B)$ (*i.e.* m^* is monotone).

(c) For all $A \subset \mathbb{R}$ and for all $x \in \mathbb{R}$, $m^*(A + x) = m^*(A)$ (*i.e.* m^* is translation invariant).

(d) For every sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of $\mathbb{R}$, $m^*\Big(\bigcup\limits_{n=1}^{\infty} A_n\Big) \le \sum\limits_{n=1}^{\infty} m^*(A_n)$ (*i.e.* m^* is countably subadditive).
In particular, m^* is finitely subadditive, *i.e.* if $A_1, ..., A_n \subset \mathbb{R}$, then $m^*\Big(\bigcup\limits_{i=1}^{n} A_i\Big) \le \sum\limits_{i=1}^{n} m^*(A_i)$.

(e) For every interval I of $\mathbb{R}$, $m^*(I) = \ell(I)$.

The Cantor set: The Cantor (ternary or middle third) set C is obtained by successively removing from $[0, 1]$ the middle third open intervals. Thus if $F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, and more generally $F_n = [0, \frac{1}{3^n}] \cup [\frac{2}{3^n}, \frac{1}{3^{n-1}}] \cup \cdots \cup [\frac{3^n-1}{3^n}, 1]$ for all $n \in \mathbb{N}$, then $C = \bigcap\limits_{n=1}^{\infty} F_n$.
It can be shown that C is uncountable, closed, bounded and $m^*(C) = 0$. Moreover,
$C = \Big\{ \sum\limits_{n=1}^{\infty} \frac{a_n}{3^n} : a_n \in \{0, 2\} \text{ for all } n \in \mathbb{N} \Big\}$.

Failure of countable additivity for m^*: We give examples below to show that m^* does not

satisfy the countable additivity (even, finite additivity) property on $\mathcal{P}(\mathbb{R})$.
In fact, it can be proved (in a similar way as the proof of the existence of Lebesgue non-measurable sets) that (in the presence of axiom of choice of set theory) it is impossible to define a set function $\mu : \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ satisfying all the following properties:

(a) For every sequence $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoint subsets of $\mathbb{R}$, $\mu\Big(\bigcup\limits_{n=1}^{\infty} A_n\Big) = \sum\limits_{n=1}^{\infty} \mu(A_n)$.
(b) For all $A \subset \mathbb{R}$ and for all $x \in \mathbb{R}$, $\mu(A + x) = \mu(A)$.
(c) For every interval I of $\mathbb{R}$, $\mu(I) = \ell(I)$.

Lebesgue measurable set: A subset E of $\mathbb{R}$ is called Lebesgue measurable if for each $A \subset \mathbb{R}$, $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$.
(The above condition for the Lebesgue measurability of a set is called the Carathéodory condition.)
Since $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$ for all $A, E \subset \mathbb{R}$, $E \subset \mathbb{R}$ is Lebesgue measurable iff $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$ for all $A \subset \mathbb{R}$.

Remarks:

(a) A subset E of $\mathbb{R}$ is Lebesgue measurable iff E^c is Lebesgue measurable.
(b) If $E \subset \mathbb{R}$ with $m^*(E) = 0$, then E is Lebesgue measurable.

Using the above two facts, we find that all countable subsets of $\mathbb{R}$ and their complements are Lebesgue measurable. Thus, in particular, $\emptyset$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are Lebesgue measurable.

Result: If E and F are Lebesgue measurable subsets of $\mathbb{R}$, then $E \cup F$, $E \cap F$ and $E \setminus F$ are Lebesgue measurable.
Hence if $E_1, ..., E_n$ are Lebesgue measurable subsets of $\mathbb{R}$, then $\bigcup\limits_{i=1}^{n} E_i$ and $\bigcap\limits_{i=1}^{n} E_i$ are Lebesgue measurable.

Result: If $E_1, ..., E_n$ are pairwise disjoint Lebesgue measurable subsets of $\mathbb{R}$ and if $A \subset \mathbb{R}$, then $m^*\Big(A \cap \Big(\bigcup\limits_{i=1}^{n} E_i\Big)\Big) = \sum\limits_{i=1}^{n} m^*(A \cap E_i)$.
In particular, taking $A = \mathbb{R}$, we find that $m^*\Big(\bigcup\limits_{i=1}^{n} E_i\Big) = \sum\limits_{i=1}^{n} m^*(E_i)$ (*i.e.* m^* is finitely additive for Lebesgue measurable subsets of $\mathbb{R}$).

Result: If $\{E_n\}_{n=1}^{\infty}$ is a sequence of Lebesgue measurable subsets of $\mathbb{R}$, then $\bigcup\limits_{n=1}^{\infty} E_n$ and $\bigcap\limits_{n=1}^{\infty} E_n$ are Lebesgue measurable.
If moreover E_n $(n = 1, 2, ...)$ are pairwise disjoint, then $m^*\Big(\bigcup\limits_{n=1}^{\infty} E_n\Big) = \sum\limits_{n=1}^{\infty} m^*(E_n)$ (*i.e.* m^* is countably additive for Lebesgue measurable subsets of $\mathbb{R}$).

Result: Every interval of $\mathbb{R}$ is Lebesgue measurable.
Since every open set in $\mathbb{R}$ is a countable union of open intervals in $\mathbb{R}$, it follows that every open set in $\mathbb{R}$ is Lebesgue measurable. Consequently every closed set in $\mathbb{R}$ is also Lebesgue measurable.

Lebesgue measure: If $\mathcal{M}(\mathbb{R})$ denotes the class of all Lebesgue measurable subsets of $\mathbb{R}$, then the set function $m = m^*|_{\mathcal{M}(\mathbb{R})} : \mathcal{M}(\mathbb{R}) \to [0, +\infty]$ is called the Lebesgue measure on $\mathbb{R}$.
Thus the Lebesgue measure of every Lebesgue measurable subset E of $\mathbb{R}$ is $m(E) = m^*(E)$.

Lebesgue non-measurable set: There exists a subset N of $[0, 1]$ which is not Lebesgue measurable.
(The proof of the existence of a Lebesgue non-measurable subset of $\mathbb{R}$ requires the use of axiom of choice of set theory.)
More generally, if $A \subset \mathbb{R}$ with $m^*(A) > 0$, then there exists a subset B of A which is not Lebesgue

measurable.

Remarks: If N is a Lebesgue non-measurable subset of $\mathbb{R}$, then there exists $X \subset \mathbb{R}$ such that $m^*(X) < m^*(X \cap N) + m^*(X \cap N^c)$. Thus $A = X \cap N$ and $B = X \cap N^c$ are disjoint subsets of $\mathbb{R}$ such that $m^*(A \cup B) < m^*(A) + m^*(B)$.

Algebra of sets: A class $\mathcal{A}$ of subsets of a nonempty set X is called an algebra of subsets of X (or, an algebra on X) if
(a) $X \in \mathcal{A}$,
(b) for every $A \in \mathcal{A}$, $A^c \in \mathcal{A}$, and
(c) for all $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$.

σ-algebra of sets: A class $\mathcal{A}$ of subsets of a nonempty set X is called a σ-algebra of subsets of X (or, a σ-algebra on X) if
(a) $X \in \mathcal{A}$,
(b) for every $A \in \mathcal{A}$, $A^c \in \mathcal{A}$, and
(c) for every sequence $\{A_n\}_{n=1}^{\infty}$ of sets in $\mathcal{A}$, $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{A}$.

Remarks:
(a) In both the above definitions, condition (a) can be replaced by either the condition (a)$'$ $\emptyset \in \mathcal{A}$, or the condition (a)$''$ $\mathcal{A} \neq \emptyset$. Similarly the condition (c) in both the definitions can be replaced by the corresponding condition involving intersections.
(b) Every σ-algebra on a nonempty set X is an algebra on X, but the converse need not be true as can be seen in an example below.
(c) If $\mathcal{A}$ is an algebra on a nonempty set X and if $A_1, ..., A_n \in \mathcal{A}$, then $A_1 \setminus A_2$, $\bigcup\limits_{i=1}^{n} A_i$, $\bigcap\limits_{i=1}^{n} A_i \in \mathcal{A}$.

Examples:
(a) For every nonempty set X, $\{\emptyset, X\}$ and $\mathcal{P}(X)$ are σ-algebras on X.
(b) If X is an infinite set, then $\{A \subset X : A \text{ is finite}\}$ is not an algebra on X.
(c) If X is an infinite set, then $\{A \subset X : A \text{ is finite or } A^c \text{ is finite}\}$ is an algebra but not a σ-algebra on X.
(d) If X is an uncountable set, then $\{A \subset X : A \text{ is countable or } A^c \text{ is countable}\}$ is a σ-algebra on X.
(e) $\mathcal{M}(\mathbb{R})$ is a σ-algebra on $\mathbb{R}$.

Intersection and union of σ-algebras: If $\{\mathcal{A}_\alpha\}_{\alpha \in \Lambda}$ is a family of σ-algebras on a nonempty set X, then $\bigcap\limits_{\alpha \in \Lambda} \mathcal{A}_\alpha$ is a σ-algebra on X.
However, the union of two σ-algebras on a nonempty set X need not be a σ-algebra on X.

σ-algebra generated by a class: If $\mathcal{S} \subset \mathcal{P}(X)$, where X is a nonempty set, then $\bigcap\{\mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X, \mathcal{S} \subset \mathcal{A}\}$ is the smallest σ-algebra on X containing $\mathcal{S}$. It is called the σ-algebra on X generated by $\mathcal{S}$ and it is denoted by $\sigma(\mathcal{S})$.

Borel σ-algebra: If X is a metric space, then the σ-algebra on X generated by the class of all open subsets of X is called the Borel σ-algebra of X, which is denoted by $\mathcal{B}(X)$.
Each set in $\mathcal{B}(X)$ is called a Borel subset of X.

Remarks: Note that $\mathcal{B}(\mathbb{R})$ is also generated by each of the classes $\{F \subset \mathbb{R} : F \text{ is closed in } \mathbb{R}\}$ and $\{(a, b) : a, b \in \mathbb{R}, a < b\}$.

Result: Every Borel subset of $\mathbb{R}$ is Lebesgue measurable (*i.e.* $\mathcal{B}(\mathbb{R}) \subset \mathcal{M}(\mathbb{R})$).

Remarks: The method used in proving the above result is usually used in proving results involving

Borel sets. For example, in order to show that every Borel subset of $\mathbb{R}$ has certain property P, it is sufficient to show that

(a) $\{A \subset \mathbb{R} : A \text{ has property } P\}$ is a σ-algebra on $\mathbb{R}$, and
(b) every open subset (or, every member of a generating class of $\mathcal{B}(\mathbb{R})$) of $\mathbb{R}$ has property P.

Lebesgue measurable function: If E is a nonempty Lebesgue measurable subset of $\mathbb{R}$, then $f : E \to [-\infty, +\infty]$ is called Lebesgue measurable if $\{x \in E : f(x) > c\}$ is Lebesgue measurable for each $c \in \mathbb{R}$.

Result: If E is a nonempty Lebesgue measurable subset of $\mathbb{R}$ and $f : E \to [-\infty, +\infty]$, then the following statements are equivalent.

(a) f is Lebesgue measurable.
(b) $\{x \in E : f(x) \le c\}$ is Lebesgue measurable for each $c \in \mathbb{R}$.
(c) $\{x \in E : f(x) < c\}$ is Lebesgue measurable for each $c \in \mathbb{R}$.
(d) $\{x \in E : f(x) \ge c\}$ is Lebesgue measurable for each $c \in \mathbb{R}$.

Result: If E is a nonempty Lebesgue measurable subset of $\mathbb{R}$ and if $f : E \to [-\infty, +\infty]$ is Lebesgue measurable, then

(a) $\{x \in E : f(x) = c\}$ is Lebesgue measurable for each $c \in [-\infty, +\infty]$, and
(b) $\{x \in E : f(x) \in \mathbb{R}\}$ is Lebesgue measurable.

Remarks: If E is a nonempty Lebesgue measurable subset of $\mathbb{R}$, then a function $f : E \to [-\infty, +\infty]$ satisfying (a) above need not be Lebesgue measurable.
For example, consider a Lebesgue non-measurable set $N \subset (0, 1)$ and define $f : (0, 1) \to \mathbb{R}$ by
$f(x) = \begin{cases} x & \text{if } x \in N, \\ -x & \text{if } x \in (0,1) \setminus N. \end{cases}$

Result: If E is a nonempty Lebesgue measurable subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$, then the following statements are equivalent.

(a) f is Lebesgue measurable.
(b) $f^{-1}((a, b))$ is Lebesgue measurable for all $a, b \in \mathbb{R}$ with $a < b$.
(c) $f^{-1}(G)$ is Lebesgue measurable for every open set G in $\mathbb{R}$.
(d) $f^{-1}(B)$ is Lebesgue measurable for every Borel set B in $\mathbb{R}$.

Examples of Lebesgue measurable functions:

(a) If $E(\neq \emptyset) \in \mathcal{M}(\mathbb{R})$, then every continuous function $f : E \to \mathbb{R}$ is Lebesgue measurable.
(b) If I is an interval of $\mathbb{R}$, then every monotonic (increasing or decreasing) function $f : I \to \mathbb{R}$ is Lebesgue measurable.
(c) If $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$
then $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable.
(d) If $f(x) = \begin{cases} \frac{1}{x} & \text{if } x(\neq 0) \in \mathbb{R}, \\ 1 & \text{if } x = 0, \end{cases}$
then $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable.

Almost everywhere concept: Let E be a nonempty Lebesgue measurable subset of $\mathbb{R}$. A property $P(x)$, associated with points $x \in \mathbb{R}$, is said to hold almost everywhere (a.e.) on E (or, for almost all $x \in E$) if $m^*(\{x \in E : P(x) \text{ does not hold}\}) = 0$.

Example: If $f, g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = 0$ for all $x \in \mathbb{R}$ and $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$
then $f = g$ a.e. on $\mathbb{R}$.

Result: Let E be a nonempty Lebesgue measurable subset of $\mathbb{R}$ and let $f, g : E \to [-\infty, +\infty]$ be such that $f = g$ a.e. on E. If f is Lebesgue measurable, then g is Lebesgue measurable.

Result: Let E be a nonempty Lebesgue measurable subset of $\mathbb{R}$ and let $f, g : E \to [-\infty, +\infty]$ be Lebesgue measurable. Then the sets $\{x \in E : f(x) > g(x)\}$, $\{x \in E : f(x) \geq g(x)\}$ and $\{x \in E : f(x) = g(x)\}$ are Lebesgue measurable.

Combination of Lebesgue measurable functions: Let E be a nonempty Lebesgue measurable subset of $\mathbb{R}$.

(a) If $f, g : E \to \mathbb{R}$ are Lebesgue measurable, then $f + g : E \to \mathbb{R}$ is Lebesgue measurable.
(b) If $f, g : E \to [-\infty, +\infty]$ are Lebesgue measurable and if $\alpha \in \mathbb{R}$, then $\alpha f, f^2, fg : E \to [-\infty, +\infty]$ are Lebesgue measurable.
(c) If $f_1, ..., f_n : E \to [-\infty, +\infty]$ are Lebesgue measurable, then $\max(f_1, ..., f_n) : E \to [-\infty, +\infty]$ and $\min(f_1, ..., f_n) : E \to [-\infty, +\infty]$ are Lebesgue measurable, where $\max(f_1, ..., f_n)(x) = \max\{f_1(x), ..., f_n(x)\}$ and $\min(f_1, ..., f_n)(x) = \min\{f_1(x), ..., f_n(x)\}$ for all $x \in E$.
(d) If $f : E \to [-\infty, +\infty]$ is Lebesgue measurable, then $f^+ : E \to [0, +\infty]$, $f^- : E \to [0, +\infty]$ and $|f| : E \to [0, +\infty]$ are Lebesgue measurable, where $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$ and $|f| = f^+ + f^-$.
(e) If for each $n \in \mathbb{N}$, $f_n : E \to [-\infty, +\infty]$ is Lebesgue measurable, then $\sup\limits_{n \in \mathbb{N}} f_n$ and $\inf\limits_{n \in \mathbb{N}} f_n$ are Lebesgue measurable, where each of these two functions are defined pointwise.
(f) For each $n \in \mathbb{N}$, let $f_n : E \to [-\infty, +\infty]$ be Lebesgue measurable and let $\lim\limits_{n \to \infty} f_n(x)$ exist in $[-\infty, +\infty]$ for each $x \in E$. If $f(x) = \lim\limits_{n \to \infty} f_n(x)$ for all $x \in E$, then $f : E \to [-\infty, +\infty]$ is Lebesgue measurable.

Characteristic function: Let X be a nonempty set and $A \subset X$. The characteristic function $\chi_A : X \to \mathbb{R}$ is defined by $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in X \setminus A. \end{cases}$
If $E \subset \mathbb{R}$, then $\chi_E : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable iff E is Lebesgue measurable.

Simple function: If E is a nonempty subset of $\mathbb{R}$, then a function $\varphi : E \to \mathbb{R}$ is called simple if Range(φ) is a finite set.
If $\varphi : E \to \mathbb{R}$ is simple with Range$(\varphi) = \{c_1, ..., c_n\}$, where $c_1, ..., c_n$ are distinct, and if $E_i = \{x \in E : \varphi(x) = c_i\}$ for $i = 1, ..., n$, then $E_1, ..., E_n$ are pairwise disjoint and $\varphi = \sum\limits_{i=1}^{n} c_i \chi_{E_i}$, which is called the canonical representation of φ.
If E is a nonempty Lebesgue measurable subset of $\mathbb{R}$, then a simple function $\varphi : E \to \mathbb{R}$ with canonical representation $\varphi = \sum\limits_{i=1}^{n} c_i \chi_{E_i}$ is Lebesgue measurable iff E_i is Lebesgue measurable for each $i \in \{1, ..., n\}$.

Remarks: If E is a nonempty subset of $\mathbb{R}$ and $\varphi : E \to \mathbb{R}$ is a simple function, then φ can have more than one representation of the form $\varphi = \sum\limits_{i=1}^{m} a_i \chi_{A_i}$ on E, where for $i = 1, ..., m$, $a_i \in \mathbb{R}$ and $A_i \subset E$ such that $A_1, ..., A_m$ are pairwise disjoint. (Note that $a_1, ..., a_m$ need not be distinct.)
For example, consider $\varphi = 2\chi_{[0,2]} + 3\chi_{(2,3]} = 2\chi_{[0,1]} + 2\chi_{(1,2]} + 3\chi_{(2,3]}$ on $[0, 3]$.

Integral of non-negative simple Lebesgue measurable function: Let E be a nonempty Lebesgue measurable subset of $\mathbb{R}$ and let $\varphi : E \to [0, +\infty)$ be simple Lebesgue measurable with canonical representation $\varphi = \sum\limits_{i=1}^{n} c_i \chi_{E_i}$. The Lebesgue integral of φ over E is defined by

$$\int_E \varphi \, dm = \sum_{i=1}^{n} c_i m(E_i).$$

(We follow the convention that $0.(+\infty) = 0 = (+\infty).0.$)

Result: Let E be a nonempty Lebesgue measurable subset of $\mathbb{R}$ and let $E_1, ..., E_m$ be pairwise disjoint Lebesgue measurable subsets of E. If $\varphi = \sum\limits_{i=1}^{m} a_i \chi_{E_i}$ on E, where $a_1, ..., a_m$ are non-negative real numbers, then $\int\limits_E \varphi\, dm = \sum\limits_{i=1}^{m} a_i m(E_i)$.

Example: Let C denote the Cantor set and $\varphi : [0,1] \to \mathbb{R}$ be defined by

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap C, \\ 2 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap C, \\ 3 & \text{if } x \in [0,1] \setminus C. \end{cases}$$

Then $\int\limits_{[0,1]} \varphi\, dm = 3$.

Integral of non-negative Lebesgue measurable function: Let E be a nonempty Lebesgue measurable subset of $\mathbb{R}$ and let $f : E \to [0, +\infty]$ be Lebesgue measurable. The Lebesgue integral of f over E is defined by

$$\int\limits_E f\, dm = \sup \Big\{ \int\limits_E \varphi\, dm : \varphi : E \to [0, +\infty) \text{ is simple, Lebesgue measurable and } \varphi \leq f \Big\}.$$

Remarks: If E is a nonempty Lebesgue measurable subset of $\mathbb{R}$ and if $\varphi : E \to [0, +\infty)$ is simple, Lebesgue measurable, then the above definition for $\int\limits_E \varphi\, dm$ agrees with its earlier definition.

Monotone convergence theorem (MCT): Let E be a nonempty Lebesgue measurable subset of $\mathbb{R}$ and let $f, f_n : E \to [0, +\infty]$ be Lebesgue measurable for each $n \in \mathbb{N}$. If $f_1(x) \leq f_2(x) \leq \cdots$ for almost all $x \in E$ and $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for almost all $x \in E$, then $\int\limits_E f\, dm = \lim\limits_{n\to\infty} \int\limits_E f_n\, dm$.

Remarks: The conclusion of the MCT need not hold if the sequence $\{f_n\}_{n=1}^{\infty}$ is assumed to be decreasing (instead of increasing), *i.e.* if $f_1(x) \geq f_2(x) \geq \cdots$ for almost all $x \in E$. For example, consider $f_n = \chi_{[n,+\infty)}$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} f_n(x) = 0$ for all $x \in \mathbb{R}$ and $\int\limits_{\mathbb{R}} \lim\limits_{n\to\infty} f_n\, dm = 0 \neq +\infty = \lim\limits_{n\to\infty} \int\limits_{\mathbb{R}} f_n\, dm$.

Integral of measurable function: Let E be a nonempty Lebesgue measurable subset of $\mathbb{R}$ and let $f : E \to [-\infty, +\infty]$ be Lebesgue measurable. The Lebesgue integral of f over E is said to exist if at least one of $\int\limits_E f^+\, dm$ and $\int\limits_E f^-\, dm$ is finite and in such case, we define the Lebesgue integral of f over E as $\int\limits_E f\, dm = \int\limits_E f^+\, d\mu - \int\limits_E f^-\, dm$.
Also, f is said to be Lebesgue integrable over E if (both) $\int\limits_E f^+\, dm < +\infty$ and $\int\limits_E f^-\, dm < +\infty$.

Remarks: Let E be a nonempty Lebesgue measurable subset of $\mathbb{R}$ and let $f : E \to [-\infty, +\infty]$ be Lebesgue measurable. If $f(x) \geq 0$ for all $x \in E$, then the above definition for $\int\limits_E f\, dm$ agrees with its earlier definition and f is Lebesgue integrable over E iff $\int\limits_E f\, dm < +\infty$.

Result: Let E be a nonempty Lebesgue measurable subset of $\mathbb{R}$ and let $f, g : E \to \mathbb{R}$ be Lebesgue measurable functions such that the Lebesgue integrals $\int\limits_E f\, dm$ and $\int\limits_E g\, dm$ exist. Then

(a) $\int\limits_E \alpha f\, dm = \alpha \int\limits_E f\, dm$ if $\alpha \in \mathbb{R}$.
(b) $\int\limits_E (f+g)\, dm = \int\limits_E f\, dm + \int\limits_E g\, dm$.
(c) $\int\limits_E f\, dm \leq \int\limits_E g\, dm$ if $f \leq g$ a.e. on E.

(d) $\int\limits_E f\,dm = \int\limits_A f\,dm + \int\limits_{E\setminus A} f\,dm$ if $A \subset E$ is Lebesgue measurable.

Consequently, if $f = g$ a.e. on E, then $\int\limits_E f\,dm = \int\limits_E g\,dm$.

Dominated convergence theorem (DCT): Let E be a nonempty Lebesgue measurable subset of $\mathbb{R}$ and let $f, f_n : E \to [-\infty, +\infty]$ be Lebesgue measurable for each $n \in \mathbb{N}$ such that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for almost all $x \in E$. If $g : E \to [0, +\infty]$ is Lebesgue integrable such that for each $n \in \mathbb{N}$, $|f_n(x)| \leq g(x)$ for almost all $x \in E$, then f, f_n ($n \in \mathbb{N}$) are Lebesgue integrable over E and $\int\limits_E f\,dm = \lim\limits_{n\to\infty} \int\limits_E f_n\,dm$.

Lebesgue integral and Riemann integral: If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then f is Lebesgue integrable on $[a,b]$ and $\int\limits_{[a,b]} f\,dm = \int\limits_a^b f(x)\,dx$.

(This result is useful in evaluating several Lebesgue integrals.)

However, the function $\chi_{\mathbb{Q}\cap[0,1]} : [0,1] \to \mathbb{R}$ is Lebesgue integrable but not Riemann integrable on $[0,1]$.

Example: (Evaluation of Lebesgue integral) If $f(x) = \begin{cases} x^2 - x & \text{if } x \in \mathbb{Q} \cap [0,1], \\ \cos x & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1], \end{cases}$ then $f = g$ a.e. on $[0,1]$, where $g(x) = \cos x$ for all $x \in [0,1]$. Since $g : [0,1] \to \mathbb{R}$ is Riemann integrable, f is Lebesgue integrable on $[0,1]$ and $\int\limits_{[0,1]} f\,dm = \int\limits_{[0,1]} g\,dm = \int\limits_0^1 g(x)\,dx = \sin 1$.

Examples: (Application of MCT and evaluation of Lebesgue integral)

(a) To evaluate the Lebesgue integral $\int\limits_{[1,+\infty)} \frac{1}{x^2}\,dx$, we consider $f_n(x) = \begin{cases} \frac{1}{x^2} & \text{if } 1 \leq x \leq n, \\ 0 & \text{if } n < x < +\infty, \end{cases}$ for each $n \in \mathbb{N}$. Then by the MCT, we get $\int\limits_{[1,+\infty)} \frac{1}{x^2}\,dx = \int\limits_{[1,+\infty)} \lim\limits_{n\to\infty} f_n\,dm = \lim\limits_{n\to\infty} \int\limits_{[1,+\infty)} f_n\,dm = \lim\limits_{n\to\infty} \Big(\int\limits_{[1,n]} f_n\,dm + \int\limits_{[n,+\infty)} f_n\,dm \Big) = \lim\limits_{n\to\infty} \Big(\int\limits_1^n f_n(x)\,dx + 0 \Big) = \lim\limits_{n\to\infty} (1 - \frac{1}{n}) = 1$.

(b) To evaluate the Lebesgue integral $\int\limits_{(0,1]} \frac{1}{\sqrt{x}}\,dx$, we consider $f_n(x) = \begin{cases} 0 & \text{if } 0 < x < \frac{1}{n^2}, \\ \frac{1}{\sqrt{x}} & \text{if } \frac{1}{n^2} \leq x \leq 1, \end{cases}$ for each $n \in \mathbb{N}$. Then by the MCT, we get $\int\limits_{(0,1]} \frac{1}{\sqrt{x}}\,dx = \int\limits_{(0,1]} \lim\limits_{n\to\infty} f_n\,dm = \lim\limits_{n\to\infty} \int\limits_{(0,1]} f_n\,dm = \lim\limits_{n\to\infty} \Big(\int\limits_{(0,\frac{1}{n^2})} f_n\,dm + \int\limits_{[\frac{1}{n^2},1]} f_n\,dm \Big) = \lim\limits_{n\to\infty} \Big(0 + \int\limits_{\frac{1}{n^2}}^1 f_n(x)\,dx \Big) = \lim\limits_{n\to\infty} 2(1 - \frac{1}{n}) = 2$.

Example: (Application of DCT) We have $\lim\limits_{n\to\infty} \frac{n\sqrt{x}}{1+n^2x^2} = 0$ for all $x \in [0,1]$, $\frac{n\sqrt{x}}{1+n^2x^2} \leq \frac{1}{2\sqrt{x}}$ for all $x \in (0,1]$ and $\int\limits_{[0,1]} \frac{1}{2\sqrt{x}}\,dx = 1 < +\infty$. Hence by the DCT, we obtain $\lim\limits_{n\to\infty} \int\limits_0^1 \frac{n\sqrt{x}}{1+n^2x^2}\,dx = \lim\limits_{n\to\infty} \int\limits_{[0,1]} \frac{n\sqrt{x}}{1+n^2x^2}\,dx = \int\limits_{[0,1]} \lim\limits_{n\to\infty} \frac{n\sqrt{x}}{1+n^2x^2}\,dx = 0$.