MA 373 : Financial Engineering II

January - May 2021

Department of Mathematics, Indian Institute of Technology Guwahati

Total Marks: 60 Mid-Semester Examination Duration: 4 PM - 5-30 PM

- Answer **all** questions.
- Justify all your answers. Answers without justification carry no marks.

Consider the standard Black-Scholes model. Our underlying risky asset is geometric Brownian motion

$$dS(t) = 0.2\, S(t)dt + 0.4\, S(t)d\tilde{W}(t),\ S(0) = 100,$$

where $\tilde{W}(t), 0 \leq t \leq T$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$.

1. Let $Y(t) = \min_{0\leq u\leq t} S(u)$ and $0 = t_0 < t_1 < \cdots < t_m = T$ be a partition of $[0, T]$. Find

$$\lim_{\|\pi_m\|\to 0} \sum_{j=1}^{m} (Y(t_j) - Y(t_{j-1}))(S(t_j) - S(t_{j-1})),$$

where $\|\pi_m\| = \max_{j=1,2,\cdots,m}(t_j - t_{j-1})$. [5]

2. The stochastic average of stock prices between 0 and t is defined by

$$X(t) = \frac{1}{t}\int_0^t S(u)d\tilde{W}(u)$$

(i) Show that $\sigma X(t) = R(t) - rA(t)$, where $R(t) = \frac{S(t)-S(0)}{t}$ is the raw average of the stock price and

$$A(t) = \frac{1}{t}\int_0^t S(u)du$$

is the continuous arithmetic average.

(ii) Let $I(t) = tA(t)$. Find $d(I(t)S(t))$, $\tilde{\mathbb{E}}[I(t)S(t)]$ and $\tilde{\mathbb{E}}[A(t)]$.

(iii) Suppose at time t we have $S(t) = x \geq 0$ and $\int_0^t S(u)du = y \geq 0$. Find the price at time t of a derivative which pays at maturity $X(T)$.

(iv) Are the processes $I(t)$ and $S(t)$ independent ?

[4+15+6+5]

3. Consider the continuously sampled a derivative security with payoff function

$$V(T) = \frac{1}{T} \int_0^T 2\, S(u)du - 100,$$

but assume now that the interest rate is $r = 0$.
(i) Find an initial capital $X(0)$ and a nonrandom function $\gamma(t), 0 \le t \le T$, which will be the number of shares of risky asset held by our portfolio so that

$$X(T) = \frac{1}{T} \int_0^T 2\, S(u)du - 100$$

still holds.
(ii) Give the formula for the resulting process $X(t), 0 \le t \le T$, in term of underlying asset price and determine the arbitrage free price, at time t, of the derivative security $V(T)$ where $t < T$.

[6+6]

4. Fix the time of maturity T and consider the following derivative security which pays at maturity

$$V(T) = \begin{cases} 100 & \text{if } S(T) \le 50 \\ 150 - S(T) & \text{if } 50 < S(T) < 150 \\ 0 & \text{if } S(T) \ge 150 \end{cases}$$

Determine the arbitrage free price of the derivative security at time $t < T$.

[9]

5. Let $X(t) = 3t + 3W(t)$ and $Y(t) = 4t + 3W(t)$. If $X(t)$ and $Y(t)$ model the prices of two stocks, which one would you like to own?

[4]