Statistical Inference and Multivariate Analysis (MA 324)

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Instructor
Ayon Ganguly
Department of Mathematics
IIT Guwahati

Contents

Chapter 4

Interval Estimation

4.1 Confidence Interval

In this section, we assume that the parameter under consideration is a real valued parameter. We are interested to find an interval in $\Theta \subseteq \mathbb{R}$ such that the interval covers the unknown parameter with a specified probability. Of course, the interval will be based on a RS. Interval estimation is quite useful in practice. For example, one may be interested to find a upper limit of mean of toxic level of some drug or food.

Note that for a RV X and two real constants $a > 0$ and $b > 0$,

$$P\left(a < X < b\right) = P\left(X < b < \frac{bX}{a}\right).$$

Though, these two probabilities are same, there is a basic difference in these two probability statements. For the LHS, we are taking about probability that a random quantity X belongs to a fixed interval $(a,\, b)$. For the RHS, we are taking about probability that a random interval $(X,\, \frac{bX}{a})$ contains a fixed point b. For example, let $X \sim U(0,\, 1)$, $a = 0.5$, and $b = 1$. In this case, $P\left(X < 1 < 2X\right) = P(0.5 < X < 1) = 0.5$.

Definition 4.1. *An interval estimate of a real valued parameter θ is any pair of functions $L(\boldsymbol{x})$ and $U(\boldsymbol{x})$ of random sample only (do not involve any unknown parameters) that satisfy $L(\boldsymbol{x}) \leq U(\boldsymbol{x})$ for all $\boldsymbol{x}$ in the support of the RS. The random interval $[L(\boldsymbol{X}),\, U(\boldsymbol{X})]$ is called an interval estimator of θ.*

Remark 4.1. Though in the definition, the closed interval $[L(\boldsymbol{X}),\, U(\boldsymbol{X})]$ is written, the interval may be closed, open or semi-open based on the problem. †

If $L(\boldsymbol{x}) = -\infty$, then $U(\boldsymbol{x})$ provides an upper limit and $(-\infty,\, U(\boldsymbol{X}))$ is called upper interval estimator. Similarly, if $U(\boldsymbol{x}) = \infty$, then $L(\boldsymbol{x})$ provides a lower limit, and $(L(\boldsymbol{X}),\, \infty)$ is called lower interval estimator.

Example 4.1. Let $X_1,\, \ldots,\, X_n \overset{i.i.d.}{\sim} N(\mu,\, 1)$. Consider $L_1(\boldsymbol{x}) = x_1 - 1$, $U_1(\boldsymbol{x}) = x_1 + 1$, $L_2(\boldsymbol{x}) = \bar{x} - 1$, and $U_2(\boldsymbol{x}) = \bar{x} + 1$. Then both $[L_1(\boldsymbol{X}),\, U_1(\boldsymbol{X})]$ and $[L_2(\boldsymbol{X}),\, U_2(\boldsymbol{X})]$ are interval estimator of μ. Which one should we use? Note that here the lengths of both intervals are same, hence, one should use the interval estimator which has higher probability that the random interval includes μ.

$$P\left(X_1 - 1 \leq \mu \leq X_1 + 1\right) = P\left(-1 \leq X_1 - \mu \leq 1\right) = 2\Phi(1) - 1,$$

$$P\left(\overline{X}-1 \le \mu \le \overline{X}+1\right) = P\left(-\sqrt{n} \le \sqrt{n}\left(\overline{X}-\mu\right) \le \sqrt{n}\right) = 2\Phi(\sqrt{n})-1.$$

Now, as $\Phi(\cdot)$ is an increasing function, we should prefer $[L_2(\boldsymbol{X}) = \overline{X}-1,\, U_2(\boldsymbol{X}) = \overline{X}+1]$ over $[L_1(\boldsymbol{X}) = X_1 - 1,\, U_1(\boldsymbol{X}) = X_1 + 1]$. ||

Remark 4.2. In the previous example, as the length of the intervals are same, we prefer an interval for which the probability that the random interval covers the parameter μ is highest. In other cases, we may have interval estimators that have equal probability of covering the parameter. In such cases, we should prefer an interval which has minimum length. We will not study such optimality issues in this course. †

Definition 4.2 (Coverage Probability)**.** *Coverage probability associated with an interval estimator* $[L(\boldsymbol{X}),\, U(\boldsymbol{X})]$ *for* θ *is measured by*

$$P_\theta\left(L(\boldsymbol{X}) \le \theta \le U(\boldsymbol{X})\right).$$

Definition 4.3 (Confidence Coefficient)**.** *The confidence coefficient associated with an interval estimator* $[L(\boldsymbol{X}),\, U(\boldsymbol{X})]$ *is defined by*

$$\inf_{\theta\in\Theta} P_\theta\left(L(\boldsymbol{X}) \le \theta \le U(\boldsymbol{X})\right).$$

Definition 4.4 (Confidence Interval)**.** *Let* $\alpha \in (0,\, 1)$. *An interval estimator* $[L(\boldsymbol{X}),\, U(\boldsymbol{X})]$ *is said to be a confidence interval (CI) of level* $1-\alpha$ *(or a* $100(1-\alpha)\%$ *confidence interval) if*

$$P_\theta\left(L(\boldsymbol{X}) \le \theta \le U(\boldsymbol{X})\right) \ge 1-\alpha \textit{ for all } \theta\in\Theta.$$

Remark 4.3. Typical values of α are 0.1, 0.05, 0.01. †

Remark 4.4. Clearly, we are loosing precision in interval estimation compared to point estimation. Do we have any gain? Consider the previous example. A reasonable point estimator of μ is $\overline{X}$. However, $P\left(\overline{X}=\mu\right)=0$ as $\overline{X}$ is a CRV. On the other hand,

$$P\left(\overline{X}-1 \le \mu \le \overline{X}+1\right) > 0.$$

Hence, in interval estimation we have some confidence which we gain by loosing precision. †

4.1.1 Interpretation of Confidence Interval

Let us try to interpret a CI. Let $[L(\boldsymbol{X}),\, U(\boldsymbol{X})]$ be an interval estimator of the parameter θ. Once we observe $\boldsymbol{X}=\boldsymbol{x}$, an interval estimate $[L(\boldsymbol{x}),\, U(\boldsymbol{x})]$ is a fixed interval. Also, recall that the parameter θ is an unknown but fixed entity. Therefore, no probability is attached to these observed interval estimate. The interpretation of the phrase "$(1-\alpha)$ confidence" can be discussed as follows. Suppose that the RS is drawn repeatedly. For the first observation $\boldsymbol{X}=\boldsymbol{x}_1$, the interval estimate is $[L(\boldsymbol{x}_1),\, U(\boldsymbol{x}_1)]$. For the second observation $\boldsymbol{X}=\boldsymbol{x}_2$, the interval estimate is $[L(\boldsymbol{x}_2),\, U(\boldsymbol{x}_2)]$, and so on. If we keep on repeating this procedure, we will have interval estimates

$$\left[L(\boldsymbol{x}_1),\, U(\boldsymbol{x}_1)\right],\, \left[L(\boldsymbol{x}_2),\, U(\boldsymbol{x}_2)\right],\, \left[L(\boldsymbol{x}_3),\, U(\boldsymbol{x}_3)\right],\, \left[L(\boldsymbol{x}_4),\, U(\boldsymbol{x}_4)\right],\, \ldots.$$

In a long haul, out of these conceptual interval estimates found, approximately $100(1-\alpha)\%$ would include the unknown value of the parameter θ. This interpretation goes hand in hand with the relative frequency definition of probability.

4.2 Method of Finding CI

There are several ways of construction of CI. In this section, we will discuss the construction of CI based on pivot. The definition of pivot in given below.

Definition 4.5. *A random variable $T = T(\boldsymbol{X}, \theta)$ is called a pivot (or a pivotal quantity) if the distribution of T does not involve any unknown parameters.*

Remark 4.5. Pivot is a function of random sample and unknown parameters, but its' distribution is independent of all unknown parameters. Hence, pivot is not a statistic in general. †

Remark 4.6. In general, we want to find a pivot that is a function of minimal sufficient statistic. †

Example 4.2. Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, 1)$. Then $\overline{X} - \mu$ is a pivot as $\overline{X} - \mu \sim N(0, 1/n)$. ||

Example 4.3. Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$ and μ and σ both are unknown. Then $\overline{X} - \mu$ is not a pivot as $\overline{X} - \mu \sim N(0, \sigma^2/n)$. However, $\frac{\sqrt{n}}{\sigma}\left(\overline{X} - \mu\right) \sim N(0, 1)$ and $\frac{\sqrt{n}}{S}\left(\overline{X} - \mu\right) \sim t_{n-1}$. Therefore, these are pivots. ||

Example 4.4. Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} Exp(\lambda)$. Then $2\lambda \sum_{i=1}^{n} X_i \sim \chi^2_{2n}$ (why?), and hence, is a pivot. ||

Once an appropriate pivot is found, the CI for a parameter θ can be obtained as follows. Let T be a pivot. Find two real numbers a and b such that

$$P_\theta\left(a \leq T(\boldsymbol{X}, \theta) \leq b\right) \geq 1 - \alpha.$$

Note that a and b are independent of all unknown parameters as the distribution of T does not involve any unknown parameter. Let us denote the set

$$C(\boldsymbol{x}) = \left\{\theta \in \Theta : a \leq T(\boldsymbol{x}; \theta) \leq b\right\}.$$

Then, $C(\boldsymbol{X})$ is a $100(1-\alpha)\%$ CI for θ. Note that $C(\boldsymbol{x})$ does not involve any unknown parameters as a and b are independent of all unknown parameters. Also notice that if $T(\boldsymbol{x}; \theta)$ is monotone in $\theta \in \Theta$ for each $\boldsymbol{x}$, then $C(\boldsymbol{x})$ is an interval. Otherwise it could be a general set.

4.2.1 One-sample Problems

Example 4.5. Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ is unknown and $\sigma > 0$ is known. We are interested in μ. A pivot based on minimal sufficient statistics is $\frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} \sim N(0, 1)$. Let z_α be the upper α-point of the standard normal distribution. We can take $a = z_{1-\alpha/2} = -z_{\alpha/2}$ (as $N(0, 1)$ distribution is symmetric about zero) and $b = z_{\alpha/2}$. Now,

$$P\left(-z_{\frac{\alpha}{2}} \leq \frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} \leq z_{\frac{\alpha}{2}}\right) = 1 - \alpha \implies P\left(\overline{X} - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} \leq \mu \leq \overline{X} + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}\right) = 1 - \alpha.$$

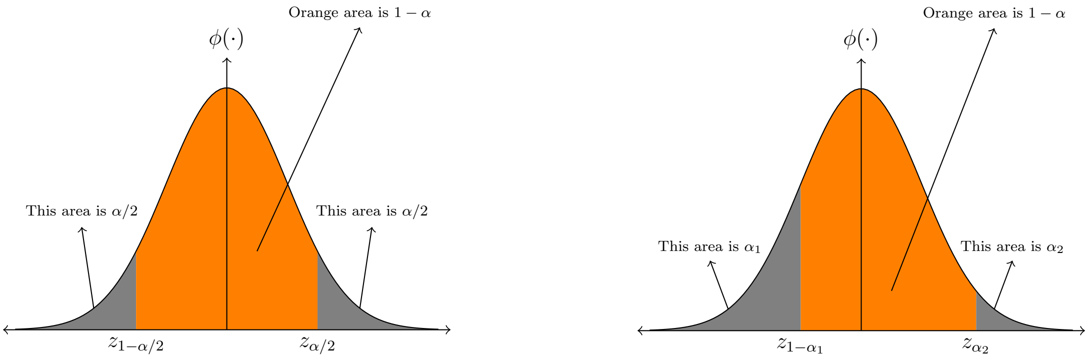

Figure 4.1: Symmetric and asymmetric CIs

Hence, a $100(1-\alpha)\%$ symmetric CI for μ is

$$C(\boldsymbol{X}) = \left[\overline{X} - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}, \overline{X} + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}\right]. \qquad ||$$

Note that the choice of $a = -z_{\frac{\alpha}{2}}$ and $b = z_{\frac{\alpha}{2}}$ corresponds to symmetric CI, as we leave $\frac{\alpha}{2}$ probability on both sides and take the middle part of the probability distribution. Of course, there are infinite number of choices for a and b. For example, let $\alpha_1 > 0$, $\alpha_2 > 0$ are to real numbers such that $\alpha_1 + \alpha_2 = \alpha$. Then, $a = z_{1-\alpha_1}$ and $b = z_{\alpha_2}$ can be considered (see the right panel of Figure 4.1).

Example 4.6. Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ is known and $\sigma > 0$ is unknown. We are interested in CI of σ^2. A pivot based on minimal sufficient statistics is $\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 \sim \chi^2_n$. Let $\chi^2_{n,\alpha}$ be the upper α-point of a χ^2-distribution with degrees of freedom n. We can take $a = \chi^2_{n,1-\alpha/2}$ and $b = \chi^2_{n,\alpha/2}$. Hence, a $100(1-\alpha)\%$ symmetric CI for σ^2 is

$$C(\boldsymbol{X}) = \left[\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{\chi^2_{n,\alpha/2}}, \frac{\sum_{i=1}^{n}(X_i - \mu)^2}{\chi^2_{n,1-\alpha/2}}\right]. \qquad ||$$

Example 4.7. Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are unknown. We are interested in CI of μ. A pivot based in minimal sufficient statistic is $\frac{\sqrt{n}(\overline{X}-\mu)}{S} \sim t_{n-1}$, where $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$. Let $t_{n,\alpha}$ be the upper α-point of a t-distribution with degrees of freedom n. Then, we can take $a = t_{n-1,1-\alpha/2} = -t_{n-1,\alpha/2}$ (as t-distribution is symmetric about zero) and $b = t_{n-1,\alpha/2}$. Hence, a $100(1-\alpha)\%$ symmetric CI for μ is

$$C(\boldsymbol{X}) = \left[\overline{X} - \frac{S}{\sqrt{n}} t_{n-1,\alpha/2}, \overline{X} + \frac{S}{\sqrt{n}} t_{n-1,\alpha/2}\right]. \qquad ||$$

Example 4.8. Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are unknown. We are interested in CI for σ^2. A pivot based on minimal sufficient statistic is $\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \sim \chi^2_{n-1}$. We can take $a = \chi^2_{n-1,1-\alpha/2}$ and $b = \chi^2_{n-1,\alpha/2}$. Hence, a $100(1-\alpha)\%$ symmetric CI for σ^2 is

$$C(\boldsymbol{X}) = \left[\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\chi^2_{n-1,\frac{\alpha}{2}}}, \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}\right]. \qquad ||$$

4.2.2 Two-sample Problems

Example 4.9. Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu_1, \sigma^2)$ and $Y_1, Y_2, \ldots, Y_m \overset{i.i.d.}{\sim} N(\mu_2, \sigma^2)$. Also, assume that X_i's and Y_j's are independent. Here, μ_1, μ_2, and σ are assumed to be unknown and we are interested to construct a $100(1-\alpha)\%$ CI for $\mu_1 - \mu_2$. Let us first try to construct a pivot based on minimal sufficient statistic $\left(\overline{X}, \overline{Y}, S^2\right)$, where the pooled sample variance S^2 is defined by

$$S^2 = \frac{1}{n+m-2}\left[\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 + \sum_{i=1}^{m}\left(Y_i - \overline{Y}\right)^2\right].$$

Now, notice that

$$\overline{X} - \overline{Y} \sim N\left(\mu_1 - \mu_2, \sigma^2\left(\frac{1}{n} + \frac{1}{m}\right)\right) \implies T = \frac{\left(\overline{X} - \overline{Y}\right) - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim N(0, 1).$$

Of course, T is a pivot, but we cannot use it to construct the required confidence interval due to the presence of unknown σ in T. Also, note

$$\frac{(n+m-2)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 + \frac{1}{\sigma^2}\sum_{i=1}^{m}\left(Y_i - \overline{Y}\right)^2 \sim \chi^2_{n+m-2}.$$

Moreover, S^2 and $\left(\overline{X}, \overline{Y}\right)$ are independent. Therefore,

$$T_1 = \frac{\frac{\left(\overline{X} - \overline{Y}\right) - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}}}{\sqrt{\frac{(n+m-2)S^2}{(n+m-2)\sigma^2}}} = \frac{\left(\overline{X} - \overline{Y}\right) - (\mu_1 - \mu_2)}{S\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{n+m-2}.$$

Thus, T_1 is a pivot and we will use it to construct CI for $\mu_1 - \mu_2$.

$$P\left(-t_{n+m-2,\,\frac{\alpha}{2}} \le \frac{\left(\overline{X} - \overline{Y}\right) - (\mu_1 - \mu_2)}{S\sqrt{\frac{1}{n} + \frac{1}{m}}} \le t_{n+m-2,\,\frac{\alpha}{2}}\right) = 1 - \alpha$$

Therefore, the symmetric $100(1-\alpha)\%$ CI for $\mu_1 - \mu_2$ is

$$C(\boldsymbol{X}) = \left[\left(\overline{X} - \overline{Y}\right) - S\sqrt{\frac{1}{n} + \frac{1}{m}}\,t_{n+m-2,\,\frac{\alpha}{2}},\ \left(\overline{X} - \overline{Y}\right) + S\sqrt{\frac{1}{n} + \frac{1}{m}}\,t_{n+m-2,\,\frac{\alpha}{2}}\right] ||$$

Example 4.10. Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu_1, \sigma_1^2)$ and $Y_1, Y_2, \ldots, Y_m \overset{i.i.d.}{\sim} N(\mu_2, \sigma_2^2)$. Also, assume that X_i's and Y_j's are independent. Here, μ_1, μ_2, σ_1, and σ_2 are assumed to be unknown and we are interested to construct a $100(1-\alpha)\%$ CI for $\frac{\sigma_2^2}{\sigma_1^2}$. In this case minimal sufficient statistic is $\left(\overline{X}, \overline{Y}, S_1^2, S_2^2\right)$, where S_1^2 and S_2^2 are sample variances based on the samples X_i's and Y_j's, respectively. In this case,

$$T = \frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2} = \frac{\frac{(n-1)S_1^2}{(n-1)\sigma_1^2}}{\frac{(m-1)S_2^2}{(m-1)\sigma_2^2}} \sim F_{n-1,\,m-1}.$$

Thus,

$$P\left(F_{n-1,\,m-1,\,1-\frac{\alpha}{2}} \leq \frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2} \leq F_{n-1,\,m-1,\,\frac{\alpha}{2}}\right) = 1-\alpha$$

so that a $100(1-\alpha)\%$ CI for $\dfrac{\sigma_2^2}{\sigma_1^2}$ is

$$C(\boldsymbol{X}) = \left[\frac{S_2^2}{S_1^2} F_{n-1,\,m-1,\,1-\frac{\alpha}{2}},\ \frac{S_2^2}{S_1^2} F_{n-1,\,m-1,\,\frac{\alpha}{2}}\right]. \qquad ||$$

4.3 Asymptotic CI

In many cases it is very difficult to find pivot for a small sample. For example, it is difficult to find a pivot to construct CI for successes probability of a Bernoulli distribution. However, we may able to find CI quite easily if the sample size is sufficiently large. This CI is called asymptotic confidence interval. For this purpose, convergence in distribution (mainly CLT or large sample distribution of MLE) and convergence in probability (consistent estimator) are handy tools.

4.3.1 Distribution Free Population Mean

Let $X_1,\, X_2,\, \ldots$ be i.i.d. random variables with mean μ and finite variance σ^2. Then, using CLT

$$\frac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{\sigma} \xrightarrow{\mathscr{L}} Z \sim N(0,\,1).$$

Thus, if we have a RS with large sample size n, we can approximate the distribution of $\frac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{\sigma}$ using a standard normal distribution. Hence,

$$P\left(-z_{\alpha/2} \leq \frac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{\sigma} \leq z_{\alpha/2}\right) \approx 1-\alpha.$$

If σ is known and n is sufficiently large, we can use the last statement to find an asymptotic CI for μ and it is given by

$$\left[\overline{X}_n - \frac{\sigma}{\sqrt{n}} z_{\alpha/2},\, \overline{X}_n + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right].$$

If σ is unknown, we can proceed as follows. Using WLLN, we have $\frac{S_n}{\sigma} \xrightarrow{P} 1$, and hence,

$$\frac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{S_n} \xrightarrow{\mathscr{L}} Z \sim N(0,\,1).$$

Hence, $P\left(-z_{\alpha/2} \leq \frac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{S_n} \leq z_{\alpha/2}\right) \approx 1-\alpha$. An asymptotic CI for μ is given by

$$\left[\overline{X}_n - \frac{S_n}{\sqrt{n}} z_{\alpha/2},\, \overline{X}_n + \frac{S_n}{\sqrt{n}} z_{\alpha/2}\right].$$

Note that this method can be used for any distribution of $X_1,\, X_2,\, \ldots,\, X_n$, as long as the conditions of CLT hold true. Therefore, it is called distribution free.

4.3.2 Using MLE

Let $\widehat{\theta}_n$ be a consistent estimator of θ and $\sqrt{n}\left(\widehat{\theta}_n - \theta\right) \xrightarrow{\mathscr{L}} N(0,\, b^2(\theta))$, where $b(\theta) > 0$ for all $\theta \in \Theta$. Assume that $b(\cdot)$ is a continuous function. Then, $\frac{b(\widehat{\theta}_n)}{b(\theta)} \xrightarrow{P} 1$, and hence, $\frac{\sqrt{n}\left(\widehat{\theta}_n - \theta\right)}{b(\widehat{\theta}_n)} \xrightarrow{\mathscr{L}} N(0,\, 1)$. A $100(1-\alpha)\%$ asymptotic CI for θ is given by

$$\left[\widehat{\theta}_n - \frac{b(\widehat{\theta}_n)}{\sqrt{n}} z_{\alpha/2},\, \widehat{\theta}_n + \frac{b(\widehat{\theta}_n)}{\sqrt{n}} z_{\alpha/2}\right].$$

Under some regularity conditions, we may use MLE of θ in place of $\widehat{\theta}_n$.

Example 4.11. Let $X_1,\, X_2,\, \ldots,\, X_n \overset{i.i.d.}{\sim} Bernaoulli(p)$, where $p \in (0,\, 1)$. We are interested to construct asymptotic CI for p. We know that $\widehat{p}_n = \overline{X}_n \xrightarrow{P} p$ and $\frac{\sqrt{n}\left(\overline{X}_n - p\right)}{\sqrt{p(1-p)}} \xrightarrow{\mathscr{L}} N(0,\, 1)$. Here, $b(p) = \sqrt{p(1-p)}$, which is a continuous function in $p \in (0,\, 1)$. Hence, $\frac{\sqrt{n}\left(\overline{X}_n - p\right)}{\sqrt{\overline{X}_n(1-\overline{X}_n)}} \xrightarrow{\mathscr{L}} N(0,\, 1)$. A $100(1-\alpha)\%$ asymptotic CI for p is

$$\left[\overline{X}_n - \sqrt{\frac{\overline{X}_n(1-\overline{X}_n)}{n}} z_{\alpha/2},\, \overline{X}_n + \sqrt{\frac{\overline{X}_n(1-\overline{X}_n)}{n}} z_{\alpha/2}\right]. \qquad ||$$