Problem Set - 1

1. $f(x;p) = \begin{cases} p(1-p)^x & , x = 0, 1, 2, \dots \\ 0 & \text{other} \end{cases}$

Claim : $T = X_1 + X_2 + \dots + X_n$ is sufficient

Proof : $\mathbb{P}(X = x_1, \dots, X_n = x_n \mid T = t)$

$$= \begin{cases} \dfrac{\mathbb{P}(X = x_1, x, \dots, X_n = x_n)}{\mathbb{P}(T = t)} & , \sum_{i=1}^{n} x_i = t \\ & , \text{ow} \end{cases}$$

$$= \frac{\prod_{i=1}^{n} p(1-p)^{x_i}}{}$$

$\Downarrow$

But can't proceed easily (see negative binom dist)

M-2 : Using Neyman-Fisher Factorization

$\therefore f(x;p) = p(1-p)^x \quad , x = 0, 1, 2, \dots$

$$\therefore f_X(\vec{x};p) = \prod_{i=1}^{n} p(1-p)^{x_i}$$
$$= p^n (1-p)^{n\bar{x}}$$

$h(x) = 1$

$g_p(T(x)) = g_p(t) = p^n(1-p)^{nt}$

$\therefore T(x) = \bar{x}$ is sufficient.

2. $\mathbb{P}(X_1 = x_1, \dots X_n = x_n, Y_1 = y_1, \dots Y_n = y_n \mid T = t)$

$$= \frac{\mathbb{P}(X_1 = x_1, \dots Y_n = y_n, T = t)}{\mathbb{P}(T = t)}$$

$$= \begin{cases} \dfrac{\mathbb{P}(X_1 = x_1, \dots Y_n = y_n)}{\mathbb{P}(T = t)} & , \Sigma \\ 0 & , \end{cases}$$

M-2 : Note

$\Rightarrow$

Let, $T =$

where,

Now, we'll

$\therefore T \sim$ Poi

$\sim$ Poi

$\therefore f_X(x;$

3. Claim : $T =$

Here,

$\therefore h(x, y,$

But $f(x;\theta$

$\therefore h(x, y, \theta$

Now $T = X$

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M-2 : Note, if $X_i \sim Poi(\lambda_i)$ iid

$\Rightarrow \sum_{i=1}^{n} X_i \sim Poi(\sum_{i=1}^{n} \lambda_i)$

Let $T = T_1 + T_2$

where $T_1 = \sum_{i=1}^{m} X_i \sim Poi(\sum_{i=1}^{m} \lambda_i)$

$T_2 = \sum_{i=1}^{n} Y_i \sim Poi(\sum_{i=1}^{n} \lambda_i)$

Now we'll find distribution of $X+Y$

$\therefore T \sim Poi(\sum_{i=1}^{m} \lambda_i + 2\sum_{i=1}^{n} \lambda_i)$

$\sim Poi(\lambda')$

$\therefore f_X(x; p) = \prod_{i=1}^{m+n} \frac{e^{-\lambda'} (\lambda')^{x_i}}{(x_i)!}$

$= \frac{e^{-n\lambda'} (\lambda')^{n\bar{x}}}{\prod_{i=1}^{m+n} (x_i!)}$

3. Claim : $T = \sum_{i=1}^{n} X_i$ is minimal sufficient

Here, $n=1$. $T = X$ is min sufficient.

$\therefore h(x, y, \theta) = \cancel{\dots}\ \frac{f(x; \theta)}{f(y; \theta)}$

But $f(x; \theta) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{x-\mu}{2\sigma^2}\}$

$= \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x-\theta}{2}\}$

$\therefore h(x, y, \theta) = \frac{\exp\{-(x-\theta)\}}{\exp\{-(y-\theta)\}} = \exp\{y-x\}$

$T = X$ is minimally sufficient. (But it is actually trivial LOL)

Now, disprove $|X|$ is sufficient using uniqueness

4. First find minimal sufficient statistic

$T = \sum_{i=1}^{4} X_i$

Let U be the sufficient statistic

$\therefore$ T is a function of U.

i.e. U uniquely determines T.

$\therefore$ Let's see 2 cases:-

(i) $X_1 = X_2 = 0,\ X_3 = X_4 = 1 \Rightarrow U = 0$

(ii) $X_1 = X_2 = X_3 = X_4 = 0 \Rightarrow U = 0$

But $T = 2$ in (i) & $T = 0$ in (ii) $\Rightarrow \Leftarrow$

5. $$f(x;\sigma,\mu) = \begin{cases} \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}}, & x > \mu \\ 0, & \text{o.w.} \end{cases}$$

$\mu \in \mathbb{R},\ \sigma > 0$

(a) <u>To show</u>: $X_{(1)} = \min\{X_1, \dots, X_n\}$ is minimally sufficient for μ if σ is known.

<u>Pf</u>: $$f(x,\mu) = \begin{cases} e^{-(x-\mu)}, & x > \mu \\ 0 & \text{o.w.} \end{cases}$$

Let $T(X_1, \dots, X_n) = X_{(1)}$

$F_T(t) = P(T \leq t)$

$= P(X_1 \leq t, \dots, X_n \leq t)$

$= \prod_{i=1}^{n} P(X_i \leq t)$

$\cancel{= \prod_{i=1}^{n} (1 - e^{-t^2/2})}$

$=$

$\Rightarrow 1 - F_T(t) = \prod_{i=1}^{n} P(X > t)$

$= \prod_{i=1}^{n} e^{-(t-\theta)}$

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Joint pdf

(c) 1-1 mapp.

$\therefore 1 - F_T(t) = e^{-n(t-\mu)}, \quad t \geq \mu$

$\therefore f_T(t, \mu) = n e^{-n(t-\mu)}$

$= \cancel{n e^{n\mu} e^{-nt}} \quad \underbrace{n e^{-nt}}_{h} \; \underbrace{e^{n\mu}}_{g}$

$\therefore$ Using Fisher-Neyman Th, we prove.

(b) $T = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)$ is minimal sufficient for σ if μ is known.

$\Rightarrow f(\vec{x} \mid \mu, \sigma) = \frac{1}{\sigma^n} e^{-\frac{n}{\sigma}(\bar{x} - \mu)} I_{(\mu, \infty)}(x_{(1)})$

{ This one will alone handle all the sub-cases (a), (b) & (c) }

(c) $f(x,\theta) = a(\theta) g(x) \exp\left\{ \sum_{j=1}^{k} b_j(\theta) R_j(x) \right\}$

Denote $T_j = \sum_{i=1}^{n} R_j(X_i)$

Then $T = (T_1, T_2, \ldots, T_k)$ is minimal suff.

$\Rightarrow$ $f(x,\mu,\sigma) = \begin{cases} \frac{1}{\sigma} \exp\left\{ \frac{-(x-\mu)}{\sigma} \right\} & , x > \mu \\ 0 & , \text{o.w.} \end{cases}$

$\therefore$ $a(\sigma) = \frac{1}{\sigma}$, $g(x) = 1$

$b_j(\sigma) = b_1(\mu,\sigma) = \frac{\mu}{\sigma}$ $R_1(x) = \frac{-x}{\sigma}$

$\therefore$ T.X= Not applicable

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6. $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\theta, \theta^2)$, $\theta > 0$

$f(x, \theta, \theta^2) = \frac{1}{\theta\sqrt{2\pi}} \exp\left\{-\frac{(x-\theta)^2}{2\theta^2}\right\}$

$\therefore h(x, y, \theta) = \exp\left\{\frac{(y-\theta)^2}{2\theta^2} - \frac{(x-\theta)^2}{2\theta^2}\right\}$

it is independent of θ

iff. $(y-\theta)^2 = (x-\theta)^2$

$\Rightarrow \sum_{i=1}^{n}(y-\theta)^2 = \sum_{i=1}^{n}(x-\theta)^2$

Using ideas of Example 2.14

$T = (\sum_{i=1}^{n} X_i^2, \sum_{i=1}^{n} X_i)$

7. $h(x, y, \theta) = \exp\left[-\frac{1}{2\theta}\left\{\left(\sum_{i=1}^{n} x^2/\theta - \sum_{i=1}^{n} y_i^2\right) - 2\theta \ldots\right\}\right]$

$\therefore T = \sum_{i=1}^{n} X_i^2$

8. $X_1, X_2, \ldots, X_n \overset{iid}{\sim} U(-\theta, \theta)$, $\theta > 0$

$\therefore f(x, \theta) = \frac{1}{(2\theta)^n} I_{(-\theta,\theta)}(x_{(1)}) I_{(-\theta,\theta)}(x_{(n)})$

$\therefore h(x, y, \theta) = \frac{I_{(-\theta,\theta)}(x_{(1)})}{I_{(-\theta,\theta)}(y_{(1)})} \cdot \frac{I_{(-\theta,\theta)}(x_{(n)})}{I_{(-\theta,\theta)}(y_{(n)})}$

Let's redefine density,

$f(x, \theta) = \frac{1}{(2\theta)^n} I[\max\{-X_{(1)}, X_{(n)}\} < \theta]$

$\therefore T = \max\{-X_{(1)}, X_{(n)}\}$ is minimal sufficient.

$\begin{cases} \frac{1}{(2\theta)^n}, & -\theta < x_1, x_2, \ldots x_n < \theta \\ 0, & o.w. \end{cases}$

also can directly be written