Name: Naman Goyal
Roll No: 180123029

Sol^n (1)

(i) $dr(t) = 4\,dt - 2r(t)\,dt + 3\,dW(t)$
using $r(0) = 2$

In Vasicek model, we have:

$$dr(t) = (b - a\,r)\,dt + \sigma\,dW(t). \qquad a > 0$$
$$b = 4,\ a = 2,\ \sigma = 3.$$

↳ Multiply both sides by $e^{2t}$.

$$e^{2t}dr(t) + r(t)2e^{2t}dt = 4e^{2t}dt + 3e^{2t}dW(t)$$
$$\Rightarrow d(e^{2t}r(t)) = 4e^{2t}dt + 3e^{2t}dW(t).$$

$$e^{2t}r(t) = r(0) + \int_0^t 4e^{2s}ds + 3\int_0^t e^{2s}dW(s)$$
$\therefore r(0) = 2$
$$\Rightarrow r(t) = 2e^{-2t} + \underbrace{4e^{-2t}\int_0^t e^{2s}ds}_{II} + \underbrace{3e^{-2t}\int_0^t e^{2s}dW(s)}_{I}.$$

$$\Rightarrow r(t) = 2 + 3e^{-2t}\int_0^t e^{+2s}dW(s)$$

II const value (deterministic val)

and I is a Ito integral. ⇒ Gaussian process.
Thus $r(t)$ is a gaussian, I has mean 0.

$$E(r(t)) = 2e^{-2t} + \frac{4}{2}(1 - e^{-2t}) = 2.$$

$$Var(r(t)) = \sigma^2 e^{-2(2)t}\int_0^t e^{4s}ds = \frac{9}{2(2)}(1 - e^{-4t}).$$

(ii)

$\therefore \quad r(t) = 2 + 3e^{-2t}\int_0^t e^{2s}\,dW(s)$

Here $\Rightarrow$

$$\int_0^t 3e^{-2(t-s)}\,dW(s) \sim N(0, \text{Var } V)$$

where $\text{Var } V = \frac{9}{4}(1-e^{-4t})$

(calculated previously)

$$r(t) \sim N\left(2, \frac{9}{4}(1-e^{-4t})\right)$$

as $t \to \infty \Rightarrow$ ~~r(t) ~~ the distribut^n tends to

$$\sim N(2, 9/4).$$

(iii) when $r(0) \sim N(2, 3/2)$

since $r(0)e^{-2t}$ will contain

~~$E(r(t))$~~ = same mean as that of previous.

now $E(r(t)) = 2e^{-2t} + 2(1-e^{-4t})$

$= 2.$

$$\text{Var}(r(t)) = \frac{9}{4}e^{-4t} + \frac{9}{4}(1-e^{-4t})$$

because $\text{Var}(r(0)) = 9/4\, e^{-4t}$

because $r(0)$ has a standard deviation of $3/2$.

$\text{Var}(r(t)) = 9/4.$

Hence in limiting case dist. $\sim N(2, 9/4)$

(iv) $x(0) = 2$.

$$x(t) = 2 + 3e^{-2t}\int_0^t e^{2s}\,dw(s)$$

$$x(t) \sim N\left(2, \frac{9}{4}(1-e^{-4t})\right)$$

$$P(x(t)<0) = P\left(\frac{x(t)-2}{3/2\,(1-e^{-4t})^{1/2}} < \frac{-2}{3/2\,(1-e^{-4t})^{1/2}}\right)$$

$$= P\left(Z < -4/3\,(1-e^{-4t})^{-1/2}\right)$$

$$= \phi\left(-4/3\,(1-e^{-4t})^{-1/2}\right)$$

$$= 1-\phi\left(4/3\,(1-e^{-4t})^{-1/2}\right)$$

$$\Rightarrow \lim_{t\to\infty} P(x(t)<0) = 1-\phi(4/3)$$

Scanned by CamScanner

Sol$^n$ 2 : $dS(t) = r S(t)\,dt + \sigma S(t)\,d\tilde{W}(t)$, $S(0) = 5$

given $b > 0$, $b < 5$.

$\therefore S(t) = S(0)\, e^{(r - \delta - \sigma^2/2)t + \sigma \tilde{W}(t)}$. dividend paying stock.

where $\mu = r - \delta - \sigma^2/2$ = Drift parameter.

$\sigma' = \sigma$ (Volatility param).

Here the barrier $b$ is $< S(0)$

Now hitting time ($\tau$):

$$E[e^{-\lambda\tau}] = e^{x/\sigma^2\left(\mu + \sqrt{2\lambda\sigma^2 + \mu}\right)}$$

where $x = \log(b/S(0))$, $\mu = r - \delta - \sigma^2/2$.

$\sigma' = \sigma$.

so the discounted price using solving above expected value:-

$$\Rightarrow g(b) = (K - b)\left(5/b\right)^{h_2}.$$

where $h_2 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$.

(ii) $g(b) = (k-b)\left(\frac{5}{b}\right)^{h_2}$

$$g'(b) = -(5/b)^{h_2} + \frac{(k-b)\, 5^{h_2}(-h_2)}{b^{h_2+1}}$$

$\Rightarrow \quad g'(b) = 0$

$$-\left(\frac{5}{b}\right)^{h_2}\left[1 + \frac{h_2(k-b)}{b}\right) = 0$$

$$\Rightarrow \frac{k}{b} = 1 - \frac{1}{h_2}$$

$$\left(b^* = \frac{k h_2}{h_2 - 1}\right) \longrightarrow 1$$

using $g''(b) \Rightarrow \left(-5/b\right)^{h_2}\left(1 + \frac{h_2}{b}(k-b)\right)$

$\therefore \; h_2 < 0$

Hence $g'(b) < 0 \;\forall$ higher value of b.

$g'(b) > 0 \;\forall$ lower value of b.

$$\boxed{b^* = \frac{k h_2}{h_2 - 1}}$$ achieves maximum.

(ii) $g(b^*) = (K - b^*)\left(S/b\right)^{h_2}$. Putting value of $b^*$

$$g(b^*) = \frac{K}{1-h_2}\left(S/b\right)^{h_2}.$$

Since, this is the max value of discounted payoff, $g(b^*)$ is the price of the perpetual put.