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Solⁿ (1): using Cauchy's inequality & monotonicity of Y:-

$$\left|\sum_{j=1}^{m}(Y(t_j)-Y(t_{j-1}))(S(t_j)-S(t_{j-1}))\right| \le$$

$$\left|\sum_{j=1}^{m}|Y(t_j)-Y(t_{j-1})||S(t_j)-S(t_{j-1})|\right| \le$$

$$\sqrt{\max|Y(t_j)-Y(t_{j-1})|(Y_0-Y_t)}\sqrt{\Sigma(S(t_j)-S(t_{j-1}))^2}$$

as no of partitions $\longrightarrow \infty$.
or length $\to 0$ or $\|\pi_m\| \to 0$.

then $\sqrt{\sum_{j=1}^{m}(S(t_j)-S(t_{j-1}))^2} \longrightarrow \sqrt{S(T)-S(0)}$.

and $\max_{1\le j\le m}|Y(t_j)-Y(t_{j-1})| \to 0$ as Y is continous fx.

Hence, $\sum_{j=1}^{m}(Y(t_j)-Y(t_{j-1}))(S(t_j)-S(t_{j-1})) \to 0$

as $\|\pi_m\| \to 0$.

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Sol^n 2 (i) To st. $\sigma X(t) = R(t) - \alpha A(t)$.

given $R(t) = \frac{S(t) - S(0)}{t}$.

$A(t) = \frac{1}{t}\int_0^t S(u)\,du$.

making $R(t)$ of the same form as $A(t)$.

$$R(t) = \frac{1}{t}\int_0^t dS(u) = \frac{1}{t}\int_0^t (\alpha S(u)du + \sigma S(u)dW(u))$$

$$= \frac{1}{t}\int_0^t (\alpha S(u)du + \sigma S(u)dW(u))$$

$R(t) = \alpha A(t) + \sigma X(t)$.

$\Rightarrow \sigma X(t) = R(t) - \alpha A(t)$.

(iii) $\because e^{-ru} S(u) \rightarrow$ is a martingale

$$V(t) = e^{-r(T-t)}\tilde{E}\left[\frac{1}{T}\int_0^T S(u)du \,\Big|\, F_t\right]$$

$$= \frac{e^{-r(T-t)}}{T}\tilde{E}\left[\left(\int_0^t S(u)du + \int_t^T S(u)du\right)\Big| F_t\right]$$

$$V(t) = \frac{e^{-r(T-t)}}{T}\left[\int_0^t S(u)du + \int_t^T \tilde{E}(S(u))du\right]$$

for $t \le u \le T$:

$$\tilde{E}(S(u)) = \tilde{E}\left(\tilde{E}(S(u)|F_t)\right]$$

$$= \tilde{E}(e^{ru} e^{-rt} S(t))$$

$$= \tilde{E}\left[S(t)\, e^{-r(t-u)}\right]$$

$$\tilde{E}(S(u)) = S(t)\, e^{-r(t-u)}$$

$$= \frac{S(t)}{\lambda}\left(e^{r(T-t)} - 1\right).$$

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Sol^n 3 : (i) $V(T) = \frac{1}{T}\int_0^T 2S(u)\,du - 100$

Consider $\hat{V}(T) = \frac{1}{T}\int_0^T 2S(u)\,du$.

Then, $\hat{v}(T) = \tilde{E}\left[\left(e^{-r(T-t)}\hat{V}(t)\right)\Big| F_t\right]$

$$= \frac{2e^{-r(T-t)}}{T}\tilde{E}\left[\int_0^T S(u)\,du \Big| F_t\right] \longrightarrow (1)$$

Now, $\tilde{E}\left[\int_0^T S(u)du \Big| F_t\right] = \int_0^t S(u)\,du + \int_t^T \tilde{E}(S(u))\,du$.

$\therefore$ as we know that discounted stock price is martingale.

$$\Rightarrow \tilde{E}\left[S(u)\big| F_t\right] = S(t)e^{-r(t-u)}$$

$$\Rightarrow \tilde{E}\left[\int_0^T S(u)\,du \Big| F_t\right] = \int_0^t S(u)\,du + S(t)\int_t^T e^{-r(t-u)}du.$$

$$= \int_0^t S(u)\,du + \frac{S(t)}{r}e^{-rt}\left[e^{rT} - e^{rt}\right]$$

$$= \int_0^t S(u)\,du + \frac{S(t)}{r}\left[e^{r(T-t)} - 1\right] \longrightarrow (2)$$

from (1) && (2):

$$\hat{V}(t) = \frac{2e^{-r(T-t)}}{T}\left[\int_0^t S(u)du + \frac{S(t)}{r}\left(e^{r(T-t)} - 1\right)\right]$$

given $x = S(t)$, $y = \int_0^t S(u)\,du$.

$$v(t,x,y) = \frac{2e^{-r(T-t)}}{T} y + x \frac{2}{rT}\left(1-e^{-r(T-t)}\right)$$

$$\gamma(t) = \frac{2}{rT}\left(1-e^{-r(T-t)}\right)$$

Now,

$$\lim_{r\to 0} \gamma(t) = \lim_{r\to 0} \frac{2}{rT}\left(1-e^{-r(T-t)}\right) = 2\left(1-t/T\right)$$

Now $X(0) = 2\left(1-0/T\right) S(0) - 100$

$$= \left(2S(0) - 100\right).$$

$X(0) \to$ initial capital.

and hence $X(T) = \frac{1}{T}\int_0^T 2\,S(u)\,du - 100$.

(ii) for $r>0$,

$$X_t = 2\left(\left[\frac{1}{rT}\left(1-e^{-r(T-t)}\right)\right] S_t + \frac{e^{-r(T-t)}}{T}\int_0^t S(u)\,du\right)$$

$$- e^{-r(T-t)} \cdot K.$$

$$\Rightarrow \lim_{r\to 0}(X_t) = \frac{2S_t}{t} \lim_{r\to 0} \frac{\left(1-e^{-r(T-t)}\right)}{r} + 2\int_0^t \frac{S_u du}{T} \lim_{r\to 0} e^{-r(T-t)}.$$

$$- \lim_{r\to 0} 2k e^{-r(T-t)}$$

$$= \frac{2S(t)}{t}\left(r(T-t)\right) + \frac{2}{T}\int_0^t s(u)\,du - 2k.$$

$$= \frac{2S(t)}{t} r(T-t) + \frac{1}{T}\int_0^t 2S(u)\,du - 100.$$

$$\therefore X_T = \frac{1}{T}\int_0^T 2S(u)\,du.$$

$\therefore$ Discounted price is X_t.

Sol^n (4): $V(T) = \begin{cases} 100 & S(T) \le 50 \\ 150 - S(T) & 50 < S(T) \le 150 \\ 0 & S(T) \ge 150 \end{cases}$

$X(T) = (150 - S(T))^+ + (50 - S(T))^+$

$\Rightarrow X(T) = \begin{cases} 100 & S(T) \le 50 \\ 150 - S(T) & 50 < S(T) < 150 \\ 0 & S(T) \ge 150 \end{cases}$

So, $X(T) = V(T)$

$X(T) = \underbrace{(150 - S(T))^+}_{\text{European put off } (K=150)} + \underbrace{(50 - S(T))^+}_{\text{European put of } (K=50)}$

for $V_1(t) = 150 e^{-r(T-t)} N(-d_2) - S(t) N(-d_1)$
$d_1, d_2 \rightarrow V_1(t)$.

$V_2(t) = 50 e^{-r(T-t)} N(-d_2') - S(t) N(-d_1')$
where $d_1' d_2'$ are for $V_2(t)$

since $V(t) = V_1(t) - V_2(t)$ [using superimposition principle].

$V(t) = 150 e^{-r(T-t)} N(-d_2) - 50 e^{-r(T-t)} N(-d_2') + S(t) N(-d_1') - S(t) N(-d_1)$.

where $d_1, d_2, d_1', d_2' \longrightarrow$ ~~repe~~ repe respectively.

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(S(t)/150\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right)$$

$$d_2 = \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(S(t)/150\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)$$

$$d_1' = \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(S(t)/50\right) + \left(r + \sigma^2/2\right)(T-t)\right)$$

$$d_2' = \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(S(t)/50\right) + \left(r - \sigma^2/2\right)(T-t)\right).$$

where $\sigma = 0.4$

$r = 0.08 + 0.2 = 0.28$

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Sol^n (5):

$$X(t) = 3t + 3W(t)$$
$$Y(t) = 4t + 3W(t)$$

$E(X(t)) = 3t$ $\quad$ $Var(X(t)) = 9t$
$E(Y(t)) = 4t$ $\quad$ $Var(Y(t)) = 9t$.

Since, the risk is same both for X and Y, expected returns matter.
That's why Y is preferable over X.