Name: Naman Goyal Roll No: 180123029

Sol^n 1

q is a langrange poly of deg n for $\{(x_i, y_i)$ $i = 0 \ldots n\}$.

r is a langrange poly of deg n for $\{(x_i, y_i)$ $i = 0, 1 \ldots n\}$.

$$p(x) = \frac{(x - x_0) r(x) - (x - x_{n+1}) q(x)}{x_{n+1} - x_0}.$$

$q(x) = \sum_{j=0}^{n} f_j \, l_j(x)$ where $l_j(x) = \prod_{\substack{i \neq j \\ i = 0 \ldots n}} \frac{(x - x_i)}{(x_j - x_i)}$

$$\therefore q = f_0 \frac{(x - x_1) \ldots (x - x_n)}{(x_0 - x_1) \ldots (x_0 - x_n)} + f_1 \frac{(x - x_0) \ldots (x - x_n)}{(x_1 - x_0) \ldots (x_1 - x_n)}$$

$$\ldots + f_n \frac{(x - x_0) \ldots (x - x_{n-1})}{(x_n - x_0) \ldots (x_n - x_{n-1})}$$

$$\Rightarrow \frac{(x - x_{n+1})}{(x_{n+1} - x_0)} q = f_0 \frac{(x - x_1) \ldots (x - x_{n+1})}{(x_0 - x_1) \ldots (x_{n+1} - x_0)} + \ldots$$

Similarly; $$\frac{(x - x_0) r(x)}{(x_{n+1} - x_0)} = f_1 \frac{(x - x_0) \ldots (x - x_{n+1})}{(x_1 - x_2) \ldots (x_{n+1} - x_0)} + \ldots$$

$$\ldots + f_{n+1} \frac{(x - x_0) \ldots (x - x_{n+1})}{(x_{n+1} - x_1) \ldots (x_{n+1} - x_0)}$$

$$\therefore \text{In } p(x) \Rightarrow f_i \frac{(x - x_0) \ldots (x - x_{n+1})}{(x_i - x_1) \ldots (x_i - x_n)} \left[\frac{1}{x_i - x_{n+1}} - \frac{1}{x_i - x_0} \right] \times \frac{1}{(x_{n+1} - x_0)}$$

$\forall i \neq 0, n+1$

$$p(x) = \sum \frac{f_i (x-x_0) \cdots (x-x_{n+1})}{(x_i - x_0) \cdots (x_i - x_{n+1})}$$

$\forall\ i \neq 0$ and $i \neq n+1$.

which follows langrange form,
also for $i = 0$, $i = n+1$ the terms follows format of langrange polynomial.

Now using Taylor's series Expansion:

$$f(h) = f(0+h) = f(0) + h f'(0) + \frac{h^2}{\lfloor 2} f''(0) + R_n \rightarrow (1)$$

where $R_n = \frac{h^3}{\lfloor 3} f'''(\varepsilon)$ since $f \in C^3[-h, h]$
for some $\varepsilon \in (-h, h)$

also $f(-h) = f(0-h)$
$$= f(0) - h f'(0) + \frac{h^2}{\lfloor 2} f''(0) - \frac{h^3}{6} f'''(\varepsilon) \rightarrow (2)$$

Now (1) − (2):

$$f(h) - f(-h) = 2\left[h f'(0) + \frac{h^3}{6} f'''(\varepsilon)\right]$$

$$\frac{f(h) - f(-h)}{2h} = f'(0) + \frac{h^2}{6} f'''(\varepsilon) \rightarrow (3)$$

$\therefore \ E(h) = \dfrac{f(h) + \varepsilon_+ - (f(-h) + \varepsilon_-)}{2h} - f'(0)$

$= \dfrac{f(h) - f(-h)}{2h} - f'(0) + \dfrac{(e_+ - e_-)}{2h}$.

$\Rightarrow E(h) = \dfrac{h^2}{6} f'''(\xi) + \dfrac{(e_+ - e_-)}{2h}$. for $\xi \in (-h, h)$ $\rightarrow$ (4)

Hence, proved.

using (4) :

$|E(h)| \leq \dfrac{h^2}{6} |f'''(\xi)| + \cancel{\left|\dfrac{e_+ - e_-}{2h}\right|} \left|\dfrac{e_+ - e_-}{2h}\right|$

$\leq \dfrac{h^2}{6} M + \dfrac{|e_+| + |e_-|}{2h}$. (using Triangular Inequality)

where $M = \max\limits_{\xi \in (-h,h)} |f'''(\xi)|$ $(\because f \in C^3[-h, h])$

Let $E = \max\{|e_+| ; |e_-|\}$

$|E(h)| \leq h^2/_6 M + \dfrac{2E}{2h}$

$|E(h)| \leq h^2 M/_6 + E/h$.

Let call $E = e$.

$\boxed{|E(h)| \leq h^2 M/_6 + e/h.}$ Hence, proved.

let $g(h) = \frac{h^2}{6} M + e/h$.

$g'(h) = \frac{2h}{6} M - e/h^2$

If $g'(h) = 0 \Rightarrow h^3 = \frac{3e}{M} \Rightarrow \boxed{h^* = \left(\frac{3e}{M}\right)^{1/3}}$

Also $g''(h) = M/3 + 2e/h^3$.

$g''(h^*) = M/3 + \frac{2eM}{3e} = M$.

$\therefore$ $M > 0$ and $g''(h^*) > 0$, $g'(h^*) = 0$

$\cancel{g'(h^*) < 0}$

Hence $g(h)$ attains it's min. value at $\boxed{h^* = \left(\frac{3e}{M}\right)^{1/3}}$.

$$\text{Min}(g(h)) = M/6 \times \frac{(3e)^{2/3}}{M^{2/3}} + \frac{e M^{1/3}}{(3e)^{1/3}}.$$

at $h = (3e/M)^{1/3}$.

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Sol^n (2) : given

$Q(f) = C_{-1} f(-1) + C_0 f(0) + C_1 f(1) + C_2 f(2)$

$\therefore \quad Q(f) = \int_0^1 f(x)\,dx$ where $f \to$ polynomial of degree 3.

Consider $P_1(x) = (x-1)x(x+1)$

we have $Q(P_1) = \int_0^1 P_1(x)\,dx = \int_0^1 (x^2-1)x\,dx = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

$\Rightarrow C_{-1} P_1(-1) + C_0 P_1(0) + C_1 P_1(1) + C_2 P_1(2) = -\frac{1}{4}$.

$= \cancel{6 \times P_1(2)} \quad 6C_2 = -\frac{1}{4} \Rightarrow \boxed{C_2 = -\frac{1}{24}}$.

Now consider $P_2(x) = x(x-1)(x-2)$

$Q(P_2) = \int_0^1 P_2(x)\,dx = \int_0^1 (x^3 - 3x^2 + 2x)\,dx = \frac{1}{4}$.

$= (-1)(-2)\times(-3)\times C_{-1} = \frac{1}{4} \Rightarrow \boxed{C_{-1} = -\frac{1}{24}}$

Similarly consider $P_3(x) = (x-2)x(x+1)$

$Q(P_3) = \int_0^1 P_3(x)\,dx = \int_0^1 (x^3 - x^2 - 2x)\,dx = -\frac{13}{12}$.

$\Rightarrow \quad 2(1)(-1)C_1 = \frac{-13}{12} \Rightarrow \boxed{C_1 = \frac{13}{24}}$

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Consider $P_4(x) = (x-2)(x-1)(x+1)$

$$Q(P_4) = \int_0^1 P_4(x)\,dx = \int_0^1 (x^3 - 2x^2 - x + 2)\,dx = \frac{13}{12}$$

$$\Rightarrow c_0(1)(-1)(-2) = 13/12$$

$$\Rightarrow \boxed{c_0 = 13/24}$$

(b) $\left| \int_0^1 f(x)\,dx - Q(f) \right| \leq 11/720 M$

Let $f(x) = \underline{x^!}$