MA 322: Scientific Computing

Lecture - 5

Linear Curve Fitting

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Linear Curve Fitting Contd..

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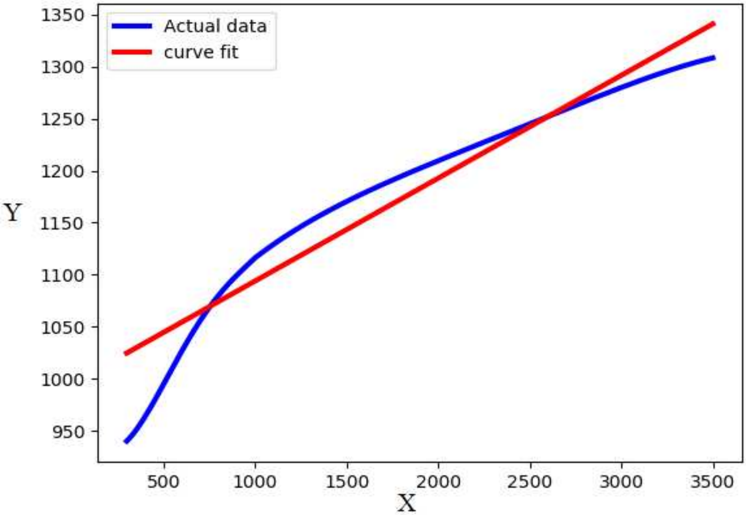
Geometrically, linear curve fitting can be described by following figure



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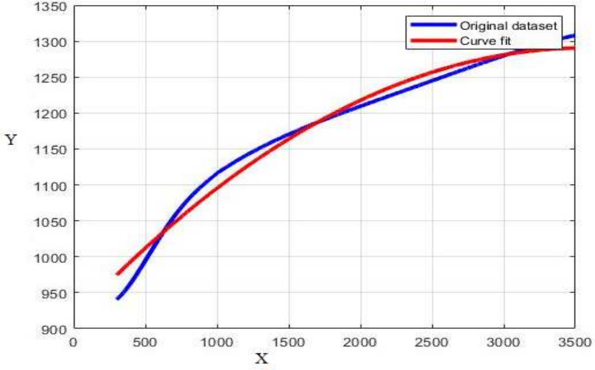

Cubic Curve Fitting

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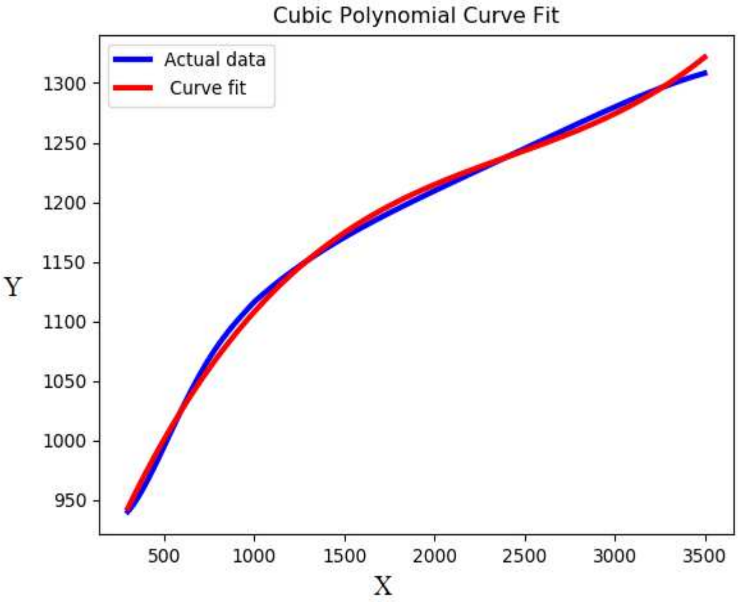

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  The value $P_n(x)$, for $x \neq x_i$, is called estimate of $f(x)$.

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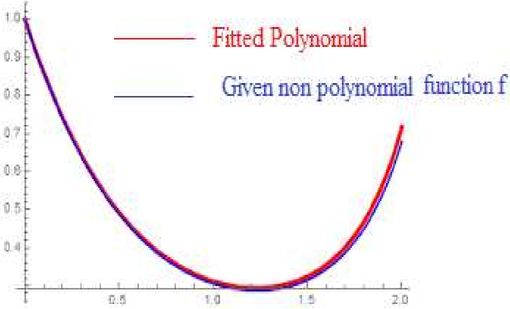

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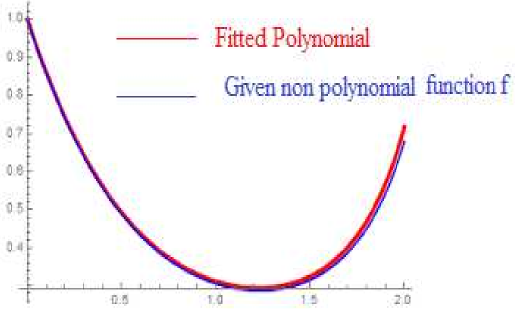

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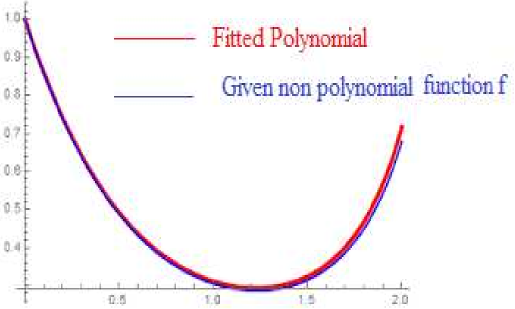

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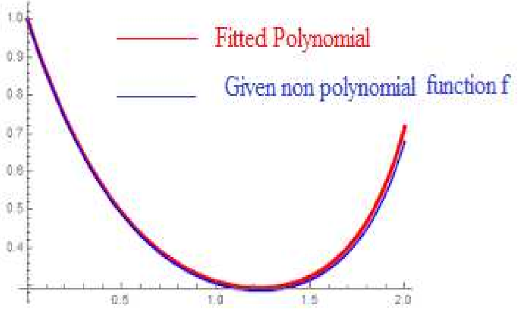

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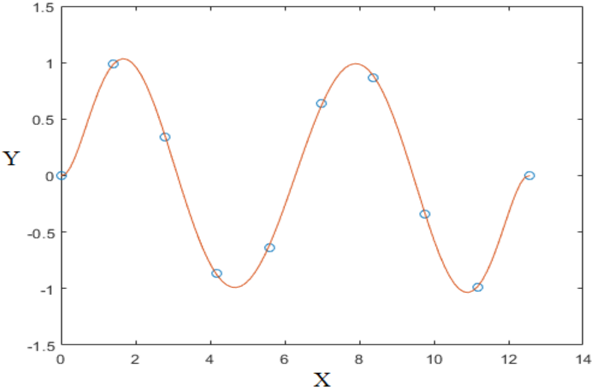

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Problem: Given a data set $(x_0, f_0), \ldots, (x_n, f_n)$ consisting of

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Since the coefficient matrix is nonsingular (Check), the linear system has a unique solution. Thus the polynomial interpolation problem has a unique solution.

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Theorem:

Suppose $x_0,\ x_1, \ldots, x_n$ are distinct real numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each $x \in [a, b]$, a number ξ_x between $x_0,\ x_1, \ldots, x_n$ and hence in (a, b) exists such that

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n), \quad (1)$$

where $p(x)$ is an interpolating polynomial of f with degree n.