Sol^n (4) The IVP $\frac{dy}{dx} = f(x,y)$ $y(x_0) = y_0$

given, $\frac{\partial f(x,y)}{\partial y} \leq 0$ $\forall x \in [x_0, x_n]$

Part (i)

Let $h > 0$ and $x_j = x_0 + j^* h$ $(1 \leq j \leq n)$

We can see that: $|e_n| \leq |y(x_n) - y_n|$
$= |y(x_{n-1} + h) - y_n|$

Now use taylor expansion for $y(x_{n-1} + h)$

$\Rightarrow |e_n| = |y(x_{n-1}) + hy'(x_{n-1}) + \frac{h^2}{2} y''(\xi_k) - (y_{n-1} + h f(x_{n-1}, y_{n-1}))|$.

$= |(y(x_{n-1}) - y_{n-1}) + h(y'(x_{n-1}) - f(x_{n-1}, y_{n-1})) + \frac{h^2}{2} y''(\xi_k)|$.

$\Rightarrow |e_{n-1} + h(f(x_{n-1}, Y_{n-1}) - f(x_{n-1}, y_{n-1})) + \frac{h^2}{2} y''(\xi_{n-1})|$. $\because Y_{n-1} = y(x_{n-1})$,

and $\xi_{n-1} \in (x_{n-1}, x_n)$

using MVT for fxn of 2 variables,

$f(x_k, Y_k) - f(x_k, y_k) = (Y_k - y_k) \frac{\partial f}{\partial y}(x_k, \eta_k)$
$= e_k \frac{\partial f}{\partial y}(x_k, \eta_k)$

Thus, $f(x_{n-1}, Y_{n-1}) = f(x_{n-1}, y_{n-1}) + e_{n-1} \frac{\partial f}{\partial y}(x_{n-1}, \eta_{n-1})$

$$f(x_{n-1}, Y_{n-1}) - f(x_{n-1}, y_{n-1}) = e_{n-1} \frac{\partial f}{\partial y}(x_{n-1}, \eta_{n-1})$$

where $\eta_{n-1} \rightarrow$ real no. b/w y_{n-1}, Y_{n-1}.

$$\Rightarrow |e_n| = |e_{n-1} + he_{n-1} \frac{\partial f}{\partial y}(x_{n-1}, \eta_{n-1}) + \frac{h^2}{2} y''(\varepsilon_{n-1})|.$$

$$\Rightarrow |e_n| \le |e_{n-1}| \left(\left| 1 + h \frac{\partial f}{\partial y}(x_{n-1}, \eta_{n-1}) \right| \right) + \frac{h^2}{2} |y''(\varepsilon_{n-1})|$$

$\because h > 0, \frac{\partial f}{\partial y}(x_{n-1}, \eta_{n-1}) \le 0$

$\Rightarrow 1 + h \frac{\partial f}{\partial y}(x_{n-1}, \eta_{n-1}) \le 1 \implies$

$$|e_n| \le |e_{n-1}| + \frac{h^2}{2} |y''(\varepsilon_{n-1})|.$$

for some $\varepsilon \in (x_{n-1}, x_n)$.

Hence, proved.

Part (ii) using the previously derived eq^n:

$$|e_1| \le |e_0| + h^2/2 \, |y''(\varepsilon_0)|.$$

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$$|e_2| \leq |e_1| + \frac{h^2}{2} |(y''(\varepsilon_1))|.$$

$\vdots$

$$|e_n| \leq |e_{n-1}| + \frac{h^2}{2} |y''(\varepsilon_{n-1})|.$$

Summing up all eq^{n}s.

$$\sum_{i=1}^{n} |e_i| \leq \sum_{i=0}^{n-1} |e_i| + \frac{h^2}{2} \sum_{i=0}^{n-1} |y''(\varepsilon_i)|.$$

$$\Rightarrow \sum_{i=1}^{n} |e_i| - \sum_{i=0}^{n-1} |e_i| \leq \frac{h^2}{2} \sum_{i=0}^{n-1} |y''(\varepsilon_i)|$$

Now $\varepsilon_i \in (x_{i-1}, x_i)$

$$\Rightarrow |y''(\varepsilon_i)| \leq \max_{x_0 \leq x \leq x_n} |y''(x)|.$$

$$|e_n| - |e_0| \leq \frac{h^2}{2} \sum_{i=0}^{n-1} \max_{x_0 \leq x \leq x_n} |y''(x)|.$$

now $\max_{x_0 \leq x \leq x_n} |y''(x)| \to$ const. $\Rightarrow \sum_{i=0}^{n-1} \max_{x_0 \leq x \leq x_n} |y''(x)|$

$$= n \max_{x_0 \leq x \leq x_n} |y''(x)|.$$

$$\Rightarrow |e_n| - |e_0| \leq nh^2 Y.$$

$$\Rightarrow |e_n| \leq |e_0| + nh^2 Y$$

where $Y = \frac{1}{2} \max_{x_0 \leq x \leq x_n} |y''(x)|.$

Hence, proved.