Lab 6: MA 322

Date: 2/03/2021

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1. Use Euler's method to approximate the solutions for each of the following initial-value problems.
   (a) $\frac{dy}{dt} = e^{t-y}, \;\; 0 \le t \le 1, \;\; y(0) = 1, \;\;$ with $\;h = 0.5$
   (b) $\frac{dy}{dt} = \frac{1+t}{1+y}, \;\; 1 \le t \le 2, \;\; y(1) = 2, \;\;$ with $\;h = 0.5$
   (c) $\frac{dy}{dt} = -y + ty^{1/2}, \;\; 2 \le t \le 3, \;\; y(2) = 2, \;\;$ with $\;h = 0.25$
   (d) $\frac{dy}{dt} = t^{-2}(\sin 2t - 2ty), \;\; 1 \le t \le 2, \;\; y(1) = 2, \;\;$ with $\;h = 0.25$

2. The actual solutions to the initial-value problems in problem 1 are given here. Compute the actual error
   (a) $y(t) = \log(e^t + e - 1)$
   (b) $y(t) = \sqrt{t^2 + 2t + 6} - 1$
   (c) $y(t) = \left(t - 2 + \sqrt{2}ee^{-t/2}\right)^2$
   (d) $y(t) = \frac{4+\cos 2 - \cos 2t}{2t^2}$

3. Given the initial-value problem
   $$\frac{dy}{dt} = \frac{1}{t^2} - \frac{y}{t} - y^2,, \quad 1 \le t \le 2, \;\; y(1) = -1$$
   with exact solution $y(t) = -\frac{1}{t}$.
   (a) Use Euler's method with $h = 0.05$ to approximate the solution, and compare it with the actual values of $y(t)$.
   (b) Use the answers generated in part (a) and linear interpolation to approximate the following values of $y(t)$, and compare them to the actual values.
   (I) $y(1.052)$ $\quad (II)$ $y(1.555)$ $\quad (III)$ $y(1.978)$

4. Let $h > 0$ and let $x_j = x_0 + jh$ $(j = 1, 2, \ldots, n)$ be given nodes. Consider the initial value problem $\frac{dy}{dx} = f(x, y)$ $\;y(x_0) = y_0$, with
   $$\frac{\partial f(x,y)}{\partial y} \le 0$$
   for all $x \in [x_0, x_n]$ and for all y.

i. Using error analysis of the Euler's method, show that there exists an $h > 0$ such that

$$|e_n| \le |e_{n-1}| + \frac{h^2}{2} f''(\xi)$$

for some $\xi \in (x_{n-1}, x_n)$, where $e_n = y(x_n) - y_n$ with y_n obtained using Euler method.

ii. Applying the conclusion of (i) above recursively, prove that

$$|e_n| \le |e_0| + nh^2 Y, \quad \text{where,} \quad Y = \frac{1}{2} \max_{x_0 \le x \le x_n} |y''(x)|. \tag{1}$$

5. The solution of the initial value problem

$$\frac{dy}{dx} = \lambda y(x) + \cos x - \lambda \sin x, \quad y(0) = 0$$

is $y(x) = \sin x$. For $\lambda = -20$, find the approximate value of $y(3)$ using the Euler's method with $h = 0.5$. Compute the error bound given in (1), and Show that the actual absolute error exceeds the computed error bound given in (1). Explain why it does not contradict the validity of (1).

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