CS 4980/6980: Predictive Data Analysis © Spring 2018

Lecture 20: Vector Spaces and Linear Combinations

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20.1 Introduction

A **Vector Space** *informally* is a set of objects (vectors or points) that can be added and multiplied without **"falling"** outside of the set.

Falling Outside of set : A point or vector does not qualify to be an element of the set. For example:

Suppose $\mathbb{R}_+ :=$ {Set of all positive numbers}

$$\begin{aligned} \text{Then: } 5 &\in \mathbb{R}_+ \\ 7 &\in \mathbb{R}_+ \\ 5 - 7 = -2 &\notin \mathbb{R}_+ \end{aligned}$$

Therefore, -2 fall outside of set and $\mathbb{R}_+$ is not a **Vector Space**

20.2 Vector Space

A Formal Definition of a Vector Space

A set $\mathcal{V}$ is a vector space if $\forall v, w \in V$, and for every scalar values $a, b \in \mathbb{R}$

$$\mathrm{a}V + \mathrm{b}V \in V$$

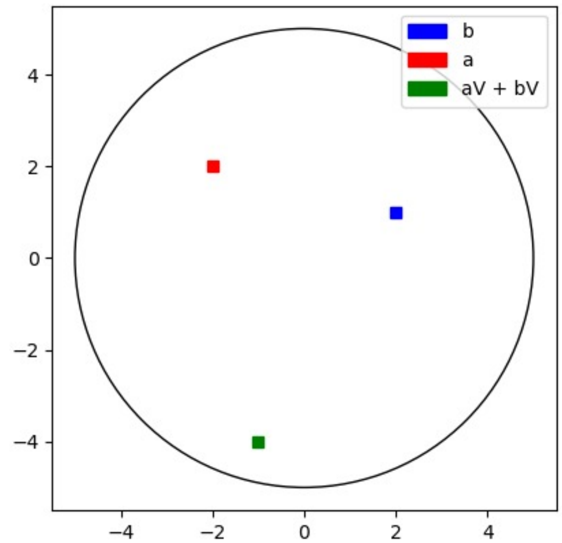

Figure 20.1: Looking at the nearest point only, the new point is classified as green.

A set $\mathcal{V}$ is also a vector space, if it is closed under linear combination (Will be discussed later).

Now the question arises:
Is $\mathbb{R}$ a vector space?

Proof. $\mathbb{R}$ is a Vector Space

Let v, w $\in \mathbb{R}$

Let a, b be scalars   every scalar a, b $\in \mathbb{K}$

Let u = a$\mathcal{V}$ + b$\mathcal{V} \in \mathbb{R}$

Then u $\in \mathbb{R}$ □

Some Examples:

Example 1: If $W = \mathbb{R}$. Is $\mathbb{W}$ under $\sqrt{*}$

Answer: No, here's a a counter example:

$$-4 \in \mathbb{W}$$
$$\sqrt{-4} = 2\iota \notin \mathbb{W}$$

Example 2: If $\mathbb{W} = \mathbb{Z}$ and $\mathbb{K} = \mathbb{R}$. Is $\mathcal{V}$ a vector space?

Answer: No, here's a a counter example:

$$\text{Take v} = 1 \text{ and w} = 2 \in \mathbb{V}$$
$$\text{Take a} = 0.5 \text{ and b} = 0.5 \in \mathbb{K}$$
$$\text{Let u} = \text{av} + \text{bw}$$
$$\text{u} = 1.5 \in \mathbb{V} \notin \mathbb{R}$$

20.3 More about Vector Space

A vector space satisfies the following additive properties:

(A1) Closure: $\forall$ v,w $\in \mathbb{V}$, v + w $\in \mathbb{V}$
(A2) Commutativity: $\forall$ v,w $\in \mathbb{V}$, v + w = w + v
(A3) Associativity: $\forall$ u,v,w $\in \mathbb{V}$, (u + v) + w = u + (v + w)
(A4) Additive Identity: $\exists$ an element in $\mathbb{V}$, denoted by 0 | v $\in \mathbb{V}$, v + 0 = v
(A5) Additive Inverse: $\forall$ v $\in \mathbb{V}$, $\exists!$ element in $\mathbb{V}$, denoted by -v | v + (-v) = 0

Example 3: If $\mathbb{V} = \mathbb{N}$. Is $\mathbb{V}$ a vector space?

Answer: No, because of **(A5) Additive Inverse** property

A vector space also has the following multiplicative properties:
(M1) Closure: $\forall$ a $\in \mathbb{R}$, and $\forall$ v $\in \mathbb{V}$, av $\in \mathbb{V}$
(M2) Associativity: $\forall$ a, b $\in \mathbb{R}$, and $\forall$ v $\in \mathbb{V}$, a(bv) = (ab)v
(M3) 1^{st}**Distributive Law:** $\forall$ a $\in \mathbb{R}$, and $\forall$ v, w $\in \mathbb{V}$, a(v+w) = av + aw

(M4) 2^{nd}**Distributive Law:** $\forall$ a, b $\in \mathbb{R}$, and $\forall$ v, w $\in \mathbb{V}$, (a+b)v = av + bv
(M5) Multiplicative Identity: $\forall$ v $\in \mathbb{V}$, 1v= v

Example 4: If $\mathbb{V} = \mathbb{R}^2$. Is $\mathbb{V}$ a vector space?

Proof. $\mathbb{V}$ is a Vector Space
Let $\mathbb{V} = \mathbb{R}^2$ and $\mathbb{K} = \mathbb{R}$
(A1) Let vectors v, w $\in \mathbb{R}^2$
Let u = v + w
Then u $\in \mathbb{R}^2$

□

Example 5: Prove if **(A2)** holds?

Proof. Let v, w $\in \mathbb{R}^2$

Let u = v + w = $\begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$

$\begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ = w + v

□

20.4 Linear Combination

Let $\mathbb{V}$ be a vector space
Let $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_r \in \mathbb{V}$
We say that $\mathbf{u}$ is a linear combination of the $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_r$ if:

$$\mathbf{u} = \sum_{i=1}^{r} c_i v_i$$

$\mathbf{u} = c_1 v_1 + c_2 v_2 + ... + c_r v_r$

For some scalars $c_1, c_2, ...c_r$. In this case $c_1, c_2, ...c_r$ are called the coefficients of the $\mathbf{u}$ w.r.t $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_r$

$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\mathbf{u} = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

= 7v + 5w + 1x $\quad c_1 = 7 \quad c_2 = 5 \quad c_3 = 1$