10.7.89

EE25BTECH11041 - Naman Kumar

Question:
Lines $5x + 12y - 10 = 0$ and $5x - 12y - 40 = 0$ touch a Circle C_1 of diameter 6. If the centre of C_1 lies in the first quadrant, find the equation of circle C_2 which is concentric with C_1 and cuts intercepts of length 8 on these lines.

Solution:
Given lines are tangents, their equations

$$\mathbf{n_1}\mathbf{x} = c_1, \mathbf{n_2}\mathbf{x} = c_2 \tag{1}$$

$\mathbf{c_1}$	10	Constant for line 1
$\mathbf{c_2}$	40	Constant for line 2
$\mathbf{n_1}$	$\begin{pmatrix} 5 \\ 12 \end{pmatrix}$	normal of line 1
$\mathbf{n_2}$	$\begin{pmatrix} 5 \\ -12 \end{pmatrix}$	normal of line 2

(2)

Distance of point from a line

$$d = \frac{|\mathbf{n}^T\mathbf{x} - c|}{||\mathbf{n}||} \tag{3}$$

Center must lie on one of the angle bisector of tangents

d_1	Distance of center from tangent 1
d_2	Distance of center from tangent 2
$\mathbf{x}$	center of circle C_1

(4)

$$\frac{|\mathbf{n_1}^T\mathbf{x} - c_1|}{||\mathbf{n_1}||} = \frac{|\mathbf{n_2}^T\mathbf{x} - c_2|}{||\mathbf{n_2}||} \tag{5}$$

$$\frac{|\mathbf{n_1}^T\mathbf{x} - 10|}{13} = \frac{|\mathbf{n_2}^T\mathbf{x} - 40|}{13} \tag{6}$$

$$\mathbf{n_1}^T\mathbf{x} - 10 = \pm(\mathbf{n_2}^T\mathbf{x} - 40) \tag{7}$$

$$\mathbf{n_1}^T\mathbf{x} - 10 = \mathbf{n_2}^T\mathbf{x} - 40,\ \mathbf{n_1}^T\mathbf{x} - 10 = -\mathbf{n_2}^T\mathbf{x} + 40 \tag{8}$$

$$(\mathbf{n_2}^T - \mathbf{n_1}^T)\mathbf{x} = 30, (\mathbf{n_2}^T + \mathbf{n_1}^T)\mathbf{x} = 50 \tag{9}$$

$$\begin{pmatrix} 0 & -24 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 30 \tag{10}$$

$$-24y = 30 \implies y = \frac{-5}{6} \tag{11}$$

$$\begin{pmatrix} 10 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 50 \tag{12}$$

$$10x = 50 \implies x = 5 \tag{13}$$

Since center is in I quadrant so

$$Case : y = \frac{-5}{6}, \text{rejected} \tag{14}$$

$$Case : x = 5, \text{accepted} \tag{15}$$

Now

$$\frac{|\mathbf{n_1}^T \mathbf{x} - c_1|}{\|\mathbf{n_1}\|} = 3 \tag{16}$$

$$\mathbf{n_1}^T \mathbf{x} - c_1 = \pm 39 \tag{17}$$

$$5x + 12y - 10 = \pm 39 \tag{18}$$

$$at, x = 5 \tag{19}$$

$$y = 2, -\frac{54}{12} \tag{20}$$

$$so, center = \mathbf{c} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \tag{21}$$

General equation of conic

$$g(\mathbf{x}) = \mathbf{x^T V x} + 2\mathbf{u^T x} + f \tag{22}$$

Intercept by a circle on line

$$r^2 = p^2 + d^2 \tag{23}$$

d	Distance of center from line
p	intercept by circle on line

(24)

$$d = 3, p = \frac{8}{2} = 4 \tag{25}$$

So,

$$r^2 = 4^2 + 3^2 \tag{26}$$

$$r = 5 \tag{27}$$

Equation of circle C_2,

$$\mathbf{x^T} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -5 \\ -2 \end{pmatrix}^T \mathbf{x} - 5^2 = 0 \tag{28}$$

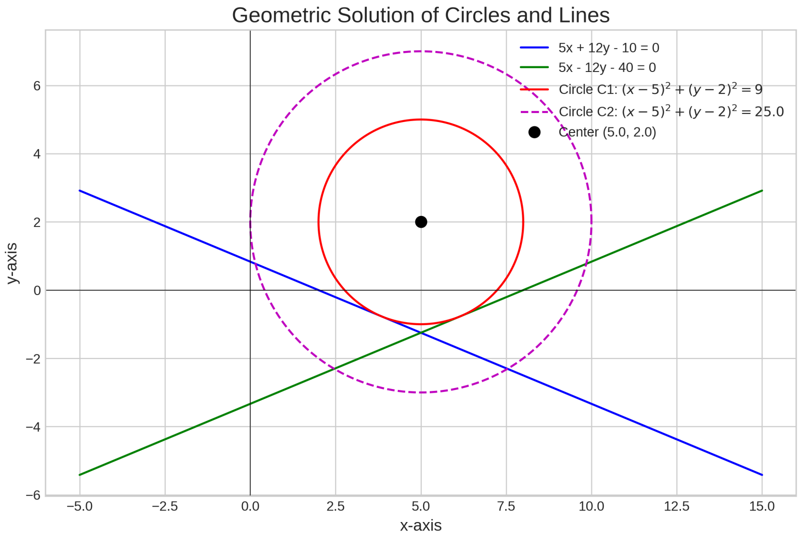
Geometric Solution of Circles and Lines
5x + 12y - 10 = 0
5x - 12y - 40 = 0
Circle C1: $(x-5)^2 + (y-2)^2 = 9$
Circle C2: $(x-5)^2 + (y-2)^2 = 25.0$
Center (5.0, 2.0)
y-axis
x-axis
6
4
2
0
−2
−4
−6
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
12.5
15.0

Figure 1