12.697

EE25BTECH11060 - V.Namaswi

Question
Given the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 2 & 0 \end{pmatrix},$$

find the value of

$$\det(\mathbf{A^4} - \mathbf{5A^3} + \mathbf{6A^2} + \mathbf{21I})$$

Solution
The characteristic equation of $\mathbf{A}$ is

$$\det(A - \lambda I) = 0 \tag{1}$$

$$\det\begin{pmatrix} 5-\lambda & 3 \\ 2 & -\lambda \end{pmatrix} = 0 \tag{2}$$

$$(5-\lambda)(-\lambda) - (3 \cdot 2) = -5\lambda + \lambda^2 - 6 = 0 \tag{3}$$

$$\lambda^2 - 5\lambda - 6 = 0 \tag{4}$$

Factorizing:

$$(\lambda - 6)(\lambda + 1) = 0 \tag{5}$$

Hence, the eigenvalues are:

$$\lambda_1 = 6, \quad \lambda_2 = -1 \tag{6}$$

According to eigenvalue property
If $f(\mathbf{A}) = \mathbf{A^4} - \mathbf{5A^3} + \mathbf{6A^2} + \mathbf{21I}$, then the eigenvalues of $f(A)$ are:

$$f(\lambda_i) = \lambda_i^4 - 5\lambda_i^3 + 6\lambda_i^2 + 21$$

So Find,

$$f(6) = 6^4 - 5 \cdot 6^3 + 6 \cdot 6^2 + 21 \tag{7}$$

$$= 1296 - 1080 + 216 + 21 \tag{8}$$

$$= 453 \tag{9}$$

$$f(-1) = (-1)^4 - 5(-1)^3 + 6(-1)^2 + 21 \tag{10}$$

$$= 1 + 5 + 6 + 21 \tag{11}$$

$$= 33 \tag{12}$$

Determinant of $f(A)$

$$\det(f(A)) = f(\lambda_1) \cdot f(\lambda_2) \tag{13}$$
$$= 453 \cdot 33 \tag{14}$$
$$= 14949 \tag{15}$$