PHYS 4322 — Intermediate Electromagnetic Theory II
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Name: Jonathan Gonzales
Student ID: 1977384
Email: Jagonz84@cougarnet.uh.edu

Assignment: Chapter 12
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Problem 1: A particle moves with constant velocity from point $x'_A = (5, 0, 0)$ to point $x'_B = (7, 0, 0)$ in time $c\Delta t' = 5$, measured with respect to frame S'. Suppose frame S moves with velocity $v = (1/2)c$ in the positive x-direction with respect to frame S.

(a) What is the particle's velocity u' in frame S'?

Solution

$$\boxed{u' = \frac{\Delta x'}{\Delta t'} = \frac{2}{5}c}$$

(b) How far does the particle move in frame S?

Solution

$$\boxed{\Delta x = \gamma\big(\Delta x' + v\Delta t'\big) = 3\sqrt{3} \approx 5.196}$$

(c) How long does it take the particle to move in frame S?

Solution

$$\boxed{\Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right) = \frac{1}{c}4\sqrt{3}}$$

(d) Calculate the particle's velocity u in frame S using your answers from parts (b) and (c).

Solution

$$\boxed{u = \frac{\Delta x}{\Delta t} = \frac{3}{4}c}$$

(e) Calculate the particle's velocity u in frame S using velocity addition, Equation 12.20 in the textbook.

Solution

$$\boxed{u = \frac{u' + v}{1 + (u'v/c^2)} = \frac{3}{4}c}$$

Problem 2: In frame S, a particle is moving with (ordinary) velocity $\vec{\mathbf{u}} = (3 \text{ m/s})\vec{\mathbf{e}}_y + (4 \text{ m/s})\vec{\mathbf{e}}_z$.

(**a**) Write out the components of the velocity four-vector U^μ for this particle in frame S. Approximate very small numbers (numbers of order 10^{-8} or smaller) as zero.

Solution

$$\boxed{\begin{aligned} U^0 &= \gamma c \approx c \\ U^1 &= \gamma u_x = 0 \\ U^2 &= \gamma u_y \approx 3 \text{ m/s} \\ U^3 &= \gamma u_z \approx 4 \text{ m/s} \end{aligned}}$$

(**b**) Frame S' moves with velocity $v = (3/5)c$ in the positive x-direction with respect to frame S. Using the Lorentz transformation for a four-vector (Equation 12.43 in the textbook), find the velocity four-vector $U^{\mu\prime}$ for this particle in frame S'.

Solution

$$\boxed{\begin{aligned} U^{0\prime} &= \gamma'\left(U^0 - \beta' U^1\right) \approx \frac{5}{4}c \\ U^{1\prime} &= \gamma'\left(U^1 - \beta' U^0\right) \approx -\frac{3}{4}c \\ U^{2\prime} &= U^2 \approx 3 \text{ m/s} \\ U^{3\prime} &= U^3 \approx 4 \text{ m/s} \end{aligned}}$$

(**c**) Use velocity addition (Equation 12.45 in the textbook) to find the components of the ordinary velocity in frame S', and then use those components to find the velocity four-vector $U^{\mu\prime}$ in frame S'.

Solution

$$\boxed{\begin{aligned} {u_x}' &= \frac{u_x - v}{1 - v u_x/c^2} = -\frac{3}{5}c \\ {u_y}' &= \frac{u_y}{\gamma'\left(1 - v u_x/c^2\right)} = \frac{12}{5} \text{ m/s} \\ {u_z}' &= \frac{u_y}{\gamma'\left(1 - v u_x/c^2\right)} = \frac{16}{5} \text{ m/s} \end{aligned}}$$

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$$\boxed{\begin{aligned} U^{0\prime} &= \gamma' c = \frac{5}{4}c \\ U^{1\prime} &= \gamma' u_x = -\frac{3}{4}c \\ U^{2\prime} &= \gamma' u_y \approx 3 \text{ m/s} \\ U^{3\prime} &= \gamma' {u_z}' \approx 4 \text{ m/s} \end{aligned}}$$

Problem 3: A wire has a constant linear charge density λ_0 in the rest frame of the wire. Call this frame S'. Let the wire be along the z' axis. In the lab frame (frame S), the wire moves in the z direction at speed v.

(**a**) Use Gauss's law to find the electric field in the rest frame of the wire (frame S').

Solution

$$\oint \vec{\mathbf{E}}_0 \cdot d\vec{\mathbf{A}} = \frac{\lambda_0 L_0}{\varepsilon_0} \implies \boxed{\vec{\mathbf{E}} = \frac{\lambda_0}{2\pi\varepsilon_0 r}\,\vec{\mathbf{e}}_r}$$

(**b**) Use Ampere's law to find the magnetic field in the rest frame of the wire (frame S').

Solution

$$\oint \vec{\mathbf{B}}_0 \cdot d\vec{\ell} = \mu_0 I_0 \implies \boxed{\vec{\mathbf{B}} = \frac{\mu_0 I_0}{2\pi r}\,\vec{\mathbf{e}}_\phi}$$

(**c**) What is the linear charge density λ in the lab frame (frame S)?

Solution

$$\lambda = \frac{q}{L} = \frac{q}{L_0/\gamma} = \gamma\frac{q}{L_0} = \gamma\lambda_0$$

(**d**) What is the current I in the lab frame (frame S)?

Solution

$$I = \frac{dq}{d\tau} = \frac{dq}{dt}\frac{dt}{d\tau} = \gamma I_0$$

(**e**) Use Gauss's law to find the electric field in the lab frame (frame S).

Solution

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{\lambda L}{\varepsilon_0} \implies \boxed{\vec{\mathbf{E}} = \gamma\frac{\lambda_0}{2\pi\varepsilon_0 r}\,\vec{\mathbf{e}}_r = \gamma\vec{\mathbf{E}}_0}$$

(**f**) Use Ampere's law to find the magnetic field in the lab frame (frame S).

Solution

$$\oint \vec{\mathbf{B}} \cdot d\vec{\ell} = \mu_0 I \implies \boxed{\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi r}\,\vec{\mathbf{e}}_\phi = \gamma\vec{\mathbf{B}}_0}$$

Problem 4: Suppose in frame S, there is an electric field, $\vec{\mathbf{E}} = a_1\vec{\mathbf{e}}_x + a_2\vec{\mathbf{e}}_y$, and a magnetic field $\vec{\mathbf{B}} = b\,\vec{\mathbf{e}}_z$, where a_1, a_2, and b are constants. Frame S' moves with constant speed $v = (4/5)c$ in the positive x-direction with respect to frame S.

(**a**) Find γ.

Solution

$$\boxed{\gamma = \left(1 - \left(\frac{v}{c}\right)^2\right)^{-1/2} = 5/3}$$

(**b**) Find the electric field in frame S'.

Solution

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} \qquad \Lambda^{\mu'}_{\mu} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$F^{\mu'\nu'} = \Lambda^{\mu'}_{\mu}\Lambda^{\nu'}_{\nu}F^{\mu\nu}$$

$$\boxed{\begin{aligned} E'_x &= cF^{0'1'} = \Lambda^{0'}_{\mu}\Lambda^{1'}_{\nu}F^{\mu\nu} = E_x = a_1 \\ E'_y &= cF^{0'2'} = \Lambda^{0'}_{\mu}\Lambda^{2'}_{\nu}F^{\mu\nu} = \gamma E_y = (5/3)a_2 \\ E'_z &= cF^{0'3'} = \Lambda^{0'}_{\mu}\Lambda^{3'}_{\nu}F^{\mu\nu} = \gamma(E_z - c\beta B_y) = 0 \end{aligned}}$$

(**c**) Find the magnetic field in frame S'.

Solution

$$\boxed{\begin{aligned} B'_x &= F^{2'3'} = \Lambda^{2'}_{\mu}\Lambda^{3'}_{\nu}F^{\mu\nu} = B_x = 0 \\ B'_y &= F^{1'3'} = \Lambda^{1'}_{\mu}\Lambda^{3'}_{\nu}F^{\mu\nu} = \gamma(B_y - \beta E_z/c) = 0 \\ B'_z &= F^{1'2'} = \Lambda^{1'}_{\mu}\Lambda^{2'}_{\nu}F^{\mu\nu} = \gamma(B_z - \beta E_y/c) = (5/3)b - (4/3c)a_2 \end{aligned}}$$

Problem 5: Suppose a neutron at rest decays into a proton and an electron. Use conservation of energy and momentum to find the energy of the outgoing electron in terms of the masses of the particles. Note that (unlike the example we worked in lecture) you cannot neglect the mass of any of these particles.

Solution

$$||\mathbf{P}_p||^2 = ||\mathbf{P}_n - \mathbf{P}_e||^2 = ||\mathbf{P}_n||^2 + ||\mathbf{P}_e||^2 - 2||\mathbf{P}_n|| \cdot ||\mathbf{P}_e|| = m_n^2c^2 + m_e^2c^2 - 2m_nE_e = m_p^2c^2$$

$$\Downarrow$$

$$\boxed{E_e = \frac{m_n^2 + m_e^2 - m_p^2}{2m_n}c^2}$$