Approach:
I created the program in Python and used Google Colab as my IDE. The code can be run by following the link Project 2.ipynb or by running the .ipynb file that was attached to this Canvas Submission.

Question 1:
Took N=10,000 samples. I used NumPy to generate N staples of the uniform and gaussian distributions, and then I summed them up. I developed a null hypothesis that the mean is 0 and performed a test to either accept or reject it with a significance level of alpha=0.05. Since I am trying to see whether the mean is equal to 0 or not, this is a two-tailed test meaning that the z-score is around 1.96. I calculated a 95% confidence interval using this z-score divided by the square root of the sample size N. I then checked whether the null hypothesis mean is within this confidence interval in which case I accept the null hypothesis, otherwise I reject the null hypothesis.

Question 2:
I plotted 100 different realizations of Xn versus n from n = 1 to n = 20. I generated the random uniform phase shifts using the NumPy library and plotted using Matplotlib. I marked the dots for every (n, Xn) pair in the 2D plane. I made the lines lighter, the dots darker, and colored each line differently so that it would be less cluttered and easier to see.

Question 3:
I plotted 5 realizations from n=1 to n=100 for each alpha=0.2 and alpha=0.9. Each realization has a different color and the colors are light so that it is easier to see.

I simulated and plotted the theoretical and empirical autocorrelation functions. To calculate the kth autocorrelation, I used the following formula:

$$\hat{\rho}_k = \frac{\sum_{t=k+1}^{T} (r_t - \bar{r})(r_{t-k} - \bar{r})}{\sum_{t=1}^{T} (r_t - \bar{r})^2}$$

I calculated the theoretical ACFs, which for a 1st order autoregressive random process is (alpha)^k. I overlaid both theoretical and empirical plots.

I plotted the Power Spectral Density by taking the Fast Fourier Transform of the autocorrelation functions. The theoretical PSD is:

$$S_Y(f) = \frac{\sigma^2}{1 - 2\alpha\cos(2\pi f) + \alpha^2}$$

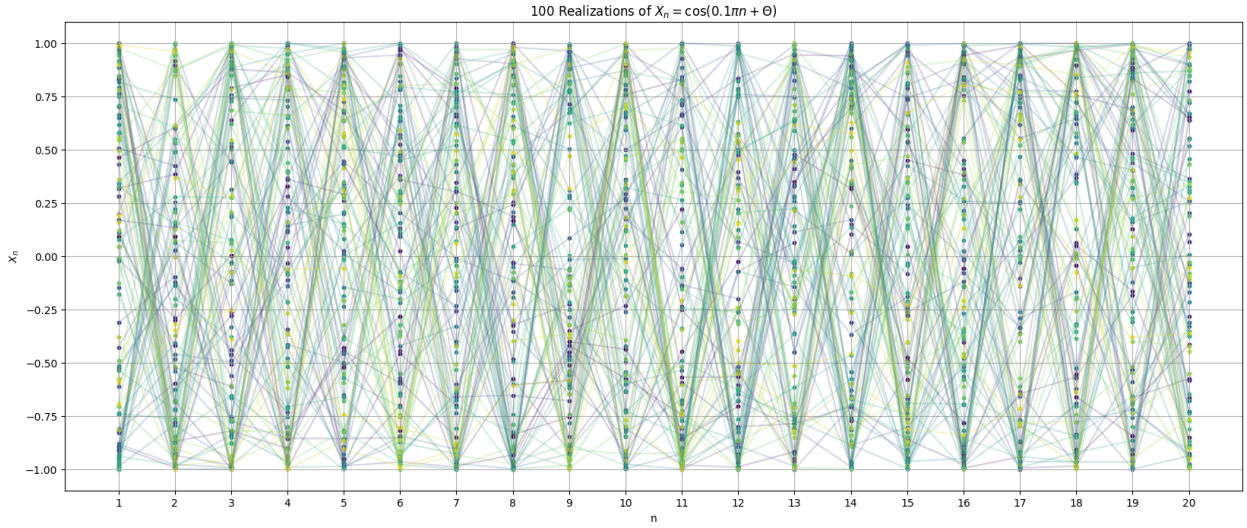
where the variance is (1-alpha)^2. I overlaid both theoretical and empirical plots.

Results:

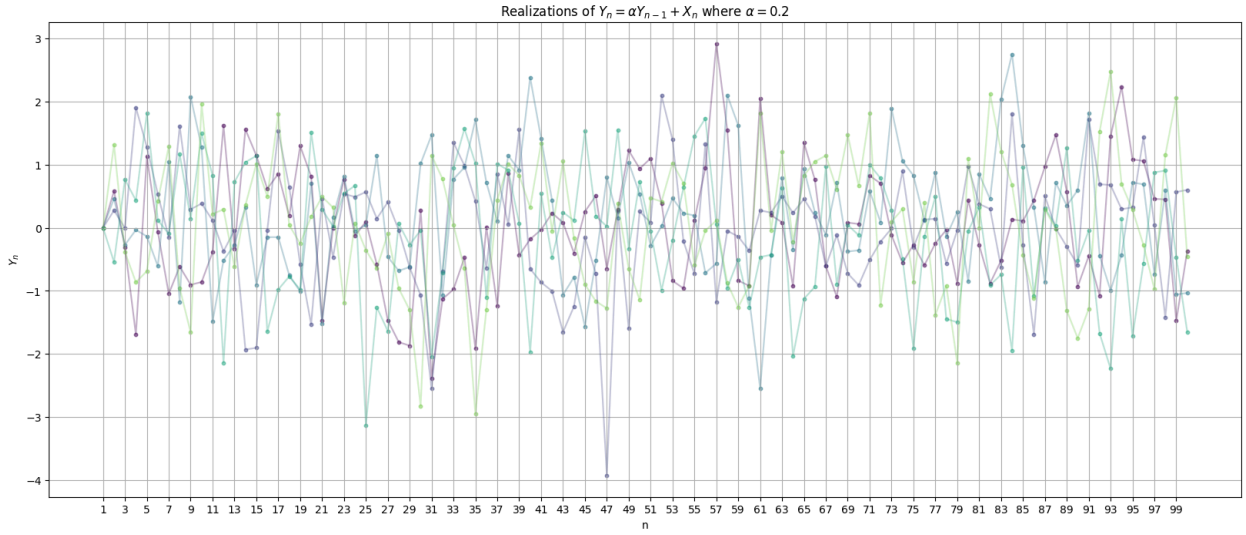
Question 1:
The 95% Confidence Interval of the mean is (-0.0155722687370347, 0.0155722687370347). The sample mean when N = 10,000 is 0.09680377456406018. Since the sample mean is outside the confidence interval, we can reject the null hypothesis.

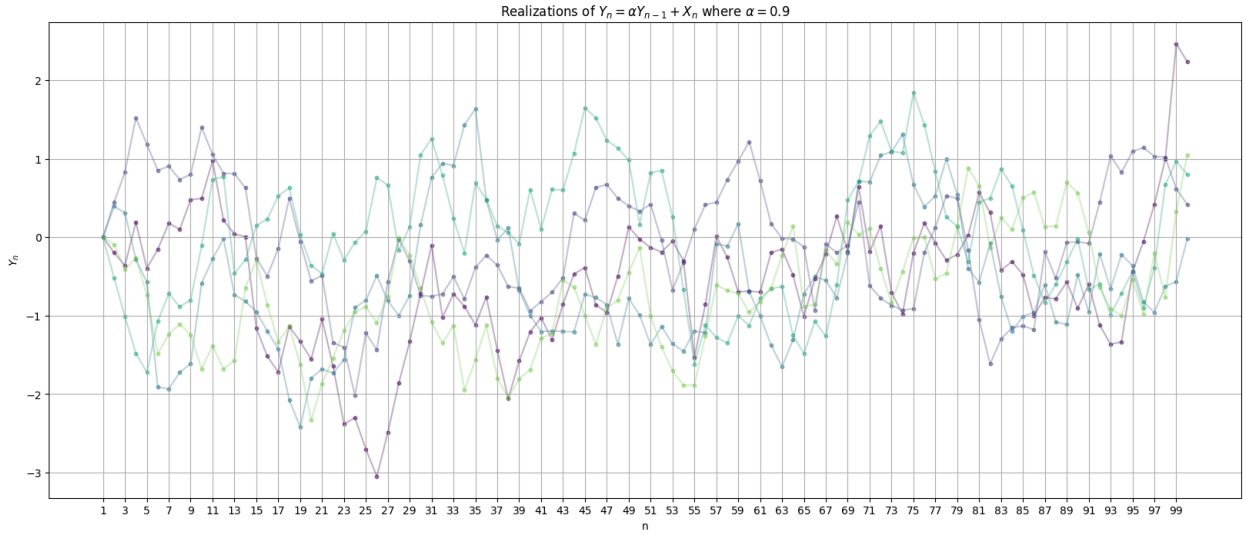
Question 2:

- The values of X_n always remain between -1 and 1 since cosine is bounded by these values
- We see dense patches of points in certain regions due to cos(0.1*pi*n), but most values are evenly distributed above and below zero due to the uniform distribution

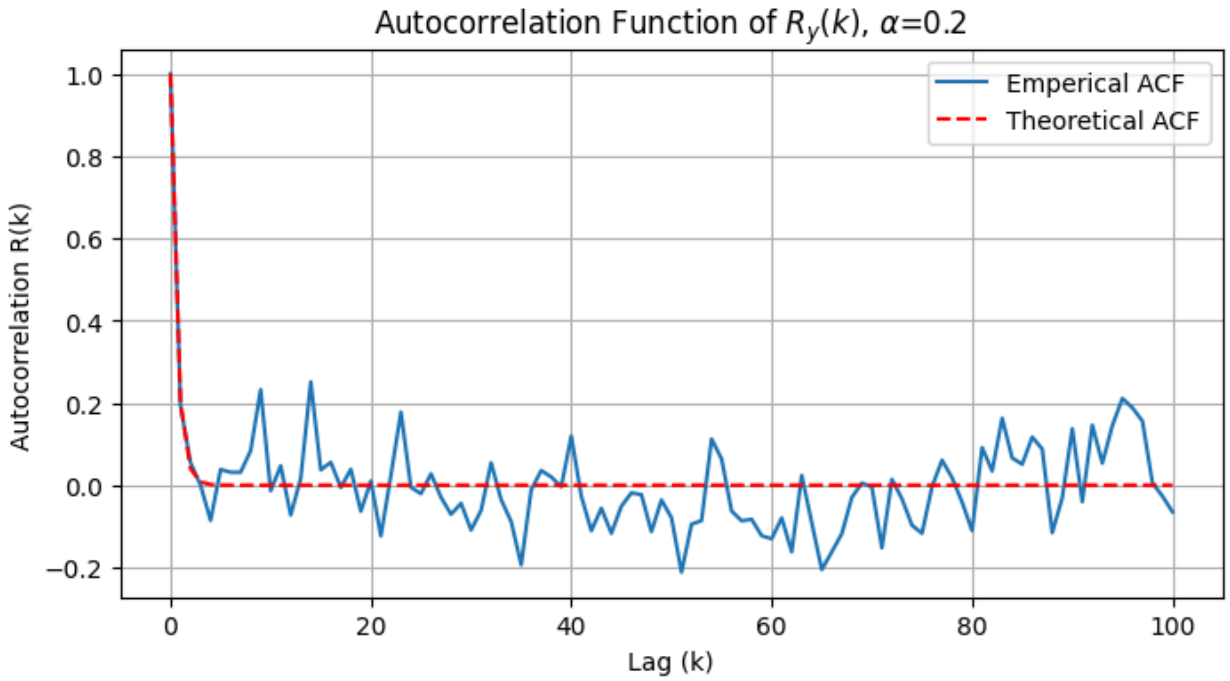


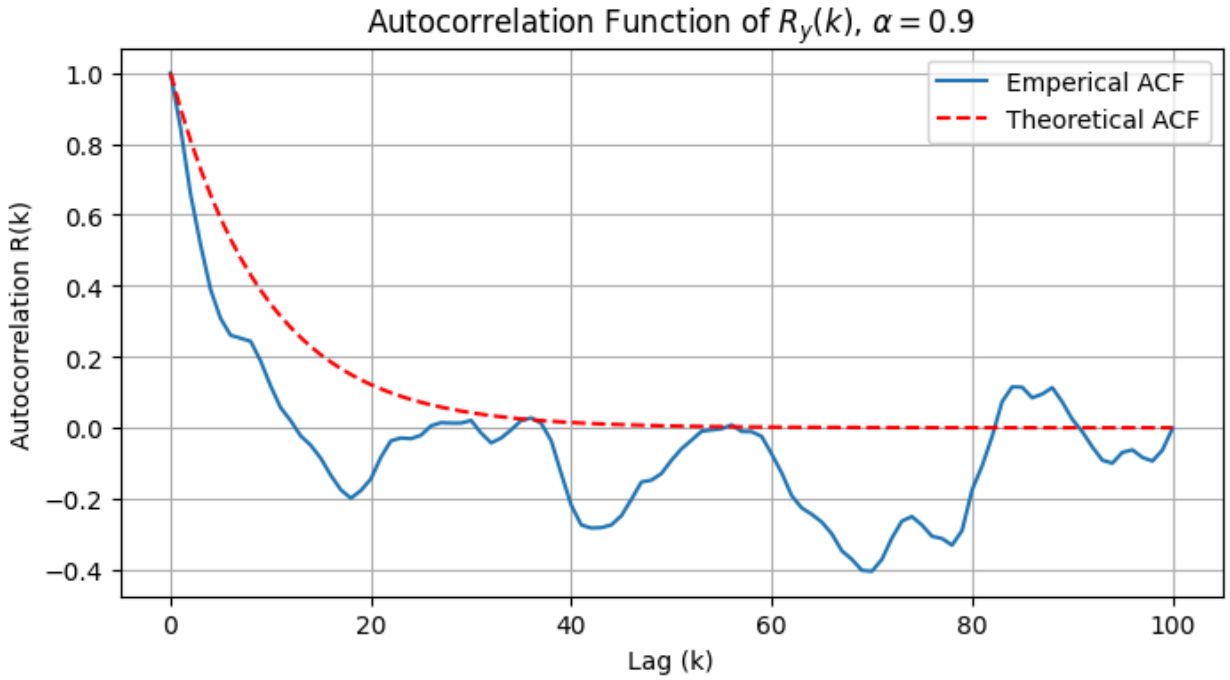
Question 3:
The realizations of alpha=0.2 are more frequent and random compared to alpha=0.9 which is smoother. This can be explained by the fact that a higher alpha value means that there is more weight on previous values which means that the process will be smoother compared to lower values.



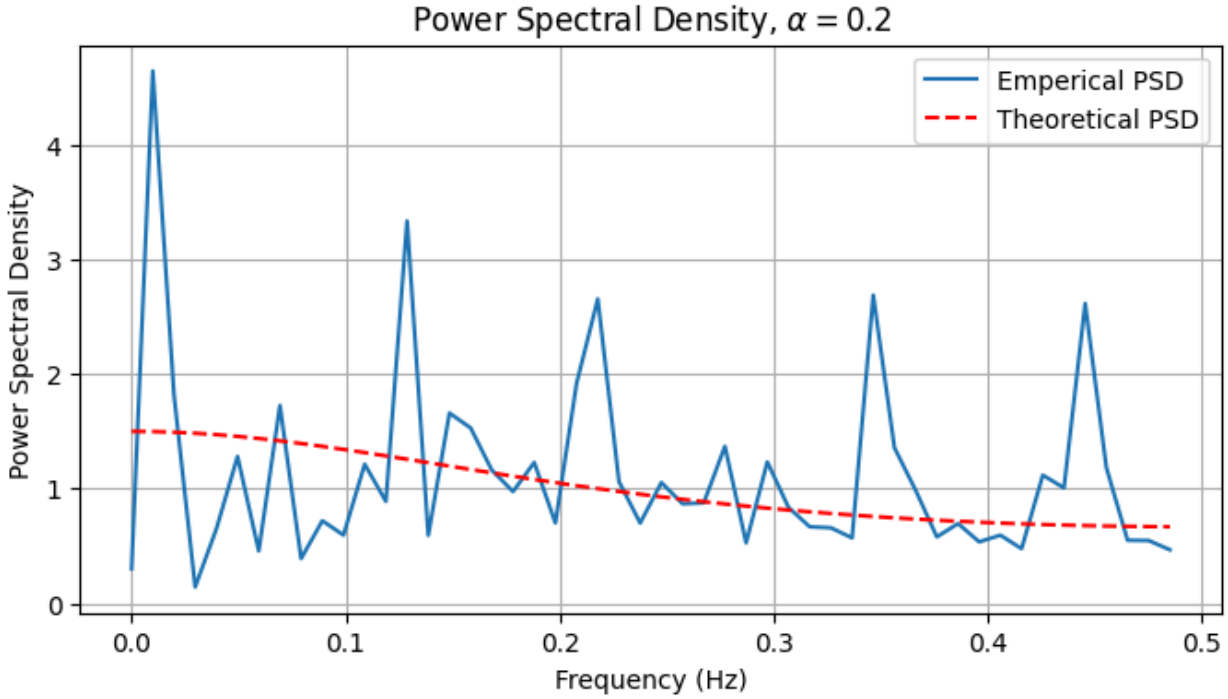


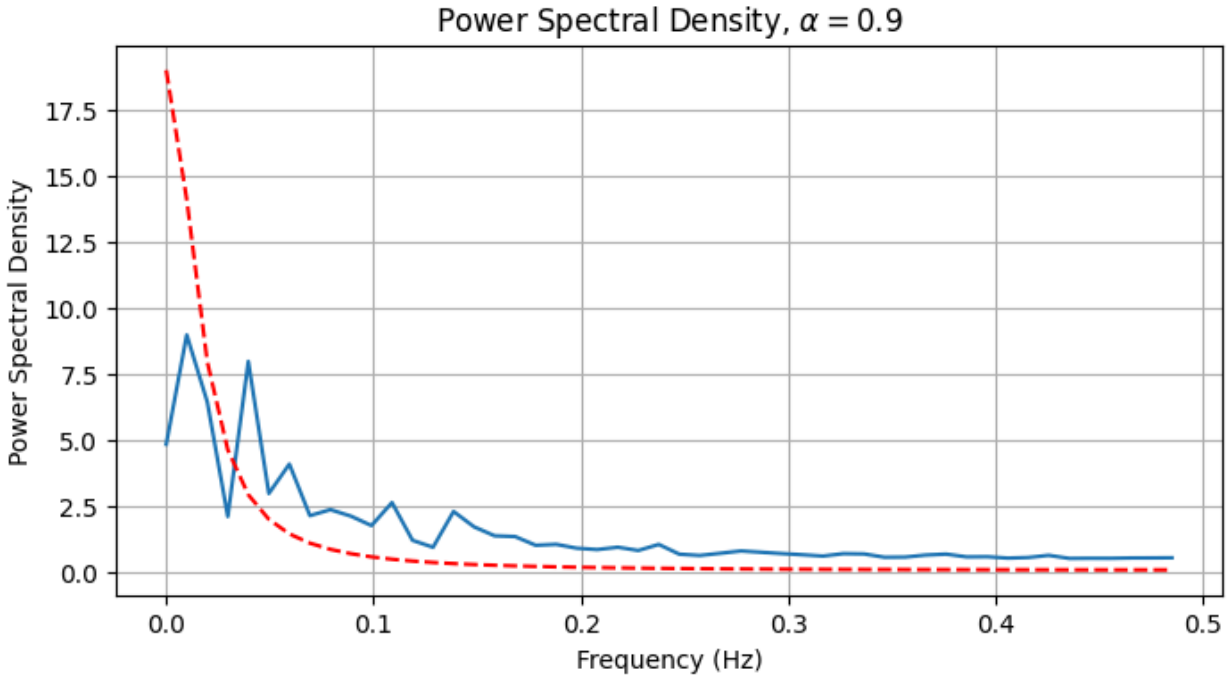
The autocorrelation function for 0.2 decays rapidly while 0.9 decays much more slowly. This can be explained by the fact that the previous value is more persistent when alpha=0.9 compared to when the Guassian noise is more dominant when alpha=0.2. Both empirical ACFs match closely to the theoretical ACF. Both empirical ACFs have oscillatory artifacts, with alpha=0.2 being the most significant.





For alpha=0.9, the power is concentrated at low frequencies which means the process exhibits longer term trends. This matches our intuition that the realizations for alpha=0.9 are smoother since they exhibit lower frequency fluctuations as opposed to higher frequency fluctuations. For alpha=0.2, the power is more evenly spread across frequencies. Since lower frequencies no longer dominate, this aligns with the erratic behavior observed by the time series for alpha=0.2.


Power Spectral Density, $\alpha = 0.2$
Emperical PSD
Theoretical PSD
Power Spectral Density
Frequency (Hz)
0
1
2
3
4
0.0
0.1
0.2
0.3
0.4
0.5


Power Spectral Density, $\alpha = 0.9$
Power Spectral Density
Frequency (Hz)
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
0.0
0.1
0.2
0.3
0.4
0.5