Complexity Of Algorithms (CA)-AI3011

Basic introduction, time and space complexity analysis

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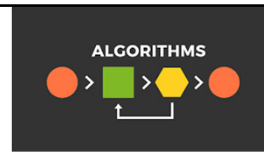
Syllabus

- **Basic introduction and time and space complexity analysis:**

Asymptotic notations (Big Oh, small oh, Big Omega, Theta notations). Best case, average case, and worst-case time and space complexity of algorithms. Overview of searching, sorting algorithms. Using Recurrence relations and Mathematical Induction to get asymptotic bounds on time complexity. Proving correctness of algorithms.

What is an Algorithm?

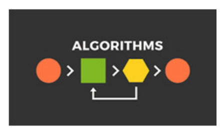


- **Definition Of Algorithm**

(Ref-https://en.wikipedia.org/wiki/Algorithm

- In mathematics and computer science, an **algorithm** is a finite sequence of well-defined, computer-implementable instructions, typically to solve a class of problems or to perform a computation.
- Algorithms are always unambiguous and are used as specifications for performing calculations, data processing, automated reasoning, and other tasks.
- It has an input (can also be empty) and produces output (goal)
- Algorithms are used in Babylonian mathematics 2500 BC (wikipedia) (oldest reference)
- ***The word algorithm is derived from the name of the 9th-century Persian mathematician Muhammad ibn Musa al-Khwarizmi in 825.***

What is an Algorithm?



- Informally, an ***algorithm*** is any well-defined computational procedure that takes some value, or set of values, as ***input*** and produces some value, or set of values, as ***output***.
- An **algorithm** is thus a sequence of computational steps that transform the input into the output- *Book of Cormen and et.all*
- Algorithm can be used as a *tool* solve a ***computational problem***.

Computational problem Example

- **Sorting problem**

$Input:-$ A sequence of n numbers $< a_1, a_2, ..., a_n >$

$Output:-$ A permutation (reordering) $< a_1', a_2', ..., a_n' >$

such that $a_1' \le a_2' \le, ..., \le a_n'$

- Input can be **<31; 41; 59; 26; 41; 58>** and output produced by algorithm will be

  <26; 31; 41; 41; 58; 59>

- Input sequence is call ***instance*** of sorting program Or ***instance of a problem.***

Correctness of Algorithm

- Algorithm is said to be ***correct***, if for **every** input instance it **halts** with **correct output**.
- We say that the **correct algorithm** solves computational problem.
- **Incorrect Algorithms**
  - **May not halt at all for some inputs**
  - **Or halts with incorrect output**
- Algorithm can be specified in English, a Computer program or even as a hardware design.
- Generally a **pseudocode** is commonly used which looks like C, C++ code. (pseudo –unreal)

What is Analysis of the algorithm?

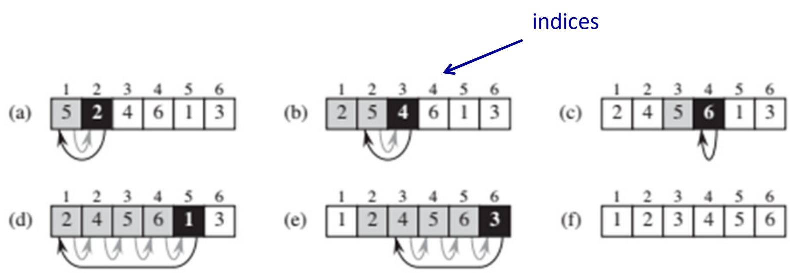
- **Analyzing** an algorithm means finding out
  - **Execution time-** arithmetic operations
  - **Memory requirement-** space requirement
- Since, execution time is machine dependent (depends on RAM, Processor speed, Cache, OS and etc.), we are interested in finding number of basic operations in terms of input size (n)
- The amount of memory needed is also counted in terms of input size n.

Insertion sort Analysis

- It works like **how we add a new card in left hand ?** at its appropriate location. We are inserting the card at correct place by comparing it with previous sorted cards.

Insertion sort Working

- Sort Array A=[5,2,4,6,1,3]



Ref- introduction to algorithms- By Cormen et.al

Insertion sort Algorithm

A.length=n=6

```
INSERTION-SORT(A)
for j = 2 to A.length
    key = A[j]        // j th element to be placed at its proper place
    // Insert A[j] into the sorted sequence A[1 . . j - 1].
    i = j - 1
    while i > 0 and A[i] > key
        A[i + 1] = A[i]   } shift item A[i] at A[i+1]
        i = i - 1         }
    A[i + 1] = key    Copy key at its proper location in A
```

Insertion sort time analysis

INSERTION-SORT(A)	*cost*	*times*
1 **for** $j = 2$ **to** $A.length$	c_1	n
2 $\quad key = A[j]$	c_2	$n-1$
3 $\quad$ // Insert $A[j]$ into the sorted sequence $A[1..j-1]$.	0	$n-1$
4 $\quad i = j-1$	c_4	$n-1$
5 $\quad$ **while** $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
6 $\qquad A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
7 $\qquad i = i-1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
8 $\quad A[i+1] = key$	c_8	$n-1$

c_i – is the execution time needed for line i

t_j – denote the number of times the while loop runs in line 5 for that value of j

$$\sum_{j=2}^{n} t_j = t_2 + t_3 + \ldots + t_n$$

Insertion sort time analysis

- The *running time* of insertion sort *T(n)* is then

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1) .$$

- For **the best case** (A is sorted) the while loop will run once for each value of j, thus $t_j=1$. So the T(n) can be

$$\begin{aligned} T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1) \\ &= (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8) . \end{aligned}$$

- The above formula is in the type ***an+b,*** where a and b are constants, which is a *linear function*

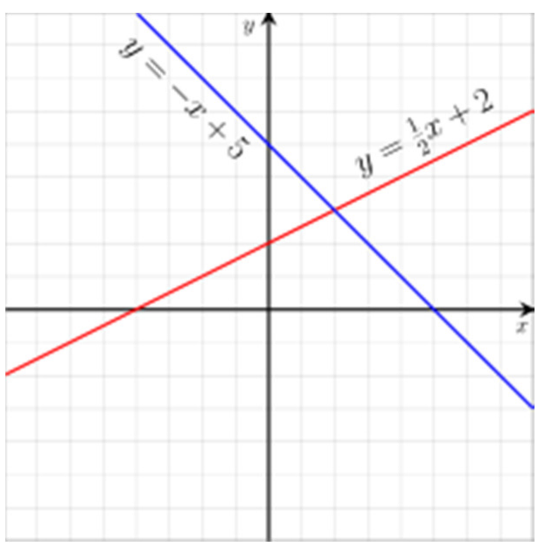
What is a linear function?

- A **linear function** is an algebraic equation in which each term is either a constant or the **product of a constant and (the first power of) a single variable**.
- Its graph is a line of form ***y=mx+c***, for constants m and c.
- Mathematically, algebraic equation that satisfy superposition theorem/principle

For constants a and b, and variables x and y function f is linear iff,

$$f(ax+by)=a\cdot f(x)+b\cdot f(y)$$

Graph of linear function



Insertion sort time analysis

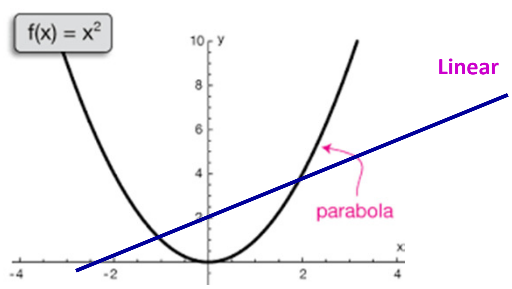
- **The worst case** is, A is in decreasing order, we must compare each element A[j] with each element of A[1,..,j-1] so tj=j. Using $\sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1$ $\quad \sum_{j=2}^{n}(j-1) = \frac{n(n-1)}{2}$

Worst case running time can be

$$\begin{aligned} T(n) &= c_1 n + c_2(n-1) + c_4(n-1) + c_5\left(\frac{n(n+1)}{2} - 1\right) \\ &\quad + c_6\left(\frac{n(n-1)}{2}\right) + c_7\left(\frac{n(n-1)}{2}\right) + c_8(n-1) \\ &= \left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2}\right)n^2 + \left(c_1 + c_2 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8\right)n \\ &\quad - (c_2 + c_4 + c_5 + c_8) . \end{aligned}$$

Worst case running time is in the form $an^2 + bn + c$,
for constants a, b and c, which is quadratic in nature

Quadratic function



$$Linear --- f(x) = ax + b$$

$$Quadratic -- f(x) = ax^2 + bx + c$$

Insertion sort time analysis

- The **Average case running time**.
- In this case half of the elements in **A [1,…j-1]** are less than A[j] and remaining are greater than A[j], Thus, tj=j/2. It also turns out to be a **Quadratic function**.
- We generally interested in "*rate of growth*" or "*order of growth*" of running time functions.

Thus, from $an^2 + bn + c$, we remove the lower ordered term and eliminate even the constant of highest ordered term and use it as n^2. we write that insertion sort has worst case running time of $\Theta(n^2)$. Read it as Theta of n square.

Linear searching from unsorted array

- We have an unsorted array of n elements and we want to search a key=x. Find the best case, average case and worst case running time, assuming that k is constant time required for a single comparison.

A=[4,6,1,3,8,4] and x=4

Asymptotic notations

(Batchmann-Landau notation, 1894)-wikipedia

- **Asymptotic** – means approaching a **value** or **curve** arbitrary closely. (also called limiting behavior)
- We are interested in understanding **3 notations**

1. Θ – Theta - provide asymptotic tight bounds (lower and upper both)
2. O - Big Oh - provide asymptotic upper bound
3. Ω - Big Omega - provide asymptotic lower bound

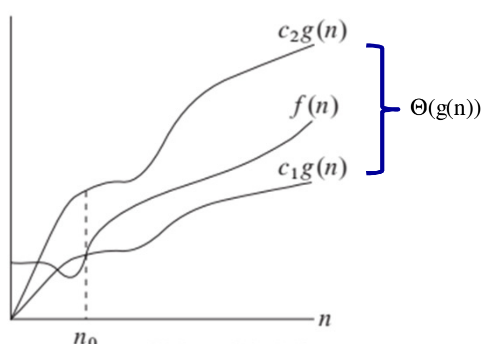
Asymptotic notations

$\Theta(g(n)) = \{f(n) :$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$.[1]

Read it as $\Theta(g(n))$ is set of all the functions f(n) such that.....

We will write $f(n) \in \Theta(g(n))$ as $f(n) = \Theta(g(n))$

n is input size of Algorithm



$f(n) = \Theta(g(n))$

Asymptotic notations

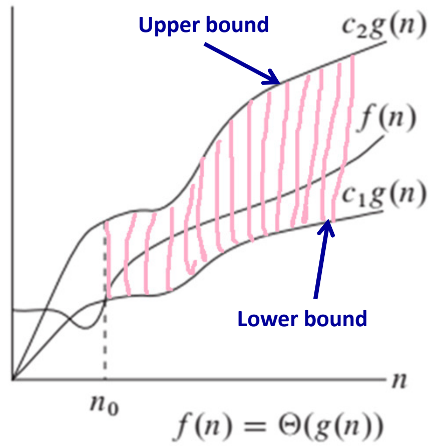
$f(n) = \Theta(g(n))$ means $f(n)$ can be any function that lie in marked pink region. There can be many such functions

$f(n)$ must be greater $\geq c_1 g(n)$ and $\leq c_2 g(n)$ for all $n \geq n_0$

For insertion sort running time

$T(n) = an^2 + bn + c$

Thus, $T(n) = \Theta(n^2)$



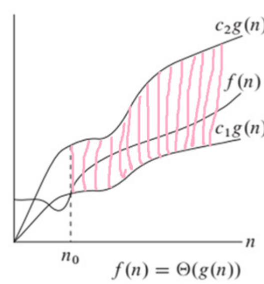
Asymptotic notations

Following all the functions are belonging to $\Theta(n^2)$

$f(n) = 8n^2 + 3n + 4$

$f(n) = 106n^2 + 300n + 56$

$f(n) = n^2 + 33n + 400$



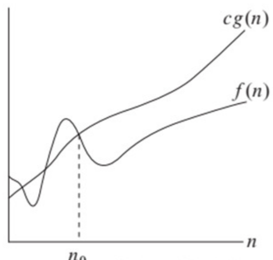
We say that $g(n)$ is asympotically tight bound for $f(n)$ for sufficiently large values of n and $f(n)$ is non-negative

Solve problems from notes- EX-1

Asymptotic notation- Big-Oh

$$O(g(n)) = \{f(n) : \text{ there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\} .$$

Big-Oh (O) represents asymptotic upper bound (may not be the tight)

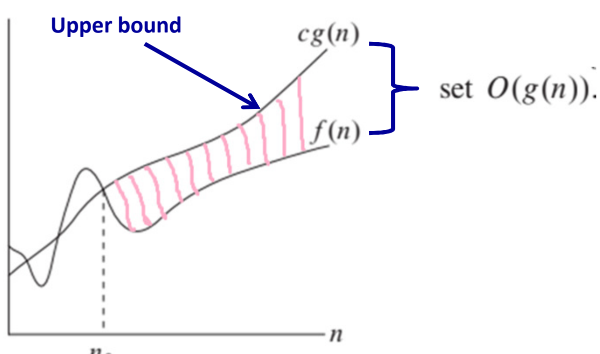


$f(n) = O(g(n))$

Big –O stands for “ordunung” in German, means order of approximation or “order of”

We use O-notation to give an upper bound on a function, to within a constant factor. Figure shows the intuition behind O-notation. For all values n at and to the right of n_0, the value of the function $f(n)$ is on or below $cg(n)$.

Asymptotic notation- Big-Oh

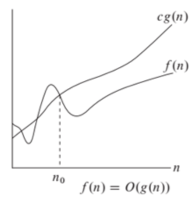


$f(n) = O(g(n))$

We write $f(n) = O(g(n))$ to indicate that a function $f(n)$ is a member of the set $O(g(n))$. Note that $f(n) = \Theta(g(n))$ implies $f(n) = O(g(n))$, since Θ-notation is a stronger notion than O-notation. Written set-theoretically, we have $\Theta(g(n)) \subseteq O(g(n))$. Thus, our proof that any quadratic function $an^2 + bn + c$, where $a > 0$, is in $\Theta(n^2)$ also shows that any such quadratic function is in $O(n^2)$.

Asymptotic notation- Big-Oh

Using O- notation , we can often describe the running time of an algorithm merely by inspecting its overall structure. E.g. The doubly nested loop structure of insertion sort Indicates that its worst case upper bound is *O(n^2)*.



generalization

Since O-notation describes an upper bound, when we use it to bound the worst-case running time of an algorithm, we have a bound on the running time of the algorithm on every input—the blanket statement we discussed earlier. Thus, the $O(n^2)$ bound on worst-case running time of insertion sort also applies to its running time

Asymptotic notation- Big-Oh

input of size n. When we say "the running time is $O(n^2)$," we mean that there is a function $f(n)$ that is $O(n^2)$ such that for any value of n, no matter what particular input of size n is chosen, the running time on that input is bounded from above by the value $f(n)$. Equivalently, we mean that the worst-case running time is $O(n^2)$.

```
for  (i=1 to n)
     for (j=1 to n)
         for (k=1 to n)
           ……….
         end
      end
end
```

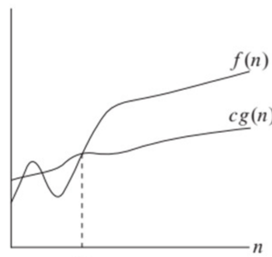
$\Rightarrow$ $O(n^3)$

Solve problems from notes- EX-2

Asymptotic notation- Big-Omega

Just as O-notation provides an asymptotic *upper* bound on a function, Ω-notation provides an ***asymptotic lower bound***. For a given function $g(n)$, we denote by $\Omega(g(n))$ (pronounced "big-omega of g of n" or sometimes just "omega of g of n") the set of functions

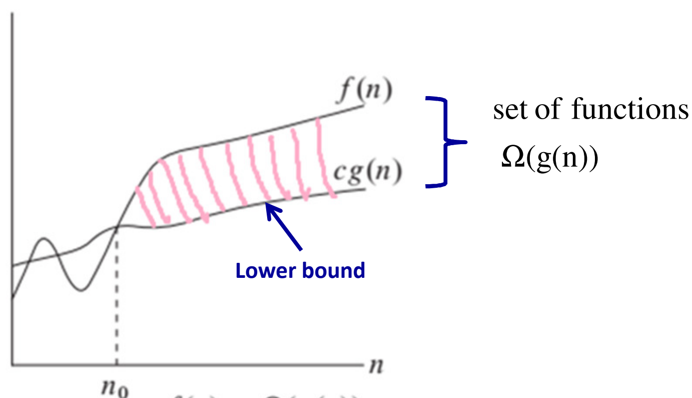
$$\Omega(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\} .$$



$f(n) = \Omega(g(n))$

Figure shows the intuition behind Ω-notation. For all values n at or to the right of n_0, the value of $f(n)$ is on or above $cg(n)$.

Asymptotic notation- Big-Omega



$f(n) = \Omega(g(n))$

For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. ■

Asymptotic notation- Big-Omega

When we say that the *running time* (no modifier) of an algorithm is $\Omega(g(n))$, we mean that *no matter what particular input of size n is chosen for each value of* n, the running time on that input is at least a constant times $g(n)$, for sufficiently large n. Equivalently, we are giving a lower bound on the best-case running time of an algorithm. For example, the best-case running time of insertion sort is $\Omega(n)$, which implies that the running time of insertion sort is $\Omega(n)$.

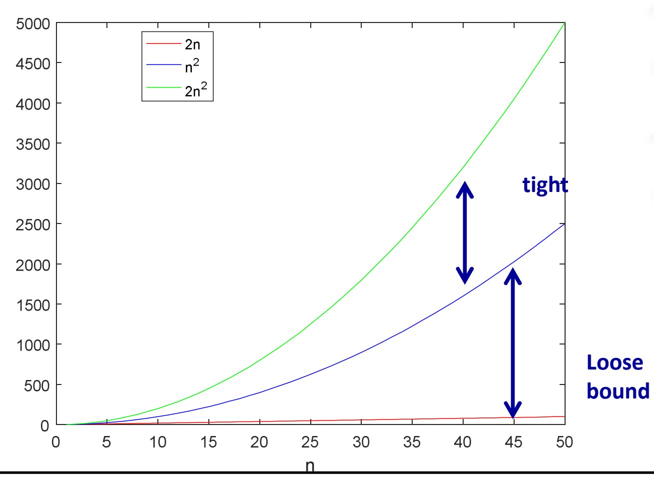
contradictory, however, to say that the *worst-case* running time of insertion sort is $\Omega(n^2)$, since there exists an input that causes the algorithm to take $\Omega(n^2)$ time.

Solve problems from notes- EX-3 and 4

Asymptotic notation- little - o

The bound $2n^2 = O(n^2)$ is asymptotically tight

but $2n = O(n^2)$ is not tight



Asymptotic notation- little - oh

The asymptotic upper bound provided by O-notation may or may not be asymptotically tight. The bound $2n^2 = O(n^2)$ is asymptotically tight, but the bound $2n = O(n^2)$ is not. We use o-notation to denote an upper bound that is not asymptotically tight. We formally define $o(g(n))$ ("little-oh of g of n") as the set

$$o(g(n)) = \{ f(n) : \text{for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le f(n) < cg(n) \text{ for all } n \ge n_0 \} .$$

The definitions of O-notation and o-notation are similar. The main difference is that in $f(n) = O(g(n))$, the bound $0 \le f(n) \le cg(n)$ holds for *some* constant $c > 0$, but in $f(n) = o(g(n))$, the bound $0 \le f(n) < cg(n)$ holds for *all* constants $c > 0$.

Asymptotic notation- little - oh

- Some authors define it using limit

$$f(n) = o(g(n))$$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 .$$

If f(n) = n^2 and g(n) = n^3 then check whether f(n) = o(g(n)) or not.

$$\lim_{n \to \infty} \frac{n^2}{n^3}$$

$$= \lim_{n \to \infty} \frac{1}{n}$$

$$= \frac{1}{\infty}$$

$$= 0$$

Asymptotic notation- little – oh provides loose upper bound

Which bound we generally refers????

Mostly we are interested in computing worst case upper bound *O(g(n))* to compare algorithms and call it as worst case **time complexity**

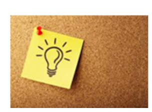
- Complexity-Refers- **quality of difficulty** or **complications, quality of being complex**
- **Time complexity????**

Computing time complexities

```
int Add(n, m)
{
   int sum=0;
   sum=n+m;
  return sum;
}
```

Total operations needed
T= 1 Addition
+ 1 assignment
+1 return
T=3=constant number of operations for every possible values of *n* and *m*.

Thus, the time complexity is O(1)



Computing time complexities

Find an element in array A of size n=5

```
int Search(m)
{
   int i;
 for (i=I to  n)
     if (A[i]==m)
        Break;
     return i;
 end
}
```

Total operations needed in worst case

T= n comparisons

+ n assignments in for loop

+1 return

T=2n+1

Thus, the time complexity is O(n)

Computing time complexities

Addition of two matrices of size (n x n)

```
MatAdd (A,B,C)
{
   int I, j;
 for (i=I to  n)
       for (j=I to  n)
              C(I,j)=A(I,j)+B(I,j);
       end

end
}
```

Total number of operations needed

T= (n x n) assignments for loops i, and j

+ (nxn) additions in loop A+B

+(n x n) assignments in C=A+B

T=3*(nx n)

$$T = 3n^2$$

$$Thus, O(n^2)$$

Even if we ignore assignment operations it has the same time complexity

Addition of two matrices of size (m x n) is thus, O(m x n)

Find the time complexity

- *Pseudocode*:

```
list_Sum(A,n) //A->array and n->elements
{ total =0
for i=0 to n-1
   sum = sum + A[i]
return sum}
```

O(n)

DO THE
WORK

```
for (int i = 1; i <=m; i += c) {
// some work
 }
for (int i = 1; i <=n; i += c) {
// some work
 }
```

If m!= n O(m+n)
If m=n O(2n)=O(n)

Computing time complexities

```
SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
```

-Innermost loop with k runs n times
- In this loop we require n additions and n multiplications i.e (n+n)
- This loop runs up to (nxn) times for above two loops
- Thus, the total number of operations needed
- T=(nxn)*(n+n)

$$T = (n \times n) * (n + n) = n^3 + n^3 = 2n^3$$

$$Thus, \text{is } O(n^3)$$

What is space complexity of an algorithm?

- The **space complexity** of an algorithm or a computer program is the amount of memory space required to solve an instance of the computational problem as a function of the input (size). It is the memory required by an algorithm to execute a program and produce output.- Wikipedia
- An algorithm/ Program need memory (main/RAM) for
  - Variables
  - Input and output data
  - Program stack for function calls (Auxiliary memory)
  - Heap for dynamic allocations
  - Instructions

What is space complexity of an algorithm?

- The space complexity is also expressed asymptotically in big O-h notation
- Space complexity is computed using

  input size+ **auxiliary memory (additional)** required

Computing space complexity

```
int Add(n, m)
{
   int sum=0;
   sum=n+m;
  return sum;
}
```

-We need to store 3 integers *n*, *m* and *sum*.
- If each integer requires 4 bytes to store , hence we need
- 3X4=12 bytes+ some constant auxiliary memory (k)
- Thus, total space needed
- ***S=(12 +k) bytes*** = constant, irrespective of any values of *n* and m
- Thus, space complexity is *O(1)* i.e constant

Computing space complexity

Find an element in array A of size n=5

```
int Search(m)
{
   int i;
 for (i=I to  n)
     if (A[i]==m)
      break
end
     return i;

}
```

- We need to store *n* integers in A and thus need= 4xn bytes
- 4 bytes are needed to store input *m*
- 4 bytes to store integer *i*
- Thus, total memory needed is
- **S=(4n+8) byes**
- Thus, the *space complexity* is **O(n)**

Computing space complexity

```
SQUARE-MATRIX-MULTIPLY(A, B)
1  n = A.rows
2  let C be a new n × n matrix
3  for i = 1 to n
4      for j = 1 to n
5          c_ij = 0
6          for k = 1 to n
7              c_ij = c_ij + a_ik · b_kj
8  return C
```

- 4 bytes for storing n of line 1
- 4*(nxn) bytes for storing matrix C in line2
- 4*(nxn) bytes for storing matrix A
- 4*(nxn) bytes for storing matrix B
- 4x3=12 bytes needed to store loop counters ***i, j, k***
- Thus, *total memory* needed *S* can be computed as

$$S = 4 + 12 * (nxn) + 12$$

$$S = 12n^2 + 16$$

Thus, space complexity is $O(n^2)$

- https://www.youtube.com/watch?v=yOb0BL-84h8
- Space complexity

Computing space complexity

- **Iterative version of factorial**

```
Int factorial (int n)
{
   int  i, fact = 1;
    for ( i = 1; i <= n; i++)
        fact = fact * i;
   return fact;
}
```

- We need 4 bytes for saving ***fact***
- 4 bytes for storing ***i***
- 4 bytes for storing **n**
- Some constant bytes **K**, as auxiliary space for initializing for loop and return statement
- Thus, total space needed
- **S=12 bytes+ Auxiliary space(K)**
- **S=constant bytes**, irrespective of **value of n**

Thus, space complexity is O(1)

Yannis- binomial coefficients computation- Iteration or recursion?

Computing space complexity

- **Recursive version of factorial**

```
int fact(int n)
{
   if(n<=1)
    return(1);
else
   // recursion
    return(n*fact(n-1));
}
```

fact(5) call stack

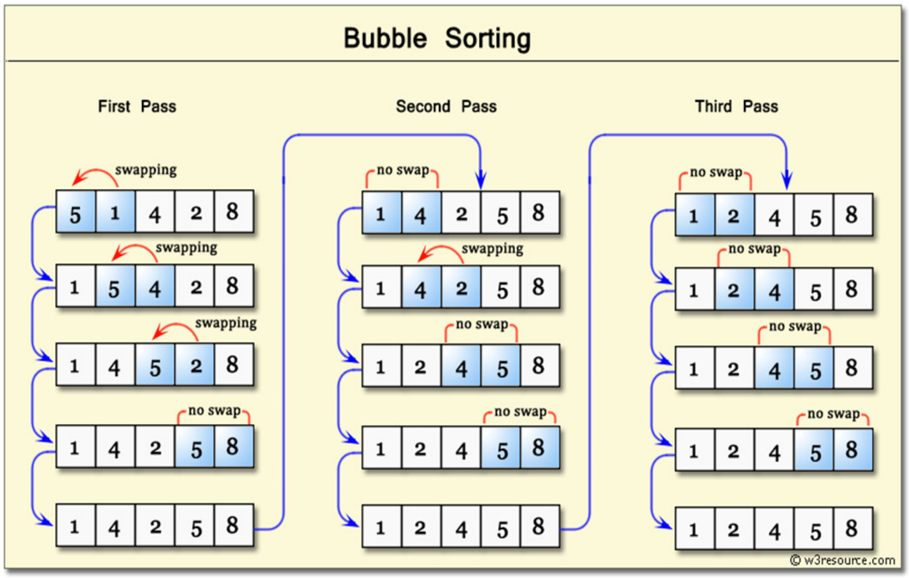
1
2*fact(1)
3*fact(2)
4*fact(3)
5*fact(4)

-We need some constant K bytes for the <u>**stack element**</u> for each call and have n calls
- 4Bytes for storing value of n

Thus total space **S = input size + Auxiliary space**
= 4 bytes+ (K*n)

Hence, space complexity is O(n). Input size don't have direct impact on space complexity.

Time complexity of bubble sort



Bubble sort Time Analysis

```
integer i, j,n;
n=A.length (Array  length )
for (i=0;i<n;i++)
   for(j=0;j<n-i-1;j++)
          if(A[j]>A[j+1])
              temp=A[j];
              A[j]=A[j+1];
              A[j+1]=temp;
          endIf
     endFor
endFor
```

Computing number of operations
-for each value of i and j we need
Comparison=1
Assignments=3
Add=3
Total=3+3+1=7

For i=0 inner loop runs upto (n-1) times. Since outer loop runs for n times. Total work done is n*(n-1). Thus, the order of operations required will be, roughly, 7*n^2. Hence it is O(n^2)

```
integer i, j, n;
n=A.length (Array  length )
for (i=0;i<n;i++)
   for(j=0;j<n-i-1;j++)
        if(A[j]>A[j+1])
            temp=A[j];
            A[j]=A[j+1];
            A[j+1]=temp;
        endIf
    endFor
endFor
```

For n=5 Analysis

Outer loop value i	Inner loop runs up to
i=0	(n-1)=4
i=1	(n-2)=3
i=2	(n-3)=2
i=3	(n-4)=1
i=4	(n-5)=0
Max- n times	Max-(n-1) times

Rate of growth =n*(n-1)

Total operations=7(n-1)+7(n-2)+………+7(n-4)+…7x1

Bubble sort requires maximum (n-1) passes

Modified Bubble sort

```
integer i, j, n,swap;
n=A.length (Array  length )
for (i=0;i<n;i++)
    swap=0;
   for(j=0;j<n-i-1;j++)
        if(A[j]>A[j+1])
            temp=A[j];
            A[j]=A[j+1];
            A[j+1]=temp;
            swap=1;
        endIf
    endFor
   if (swap==0)
    break;
endFor
```

Worst case:- Array requires all the **(n-1)** passes,
Thus, the order of work done is **n*(n-1)**=**n^2-n**.
Thus, it is **O(n^2)**

Best case:- Array (already sorted) requires only **1** pass,
thus the order of work done is **1x (n-1).**
Thus, it is **O(n)**

Average case:- Array requires half of the passes (n-1)/2,
thus the order of work done is **n x ((n-1)/2).**
Thus, it is **O(n^2)**

Find the space complexity of both bubble sorts

```
integer i, j,n;
n=A.length (Array  length )
for (i=0;i<n;i++)
   for(j=0;j<n-i-1;j++)
         if(A[j]>A[j+1])
             temp=A[j];
             A[j]=A[j+1];
             A[j+1]=temp;
         endIf
     endFor
endFor
```

A

```
integer i, j, n,swap;
n=A.length (Array  length )
for (i=0;i<n;i++)
     swap=0;
   for(j=0;j<n-i-1;j++)
         if(A[j]>A[j+1])
             temp=A[j];
             A[j]=A[j+1];
             A[j+1]=temp;
             swap=1;
         endIf
     endFor
   if (swap==0)
     break;
endFor
```

B

Compare the space needed for both the versions

Time memory trade off, important principle in computer science

Find total number Add, Mult, and Assignment operations needed

SQUARE-MATRIX-MULTIPLY(A, B)

$n = A.rows$
let C be a new $n \times n$ matrix
for $i = 1$ **to** n
  for $j = 1$ **to** n
    $c_{ij} = 0$
    for $k = 1$ **to** n
      $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$
return C

Recurrence relations

- Running time of many recursive algorithms is, naturally, written by using recurrence relations.
- Recurrence is equation or inequality that describes a function in terms of its value on smaller inputs.
- E.g. Running Time T(n) of factorial method is given by

```
Int fact(int n)
{
    if(n<=1)
      return  1;
    Else
      return (n*fact(n-1));
}
```

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n-1)+1, & \text{if } n > 1 \end{cases}$$

Recurrence relations

```
void test(int n)
{
    if(n>0)
    {  printf(n/2);
       printf(n);
       test(n-1);
    }
}
```

$$T(n) = \begin{cases} 1 & \text{if } n = 0 \\ T(n-1)+1, & \text{if } n > 0 \end{cases}$$

Recurrence relations-QuickSort-Bestcase

l-low and h-high index

```
void QuickSort(int l, int h)
{
   if(l<h)
   {
      j=partition(l,h);        O(n)
      QuickSort(l, j);         O(n/2)
      QuickSort(j+1, h);       O(n/2)
   }
}
```

$$T(n) = \begin{cases} 1 & \text{if } l = h \\ 2T(n/2) + cn, & \text{if } l \neq h \end{cases}$$

Best case- partitions are highly balanced and has nearly (n/2) elements each.

Recurrence relations

- There are **3 approaches** to solve the recurrence relations, for obtaining the **asymptotic bounds** on the solutions (Time complexity)
- **1. Substitution method**
- **2. Recursion tree method**
- **3. Master method (theorem)**

Recurrence relations- Substitution method

- In this method, we **substitute** the value of a term in **terms of smaller input size and guess the form of solution** and using **induction** find the constants and show that solution works.

```
Fact(n)
{   if (n<=1)
        return 1;
Else
      return n*fact(n-1);
}
```

$$T(n)=\begin{cases} 1 & \text{if } n=1 \\ T(n-1)+1, & \text{if } n>1 \end{cases}$$

Also called as back-substitution

Recurrence relations- Substitution method

$$\begin{aligned} T(n) &= T(n-1)+1 \\ &= (T(n-2)+1)+1 \\ &= T(n-2)+2 \\ &= (T(n-3)+1)+2 \\ &= T(n-3)+3 \\ &= \ldots\ldots \\ &= \underline{T(n-(n-1))}+(n-1) \\ &= T(1)+(n-1) = 1+n-1 = n \\ &\Rightarrow O(n) \end{aligned}$$

To make this T(1)

solution

$T(n) = O(n)$

Prove by induction that $T(n) \le c * n$

Using induction

1. Assume that it is true for $T(1)$
2. Assume it is true for some n
3. prove that it is true for $c*n$

$$
\begin{aligned}
T(n) &= T(n-1)+1 \\
&\le T(cn\text{-}1)+1 \quad \textit{putting } n = cn \\
&\le (T(cn\text{-}2)+1)+1 \\
&\le T(cn\text{-}2)+2 \\
&\ldots\ldots \\
&\le T(cn\text{-}(cn\text{-}1))+(cn\text{-}1) \\
&\le T(1)+(cn\text{-}1) \\
&\le 1+cn\text{-}1 \\
&\le c*n \quad \textit{hence proved}
\end{aligned}
$$

Recurrence relations- Substitution method

Binary search

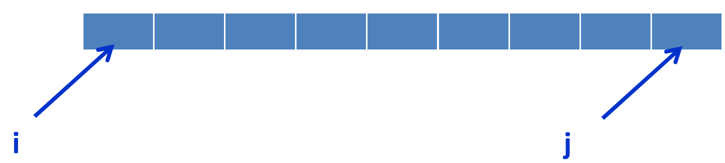
```
BS(a, i, j, x)
{ mid=(i+j)/2;
  if (a[mid]==x)
     return mid;
Else
   if (a[mid]>x)
      BS(a,i,mid-1,x);
   else
      BS(a,mid+1,j,x);
}
```

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + C & \text{if } n > 1 \end{cases}$$

C – constant time needed for comparison and computing mid can be taken as 1

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 1 & \text{if } n > 1 \end{cases}$$

Array a



Recurrence relations- Substitution method

$$\begin{aligned} T(n) &= T(n/2) + 1 \\ &= (T(n/4) + 1) + 1 \\ &= 2 + T(n/4) \\ &= 2 + (T(n/8) + 1) \\ &= 3 + T(n/8) \\ &\ldots\ldots\ldots \\ &= k + T(n/2^k) \qquad \text{max } \textit{vaue of k can be } \log n \\ &= \log n + T(n/2^{\log n}) \\ &= \log n + T(1) \\ &= \log n \\ &\Rightarrow O(\log n) \end{aligned}$$

Recurrence relations- Substitution method

To prove that our guess is correct $T(n) = O(\log n)$

we have to prove that $T(n) \le c * \log n$ using induction

$T(n) = T(n/2) + 1$ known, since $T(n/2) \le c * \log n/2$

$$\le c * \log n/2 + 1$$

$$= c * (\log n - \log 2) + 1$$

$$= c * \log n - c * \log 2 + 1$$

$$\boxed{T(n) \le c * \log n}$$

Recurrence relations- Substitution method

Solve recurrence relation and find time complexity of recursive algorithm

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/2) + n & \text{if } n > 1 \end{cases}$$

Recurrence relations- Recursion tree method

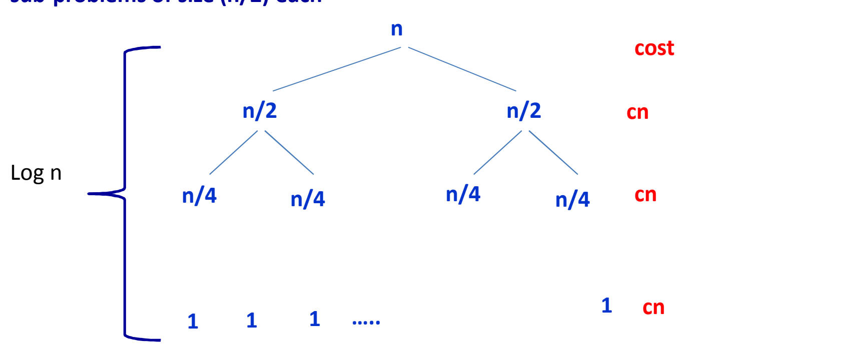
- ***Recursion tree***
- It is a **diagrammatic approach** of finding **asymptotic bounds**.
- Each node represents size of a **problem**/**single sub-problem** somewhere in the set of recursive function calls.
- For solving a given **sub-problem** we also need to **pay some cost.**
- We sum the costs of all the sub-problems within each level of the tree to obtain the **per level costs**.
- The **Total cost** required to solve that problem is then, sum of all the **per level costs**.
- From **Total cost** we can infer asymptotic bounds.

Recursion tree method

$$T(n) = 2T(n/2) + cn$$

A problem of size n is divided into 2 sub-problems of size (n/2) each

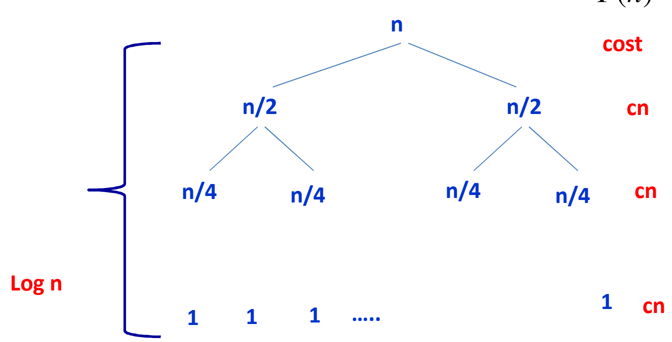
The cost of dividing and combining is cn



let k be the max height of tree where size of each problem is 1, thus $\left(n/2^k\right) = 1$, so $k = \log_2 n$

Recursion tree method

$$T(n) = 2T(n/2) + cn$$



Total cost= sum of costs at all the levels=cn + cn + cn +…log n times
= c n log n
=O(n logn)

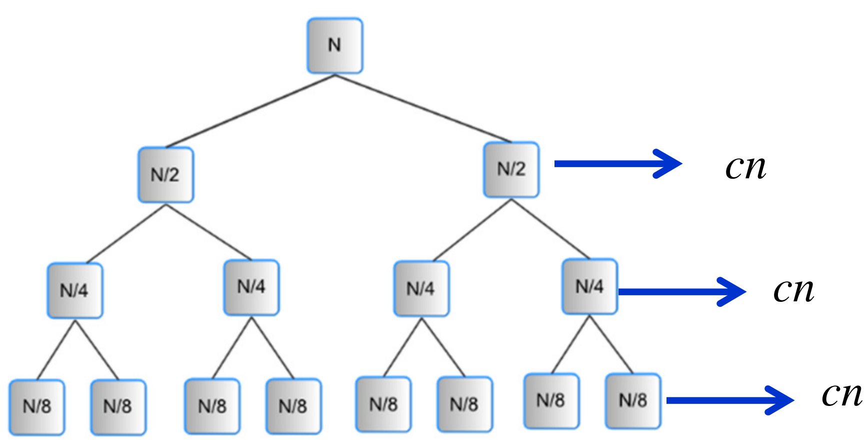
$T(n) = 2T(n/2) + cn$

is O(n logn) or θ(n logn)

Recursion tree method –merge sort

$T(n) = 2T(n/2) + cn$

Time needed to combine



Since we are dividing the problem into 2, tree will have height of log2 to the base 2

https://www.youtube.com/watch?v=C4JjXc0htp0

Recursion tree method –merge sort

Since there are $\log_2 n$ levels are there and we need cn time to each layer, thus total running time will be

$$T(n) = cn\log_2 n$$

$$\therefore T(n) = O(n\log n)$$

Recursion tree method

$$T(n) = 3T(\lfloor n/4 \rfloor) + cn^2$$

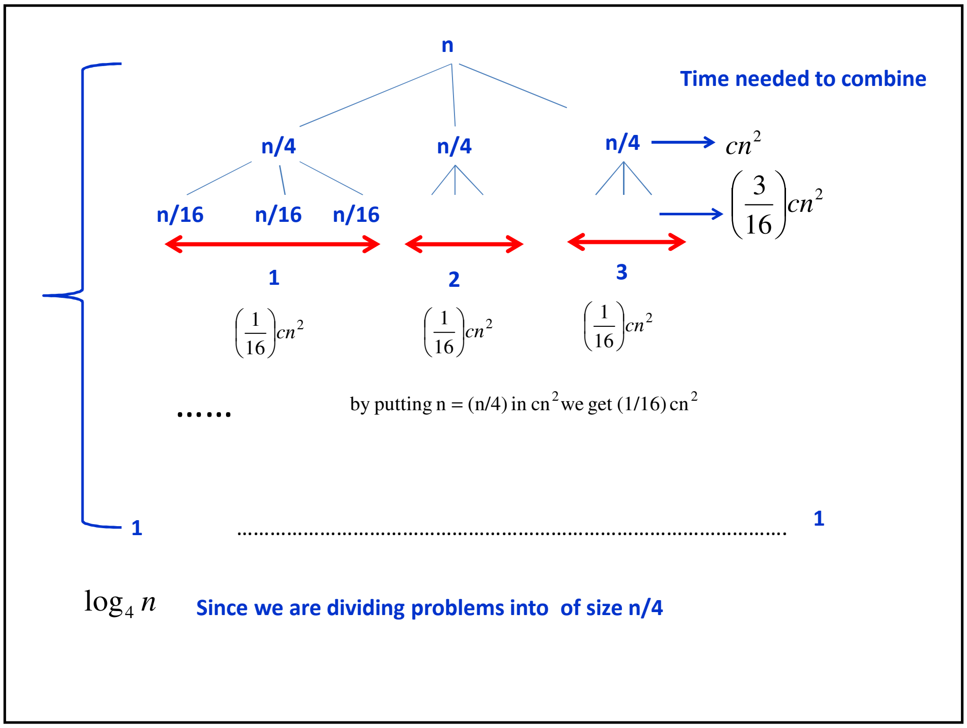
We can ignore **floor** operation since it is insignificant in finding time complexity again when n is divisible by 4, **floor(n/4)=n/4**

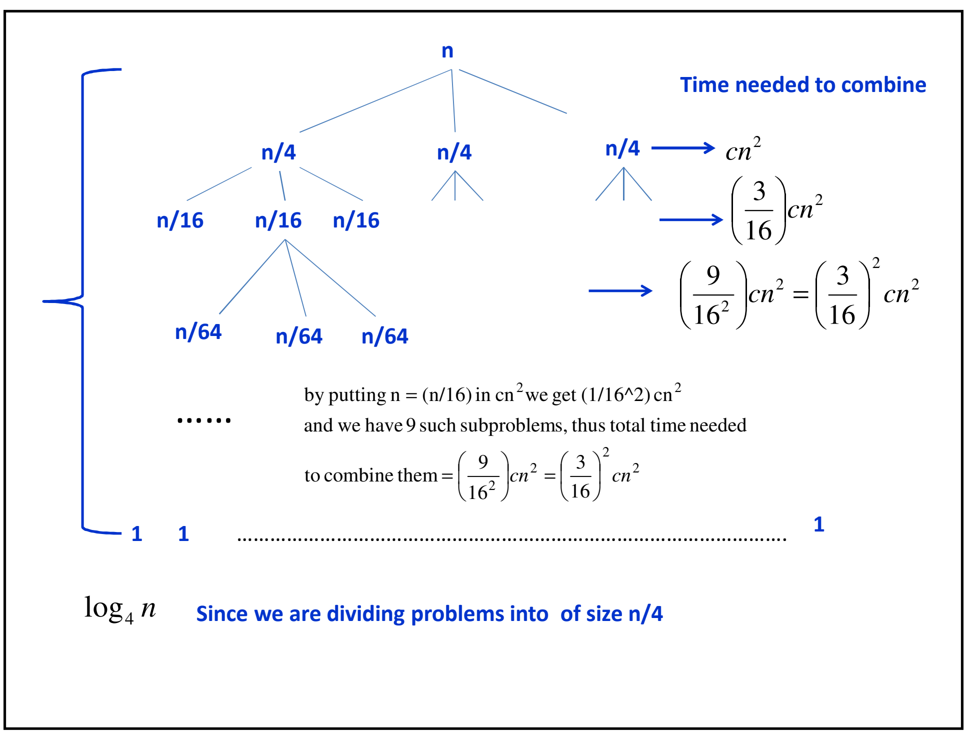
$$T(n) = 3T(n/4) + cn^2$$

Dividing a problem of size n into 3 sub-problems of size n/4 each

We need this much time to combine sub problems

https://www.youtube.com/watch?v=JPAA1FbM7jk


n
Time needed to combine
n/4
n/4
n/4
cn^2
n/16
n/16
n/16
$\left(\frac{3}{16}\right)cn^2$
1
2
3
$\left(\frac{1}{16}\right)cn^2$
$\left(\frac{1}{16}\right)cn^2$
$\left(\frac{1}{16}\right)cn^2$
……
by putting n = (n/4) in cn^2 we get (1/16) cn^2
1
1
$\log_4 n$
Since we are dividing problems into of size n/4


n
Time needed to combine
n/4
n/4
n/4
cn^2
n/16
n/16
n/16
$\left(\frac{3}{16}\right)cn^2$
n/64
n/64
n/64
$\left(\frac{9}{16^2}\right)cn^2 = \left(\frac{3}{16}\right)^2 cn^2$
……
by putting n = (n/16) in cn^2 we get (1/16^2) cn^2
and we have 9 such subproblems, thus total time needed
to combine them = $\left(\frac{9}{16^2}\right)cn^2 = \left(\frac{3}{16}\right)^2 cn^2$
1
1
1
$\log_4 n$
Since we are dividing problems into of size n/4

Sum of time required to combine solutions at each level will be

$$T(n) = cn^2 + \left(\frac{3}{16}\right)cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \left(\frac{3}{16}\right)^3 cn^2 \ldots\ldots\ldots\ldots + 1$$

$$= cn^2 \left\{1 + \left(\frac{3}{16}\right) + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \ldots\ldots\ldots \right\}$$

Geometric series

$$1 + r + r^2 + r^3 + \ldots\ldots = \frac{1}{1-r} \quad for\ r < 1 \qquad r = (3/16) < 1$$

$$T(n) = cn^2 \left(\frac{1}{1-(3/16)}\right)$$

$$= cn^2 (16/13)$$

$$T(n) = O(n^2)$$

Recursion tree method –Binary search

$$T(n) = T(n/2) + 1$$

	tree	Time needed for computing mid and comparison
	n	1
	|	|
	n/2	1
	|	|
	n/4	1
Log n	|	|
	n/8	1
	|	|
	.	.
	.	.
	1	.

T(n)=1xlog n

Thus

T(n)=**O(logn)**

Master method/theorem

- This is the direct method of solving the recurrence relations by remembering 3 cases, of the form (**cookbook approach**)

$$T(n) = aT(n/b) + f(n)$$

$$where\ a \geq 1 \text{ and } b > 1 \text{ are constants and } f(n) \text{ is positive}$$

- The problem of size ***n*** is divided into ***a*** sub-problems of size **n/b**.
- The **a** sub-problems are solved recursively , each in time ***T(n/b)***
- The cost of **dividing** the problems and **combining** the results of the sub-problems is described by function ***f(n)***

Master method/theorem- case 1

$$T(n) = aT(n/b) + f(n)$$

$$where\ a \geq 1 \text{ and } b > 1 \text{ are constants and } f(n) \text{ is positive}$$

$\boxed{\text{Case1:}}$ if $f(n) = O(n^{\log_b a-\epsilon})$ for some constant $\epsilon > 0$

Then $\underline{T(n) = \Theta(n^{\log_b a})}$

In each of the case we are comparing $f(n)$ and $n^{\log_b a}$.

In case 1 $n^{\log_b a}$ is larger than $f(n)$

Master method/theorem- case 1 example

$T(n) = 9T(n/3) + n$

$a = 9, b = 3$ satisfies $a \geq 1$ and $b > 1$

check $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$

$\log_b a = \log_3 9 = 2$

$f(n) = O(n^{2-\epsilon})$ if $\epsilon = 1$ condition can satisfy

$f(n) = O(n)$ for $\epsilon = 1$

Thus, can conclude that $T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$

Means that T(n) is lower and upper bounded by **n^2**, i.e **T(n)=O(n^2)** and **T(n)=Omega(n^2)**

Master method/theorem- case 2

$T(n) = aT(n/b) + f(n)$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is positive

$\boxed{\text{Case2}}$: if $f(n) = \Theta(n^{\log_b a})$

Then $\underline{T(n) = \Theta(n^{\log_b a} \log n)}$

In this case we are comparing $f(n)$ and $n^{\log_b a}$.

In case 2: $n^{\log_b a}$ is same as $f(n)$

Master method/theorem- case 2 example

$T(n) = T(2n/3) + 1$

where $a = 1,\ b = 3/2, f(n) = 1$; thus condition $a \geq 1$ and $b > 1$ satisfied

$n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$, since log 1 to any base is 0.

$f(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_{3/2} 1}) = \Theta(n^0) = \Theta(1)$

$1 = \Theta(1)$ is satisfied

Thus,

$T(n) = \Theta(n^{\log_b a} \log n) = \Theta(n^0 \log n)$

$\boxed{T(n) = \Theta(\log n)}$

Master method/theorem- case 3

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is positive

$\boxed{\text{Case3}}$: if $f(n) = \Omega(n^{\log_b a + \epsilon})$ for $\epsilon > 0$

and $af(n/b) \leq c \cdot f(n)$ for $c < 1$, and $\forall n$

then $T(n) = \Theta(f(n))$

Master method/theorem- case 3 example

$T(n) = 3T(n/4) + n\log n$

$a = 3,\ b = 4,\ f(n) = n\ \log\ n$

$\log_b a = \log_4 3 = 0.793$

$f(n) = n\log n\ = \Omega(n^{\log_b\ a+\in}) = \Omega(n^{0.793+\in})\ \text{for} \in > 0$

$\text{if we put} \in = 0.2, \text{then}\ (0.793 + 0.2) \approx 1$

$\therefore n\ \log n\ = \Omega(n^1) = \Omega(n) - \text{which is lower bound}$

Master method/theorem- case 3 example

checking for next condition

$af(n/b) \le c \cdot f(n)\ \ \forall n,\ c < 1$

$3\,f(n/4) \le c \cdot f(n)\ \ \forall n,\ c < 1$

$3 \cdot (n/4) \cdot \log(n/4) \le c \cdot f(n)\ \ \forall n,\ c < 1$

$(3/4) \cdot \text{n} \cdot \log\ (\text{n}/4) \le (3/4) \cdot n \cdot \log n \quad \text{for c} = 3/4$

thus $\ af(n/b) \le c \cdot f(n)$ is also satisfied.

Hence $T(n) = \Theta(f(n)) = \Theta(n\log n)$

Not all recurrences can be solved by master method

Proof of correctness of algorithms

Methods
1. **Loop invariants**
2. **Proof by counter example**

- In computer science, a **loop invariant** is a **property** of a program loop that is true before (and after) each iteration.
- It is a logical assertion (belief), sometimes checked within the code by an assertion. (wikipedia)
- Loop invariants characterizes the deeper purpose of the loop beyond the details of this implementation.
- They used to provide **correctness** of algorithm using loops.

Proof of correctness of algorithms

Finding sum of array A-using loop invariants

```
int SumArray( A, n)
{
   int i=0; sum=0;                              Empty array
                                                A[0,…,0]
   //sum will have addition of A[0,…,0]-initialization
    for i=1to n                                 Partial
          sum=sum + A[i];                       array
     // sum will have addition of A[1,…,i]
     end
   // sum will have addition of A[1,…,n]
return sum;                                     Total
}                                               array
```

Loop invariants

Proof of correctness of algorithms- sum of array A

Loop invariant:- At the start of iteration **i** of the loop, the variable **sum** should contain the sum of the numbers from the subarray A[1: (i-1)].

1. **initialization:**- At the start of the first loop the loop invariant states: 'At the start of the first iteration of the loop, the variable *sum* should contain the **sum of the numbers** from the subarray **A[0:0]**, which is an empty array. **The sum of the numbers in an empty array is 0, and this is what sum has been set to.**
2. **Maintenance:** Assume that the **loop invariant holds (true)** at the start of *iteration i.* Then it must be that **sum** contains the sum of numbers in subarray ***A[1: (i-1)].*** In the body of the loop we add **A[i]** to **sum**. Thus, at the start of **iteration i+1**, **sum** will contain the **sum of numbers in A[1: i], which is what we needed to prove.**

Proof of correctness of algorithms- sum of array A

3. Termination:

The **for**-loop terminates when i=n+1. Now the **loop invariant gives:** The variable **sum** contains the **sum of all numbers in complete array A[1: n].** This is exactly the value that the algorithm should output, and which it then outputs.

Conclusion-

Since, loop invariants are true for initialization, maintenance and termination, therefore the algorithm is **correct**.

Proof of correctness of algorithms- Max of array A

$$
\begin{aligned}
&\max = A[1];\\
&\text{for } i = 1 \text{ to } n\\
&\qquad \text{if } A[i] > \max\\
&\qquad\quad \max = A[i];
\end{aligned}
$$

Guess Loop invariants ???-

max is the maximum of A[1,….,i] for each value of i

Initialization:- For i=1, the scope of array is A[1,…,1]=A[1]=max, thus invariants holds

Maintenance:- loop invariants holds for k th iteration, for i=k, the max is maximum of A[1,…,k], thus invariants holds

Termination:- The loop execution terminates at i=n. loop invariants holds for i=n and the max is maximum of A[1,…,n], thus invariants holds here too.

Proof of correctness of algorithms

- **Proof by counter example**

$$\text{Prove or disprove} \quad \lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$$

$$\text{counter example: - } x = 1/2 \text{ and } y = 1/2$$

$$\lceil 1/2 + 1/2 \rceil = \lceil 1 \rceil = 1$$

$$\text{But } \lceil 1/2 \rceil + \lceil 1/2 \rceil = 1 + 1 = 2$$

$$\text{Hence algo is not correct since } \lceil x+y \rceil \neq \lceil x \rceil + \lceil y \rceil$$

Learn about loop invariants of insertion sort

Proof of correctness of algorithms

- **Proof by counter example**

Any integer is sum of squares of **two integers**
Counter example :- 3

3=1+2
3=1+1+1