Name:______________________________ Student Number:______________

READ ALL THE ITEMS ON THE COVER PAGE BEFORE YOU START THE EXAM

1. (a) (2.5 pts) Find the number of ways in which two A's, four B's, three C's, two D's and two F's can be distributed among 13 students.

Partitioning of letters among 13 students:

$$\underbrace{\binom{13}{2}}_{2\text{ A's}} \cdot \underbrace{\binom{11}{4}}_{4\text{ B's}} \cdot \underbrace{\binom{7}{3}}_{3\text{ C's}} \cdot \underbrace{\binom{4}{2}}_{2\text{ D's}} \cdot \underbrace{\binom{2}{2}}_{2\text{ F's}}$$

$$= \frac{13!}{2!\,\cancel{11!}} \cdot \frac{\cancel{11!}}{4!\,\cancel{7!}} \cdot \frac{\cancel{7!}}{3!\,\cancel{4!}} \cdot \frac{\cancel{4!}}{2!\,2!} \cdot 1 = \frac{13!}{2!\,4!\,3!\,2!\,2!}$$

$$= \frac{13.12.11.10.9.\overset{(2!)^3}{\cancel{8}}.7.\overset{3!}{\cancel{6}}.5.\overset{4!}{\cancel{4.3.2.1}}}{\cancel{2!.4!.3!.2!.2!}} = 13.12.11.10.9.7.5$$

(b) (2.5 pts) Show that

$$\sum_{r=0}^{n} \binom{n}{r} (a-1)^r = a^n$$

Recall: $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} . x^{n-r} . y^r$

Let $y = a-1$ and $x = 1$ in the Binomial Theorem:

$$(\underbrace{1+(a-1)}_{= a})^n = \sum_{r=0}^{n} \binom{n}{r} . (a-1)^r . \underbrace{1^{n-r}}_{= 1}$$

$$\Rightarrow a^n = \sum_{r=0}^{n} \binom{n}{r} (a-1)^r \;//$$

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2. In his pocket, a man has five coins, of which two are double headed (both sides are heads), one is double tailed (both sides are tails) and two are fair (normal) coins.

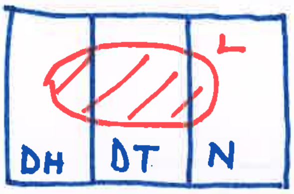
(a) (2.5 pts) Man shuts his eyes, chooses a coin at random and tosses it. What is the probability that the lower face of the coin is heads?

Define the events: DH := double headed $\perp$:= lower face is Heads

DT := double tailed

N := normal (fair)

$\to \perp = (\perp \cap DH) \cup (\perp \cap DT) \cup (\perp \cap N)$ disjoint union



$$\mathbb{P}(\perp) = \underbrace{\mathbb{P}(\perp \cap DH)}_{\parallel} + \underbrace{\mathbb{P}(\perp \cap DT)}_{\parallel} + \underbrace{\mathbb{P}(\perp \cap N)}_{\parallel}$$

$$= \underbrace{\mathbb{P}(DH)}_{\parallel} \cdot \underbrace{\mathbb{P}(\perp | DH)}_{\parallel} + \underbrace{\mathbb{P}(DT)}_{\parallel} \cdot \underbrace{\mathbb{P}(\perp | DT)}_{\parallel} + \underbrace{\mathbb{P}(N)}_{\parallel} \cdot \underbrace{\mathbb{P}(\perp | N)}_{\parallel}$$

$$= \frac{2}{5} \cdot 1 + \frac{1}{5} \cdot 0 + \frac{2}{5} \cdot \frac{1}{2} = \frac{6}{10}$$

Think about these.

(b) (2.5 pts) He opens his eyes and sees that the upper face of the coin is heads. What is the probability that the lower face is also heads?

Define a new event: $\mathcal{U}$:= upper face is heads.

we're asked to find $\mathbb{P}(\perp | \mathcal{U}) = ?$

$$\mathbb{P}(\perp | \mathcal{U}) = \frac{\mathbb{P}(\perp \cap \mathcal{U})}{\underbrace{\mathbb{P}(\mathcal{U})}} = \frac{\mathbb{P}(DH)}{\mathbb{P}(\mathcal{U})} = \frac{2/5}{6/10} = \frac{2}{3} \,//$$

similar to above

$$\mathbb{P}(\mathcal{U}) = \mathbb{P}(\mathcal{U} \cap DH) + \mathbb{P}(\mathcal{U} \cap DT) + \mathbb{P}(\mathcal{U} \cap N)$$

$$= \underbrace{\mathbb{P}(DH)}_{\parallel} \cdot \underbrace{\mathbb{P}(\mathcal{U} | DH)}_{\parallel} + \underbrace{\mathbb{P}(DT)}_{\parallel} \cdot \underbrace{\mathbb{P}(\mathcal{U} | DT)}_{\parallel} + \underbrace{\mathbb{P}(N)}_{\parallel} \cdot \underbrace{\mathbb{P}(\mathcal{U} | N)}$$

$$= \frac{2}{5} \cdot 1 + \frac{1}{5} \cdot 0 + \frac{2}{5} \cdot \frac{1}{2} = \frac{6}{10}$$

Think about these.

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3. A blood test has probability 0.95 of giving a positive result when applied to a person suffering from a certain disease and a probability 0.05 of giving a (false) positive when applied to a healthy person. It is estimated that) 0.1% of the population has the disease. Suppose that the test is administered to a randomly chosen person from this population.

(a) (2 pts) What is the probability that the test result will be positive?

$P :=$ the event that the test is positive

$H :=$ the event that the person is healthy

$\mathbb{P}(H) = \overbrace{1 - \mathbb{P}(H')}^{0.999}$

Given probabilities: $\mathbb{P}(P \mid H') = 0.95$, $\mathbb{P}(P \mid H) = 0.05$, $\mathbb{P}(H') = 0.001$

$$\mathbb{P}(P) = \mathbb{P}(P \cap H) + \mathbb{P}(P \cap H')$$

since $P = (P \cap H) \cup (P \cap H')$ disjoint union

$$= \underbrace{\mathbb{P}(H)}_{0.999} \cdot \underbrace{\mathbb{P}(P \mid H)}_{0.05} + \underbrace{\mathbb{P}(H')}_{0.001} \cdot \underbrace{\mathbb{P}(P \mid H')}_{0.95}$$

$$= 0.999 \times 0.05 + 0.001 \times 0.95$$

(b) (3 pts) What is the probability that the person is healthy given that the test result is negative?

we're asked to find $\mathbb{P}(H \mid P') = ?$

$$\mathbb{P}(H \mid P') = \frac{\mathbb{P}(P' \cap H)}{\mathbb{P}(P')}$$

From part (a): $\mathbb{P}(P') = 1 - \mathbb{P}(P) = 1 - (0.999 \times 0.05 + 0.001 \times 0.95)$

$$\mathbb{P}(P' \cap H) = \mathbb{P}(H) \cdot \underbrace{\mathbb{P}(P' \mid H)}_{=}$$

Recall: $\mathbb{P}(P \mid H) + \mathbb{P}(P' \mid H) = 1$

$$= \underbrace{\mathbb{P}(H)}_{=} \cdot \underbrace{(1 - \mathbb{P}(P \mid H))}_{=}$$

$$= 0.999 \times (0.95)$$

Hence $$\mathbb{P}(H \mid P') = \frac{0.999 \times 0.95}{1 - [0.999 \times 0.05 + 0.001 \times 0.95]}$$

4. There are two coins in an urn. One is fair, and the other is double-headed. Consider an experiment in which you select one of the coins at random and toss it independently three times. Let X denote the number of heads observed in this experiment. Determine the probability distribution $f(x)$ of X. Determine $E(X)$ and $Var(X)$.

X = # of heads observed. Range of $X = \{0, 1, 2, 3\}$

$$\text{Sample space} = \{\underbrace{HHH}_{X=3}, \underbrace{HHT, HTH, THH}_{X=2}, \underbrace{HTT, THT, TTH}_{X=1}, \underbrace{TTT}_{X=0}\}$$

we need to compute the probabilities by considering the type of the coin:

$F :=$ coin is fair

$D :=$ coin is double-headed

$$\mathbb{P}(HHH) = \underbrace{\mathbb{P}(HHH \cap F)}_{\shortparallel} + \underbrace{\mathbb{P}(HHH \cap D)}_{\shortparallel}$$

$$= \underbrace{\mathbb{P}(F)}_{\frac{1}{2}} \cdot \underbrace{\mathbb{P}(HHH \mid F)}_{\frac{1}{8}} + \underbrace{\mathbb{P}(D)}_{\frac{1}{2}} \cdot \underbrace{\mathbb{P}(HHH \mid D)}_{1}$$

$$= \frac{1}{2} \times \frac{1}{8} + \frac{1}{2} \times 1 = \frac{9}{16} \,//$$

For all the other $w \in \Omega$, we have

$$\mathbb{P}(w) = \underbrace{\mathbb{P}(w \cap F)}_{\shortparallel} + \underbrace{\mathbb{P}(w \cap D)}_{\shortparallel}$$

$$= \underbrace{\mathbb{P}(F)}_{\frac{1}{2}} \cdot \underbrace{\mathbb{P}(w \mid F)}_{\frac{1}{8}} + \underbrace{\mathbb{P}(D)}_{\frac{1}{2}} \cdot \underbrace{\mathbb{P}(w \mid D)}_{0} = \frac{1}{16}$$

$$= \frac{1}{2} \times \frac{1}{8} + \frac{1}{2} \times 0$$

Hence

$\mathbb{P}(X=0) = \mathbb{P}(\{TTT\}) = 1/16$

$\mathbb{P}(X=1) = \mathbb{P}(\{HTT, THT, TTH\}) = 3/16$

$\mathbb{P}(X=2) = \mathbb{P}(\{HHT, HTH, THH\}) = 3/16$

$\mathbb{P}(X=3) = \mathbb{P}(\{HHH\}) = \frac{9}{16}$

x	0	1	2	3
$f(x) = \mathbb{P}(X=x)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{9}{16}$

$$E(X) = \sum_x x f(x) = \cancel{0 \cdot f(0)} + 1 \cdot f(1) + 2 \cdot f(2) + 3 \cdot f(3) = \frac{3}{16} + \frac{6}{16} + \frac{27}{16} = \frac{36}{16}$$

$$E(X^2) = \sum_x x^2 f(x) = \cancel{0^2 \cdot f(0)} + 1^2 \cdot f(1) + 2^2 \cdot f(2) + 3^2 \cdot f(3) = \frac{3}{16} + \frac{12}{16} + \frac{81}{16} = \frac{96}{16}$$

$$var(X) = E(X^2) - (E(X))^2 = \frac{96}{16} - \left(\frac{36}{16}\right)^2 = \underset{\text{exercise}}{\underset{\uparrow}{=}}$$

5. Suppose you roll a 4 sided fair die, then you select at random, without replacement, as many cards from the deck as the number shown on the die. What is the probability that you get at least one Queen? (In your solution, define the corresponding events and write down the given probabilities in terms of them. You do not need to simply your final answer.)
Recall: There are 52 cards in a deck, 4 different suits (Clubs, Spades, etc.) and 13 different denominations (Ace, Two, ..., King).
Hint: Let N_i be the event that the number shown on the die is i, for $i = 1, \cdots, 4$.
Let Q be the event that you get at least one Queen.

$$\mathbb{P}(Q) = \mathbb{P}(Q\cap N_1) + \mathbb{P}(Q\cap N_2) + \mathbb{P}(Q\cap N_3) + \mathbb{P}(Q\cap N_4)$$
$$= \mathbb{P}(N_1)\times \mathbb{P}(Q|N_1) + \mathbb{P}(N_2)\times\mathbb{P}(Q|N_2) + \mathbb{P}(N_3)\times\mathbb{P}(Q|N_3) + \mathbb{P}(N_4)\times\mathbb{P}(Q|N_4)$$

$\mathbb{P}(N_i) = \frac{1}{4}$ for each $i = 1,2,3,4$.

$$\mathbb{P}(Q|\underset{\#\text{ of draws}=1}{N_1}) = 1 - \underbrace{\mathbb{P}(Q'|N_1)}_{\text{no queen in one draw}} = 1 - \frac{48}{52} \overset{\text{OR}}{=} 1 - \underbrace{\frac{\binom{4}{0}\binom{48}{1}}{\binom{52}{1}}}_{\text{hypergeometric with } n=1,\ N=52,\ M=4} = \frac{4}{52}$$

$$\mathbb{P}(Q|\underset{\#\text{ of draws}=2}{N_2}) = 1 - \underbrace{\mathbb{P}(Q'|N_2)}_{\text{no queens in two draws}} \overset{\text{sequentially}}{=} 1 - \left(\frac{48}{52}\cdot\frac{47}{51}\right) \overset{\text{OR}}{=} 1 - \underbrace{\left[\frac{\binom{4}{0}\binom{48}{2}}{\binom{52}{2}}\right]}_{\text{hypergeometric with } n=2,\ N=52,\ M=4}$$

$$\mathbb{P}(Q|\underset{\#\text{ of draws}=3}{N_3}) = 1 - \underbrace{\mathbb{P}(Q'|N_3)}_{\text{no queen in three draws}} \overset{\text{sequentially}}{=} 1 - \left(\frac{48}{52}\cdot\frac{47}{51}\cdot\frac{46}{50}\right) \overset{\text{OR}}{=} 1 - \underbrace{\left[\frac{\binom{4}{0}\binom{48}{3}}{\binom{52}{3}}\right]}_{\text{hypergeometric with } n=3,\ N=52,\ M=4}$$

$$\mathbb{P}(Q|\underset{\#\text{ of draws}=4}{N_4}) = 1 - \underbrace{\mathbb{P}(Q'|N_4)}_{\text{no queen in four draws}} \overset{\text{sequentially}}{=} 1 - \left(\frac{48}{52}\times\frac{47}{51}\times\frac{46}{50}\times\frac{45}{49}\right)$$
$$\overset{\text{OR}}{=} 1 - \underbrace{\left[\frac{\binom{4}{0}\binom{48}{4}}{\binom{52}{4}}\right]}_{\text{hypergeometric with } n=4,\ N=52,\ M=4}$$

Hence $\mathbb{P}(Q) = \mathbb{P}(N_1)\mathbb{P}(Q|N_1) + \mathbb{P}(N_2)\cdot\mathbb{P}(Q|N_2) + \mathbb{P}(N_3)\mathbb{P}(Q|N_3) + \mathbb{P}(N_4)\mathbb{P}(Q|N_4)$

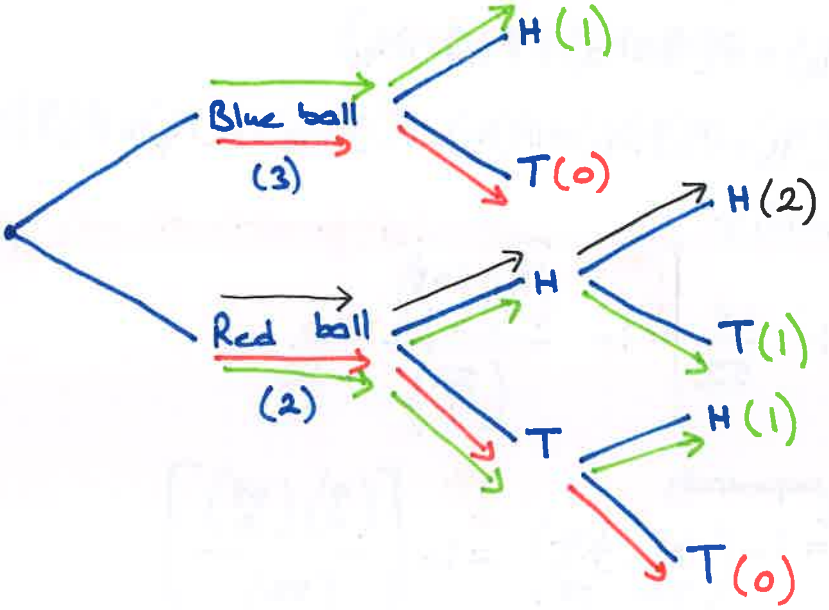
$$= \frac{1}{4}\left[\underbrace{\mathbb{P}(Q|N_1) + \mathbb{P}(Q|N_2) + \mathbb{P}(Q|N_3) + \mathbb{P}(Q|N_4)}_{\text{as found above}}\right]$$

6. A bag contains 3 identical blue and 2 identical red balls. We select a ball at random from the bag and conduct the following experiment: If the color of selected ball is blue, we toss a fair coin once. On the other hand, if it is red, we toss a fair coin twice. Let random variable, X, denote the number of heads recorded in this experiment.

(a) Find the probability distribution function (p.d.f.) of X.

(b) Calculate the mean and variance of X.

$X = \#$ of heads recorded



Range of $X = \{0,1,2\}$

$$\mathbb{P}(X=0) = \mathbb{P}(\{BT, RTT\}) = \underbrace{\mathbb{P}(BT)}_{=} + \underbrace{\mathbb{P}(RTT)}_{=}$$
$$= \frac{3}{5}\times\frac{1}{2} + \frac{2}{5}\times\frac{1}{2}\times\frac{1}{2} = \frac{3}{10}+\frac{2}{20} = \frac{8}{20}$$

$$\mathbb{P}(X=1) = \mathbb{P}(\{BH, RHT, RTH\}) = \underbrace{\mathbb{P}(BH)}_{=} + \underbrace{\mathbb{P}(RHT)}_{=} + \underbrace{\mathbb{P}(RTH)}_{=}$$
$$= \frac{3}{5}\times\frac{1}{2} + \frac{2}{5}\times\frac{1}{2}\times\frac{1}{2} + \frac{2}{5}\times\frac{1}{2}\times\frac{1}{2} = \frac{10}{20}$$

$$\mathbb{P}(X=2) = \mathbb{P}(\{RHH\}) = \frac{2}{5}\times\frac{1}{2}\times\frac{1}{2} = \frac{2}{20}$$

Hence:

x	0	1	2
$\mathbb{P}(X=x) = f(x)$	$\frac{8}{20}$	$\frac{10}{20}$	$\frac{2}{20}$

$$E(X) = \sum_x x f(x) = 0\cdot f(0) + 1\cdot f(1) + 2\cdot f(2)$$
$$= 0\cdot\frac{8}{20} + 1\cdot\frac{10}{20} + 2\cdot\frac{2}{20} = \frac{14}{20}$$

$$E(X^2) = \sum_x x^2 f(x) = \cancel{0^2\cdot f(0)} + 1^2\cdot f(1) + 2^2\cdot f(2) = 1\cdot\frac{10}{20} + 4\cdot\frac{2}{20} = \frac{18}{20}$$

$$\operatorname{var}(X) = E(X^2) - (E(X))^2 = \frac{18}{20} - \left(\frac{14}{20}\right)^2 = \frac{360-196}{400} = \frac{164}{400}$$

(20)

7. Sergio Rodriguez, who plays for Real Madrid Baloncesto (basketball club), has 90% free throw percentage in Euroleague. During the season,

(a) What is the probability that he makes his first free throw on his fifth or sixth shot?

(b) What is the probability that he makes his third free throw on his fifth shot?

(c) If he misses six shots in a row, what is the probability that he makes his first free throw on his tenth (counted from the very first) shot?

(d) How many free throws are expected to be shot until he makes eighteen of them?

(e) How many free throws are expected to be successful if he attempts fifty?

success — trial

a) $X = \#$ of shots until he makes his 1st free throw $X \sim$ geometric with $\theta = 0.9$

$$\mathbb{P}(\underbrace{X=5}_{MMMMS}) + \mathbb{P}(\underbrace{X=6}_{MMMMMS}) = (1-0.9)^4 \cdot (0.9) + (1-0.9)^5 \cdot (0.9) = (0.1)^4 \cdot (0.9) + (0.1)^5 \cdot (0.9)$$

M – misses the shot
S – makes "

b) $X = \#$ of shots until he makes his third free throw

$X \sim$ negative binomial with $\theta = 0.9$ and $k = 3$

$\underbrace{MSMS}_{\text{missed 2, made 2}} | S$ ← 3rd success on the 5th

$$\mathbb{P}(X=5) = \binom{4}{2} \cdot (0.1)^2 \cdot (0.9)^2 \cdot (0.9) = \binom{4}{2} \cdot (0.1)^2 \cdot (0.9)^3$$

OR (use formula)

$$\mathbb{P}(X=5) = b^*(\underset{x}{5}, \underset{k}{3}, \underset{\theta}{0.9}) = \underset{\binom{x-1}{k-1}}{\binom{4}{2}} \cdot \underset{(1-\theta)^{x-k}}{(0.1)^2} \cdot \underset{\theta^k}{(0.9)^3}$$

c) $X \sim$ Geometric with $\theta = 0.9$

$$\mathbb{P}(X=10 \mid X>6) = \frac{\mathbb{P}(\{X=10\} \cap \{X>6\})}{\mathbb{P}(X>6)} = \frac{\mathbb{P}(X=10)}{\mathbb{P}(X>6)} = \frac{(0.1)^9 \times 0.9}{(0.1)^6} = \underbrace{(0.1)^3 \cdot (0.9)}_{\mathbb{P}(X=4)}$$

$\mathbb{P}(X>x) = (1-\theta)^x$

recall: memoryless property of the geometric distribution

d) $X \sim$ negative binomial with $\theta = 0.9$ and $k = 18$

$$\Rightarrow E(X) = \frac{k}{\theta} = \frac{18}{0.9} = 20 \text{ shots}$$

e) $X = \#$ of free throws he makes.

$X \sim$ binomial with $n = 50$ and $\theta = 0.9 \Rightarrow E(X) = n\theta = 50 \times 0.9 = 45$ shots.

Recall: $F(x) = \mathbb{P}(X \le x)$

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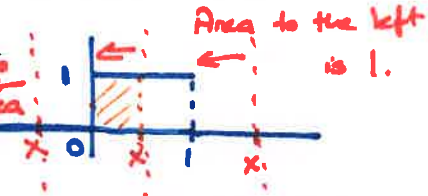
8. (5 pts) Let X be a continuous random variable with the density function



$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Let Y be another random variable defined by $Y = X + 2$. Find the cumulative distribution function $F(y)$ of Y.

C.D.F. of X: when $x \le 0$, $F(x) = \int_{-\infty}^{x} \underbrace{f(t)}_{=0} dt = 0$

when $0 < x < 1$, $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} \underbrace{f(t)}_{=0}dt + \int_{0}^{x} \underbrace{f(t)}_{=1}dt = x$

when $x \ge 1$, $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} \underbrace{f(t)}_{=0}dt + \int_{0}^{1} \underbrace{f(t)}_{=1}dt + \int_{1}^{x} \underbrace{f(t)}_{=0}dt = 1$

$$\boxed{F_X(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases} \quad \text{where } F_X(x) = \mathbb{P}(X \le x)}$$

Now we're asked to find $F_Y(y)$:

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X+2 \le y) = \mathbb{P}(X \le y-2) = F_X(\underline{y-2})$$

Hence we get $= \begin{cases} 0 & \text{if } \underline{y-2} \le 0 \\ \underline{y-2} & \text{if } 0 < \underline{y-2} < 1 \\ 1 & \text{if } \underline{y-2} \ge 1 \end{cases} = \underbrace{\begin{cases} 0 & \text{if } y \le 2 \\ y-2 & \text{if } 2 < y < 3 \\ 1 & \text{if } y \ge 3 \end{cases}}_{\text{c.d.f. of } Y}$

One can also find the density of Y using the $f_Y(y) = \frac{d}{dy} F_Y(y)$

to get $f_Y(y) = \begin{cases} 1 & \text{if } 2 < y < 3 \\ 0 & \text{elsewhere} \end{cases}$

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9. Let X be a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} \dfrac{c}{x^2} & \text{for } x > 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of the constant c.

(i) $f(x) = \frac{c}{x^2} \geq 0$ for all $x \in \mathbb{R} \Leftrightarrow c \geq 0$

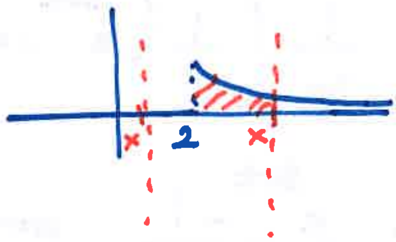
(ii) $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{2} \underset{0}{\underset{\shortparallel}{f(x)}}dx + \int_{2}^{\infty} \underset{\frac{c}{x^2}}{\underset{\shortparallel}{f(x)}}dx = c \lim_{b\to\infty}\left(\int_2^b \frac{dx}{x^2}\right)$

$= c \lim_{b\to\infty}\left(-\frac{1}{x}\Big|_{x=2}^{x=b}\right) = c \lim_{b\to\infty}\left(\cancel{-\frac{1}{b}}^{\,0} + \frac{1}{2}\right) = \frac{c}{2} \overset{\text{must hold}}{=} 1 \Rightarrow \boxed{c=2}$

b) Compute the probability $P(4 < X < 5)$.

$\mathbb{P}(4<X<5) = \int_4^5 f(x)dx = \int_4^5 \frac{2}{x^2}dx = -\frac{2}{x}\Big|_{x=4}^{x=5}$

$= -\underset{(4)}{\frac{2}{5}} + \underset{(5)}{\frac{2}{4}} = \frac{2}{20} = \frac{1}{10}$ //



c) Find the cumulative distribution function $F(x)$ of X.

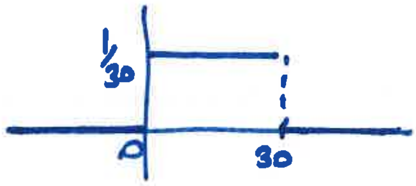
when $x \leq 2$, $F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} \underset{0}{\underset{\shortparallel}{f(t)}}dt = 0$

when $x > 2$, $F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{2} \underset{0}{\underset{\shortparallel}{f(t)}}dt + \int_{2}^{x} \underset{\frac{2}{t^2}}{\underset{\shortparallel}{f(t)}}dt$

$= -\frac{2}{t}\Big|_{t=2}^{t=x} = -\frac{2}{x} + 1$ Hence $\boxed{F(x) = \begin{cases} 0 & \text{if } x \leq 2 \\ 1 - \frac{2}{x} & \text{if } x > 2 \end{cases}}$

10. Let X be a uniform random variable on the interval $(0, 30)$. Determine the moment generating function of X, i.e. $M_X(t)$. (Write the p.d.f. of X explicitly.)

$$f(x) = u(x; 0, 30) = \begin{cases} \frac{1}{30} & \text{if } \quad 0<x<30 \\ 0 & \text{elsewhere} \end{cases}$$



$$M_X(t) = E\left[e^{tX}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$= \cancel{\int_{-\infty}^{0} e^{tx} \underbrace{f(x)}_{=0} dx} + \int_{0}^{30} e^{tx} \underbrace{f(x)}_{=\frac{1}{30}} dx + \cancel{\int_{30}^{\infty} e^{tx} \underbrace{f(x)}_{=0} dx}$$

$$= \frac{1}{30} \int_0^{30} e^{tx} dx$$

$$\int e^{tx}dx = \int \frac{e^u du}{t}$$

$u = tx$

$du = t\,dx$

$$= \frac{1}{t} e^u + C$$

$$= \frac{1}{t} e^{tx} + C$$

$$= \frac{1}{30} \left(\frac{1}{t} e^{tx} \Big|_{x=0}^{x=30} \right)$$

$$= \frac{1}{30} \left(\frac{1}{t} \left(e^{30t} - e^{0} \right) \right)$$

$$= \frac{e^{30t} - 1}{30t}$$ provided that $t \neq 0$

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11. The probability of an electrical component being defective is 0.05. The component is supplied in boxes of 60.

a) Find the probability that there are at least 3 defective components in a box. Do not simplify your answer.

X = # of defective comp. in a box

(Assuming independence and $\theta = 0.05$ being the same), then

$X \sim$ binomial with $n=60$ and $\theta = 0.05$ with

$$\mathbb{P}(X=x) = b(x; 60, 0.05) = \binom{60}{x} \cdot (0.05)^x \cdot (0.95)^{60-x} \quad x=0,\dots,60.$$

$$\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X<3) = 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1) - \mathbb{P}(X=2)$$

$$= 1 - \binom{60}{0} \cdot (0.05)^0 \cdot (0.95)^{60} - \binom{60}{1} \cdot (0.05)^1 \cdot (0.95)^{59} - \binom{60}{2} \cdot (0.05)^2 \cdot (0.95)^{58}$$

$n \geq 20$ and $\theta \leq 0.05 \Rightarrow$ we get good approximation

b) Using possion approximation, estimate the same probability asked in part a) Do not simplify your answer.

Set $\lambda = n\theta = 60 \times 0.05 = 3$ we will use Poisson with $\lambda = 3$, i.e.

$$\mathbb{P}(X=x) = p(x;3) = \frac{e^{-3} \cdot 3^x}{3!} \quad \text{for } x=0,1,\dots \text{ to approximate } b(x;60,0.05)$$

$$\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X<3) = 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1) - \mathbb{P}(X=2)$$

$$= 1 - \frac{e^{-3} \cdot \overbrace{3^0}^{=1}}{0!} - \frac{e^{-3} \cdot \overbrace{3^1}^{=3}}{1!} - \frac{e^{-3} \cdot 3^2}{2!} = 1 - e^{-3} - 3e^{-3} - \frac{9}{2}e^{-3} //$$

c) A retailer buys 2 boxes of components. Find the probability that there are at least 3 defective components in each box by using the result in **b**).

This is $[\mathbb{P}(X \geq 3)]^2$ since the event that either box having defective components is independent of each other. $(\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B))$