INSTRUCTOR'S RESOURCE GUIDE

EDWARD B. SAFF
Vanderbilt University
A. DAVID SNIDER
University of South Florida

FUNDAMENTALS OF DIFFERENTIAL EQUATIONS

SIXTH EDITION

FUNDAMENTALS OF DIFFERENTIAL EQUATIONS AND BOUNDARY VALUE PROBLEMS

FOURTH EDITION

R. Kent Nagle
Edward B. Saff
A. David Snider

PEARSON
Addison Wesley

Boston San Francisco New York
London Toronto Sydney Tokyo Singapore Madrid
Mexico City Munich Paris Cape Town Hong Kong Montreal

Reproduced by Pearson Addison-Wesley from electronic files supplied by the authors.

Publishing as Pearson Addison-Wesley, 75 Arlington Street, Boston MA 02116

ISBN 0-321-17318-X

1 2 3 4 5 6 QEP 06 05 04 03

PEARSON
Addison
Wesley

Contents

Notes to the Instructor

One goal in our writing has been to create flexible texts that afford the instructor a variety of topics and make available to the student an abundance of practice problems and projects. We recommend that the instructor read the discussion given in the preface in order to gain an overview of the prerequisites, topics of emphasis, and general philosophy of the texts.

An additional resource to accompany the texts is the on-line tool, *MyMathLab*. *MyMathLab* is ideal for lecture-based, lab-based, and on-line courses and provides students and instructors with a centralized point of access to the multimedia resources available with the texts. The pages of the actual text are loaded into *MyMathLab* and extensive course-management capabilities, including a host of communication tools for course participants, are provided to create a user-friendly and interactive on-line learning environment. Instructors can also remove, hide, or annotate Addison-Wesley preloaded content, add their own course documents, or change the order in which material is presented. A link to the Instructional Utility Software package is also included. For more information visit our Web site at www.mymathlab.com or contact your Addison-Wesley sales representative for a live demonstration.

INSTRUCTIONAL UTILITY SOFTWARE

Written specifically for the texts and available via *MyMathLab*, this utility has the following items:

GRAPHICAL METHODS

Graph, tabulate or evaluate functions
Direction field
Phase plane diagram

MATRIX OPERATIONS

Solve a system of linear equations
Determinant and inverse of a matrix
Matrix multiplication
Eigenvalues and eigenvectors

NUMERICAL SOLUTIONS OF INITIAL VALUE PROBLEMS

Euler's method subroutine
Euler's method with tolerance
Improved Euler's method subroutine
Improved Euler's method with tolerance
Fourth order Runge-Kutta method subroutine
Fourth order Runge-Kutta method with tolerance

FOURTH ORDER RUNGE-KUTTA METHOD FOR SYSTEMS

System with 2 equations
System with 3 equations
System with 4 equations

OTHER COMPUTATIONAL METHODS

Roots of a polynomial equation
Newton's method
Simpson's Rule for definite integrals
Simpson's Rule for linear differential equations

INTERACTIVE DIFFERENTIAL EQUATIONS CD-ROM

Written by Beverly West (Cornell University), Steven Strogatz (Cornell University), Jean Marie McDill (California Polytechnic State University—San Luis Obispo), John Cantwell (St. Louis University), and Hubert Hohn (Massachusetts College of Art), this CD-ROM is a revised version of a popular software directly tied to the text. It focuses on helping students visualize concepts with applications drawn from engineering, physics, chemistry, and biology. This software runs on supported Windows or Macintosh operating systems and is bundled *free* with every book.

INSTRUCTOR'S MAPLE/MATLAB/MATHEMATICA MANUAL

The Instructor's Maple/Matlab/Mathematica Manual, 0-321-17320-1, was written as an aid for anyone interested in coordinating the use of computer algebra systems with their course. This supplement, including sample worksheets, is available upon request from Addison Wesley and includes the following.

- specific instruction in the use of Maple, Matlab and Mathematica to obtain graphic (direction fields, solution curves, phase portraits), numeric (built-in and user defined), and symbolic information about differential equations;
- a sampling of techniques that can be used to ease the introduction of Maple into the differential equations classroom (including sample worksheets directly related to the text); and
- a collection of additional projects that are particularly amenable to solution using a computer algebra system.

While this supplement is written for use with MAPLE, the general ideas can be adapted for use with MATHEMATICA, MATLAB, or any other sophisticated numerical and/or computer algebra software.

Computer Labs: A computer lab in connection with a differential equations course can add a whole new dimension to the teaching and learning of differential equations. As more and more colleges and universities set up computer labs with software such as MATLAB, MAPLE, DERIVE, MATHEMATICA, PHASEPLANE, and MACMATH, there will be more opportunities to include a lab as part of the differential equations course. In our teaching and in our texts, we have tried to provide a variety of exercises, problems, and projects that encourage the student to use the computer to explore. Even one or two hours at a computer generating phase plane diagrams can provide the students with a feeling of how they will use technology together with the theory to investigate real world problems. Furthermore, our experience is that they thoroughly enjoy these activities. Of course the software provided free with the texts is especially convenient for such labs.

Group Projects: Although the projects that appear at the end of the chapters in the texts can be worked out by the conscientious student working alone, making them *group* projects adds a social element that encourages discussion and interactions that simulate a professional work place atmosphere. Group sizes of 3 or 4 seem to be optimal. Moreover, requiring that each individual student separately write up the group's solution as a formal technical report for grading by the instructor also contributes to the professional flavor.
Typically our students each work on 3 or 4 projects per semester. If class time permits, oral presentations by the groups can be scheduled and help to improve the communication skills of the students.
The role of the instructor is, of course, to help the students solve these elaborate problems on their own and to recommend additional reference material when appropriate.
Some additional Group Projects are presented in this guide (see page 8).

Technical Writing Exercises: The technical writing exercises at the end of most chapters invite students to make documented responses to questions dealing with the concepts in the chapter. This not only gives students an opportunity to improve their writing skills, but it helps them organize their thoughts and better understand the new concepts. Moreover, many questions deal with critical thinking skills that will be useful in their careers as engineers, scientists, or mathematicians.

Since most students have little experience with technical writing, it may be necessary to return *ungraded* the first few technical writing assignments with comments and have the students redo the exercise. This has worked well in our classes and is much appreciated by the students. Handing out a "model" technical writing response is also helpful for the students.

Student Presentations: It is not uncommon for an instructor to have students go to the board and present a solution to a problem. Differential equations is so rich in theory and applications that it is an excellent course to allow (require) a student to give a presentation on a special application (e.g. almost any topic from Chapters 3 and 5), on a new technique not covered in class (e.g. material from Section 2.6, Projects A, B or C in Chapter 4), or on additional theory (e.g. material from Chapter 6 which generalizes the results in Chapter 4). In addition to improving students' communication skills, these "special" topics are long remembered by the students. Here, too, working in groups of 3 or 4 and sharing the presentation responsibilities can add substantially to the interest and quality of the presentation. Students should also be encouraged to enliven their communication by building physical models, preparing part of their lectures on video cassette, etc.

Homework Assignments: We would like to share with you an obvious, nonoriginal, but effective method to encourage students to do homework problems.

An essential feature is that it requires little extra work on the part of the instructor or grader. We assign homework problems (about 10 of them) after each lecture. At the end of the week (Fridays), students are asked to turn in their homeworks (typically 3 sets) for that week. We then choose at random one problem from each assignment (typically a total of 3) that will be graded. (The point is that the student does not know in advance which problems will be chosen.) Full credit is given for any of the chosen problems for which there is evidence that the student has made an honest attempt at solving. The homework problem sets are returned to the students at the next meeting (Mondays) with grades like 0/3, 1/3, 2/3 or 3/3 indicating the proportion of problems for which the student received credit. The homework grades are tallied at the end of the semester and count as one test grade. Certainly there are variations on this theme. The point is that students are motivated to do their homework with little additional cost (= time) to the instructor.

Syllabus Suggestions: To serve as a guide in constructing a syllabus for a one-semester or two-semester course, the prefaces to the texts list sample outlines that emphasize methods, applications, theory, partial differential equations, phase plane analysis, or computation or combinations of these. As a further guide in making a choice of subject matter, we provide below a listing of text material dealing with some common areas of emphasis.

References to material in Chapters 11, 12, or 13 refer to the expanded text *Fundamentals of Differential Equations and Boundary Value Problems*, 4th edition.

Numerical, Graphical, and Qualitative Methods

The sections and projects dealing with numerical, graphical, and qualitative techniques for solving differential equations include:

Section 1.3, *Direction Fields*

Section 1.4, *The Approximation Method of Euler*

Project A for Chapter 1, *Taylor Series Method*

Project B for Chapter 1, *Picard's Method*

Project D for Chapter 1, *The Phase Line*

Section 3.6, *Improved Euler's Method*, which includes step-by-step outlines of the improved Euler's method subroutine and improved Euler's method with tolerance. These outlines are easy for the student to translate into a computer program (cf. pages 129, 130).

Section 3.7, *Higher-Order Numerical Methods: Taylor and Runge-Kutta*, which includes outlines for the Fourth Order Runge-Kutta subroutine and algorithm with tolerance (cf. pages 138, 139).

Project G for Chapter 3, *Stability of Numerical Methods*

Project H for Chapter 3, *Period Doubling and Chaos*

Section 4.7, *Qualitative Considerations for Variable-Coefficient and Nonlinear Equations*, which discusses the energy integral lemma, as well as the Airy, Bessel, Duffing, and van der Pol equations.

Section 5.3, *Solving Systems and Higher-Order Equations Numerically*, which describes the vectorized forms of Euler's Method and the Fourth-Order Runge-Kutta method, and discusses an application to population dynamics.

Section 5.4, *Introduction to the Phase Plane*, which introduces the study of trajectories of autonomous systems, critical points, and stability.

Section 5.7, *Dynamical Systems, Poincaré Maps, and Chaos*, which discusses the use of numerical methods to approximate the Poincaré map and how to interpret the results.

Project B for Chapter 5, *Designing a Landing System for Interplanetary Travel*

Project C for Chapter 5, *Things That Bob*

Project F for Chapter 5, *Strange Behavior of Competing Species-Part I*

Project D for Chapter 9, *Strange Behavior of Competing Species-Part II*

Project D for Chapter 10, *Numerical Method for* $\Delta u = f$ *on a Rectangle*

Project D for Chapter 11, *Shooting Method*

Project E for Chapter 11, *Finite-Difference Method for Boundary Value Problems*

Project C for Chapter 12, *Computing Phase Plane Diagrams*

Project D for Chapter 12, *Ecosystem of Planet GLIA-2*

Appendix A, *Newton's Method*

Appendix B, *Simpson's Rule*

Appendix D, *Method of Least Squares*

Appendix E, *Runge-Kutta Procedure for n Equations*

The instructor who wishes to emphasize numerical methods should also note that the text contains an extensive chapter on series solutions of differential equations (Chapter 8).

Engineering/Physics Applications

Since Laplace Transforms is a subject vital to engineering, we have included a detailed chapter on this topic—see Chapter 7. Stability is also an important subject for engineers, so we have included an introduction to the subject in Section 5.4 along with an entire chapter addressing this topic—see Chapter 12. Further materials dealing with engineering/physics applications include:

Project C for Chapter 1, *Magnetic "Dipole"*

Project A for Chapter 2, *Torricelli's Law of Fluid Flow*

Section 3.1, *Mathematical Modeling*

Section 3.2, *Compartmental Analysis*, which contains a discussion of mixing problems and of population models.

Section 3.3, *Heating and Cooling of Buildings*, which discusses temperature variations in the presence of air conditioning or furnace heating.

Section 3.4, *Newtonian Mechanics*

Section 3.5, *Electrical Circuits*

Project B for Chapter 3, *Curve of Pursuit*

Project C for Chapter 3, *Aircraft Guidance in a Crosswind*

Project D for Chapter 3, *Feedback and the Op Amp*

Project E for Chapter 3, *Bang-Bang Controls*

Section 4.1, *Introduction: The Mass-Spring Oscillator*

Section 4.7, *Qualitative Considerations for Variable-Coefficient and Nonlinear Equations*

Section 4.8, *A Closer Look at Free Mechanical Vibrations*

Section 4.9, *A Closer Look at Forced Mechanical Vibrations*

Project F for Chapter 4, *Apollo Reentry*

Project G for Chapter 4, *Simple Pendulum*

Chapter 5, *Introduction to Systems and Phase Plane Analysis*, which includes sections on coupled mass-spring systems, electrical circuits, and phase plane analysis.

Project B for Chapter 5, *Designing a Landing System for Interplanetary Travel*

Project C for Chapter 5, *Things that Bob*

Project E for Chapter 5, *Hamiltonian Systems*

Project B for Chapter 6, *Transverse Vibrations of a Beam*

Chapter 7, *Laplace Transforms*, which in addition to basic material includes discussions of transfer functions, the Dirac delta function, and frequency response modeling.

Projects for Chapter 8, which deal with Schrödinger's equation, buckling of a tower, and aging springs.

Project B for Chapter 9, *Matrix Laplace Transform Method*

Project C for Chapter 9, *Undamped Second-Order Systems*

Chapter 10, *Partial Differential Equations*, which includes sections on Fourier series, the heat equation, wave equation, and Laplace's equation.

Project A for Chapter 10, *Steady-State Temperature Distribution in a Circular Cylinder*

Project B for Chapter 10, *A Laplace Transform Solution of the Wave Equation*

Project A for Chapter 11, *Hermite Polynomials and the Harmonic Oscillator*

Section 12.4, *Energy Methods*, which addresses both conservative and nonconservative autonomous mechanical systems.

Project A for Chapter 12, *Solitons and Korteweg-de Vries Equation*

Project B for Chapter 12, *Burger's Equation*

Students of engineering and physics would also find Chapter 8 on series solutions particularly useful, especially Section 8.8 on Special Functions.

Biology/Ecology Applications

Project D for Chapter 1, *The Phase Line*, which discusses the logistic population model and bifurcation diagrams for population control.

Problem 32 in Exercises 2.2, which concerns radioisotopes and cancer detection.

Problem 37 in Exercises 2.3, which involves a simple model for the amount of a hormone in the blood during a 24-hour cycle.

Section 3.1, *Mathematical Modeling*

Section 3.2, *Compartmental Analysis*, which contains a discussion of mixing problems and population models.

Problem 19 in Exercises 3.6 which considers a generalization of the logistic model.

Problem 20 in Exercises 3.7, which involves chemical reaction rates.

Project A for Chapter 3, *Aquaculture*, which deals with a model for raising and harvesting catfish.

Section 5.1, *Interconnected Fluid Tanks*, which introduces systems of differential equations.

Section 5.3, *Solving Systems and Higher-Order Equations Numerically*, which contains an application to population dynamics.

Section 5.4, *Introduction to the Phase Plane*, which contains exercises dealing with the spread of a disease through a population (epidemic model).

Project A for Chapter 5, *The Growth of a Tumor*

Project D for Chapter 5, *Periodic Solutions to Volterra-Lotka Systems*

Project F for Chapter 5, *Strange Behavior of Competing Species-Part I*

Project G for Chapter 5, *Cleaning Up the Great Lakes*

Project D for Chapter 9, *Strange Behavior of Competing Species-Part II*

Problem 19 in Exercises 10.5, which involves chemical diffusion through a thin layer.

Project D for Chapter 12, *Ecosystem on Planet GLIA-2*

The basic content of the remainder of this instructor's manual consists of supplemental Group Projects along with the answers to the even numbered problems. These answers are for the most part not available any place else since the text only provides answers to odd numbered problems, and the Student Solution Manual contains only a handful of worked solutions to even numbered problems.

We would appreciate any comments you may have concerning the answers in this manual. These comments can be sent to the authors' email addresses given below. We also would encourage sharing with us (= the authors and users of the texts) any of your favorite Group Projects.

E. B. Saff
esaff@math.vanderbilt.edu

A. D. Snider
snider@eng.usf.edu

Supplemental Group Projects

Group Project for Chapter 2

Designing a Solar Collector

You want to design a solar collector that will concentrate the sun's rays at a point. By symmetry this surface will have a shape that is a surface of revolution obtained by revolving a curve about an axis. Without loss of generality, you can assume that this axis is the x-axis and the rays parallel to this axis are focused at the origin (see Figure GP-1). To derive the equation for the curve, proceed as follows:

a. The law of reflection says that the angles γ and δ are equal. Use this and results from geometry to show that $\beta = 2\alpha$.

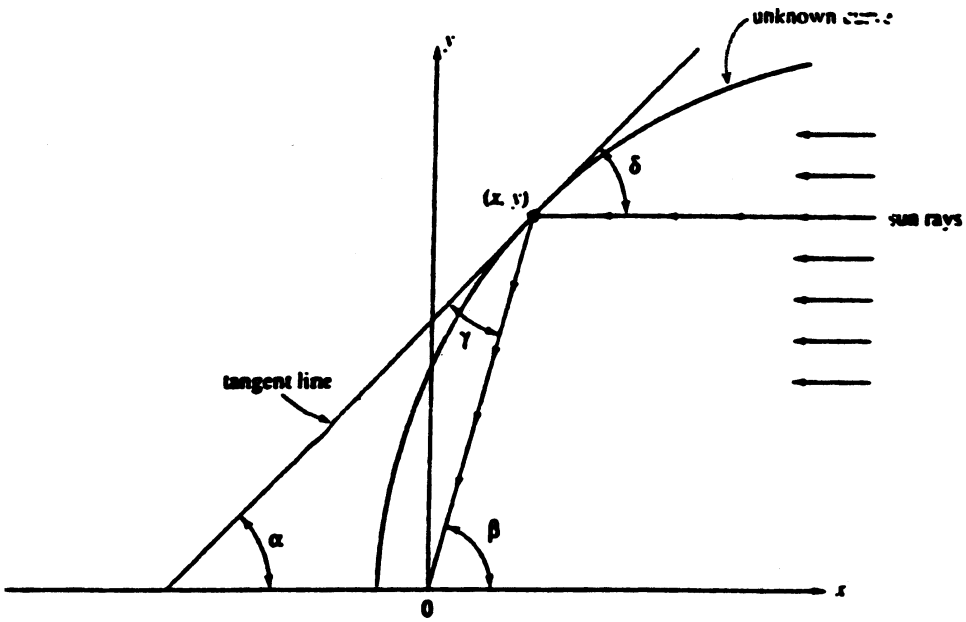


Figure GP-1: Curve that generates a solar collector

b. From calculus recall that $\dfrac{dy}{dx} = \tan\alpha$. Use this, the fact that $\dfrac{y}{x} = \tan\beta$, and the double angle formula to show that

$$\frac{y}{x} = \frac{2\frac{dy}{dx}}{1-\left(\frac{dy}{dx}\right)^2}.$$

c. Now show that the curve satisfies the differential equation.

$$\textbf{(1)} \qquad \frac{dy}{dx} = \frac{-x+\sqrt{x^2+y^2}}{y}$$

d. Solve equation (1).

e. Describe the solutions and identify the type of collector obtained.

Group Projects for Chapter 3

Delay Differential Equations

In our discussion of mixing problems in Section 3.2, we encountered the initial value problem

(1) $$x'(t) = 6 - \frac{3}{500}x(t - t_0),$$

$$x(t) = 0 \quad \text{for} \quad t \in [-t_0, 0]$$

where t_0 is a positive constant. The equation in (1) is an example of a **delay differential equation.** These equations differ from the usual differential equations by the presence of the shift $(t - t_0)$ in the argument of the unknown function $x(t)$. In general, these equations are more difficult to work with than are regular differential equations, but quite a bit is known about them.†

a. Show that the simple linear delay differential equation

(2) $$x'(t) = ax(t - b),$$

where a and b are constants, has a solution of the form $x(t) = Ce^{st}$, for any constant C, provided s satisfies the transcendental equation $s = ae^{-bs}$.

b. A solution to (2) for $t > 0$ can also be found using the **method of steps.** Assume that $x(t) = f(t)$ for $-b \le t \le 0$. For $0 \le t \le b$, equation (2) becomes

$$x'(t) = ax(t - b) = af(t - b),$$

and so

$$x(t) = \int_0^1 af(v - b)\,dv + x(0).$$

Now that we know $x(t)$ on $[0, b]$, we can repeat this procedure to obtain

$$x(t) = \int_b^1 ax(v - b)\,dv + x(b)$$

for $b \le t \le 2b$. This process can be continued indefinitely.

Use the method of steps to show that the solution to the initial value problem

$$x'(t) = -x(t - 1), \quad x(t) = 1 \text{ on } [-1, 0]$$

is given by

$$x(t) = \sum_{k=0}^{n} (-1)^k \frac{[t - (k - 1)]^k}{k!}, \quad \text{for } n - 1 \le t \le n,$$

where n is a nonnegative integer. (This problem can also be solved using the Laplace transform method of Chapter 7.)

c. Use the method of steps to compute the solution to the initial value problem given in (1) on the interval $0 \le t \le 15$ for $t_0 = 3$.

†See, for example, *Differential-Difference Equations,* by R. Bellman and K. L. Cooke, Academic Press, New York, 1963, or *Ordinary and Delay Differential Equations,* by R. D. Driver, Springer-Verlag, New York, 1977.

Extrapolation

When precise information about the *form* of the error in an approximation is known, a technique called **extrapolation** can be used to improve the rate of convergence.

Suppose the approximation method converges with rate $O(h^p)$ as $h \to 0$ (cf. Section 3.6). From theoretical considerations assume we know, more precisely, that

(1) $$y(x; h) = \phi(x) + h^p a_p(x) + O(h^{p+1}),$$

where $y(x; h)$ is the approximation to $\phi(x)$ using step size h and $a_p(x)$ is some function that is independent of h (typically we do not know a formula for $a_p(x)$, only that it exists). Our goal is to obtain approximations that converge at the *faster* rate $O(h^{p+1})$.

We start by replacing h by $\frac{h}{2}$ in (1) to get

$$y\left(x; \frac{h}{2}\right) = \phi(x) + \frac{h^p}{2^p} a_p(x) + O(h^{p+1}).$$

If we multiply both sides by 2^p and subtract equation (1), we find

$$2^p y\left(x; \frac{h}{2}\right) - y(x; h) = (2^p - 1)\phi(x) + O(h^{p+1}).$$

Solving for $\phi(x)$ yields

$$\phi(x) = \frac{2^p y\left(x; \frac{h}{2}\right) - y(x; h)}{2^p - 1} + O(h^{p+1}).$$

Hence

$$y^*\left(x; \frac{h}{2}\right) := \frac{2^p y\left(x; \frac{h}{2}\right) - y(x; h)}{2^p - 1}$$

has a rate of convergence of $O(h^{p+1})$.

a. Assuming
$$y^*\left(x; \frac{h}{2}\right) = \phi(x) + h^{p+1} a_{p+1}(x) + O(h^{p+2}),$$
show that
$$y^{**}\left(x; \frac{h}{4}\right) := \frac{2^{p+1} y^*\left(x; \frac{h}{4}\right) - y^*\left(x; \frac{h}{2}\right)}{2^{p+1} - 1}$$ has a rate of convergence of $O(h^{p+2})$.

b. Assuming
$$y^{**}\left(x; \frac{h}{4}\right) = \phi(x) + h^{p+2} a_{p+2}(x) + O(h^{p+3}),$$
show that
$$y^{***}\left(x; \frac{h}{8}\right) := \frac{2^{p+2} y^{**}\left(x; \frac{h}{8}\right) - y^{**}\left(x; \frac{h}{4}\right)}{2^{p+2} - 1}$$
has a rate of convergence of $O(h^{p+3})$.

c. The results of using Euler's method $\left(\text{with } h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$ to approximate the solution to the initial value problem $y' = y$, $y(0) = 1$, at $x = 1$ are given in Table 1.2, page 27. For Euler's method, the extrapolation procedure applies with $p = 1$. Use the results in Table 1.2 to find an approximation to $e = y(1)$ by computing $y^{***}\left(1; \frac{1}{8}\right)$. [Hint: Compute $y^*\left(1; \frac{1}{2}\right)$, $y^*\left(1; \frac{1}{4}\right)$, and $y^*\left(1; \frac{1}{8}\right)$; then compute $y^{**}\left(1; \frac{1}{4}\right)$, and $y^{**}\left(1; \frac{1}{8}\right)$.]

d. Table 1.2 also contains Euler's approximation for $y(1)$ when $h = \frac{1}{16}$. Use this additional information to compute the next step in the extrapolation procedure; that is, compute $y^{****}\left(1; \frac{1}{16}\right)$.

Group Projects for Chapter 5

Effects of Hunting on Predator~Prey Systems

As discussed in Section 5.3 (page 257), cyclic variations in the populations of predators and their prey have been studied using the Volterra-Lotka predator-prey model given by the system

(1) $$\frac{dx}{dt} = Ax - Bxy,$$

(2) $$\frac{dy}{dt} = -Cy + Dxy,$$

where *A, B, C,* and *D* are positive constants, $x(t)$ is the population of prey at time *t*, and $y(t)$ is the population of predators. It can be shown that such a system has a periodic solution (see Project D). That is, there exists some constant *T* such that $x(t) = x(t + T)$ and $y(t) = y(t + T)$ for all *t*. This periodic or cyclic variation in the populations has been observed in various systems such as sharks-food fish, lynx-rabbits, and ladybird beetles-cottony cushion scale. Because of this periodic behavior, it is useful to consider the average populations $\overline{x}$ and $\overline{y}$ defined by

$$\overline{x} := \frac{1}{T}\int_0^T x(t)\,dt, \quad \overline{y} := \frac{1}{T}\int_0^T y(t)\,dt$$

a. Show that $\overline{x} = C/D$ and $\overline{y} = A/B$. [*Hint:* Use equation (1) and the fact that $x(0) = x(T)$ to show that

$$\int_0^T \left(A - By(t)\right)dt = \int_0^T \frac{x'(t)}{x(t)}\,dt = 0.]$$

b. To determine the effect of indiscriminate hunting on the populations, assume hunting reduces the rate of change in a population by a constant times the population. Then, the predator-prey system satisfies the new set of equations

(3) $$\frac{dx}{dt} = Ax - Bxy - \varepsilon x = (A - \varepsilon)x - Bxy,$$

(4) $$\frac{dy}{dt} = -Cy + Dxy - \delta y = -(C + \delta)y + Dxy,$$

where ε and δ are positive constants with $\varepsilon < A$. What effect does this have on the average population of prey? On the average population of predators?

c. Assume the hunting is done selectively, as in shooting only rabbits (or shooting only lynx). Then we have $\varepsilon > 0$and $\delta = 0$ (or $\varepsilon = 0$ and $\delta > 0$) in (3)-(4). What effect does this have on the average populations of predator and prey?

d. In a rural county, foxes prey mainly on rabbits but occasionally include a chicken in their diet. The farmers decide to put a stop to the chicken killing by hunting the foxes. What do you predict will happen? What will happen to the farmers' gardens?

† The derivation of these equations is found in *Attitude Stabilization and Control of Earth Satellites*, by O.H. Gerlach, Space Science Reviews, #4 (1965), 541–566

Limit Cycles

In the study of triode vacuum tubes, one encounters the van der Pol equation:†

$$y'' - \mu\left(1 - y^2\right)y' + y = 0,$$

where the constant μ is regarded as a parameter. In Section 4.7 (page 205), we used the mass-spring oscillator analogy to argue that the nonzero solutions to the van der Pol equation with $\mu = 1$ should approach a periodic limit cycle. The same argument applies for any positive value of μ.

a. Recast the van der Pol equation as a system in normal form and use software to plot some typical trajectories for $\mu = 0.1$, 1, and 10. Rescale the plots if necessary until you can discern the limit cycle trajectory; find trajectories that spiral in, and ones that spiral out, to the limit cycle.

b. Now let $\mu = -0.1$, -1, and -10. Try to predict the nature of the solutions using the mass-spring analogy. Then use the software to check your predictions. Are there limit cycles? Do the neighboring trajectories spiral into, or spiral out from, the limit cycles?

c. Repeat parts (a) and (b) for the Rayleigh equation

$$y'' - \mu\left[1 - (y')^2\right]y' + y = 0.$$

†***Historical Footnote:*** Experimental research by **E. V. Appleton** and **B. van der Pol** in 1921 on the oscillations of an electrical circuit containing a triode generator (vacuum tube) led to the nonlinear equation now called **van der Pol's equation.** Methods of solution were developed by van der Pol in 1926-1927. **Mary L. Cartwright** continued research into nonlinear oscillation theory and together with **J. E. Littlewood** obtained existence results for forced oscillations in nonlinear systems in 1945.

Group Project for Chapter 13

David Stapleton, University of Central Oklahoma

Satellite Attitude Stability

In this problem, we determine the orientation at which a satellite in a circular orbit of radius *r* can maintain a relatively constant facing with respect to a spherical primary (e.g. a planet) of mass *M*. The torque of gravity on the asymmetric satellite maintains the orientation.

Suppose (x, y, z) and $(\bar{x}, \bar{y}, \bar{z})$ refer to coordinates in two systems that have a common origin at the satellite's center of mass. Fix the *xyz*-axes in the satellite as principal axes; then let the $\bar{z}$-axis point toward the primary and let the $\bar{x}$-axis point in the direction of the satellite's velocity. The *xyz*-axes may be rotated to coincide with the $\bar{x}\bar{y}\bar{z}$-axes by a rotation ϕ about the *x*-axis (roll), followed by a rotation θ about the resulting *y*-axis (pitch), and a rotation ψ about the final *z*-axis (yaw). Euler's equations from physics (with high order terms omitted† to obtain approximate solutions valid near $(\phi, \theta, \psi) = (0, 0, 0)$) show that the equations for the rotational motion due to gravity acting on the satellite are

$$\begin{aligned} I_x \ddot{\phi} &= -4\omega_0^2 (I_z - I_y)\phi - \omega_0 (I_y - I_z - I_x)\dot{\psi} \\ I_y \ddot{\theta} &= -3\omega_0^2 (I_x - I_z)\theta \\ I_z \ddot{\psi} &= -\omega_0^2 (I_y - I_x)\psi + \omega_0 (I_y - I_z - I_x)\dot{\phi}, \end{aligned}$$

(where $\omega_0 = \sqrt{\dfrac{GM}{r^3}}$ is the angular frequency of the orbit and the positive constants I_x, I_y, I_z are the moments of inertia of the satellite about the *x*, *y*, and *z*-axes).

a. Find constants $c_1, \ldots, c_5$ such that these equations can be written as two systems

$$\frac{d}{dt}\begin{bmatrix} \phi \\ \psi \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_1 & 0 & 0 & c_2 \\ 0 & c_3 & c_4 & 0 \end{bmatrix}\begin{bmatrix} \phi \\ \psi \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix}$$

and

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c_5 & 0 \end{bmatrix}\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

b. Show that the origin is asymptotically stable for the first system in (a) if $(c_2 c_4 + c_3 + c_1)^2 - 4c_1c_3 > 0,$ $c_1c_3 > 0,$ and $c_2 c_4 + c_3 + c_1 > 0$ and hence deduce that $I_y > I_x > I_z$ yields an asymptotically stable origin. Are there other conditions on the moments of inertia by which the origin is stable?

c. Show that for the asymptotically stable configuration in (b) the second system in (a) becomes a harmonic oscillator problem, and find the frequency of oscillation in terms of I_x, I_y, I_z and ω_0. Phobos maintains $I_y > I_x > I_z$ in its orientation with respect to Mars, and has angular frequency of orbit $\omega_0 = .82$ rad/hr. If $\dfrac{(I_x - I_z)}{I_y} = .23,$ show that the period of the libration for Phobos (the period with which the side of Phobos facing Mars shakes back and forth) is about 9 hours.

† The derivation of these equations is found in *Attitude Stabilization and Control of Earth Satellites*, by O.H. Gerlach, Space Science Reviews, #4 (1965), 541–566

ANSWERS TO EVEN-NUMBERED PROBLEMS

CHAPTER 1

Exercises 1.1 (page 5)

2. ODE, 2nd order, ind. var. *x*, dep. var. *y*, linear

4. PDE, 2nd order, ind. var. *x*, *y*; dep. var. *u*

6. ODE, 1st order, ind. var. *t*, dep. var. *x*, nonlinear

8. ODE, 2nd order, ind. var. *x*, dep. var. *y*, nonlinear

10. ODE, 4th order, ind. var. *x*, dep. var. *y*, linear

12. ODE, 2nd order, ind. var. *x*, dep. var. *y*, nonlinear

14. $\frac{dx}{dt} = kx^4$, where k is the proportionality constant

16. $\frac{dA}{dt} = kA^2$, where k is the proportionality constant

Exercises 1.2 (page 14)

4. Yes

6. No

8. Yes

10. Yes

12. Yes

20. **a.** $-1, -5$

b. $0, -1, -2$

22. **a.** $c_1 = \frac{5}{3},\ c_2 = \frac{1}{3}$

b. $c_1 = \frac{2e^{-1}}{3},\ c_2 = \frac{e^2}{3}$

24. Yes

26. No

28. Yes

Exercises 1.3 (page 22)

2. **a.** $y = -2 - 2x$

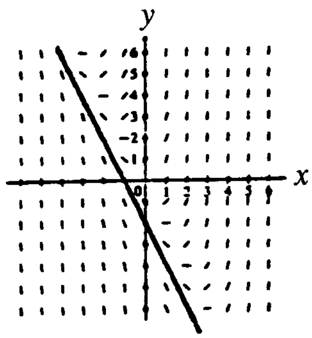


Figure 1
Solution to Problem 2(a)

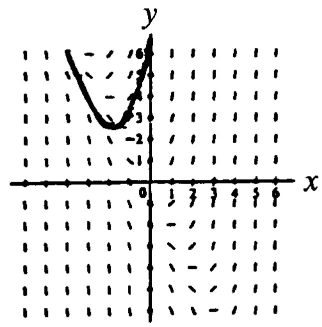


Figure 2
Solution to Problem 2(b)

c. As $x \to \infty$, solution becomes infinite. As $x \to -\infty$, solution again becomes infinite and has $y = -2 - 2x$ as an asymptote.

4. Terminal velocity is $v = 2$.

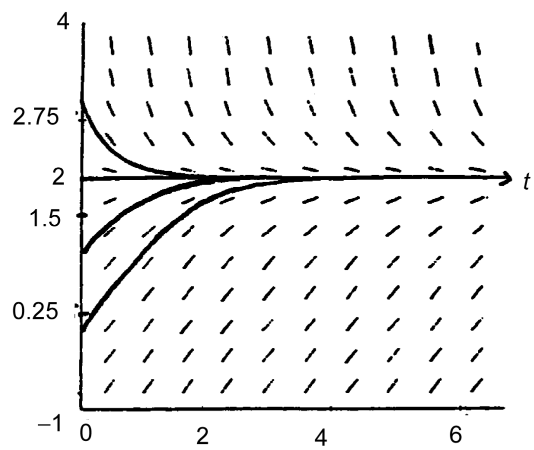


Figure 3
Solutions to Problem 4 satisfying $v(0) = 0$, $v(0) = 1$, $v(0) = 2$, $v(0) = 3$

6. **a.** 2

b. For $x > 1$ we have $x + \sin y > 0$, so $y' > 0$ and hence y is increasing.

c.
$$\begin{aligned}\frac{d^2y}{dx^2} = \frac{d}{dx}(x + \sin y) &= 1 + y' \cos y \\ &= 1 + (x + \sin y)\cos y \\ &= 1 + x\cos y + \frac{1}{2}\sin(2y)\end{aligned}$$

d. For $x = y = 0$ we have $y' = 0$ and, from the formula in part (c), it follows that $\frac{d^2y}{dx^2} = 1 > 0$ when $x = y = 0$. Thus $x = 0$ is a critical point and by the 2nd derivative test, y has a relative minimum at (0, 0).

8. **a.** 7 **b.**
$$\frac{d^2x}{dt^2} = \frac{d}{dt}(t^3 - x^3) = 3t^2 - 3x^2\frac{dx}{dt}$$
$$= 3t^2 - 3x^2(t^3 - x^3)$$
$$= 3t^2 - 3x^2t^3 + 3x^5$$

c. No, because if $1 < x < 2$ and $t > 2.5$, then $\dfrac{dx}{dt} = t^3 - x^3 > 0$, and so the particle moves to the right, away from $x = 1$.

10.

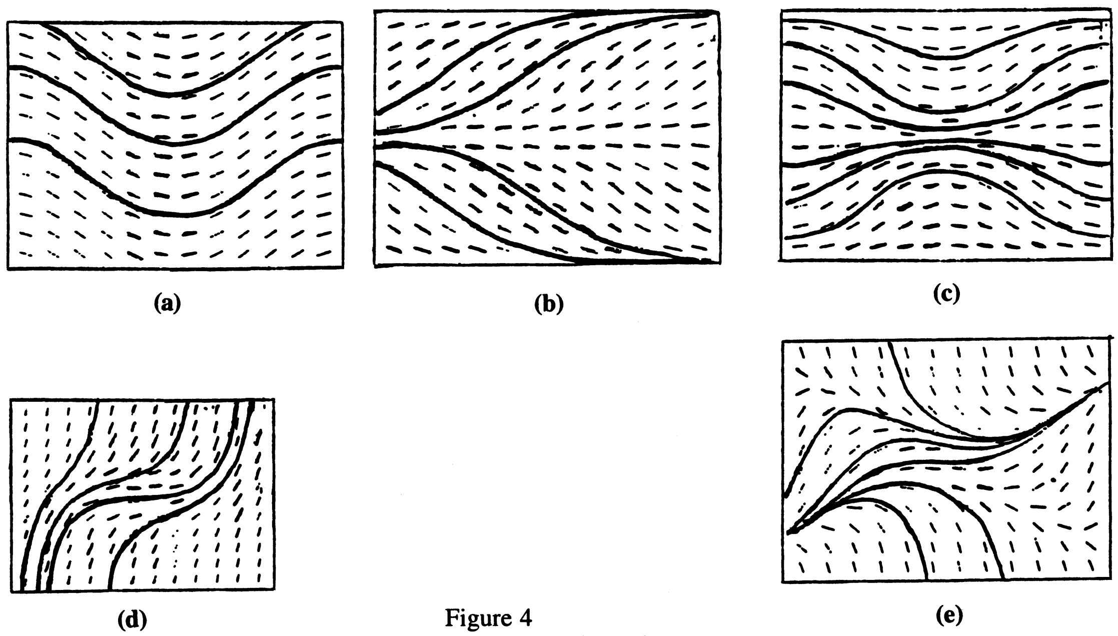


Figure 4
Direction fields for Problem 10

12.

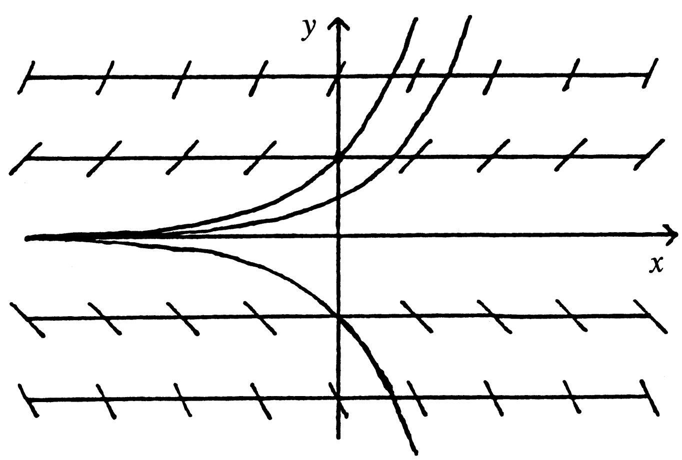


Figure 5

14.

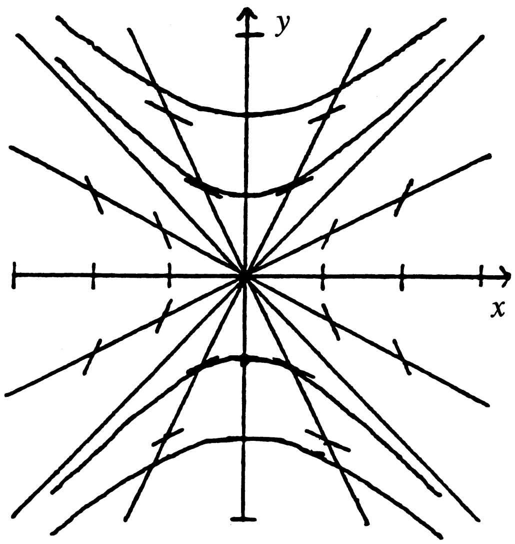


Figure 6

16.

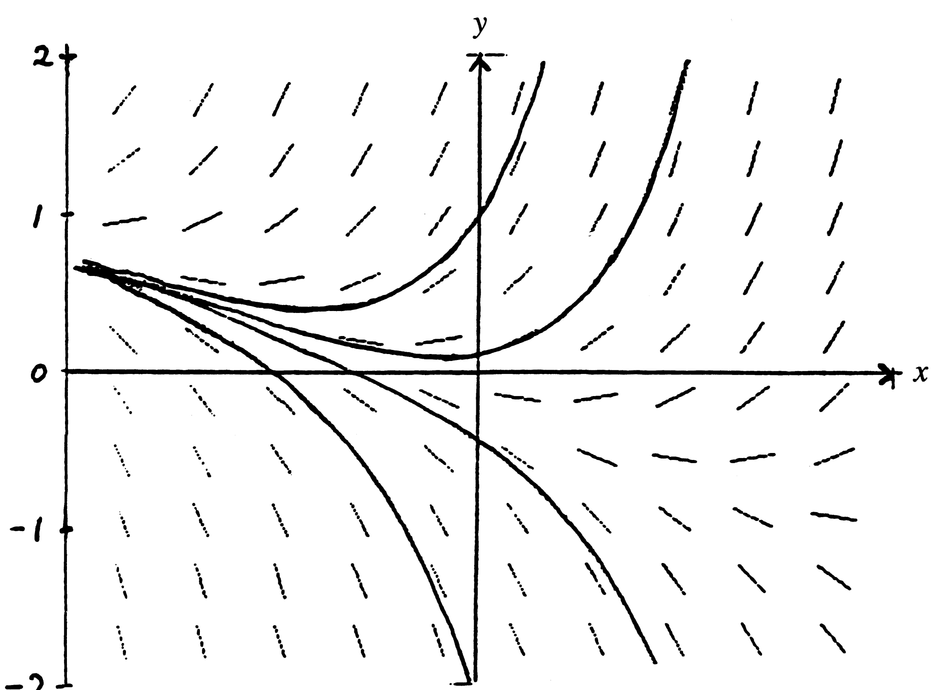


Figure 7

18. Every solution approaches zero.

Exercises 1.4 (page 28)

2.

x_n	0.1	0.2	0.3	0.4	0.5
y_n	4.000	3.998	3.992	3.985	3.975

(rounded to three decimal places)

4.

x_n	0.1	0.2	0.3	0.4	0.5
y_n	1.00	1.220	1.362	1.528	1.721

6.

x_n	1.1	1.2	1.3	1.4	1.5
y_n	0.1	0.209	0.32463	0.44409	0.56437

8.

N	1	2	4	8
$\phi(\pi)$	π	$\frac{\pi}{2}$	1.20739	1.14763

10.

x_n	y_n	actual
0	0	0
0.1	0	0.00484
0.2	0.01	0.01873
0.3	0.029	0.04082
0.4	0.0561	0.07032
0.5	0.09049	0.10653
0.6	0.13144	0.14881
0.7	0.17830	0.19658
0.8	0.23047	0.24933
0.9	0.28742	0.30657
1.0	0.34868	0.36788

14. $h = 0.5$

x_n	0.5	1.0	1.5	2.0
y_n	1.000	1.500	3.750	24.844

$h = 0.1$

x_n	0.5	1.0	1.5	2.0
y_n	1.231	3.819	1,023,041.61	overflow

$h = 0.05$

x_n	0.5	1.0	1.5	2.0
y_n	1.278	5.936	overflow	overflow

$h = 0.01$

x_n	0.5	1.0	1.5	2.0
y_n	1.321	18.299	overflow	overflow

16. $T(1) \approx 82.694°$; $T(2) \approx 76.446°$

CHAPTER 2

Exercises 2.2 (page 46)

2. No

4. Yes

6. Yes

8. $y^4 = 4\ln|x| + C$

10. $x = Ce^{t^3}$

12. $4v^2 = 1 + Cx^{-8/3}$

14. $y = \tan(x^3 + C)$

16. $\ln(1 + y^2) = e^{-x^2} + C$

18. $y = \tan\left[\frac{\pi}{3} - \ln(\cos x)\right]$

20. $y = \left(\frac{1}{2}\right)\left(-1 - \sqrt{1 + 4x^3 + 8x^2 + 8x}\right)$

22. $y = \sqrt{4 - \frac{x^3}{3}}$

24. $y = \ln\sqrt{4x^4 - 3}$

26. $y = \left(1 - \ln\sqrt{1 + x}\right)^2$

28. $y = 3e^{2t - t^2}$; maximum value is $y(1) = 3e$.

32. 79.95 mCi

34. **a.** $T = Ce^{kt} + M$

b. 70.77

36. 20°

38. $v(t) = 196 - 186e^{-t/20}$ m/sec; limiting velocity is 196 m/sec.

Exercises 2.3 (page 54)

2. Neither

4. Linear

6. Linear

8. $y = 2x^2 + x\ln|x| + Cx$

10. $y = -x^{-3} + Cx^{-2}$

12. $y = \frac{x^3 e^{-4x}}{3} + Ce^{-4x}$

14. $y = \frac{x^3}{6} - \frac{2x^2}{5} + x + Cx^{-3}$

16. $y = \dfrac{\frac{x^5}{5} + \frac{x^4}{2} + x^2 - x + C}{(x^2 + 1)^2}$

18. $y = \frac{e^{-x}}{3} + e^{-4x}$

20. $y = \frac{3x^2}{5} - \frac{x}{2} + \frac{9x^{-3}}{10}$

22. $y = 1 - x\cot x + \csc x$

24. **a.** $y(t) = 5e^{-10t} + 35e^{-20t}$; the term $5e^{-10t}$ eventually dominates.

b. $y(t) = 50te^{-10t} + 40e^{-10t}$

26. 0.860

30. **b.** $y = \left(\frac{x}{2} - \frac{1}{12} + Ce^{-6x}\right)^{1/3}$

32. $y = \begin{cases} 1 - e^{-2x} & 0 \le x \le 3 \\ -1 + (2e^6 - 1)e^{-2x} & 3 < x \end{cases}$

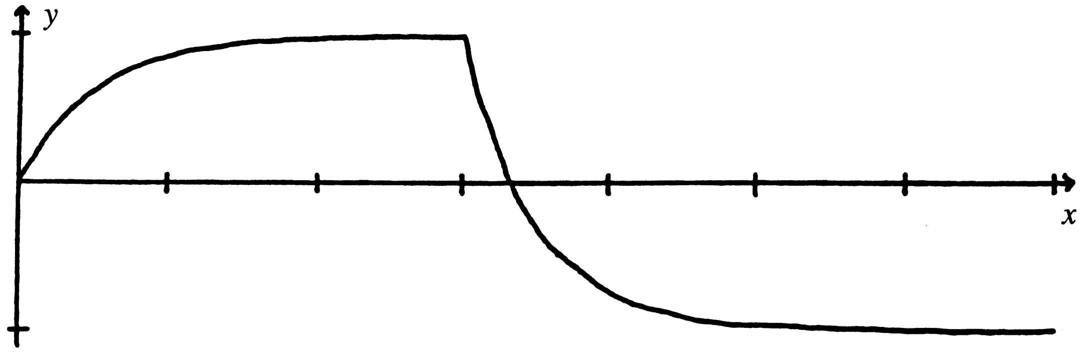


Figure 8

36. **b.** $y_h(x) = x^{-3}$

d. $v(x) = \dfrac{x^6}{6}$

e. $y(x) = Cx^{-3} + \dfrac{x^3}{6}$

Exercises 2.4 (page 65)

2. Linear with y as dep. var.

4. Separable

6. Separable and linear with y as dep. var.

8. Exact

10. $x^2 + xy - y^2 = C$

12. $e^x \sin y - x^3 + y^{1/3} = C$

14. $y = \dfrac{C + e^t(t-1)}{1 + e^t}$

16. $e^{xy} - \dfrac{x}{y} = C$

18. $x^2 + xy^2 - \sin(x+y) - e^y = C$

20. $2\arcsin x + \sin(xy) - \dfrac{3y^{2/3}}{2} = C$

22. $e^{xy} - \dfrac{x}{y} = e - 1$

24. $x = \dfrac{e - t}{e^t - 1}$

26. $x \tan y + \ln y - 2x = 0$

28. **a.** $x\cos(xy) + g(y)$

b. $xe^{xy} - x^4 + g(y)$
where g is a function of y only.

30. **b.** $x^2 y$

c. $x^5y^2 + x^6y^3 + x^4y^3 = C$

34. $\mu = x^2 y^{-2}$; $x^2 + y^2 = Cy$

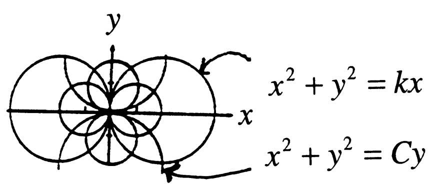


Figure 9
Orthogonal trajectures for Problem 34

Exercises 2.5 (page 71)

2. Integrating factor depending on x alone.

4. Exact

6. Linear, integrating factor depends on x alone.

8. $\mu = y^{-4}$, $x^2 y^{-3} - y^{-1} = C$ and $y \equiv 0$.

10. $\mu = x^{-2}$ (also linear); $y = \dfrac{x^4}{3} - x \ln|x| + Cx$

12. Exact; $x^2 y^3 + x - \ln|y| = C$

14. $\mu = x^3 y^2$, $3x^4 y^2 + x^5 y^3 = C$, but not $y \equiv 0$.

16. $y = Cx$

Exercises 2.6 (page 0)

2. Homogeneous

4. Bernoulli

6. Homogeneous

8. $y' = G(ax + by)$

10. $y = \dfrac{x}{C - \ln|x|}$ and $y \equiv 0$.

12. $x^3 + 3xy^2 = C$

14. $\sin\left(\dfrac{y}{\theta}\right) - \ln|\theta| = C$

16. $y = xe^{Cx}$

18. $y = -x - 2 + \tan(x + C)$

20. $\tan(x - y) + \sec(x - y) - x = C$

22. $y^{-2} = Ce^{-2x} - \dfrac{e^{2x}}{2}$ and $y \equiv 0$.

24. $y = [(x-2)^2 + C(x-2)^{-1/2}]^2$ and $y \equiv 0$

26. $y = \left(\dfrac{3e^x}{4} + Ce^{-3x}\right)^{1/3}$

28. $y^{-2} = -x - \dfrac{1}{2} + Ce^{2x}$ and $y \equiv 0$.

30. $\ln[(y-3)^2 + (x+2)^2] - 2\arctan\left(\dfrac{y-3}{x+2}\right) = C$

32. $\left(x + \dfrac{6}{5}\right)^2 + \left(x + \dfrac{6}{5}\right)\left(y + \dfrac{8}{5}\right) - \left(y + \dfrac{8}{5}\right)^2 = C$

34. $t^2 + x^2 = Ct$

36. $y = \dfrac{3}{Cx - x^4}$ and $y \equiv 0$.

38. $y = \dfrac{\theta\left(\ln|\theta| + C\right)^2}{4}$

40. $\cos(x + y) + \sin(x + y) = Ce^{x-y}$

42. $\sec\left(\dfrac{y}{x^2}\right) + \tan\left(\dfrac{y}{x^2}\right) = Cx$

46. **b.** $y = x + \dfrac{5x}{C - x^5}$

Chapter 2 Review (page 81)

2. $y = -8x^2 - 4x - 1 + Ce^{4x}$

4. $\dfrac{x^3}{6} - \dfrac{4x^2}{5} + \dfrac{3x}{4} + Cx^{-3}$

6. $y^{-2} = 2\ln\left|1 - x^2\right| + C$ and $y \equiv 0$.

8. $y = (Cx^2 - 2x^3)^{-1}$ and $y \equiv 0$.

10. $x + y + 2y^{1/2} + \arctan(x + y) = C$

12. $2ye^{2x} + y^3e^x = C$

14. $x = \dfrac{t^2(t-1)}{2} + t(t-1) + 3(t-1)\ln|t-1| + C(t-1)$

16. $y = (\cos x)\ln|\cos x| + C\cos x$

18. $y = 1 - 2x + \sqrt{2}\tan\left(\sqrt{2}x + C\right)$

20. $y = \left(C\theta^{-3} - \dfrac{12\theta^2}{5}\right)^{1/3}$

22. $(3y - 2x + 9)(y + x - 2)^4 = C$

24. $2\sqrt{xy} + \sin x - \cos y = C$

26. $y = Ce^{-x^2/2}$

28. $(y+3)^2 + 2(y+3)(x+2) - (x+2)^2 = C$

30. $y = Ce^{4x} - x - \dfrac{1}{4}$

32. $y^2 = x^2\ln(x^2) + 16x^2$

34. $y = x^2\sin x + \dfrac{2x^2}{\pi^2}$

36. $\sin(2x + y) - \dfrac{x^3}{3} + e^y = \sin 2 + \dfrac{2}{3}$

38. $y = \left[2 - \left(\dfrac{1}{4}\right)\arctan\left(\dfrac{x}{2}\right)\right]^2$

40. $\dfrac{8}{1 - 3e^{-4x} - 4x}$

CHAPTER 3

Exercises 3.2 (page 98)

2. $2.5 - 2e^{-3t/25}$, 5.78 min

4. $\dfrac{(t+100)}{5} - \dfrac{10^8(t+100)^{-3}}{5}$, 18.92 min

6. 43.8 min

8. $(0.2)(1-e^{-3t/125})$ g/cm^3; 28.88 sec

14. 187,500

16. Since $\lim\limits_{h\to 0} \dfrac{p(t+h)-p(t)}{h} = p'(t)$ and $\lim\limits_{h\to 0} \dfrac{p(t-h)-p(t)}{-h} = p'(t)$, adding these two equations and dividing by 2 yields the given formula.

18. **a.** $P(t) = \exp\left\{Ce^{-bt} + \dfrac{a}{b}\right\}$

b. $P(t) = \exp\left\{\left(\ln(P_0) - \dfrac{a}{b}\right)e^{-bt} + \dfrac{a}{b}\right\}$

c. $b > 0$: $P(t) \to e^{a/b}$;

If $b < 0$ and $\ln(P_0) > \dfrac{a}{b}$, then $P(t) \to \infty$;

If $b < 0$ and $\ln(P_0) < \dfrac{a}{b}$, then $P(t) \to 0$;

If $b < 0$ and $\ln(P_0) = \dfrac{a}{b}$, then $P(t) \equiv e^{a/b}$.

20. 6.9315 light years

22. 90 min., vanishes at $t = \infty$.

24. 41.94 years

26. **a.** 31,323 yr

b. 28,333 yr

c. percent of mass remaining

Exercises 3.3 (page 107)

2. 57.5°F

4. 19.7 min.

6. 2:26 P.M.; 1:37 P.M.

8. 3:51 A.M.; 3:51 P.M.

10. 59.5°F; never

12. The impatient friend

14. 127.5°F

Exercises 3.4 (page 115)

2. $40t + 50e^{-(0.8)t} - 50$ ft; 13.75 sec

4. 1.67 sec

6. $12.45e^{-2t} + 4.91t - 12.45$ m, 22.9 sec

8. 206.8 sec, 86.8 sec

10. $x(t) = \begin{cases} 1.96t + 0.39(e^{-5t} - 1) & 0 \le t \le 15.5 \\ 30 + 0.098(t - 15.5) + 0.035[1 - e^{-50(t-15.5)}] & 15.5 \le t \end{cases}$

0.098 m/sec

12. 16.48 sec; 1535 m

14. $\left(\dfrac{mg}{k}\right)^{1/n}$

16. $e^{\sqrt{\omega}/k}\left(k\sqrt{\omega} - T\right)^{T/k^2} = Ce^{-t/2I}$, where $C = e^{\sqrt{\omega_0}/k}\left(k\sqrt{\omega_0} - T\right)^{T/k^2}$

18. $1.61t^2$ m, 5.65 m/sec

20. $\alpha_0 = \arctan \mu$

22. $x(t) = \begin{cases} \dfrac{5}{2e^{2t}} + 6t - \dfrac{5}{2} & t \le \left(\dfrac{1}{2}\right)\ln 5 \approx 0.805 \\ 10t + \dfrac{5^{13/5}}{6e^{6t/5}} - 2\ln 5 - \dfrac{37}{6} & t > \left(\dfrac{1}{2}\right)\ln 5 \end{cases}$

10 m/sec

24. $v(t) = \beta \ln\left[\dfrac{m_0}{m_0 - \alpha t}\right] - gt,\quad x(t) = \beta t - \left(\dfrac{1}{2}\right)gt^2 - \left(\dfrac{\beta}{\alpha}\right)(m_0 - \alpha t)\ln\left[\dfrac{m_0}{m_0 - \alpha t}\right]$

Exercises 3.5 (page 122)

2. capacitor voltage = $-\dfrac{10,000}{100,000,001}\cos 100t + \dfrac{100,000,000}{100,000,001}\sin 100t + \dfrac{10,000}{100,000,001}e^{-1,000,000t}V$

resistor voltage = $\dfrac{10,000}{100,000,001}\cos 100t + \dfrac{1}{100,000,001}\sin 100t - \dfrac{10,000}{100,000,001}e^{-1,000,000t}V$

current = $\dfrac{10,000}{100,000,001}\cos 100t + \dfrac{1}{100,000,001}\sin 100t - \dfrac{10,000}{100,000,001}e^{-1,000,000t}A$

4. From (2), $I = \dfrac{1}{L}\int E(t)\,dt$. From the derivative of (4), $I = C\dfrac{dE}{dt}$.

6. Multiply (2) by I to derive $\dfrac{d}{dt}\left[\dfrac{1}{2}LI^2\right] + RI^2 = EI$ (power generated by the voltage source equals the power inserted into the inductor plus the power dissipated by the resistor). Multiply the equation above (4) by I, replace I by $\dfrac{dq}{dt}$ and then replace q by CE_C in the capacitor term, and derive $RI^2 + \dfrac{d}{dt}\left[\dfrac{1}{2}CE_C^2\right] = EI$ (power generated by the voltage source equals the power inserted into the capacitor plus the power dissipated by the resistor).

8. In cold weather, 96.27 hours. In (*extremely*) humid weather, 0.0485 seconds.

Exercises 3.6 (page 132)

6. For $k = 0, 1, 2, \ldots, n$, let $x_k = kh$ and $z_k = f(x_k)$ where $h = \dfrac{1}{n}$.

a. $h(z_0 + z_1 + z_2 + \cdots + z_{n-1})$

b. $\left(\dfrac{h}{2}\right)(z_0 + 2z_1 + 2z_2 + \cdots + 2z_{n-1} + z_n)$

c. Same as (b)

8.

x_n	1.2	1.4	1.6	1.8
y_n	1.48	2.24780	3.65173	6.88712

10.

x_n	y_n
0.1	1.15845
0.2	1.23777
0.3	1.26029
0.4	1.24368
0.5	1.20046
0.6	1.13920
0.7	1.06568
0.8	0.98381
0.9	0.89623
1.0	0.80476

12. $\phi(\pi) \approx y(\pi;\ \pi 2^{-4}) = 1.09589$

14. 2.36 at $x = 0.78$

16. $x = 1.26$

20.

t	$r = 1.0$	$r = 1.5$	$r = 2.0$
0	0.0	0.0	0.0
0.2	1.56960	1.41236	1.19211
0.4	2.63693	2.14989	1.53276
0.6	3.36271	2.51867	1.71926
0.8	3.85624	2.70281	1.84117
1.0	4.19185	2.79483	1.92743
1.2	4.42005	2.84084	1.99113
1.4	4.57524	2.86384	2.03940
1.6	4.68076	2.87535	2.07656
1.8	4.75252	2.88110	2.10548
2.0	4.80131	2.88398	2.12815
2.2	4.83449	2.88542	2.14599
2.4	4.85705	2.88614	2.16009
2.6	4.87240	2.88650	2.17126
2.8	4.88283	2.88668	2.18013
3.0	4.88992	2.88677	2.18717
3.2	4.89475	2.88682	2.19277
3.4	4.89803	2.88684	2.19723
3.6	4.90026	2.88685	2.20078
3.8	4.90178	2.88686	2.20360
4.0	4.90281	2.88686	2.20586
4.2	4.90351	2.88686	2.20765
4.4	4.90399	2.88686	2.20908
4.6	4.90431	2.88686	2.21022
4.8	4.90453	2.88686	2.21113
5.0	4.90468	2.88686	2.21186

Exercises 3.7 (page 142)

2. $y_{n+1} = y_n + h(x_n y_n - y_n^2) + \left(\dfrac{h^2}{2}\right)(y_n + (x_n - 2y_n)(x_n y_n - y_n^2))$

4. $y_{n+1} = y_n + h(x_n^2 + y_n) + \left(\dfrac{h^2}{2!}\right)(2x_n + x_n^2 + y_n) + \left(\dfrac{h^3}{3!}\right)(2 + 2x_n + x_n^2 + y_n) + \left(\dfrac{h^4}{4!}\right)(2 + 2x_n + x_n^2 + y_n)$

6. Order 2: $\phi(1) \approx 0.62747$;
order 4: $\phi(1) \approx 0.632310$

8. 0.63211

10. 0.70139 with $h = 0.25$

12. –0.928 at $x = 1.2$

14. 1.00000 with $h = \dfrac{\pi}{16}$

16.

x_n	y_n
0.5	1.17677
1.0	0.37628
1.5	1.35924
2.0	2.66750
2.5	2.00744
3.0	2.72286
3.5	4.11215
4.0	3.72111

20. $x(10) \approx 2.23597 \times 10^{-4}$

CHAPTER 4

Exercises 4.1 (page 159)

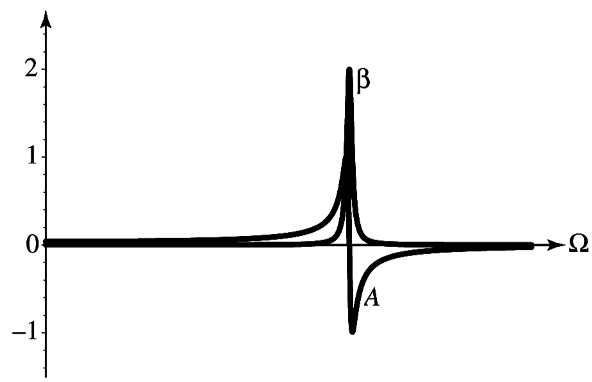
4. Both solutions approach zero as $t \to \infty$.

6. The oscillations get larger until Hooke's law becomes invalid.

8. $-\dfrac{30}{61}\cos 3t - \dfrac{25}{61}\sin 3t$

10. **a.** $$A\cos\Omega t + B\sin\Omega t = \frac{k - m\Omega^2}{(k-m\Omega^2)^2 + b^2\Omega^2}\cos\Omega t + \frac{b\Omega}{(k-m\Omega^2)^2 + b^2\Omega^2}\sin\Omega t$$

b.



c.

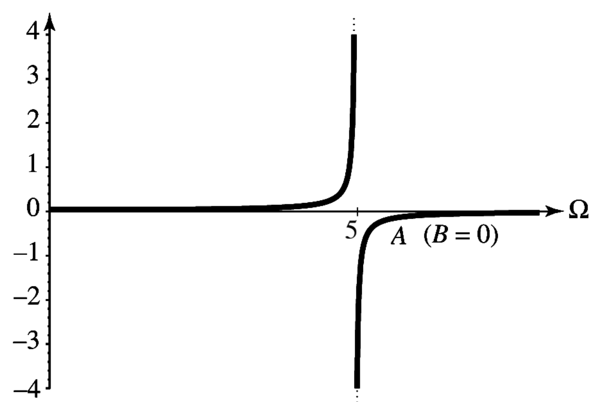


Figure 10

Exercises 4.2 (page 167)

2. $c_1e^{-t}+c_2e^{2t}$

4. $c_1e^{-3t}+c_2te^{-3t}$

6. $c_1e^{2t}+c_2e^{3t}$

8. $c_1e^{t/2}+c_2e^{-2t/3}$

10. $c_1e^{t/2}+c_2te^{t/2}$

12. $c_1e^{\left(-11+\sqrt{205}\right)t/6}+c_2e^{\left(-11-\sqrt{205}\right)t/6}$

14. $3-e^{-t}$

16. $\dfrac{4e^t}{3}-\dfrac{e^{3t}}{3}$

18. $2e^{3t}+\dfrac{7te^{3t}}{3}$

20. $(2-t)e^{2t-2}$

22. $\dfrac{ce^{7t}}{3}$

24. $ce^{-11t/3}$

26. **a.** $2\cos t$

b. $2\cos t+c_2\sin t$, where c_2 is arbitrary.

28. linearly independent

30. linearly independent

32. linearly dependent

38. If $c_1\neq 0$, then $y_1=\left[-\dfrac{c_2}{c_1}\right]y_2$.

42. $c_1e^{-t}+c_2e^{t}+c_3e^{6t}$

44. $c_1e^{-t}+c_2e^{3t}+c_3e^{5t}$

46. $c_1e^{t}+c_2e^{\left(-1+\sqrt{3}\right)t}+c_3e^{\left(-1-\sqrt{3}\right)t}$

48. $e^{t}+e^{2t}$

Exercises 4.3 (page 177)

2. $c_1\cos t+c_2\sin t$

4. $c_1e^{5t}\cos t+c_2e^{5t}\sin t$

6. $c_1e^{2t}\cos\sqrt{3}t+c_2e^{2t}\sin\sqrt{3}t$

8. $c_1e^{-t/2}\cos\left(\dfrac{\sqrt{5}t}{2}\right)+c_2e^{-t/2}\sin\left(\dfrac{\sqrt{5}t}{2}\right)$

10. $c_1e^{-2t}\cos 2t+c_2e^{-2t}\sin 2t$

12. $c_1\cos\sqrt{7}t+c_2\sin\sqrt{7}t$

14. $c_1e^{t}\cos 5t+c_2e^{t}\sin 5t$

16. $c_1e^{\left(3+\sqrt{53}\right)t/2}+c_2e^{\left(3-\sqrt{53}\right)t/2}$

18. $c_1e^{-7t}+c_2e^{t/2}$

20. $c_1e^{-t}+c_2e^{t}\cos t+c_3e^{t}\sin t$

22. $e^{-t}\cos 4t$

24. $\left(\dfrac{1}{3}\right)\sin 3t+\cos 3t$

26. $e^{t}-3te^{t}$

28. $b=5$: $y=\dfrac{4e^{-t}-e^{-4t}}{3}$

$b=4$: $y=(1+2t)e^{-2t}$

$b=2$: $y=e^{-t}\cos\sqrt{3}t+3^{-1/2}e^{-t}\sin\sqrt{3}t$

32. **a.** $y(t) = 0.3\cos 5t - 0.02\sin 5t$ m

b. $\dfrac{5}{2\pi}$

34. $I(t)=\left(\dfrac{2}{5}\right)e^{-t}\sin 25t$

36. **a.** $y(t)=\left[1-i\left(\dfrac{3}{2}\right)\right]e^{(-1+i)t}+\left[1+i\left(\dfrac{3}{2}\right)\right]e^{(-1-i)t}$

40. $c_1x+c_2x^{-7}$

42. $c_1x^2 \sin\left[\sqrt{2}\ln x\right] + c_1x^2 \cos\left[\sqrt{2}\ln x\right]$

44. $c_1x^{-1}\sin\left[2\ln x\right] + c_1x^{-1}\cos\left[2\ln x\right]$

Exercises 4.4 (page 186)

2. Yes

4. Yes

6. Yes

8. Yes

10. $y_p = -10$

12. $x_p = 3t^2 - 12t + 24$

14. $y_p = 2^x\left[(\ln 2)^2 + 1\right]^{-1}$

16. $\theta p = -\dfrac{t\sin t + \cos t}{2}$

18. $y_p = -2t\cos 2t$

20. $y_p = -2t^2\cos 2t + t\sin 2t$

22. $x_p = 2t^4e^t$

24. $y_p = x^2\sin x + x\cos x$

26. $y_p = t^2e^{-t}\sin t + te^{-t}\cos t$

28. $\left(At^4 + Bt^3 + Ct^2 + Dt + E\right)e^t$

30. $Ae^t\sin t + Be^t\cos t$

32. $t\left(At^6 + Bt^5 + Ct^4 + Dt^3 + Et^2 + Ft\right)e^{-3t}$

34. $-\dfrac{e^{-t}}{4}$

36. $-\dfrac{\sin t}{4}$

Exercises 4.5 (page 192)

2. **a.** $\dfrac{t}{4} - \dfrac{1}{8} + \left(\dfrac{1}{4}\right)\sin 2t$

b. $\dfrac{t}{2} - \dfrac{1}{4} - \left(\dfrac{3}{4}\right)\sin 2t$

c. $\dfrac{11t}{4} - \dfrac{11}{8} - 3\sin 2t$

4. $c_1 + c_2e^{-t} + t$

6. $c_1e^{-2x} + c_2e^{-3x} + e^x + x^2$

8. $c_1e^{\sqrt{2}x} + c_2e^{-\sqrt{2}x} + \tan x$

10. Yes

12. No

14. Yes

16. Yes

18. $c_1e^{3t} + c_2e^{-t} - t^2 + \dfrac{4t}{3} + \dfrac{1}{9}$

20. $c_1\cos 2\theta + c_2\sin 2\theta + \dfrac{\sin\theta - \cos\theta}{3}$

22. $c_1e^{-3x}\cos x + c_2e^{-3x}\sin x + x^4 - x^2 + 2$

24. $t^3 - t + 3$

26. $\cos 3t + 2\sin 3t + 3$

28. $\dfrac{e^{-4t}}{60} + \dfrac{1}{12} - \dfrac{e^t}{10} - \dfrac{e^{2t}}{6} + \dfrac{7e^{3t}}{6}$

30. $t^2 - 4t + 7 - \dfrac{e^t}{4} - \dfrac{27e^{-t}}{4} - \dfrac{te^{-t}}{2}$

32. $(At^2 + Bt + C)e^{2t}$

34. $A\cos t + B\sin t + C\cos 2t + D\sin 2t$

36. $e^{2t}(At^4 + Bt^3 + Ct^2)$

38. $-2\sin t + \cos t$

40. $-3t^2 + 2t$

42. **a.** $y(t) = A\sin\beta t + B\cos\beta t + y_h$, where $A = \dfrac{k - m\beta^2}{(k - m\beta^2)^2 + b^2\beta^2}$, $B = \dfrac{-b\beta}{(k - m\beta^2)^2 + b^2\beta^2}$ and

$$y_h = e^{-bt/2m}\left[c_1\cos\left(\frac{\sqrt{4mk - b^2}}{2m}t\right) + c_2\sin\left(\frac{\sqrt{4mk - b^2}}{2m}t\right)\right]$$

b. In each case, $y_h \to 0$ as $t \to \infty$. So as $t \to \infty$, $y(t)$ approaches the function $y_p(t) = A\sin\beta t + B\cos\beta t$.

44. $\sin 3t - \cos 3t + e^{-t}\sin 2t$

46. If $\lambda^2 = 1$ the general solution is $-\dfrac{t\cos t}{2} + c_1\sin t + c_2\cos t$; if $\lambda^2 = 4, 9, \ldots$ the general solution is $\dfrac{\sin t}{\lambda^2 - 1} + c_1\sin\lambda t + c_2\cos\lambda t$. In neither case is there a solution satisfying the boundary conditions.

Exercises 4.6 (page 197)

2. $c_1\cos t + c_2\sin t + t\sin t + (\cos t)\ln|\cos t|$

4. $c_1e^t + c_2e^{-t} - 2t - 4$

6. $c_1e^{-t} + c_2te^{-t} + \dfrac{t^2e^{-t}}{2}$

8. $c_1\cos 3t + c_2\sin 3t + \dfrac{(\sin 3t)\ln|\sec 3t + \tan 3t| - 1}{9}$

10. $c_1e^{-2t} + c_2te^{-2t} + \left(\dfrac{t^2e^{-2t}}{2}\right)\ln t - \dfrac{3t^2e^{-2t}}{4}$

12. $c_1\cos t + c_2\sin t + \dfrac{e^{3t}}{10} - 1 - (\cos t)\ln|\sec t + \tan t|$

14. $c_1\cos\theta + c_2\sin\theta + \sin\theta\tan\theta - \dfrac{\sec\theta}{2}$

16. $c_1e^{-2t} + c_2e^{-3t} + 3t^2 - 5t + \dfrac{19}{6}$

18. $c_1e^{3t} + c_2te^{3t} + \dfrac{e^{3t}}{2t}$

22. $c_1t^2 + c_2t^3 + t^3\ln|t| + \dfrac{1}{6}$

24. $c_1e^t + c_2e^t\ln t + \dfrac{t^2e^t}{4}$

Exercises 4.7 (page 208)

2. Inertia = 1, damping = 0, stiffness = –6y, and is negative for $y > 0$ (which explains why solutions $y = \dfrac{1}{(c-t)^2} > 0$ run away).

4. Suppose $\dfrac{a_1}{(3-t)^2} + \dfrac{a_2}{(2-t)^2} + \dfrac{a_3}{(1-t)^2} \equiv 0$ for $-1 < t < 1$. By taking limits as $t \to 1$, we conclude $a_3 = 0$.

Therefore $\dfrac{a_1}{(3-t)^2} + \dfrac{a_2}{(2-t)^2} \equiv 0$, so $a_1(2-t)^2 + a_2(3-t)^2 = (a_1+a_2)t^2 + (-4a_1 - 6a_2)t + 4a_1 + 9a_2 \equiv 0$.

Thus $a_1 + a_2 = 0$ and $-4a_1 - 6a_2 = 0$ and $4a_1 + 9a_2 = 0$, which implies $a_1 = a_2 = 0$. Hence $a_1 = a_2 = a_3 = 0$ and they are linearly independent.

6. $y'' = -\dfrac{k}{m}y = -\dfrac{d}{dy}\left(\dfrac{k}{2m}y^2\right)$, so $\dfrac{1}{2}(y')^2 + \dfrac{k}{2m}y^2 = \text{constant}$, or $m(y')^2 + ky^2 = 2m \cdot (\text{constant})$.

8. By eq. (21), $\theta'' = -\dfrac{g}{\ell}\sin\theta = \dfrac{d}{d\theta}\left(\dfrac{g}{\ell}\cos\theta\right)$. Therefore $\dfrac{(\theta')^2}{2} - \dfrac{g}{\ell}\cos\theta = \text{constant}$.

10. At $t = 0$, $\theta'(0) = 0$ and $\theta(0) = \alpha$, so
$$\frac{(\theta')^2}{2} - \frac{g}{\ell}\cos\theta = \frac{\theta'(0)^2}{2} - \frac{g}{\ell}\cos\alpha = -\frac{g}{\ell}\cos\alpha.$$

Therefore, $\cos\theta(t) = \dfrac{\ell}{g}\dfrac{\theta'(t)^2}{2} + \cos\alpha \geq \cos\alpha$, and since $\cos\theta$ is a decreasing function for $0 \leq \theta \leq \pi$, $\theta(t) \leq \alpha$.

16. $y'' = -y - y^3 = \dfrac{d}{dy}\left[-\dfrac{y^2}{2} - \dfrac{y^4}{4}\right]$, so $\dfrac{(y')^2}{2} + \dfrac{y^2}{2} + \dfrac{y^4}{4} = K$.

Therefore, $\dfrac{y^2}{2} \leq K - \dfrac{(y')^2}{2} - \dfrac{y^4}{4} \leq K$, and hence $y \leq \sqrt{2K}$.

Exercises 4.8 (page 219)

2.
$$y = -\frac{1}{4}\cos 5t - \frac{1}{5}\sin 5t = \frac{\sqrt{41}}{20}\sin(5t + \phi),$$

where $\phi = \arctan\left(\dfrac{5}{4}\right) - \pi = -2.246$.

amp. $= \dfrac{\sqrt{41}}{20}$, period $= \dfrac{2\pi}{5}$, freq. $= \dfrac{5}{2\pi}$. Passes through equilibrium at time $\dfrac{\pi - \arctan\left(\frac{5}{4}\right)}{5} \approx 0.449$ sec.

4. $b = 0$: $y(t) = \cos 8t$

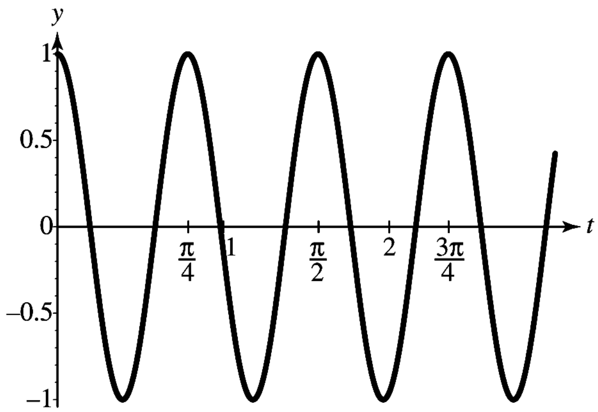


Figure 11 ($b = 0$)

$$b = 10:\ \ y(t) = e^{-5t}\cos\sqrt{39}t + \left(\frac{5}{\sqrt{39}}\right)e^{-5t}\sin\sqrt{39}t$$

$$= \left(\frac{8}{\sqrt{39}}\right)e^{-5t}\sin\left(\sqrt{39}t + \phi\right),$$

where $\phi = \arctan\left(\frac{\sqrt{39}}{5}\right) \approx 0.896.$

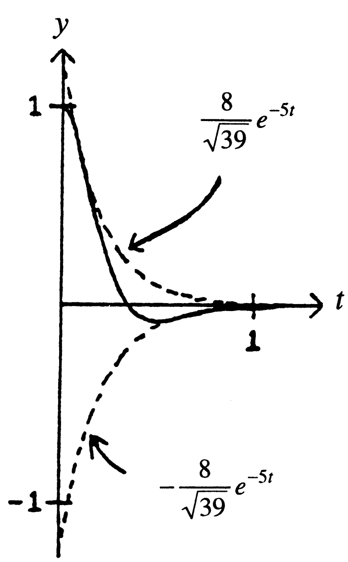


Figure 12 ($b = 10$)

$b = 16$: $y(t) = (1+8t)e^{-8t}$

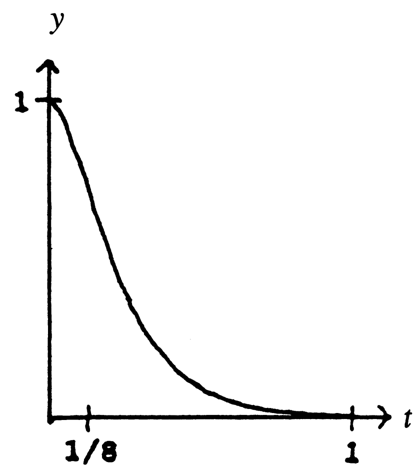


Figure 13 ($b = 16$)

$b = 20$: $y(t) = \left(\frac{4}{3}\right)e^{-4t} - \left(\frac{1}{3}\right)e^{-16t}$

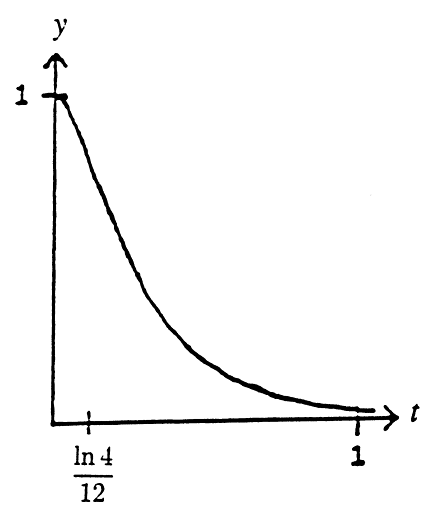


Figure 14 ($b = 20$)

6. $k = 2$: $y(t) = \left[\dfrac{1+\sqrt{2}}{2}\right] e^{\left(-2+\sqrt{2}\right)t} + \left[\dfrac{1-\sqrt{2}}{2}\right] e^{\left(-2-\sqrt{2}\right)t}$

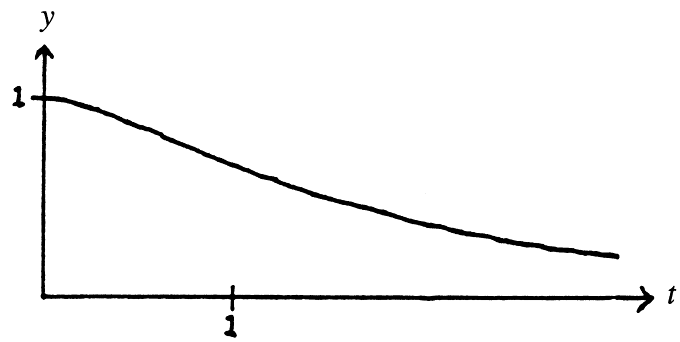


Figure 15 ($k = 2$)

$k = 4$: $y(t) = (1+2t)e^{-2t}$

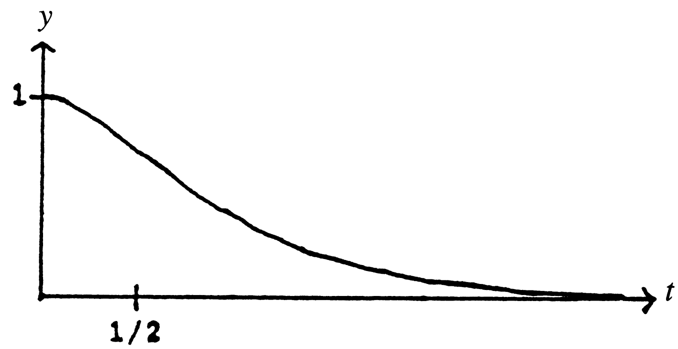


Figure 16 ($k = 4$)

$k = 6$: $y(t) = e^{-2t}\cos\sqrt{2}t + \sqrt{2}e^{-2t}\sin\sqrt{2}t$
$= \sqrt{3}\,e^{-2t}\sin\left(\sqrt{2}t + \phi\right),$

where $\phi = \arctan\left(\dfrac{\sqrt{2}}{2}\right) \approx 0.615$.

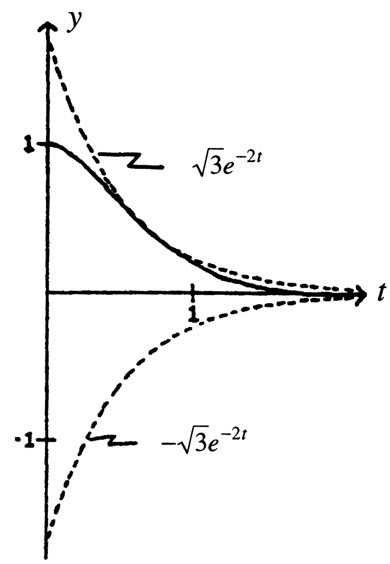


Figure 17 ($k = 6$)

8. Never **10.** $e^{-\pi/\sqrt{127}} \approx 0.755$ m **12.** 0.080 sec **16.** $\dfrac{18}{7}$ kg

Exercises 4.9 (page 227)

2. $M(\gamma) = \dfrac{1}{\sqrt{(3-2\gamma^2)^2 + 9\gamma^2}}$

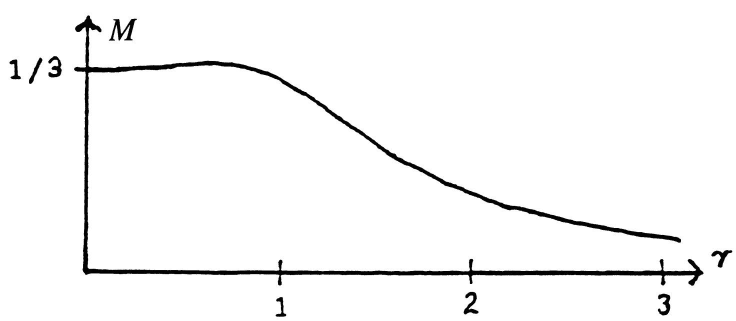


Figure 18
Response curve for Problem 2

4. $y(t) = \left[1 + \left(\dfrac{5}{2}\right)t\right]\sin t$

8. $y(t) = \left(\frac{3}{2}\right)e^{-3t} - \left(\frac{9}{2}\right)e^{-t} + 3;\ \lim_{t\to\infty} y(t) = 3,$ which is the displacement of a simple spring with spring constant $k = 6$ when a force $F_0 = 18$ is applied.

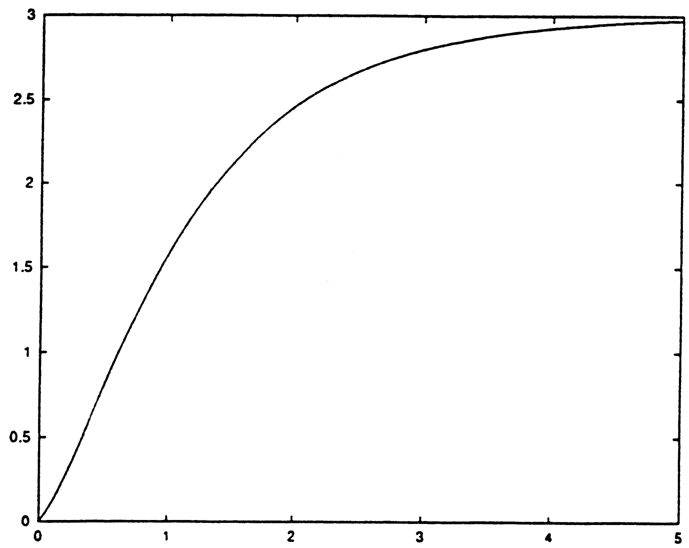


Figure 19

12. $y(t) = (0.047)e^{-5t/4}\cos\left(\frac{\sqrt{759}t}{4}\right) + (0.058)e^{-5t/4}\sin\left(\frac{\sqrt{759}t}{4}\right) + \left[\frac{3}{10\sqrt{9241}}\right]\sin(t+\theta)$ m,

where $\theta = \arctan\left(\frac{96}{5}\right) \approx 1.519;\ \frac{\sqrt{367}}{4\sqrt{2\pi}} \approx 1.078$

Chapter 4 Review (page 230)

2. $c_1e^{-t/7} + c_2te^{-t/7}$

4. $c_1e^{5t/3} + c_2te^{5t/3}$

6. $c_1e^{\left(-4+\sqrt{30}\right)t} + c_2e^{\left(-4-\sqrt{30}\right)t}$

8. $c_1e^{-2t/5} + c_2te^{-2t/5}$

10. $c_1\cos\left(\sqrt{11}t\right) + c_2\sin\left(\sqrt{11}t\right)$

12. $c_1e^{-5t/2} + c_2e^{2t} + c_3te^{2t}$

14. $c_1e^{2t}\cos\left(\sqrt{3}t\right) + c_2e^{2t}\sin\left(\sqrt{3}t\right)$

16. $e^{-t}\left[c_1 + c_2 \cos\left(\sqrt{2}t\right) + c_3 \sin\left(\sqrt{2}t\right)\right]$

18. $c_1 t^3 + c_2 t^2 + c_3 t + c_4 + t^5$

20. $c_1 e^{t/\sqrt{2}} + c_2 e^{-t/\sqrt{2}} - \left(\frac{4}{9}\right)\cos t - \left(\frac{1}{3}\right) t \sin t$

22. $c_1 e^{11t} + c_2 e^{-3t} + \left(\frac{1092}{305}\right)\cos t - \left(\frac{4641}{305}\right)\sin t$

24. $c_1 e^{t/2} + c_2 e^{-3t/5} - \left(\frac{1}{3}\right)t - \frac{1}{9} - \left(\frac{1}{11}\right) t e^{t/2}$

26. $c_1 e^{-3t} \cos\left(\sqrt{6}t\right) + c_2 e^{-3t} \sin\left(\sqrt{6}t\right) + \left(\frac{1}{31}\right) e^{2t} + 5$

28. $c_1 x^3 \cos(2 \ln x) + c_2 x^3 \sin(2 \ln x)$

30. $3e^{-\theta} + 2\theta e^{-\theta} + \sin\theta$

32. $e^{t/2} \cos t - 6e^{t/2} \sin t$

34. $e^{-7t} + 4e^{2t}$

36. $-3e^{-2t/3} + te^{-2t/3}$

38. $y(t) = -\frac{1}{4}\cos 5t + \frac{1}{5}\sin 5t$, amp. ≈ 0.320 m, period $= \frac{2\pi}{5}$, freq. $= \frac{5}{2\pi}$,
$t_{\text{equil}} = \frac{1}{5}\arctan\left(\frac{5}{4}\right) \approx 0.179$ sec.

CHAPTER 5

Exercises 5.2 (page 250)

2. $x = \left(\frac{3}{2}\right)c_1 e^{2t} - c_2 e^{-3t}$; $y = c_1 e^{2t} + c_2 e^{-3t}$

4. $x = -\left(\frac{1}{2}\right)c_1 e^{3t} + \left(\frac{1}{2}\right)c_2 e^{-t}$; $y = c_1 e^{3t} + c_2 e^{-t}$

6. $x = \left[\frac{c_1 + c_2}{2}\right]e^t \cos 2t + \left[\frac{c_2 - c_1}{2}\right]e^t \sin 2t + \left(\frac{7}{10}\right)\cos t - \left(\frac{1}{10}\right)\sin t$;
$y = c_1 e^t \cos 2t + c_2 e^t \sin 2t + \left(\frac{11}{10}\right)\cos t + \left(\frac{7}{10}\right)\sin t$

8. $x = \left(-\frac{5c_1}{4}\right)e^{11t} - \left(\frac{4}{11}\right)t - \frac{26}{121}$; $y = c_1 e^{11t} + \left(\frac{1}{11}\right)t + \frac{45}{121}$

10. $x = c_1 \cos t + c_2 \sin t$; $y = \left[\frac{c_2 - 3c_1}{2}\right]\cos t - \left[\frac{c_1 + 3c_2}{2}\right]\sin t + \left(\frac{1}{2}\right)e^t - \left(\frac{1}{2}\right)e^{-t}$

12. $u = c_1 e^{2t} + c_2 e^{-2t} + 1$; $v = -2c_1 e^{2t} + 2c_2 e^{-2t} + 2t + c_3$

14. $x = -c_1 \sin t + c_2 \cos t + 2t - 1$; $y = c_1 \cos t + c_2 \sin t + t^2 - 2$

16. $x = c_1 e^t + c_2 e^{-2t} + \left(\frac{2}{9}\right)e^{4t}$; $y = -2c_1 e^t - \left(\frac{1}{2}\right)c_2 e^{-2t} + c_3 - \left(\frac{1}{36}\right)e^{4t}$

18. $x(t) = -t^2 - 4t - 3 + c_3 + c_4 e^t - c_1 t e^t - \left(\frac{1}{2}\right)c_2 e^{-t}$; $y(t) = -t^2 - 2t - c_1 e^t + c_2 e^{-t} + c_3$

20. $x(t) = \frac{3}{2}e^t - \frac{1}{2}e^{3t}$, $y(t) = -\frac{3}{2}e^t - \frac{1}{2}e^{3t} + e^{2t}$

22. $x(t) = 1 + \frac{9}{4}e^{3t} - \frac{5}{4}e^{-t}$, $y(t) = 1 + \frac{3}{2}e^{3t} - \frac{5}{2}e^{-t}$

24. No solutions

26. $x = \left(\frac{1}{2}\right)e^t\{(c_1 - c_2)\cos t + (c_1 + c_2)\sin t\} + c_3 e^{2t}$; $y = e^t(c_1 \cos t + c_2 \sin t)$;
$z = \left(\frac{3}{2}\right)e^t\{(c_1 - c_2)\cos t + (c_1 + c_2)\sin t\} + c_3 e^{2t}$

28. $x(t) = c_1 + c_2 e^{-4t} + 2c_3 e^3 t$, $y(t) = 6c_1 - 2c_2 e^{-4t} + 3c_3 e^{3t}$, $z(t) = -13c_1 - c_2 e^{-4t} - 2c_3 e^{3t}$

30. $\lambda \le 1$

32. $x = \frac{20-10\sqrt{19}}{\sqrt{19}} e^{\left(-7+\sqrt{19}\right)t/100} + \frac{-20-10\sqrt{19}}{\sqrt{19}} e^{\left(-7-\sqrt{19}\right)t/100} + 20;$

$y = \frac{-50}{\sqrt{19}} e^{\left(-7+\sqrt{19}\right)t/100} + \frac{50}{\sqrt{19}} e^{\left(-7-\sqrt{19}\right)t/100} + 20$

34. b. $V_1 = \left(\frac{3}{2}c_2 - \frac{1}{2}c_1\right)e^{-t}\cos 3t - \left(\frac{1}{2}c_2 + \frac{3}{2}c_1\right)e^{-t}\sin 3t + 5\text{ L};\ V_2 = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t + 18\text{ L}$

c. As $t \to +\infty$, $V_1 \to 5$ L and $V_2 \to 18$ L.

36. $\frac{400}{11} \approx 36.4°\text{F}$

38. A runaway arms race.

40. **a.** $3x^2 = x^3$ **b.** $6x + 3x^2 - 2x^3$ **c.** $3x^2 + 2x^3$ **d.** $6x + 3x^2 - 2x^3$ **e.** $D^2 + D - 2$ **f.** $6x + 3x^2 - 2x^3$

Exercises 5.3 (page 261)

2. $x_1' = x_2,\ x_2' = x_1^2 + \cos(t - x_1);\ \ x_1(0) = 1,\ \ x_2(0) = 0$

4. $x_1' = x_2,\ x_2' = x_3,\ x_3' = x_4,\ x_4' = x_5,\ x_5' = x_6,\ x_6' = (x_2)^2 - \sin x_1 + e^{2t};\ \ x_1(0) = \cdots = x_6(0) = 0$

6. Setting $x_1 = x,\ x_2 = x',\ x_3 = y,\ \ x_4 = y'$, we obtain $x_1' = x_2,\ \ x_2' = -\frac{5}{3}x_1 + \frac{2}{3}x_3,\ \ x_3' = x_4,\ \ x_4' = \frac{3}{2}x_1 - \frac{1}{2}x_3$.

8. $x_1(0) = a,\ x_2(0) = p(0)b$

10.

i	t_i	$y(t_i)$
1	0.250	0.96924
2	0.500	0.88251
3	0.750	0.75486
4	1.000	0.60656

12.

i	t_i	$y(t_i)$
1	1.250	0.80761
2	1.500	0.71351
3	1.750	0.69724
4	2.000	0.74357

14. $y(8) \approx 24.01531$

16. $x(1) \approx 127.773;\ y(1) \approx -423.476$

18. Conventional troops

20. **a.** period $\approx 2(3.14)$

b. period $\approx 2(3.20)$

c. period $\approx 2(3.34)$

24. Yes, yes

26. $x(1) \approx 0.80300$; $y(1) \approx 0.59598$; $z(1) \approx 0.82316$

28. **a.**

$$\begin{aligned} x_1' &= x_2 & x_1(0) &= 1 \\ x_2' &= \frac{-x_1}{(x_1^2 + x_3^2)^{3/2}} & x_2(0) &= 0 \\ x_3' &= x_4 & x_3(0) &= 0 \\ x_4' &= \frac{-x_3}{(x_1^2 + x_3^2)^{3/2}} & x_4(0) &= 1 \end{aligned}$$

b.

i	t_i	$x_1(t_i) \approx x(t_i)$	$x_3(t_i) \approx y(t_i)$
10	0.628	0.80902	0.58778
20	1.257	0.30902	0.95106
30	1.885	–0.30902	0.95106
40	2.513	0.80902	0.58779
50	3.142	–1.00000	0.00000
60	3.770	–0.80902	–0.58778
70	4.398	–0.30903	–0.95106
80	5.027	0.30901	–0.95106
90	5.655	0.80901	–0.58780
100	6.283	1.00000	–0.00001

30. **a.**

t_i	l_i	θ_i
1.0	5.27015	0.0
2.0	4.79193	0.0
3.0	4.50500	0.0
4.0	4.67318	0.0
5.0	5.14183	0.0
6.0	5.48008	0.0
7.0	5.37695	0.0
8.0	4.92725	0.0
9.0	4.54444	0.0
10.0	4.58046	0.0

b.

t_i	l_i	θ_i
1.0	5.13916	0.45454
2.0	4.10579	0.28930
3.0	2.89358	–0.10506
4.0	2.11863	–0.83585
5.0	2.13296	–1.51111
6.0	3.18065	–1.64163
7.0	5.10863	–1.49843
8.0	6.94525	–1.29488
9.0	7.76451	–1.04062
10.0	7.68681	–0.69607

Exercises 5.4 (page 274)

2.

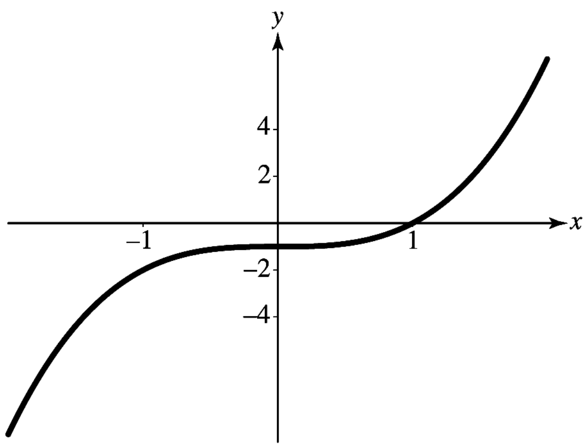


Figure 20

4. $x = -6$, $y = 1$

6. The line $y = 2$ and the point (1, 1)

8. $x^3 - x^2 y - y^{-2} = c$

10. Critical points are (1, 0) and (–1, 0).
Integral curves:
for $y > 0$, $|x| > 1$, $y = c\sqrt{x^2 - 1}$;
for $y > 0$, $|x| < 1$, $y = c\sqrt{1 - x^2}$;
for $y < 0$, $|x| > 1$, $y = -c\sqrt{x^2 - 1}$;
for $y < 0$, $|x| < 1$, $y = -c\sqrt{1 - x^2}$;
all with $c \geq 0$.
If $c = 1$, $y = \pm\sqrt{1 - x^2}$ are semicircles ending at (1, 0) and (–1, 0).

12. $9x^2 + 4y^2 = c$

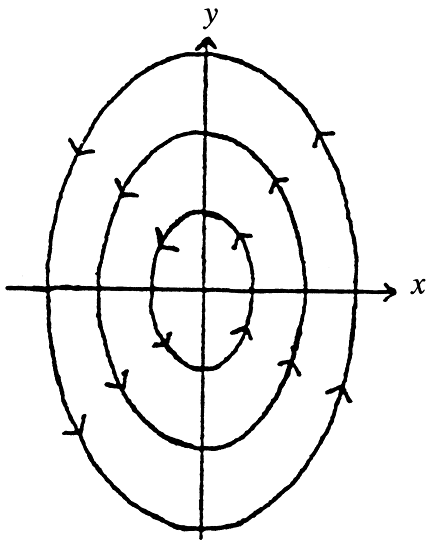


Figure 21

14. $y = cx^{2/3}$

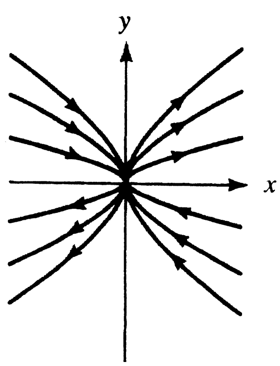


Figure 22

16. (0, 0) is a stable node.

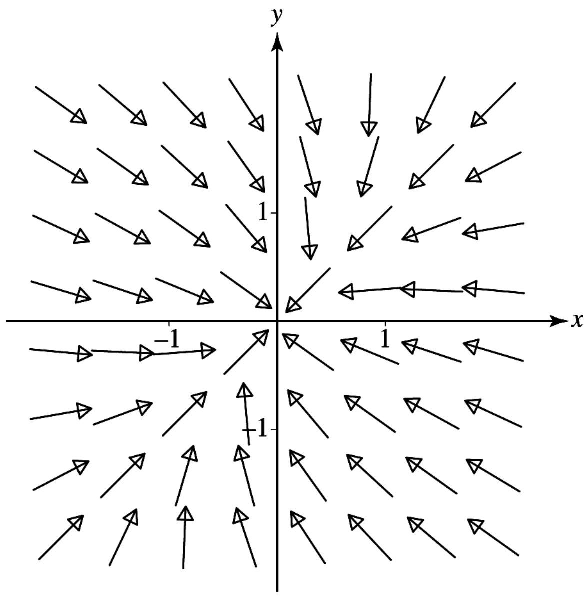


Figure 23

18. (0, 0) is an unstable node.
(0, 5) is a stable node.
(7, 0) is a stable node.
(3, 2) is a saddle point.

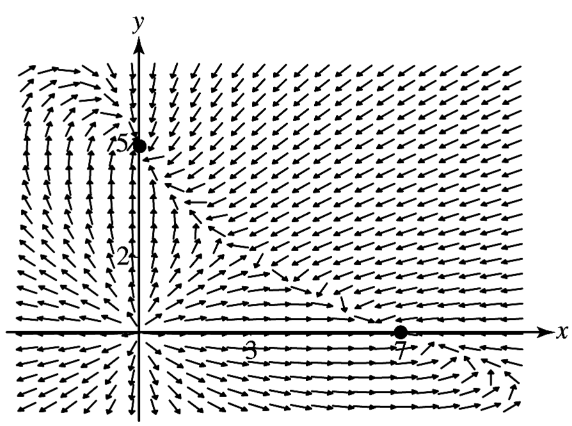


Figure 24

20. $\begin{cases} y' = v \\ v' = -y \end{cases}$; (0, 0) is a center.

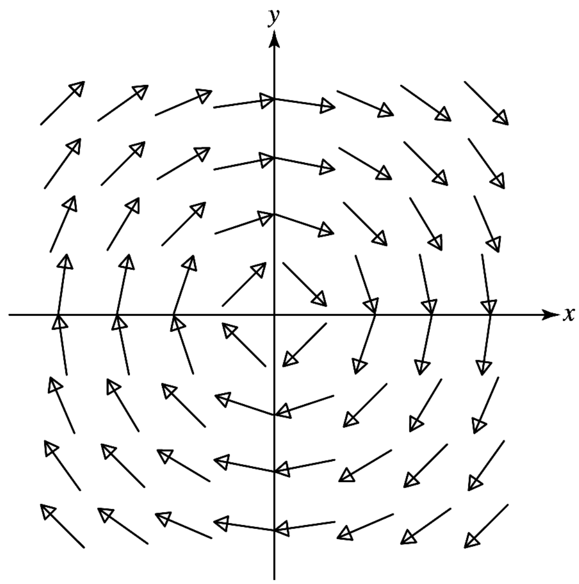


Figure 25

22. $\begin{cases} y' = v \\ v' = -y^3 \end{cases}$; (0, 0) is a center.

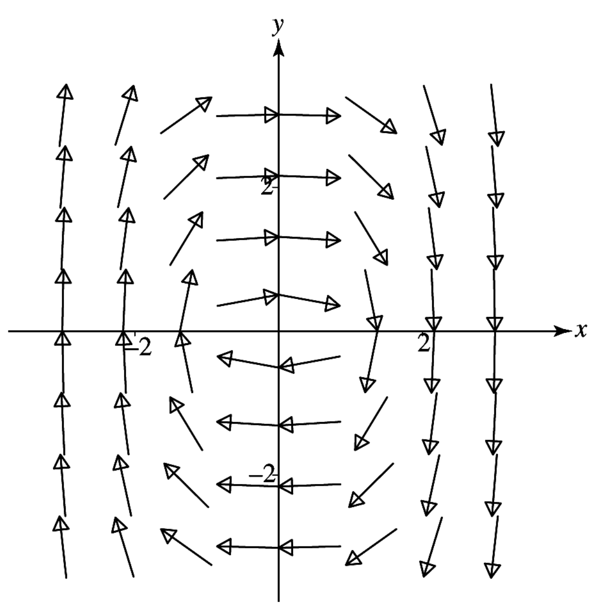


Figure 26

24. $\begin{cases} y' = v \\ v' = -y + y^3 \end{cases}$;

(0, 0) is a center.
(–1, 0) is a saddle point.
(1, 0) is a saddle point.

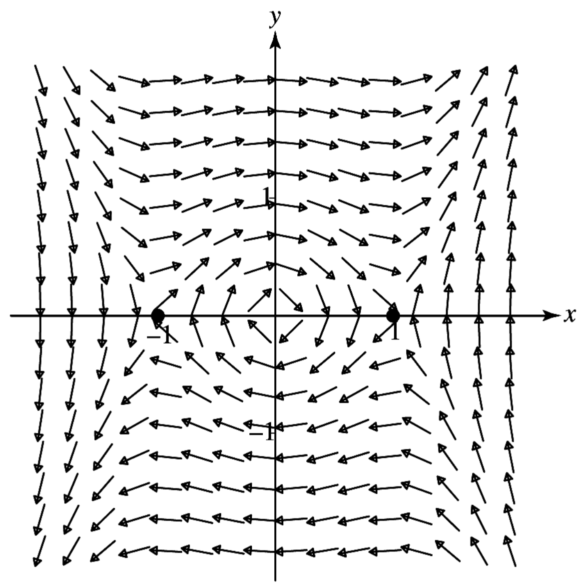


Figure 27

26. $\dfrac{x^2}{2} + \dfrac{x^4}{4} + \dfrac{y^2}{2} = c$; all solutions are bounded.

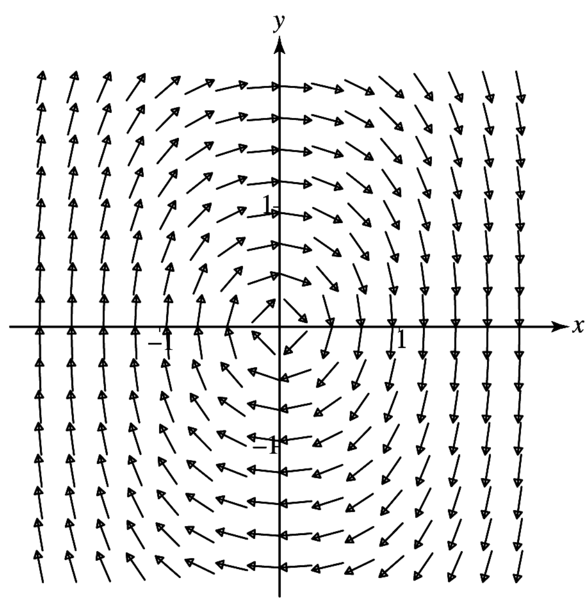


Figure 28

28. (0, 0) is a center. (1, 0) is a saddle.

30. **a.** $x(t) \to x^*$, $y(t) \to y^*$, f and g are continuous implies $x'(t) \equiv f(x(t),\ y(t)) \to f(x^*,\ y^*)$ and $y'(t) \equiv g(x(t),\ y(t)) \to g(x^*,\ y^*)$.

b.
$$\begin{aligned} x(t) &= \int_T^t x'(\tau)d\tau + x(T) \\ &> \frac{f(x^*,\ y^*)}{2}(t-T) + x(T) \\ &\equiv f(x^*,\ y^*)\frac{t}{2} + C \end{aligned}$$

c. If $f(x^*, y^*) > 0$, $f(x^*, y^*)t \to \infty$ implying $x(t) \to \infty$.

d, e. similar

32. $\dfrac{(y')^2}{2} + \dfrac{y^4}{4} = c$ by Problem 30.

Thus, $\dfrac{y^4}{4} = c - \dfrac{(y')^2}{2} \le c$, so $|y| \le \sqrt[4]{4c}$.

34. **a.** Peak will occur when $\dfrac{dI}{dt} = 0$, that is when $aSI - bI = 0$ or $S = \dfrac{b}{a}$ $\left(\text{provided } S(0) > \dfrac{b}{a}\right.$ so that S can in fact attain the value $\left.\dfrac{b}{a}\right)$.

b. $I'(t) > 0$ if $I > 0$ and $S > \dfrac{b}{a}$, while $I'(t) < 0$ if $I > 0$ and $S < \dfrac{b}{a}$.

c. It steadily decreases to zero.

36.

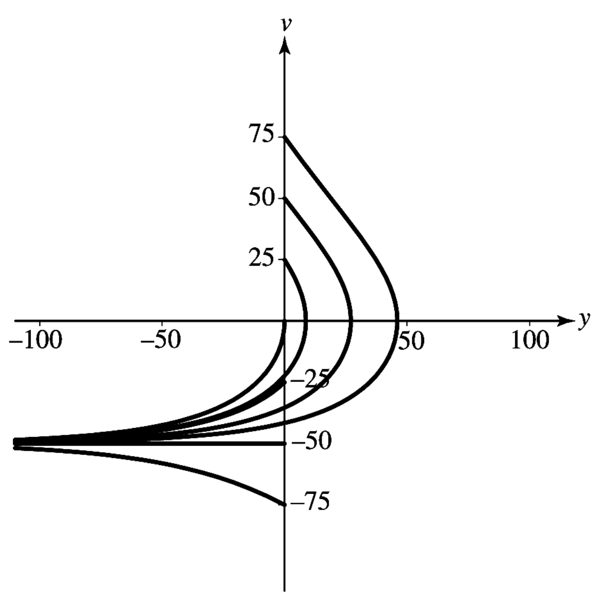


Figure 29

38. **a.** $\frac{d}{dt}[x^2+y^2+z^2]=0$; the magnitude of the angular velocity is constant.

b. All points on the axes are critical points: $(x, 0, 0)$, $(0, y, 0)$, $(0, 0, z)$.

c. From (a), $x^2+y^2+z^2=K$ (sphere). Also $\frac{dy}{dx}=\frac{-2x}{y}$, so $x^2+\frac{y^2}{2}=c$ (cylinder).

d. The solutions are periodic.

e. The critical point on the y-axis is unstable. The other two are stable.

Exercises 5.5 (page 284)

2. $x(t)=\frac{\sqrt{10}}{20}\left[\left(1-\sqrt{10}\right)\cos r_1 t-\left(1+\sqrt{10}\right)\cos r_2 t\right]$, $y(t)=\frac{3\sqrt{10}}{20}[\cos r_1 t-\cos r_2 t]$, where $r_1=\sqrt{4+\sqrt{10}}$, $r_2=\sqrt{4-\sqrt{10}}$.

4. $m_1 x''=-k_1 x+k_2(y-x)$, $m_2 y''=-k_2(y-x)-by'$

6. **b.** $x(t)=c_1\cos t+c_2\sin t+c_3\cos 2t+c_4\sin 2t+\left(\frac{37}{40}\right)\cos 3t$

c. $y(t)=2c_1\cos t+2c_2\sin t-c_3\cos 2t-c_4\sin 2t-\left(\frac{111}{20}\right)\cos 3t$

d. $x(t)=\left(\frac{23}{8}\right)\cos t-\left(\frac{9}{5}\right)\cos 2t+\left(\frac{37}{40}\right)\cos 3t$, $y(t)=\left(\frac{23}{4}\right)\cos t+\left(\frac{9}{5}\right)\cos 2t-\left(\frac{111}{20}\right)\cos 3t$

8. $\theta_1(t)=\frac{\pi}{12}\cos\left(\sqrt{\frac{9.8}{5+\sqrt{10}}}t\right)+\frac{\pi}{12}\cos\left(\sqrt{\frac{9.8}{5-\sqrt{10}}}t\right)$; $\theta_2(t)=\frac{\pi\sqrt{10}}{24}\cos\left(\sqrt{\frac{9.8}{5+\sqrt{10}}}t\right)-\frac{\pi\sqrt{10}}{24}\cos\left(\sqrt{\frac{9.8}{5-\sqrt{10}}}t\right)$

Exercises 5.6 (page 291)

2. $q(t)=\left(\frac{1}{2}\right)e^{-4t}\cos(6t)+3\cos(2t)+\sin(2t)$ coulombs

4. $I(t)=\left(\frac{10}{33}\right)\cos 5t-\left(\frac{10}{33}\right)\cos 50t$ amps

8. L = 0.01 henrys, R = 0.2 ohms, $C=\frac{25}{32}$ farads, and $E(t)=\left(\frac{2}{5}\right)\cos 8t$ volts

10. $I_1=-\left(\frac{1}{4}\right)e^{-2t}-\left(\frac{9}{4}\right)e^{-2t/3}+\frac{5}{2}$; $I_2=\left(\frac{1}{4}\right)e^{-2t}-\left(\frac{3}{4}\right)e^{-2t/3}+\frac{1}{2}$;
$I_3=-\left(\frac{1}{2}\right)e^{-2t}-\left(\frac{3}{2}\right)e^{-2t/3}+2$

12. $I_1=1-e^{-900t}$; $I_2=\left(\frac{5}{9}\right)-\left(\frac{5}{9}\right)e^{-900t}$; $I_3=\left(\frac{4}{9}\right)-\left(\frac{4}{9}\right)e^{-900t}$

Exercises 5.7 (page 301)

2. $(x_0, v_0) = (-1.5, 0.5774)$
$(x_1, v_1) = (-1.9671, -0.5105)$
$(x_2, v_2) = (-0.6740, 0.3254)$
$\vdots$
$(x_{20}, v_{20}) = (-1.7911, -0.5524)$
The limit set is the ellipse $(x+1.5)^2 + 3v^2 = 1$.

4. $(x_0, v_0) = (0, 10.9987)$
$(x_1, v_1) = (-0.00574, 10.7298)$
$(x_2, v_2) = (-0.00838, 10.5332)$
$\vdots$
$(x_{20}, v_{20}) = (-0.00029, 10.0019)$
The attractor is the point (0, 10).

12. For $F = 0.2$, attractor is the point (–0.319, –0.335). For $F = 0.28$, attractor is the point (–0.783, 0.026).

14. Attractor consists of two points: (–1.51, 0.06) and (–0.22, –0.99).

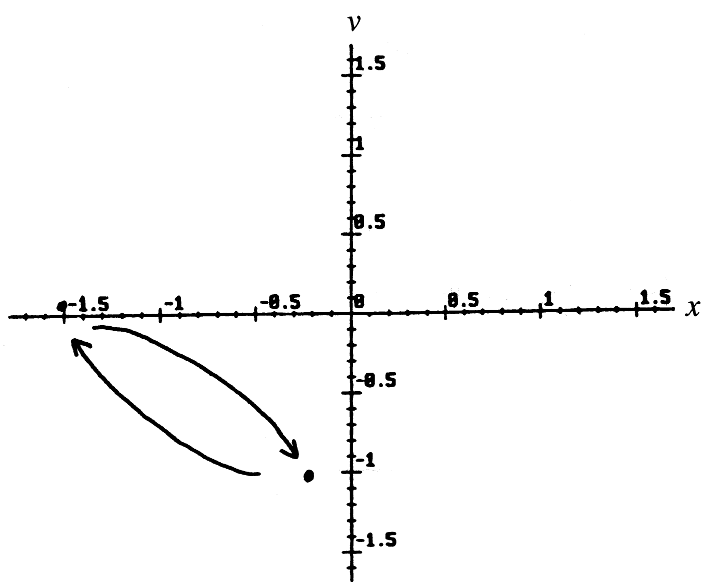


Figure 30

Chapter 5 Review Problems (page 304)

2. $x = -(c_1 + c_2)e^{-t}\cos 2t + (c_1 - c_2)e^{-t}\sin 2t$

$y = 2c_1 e^{-t}\cos 2t + 2c_2 e^{-t}\sin 2t$

4. $x = c_1 t + c_2 + e^{-t}, \quad y = \frac{c_1}{6}t^3 + \frac{c_2}{2}t^2 + c_3 t + c_4$

6. $x = e^{2t} + e^{-t}, \; y = e^{2t} + e^{-t}, \quad z = e^{2t} - 2e^{-t}$

8. With $x_1 = y, \; x_2 = y',$ we obtain $x_1' = x_2, \; x_2' = \frac{1}{2}(\sin t - 8x_1 + tx_2).$

10. With $x_1 = x, \; x_2 = x', \; x_3 = y, \; x_4 = y',$ we get $x_1' = x_2, \; x_2' = x_1 - x_3, \; x_3' = x_4, \; x_4' = -x_2 + x_3.$

12. Solutions to phase plane equation $\frac{dy}{dx} = \frac{2-x}{y-2}$ are given implicitly by $(x-2)^2 + (y-2)^2 = \text{const}.$ Critical point is at (2, 2), which is a stable center.

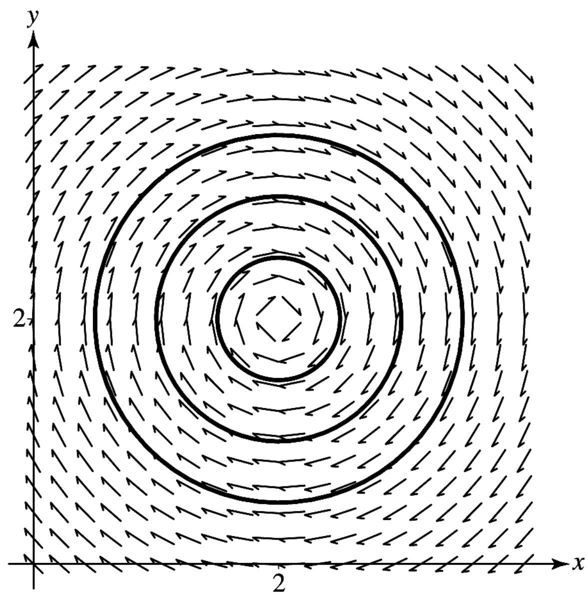


Figure 31

14. Critical points are $(m\pi, n\pi)$, m, n integers and $\left(\frac{(2j+1)\pi}{2}, \frac{(2k+1)\pi}{2}\right)$, j, k integers. Equation for integral curves is $\frac{dy}{dx} = \frac{\cos x \sin y}{\sin x \cos y} = \frac{\tan y}{\tan x}$, with solutions $\sin y = C \sin x$.

16. Origin is saddle (unstable).

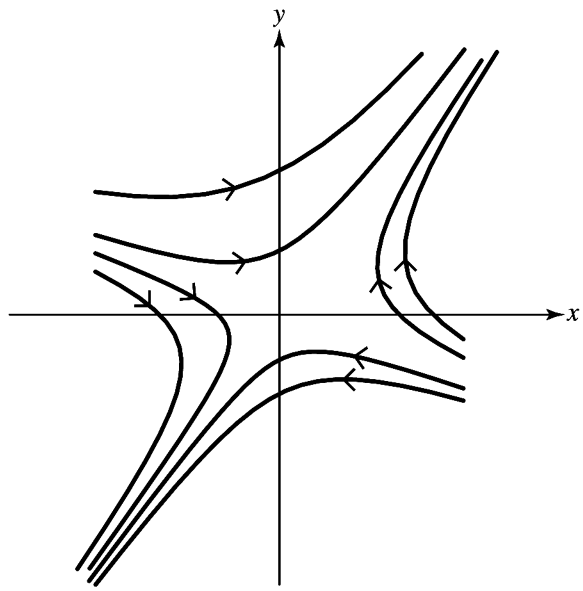


Figure 32

18. Natural angular frequencies are $\sqrt{2},\ 2\sqrt{3}$.

General solution is $x(t) = c_1 \cos\left(2\sqrt{3}t\right) + c_2 \sin\left(2\sqrt{3}t\right) + c_3 \cos\left(\sqrt{2}t\right) + c_4 \sin\left(\sqrt{2}t\right)$,

$y(t) = -\frac{1}{3} c_1 \cos\left(2\sqrt{3}t\right) - \frac{c_2}{3} \sin\left(2\sqrt{3}t\right) + 3c_3 \cos\left(\sqrt{2}t\right) + 3c_4 \sin\left(\sqrt{2}t\right)$.

CHAPTER 6

Exercises 6.1 (page 324)

2. $(0, \infty)$

4. (–1, 0)

6. (0, 1)

8. Lin. dep.; 0

10. Lin. ind.; $-2\tan^3 x - \sin x \cos x - \sin^2 x \tan x - 2\tan x$

12. Lin. dep.; 0

14. Lin. indep.; $(x+2)e^x$

16. $c_1 e^x + c_2 \cos 2x + c_3 \sin 2x$

18. $c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$

20. **a.** $c_1 + c_2 x + c_3 x^3 + x^2$

b. $2 - x^3 + x^2$

22. **a.** $c_1 e^x \cos x + c_2 e^x \sin x + c_3 e^{-x} \cos x + c_4 e^{-x} \sin x$

b. $e^x \cos x + \cos x$

24. **a.** $7 \cos 2x + 1$

b. $-6\cos 2x - \dfrac{11}{3}$

26. $y(x) = \sum_{j=1}^{n+1} \gamma_{j-1} y_j(x)$

34. The Wronskian $W[f_1, f_2, f_3](x)$

Exercises 6.2 (page 331)

2. $c_1 e^x + c_2 e^{-x} + c_3 e^{3x}$

4. $c_1 e^{-x} + c_2 e^{-5x} + c_3 e^{4x}$

6. $c_1 e^{-x} + c_2 e^x \cos x + c_3 e^x \sin x$

8. $c_1e^{x}+c_2xe^{x}+c_3e^{-7x}$

10. $c_1e^{-x}+c_2e^{\left(-1+\sqrt{7}\right)x}+c_3e^{\left(-1-\sqrt{7}\right)x}$

12. $c_1e^{x}+c_2e^{-3x}+c_3xe^{-3x}$

14. $c_1\sin x+c_2\cos x+c_3e^{-x}+c_4xe^{-x}$

16. $(c_1+c_2x)e^{-x}+(c_3+c_4x+c_5x^2)e^{6x}+c_6e^{-5x}+c_7\cos x+c_8\sin x+c_9\sin 2x+c_{10}\cos 2x$

18. $(c_1+c_2x+c_3x^2)e^{x}+c_4e^{2x}+c_5e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right)+c_6e^{-x/2}\sin\left(\frac{\sqrt{3}x}{2}\right)+(c_7+c_8x+c_9x^2)e^{-3x}\cos x$
$+(c_{10}+c_{11}x+c_{12}x^2)e^{-3x}\sin x$

20. $e^{-x}-e^{-2x}+e^{-4x}$

22. $x(t)=c_1e^{\sqrt{3}t}+c_2e^{-\sqrt{3}t}+c_3\cos 2t+c_4\sin 2t$
$y(t)=-\left(\frac{2}{5}\right)c_1e^{\sqrt{3}t}-\left(\frac{2}{5}\right)c_2e^{-\sqrt{3}t}+c_3\cos 2t+c_4\sin 2t$

28. $c_1e^{1.879x}+c_2e^{-1.532x}+c_3e^{-0.347x}$

34. $x(t)=\left(\frac{5-\sqrt{10}}{10}\right)\cos\sqrt{4+\sqrt{10}}\,t+\left(\frac{5+\sqrt{10}}{10}\right)\cos\sqrt{4-\sqrt{10}}\,t,$
$y(t)=\left(\frac{5-2\sqrt{10}}{10}\right)\cos\sqrt{4+\sqrt{10}}\,t+\left(\frac{5+2\sqrt{10}}{10}\right)\cos\sqrt{4-\sqrt{10}}\,t$

Exercises 6.3 (page 337)

2. $c_1e^{-x}+c_2\cos x+c_3\sin x$

4. $c_1+c_2xe^{x}+c_3x^2e^{x}$

6. $c_1e^{x}+c_2xe^{x}+c_3e^{-3x}+\left(\frac{1}{8}\right)e^{-x}+\left(\frac{3}{20}\right)\cos x+\left(\frac{1}{20}\right)\sin x$

8. $c_1e^{x}+c_2e^{-x}\cos x+c_3e^{-x}\sin x-\left(\frac{1}{2}\right)-\left(\frac{4}{25}\right)xe^{x}+\left(\frac{1}{10}\right)x^2e^{x}$

10. $c_1e^{2x}+c_2e^{-3x}\cos 2x+c_3e^{-3x}\sin 2x+\left(\frac{5}{116}\right)xe^{-3x}\cos 2x-\left(\frac{1}{58}\right)xe^{-3x}\sin 2x-\left(\frac{1}{26}\right)x-\left(\frac{1}{676}\right)$

12. D^3

14. $D-5$

16. $D^3(D-1)$

18. $[(D-3)^2+25]^2$

20. $D^4(D-1)^3(D^2+16)^2$

22. $c_3e^{3x}+c_4\cos x+c_5\sin x$

24. $c_3xe^x+c_4x^2e^x$

26. $c_3x^2+c_4x+c_5$

28. $c_3e^{3x}+c_4+c_5x$

30. $c_4xe^x+c_5$

32. $5+e^x\sin 2x-e^{3x}$

38. $x(t)=\left\{c_1+\frac{1}{2}+\left(\frac{1}{4}\right)t\right\}e^t+\left\{-\left(\frac{3}{2}\right)c_2-\left(\frac{\sqrt{7}}{2}\right)c_3\right\}e^{-t/2}\cos\left(\frac{\sqrt{7}t}{2}\right)+\left\{\left(\frac{\sqrt{7}}{2}\right)c_2-\left(\frac{3}{2}\right)c_3\right\}e^{-t/2}\sin\left(\frac{\sqrt{7}t}{2}\right)+t+1$

$y(t)=\left\{c_1+\left(\frac{1}{4}\right)t\right\}e^t+c_2e^{-t/2}\cos\left(\frac{\sqrt{7}t}{2}\right)+c_3e^{-t/2}\sin\left(\frac{\sqrt{7}t}{2}\right)+\frac{1}{2}$

40. $I_1(t)=\left(\frac{2187}{40}\right)\sin\left(\frac{t}{8}\right)-\left(\frac{3}{40}\right)\sin\left(\frac{t}{72}\right)-18\sin\left(\frac{t}{24}\right)$

$I_2(t)=\left(\frac{243}{40}\right)\sin\left(\frac{t}{8}\right)-\left(\frac{27}{40}\right)\sin\left(\frac{t}{72}\right)$

$I_3(t)=\left(\frac{243}{5}\right)\sin\left(\frac{t}{8}\right)+\left(\frac{3}{5}\right)\sin\left(\frac{t}{72}\right)-18\sin\left(\frac{t}{24}\right)$

Exercises 6.4 (page 341)

2. $\left(\frac{1}{2}\right)x^2 + 2x$

4. $\left(\frac{1}{6}\right)x^3 e^x$

6. $\sec\theta - \sin\theta\tan\theta + \theta\sin\theta + (\cos\theta)\ln(\cos\theta)$

8. $c_1 x + c_2 x\ln x + c_3 x^3 - x^2$

10. $\left(\frac{x^{-1}}{10}\right)\int g(x)\,dx + \left(\frac{x^4}{15}\right)\int x^{-5} g(x)dx - \left(\frac{x}{6}\right)\int x^{-2} g(x)dx$

Chapter 6 Review Problems (page 344)

2. **a.** Lin. indep.

b. Lin. indep.

c. Lin. dep.

4. **a.** $c_1 e^{-3x} + c_2 e^{-x} + c_3 e^x + c_4 x e^x$

b. $c_1 e^x + c_2 e^{\left(-2+\sqrt{5}\right)x} + c_3 e^{\left(-2-\sqrt{5}\right)x}$

c. $c_1 e^x + c_2 \cos x + c_3 \sin x + c_4 x\cos x + c_5 x\sin x$

d. $c_1 e^x + c_2 e^{-x} + c_3 e^{2x} - \left(\frac{x}{2}\right)e^x + \left(\frac{x}{2}\right) + \frac{1}{4}$

6. $c_1 e^{-x/\sqrt{2}} \cos\left(\frac{x}{\sqrt{2}}\right) + c_2 e^{-x/\sqrt{2}} \sin\left(\frac{x}{\sqrt{2}}\right) + c_3 e^{x/\sqrt{2}} \cos\left(\frac{x}{\sqrt{2}}\right) + c_4 e^{x/\sqrt{2}} \sin\left(\frac{x}{\sqrt{2}}\right) + \sin(x^2)$

8. **a.** $c_3 x e^{-x} + c_4 + c_5 x + c_6 x^2$

b. $c_4 x e^{-x} + c_5 x^2 e^{-x}$

c. $c_5 + c_6 x + c_7 x^2 + c_8 \cos 3x + c_9 \sin 3x$

d. $c_4 \cos x + c_5 \sin x + c_6 x\cos x + c_7 x\sin x$

10. **a.** $c_1 x^{1/2} + c_2 x^{-1/2} + c_3 x$

b. $c_1 x^{-1} + c_2 x\cos\left(\sqrt{3}\ln x\right) + c_3 x\sin\left(\sqrt{3}\ln x\right)$

CHAPTER 7

Exercises 7.2 (page 359)

2. $\dfrac{2}{s^3}$, $s > 0$

4. $\dfrac{1}{(s-3)^2}$, $s > 3$

6. $\dfrac{s}{s^2+b^2}$, $s > 0$

8. $\dfrac{2}{(s+1)^2+4}$, $s > -1$

10. $\dfrac{1}{s}+\dfrac{e^{-s}-1}{s^2}$, $s > 0$

12. $\dfrac{1-e^{6-3s}}{s-2}+\dfrac{e^{-3s}}{s}$, $s > 2$

14. $\dfrac{5}{s}-\dfrac{1}{s-2}+\dfrac{12}{s^3}$, $s > 2$

16. $\dfrac{2}{s^3}-\dfrac{3}{s^2}-\dfrac{6}{(s+1)^2+9}$, $s > 0$

18. $\dfrac{24}{s^5}-\dfrac{6}{s^3}-\dfrac{1}{s^2}+\dfrac{\sqrt{2}}{s^2+2}$, $s > 0$

20. $\dfrac{s+2}{(s+2)^2+3}-\dfrac{2}{(s+2)^3}$, $s > -2$

22. Piecewise continuous

24. Piecewise continuous

26. Neither

28. Continuous

30. $\lim\limits_{s\to\infty} F(s) = 0$

Exercises 7.3 (page 0)

2. $\dfrac{6}{s^3}-\dfrac{1}{s-2}$

4. $\dfrac{72}{s^5}-\dfrac{4}{s^3}+\dfrac{1}{s}$

6. $\dfrac{2}{(s+2)^2+4}+\dfrac{2}{(s-3)^3}$

8. $\dfrac{1}{s}+\dfrac{2}{s+1}+\dfrac{1}{s+2}$

10. $\dfrac{(s-2)^2-25}{[(s-2)^2+25]^2}$

12. $\dfrac{3}{s^2+36}$

14. $\dfrac{2}{(s-7)[(s-7)^2+4]}$

16. $\dfrac{3s^2+4}{s^2(s^2+4)^2}$

18. $\dfrac{s}{2[s^2+(n+m)^2]}+\dfrac{s}{2[s^2+(n-m)^2]}$

20. $\dfrac{20(s^2+9)(s^2+49)-40s^2(2s^2+58)}{(s^2+9)^2(s^2+49)^2}$

30. $H(s) = \dfrac{1}{s^2+5s+6}$

32. $\dfrac{e^{-2s}}{s}$

34. $\dfrac{e^{-\pi s}}{s^2+1}$

Exercises 7.4 (page 374)

2. $\sin 2t$

4. $\left(\frac{4}{3}\right)\sin 3t$

6. $\left(\frac{3}{16}\right)e^{-5t/2}t^2$

8. $\left(\frac{1}{24}\right)t^4$

10. $\left(\frac{1}{2}\right)e^{-t/4}\cos\left(\frac{\sqrt{47}t}{4}\right)-\left[\frac{5}{2\sqrt{47}}\right]e^{-t/4}\sin\left(\frac{\sqrt{47}t}{4}\right)$

12. $\frac{2}{s+1}-\frac{3}{s-2}$

14. $\frac{1}{s+1}+\frac{2}{s-1}-\frac{11}{s-2}$

16. $-\frac{2}{s+2}+2\frac{s}{s^2+9}-3\frac{3}{s^2+9}$

18. $\frac{1}{s}+\frac{2}{s^2}+\frac{3}{s^3}-\frac{1}{s+1}$

20. $\frac{1}{4}\frac{1}{s-1}+\frac{1}{2}\frac{1}{(s-1)^2}-\frac{1}{4}\frac{1}{s+1}$

22. $3e^t-2e^{-3t}$

24. $5e^t+2e^{2t}\cos 3t-5e^{2t}\sin 3t$

26. $1-\left(\frac{3}{2}\right)t^2+6e^{2t}$

28. $-\frac{2}{3}+\frac{5e^{-t}}{6}+\frac{4e^{2t}}{15}-\frac{13e^{-3t}}{30}$

30. $-\frac{7e^{-t}}{4}-\frac{3te^{-t}}{2}+\frac{7e^t}{4}$

32. $F_1(s)=F_2(s)=F_3(s)=\frac{1}{1-s}$;

$\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}=e^t=f_3(t)$

34. $\frac{e^{3t}-e^{4t}}{t}$

36. $\frac{\sin t}{t}$

Exercises 7.5 (page 383)

2. $e^{2t}-3e^{-t}$

4. $e^t-e^{-5t}-e^{-t}$

6. $2e^{3t}+e^{2t}\sin t$

8. $2-t+t^2-2\cos 2t+2\sin 2t$

10. $-t-e^{-2t}+2te^{-2t}+e^{2t}$

12. $10+6t+e^{t+1}+2te^{t+1}$

14. $t+\pi\cos t+\sin t$

16. $\frac{-s^3-s^2+2}{s^3(s^2+6)}$

18. $\frac{s^3-2s^2-s+3}{(s-1)(s-2)(s^2-2s-1)}$

20. $\frac{6}{s^5(s+3)}$

22. $\frac{2s^3-17s^2+28s-12}{(s-1)^3(s-5)}$

24. $\frac{s^3+2s^2+s+2se^{-3s}+e^{-3s}}{s^2(s-1)(s+1)}$

26. $2+e^t-3e^{-2t}+e^{-3t}$

28. $-2e^{-t}+e^{t}+e^{-t}\cos 2t$

30. $-\frac{6}{25}+\frac{t}{5}+\left[\frac{5a}{4}+\frac{b}{4}+\frac{1}{4}\right]e^{-t}-\left[\frac{a}{4}+\frac{b}{4}+\frac{1}{100}\right]e^{-5t}$

32. $(3a-b+6)e^{2t}+6te^{2t}+3t^2e^{2t}+(b-2a-6)e^{3t}$

36. $2-t$

38. $3t$

40. $-ae^{\mu t/2I}\left\{\cos\left(\frac{\sqrt{4Ik-\mu^2}\,t}{2I}\right)+\left(\frac{\mu}{\sqrt{4Ik-\mu^2}}\right)\sin\left(\frac{\sqrt{4Ik-\mu^2}\,t}{2I}\right)\right\}$

Exercises 7.6 (page 395)

2. $\frac{e^{-s}-e^{-4s}}{s}$

4. $\frac{e^{-s}(s+1)}{s^2}$

6. $(t+1)u(t-2)$; $\frac{e^{-2s}(3s+1)}{s^2}$

8. $(\sin t)u\left(t-\frac{\pi}{2}\right)$; $\frac{e^{-\pi s/2}s}{s^2+1}$

10. $(t-1)^2u(t-1)$; $\frac{e^{-s}2}{s^3}$

12. $(t-3)u(t-3)$

14. $\left\{\frac{\sin 3(t-3)}{3}\right\}u(t-3)$

16. $\left(\frac{1}{2}\right)[\sin 2(t-1)]u(t-1)$

18. $[\cos(t-1)+2e^{t-1}]u(t-1)$

20. $\sin t+\sin 2t+\cos 2t+\left[\left(\frac{1}{2}\right)\sin 2t-\sin t\right]u(t-2\pi)$

22. $\dfrac{1-e^{-(s-1)}}{(s-1)(1-e^{-s})}$

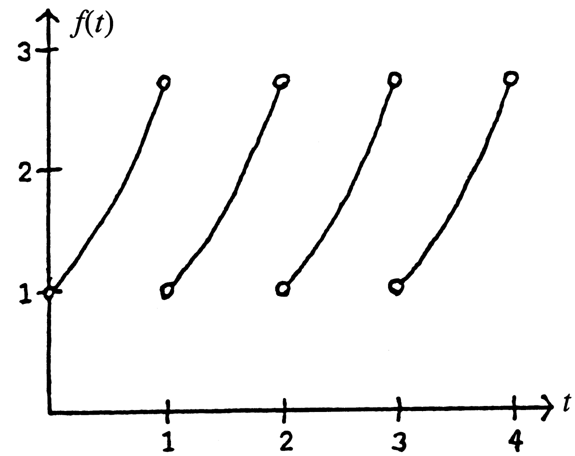


Figure 33

24. $\dfrac{se^{-2s}+e^{-2s}-2e^{-s}-se^{-s}+1}{s^2(1-e^{-2s})}$

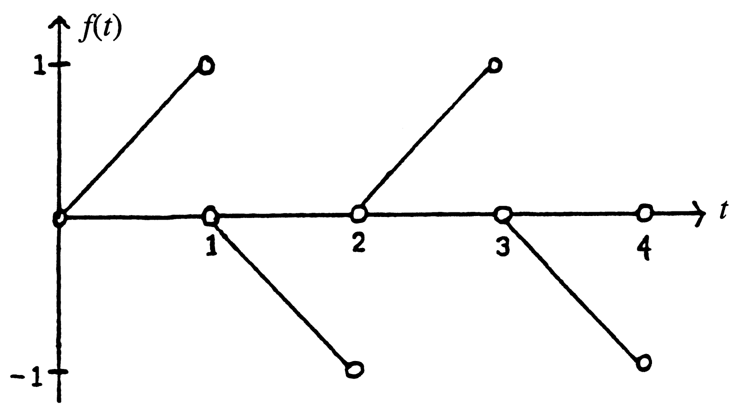


Figure 34

26. $\dfrac{1-e^{-as}-ase^{-as}}{as^2(1-e^{-as})}$

28. $\dfrac{1}{(s^2+1)(1-e^{-\pi s})}$

30. $\cos t + [1 - \cos(t-2)]u(t-2) + [1 - \cos(t-4)]u(t-4)$

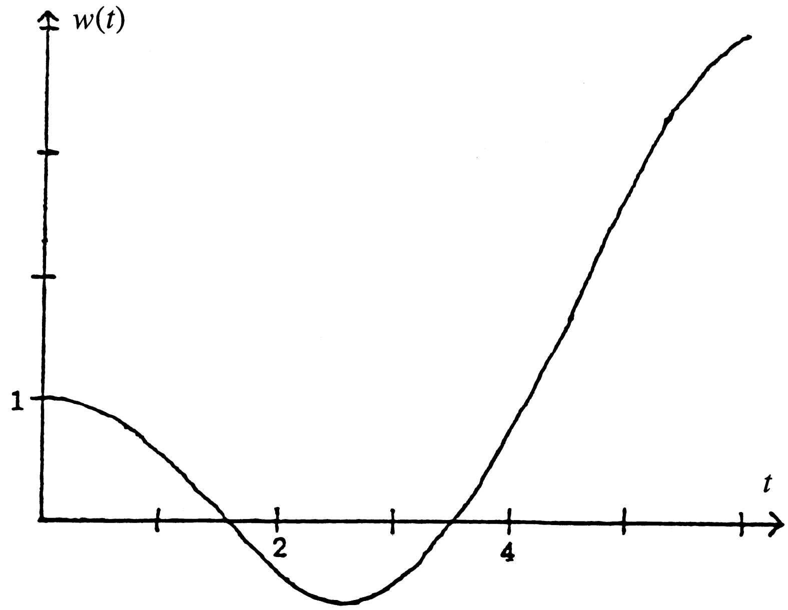


Figure 35

32. $\cos t - \sin 2t - 2(\sin t)\,u(t-2\pi) + (\sin 2t)\,u(t-2\pi)$

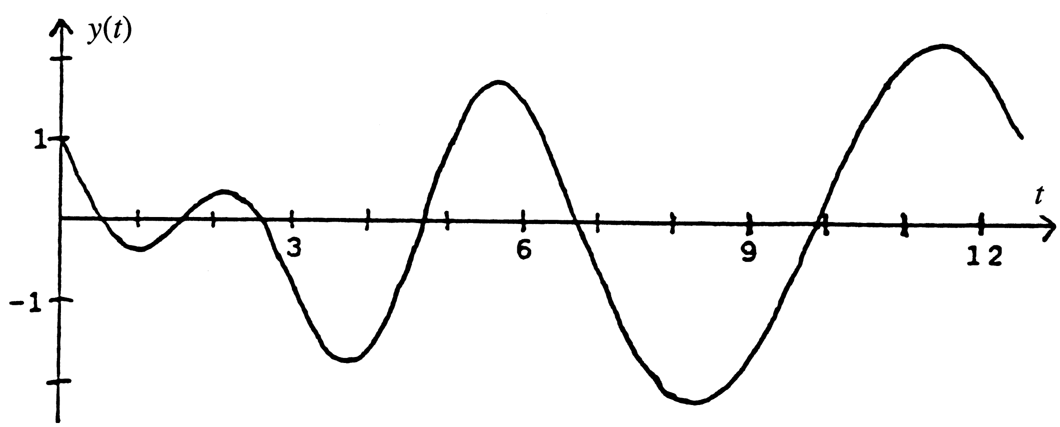


Figure 36

34. $y(t) = \left(\frac{1}{4}\right)[1 - e^{-2(t-\pi)} - 2(t-\pi)e^{-2(t-\pi)}]u(t-\pi) - \left(\frac{1}{4}\right)[1 - e^{-(t-2\pi)} - 2(t-2\pi)e^{-2(t-2\pi)}]u(t-2\pi)$

36. $e^{-2t} - e^{-3t} + \left[\left(\frac{7}{16}\right) + \left(\frac{1}{6}\right)(t-2) - \left(\frac{3}{4}\right)e^{-2(t-2)} + \left(\frac{5}{9}\right)e^{-3(t-2)}\right]u(t-2)$

38. $1 - 2e^{-t}\cos 3t - \left(\frac{2}{3}\right)e^{-t}\sin 3t + \left\{1 - e^{-(t-10)}\cos[3(t-10)] - \left(\frac{1}{3}\right)e^{-(t-10)}\sin[3(t-10)]\right\}u(t-10)$
$- \left\{2 - 2e^{-(t-20)}\cos[3(t-20)] - \left(\frac{2}{3}\right)e^{-(t-20)}\sin[3(t-20)]\right\}u(t-20)$

40. $2e^{-t} + te^{-t} + [(1-2e^{-3})e^{-(t-3)} + e^{-3}(t-3)e^{-(t-3)} + (2e^{-3}-1)e^{-2(t-3)}]u(t-3)$

42. c.

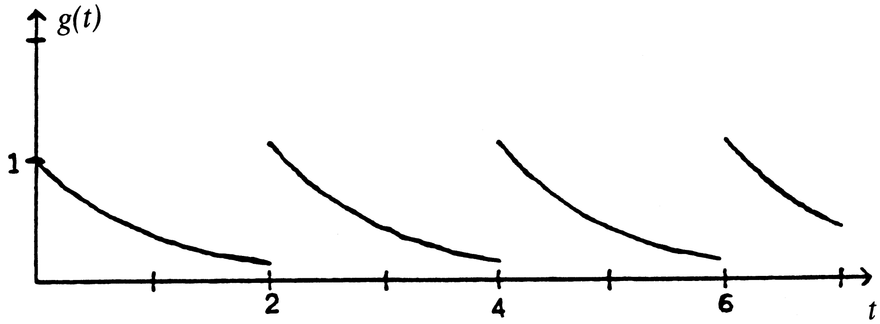


Figure 37

46. $h(t) - h(t-1)u(t-1)$, where $h(t) = \dfrac{1+n}{2} - \dfrac{e^{-t}(e^{2(n+1)}-1)}{e^2-1} + \dfrac{e^{-2t}(e^{4(n+1)}-1)}{2(e^4-1)}$, for $2n < t < 2(n+1)$.

48. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{s^2}\right)^n$

50. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{k!\,s^{2k+1}}$

60. $\left[\dfrac{e^{t-3}}{2} + \dfrac{e^{-(t-3)}}{2} - 1\right]u(t-3) - \left[\dfrac{e^{t-7}}{2} + e^{-(t-7)} - 1\right]u(t-7) + e^{-t}$

62. The concentration of salt: $\begin{cases} 0.6 - 0.4e^{-3t/125} & 0 \le t \le 10 \\ 0.4 - 0.4e^{-3t/125} + 0.2e^{(-3/125)(t-10)} & 10 \le t \le 20 \\ 0.6 - 0.4e^{-3t/125} + 0.2e^{(-3/125)(t-10)} - 2e^{(-3/125)(t-20)} & t \ge 20 \end{cases}$

Exercises 7.7 (page 405)

2. $\cos 3t + \left(\dfrac{1}{3}\right)\displaystyle\int_0^t \sin[3(t-v)]g(v)\,dv$

4. $\displaystyle\int_0^t g(v)\sin(t-v)dv + \sin t$

6. $e^{-t} - e^{-2t}$

8. $\left(\dfrac{1}{16}\right)\sin 2t - \left(\dfrac{t}{8}\right)\cos 2t$

10. $\cos t - 1 + \dfrac{t^2}{2}$

12. $\left(\dfrac{1}{2}\right)(t\sin t + \sin t - t\cos t)$

14. $(s^2+1)^{-1}(s-1)^{-1}$

16. $\cos t + \sin t - 1$

18. $2 - 2\cos t$

20. $\left(\frac{1}{3}\right)e^{-t} - \left(\frac{1}{3}\right)e^{t/2}\cos\left(\frac{\sqrt{3}t}{2}\right) + \frac{1}{\sqrt{3}}e^{t/2}\sin\left(\frac{\sqrt{3}t}{2}\right)$

22. $\frac{t}{2} + 2e^{-t} + \left(\frac{3}{4}\right)e^{2t} - \frac{3}{4}$

24. $H(s) = \frac{1}{s^2 - 9}$; $h(t) = \frac{e^{3t} - e^{-3t}}{6}$; $y_k(t) = e^{3t} + e^{-3t}$; $y(t) = \frac{1}{6}\int_0^t [e^{3(t-v)} - e^{-3(t-v)}]g(v)dv + e^{3t} + e^{-3t}$

26. $H(s) = \frac{1}{(s-3)(s+5)}$; $h(t) = \frac{e^{3t} - e^{-5t}}{8}$; $y_k(t) = e^{3t} - e^{-5t}$; $y(t) = \frac{1}{8}\int_0^t [e^{3(t-v)} - e^{-5(t-v)}]g(v)dv + e^{3t} - e^{-5t}$

28. $H(s) = \frac{1}{(s-2)^2 + 1}$; $h(t) = e^{2t}\sin t$; $y_k(t) = e^{2t}\sin t$; $y(t) = \int_0^t e^{2(t-v)}(\sin(t-v))g(v)dv + e^{2t}\sin t$

30. $\frac{1}{50}\int_0^t e^{-4(t-v)}(\sin 5(t-v))e(v)dv + 2e^{-4t}\cos 5t$

32. $\frac{t^5}{60}$

38. **d.** $87.403 \approx 100\sqrt{3-\sqrt{5}} \le b \le 100\sqrt{3+\sqrt{5}} \approx 228.825$

Exercises 7.8 (page 412)

2. 1

4. e^2

6. 1

8. $3e^{-s}$

10. $27e^{-3s}$

12. e^{3-3s}

14. $e^{-t}\cos t + 2\sin t - e^{-(t-\pi)}(\sin t)u(t-\pi)$

16. $e^{3t} + e^{-t} + \left(\frac{1}{2}\right)[e^{3(t-1)} - e^{-(t-1)}]u(t-1) + \left(\frac{1}{4}\right)[e^{-(t-3)} - e^{3(t-3)}]u(t-3)$

18. $\left(\frac{4}{3}\right)e^{2t} - \left(\frac{1}{2}\right)e^{t} - \left(\frac{5}{6}\right)e^{-t} + (e^{2t-2} - e^{1-t})u(t-1)$

20. $e^{-2t} + e^{-3t} + \left[e^{-2(t-1)} - e^{-3(t-4/3)}\right]u(t-2)$

22. $\sin t - (\cos t)u\left(t - \frac{\pi}{2}\right)$

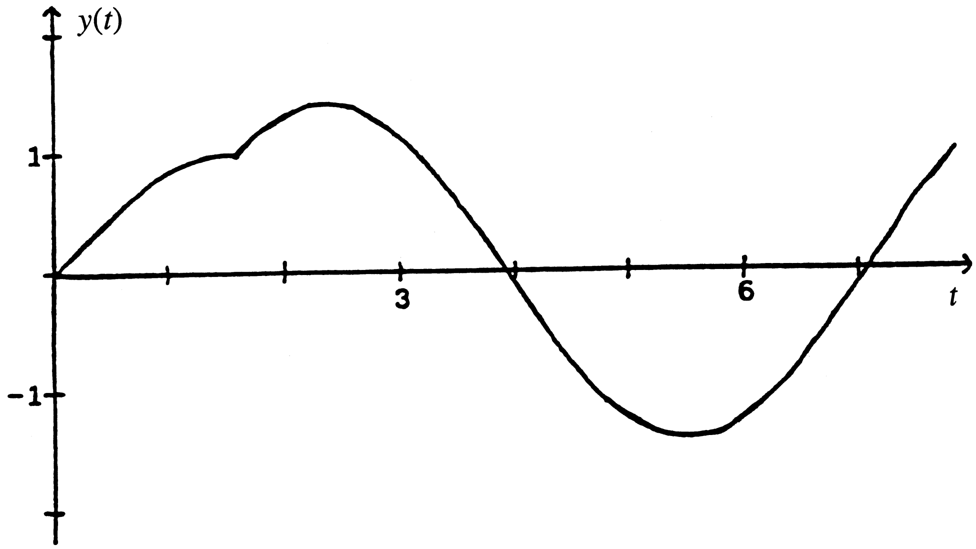


Figure 38

24. $\sin t - (\sin t)u(t-\pi) - (\sin t)u(t-2\pi)$

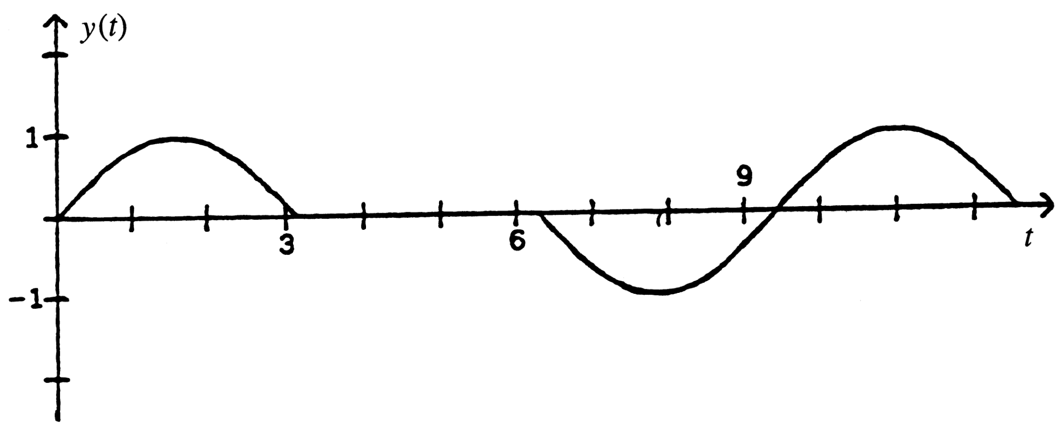


Figure 39

26. $\left(\frac{1}{2}\right)e^{3t}\sin 2t$

28. $\dfrac{(e^{t} - e^{-t})}{2}$

30. $y(t) = \sum_{k=1}^{\infty}(\sin t)u(t - 2k\pi)$

Notice that the magnitude of the oscillations of $y(t)$ becomes unbounded as $t \to \infty$.

Exercises 7.9 (page 416)

2. $x = e^{3t} - 2e^{2t}$; $y = -2e^{3t} + 2e^{2t}$

4. $x = -\left(\frac{7}{10}\right)e^t \cos 2t + \left(\frac{2}{5}\right)e^t \sin 2t + \left(\frac{7}{10}\right)\cos t - \left(\frac{1}{10}\right)\sin t$;
$y = -\left(\frac{11}{10}\right)e^t \cos 2t - \left(\frac{3}{10}\right)e^t \sin 2t + \left(\frac{11}{10}\right)\cos t + \left(\frac{7}{10}\right)\sin t$

6. $x = -e^{2t} + \left(\frac{1}{2}\right)t + 1$; $y = -e^{2t} - \left(\frac{1}{2}\right)t - \frac{3}{2}$

8. $x = \left(\frac{3}{2}\right)e^{-2t} - \left(\frac{1}{2}\right)e^{2t} + \frac{3}{4}$; $y = 3e^{-2t} + e^{2t}$

10. $x = e^t + \cos t - 1$; $y = -e^t + \cos t + 1$

12. $x = -\left(\frac{1}{2}\right) - \left(\frac{1}{6}\right)e^{2t} + \left(\frac{2}{3}\right)e^{-t} + \left[\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)(t-3) + \left(\frac{1}{12}\right)e^{2t-6} - \left(\frac{1}{3}\right)e^{-t+3}\right]u(t-3)$;
$y = -\left(\frac{1}{2}\right) + \left(\frac{1}{6}\right)e^{2t} + \left(\frac{4}{3}\right)e^{-t} + \left[-\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)(t-3) - \left(\frac{1}{12}\right)e^{2t-6} - \left(\frac{2}{3}\right)e^{-t+3}\right]u(t-3)$

14. $x = \cos t + \cosh t - 1 - [\cosh(t-1) - 1]u(t-1)$; $y = -\cos t + \cosh t - [\cosh(t-1) + 1]u(t-1)$

16. $x(t) = \cos t + \sin t + \frac{13+\sqrt{17}}{2\sqrt{17}} e^{\frac{3+\sqrt{17}}{2}(t-\pi)} - \frac{13-\sqrt{17}}{2\sqrt{17}} e^{\frac{3-\sqrt{17}}{2}(t-\pi)}$;
$y(t) = \cos t + \frac{-12+2\sqrt{17}}{\sqrt{17}} e^{\frac{3+\sqrt{17}}{2}(t-\pi)} + \frac{12+2\sqrt{17}}{\sqrt{17}} e^{\frac{3-\sqrt{17}}{2}(t-\pi)}$

18. $x(t) = t^2$; $y(t) = t$; $z(t) = t^2 - 2$

20. $x = e^{-2t} + e^{2t} + 1$; $y = 2e^{-2t} - 2e^{2t} + 2t + 2e^2 - 2e^{-2} + 2$

22. See answer to Exercises 5.5, Problem 1 on pages B-14 of text.

24. $I_1 = \left(\frac{15}{2}\right) - \left(\frac{20}{3}\right)e^{-1000t} - \left(\frac{5}{6}\right)e^{-4000t}$; $I_2 = 5 - \left(\frac{10}{3}\right)e^{-1000t} - \left(\frac{5}{3}\right)e^{-4000t}$;
$I_3 = \left(\frac{5}{2}\right) - \left(\frac{10}{3}\right)e^{-1000t} + \left(\frac{5}{6}\right)e^{-4000t}$

Chapter 7 Review Problems (page 418)

2. $\dfrac{1-e^{-5(s+1)}}{s+1}-\dfrac{e^{-5s}}{s}$

4. $\dfrac{4}{(s-3)^2+16}$

6. $\dfrac{7(s-2)}{(s-2)^2+9}-\dfrac{70}{(s-7)^2+25}$

8. $2s^{-3}+6s^{-2}-(s-2)^{-1}-6(s-1)^{-1}$

10. $\dfrac{s}{s^2+1}+\dfrac{e^{-\pi s/2}}{(1+s^2)(1-e^{-\pi s})}$

12. $2e^{2t}\cos\left(\sqrt{2}t\right)+\left(\dfrac{3}{\sqrt{2}}\right)e^{2t}\sin\left(\sqrt{2}t\right)$

14. $e^{-t}-3e^{-3t}+3e^{2t}$

16. $\dfrac{\sin 3t-3t\cos 3t}{54}$

18. $f(t)=\sum_{n=0}^{\infty}\dfrac{(-1)^n t^{2n}}{n!}$; $F(s)=\sum_{n=0}^{\infty}\dfrac{(-1)^n(2n)!}{n!}\dfrac{1}{s^{2n+1}}$

20. $(t-3)e^{-3t}$

22. $\left(\dfrac{10}{13}\right)e^{2t}-\left(\dfrac{23}{13}\right)\cos 3t+\left(\dfrac{15}{13}\right)\sin 3t$

24. $6e^t+4te^t+t^2e^t+2te^{2t}-6e^{2t}$

26. $(1+Ct^3)e^{-t}$, where C is an arbitrary constant

28. $\left[\dfrac{3}{2\sqrt{7}}\right]e^{-t/2}\sin\left(\dfrac{\sqrt{7}t}{2}\right)-\left(\dfrac{1}{2}\right)e^{-t/2}\cos\left(\dfrac{\sqrt{7}t}{2}\right)-\dfrac{1}{2}$

30. $\left(\dfrac{1}{2}\right)\left[\sin 2t+\left(\sin 2\left(t-\dfrac{\pi}{2}\right)\right)u\left(t-\dfrac{\pi}{2}\right)\right]$

32. $x=-\left(\dfrac{1}{2}\right)+\left(\dfrac{1}{6}\right)e^{2t}+\left(\dfrac{4}{3}\right)e^{-t}+\left[-\left(\dfrac{1}{4}\right)+\left(\dfrac{1}{2}\right)(t-3)-\left(\dfrac{1}{12}\right)e^{2t-6}-\left(\dfrac{2}{3}\right)e^{-t+3}\right]u(t-3)$

$y=-\left(\dfrac{1}{2}\right)-\left(\dfrac{1}{6}\right)e^{2t}+\left(\dfrac{2}{3}\right)e^{-t}+\left[\left(\dfrac{1}{4}\right)+\left(\dfrac{1}{2}\right)(t-3)+\left(\dfrac{1}{12}\right)e^{2t-6}-\left(\dfrac{1}{3}\right)e^{-t+3}\right]u(t-3)$

CHAPTER 8

Exercises 8.1...(page 430)

2. $2+4x+8x^2+\cdots$

4. $\left(\frac{1}{2}\right)x^2+\left(\frac{1}{6}\right)x^3-\left(\frac{1}{20}\right)x^5+\cdots$

6. $x-\left(\frac{1}{6}\right)x^3+\left(\frac{1}{120}\right)x^5+\cdots$

8. $1-\frac{(\sin 1)x^2}{2}+\frac{(\cos 1)(\sin 1)x^4}{24}+\cdots$

10. **a.** $p_3(x)=\frac{1}{2}+\frac{x}{4}+\frac{x^2}{8}+\frac{x^3}{16}$

b.
$$\begin{aligned}\varepsilon_3=\frac{1}{4!}\frac{24}{(2-\xi)^5}\left(\frac{1}{2}\right)^4 &\le \frac{1}{\left(\frac{3}{2}\right)^5}\frac{1}{2^4}\\ &=\frac{2}{3^5}\\ &\approx 0.00823\end{aligned}$$

c. $\left|\frac{2}{3}-p_3\left(\frac{1}{2}\right)\right|=\frac{1}{384}\approx 0.00260$

d.

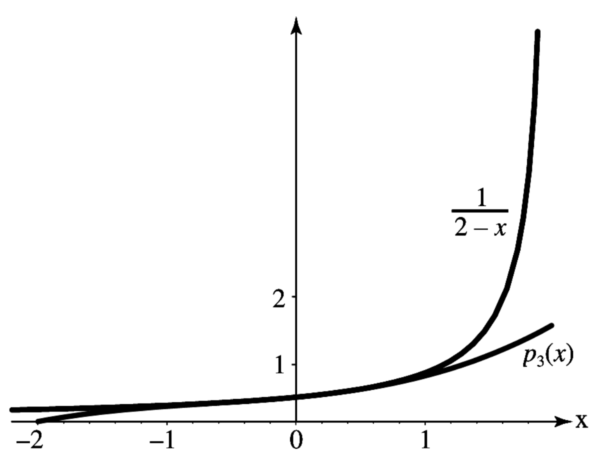


Figure 40

12. The differential equation implies $y(x)$, $y'(x)$, and $y''(x)$ exist and are continuous. Furthermore $y'''(x)$ can be obtained by differentiating the other terms: $y'''=-py''-p'y'-qy'-q'y+g'$. Since p, q, and g have derivatives of all orders, subsequent differentiations display the fact that, in turn, y''', $y^{(\text{iv})}$, $y^{(\text{v})}$, etc. all exist.

14. **a.** $t+\left(\frac{1}{2}\right)t^2-\left(\frac{1}{6}\right)t^3+\cdots$

b. For $r = 1$, $y(t)=1-\left(\frac{1}{2}\right)t^2-4t^4+\cdots$.

For $r = -1$, $y(t)=1+\left(\frac{1}{2}\right)t^2-\left(\frac{49}{12}\right)t^4+\cdots$.

c. For small t in part (b), the hard spring recoils but the soft spring extends.

16. $1-\frac{t^2}{2}+\frac{t^4}{4!}+\cdots$

Exercises 8.2 (page 438)

2. $(-\infty,\infty)$

4. [2, 4]

6. (–3, –1)

8. **a.** $\left(-\frac{1}{2},\frac{1}{2}\right)$

b. $\left(-\frac{1}{2},\frac{1}{2}\right)$

c. $(-\infty,\infty)$

d. $(-\infty,\infty)$

e. $(-\infty,\infty)$

f. $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

10. $\sum_{n=0}^{\infty}\left[\frac{2^{n+3}}{(n+3)!}+\frac{(n+2)^2}{2^{n+1}}\right](x-1)^n$

12. $x-\left(\frac{2}{3}\right)x^3+\left(\frac{2}{15}\right)x^5+\cdots$

14. 1

16. $1-\left(\frac{1}{2}\right)x+\left(\frac{1}{4}\right)x^2-\left(\frac{1}{24}\right)x^3+\cdots$

18. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \cos x$

20. $\displaystyle\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

22. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!(2k+1)}$

24. $\displaystyle\sum_{k=4}^{\infty} (k-2)(k-3)a_{k-2} x^k$

26. $\displaystyle\sum_{k=4}^{\infty} \frac{a_{k-3}}{k} x^k$

30. $\displaystyle\sum_{n=0}^{\infty} (-1)^n (x-1)^n$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$

34. $1+\left(\frac{1}{2}\right)(x-1)-\left(\frac{1}{8}\right)(x-1)^2+\left(\frac{1}{16}\right)(x-1)^3-\left(\frac{5}{128}\right)(x-1)^4+\cdots$

36. **a.** Always true

b. Sometimes false

c. Always true

d. Always true

38. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+2} = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \cdots$

Exercises 8.3 (page 449)

2. 0

4. −1, 0

6. −1

8. No singular points

10. $x \le 1$ and $x = 2$

12. $y = a_0\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots\right)$
$= a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}$
$= a_0 e^x$

14. $a_0\left(1 - \frac{x^2}{2} + \cdots\right) + a_1\left(x - \frac{x^3}{6} + \cdots\right)$

16. $a_0\left(1 - \frac{x^2}{2} - \frac{x^3}{3} + \cdots\right) + a_1\left(x + x^2 + \frac{x^3}{2} + \cdots\right) = a_0 e^x + (a_1 - a_0)xe^x$

18. $a_0\left(1 + \frac{1}{6}x^2 + \cdots\right) + a_1\left(x + \frac{1}{27}x^3 + \cdots\right)$

20. $a_0 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = a_0 \cos x + a_1 \sin x$

22. $a_{3k+2} = 0,\ k = 0, 1, \ldots$

$a_0\left(1 + \sum_{k=1}^{\infty} \frac{1 \cdot 4 \cdots (3k-5)(3k-2)}{(3k)!} x^{3k}\right) + a_1\left(x + \sum_{k=1}^{\infty} \frac{2 \cdot 5 \cdots (3k-4)(3k-1)}{(3k+1)!} x^{3k+1}\right)$

24. $a_0\left(1 - \left(\frac{1}{2}\right)x^2 + \left(\frac{1}{24}\right)x^4 + \sum_{k=3}^{\infty} \frac{[(-1)^k (2k-3)^2 (2k-5)^2 \cdots 3^2]}{(2k)!} x^{2k}\right) + a_1 x$

26. $x + \left(\frac{1}{2}\right)x^2 + \left(\frac{1}{2}\right)x^3 + \left(\frac{1}{3}\right)x^4 + \cdots$

28. $-1 - \left(\frac{1}{2}\right)x^2 - \left(\frac{1}{3}\right)x^3 - \left(\frac{1}{8}\right)x^4 + \cdots$

30. $-1 + x + \left(\frac{5}{2}\right)x^2 - \left(\frac{1}{6}\right)x^3$

32. **a.** If $a_1 = 0$, then $y(x)$ is an even function.

d. $a_0 = 0,\ a_1 > 0$

36. $3 - \left(\frac{9}{2}\right)t^2 + t^3 + t^4 + \cdots$

Exercises 8.4 (page 456)

2. infinite

4. 2

6. 1

8. $a_0\left[1-2(x+1)+3(x+1)^2-\left(\frac{10}{3}\right)(x+1)^3+\cdots\right]$

10. $a_0\left[1-\left(\frac{1}{4}\right)(x-2)^2-\left(\frac{1}{24}\right)(x-2)^3+\cdots\right]+a_1\left[(x-2)+\left(\frac{1}{4}\right)(x-2)^2-\left(\frac{1}{12}\right)(x-2)^3+\cdots\right]$

12. $a_0\left[1+\left(\frac{1}{2}\right)(x+1)^2+\left(\frac{2}{3}\right)(x+1)^3+\cdots\right]+a_1\left[(x+1)+2(x+1)^2+\left(\frac{7}{3}\right)(x+1)^3+\cdots\right]$

14. $1+x+x^2+\left(\frac{5}{6}\right)x^3+\cdots$

16. $-t+\left(\frac{1}{3}\right)t^3+\left(\frac{1}{12}\right)t^4+\left(\frac{1}{24}\right)t^5+\cdots$

18. $1+\left(x-\frac{\pi}{2}\right)+\left(\frac{1}{2}\right)\left(x-\frac{\pi}{2}\right)^2-\left(\frac{1}{24}\right)\left(x-\frac{\pi}{2}\right)^4+\cdots$

22. $a_0\left[1-\left(\frac{1}{2}\right)x^2+\cdots\right]+\left[x+\left(\frac{1}{2}\right)x^2-\left(\frac{1}{6}\right)x^3+\cdots\right]$

24. $a_0\left[1-\left(\frac{3}{2}\right)x^2+\cdots\right]+a_1\left[x-\left(\frac{1}{6}\right)x^3+\cdots\right]+\left[\frac{2}{3}-x^2\right]$

26. $a_0[1-x^2]+a_1\left[x-\left(\frac{1}{6}\right)x^3+\cdots\right]+\left[\left(\frac{1}{2}\right)x^2+\cdots\right]$

28. $a_0\left[1+\left(\frac{1}{6}\right)x^3+\cdots\right]+a_1[x+\cdots]+\left[\left(\frac{1}{2}\right)x^2+\cdots\right]$

30. $1-\left(\frac{1}{2}\right)t^2+\left(\frac{1}{2}\right)t^3-\left(\frac{1}{4}\right)t^4+\cdots$

Exercises 8.5 (page 460)

2. $c_1 x^{-5/2} + c_2 x^{-3}$

4. $c_1 x^{-\left(1+\sqrt{13}\right)/2} + c_2 x^{-\left(1-\sqrt{13}\right)/2}$

6. $c_1 x \cos\left(\sqrt{3}\ln x\right) + c_2 x \sin\left(\sqrt{3}\ln x\right)$

8. $c_1 + c_2 x^{-1/2} \cos\left[\left(\dfrac{\sqrt{5}}{2}\right)\ln x\right] + c_3 x^{-1/2} \sin\left[\left(\dfrac{\sqrt{5}}{2}\right)\ln x\right]$

10. $c_1 x^{-2} + c_2 x^{-2} \ln x + c_3 x^{-2} (\ln x)^2$

12. $c_1 (x+2)^{1/2} \cos[\ln(x+2)] + c_2 (x+2)^{1/2} \sin[\ln(x+2)]$

14. $c_1 x + c_2 x^{-1/2} + x \ln x - 2x^{-2} \ln x$

16. $3x^{-2} + 13x^{-2} \ln x$

Exercises 8.6 (page 472)

2. 0 is regular.

4. 0 is regular.

6. ±2 are regular.

8. 0 and 1 are regular.

10. 0 and 1 are regular.

12. $r^2 + 3r + 2 = 0;\ r_1 = -1,\ r_2 = -2$

14. $r^2 - r = 0;\ r_1 = 1,\ r_2 = 0$

16. $r^2 - \dfrac{5r}{4} - \dfrac{3}{4} = 0;\ \ r_1 = \dfrac{5+\sqrt{73}}{8},\ r_2 = \dfrac{5-\sqrt{73}}{8}$

18. $r^2 - r - \left(\dfrac{3}{4}\right) = 0;\ r_1 = \dfrac{3}{2},\ r_2 = -\dfrac{1}{2}$

20. $a_0\left[1-\left(\frac{1}{3}\right)x-\left(\frac{1}{15}\right)x^2-\left(\frac{1}{35}\right)x^3+\cdots\right]$

22. $a_0\left[1+4x+4x^2+\left(\frac{16}{9}\right)x^3+\cdots\right]$

24. $a_0\left[x^{1/3}+\left(\frac{1}{3}\right)x^{4/3}+\left(\frac{1}{18}\right)x^{7/3}+\left(\frac{1}{162}\right)x^{10/3}+\cdots\right]$

26. $a_0\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n!}=a_0xe^{-x}$

28. $a_0\left[x^{1/3}+\sum_{n=1}^{\infty}\frac{(-1)^n x^{n+1/3}}{n![10\cdot 13\cdots(3n+7)]}\right]$

30. $a_0\left[1+\left(\frac{4}{5}\right)x+\left(\frac{1}{5}\right)x^2\right]$

32. $a_0\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}=a_0xe^x$; yes, $a_0<0$

34. $a_0\left[1+\left(\frac{1}{2}\right)x\right]$; yes, $a_0<0$

36. $a_0\left[x+\left(\frac{1}{20}\right)x^2+\left(\frac{1}{1960}\right)x^3+\left(\frac{1}{529,200}\right)x^4+\cdots\right]$

38. $a_0\left[x^{5/6}+\left(\frac{31}{726}\right)x^{11/6}+\left(\frac{2821}{2517768}\right)x^{17/6}+\left\{\frac{629083}{(9522)(2517768)}\right\}x^{23/6}+\cdots\right]$

40. $a_0[1+2x+2x^2]$

42. The transformed equation is $18z(4z-1)^2(6z-1)\frac{d^2y}{dz^2}+9(4z-1)(96z^2-40z+3)\frac{dy}{dz}+32y=0$.

Also, $zp(z)=\frac{96z^2-40z+3}{2(4z-1)(6z-1)}$ and $z^2q(z)=\frac{32z}{18(4z-1)^2(6z-1)}$ are analytic at $z = 0$; hence $z = 0$ is a regular singular point.

$y_1(x)=a_0\left[1+\left(\frac{32}{27}\right)x^{-1}+\left(\frac{1600}{243}\right)x^{-2}+\left(\frac{241664}{6561}\right)x^{-3}+\cdots\right]$

Exercises 8.7 (page 482)

2. $c_1y_1(x)+c_2y_2(x)$, where $y_1(x)=1-\left(\frac{1}{3}\right)x-\left(\frac{1}{15}\right)x^2+\cdots$ and $y_2(x)=x^{-1/2}-x^{1/2}$.

4. $c_1y_1(x)+c_2y_2(x)$, where $y_1(x)=1+4x+4x^2+\cdots$ and $y_2(x)=y_1(x)\ln x-8x-12x^2-\left(\frac{176}{27}\right)x^3+\cdots$.

6. $c_1\left[x^{1/3}+\left(\frac{1}{3}\right)x^{4/3}+\left(\frac{1}{18}\right)x^{7/3}+\cdots\right]+c_2\left[1+\left(\frac{1}{2}\right)x+\left(\frac{1}{10}\right)x^2+\cdots\right]$

8. $c_1y_1(x)+c_2y_2(x)$, where $y_1(x)=x-x^2+\left(\frac{1}{2}\right)x^3+\cdots$ and $y_2(x)=y_1(x)\ln x+x^2-\left(\frac{3}{4}\right)x^3+\left(\frac{11}{36}\right)x^4+\cdots$.

10. $c_1\left[x^{1/3}-\left(\frac{1}{10}\right)x^{4/3}+\left(\frac{1}{260}\right)x^{7/3}+\cdots\right]+c_2\left[x^{-2}+\left(\frac{1}{4}\right)x^{-1}+\left(\frac{1}{8}\right)+\cdots\right]$

12. $c_1y_1(x)+c_2y_2(x)$, where $y_1(x)=1+\left(\frac{4}{5}\right)x+\left(\frac{1}{5}\right)x^2$ and $y_2(x)=x^{-4}+4x^{-3}+5x^{-2}$.

14. $c_1y_1(x)+c_2y_2(x)$, where $y_1(x)=x+x^2+\left(\frac{1}{2}\right)x^3+\cdots$ and $y_2(x)=y_1(x)\ln x-x^2-\left(\frac{3}{4}\right)x^3-\left(\frac{11}{36}\right)x^4+\cdots$.

16. $c_1\left[1+\left(\frac{1}{2}\right)x\right]+c_2\left[-x^{-1}-x\ln x-2\ln x-\left(\frac{1}{2}\right)+\left(\frac{9}{4}\right)x+\cdots\right]$; has a bounded solution near the origin, but not all solutions are bounded near the origin.

18. $c_1y_1(x)+c_2y_2(x)+c_3y_3(x)$, where $y_1(x)=x+\left(\frac{1}{20}\right)x^2+\left(\frac{1}{1960}\right)x^3+\cdots$,

$y_2(x)=x^{2/3}+\left(\frac{3}{26}\right)x^{5/3}+\left(\frac{9}{4940}\right)x^{8/3}+\cdots$, $y_3(x)=x^{-1/2}+2x^{1/2}+\left(\frac{2}{5}\right)x^{3/2}+\cdots$.

20. $c_1y_1(x)+c_2y_2(x)+c_3y_3(x)$, where $y_1(x)=x^{5/6}+\left(\frac{31}{726}\right)x^{11/6}+\left(\frac{2821}{2517768}\right)x^{17/6}+\cdots$,

$y_2(x)=1+x+\left(\frac{1}{28}\right)x^2+\cdots$, and $y_3(x)=y_2(x)\ln x-9x-\left(\frac{3}{98}\right)x^2+\left(\frac{437}{383292}\right)x^3+\cdots$.

22. $c_1y_1(x)+c_2y_2(x)$, where $y_1(x)=x-\frac{\alpha^2}{2}x^2+\frac{\alpha^4}{12}x^3-\frac{\alpha^6}{144}x^4+\cdots$, and

$y_2(x)=-\alpha^2y_1(x)\ln x+1+\alpha^2x-\frac{5\alpha^4}{4}x^2+\frac{5\alpha^6}{18}x^3+\cdots$.

26. d. $a_1=-\frac{2+i}{5}$, $a_2=\frac{2+i}{20}$

Exercises 8.8 (page 493)

2. $c_1 F\left(3,\ 5;\ \frac{1}{3};\ x\right) + c_2 x^{2/3} F\left(\frac{11}{3},\ \frac{17}{3};\ \frac{5}{3};\ x\right)$

4. $c_1 F\left(1,\ 3;\ \frac{3}{2};\ x\right) + c_2 x^{-1/2} F\left(\frac{1}{2},\ \frac{5}{2};\ \frac{1}{2};\ x\right)$

6. $F(\alpha,\ \beta;\ \beta;\ x) = \sum_{n=0}^{\infty} (\alpha)_n \frac{x^n}{n!} = (1-x)^{-\alpha}$

8. $$F\left(\frac{1}{2},\ 1;\ \frac{3}{2};\ -x^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n} = x^{-1} \arctan x$$

10. $$y_1(x) = F\left(\frac{1}{2},\ \frac{1}{2};\ 2;\ x\right) = 1 + \left(\frac{1}{8}\right)x + \left(\frac{3}{64}\right)x^2 + \left(\frac{25}{1024}\right)x^3 + \cdots$$

and

$$y_2(x) = y_1(x)\ln x + 4x^{-1} + \left(\frac{5}{16}\right)x + \left(\frac{1}{8}\right)x^2 + \cdots.$$

14. $c_1 J_{4/3}(x) + c_2 J_{-4/3}(x)$

16. $c_1 J_0(x) + c_2 Y_0(x)$

18. $c_1 J_4(x) + c_2 Y_4(x)$

20. $J_2(x)\ln x - 2x^{-2} - \left(\frac{1}{2}\right) - \left(\frac{3}{16}\right)x^2 + \left(\frac{17}{1152}\right)x^4 + \cdots$

26. $$J_{-3/2}(x) = -x^{-1}J_{-1/2}(x) - J_{1/2}(x) = -\sqrt{\frac{2}{\pi}}x^{-3/2}\cos x - \sqrt{\frac{2}{\pi}}x^{-1/2}\sin x$$

$$J_{5/2}(x) = \frac{3-x^2}{x^2}J_{1/2}(x) - 3x^{-1}J_{-1/2}(x) = 3\sqrt{\frac{2}{\pi}}x^{-5/2}\sin x - 3\sqrt{\frac{2}{\pi}}x^{-3/2}\cos x - \sqrt{\frac{2}{\pi}}x^{-1/2}\sin x$$

36. $y_1(x) = x$ and $y_2(x) = 1 - \sum_{k=1}^{\infty} \frac{1}{2k-1} x^{2k}$.

38. $2x^2 - 1,\ 4x^3 - 3x,\ 8x^4 - 8x^2 + 1$

40. **c.** $c_1 J_{(n+2)^{-1}}\left(\dfrac{2\sqrt{c}}{n+2}\, x^{n/2+1}\right) + c_2 Y_{(n+2)^{-1}}\left(\dfrac{2\sqrt{c}}{n+2}\, x^{n/2+1}\right)$

Chapter 8 Review (page 497)

2. **a.** ± 2 are irregular singular points.

b. $n\pi,$ where n is an integer, are regular singular points.

4. **a.** $a_0\left[1 + \sum_{k=1}^{\infty} \dfrac{[(-3)(-1)(1)\cdots(2k-5)]}{2^k\, k!} x^{2k}\right] + a_1\left[x - \left(\dfrac{2}{3}\right)x^3\right]$

b. $a_0\left[1 + \sum_{k=1}^{\infty} \dfrac{[3\cdot 5\cdot 15\cdots(4k^2 - 10k + 9)]}{2^k\,(2k)!} x^{2k}\right] + a_1\left[x + \sum_{k=1}^{\infty} \dfrac{[3\cdot 9\cdot 23\cdots(4k^2 - 6k + 5)]}{2^k\,(2k+1)!} x^{2k+1}\right]$

6. **a.** $c_1 x^{\left(-3+\sqrt{105}\right)/4} + c_2 x^{\left(-3-\sqrt{105}\right)/4}$

b. $c_1 x^{-1} + c_2 x^{-1} \ln x + c_3 x^2$

8. **a.** $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$ and $y_2(x) = y_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n-2}$.

b. $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n}$ and $y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-(3/2)}$.

c. $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n}$ and $y_2(x) = y_1(x)\ln x + \sum_{n=1}^{\infty} b_n x^{n}$.

10. **a.** $c_1 F\left(3,\ 2;\ \dfrac{1}{2};\ x\right) + c_2 x^{1/2} F\left(\dfrac{7}{2},\ \dfrac{5}{2};\ \dfrac{3}{2};\ x\right)$

b. $c_1 J_{1/3}(\theta) + c_2 J_{-1/3}(\theta)$

CHAPTER 9

Exercises 9.1 (page 507)

2. $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

4. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ \sqrt{\pi} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

6. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} \cos 2t & 0 & 0 \\ 0 & \sin 2t & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

8. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ \frac{-2}{1-t^2} & \frac{2t}{1-t^2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

10. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1+\frac{n^2}{t^2} & -\frac{1}{t} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

12. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

Exercises 9.2 (page 512)

2. $x_1 = 0,\ x_2 = \frac{1}{3},\ x_3 = \frac{2}{3},\ x_4 = 0$

4. $x_1 = \frac{1}{3},\ x_2 = \frac{2}{3},\ x_3 = 0,\ x_4 = 0$

6. $x_1 = -\frac{s}{4},\ x_2 = \frac{s}{4},\ x_3 = s\ (-\infty < s < \infty)$

8. $x_1 = -s+t,\ x_2 = s,\ x_3 = t\ (-\infty < s, t < \infty)$

10. $x_1 = \frac{-2+i}{5},\ x_2 = 0,\ x_3 = \frac{2+4i}{5}$

12. a. The equations produce the format

$$\begin{aligned} x_1 - \tfrac{1}{2}x_2 &= 0 \\ 0 &= 1. \end{aligned}$$

b. The equations produce the format

$$\begin{aligned} x_1 \qquad + \tfrac{1}{2}x_3 &= 0 \\ x_2 + \tfrac{11}{2}x_3 &= 0 \\ 0 &= 1. \end{aligned}$$

14. For $r = -1$, the unique solution is $x_1 = x_2 = x_3 = 0$. For $r = 2$ the solutions are $x_1 = -\frac{s}{2},\ x_2 = \frac{s}{4},\ x_3 = s\ (-\infty < s < \infty)$.

Exercises 9.3 (page 521)

2. a. $\mathbf{A}+\mathbf{B} = \begin{bmatrix} 3 & -1 & 7 \\ 2 & 4 & -1 \end{bmatrix}$

b. $7\mathbf{A} - 4\mathbf{B} = \begin{bmatrix} 10 & 4 & 27 \\ 14 & -5 & 15 \end{bmatrix}$

4. a. $\mathbf{AB} = \begin{bmatrix} 2 & 5 & -1 \\ 0 & 12 & 4 \\ -1 & 8 & 4 \end{bmatrix}$

b. $\mathbf{BA} = \begin{bmatrix} 3 & 2 \\ -1 & 15 \end{bmatrix}$

6. a. $\mathbf{AB} = \begin{bmatrix} 2 & 7 \\ 1 & 5 \end{bmatrix}$

b. $(\mathbf{AB})\mathbf{C} = \begin{bmatrix} 9 & -1 \\ 6 & 1 \end{bmatrix}$

c. $(\mathbf{A}+\mathbf{B})\mathbf{C} = \begin{bmatrix} 6 & 1 \\ 5 & -5 \end{bmatrix}$

10. $\begin{bmatrix} \frac{9}{31} & -\frac{1}{31} \\ -\frac{5}{31} & \frac{4}{31} \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$

16. c. $x = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, with c arbitrary.

18. $\begin{bmatrix} \sin 2t & \left(\frac{1}{2}\right)\cos 2t \\ \cos 2t & -\left(\frac{1}{2}\right)\sin 2t \end{bmatrix}$

20. $\begin{bmatrix} 0 & 0 & \left(\frac{1}{9}\right)e^{-3t} \\ 1 & -t & \frac{t}{3}-\frac{1}{9} \\ 0 & 1 & -\frac{1}{3} \end{bmatrix}$

22. 0

24. 11

26. 54

28. 1, 6

30. b. 0

c. $\mathbf{x} = c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

d. $c_1 = c_2 = c_3 = -1$

32. $\begin{bmatrix} 3e^{-t}\cos 3t - e^{-t}\sin 3t \\ -3e^{-t}\cos 3t + e^{-t}\sin 3t \end{bmatrix}$

34. $\begin{bmatrix} 2\cos 2t & -2\sin 2t & -2e^{-2t} \\ -2\cos 2t & -4\sin 2t & -6e^{-2t} \\ 6\cos 2t & -2\sin 2t & -2e^{-2t} \end{bmatrix}$

40. a. $\begin{bmatrix} t & -\left(\frac{1}{2}\right)e^{-2t} \\ 3t & -\left(\frac{1}{2}\right)e^{-2t} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

b. $\begin{bmatrix} -e^{-1}+1 & -e^{-1}+1 \\ e^{-1}-1 & -3e^{-1}+3 \end{bmatrix}$

c. $\begin{bmatrix} -e^{-t}+3e^{-3t} & -3e^{-t}-9e^{-3t} \\ -3e^{-t}+3e^{-3t} & -3e^{-t}-9e^{-3t} \end{bmatrix}$

42. In general, $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$. Thus $(\mathbf{A}^T\mathbf{A})^T = \mathbf{A}^T(\mathbf{A}^T)^T = \mathbf{A}^T\mathbf{A}$, so $\mathbf{A}^T\mathbf{A}$ is symmetric. Similarly, one shows that $\mathbf{AA}^T$ is symmetric.

Exercises 9.4 (page 530)

2. $\begin{bmatrix} r'(t) \\ \theta'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} r(t) \\ \theta(t) \end{bmatrix} + \begin{bmatrix} \sin t \\ 1 \end{bmatrix}$

4. $\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

6. $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ t^2 \end{bmatrix}$

8. $\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix}$

10. $x_1'(t) = 2x_1(t) + x_2(t) + te^t$,
$x_2'(t) = -x_1(t) + 3x_2(t) + e^t$

12. $x_1'(t) = x_2(t) + t + 3$
$x_2'(t) = x_3(t) - t + 1$
$x_3'(t) = -x_1(t) + x_2(t) + 2x_3(t) + 2t$

14. Linearly independent

16. Linearly dependent

18. Linearly independent

20. Yes; $\begin{bmatrix} 3e^{-t} & e^{4t} \\ 2e^{-t} & -e^{4t} \end{bmatrix}$; $c_1 \begin{bmatrix} 3e^{-t} \\ 2e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{4t} \\ -e^{4t} \end{bmatrix}$

22. Linearly independent; fundamental matrix is

$$\begin{bmatrix} e^t & \sin t & -\cos t \\ e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \end{bmatrix}.$$

The general solution is

$$c_1 \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \\ -\sin t \end{bmatrix} + c_3 \begin{bmatrix} -\cos t \\ \sin t \\ \cos t \end{bmatrix}.$$

24. $c_1 \begin{bmatrix} e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{3t} \\ e^{3t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -e^{-3t} \\ -e^{-3t} \\ e^{-3t} \end{bmatrix} + \begin{bmatrix} 5t+1 \\ 2t \\ 4t+2 \end{bmatrix}$

28. $\mathbf{X}^{-1}(t) = \begin{bmatrix} \left(\frac{1}{2}\right)e^{t} & -\left(\frac{1}{2}\right)e^{t} \\ \left(\frac{1}{2}\right)e^{-5t} & \left(\frac{1}{2}\right)e^{-5t} \end{bmatrix}$;

$\mathbf{x}(t) = \begin{bmatrix} 2e^{-t} + e^{5t} \\ -2e^{-t} + e^{5t} \end{bmatrix}$

32. Choosing $\mathbf{x}_0 = \text{col}(1, 0, 0, \ldots, 0)$, then $\mathbf{x}_0 = \text{col}(0, 1, 0, \ldots, 0)$, and so on, the corresponding solutions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ will have a nonvanishing Wronskian at the initial point t_0. Hence $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a fundamental solution set.

Exercises 9.5 (page 541)

2. Eigenvalues are $r_1 = 3$ and $r_2 = 4$ with associated eigenvectors $\mathbf{u}_1 = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = s\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

4. Eigenvalues are $r_1 = -4$ and $r_2 = 2$ with associated eigenvectors $\mathbf{u}_1 = s\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{u}_2 = s\begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

6. Eigenvalues are $r_1 = r_2 = -1$ and $r_3 = 2$ with associated eigenvectors $\mathbf{u}_1 = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = v\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{u}_3 = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

8. Eigenvalues are $r_1 = -1$ and $r_2 = -2$ with associated eigenvectors $\mathbf{u}_1 = s\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\mathbf{u}_2 = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

10. Eigenvalues are $r_1 = 1$, $r_2 = 1+i$, and $r_3 = 1-i$, with associated eigenvectors $\mathbf{u_1} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u_2} = s\begin{bmatrix} -1-2i \\ 1 \\ i \end{bmatrix}$, and $\mathbf{u_3} = s\begin{bmatrix} -1+2i \\ 1 \\ -i \end{bmatrix}$, where s is any complex constant.

12. $c_1 e^{7t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

14. $c_1 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

16. $c_1 e^{-10t} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{5t} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

18. **a.** Eigenvalues are $r_1 = -1$, and $r_2 = -3$ with associated eigenvectors $\mathbf{u_1} = s\begin{bmatrix}1\\1\end{bmatrix}$, and $\mathbf{u_2} = s\begin{bmatrix}1\\-1\end{bmatrix}$.

b. $\begin{cases} x_1 = e^{-t} \\ x_2 = e^{-t} \end{cases}$

c. $\begin{cases} x_1 = -e^{-3t} \\ x_2 = e^{-3t} \end{cases}$

d. $\begin{cases} x_1 = e^{-t} - e^{-3t} \\ x_2 = e^{-t} + e^{-3t} \end{cases}$

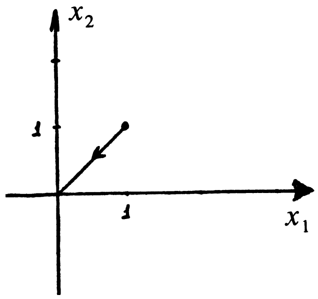


Figure 41

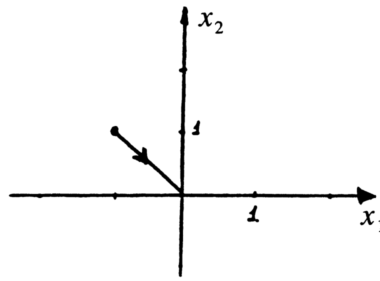


Figure 42

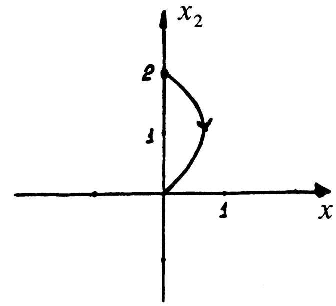


Figure 43

20. $\begin{bmatrix} e^t & 4e^{4t} \\ -e^t & -e^{4t} \end{bmatrix}$

22. $\begin{bmatrix} e^{2t} & -e^{2t} & e^t \\ 0 & e^{2t} & e^t \\ e^{2t} & 0 & 3e^t \end{bmatrix}$

24. $\begin{bmatrix} 1 & e^{4t} & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 3e^t & e^{-t} \\ 0 & 0 & e^t & e^{-t} \end{bmatrix}$

26. $x(t) = c_1 e^{-5t} + 2c_2 e^t$
$y(t) = 2c_1 e^{-5t} + c_2 e^t$

28. $\begin{bmatrix} -0.2931e^{0.4679t} & 0.4491e^{3.8794t} & -0.6527e^{1.6527t} \\ -0.5509e^{0.4679t} & -0.1560e^{3.8794t} & e^{1.6527t} \\ e^{0.4679t} & e^{3.8794t} & -0.7733e^{1.6527t} \end{bmatrix}$

30. $\begin{bmatrix} e^{0.6180t} & -0.6180e^{-1.6180t} & 0 & 0 \\ 0.6180e^{0.6180t} & e^{-1.6180t} & 0 & 0 \\ 0 & 0 & e^{0.5858t} & 0.2929e^{3.4142t} \\ 0 & 0 & 0.5858e^{0.5858t} & e^{3.4142t} \end{bmatrix}$

32. $\begin{bmatrix} 2e^{3t} - 12e^{4t} \\ 2e^{3t} - 8e^{4t} \end{bmatrix}$

34. $\begin{bmatrix} e^{2t} - 2e^{-t} \\ e^{2t} + 3e^{-t} \\ e^{2t} - e^{-t} \end{bmatrix}$

36. $c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \left\{ te^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix} \right\}$

38. $c_1 e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \left\{ te^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + e^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} + c_3 \left\{ \frac{t^2 e^t}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + te^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} -\frac{1}{4} \\ 0 \\ \frac{1}{2} \end{bmatrix} \right\}$

40. $c_1 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + c_3 \left\{ te^t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + e^t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

44. $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 t^{-5} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

46. $x_1(t) = \dfrac{e^{-3t}}{10}$ kg, $x_2(t) = \dfrac{3-\alpha}{10\alpha}(e^{\alpha t} - 1)e^{-3t}$ kg

The mass of salt in tank A is independent of α. The maximum mass of salt in tank B is $0.1\left(\dfrac{3-\alpha}{3}\right)^{3/\alpha}$ kg.

50. **b.** $x(t) = 1 + \dfrac{3}{2}e^{-t} + \dfrac{1}{2}e^{-3t}$

$y(t) = 1 - e^{-3t}$

$z(t) = 1 - \dfrac{3}{2}e^{-t} + \dfrac{1}{2}e^{-3t}$

Exercises 9.6 (page 549)

2. $c_1\begin{bmatrix} -5\cos t \\ 2\cos t - \sin t \end{bmatrix} + c_2\begin{bmatrix} -5\sin t \\ 2\sin t + \cos t \end{bmatrix}$

4. $c_1\begin{bmatrix} 5e^{2t}\cos t \\ -2e^{2t}\cos t + e^{2t}\sin t \\ 5e^{2t}\cos t \end{bmatrix} + c_2\begin{bmatrix} 5e^{2t}\sin t \\ -2e^{2t}\sin t - e^{2t}\cos t \\ 5e^{2t}\sin t \end{bmatrix} + c_3\begin{bmatrix} 0 \\ e^{2t} \\ -e^{2t} \end{bmatrix}$

6. $\begin{bmatrix} \cos 2t & \sin 2t \\ \sin 2t - \cos 2t & -\sin 2t - \cos 2t \end{bmatrix}$

8. $\begin{bmatrix} e^t & e^{-t} & 0 & 0 \\ e^t & -e^{-t} & 0 & 0 \\ 0 & 0 & e^{2t}\cos 3t & e^{2t}\sin 3t \\ 0 & 0 & e^{2t}(2\cos 3t - 3\sin 3t) & e^{2t}(2\sin 3t + 3\cos 3t) \end{bmatrix}$

10. $\begin{bmatrix} -0.0209e^{2t}\cos 3t + 0.0041e^{2t}\sin 3t & -0.0209e^{2t}\sin 3t - 0.0041e^{2t}\cos 3t & -e^{-t} & e^t \\ -0.0296e^{2t}\cos 3t + 0.0710e^{2t}\sin 3t & -0.0296e^{2t}\sin 3t - 0.0710e^{2t}\cos 3t & e^{-t} & e^t \\ 0.1538e^{2t}\cos 3t + 0.2308e^{2t}\sin 3t & 0.1538e^{2t}\sin 3t - 0.2308e^{2t}\cos 3t & -e^{-t} & e^t \\ e^{2t}\cos 3t & e^{2t}\sin 3t & e^{-t} & e^t \end{bmatrix}$

12. $\begin{bmatrix} 0 & 0 & 0 & 0 & e^t \\ 0 & 0 & e^{-t} & e^t & 0 \\ 0 & 0 & -e^{-t} & e^t & 0 \\ -0.0690e^{-2t}\cos 5t + 0.1724e^{-2t}\sin 5t & -0.0690e^{-2t}\sin 5t - 0.1724e^{-2t}\cos 5t & 0 & 0 & 0 \\ e^{-2t}\cos 5t & e^{-2t}\sin 5t & 0 & 0 & 0 \end{bmatrix}$

14. a. $\begin{bmatrix} e^t\sin t - 2e^t\cos t \\ 2e^{2t} \\ -e^t\cos t - 2e^t\sin t \end{bmatrix}$

b. $\begin{bmatrix} e^{t+\pi}\sin t \\ e^{2(t+\pi)} \\ -e^{t+\pi}\cos t \end{bmatrix}$

18. $c_1 t^{-1}\begin{bmatrix} \cos(3\ln t) \\ 3\sin(3\ln t) \end{bmatrix} + c_2 t^{-1}\begin{bmatrix} \sin(3\ln t) \\ -3\cos(3\ln t) \end{bmatrix}$

20. $x_1(t) = \cos t - \cos\sqrt{3}t$

$x_2(t) = \cos t + \cos\sqrt{3}t$

22. $I_1(t) = \frac{16}{5}e^{-2t} - \frac{1}{5}e^{-8t} + 2$

$I_2(t) = 4e^{-2t} - e^{-8t} + 2$

$I_3(t) = -\frac{4}{5}e^{-2t} + \frac{4}{5}e^{-8t}$

Exercises 9.7 (page 555)

2. $c_1\begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} + c_2\begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} + \begin{bmatrix} t \\ 2 \end{bmatrix}$

4. $c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix} + \begin{bmatrix} -2\sin t \\ 2\sin t + \cos t \end{bmatrix}$

6. $\mathbf{x}_p = t\mathbf{a} + \mathbf{b} + e^{3t}\mathbf{c}$

8. $\mathbf{x}_p = t^2\mathbf{a} + t\mathbf{b} + \mathbf{c}$

10. $\mathbf{x}_p = e^{-t}\mathbf{a} + te^{-t}\mathbf{b}$

12. $c_1e^{4t}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

14. $c_1\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + c_2\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} 2t - 1 \\ t^2 - 2 \end{bmatrix}$

16. $c_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} 4t\sin t \\ 4t\cos t - 4\sin t \end{bmatrix}$

18. $c_1e^t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2e^t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3e^t\begin{bmatrix} t \\ t \\ 1+t \end{bmatrix} + \begin{bmatrix} -te^t - e^t \\ 0 \\ -e^t \end{bmatrix}$

20. $\begin{bmatrix} -c_1\sin 2t + c_2\cos 2t - c_3e^{-2t} + 8c_4e^t + \left(\frac{8}{15}\right)te^t + \left(\frac{17}{225}\right)e^t - \left(\frac{1}{8}\right) \\ -2c_1\cos 2t - 2c_2\sin 2t + 2c_3e^{-2t} + 8c_4e^t + \left(\frac{8}{15}\right)te^t - \left(\frac{88}{225}\right)e^t \\ 4c_1\sin 2t - 4c_2\cos 2t - 4c_3e^{-2t} + 8c_4e^t + \left(\frac{8}{15}\right)te^t + \left(\frac{32}{225}\right)e^t \\ 8c_1\cos 2t + 8c_2\sin 2t + 8c_3e^{-2t} + 8c_4e^t + \left(\frac{8}{15}\right)te^t + \left(\frac{152}{225}\right)e^t - 1 \end{bmatrix}$

22. **a.** $\begin{bmatrix} 3e^{-4t} + e^{2t} \\ -6e^{-4t} + e^{2t} - 2t \end{bmatrix}$ **b.** $\begin{bmatrix} \left(-\frac{4}{3}\right)e^{-4(t-2)} + \left(\frac{7}{3}\right)e^{2(t-2)} \\ \left(\frac{8}{3}\right)e^{-4(t-2)} + \left(\frac{7}{3}\right)e^{2(t-2)} - 2t \end{bmatrix}$

24. $x(t) = 3e^{-4t} + e^{2t}$

$y(t) = -6e^{-4t} + e^{2t} - 2t$

26. **a.** $\begin{bmatrix} e^t \\ -2e^t \end{bmatrix}$ **b.** $\begin{bmatrix} te^t \\ te^t \end{bmatrix}$

28. $\begin{bmatrix} -t+1 \\ -t-1 \end{bmatrix}$

30. $c_1 t^{-2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

34. **a.** Neither wins.

b. The x_1 force wins.

c. The x_2 force wins.

Exercises 9.8 (page 566)

2. **a.** $r = 2; k = 2$ **b.** $e^{2t} \begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}$

4. **a.** $r = 2; k = 3$ **b.** $e^{2t} \begin{bmatrix} 1 & t & 3t - \frac{t^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}$

6. **a.** $r = -1; k = 3$ **b.** $e^{-t} \begin{bmatrix} 1+t+\frac{t^2}{2} & t+t^2 & \frac{t^2}{2} \\ -\frac{t^2}{2} & 1+t-t^2 & t-\frac{t^2}{2} \\ -t+\frac{t^2}{2} & -3t+t^2 & 1-2t+\frac{t^2}{2} \end{bmatrix}$

8. $\begin{bmatrix} \left(\frac{1}{2}\right)e^{3t} + \left(\frac{1}{2}\right)e^{-t} & \left(\frac{1}{4}\right)e^{3t} - \left(\frac{1}{4}\right)e^{-t} \\ e^{3t} - e^{-t} & \left(\frac{1}{2}\right)e^{3t} + \left(\frac{1}{2}\right)e^{-t} \end{bmatrix}$

10. $\frac{1}{3}\begin{bmatrix} e^{4t}+2e^{-2t} & e^{4t}-e^{-2t} & e^{4t}-e^{-2t} \\ e^{4t}-e^{-2t} & e^{4t}+2e^{-2t} & e^{4t}-e^{-2t} \\ e^{4t}-e^{-2t} & e^{4t}-e^{-2t} & e^{4t}+2e^{-2t} \end{bmatrix}$

12. $\frac{1}{9}\begin{bmatrix} 3e^{-t}+6e^{2t} & -3e^{-t}+3e^{2t} & -3e^{-t}+3e^{2t} \\ -4e^{-t}+4e^{2t}+6te^{2t} & 4e^{-t}+5e^{2t}+3te^{2t} & 4e^{-t}-4e^{2t}+3te^{2t} \\ -2e^{-t}+2e^{2t}-6te^{2t} & 2e^{-t}-2e^{2t}-3te^{2t} & 2e^{-t}+7e^{2t}-3te^{2t} \end{bmatrix}$

14. $\begin{bmatrix} e^{t} & 0 & 0 & 0 & 0 \\ 0 & e^{t}+te^{-t} & te^{-t} & 0 & 0 \\ 0 & -te^{-t} & e^{-t}-te^{-t} & 0 & 0 \\ 0 & 0 & 0 & \cos t & \sin t \\ 0 & 0 & 0 & -\sin t & \cos t \end{bmatrix}$

16. $\begin{bmatrix} e^{-t} & 0 & 0 & 0 & 0 \\ 0 & e^{-t}+te^{-t} & te^{-t} & 0 & 0 \\ 0 & -te^{-t} & e^{-t}-te^{-t} & 0 & 0 \\ 0 & 0 & 0 & e^{-2t}+2te^{-2t} & te^{-2t} \\ 0 & 0 & 0 & -4te^{-2t} & e^{-2t}-2te^{-2t} \end{bmatrix}$

18. $c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}+c_2e^{t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}+c_3e^{t}\begin{bmatrix} 1 \\ 2t \\ 1 \end{bmatrix}$

20. $c_1e^{t}\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}+c_2e^{t}\begin{bmatrix} 3-4t \\ t \\ 2 \end{bmatrix}+c_3e^{t}\begin{bmatrix} 1-2t+4t^2 \\ -t-t^2 \\ -4t \end{bmatrix}$

22. $\begin{bmatrix} -\left(\frac{4}{3}\right)e^{-t}+\left(\frac{1}{3}\right)e^{2t} \\ \left(\frac{16}{9}\right)e^{-t}-\left(\frac{16}{9}\right)e^{2t}+\left(\frac{1}{3}\right)te^{2t} \\ \left(\frac{8}{9}\right)e^{-t}+\left(\frac{19}{9}\right)e^{2t}-\left(\frac{1}{3}\right)te^{2t} \end{bmatrix}$

24. $\begin{bmatrix} e^{t}+\cos t-\sin t-1-t \\ e^{t}-\sin t-\cos t-1 \\ e^{t}-\cos t+\sin t \end{bmatrix}$

Chapter 9 Review (page 570)

2. $c_1e^{2t}\begin{bmatrix}-2\cos 3t\\ \cos 3t+3\sin 3t\end{bmatrix}+c_2e^{2t}\begin{bmatrix}-2\sin 3t\\ \sin 3t-3\cos 3t\end{bmatrix}$

4. $c_1e^{2t}\begin{bmatrix}0\\0\\1\end{bmatrix}+c_2e^{t}\begin{bmatrix}1\\0\\0\end{bmatrix}+c_3\begin{bmatrix}t\\1\\0\end{bmatrix}$

6. $\begin{bmatrix}0 & e^{5t} & 0\\ 3e^{-5t} & 0 & e^{5t}\\ -e^{-5t} & 0 & 3e^{5t}\end{bmatrix}$

8. $c_1\begin{bmatrix}1\\2\end{bmatrix}+c_2\begin{bmatrix}2e^{-5t}\\ -e^{-5t}\end{bmatrix}+e^{4t}\begin{bmatrix}\frac{11}{36}\\ \frac{13}{18}\end{bmatrix}$

10. $$c_1e^{2t}\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\left\{e^{5t/2}\cos\left(\frac{\sqrt{7}t}{2}\right)\begin{bmatrix}11\\-2\\4\end{bmatrix}-e^{5t/2}\sin\left(\frac{\sqrt{7}t}{2}\right)\begin{bmatrix}-3\sqrt{7}\\-2\sqrt{7}\\0\end{bmatrix}\right\}$$
$$+c_3\left\{e^{5t/2}\sin\left(\frac{\sqrt{7}t}{2}\right)\begin{bmatrix}11\\-2\\4\end{bmatrix}+e^{5t/2}\cos\left(\frac{\sqrt{7}t}{2}\right)\begin{bmatrix}-3\sqrt{7}\\-2\sqrt{7}\\0\end{bmatrix}\right\}+\begin{bmatrix}-\left(\frac{1}{3}\right)e^{-t}+\frac{11}{16}\\ -\frac{1}{4}\\ -\frac{5}{8}\end{bmatrix}$$

12. $\begin{bmatrix}e^{2t}\sin 2t+\left(\frac{3}{2}\right)e^{2t}\cos 2t+\left(\frac{1}{2}\right)e^{2t}\\ 2e^{2t}\cos 2t-3e^{2t}\sin 2t-te^{2t}\end{bmatrix}$

14. $c_1t\begin{bmatrix}-1\\1\\2\end{bmatrix}+c_2t^3\begin{bmatrix}1\\1\\0\end{bmatrix}+c_3t^{-2}\begin{bmatrix}1\\-1\\1\end{bmatrix}$

16. $\begin{bmatrix}1 & t & 4t+t^2\\ 0 & 1 & 2t\\ 0 & 0 & 1\end{bmatrix}$

CHAPTER 10

Exercises 10.2 (page 587)

2. $y = \dfrac{(e^{10}-1)e^{x} + (1-e^{2})e^{5x}}{e^{10}-e^{2}}$

4. $y = 2\sin 3x$

6. No solution

8. $y = e^{x-1} + xe^{x-1}$

10. $\lambda_n = \dfrac{(2n-1)^2}{4}$ and $y_n = c_n \cos\left[\dfrac{(2n-1)x}{2}\right]$, where n = 1, 2, 3, … and the c_n's are arbitrary.

12. $\lambda_n = 4n^2$ and $y_n = c_n \cos(2nx)$, where n = 0, 1, 2, … and the c_n's are arbitrary.

14. $\lambda_n = n^2 + 1$ and $y_n = c_n e^{x} \sin(nx)$, where n = 1, 2, 3, … and the c_n's are arbitrary.

16. $u(x,\ t) = e^{-27t}\sin 3x + 5e^{-147t}\sin 7x - 2e^{-507t}\sin 13x$

18. $u(x,\ t) = e^{-48t}\sin 4x + 3e^{-108t}\sin 6x - e^{-300t}\sin 10x$

20. $u(x,\ t) = -\left(\dfrac{2}{9}\right)\sin 9t \sin 3x + \left(\dfrac{3}{7}\right)\sin 21t \sin 7x - \left(\dfrac{1}{30}\right)\sin 30t \sin 10x$

22. $u(x,\ t) = \cos 3t \sin x - \cos 6t \sin 2x + \cos 9t \sin 3x + \left(\dfrac{2}{3}\right)\sin 9t \sin 3x - \left(\dfrac{7}{15}\right)\sin 15t \sin 5x$

24. $u(x,\ t) = \displaystyle\sum_{n=1}^{\infty}\left\{\left(\frac{1}{n^2}\right)\cos 4nt + \left[\frac{(-1)^{n+1}}{4n^2}\right]\sin 4nt\right\}\sin nx$

Exercises 10.3 (page 603)

2. Even

4. Neither

6. Odd

10. $f(x) \sim \dfrac{\pi}{2} - \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(2k+1)x$

12. $f(x) \sim \dfrac{\pi^2}{6} + \displaystyle\sum_{n=1}^{\infty}\left\{\frac{2(-1)^n}{n^2}\cos nx + \left[\frac{2(-1)^n - n^2\pi^2(-1)^n - 2}{n^3\pi}\right]\sin nx\right\}$

14. $f(x) \sim \dfrac{\pi}{2} + \sum_{n=1}^{\infty} \dfrac{1}{n} \sin 2nx$

16. $f(x) \sim \sum_{n=1}^{\infty} \dfrac{2}{n\pi}\left[1 - \cos\left(\dfrac{n\pi}{2}\right)\right] \sin nx$

18. The 2π periodic function $g(x)$, where $g(x) = |x|,\ -\pi \le x \le \pi$

20. The 2π periodic function $g(x)$, where $g(x) = \begin{cases} 0 & -\pi < x \le 0 \\ x^2 & 0 \le x < \pi \\ \dfrac{\pi^2}{2} & x = \pm\pi \end{cases}$

22. The 2π periodic function $g(x)$, where $g(x) = \begin{cases} x + \pi & -\pi < x < 0 \\ x & 0 < x < \pi \\ \dfrac{\pi}{2} & x = 0,\ \pm\pi \end{cases}$

24. The 2π periodic function $g(x)$, where $g(x) = \begin{cases} 0 & -\pi \le x < \dfrac{-\pi}{2} \\ -\dfrac{1}{2} & x = -\dfrac{\pi}{2} \\ -1 & \dfrac{-\pi}{2} < x < 0 \\ 0 & x = 0 \\ 1 & 0 < x < \dfrac{\pi}{2} \\ \dfrac{1}{2} & x = \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} < x \le \pi \end{cases}$

30. $a_0 = \dfrac{1}{2},\ a_1 = 0,\ a_2 = \dfrac{5}{8}$

Exercises 10.4 (page 611)

2. **a.** The π periodic function $\tilde{f}(x) = \sin 2x$ for $x \neq k\pi$ where k is an integer.

b. The 2π periodic function $f_0(x) = \sin 2x$ for $x \neq k\pi$ where k is an integer.

c. The 2π periodic function $f_e(x)$, where $f_e(x) = \begin{cases} \sin 2x & 0 < x < \pi \\ -\sin 2x & -\pi < x < 0 \end{cases}$

4. **a.** The π periodic function $\tilde{f}(x)$, where $\tilde{f}(x) = \pi - x,\ 0 < x < \pi$

b. The 2π periodic function $f_0(x)$, where $f_0(x) = \begin{cases} \pi - x & 0 < x < \pi \\ -\pi - x & -\pi < x < 0 \end{cases}$

c. The 2π periodic function $f_e(x)$, where $f_e(x) = \begin{cases} \pi - x & 0 < x < \pi \\ \pi + x & -\pi < x < 0 \end{cases}$

6. $f(x) \sim \displaystyle\sum_{k=1}^{\infty} \frac{8k}{\pi(4k^2 - 1)} \sin(2kx)$

8. $f(x) \sim \displaystyle\sum_{n=1}^{\infty} \frac{2}{n} \sin nx$

10. $f(x) \sim \displaystyle\sum_{n=1}^{\infty} \frac{2\pi n[1 - e(-1)^n]}{1 + \pi^2 n^2} \sin(n\pi x)$

12. $f(x) \sim 1 + \dfrac{\pi}{2} - \left(\dfrac{4}{\pi}\right) \displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos(2k+1)x$

14. $f(x) \sim 1 - e^{-1} + \displaystyle\sum_{n=1}^{\infty} \frac{2}{1 + \pi^2 n^2} [1 - (-1)^n e^{-1}] \cos(n\pi x)$

16. $f(x) \sim \dfrac{1}{6} - \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2} \cos(2k\pi x)$

18. $u(x,\ t) = \displaystyle\sum_{k=0}^{\infty} \frac{8}{(2k+1)^3 \pi} e^{-5(2k+1)^2 t} \sin(2k+1)x$

Exercises 10.5 (page 624)

2. $u(x,\ t) = \sum_{n=1}^{\infty} \left[\frac{2\pi}{n}(-1)^{n+1} + \frac{4}{n^3\pi}[(-1)^n - 1] \right] e^{-n^2 t} \sin nx$

4. $u(x,\ t) = \frac{1}{6} - \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} e^{-8k^2\pi^2 t} \cos 2\pi kx$

6. $u(x,\ t) = 1 - \frac{2}{\pi} + 2e^{-7t} \cos x + \sum_{k=1}^{\infty} \frac{4}{\pi(4k^2 - 1)} e^{-28k^2 t} \cos 2kx$

8. $u(x,\ t) = 3x + 6\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2 t} \sin nx$

10. $u(x,\ t) = \left(\frac{\pi^2}{18} \right) x - \left(\frac{1}{18} \right) x^3 + \left(\frac{1}{3} \right) e^{-3t} \sin x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{3n^3} e^{-3n^2 t} \sin nx$

12. $u(x,\ t) = \sum_{n=1}^{\infty} a_n e^{-\mu_n^2 t} \sin \mu_n x,$ where $\{\mu_n\}_{n=1}^{\infty}$ is the increasing sequence of positive real numbers that are solutions to $\tan \mu_n \pi = -\mu_n$, and $a_n = \frac{1}{\left[\frac{\pi}{2} - \frac{\sin(2\pi\mu_n)}{4} \right]} \int_0^{\pi} f(x) \sin \mu_n x \, dx$.

14. $u(x,\ t) = 1 + \left(\frac{5\pi}{6} \right) x - \left(\frac{5}{6} \right) x^2 - \frac{20}{3\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} e^{-3(2k+1)^2 t} \sin(2k+1)x$

16. $u(x,\ y,\ t) = e^{-2t} \cos x \sin y + 4e^{-5t} \cos 2x \sin y - 3e^{-25t} \cos 3x \sin 4y$

18. $u(x,\ y,\ t) = \left(\frac{\pi}{2} \right) e^{-t} \sin y - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} e^{-[(2k+1)^2 + 1]t} \cos[(2k+1)x] \sin y$

Exercises 10.6 (page 636)

2. $u(x,\ t)=\sum_{n=1}^{\infty}[a_n \cos 4nt+b_n \sin 4nt]\sin nx,$ where

$$a_n=\begin{cases}0 & n \text{ even}\\ \dfrac{2}{\pi}\left[\dfrac{n}{n^2-4}-\dfrac{1}{n}\right] & n \text{ odd}\end{cases}$$

and

$$b_n=\begin{cases}\dfrac{-1}{4\pi(n^2-1)} & n \text{ even}\\ \dfrac{1}{\pi n^2} & n \text{ odd}\end{cases}$$

4. $u(x,\ t)=\cos 12t \sin 4x+7\cos 15t \sin 5x+\dfrac{4}{3\pi}\sum_{k=0}^{\infty}\dfrac{(-1)^k}{(2k+1)^3}\sin 3(2k+1)t \sin(2k+1)x$

6. $u(x,\ t)=\dfrac{2v_0L^3}{\pi^3\alpha a(L-a)}\sum_{n=1}^{\infty}\dfrac{1}{n^3}\sin\dfrac{n\pi a}{L}\sin\dfrac{n\pi x}{L}\sin\dfrac{n\pi\alpha t}{L}$

8. $u(x,\ t)=(\sin t-t\cos t)\sin x+\sum_{n=2}^{\infty}\dfrac{2(-1)^n}{n^2(n^2-1)}[\sin nt-n\sin t]\sin nx$

10. $u(x,\ t)=U_1+(U_2-U_1)\left(\dfrac{x}{L}\right)+\sum_{n=1}^{\infty}\left[a_n \cos\dfrac{n\pi\alpha t}{L}+b_n \sin\dfrac{n\pi\alpha t}{L}\right]\sin\dfrac{n\pi x}{L},$ where a_n's and b_n's are chosen that

$$f(x)-U_1-(U_2-U_1)\left(\frac{x}{L}\right)=\sum_{n=1}^{\infty}a_n \sin\frac{n\pi x}{L}$$

$$g(x)=\sum_{n=1}^{\infty}b_n\left[\frac{n\pi\alpha}{L}\right]\sin\frac{n\pi x}{L}$$

14. $u(x,\ t)=x^2+\alpha^2t^2$

16. $u(x,\ t)=\sin 3x \cos\ 3\alpha t\ +t$

18. $u(x,\ t)=\cos 2x \cos\ 2\alpha t+t-xt$

Exercises 10.7 (page 649)

2. $u(x,\ y)=\dfrac{\cos x \sinh(y-\pi)}{\sinh(-\pi)}-\dfrac{2\cos 4x \sinh(4y-4\pi)}{\sinh(-4\pi)}$

4. $u(x,\ y)=\dfrac{\sin x \sinh(y-\pi)}{\sinh(-\pi)}+\dfrac{\sin 4x \sinh(4y-4\pi)}{\sinh(-4\pi)}$

8. $u(r,\ \theta)=\dfrac{1}{2}+\dfrac{r^2}{8}\cos 2\theta$

12. $u(r,\ \theta)=\dfrac{27}{3^6-1}[r^3-r^{-3}]\cos 3\theta+\dfrac{3^5}{3^{10}-1}[r^5-r^{-5}]\cos 5\theta$

14. $u(r,\ \theta)=C+\sum_{n=1}^{\infty} r^{-n}[a_n \cos n\theta+b_n \sin n\theta]$, where C is arbitrary and for n = 1, 2, …

$$a_n=\frac{-1}{n\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta$$

$$b_n=\frac{-1}{n\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta$$

16. $u(r,\ \theta)=\sum_{n=1}^{\infty} a_n \sinh\left[\dfrac{n\pi(\theta-\pi)}{\ln 2}\right]\sin\left[\dfrac{n\pi(\ln r-\ln \pi)}{\ln 2}\right]$, where

$$a_n=\frac{-2}{\ln 2\sinh\left(\frac{n\pi^2}{\ln 2}\right)}\int_{\pi}^{2\pi}\sin r \sin\left[\frac{n\pi(\ln r-\ln\pi)}{\ln 2}\right]\frac{1}{r}\,dr\,.$$

24. $u(x,\ y)=\sum_{n=1}^{\infty} A_n e^{-ny}\sin nx$, where $A_n=\dfrac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx.$

CHAPTER 11

Exercises 11.2 (page 671)

2. $\sin x - \cos x + x + 1$

4. $ce^{-2x}(\sin 2x + \cos 2x)$

6. $2e^{x} + e^{-x} - x$

8. $c_1 \sin 2x + c_2 \cos 2x + 1$

10. No solution

12. No solution

14. $\lambda_n = \dfrac{\pi^2 n^2}{9}$; $y_n(x) = b_n \sin\left(\dfrac{\pi n x}{3}\right) + c_n \cos\left(\dfrac{\pi n x}{3}\right)$, $n = 0, 1, 2, \ldots$

16. $\lambda_n = 2 + \dfrac{(2n+1)^2}{4}$; $y_n(x) = c_n \sin\left[\dfrac{(2n+1)x}{2}\right]$, $n = 0, 1, 2, \ldots$

18. $\lambda_0 = -\mu_0^2$, where $\tanh(\mu_0 \pi) = 2\mu_0$; $y_0(x) = c_0 \sinh(\mu_0 x)$,

$\lambda_n = \mu_n^2$, where $\tan(\mu_n \pi) = 2\mu_n$; $y_n(x) = c_n \sin(\mu_n x)$, $n = 1, 2, 3, \ldots$

20. $\lambda_n = \pi^2 n^2$; $y_n(x) = c_n \cos(\pi n \ln x)$, $n = 0, 1, 2, \ldots$

22. $\lambda_1 = 4.116$, $\lambda_2 = 24.139$, $\lambda_3 = 63.659$

24. No nontrivial solutions

26. $\lambda_n = \mu_n^2$, where $\cot \pi\mu_n = \mu_n$ for $\mu_n > 0$; $y_n(x) = c_n \sin \mu_n x$, $n = 1, 2, 3, \ldots$

28. $\lambda_n = -\mu_n^4$; $\mu_n > 0$ and $\cos(\mu_n L)\cosh(\mu_n L) = -1$;

$$y_n = c_n \left[\sin \mu_n x - \sinh \mu_n x - \left(\frac{\sin \mu_n L + \sinh \mu_n L}{\cos \mu_n L + \cosh \mu_n L}\right)(\cos \mu_n x - \cosh \mu_n x)\right], \quad n = 1, 2, \ldots$$

34. b. $\lambda_0 = -1$, $X_0(x) = c$ and $\lambda_n = -1 + n^2\pi^2$, $X_n(x) = c\cos(\pi n x)$

Exercises 11.3 (page 682)

2. $y'' + \lambda x^{-1} y = 0$

4. $(xy')' + xy - \lambda x^{-1} y = 0$

6. $((1-x^2)y')' + \lambda y = 0$

8. Yes

10. No

18. a. $\dfrac{1}{\sqrt{6}},\ \dfrac{1}{\sqrt{3}}\sin\left(\dfrac{n\pi x}{3}\right),\ \dfrac{1}{\sqrt{3}}\cos\left(\dfrac{n\pi x}{3}\right),\ n = 1, 2, 3, \ldots$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}6}{\pi n}\sin\left(\frac{n\pi x}{3}\right)$

20. a. $\sqrt{\dfrac{2}{\pi}}\sin\left[\left(n+\dfrac{1}{2}\right)x\right],\ n = 0, 1, 2, \ldots$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}8}{\pi(2n+1)^2}\sin\left[\left(n+\frac{1}{2}\right)x\right]$

22. a. $2\sqrt{\dfrac{\mu_0}{\sinh(2\mu_0\pi) - 2\mu_0\pi}}\sinh(\mu_0 x)$ where $\tanh(\mu_0\pi) = 2\mu_0$;

$2\sqrt{\dfrac{\mu_n}{2\mu_n\pi - \sin(2\mu_n\pi)}}\sin(\mu_n x)$ where $\tan(\mu_n\pi) = 2\mu_n$, $n = 1, 2, 3, \ldots$

b. $\displaystyle\frac{4(\pi-2)\cosh(\mu_0\pi)}{\sinh(2\mu_0\pi) - 2\mu_0\pi}\sinh(\mu_0 x) + \sum_{n=1}^{\infty}\frac{4(2-\pi)\cos(\mu_n\pi)}{2\mu_n\pi - \sin(2\mu_n\pi)}\sin(\mu_n x)$

24. a. $y_0(x) = \dfrac{1}{\sqrt{e-1}}$; $y_n(x) = \sqrt{2}\cos(\pi n \ln x)$, $n = 1, 2, 3, \ldots$

b. $\displaystyle 1 + \sum_{n=1}^{\infty}\frac{2[(-1)^n e - 1]}{1+\pi^2 n^2}\cos(\pi n \ln x)$

Exercises 11.4 (page 692)

2. $L^+[y] = x^2 y'' + (4x - \sin x)y' + (2x + 2 - \cos x)y$

4. $L^+[y] = x^2 y'' + 6xy' + 7y$

6. $L^+[y] = (\sin x)y'' + (2\cos x + e^x)y' + (-\sin x + e^x + 1)y$

8. $L^+[y] = y'' + 4y' + 5y;$
$D(L^+) = \{y \in C^2[0,\ 2\pi]:\ y(0) = y(2\pi) = 0\}$

10. $L^+[y] = x^2 y'' + 2xy' + \left(\frac{5}{4}\right) y$
$D(L^+) = \{y \in C^2[1,\ e^\pi]:\ y(1) = y(e^\pi) = 0\}$

12. $y'' - y' + y = 0;\ y(0) = y(\pi),\ \ y'(0) = y'(\pi)$

14. $y'' = 0;\ y(0) = y\left(\frac{\pi}{2}\right);\ y'(0) = y'\left(\frac{\pi}{2}\right)$

16. $x^2 y'' + 2xy' = 0;\ y(1) = 4y(2),\ \ y'(1) = 4y'(2)$

18. $\int_0^{2\pi} h(x) e^{-2x} \sin x\, dx = 0$

20. $\int_1^{e^\pi} h(x) x^{-1/2} \sin(\ln x) dx = 0$

22. Unique solution for each h

24. $\int_0^{\pi/2} h(x) dx = 0$

26. $\int_1^2 h(x)\left(1 - \frac{3}{x}\right) = 0$

Exercises 11.5 (page 698)

2. $\frac{1}{122} \sin 11x - \frac{1}{17} \sin 4x$

4. $\frac{1}{\pi - 49} \cos 7x + \frac{5}{\pi - 100} \cos 10x$

6. $\frac{1}{19} \cos 5x - \frac{1}{10} \cos 4x$

8. $\sum_{n=0}^{\infty} \frac{-8}{\pi(2n+1)^3 (4n^2 + 4n - 1)} \sin[(2n+1)x]$

10. $\sum_{n=0}^{\infty} \frac{\gamma_n}{7 - \left(n + \frac{1}{2}\right)^2} \sin\left[\left(n + \frac{1}{2}\right)x\right]$, where
$\gamma_n = \frac{2}{\pi} \int_0^\pi f(x) \sin\left[\left(n + \frac{1}{2}\right)x\right] dx .$

12. Let $\gamma_n = \frac{2}{\pi} \int_1^{e^\pi} f(x) \sin(n \ln x) dx.$ If $\gamma_1 \neq 0,$ there is no solution. If $\gamma_1 = 0,$ then
$c \sin(\ln x) + \sum_{n=2}^{\infty} \frac{\gamma_n}{1 - n^2} \sin(n \ln x)$ is a solution.

14. $\sum_{n=0}^{\infty} \frac{\gamma_n}{-1 - \pi^2 n^2} \cos(\pi n \ln x),$ where
$\gamma_n = \frac{\int_1^e f(x) \cos(\pi n \ln x) dx}{\int_1^e x^{-1} \cos^2(\pi n \ln x) dx}.$

Exercises 11.6 (page 706)

2. $G(x,\ s) = \begin{cases} -\dfrac{\sinh s \sinh(x-1)}{\sinh 1} & 0 \le s \le x \\ -\dfrac{\sinh x \sinh(s-1)}{\sinh 1} & x \le s \le 1 \end{cases}$

4. $G(x,\ s) = \begin{cases} -\cos s \sin x & 0 \le s \le x \\ -\cos x \sin s & x \le s \le \pi \end{cases}$

6. $G(x, s) = \begin{cases} (\sin s - \cos s)\sin x & 0 \le s \le x \\ (\sin x - \cos x)\sin s & x \le s \le \pi \end{cases}$

8. $$G(x, s) = \begin{cases} -\dfrac{(s^2 + s^{-2})(x^2 - 16x^{-2})}{68} & 1 \le s \le x \\ -\dfrac{(x^2 + x^{-2})(s^2 - 16s^{-2})}{68} & x \le s \le 2 \end{cases}$$

10. $$G(x, s) = \begin{cases} \dfrac{(e^{5s} - e^{-s})(5e^{6-x} + e^{5x})}{30e^6 + 6} & 0 \le s \le x \\ \dfrac{(e^{5x} - e^{-x})(5e^{6-s} + e^{5s})}{30e^6 + 6} & x \le s \le 1 \end{cases}$$

12. $$G(x, s) = \begin{cases} \dfrac{-s(x - \pi)}{\pi} & 0 \le s \le x \\ \dfrac{-x(s - \pi)}{\pi} & x \le s \le \pi \end{cases}$$

$$y = \frac{x^4 - \pi^3 x}{12}$$

14. $G(x, s) = \begin{cases} s & 0 \le s \le x \\ x & x \le s \le \pi \end{cases}$

$$y = \frac{4\pi^3 x - x^4}{12}$$

16. $$G(x, s) = \begin{cases} -\dfrac{\sin s \sin(x - 2)}{\sin 2} & 0 \le s \le x \\ -\dfrac{\sin x \sin(s - 2)}{\sin 2} & x \le s \le 2 \end{cases}$$

$$y = \frac{12}{\sin 2}[\sin x - \sin(x - 2) - \sin 2]$$

18. $$G(x, s) = \begin{cases} \dfrac{\cosh s \cosh(x - 1)}{\sinh 1} & 0 \le s \le x \\ \dfrac{\cosh x \cosh(s - 1)}{\sinh 1} & x \le s \le 1 \end{cases}$$

$y = -24$

20. $$G(x, s) = \begin{cases} -\left(1 - \dfrac{1}{s}\right)\left(1 - \dfrac{2}{x}\right) & 1 \le s \le x \\ -\left(1 - \dfrac{1}{x}\right)\left(1 - \dfrac{2}{s}\right) & x \le s \le 2 \end{cases}$$

$$y = \frac{2\ln 2}{x} + \ln\left(\frac{x}{4}\right)$$

24. a. $K(x, s) = \begin{cases} e^{x-s} s(1 - x) & 0 \le s \le x \\ e^{x-s} x(1 - s) & x \le s \le 1 \end{cases}$

b. $y = (x - 1)e^x - xe^{x-1} + 1$

26. a. $$K(x, s) = \begin{cases} \dfrac{(s^2 - s^{-2})(3x + 16x^{-3})}{76} & 1 \le s \le x \\ \dfrac{(x - x^{-3})(3s^2 + 16s^{-2})}{76} & x \le s \le 2 \end{cases}$$

b. $$y = \frac{1}{4} x \ln x + \frac{4(1 + \ln 2)}{19}(x^{-3} - x)$$

28. $H(x,\ s)=\begin{cases}\dfrac{x(\pi-s)(s^2-2\pi s+x^2)}{6\pi} & 0\le s\le x\\[2mm] \dfrac{s(\pi-x)(x^2-2\pi x+s^2)}{6\pi} & x\le s\le \pi\end{cases}$

30. $H(x,\ s)=\begin{cases}\dfrac{x^2[(x-3\pi)(s^3-3\pi s^2)+2\pi^3(x-3s)]}{12\pi^3} & 0\le s\le x\\[2mm] \dfrac{s^2[(s-3\pi)(x^3-3\pi x^2)+2\pi^3(s-3x)]}{12\pi^3} & x\le s\le \pi\end{cases}$

Exercises 11.7 (page 715)

2. $\sum_{n=1}^{\infty} b_n J_3(\alpha_{3n}x)$ where $\{\alpha_{3n}\}$ is the increasing sequence of real zeros of J_3 and $b_n=\dfrac{\int_0^1 f(x)J_3(\alpha_{3n}x)\,dx}{(\mu-\alpha_{3n}^2)\int_0^1 J_3^2(\alpha_{3n}x)x\,dx}$

4. $\sum_{n=0}^{\infty} b_n P_n(x)$ where $b_n=\dfrac{\int_{-1}^1 f(x)P_n(x)\,dx}{[\mu-n(n+1)]\int_{-1}^1 P_n^2(x)\,dx}$

6. $\sum_{n=0}^{\infty} b_n P_{2n}(x)$ where $b_n=\dfrac{\int_0^1 f(x)P_{2n}(x)\,dx}{[\mu-2n(2n+1)]\int_0^1 P_{2n}^2(x)\,dx}$

16. c. $\sum_{n=0}^{\infty} b_n L_n(x)$ where $b_n=\dfrac{\int_0^\infty f(x)L_n(x)\,dx}{(\mu-n)\int_0^\infty L_n^2(x)e^{-x}\,dx}$

Exercises 11.8 (page 725)

2. No; sin x has a finite number of zeros on any closed bounded interval.

4. No; it has an infinite number of zeros on the interval $\left[-\dfrac{1}{2},\ \dfrac{1}{2}\right]$.

8. $\dfrac{\pi}{3}$

10. Between $\pi\sqrt{\dfrac{e^{-25}}{\lambda+1}}$ and $\pi\sqrt{\dfrac{26}{\lambda+26\sin 5}}$

Chapter 11 Review (page 729)

2. **a.** $(e^{7x}y')' + \lambda e^{7x}y = 0$

b. $\left(e^{-3x^2/2}y'\right)' + \lambda e^{-3x^2/2}y = 0$

c. $(xe^{-x}y')' + \lambda e^{-x}y = 0$

4. **a.** $y'' - xy' = 0;\ y'(0) = 0,\ y'(1) = 0$

b. $x^2y'' + 2xy' - 3y = 0;\ y(1) = 0,\ \ y'(e) = 0$

6. $\dfrac{\pi}{6} + 4\cos 2x + \dfrac{2}{\pi}\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k^2+2k-1)(2k+1)^2}\cos[(2k+1)x]$

8. **a.** $\displaystyle\sum_{n=1}^{\infty} b_n J_7(\alpha_{7n}x)$ where $\{\alpha_{7n}\}$ is the increasing sequence of real zeros of J_7 and

$$b_n = \frac{\int_0^1 f(x)J_7(\alpha_{7n}x)dx}{(\mu - \alpha_{7n}^2)\int_0^1 J_7^2(\alpha_{7n}x)x\,dx}$$

b. $\displaystyle\sum_{n=0}^{\infty} b_n P_n(x)$ where $b_n = \dfrac{\int_{-1}^{1} f(x)P_n(x)\,dx}{[\mu - n(n+1)]\int_{-1}^{1} P_n^2(x)\,dx}$

10. Between $\dfrac{\pi}{\sqrt{6}}$ and $\pi\sqrt{\dfrac{3}{5}}$.

CHAPTER 12

Exercises 12.2 (page 753)

2. Unstable proper node

4. Unstable improper node

6. Stable center

8. (–1, –1) is an asymptotically stable spiral point.

10. (2, –2) is an unstable spiral point.

12. (5, 1) is an asymptotically stable improper node.

14. Unstable proper node

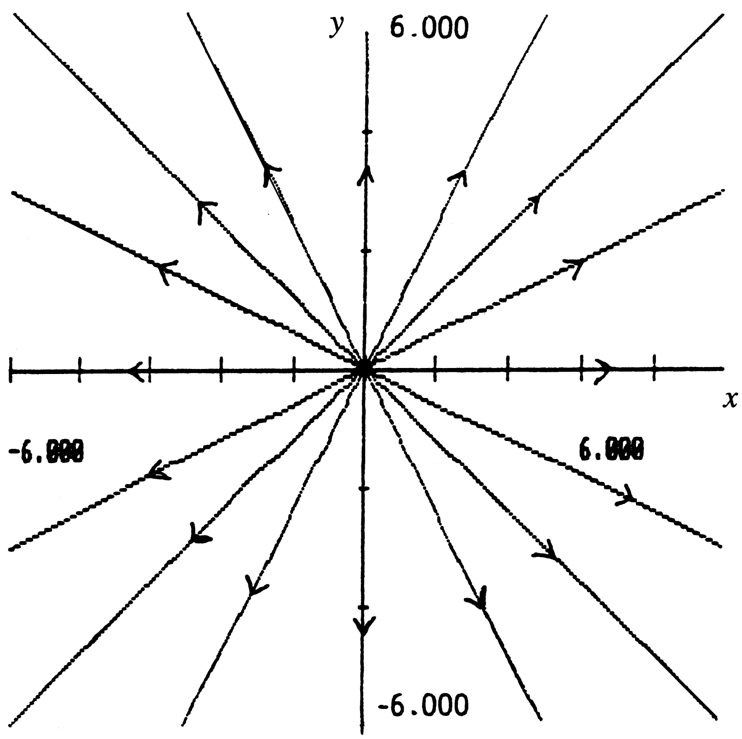


Figure 44

16. Asymptotically stable spiral point.

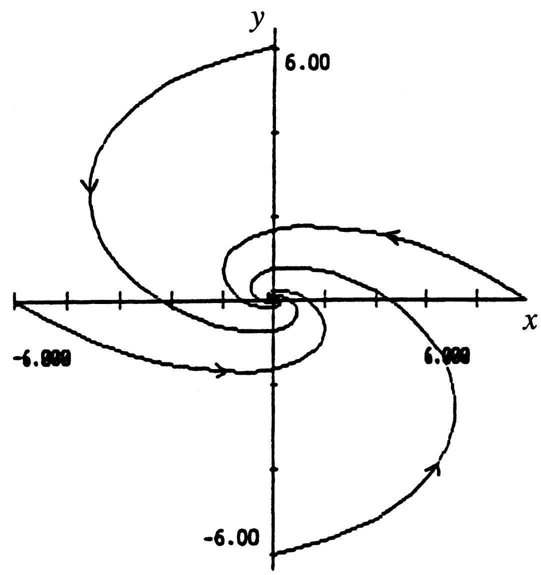


Figure 45

18. Unstable improper node

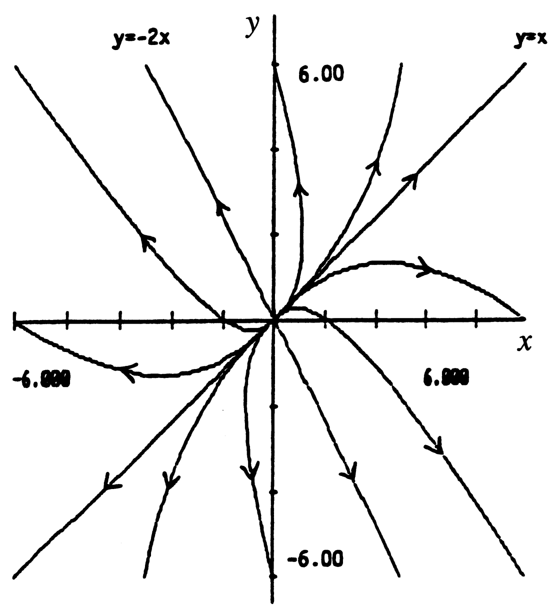


Figure 46

20. Stable center

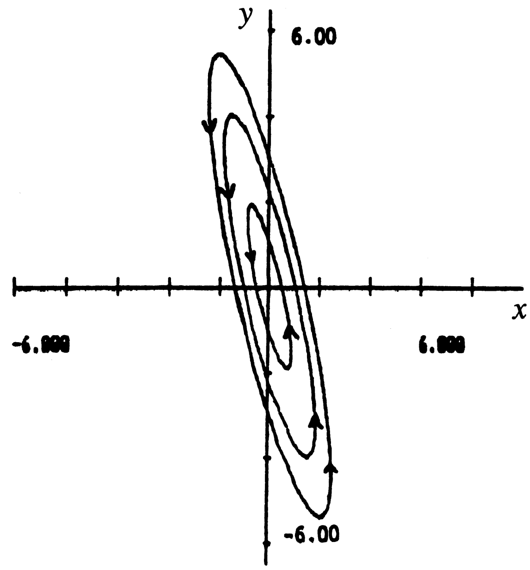


Figure 47

Exercises 12.3 (page 764)

2. Asymptotically stable improper node

4. Asymptotically stable spiral point

6. Unstable saddle point

8. Asymptotically stable improper node

10. (0, 0) is indeterminant; (–1, 1) is an unstable saddle point.

12. (2, 2) is an asymptotically stable spiral point; (–2, –2) is an unstable saddle point.

14. (3, 3) is an unstable saddle point; (–2, –2) is an asymptotically stable spiral point.

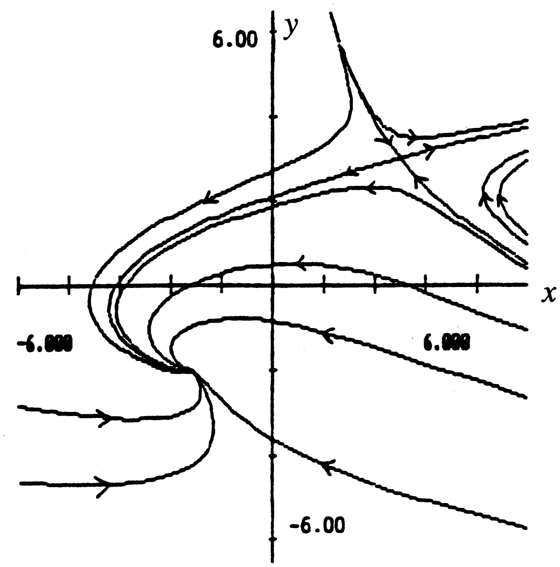


Figure 48

16. (0, 0) is an unstable saddle point; (–4, –2) is an asymptotically stable spiral point.

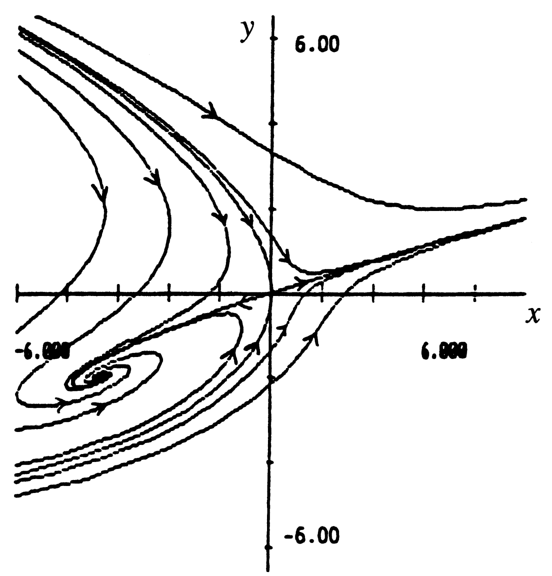


Figure 49

Exercises 12.4 (page 774)

2. $G(x) = \sin x + C$; $E(x, v) = \dfrac{1}{2}v^2 + \sin x$

4. $G(x) = \dfrac{1}{2}x^2 - \dfrac{1}{24}x^4 + \dfrac{1}{720}x^6 + C$; $E(x, v) = \dfrac{1}{2}v^2 + \dfrac{1}{2}x^2 - \dfrac{1}{24}x^4 + \dfrac{1}{720}x^6$

6. $G(x) = e^x - x + C$; $E(x, v) = \dfrac{1}{2}v^2 + e^x - x - 1$

8.

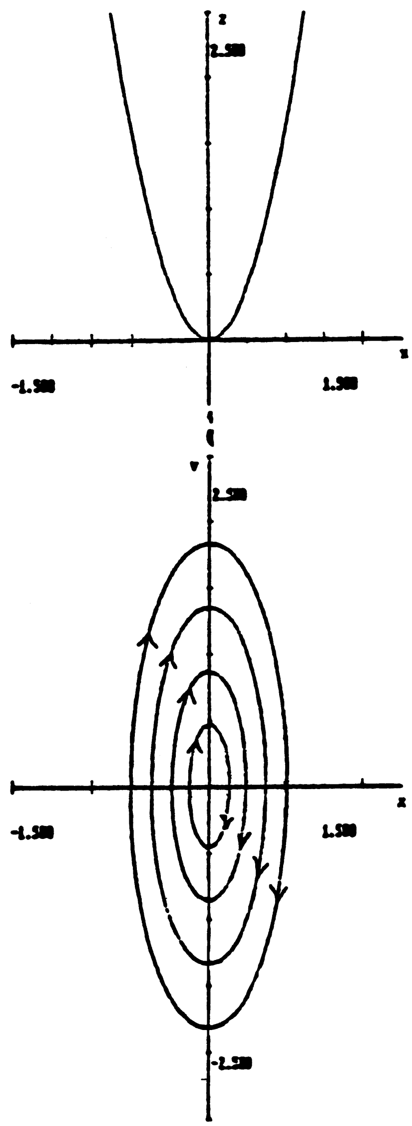


Figure 50

10.

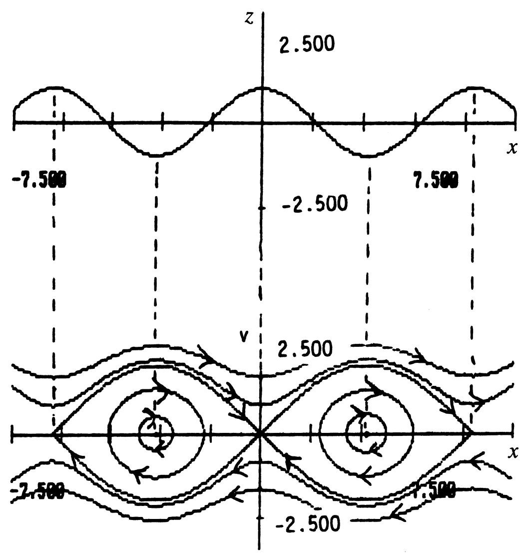


Figure 51

12.

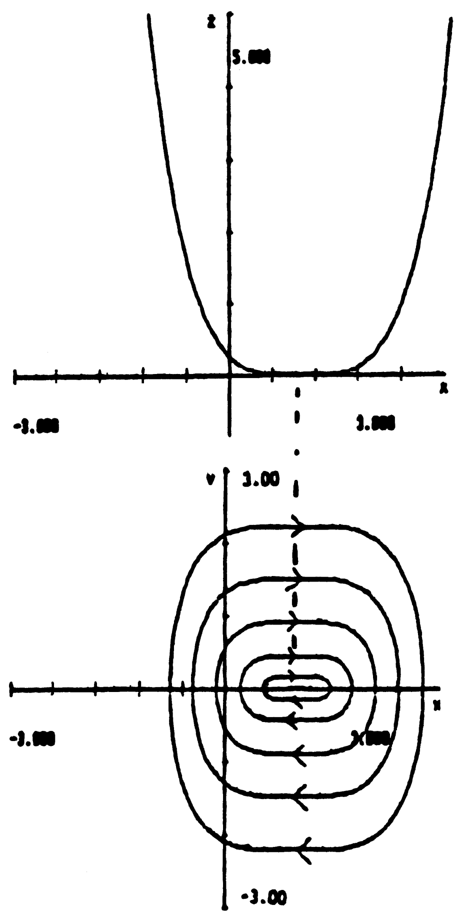


Figure 52

14. $vh(x,\ v) = v^2$, so energy is decreasing along a trajectory.

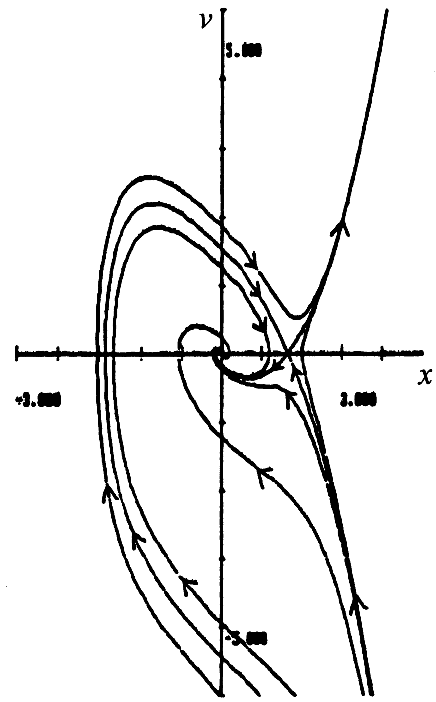


Figure 53

16. $vh(x,\ v) = v^2$, so energy is decreasing along a trajectory.

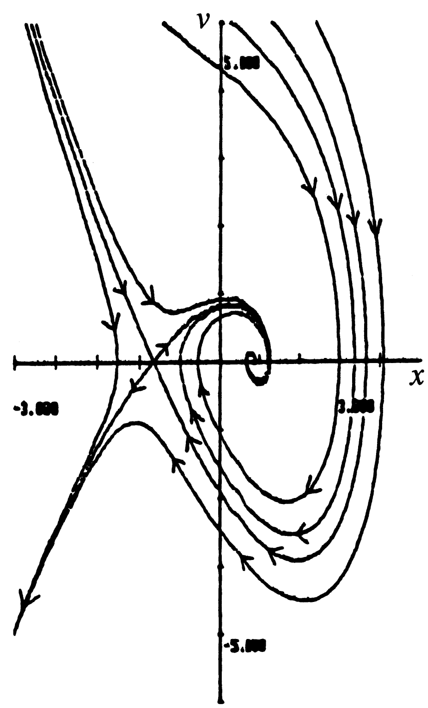


Figure 54

18.

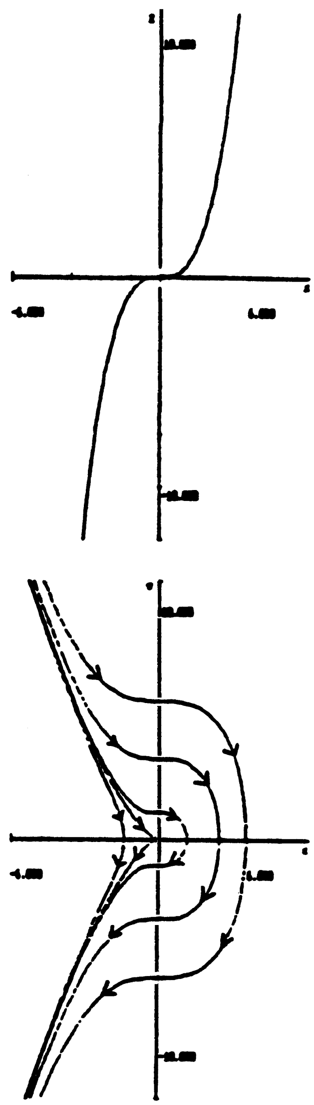


Figure 55

Exercises 12.5 (page 782)

2. Asymptotically stable

4. Stable

6. Unstable

8. Asymptotically stable

10. Stable

12. Stable

14. Stable

Exercises 12.6 (page 791)

4. b. Clockwise

6. $r = 0$ is an unstable spiral point.
$r = 2$ is a stable limit cycle.
$r = 5$ is an unstable limit cycle.

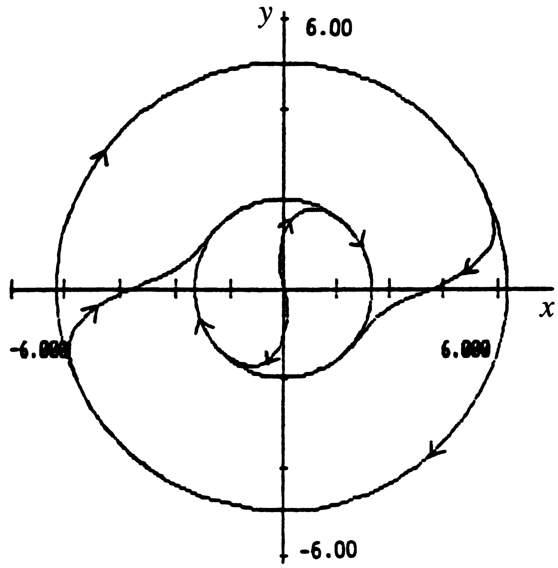


Figure 56

8. $r = 0$ is an unstable spiral point.

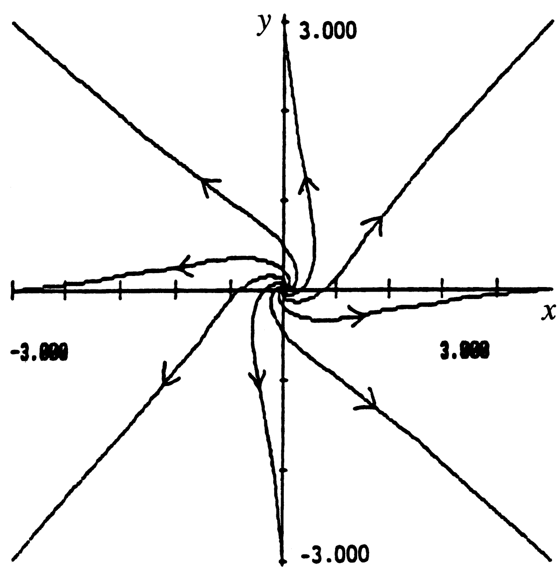


Figure 57

10. $r = 0$ is an unstable spiral point.
$r = 2$ is a stable limit cycle.
$r = 3$ is an unstable limit cycle.

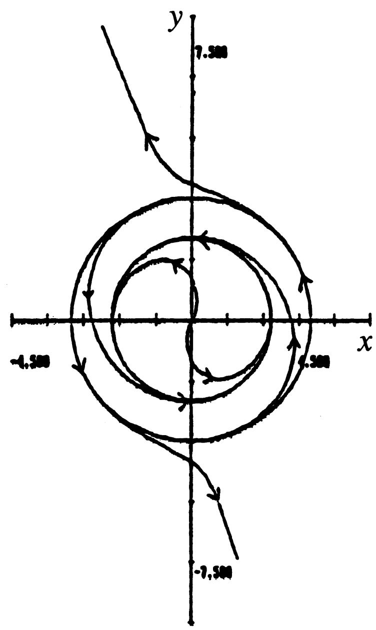


Figure 58

12. $r = 0$ is an unstable spiral point.
$r = n\pi$, $n = 1, 2, 3, \ldots$, is a limit cycle that is stable for n odd and unstable for n even.

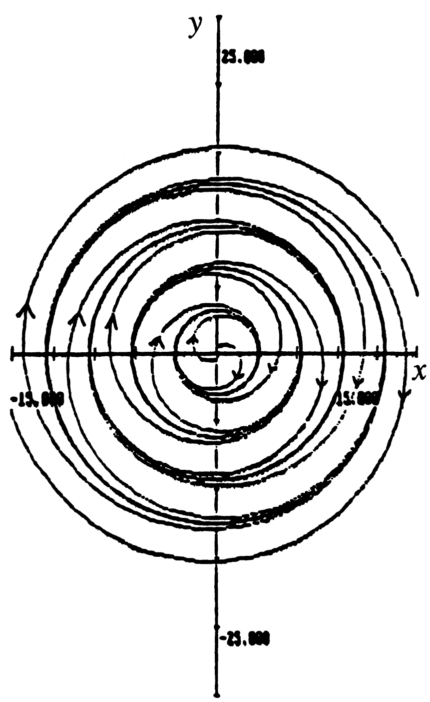


Figure 59

Exercises 12.7 (page 798)

2. Asymptotically stable

4. Unstable

6. Asymptotically stable

8. a. $\mathbf{x}(t) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

b. $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 1 \end{bmatrix}$

10. Stable

12. Asymptotically stable

14. Asymptotically stable

16. The equilibrium solution corresponding to the critical point (–3, 0, 1) is unstable.

18. The equilibrium solutions corresponding to the critical points (0, 0, 0) and (0, 0, 1) are unstable.

Chapter 12 Review (page 801)

2. (0, 0) is an unstable saddle point.

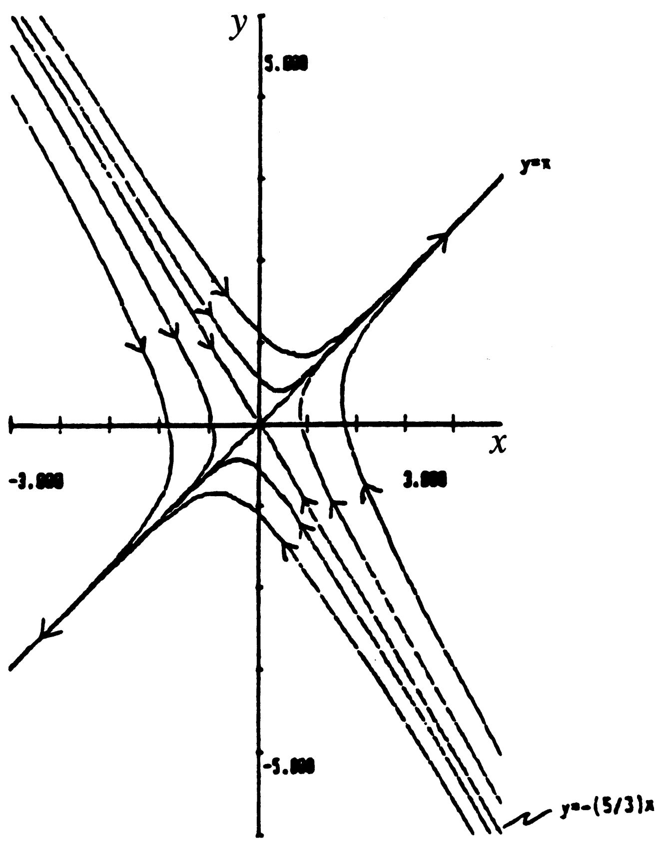


Figure 60

4. (0, 0) is a stable center.

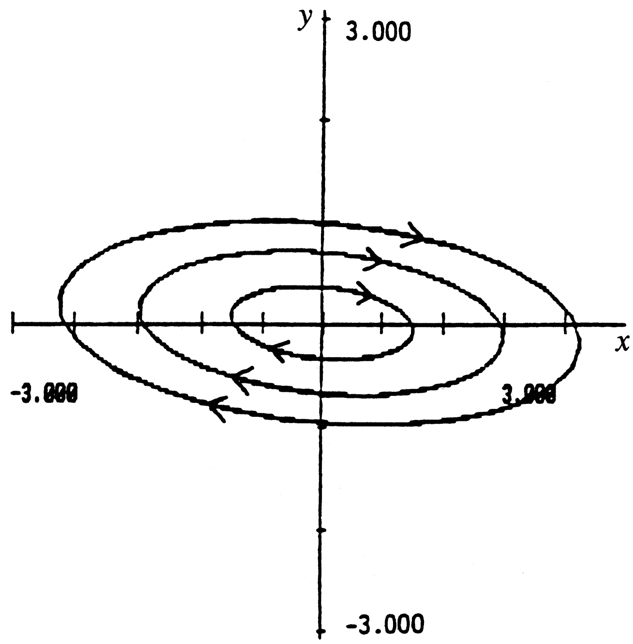


Figure 61

6. (0, 0) is an unstable improper node.

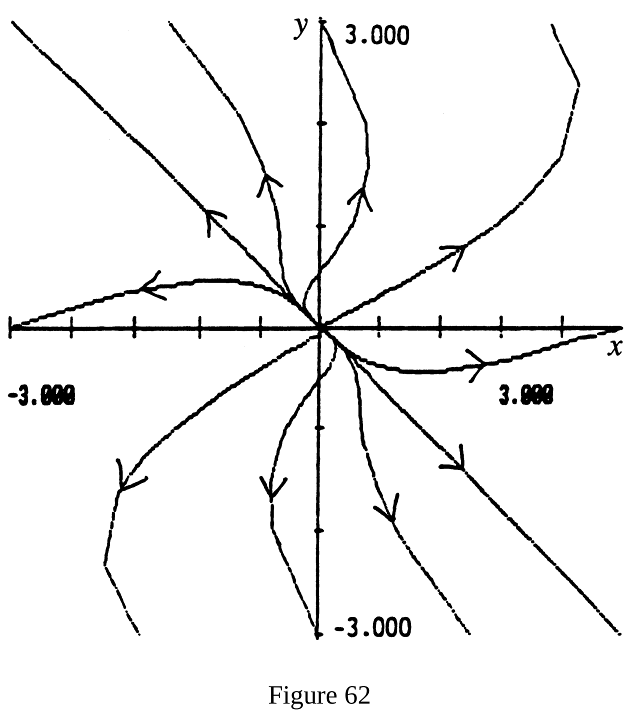


Figure 62

8.

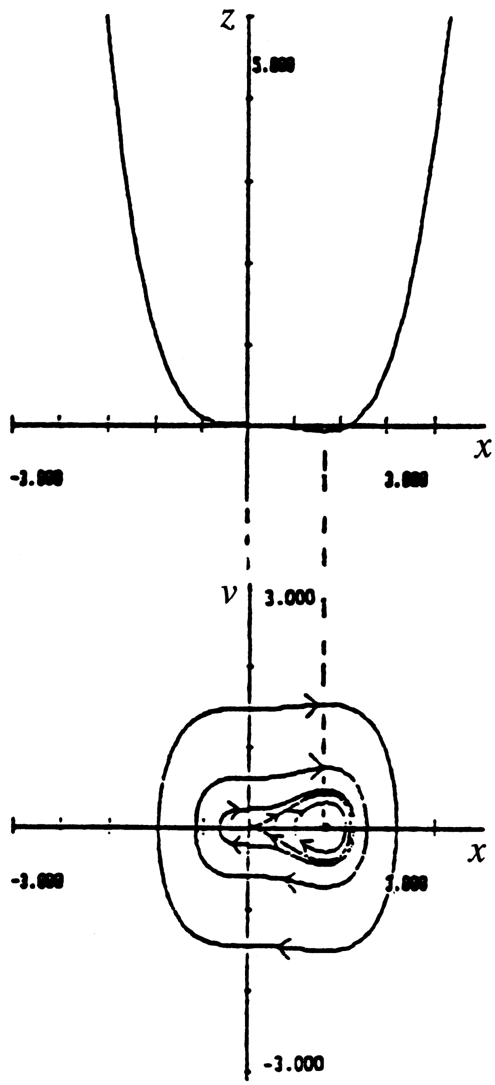


Figure 63

10. Unstable

12. Asymptotically stable

14. $r = 0$ is an asymptotically stable spiral point.
$r = 2$ is an unstable limit cycle.
$r = 3$ is a stable limit cycle.
$r = 4$ is an unstable limit cycle.

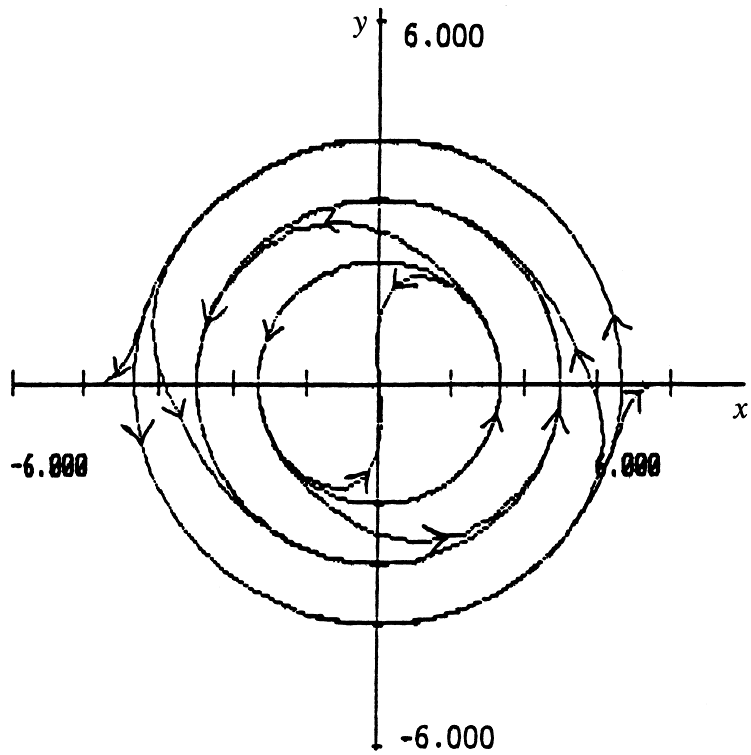


Figure 64

16. No

18. Asymptotically stable

CHAPTER 13

Exercises 13.1 (page 812)

2. $y(x) = \int_{\pi}^{x} \sin(t + y(t))dt$

4. $y(x) = 1 + \int_{0}^{x} e^{y(t)}dt$

6. 0.3775396

8. 2.2360680

10. 1.9345632

12. $y_1(x) = 1 + x, \quad y_2(x) = 1 + x + x^2 + \left(\frac{1}{3}\right)x^3$

14. $y_1(x) = y_2(x) = \sin x$

16. $y_1(x) = \left(\frac{3}{2}\right) - x + \left(\frac{1}{2}\right)x^2,$

$y_2(x) = \left(\frac{5}{3}\right) - \left(\frac{3}{2}\right)x + x^2 - \left(\frac{1}{6}\right)x^3$

Exercises 13.2 (page 820)

2. No

4. Yes

6. Yes

14. No; let $y_n(x) = \begin{cases} n^2 x & 0 \le x \le \frac{1}{n} \\ 2n - n^2 x & \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \frac{2}{n} \le x \le 1 \end{cases}$.

Then, $\lim_{n\to\infty} y_n(x) = 0$, but

$\lim_{n\to\infty} \int_0^1 y_n(x)\,dx = 1 \ne 0.$

Exercises 13.3 (page 826)

2. [–2, 1)

4. (0, 3]

6. $(-\infty, \infty)$

Exercises 13.4 (page 832)

2. $10^{-2}e$

4. $10^{-2}e^{\sqrt{2}e^{-1/2}}$

6. $10^{-2}e$

8. $\left(\frac{1}{24}\right)e^{\sin 1}$

10. $\frac{e}{6}$

Chapter 13 Review (page 835)

2. 0.7390851

4. $9 + \int_0^x [t^2 y^3(t) - y^2(t)]dt$

6. $y_1(x) = -1 + 2x, \quad y_2(x) = -1 + 2x - 2x^2$

8. No

10. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

12. $\frac{e}{6}$