Finite Mixture Models - HW2

2021712949 권남택

1.

2. $X \sim p N(\mu_1, \sigma_1^2) + (1-p) N(\mu_2, \sigma_2^2)$

- $f(x|z=1) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{1}{2\sigma_1^2}(x-\mu_1)^2\right)$, $f(x|z=0) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left\{-\frac{1}{2\sigma_2^2}(x-\mu_2)^2\right\}$

- $f(x,z) = \left\{\frac{p}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{1}{2\sigma_1^2}(x-\mu_1)^2\right)\right\}^z \left\{\frac{1-p}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{1}{2\sigma_2^2}(x-\mu_2)^2\right)\right\}^{1-z}$

- $f(z=1|x) = \quad p f(x|z=1) \,/\, \{p f(x|z=1) + (1-p) f(x|z=0)\}$

- $l_c(\theta) = \sum_i \left[z_i \left\{\log p - \log\sigma_1^2 - \frac{1}{2\sigma_1^2}(x-\mu_1)^2\right\} + (1-z_i)\left\{\log(1-p) - \log\sigma_2^2 - \frac{1}{2\sigma_2^2}(x-\mu_2)^2\right\}\right]$

- E-step : $Q(\theta|\theta^{(t)}) = \sum_i \left[\hat{z}_i^{(t)}\left\{\log p - \frac{1}{2}\log\sigma_1^2 - \frac{1}{2\sigma_1^2}(x-\mu_1)^2\right\} + (1-\hat{z}_i^{(t)})\left\{\log(1-p) - \frac{1}{2}\log\sigma_2^2 - \frac{1}{2\sigma_2^2}(x-\mu_2)^2\right\}\right]$

where
$$\hat{z}_i^{(t)} = \frac{\frac{p^{(t)}}{\sqrt{2\pi\sigma_1^{2(t)}}} \exp\left\{-\frac{1}{2\sigma_1^{2(t)}}(x-\mu_1^{(t)})^2\right\}}{\frac{p^{(t)}}{\sqrt{2\pi\sigma_1^{2(t)}}} \exp\left\{-\frac{1}{2\sigma_1^{2(t)}}(x-\mu_1^{(t)})^2\right\} + \frac{1-p^{(t)}}{\sqrt{2\pi\sigma_2^{2(t)}}} \exp\left\{-\frac{1}{2\sigma_2^{2(t)}}(x-\mu_2^{(t)})^2\right\}}$$

- M-step : $\frac{\partial Q}{\partial p} = \frac{1}{p}\sum_i \hat{z}_i^{(t)} - \frac{1}{1-p}\sum_i(1-\hat{z}_i^{(t)}) = 0$, $\quad \underline{p^{(t+1)} = \frac{1}{n}\sum_{i=1}^{n} \hat{z}_i^{(t)}}$

$\frac{\partial Q}{\partial \mu_1} = \sum_i \hat{z}_i^{(t)} \frac{1}{\sigma_1^2}(x_i-\mu_1) = 0$, $\quad \underline{\mu_1^{(t+1)} = \frac{\sum_i \hat{z}_i^{(t)} x_i}{\sum_i \hat{z}_i^{(t)}}}$. $\quad \frac{\partial Q}{\partial \mu_2} = 0 \Rightarrow \underline{\mu_2^{(t+1)} = \frac{\sum(1-\hat{z}_i^{(t)})x_i}{\sum(1-\hat{z}_i^{(t)})}}$

$\frac{\partial Q}{\partial \sigma_1^2} = \sum_i\left(-\frac{\hat{z}_i^{(t)}}{2\sigma_1^2} + \frac{\hat{z}_i^{(t)}}{2\sigma_1^4}(x_i-\mu_1)^2\right) = 0$, $\quad \sigma_1^2 \sum_i \hat{z}_i^{(t)} = \sum_{i=1} \hat{z}_i^{(t)}(x_i-\mu_1)^2$

$\underline{\sigma_1^{2(t+1)} = \frac{\sum_i \hat{z}_i^{(t)}(x_i-\mu_1)^2}{\sum_i \hat{z}_i^{(t)}}}$. $\quad$ Also, $\underline{\sigma_2^{2(t+1)} = \frac{\sum_i(1-\hat{z}_i^{(t)})(x_i-\mu_2)^2}{\sum_i(1-\hat{z}_i^{(t)})}}$.

- Set EM :
  i) Set initial values. $p^{(0)}, \mu_1^{(0)}, \mu_2^{(0)}, \sigma_1^{2(0)}, \sigma_2^{2(0)}$.
  ii) E-step.
  iii) M-step
  iv) Iterate ii) & iii) until converged.