

Lecture Note 4: Maps between Topological Spaces

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Definition 1 (Continuous Map). A *continuous map* f is a continuous function between two topological spaces \mathbf{M} and \mathbf{N} , denoted as,

$$f : \mathbf{M} \rightarrow \mathbf{N}$$

Definition 2 (Simplicial Map). A *simplicial map* φ between simplicial complexes K and L is a function

$$\varphi : \text{Vert}(K) \rightarrow \text{Vert}(L)$$

from the vertex set of K , to that of L such that whenever v_0, v_1, \dots, v_q span a q -simplex of K , $\varphi(v_0), \varphi(v_1), \dots, \varphi(v_q)$ span a p -simplex ($p \leq q$) of L . Of course, repetitions among $\varphi(v_0), \varphi(v_1), \dots, \varphi(v_q)$ are allowed.

Note that the simplicial map φ can be regarded as a function from K to L : this function sends a simplex σ of K with vertices v_0, v_1, \dots, v_q to the simplex $\varphi(\sigma)$ of L spanned by vertices $\varphi(v_0), \varphi(v_1), \dots, \varphi(v_q)$, so we also write φ as

$$\varphi : K \rightarrow L$$

Note that the simplicial map φ also induces a continuous map

$$\varphi : |K| \rightarrow |L|$$

between $|K|$ and $|L|$ (the polyhedra¹ of K and L), where a point inside $|K|$ spanned by vertex v_0, v_1, \dots, v_q is sent to a point inside $|L|$ continuously by

$$\varphi \left(\sum_{j=0}^q t_j v_j \right) = \sum_{j=0}^q t_j \varphi(v_j) \quad \text{whenever} \quad 0 \leq t_j \leq 1 \quad \text{for} \quad j = 0, 1, \dots, q \quad \text{and} \quad \sum_{j=0}^q t_j = 1$$

As a closing remark, there are thus three equivalent ways of describing a simplicial map:

1. as a function between the vertex sets of two simplicial complexes, e.g. $\varphi : \text{Vert}(K) \rightarrow \text{Vert}(L)$
2. as a function from one simplicial complex to another, e.g. $\varphi : K \rightarrow L$
3. as a continuous map between the polyhedra of two simplicial complexes, e.g. $\varphi : |K| \rightarrow |L|$

We shall describe a simplicial map using the representation that is most appropriate in the given context.

¹ $|K|$ always denotes the polyhedra of simplicial complex K .

- for 0-simplex σ_0 , $|\sigma_0|$ is itself
- for 1-simplex σ_1 , $|\sigma_1|$ is itself
- for 2-simplex σ_2 , $|\sigma_2|$ is the triangle it contains
- for 3-simplex σ_3 , $|\sigma_3|$ is the tetrahedron it contains

1 Simplicial Approximation Theorem

One may have experience with **Minecraft** game or **Lego** toy. Any real world object can be discretized in lattice. The mathematical theorem behind is *simplicial approximation theorem*, which guarantees that a continuous manifold can be (by a slight deformation) approximated by a simplicial complex of the simplest kind given its embedded simplicial complex space.

Example 3 (Manifold Embedded in Simplicial Complex). See figure 1, The manifold \mathbf{M} embedded in a given simplicial complex L described by a continuous map from $|K|$, the “parameter”, to $|L|$:

$$\mathbf{M} : |K| \rightarrow |L|$$

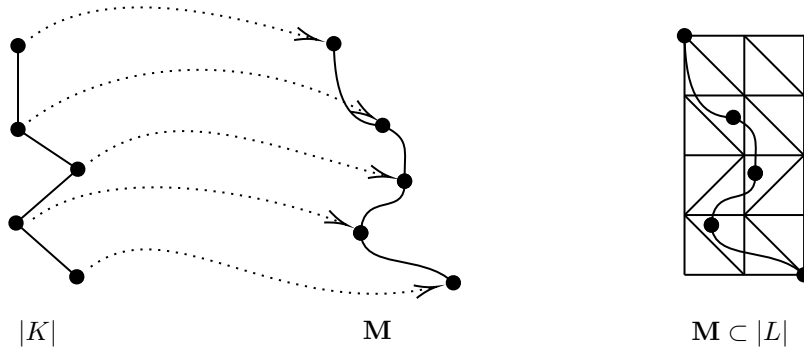


Figure 1: manifold \mathbf{M} was represented by a continuous map: $\mathbf{M} : |K| \rightarrow |L|$

Definition 4 (Star of a Vertex). Let K be a simplicial complex and let $p \in \text{Vert}(K)$. Then the *star* of p , the “discrete version of the neighbor of a point”, denoted by $\text{st}(p)$, is defined by

$$\text{st}(p) = \bigcup s^\circ \subset |K| \quad \text{where simplex } s \in K \quad \text{such that } p \in \text{Vert}(s)$$

Example 5 (Star of a Vertex). As shown in figure 2, $\text{st}(p)$ consists of the open shaded region, all the open simplices of which p is a neighbor.

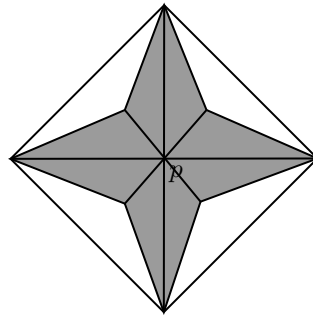


Figure 2: star of p , denoted by $\text{st}(p)$

Definition 6 (Simplicial Approximation). Let \mathbf{M} be a manifold represented by a continuous map $\mathbf{M} : |K| \rightarrow |L|$. Its approximating candidate \mathbf{M}_Δ , represented by a simplicial map $\mathbf{M}_\Delta : K \rightarrow L$, is *simplicial approximation* to \mathbf{M} if, for every vertex p of K ,

$$\mathbf{M}(\text{st}(p)) \subset \text{st}(\mathbf{M}_\Delta(p))$$

which means \mathbf{M} carries neighboring simplices of p inside the union of the simplices near $\mathbf{M}_\Delta(p)$. \mathbf{M}_Δ and \mathbf{M} are close up to a meshing unit.

Example 7 (Simplicial Approximation). See figure 3, \mathbf{M}_Δ is an simplicial approximation to \mathbf{M} . \mathbf{M}_Δ (red) is the simplest approximation to \mathbf{M} , achieved by $\text{Sd}^1 K$, the *first-order barycentric subdivision* of K . Barycentric subdivision and simplicial approximation theorem will be explained right away.

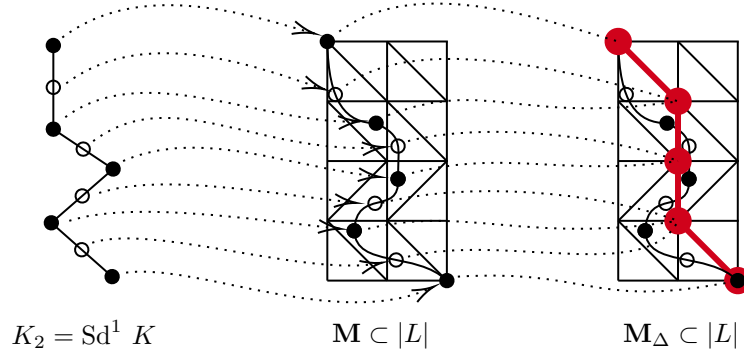


Figure 3: The simplicial approximation theorem guarantees that a simplest approximation in a given embedding mesh will be achieved by sufficient iterations of barycentric subdivision of “parameter”.

Definition 8 (Barycentric Subdivision). If s is a simplex, let b^s denote its barycenter. If K is a simplicial complex, define $\text{Sd } K$, the *barycentric subdivision* of K , to be the simplicial complex with

$$\text{Vert}(\text{Sd } K) = \{b^s : s \in K\}$$

note that here $s \in K$ are simplex of all dimensions in K . Recall that if s is a 0-simplex then trivially $b^s = s$; if s is a 1-simplex then b^s is the central point of two vertices; and so on. The q times iteration of barycentric subdivision is denoted by

$$\text{Sd}^q K$$

Example 9 (Barycentric Subdivision). See figure 4, if simplex $\sigma = [p_0, p_1, p_2]$, then $\text{Vert}(\text{Sd } \sigma) = \{p_0, p_1, p_2, b_0, b_1, b_2, b^\sigma\}$.

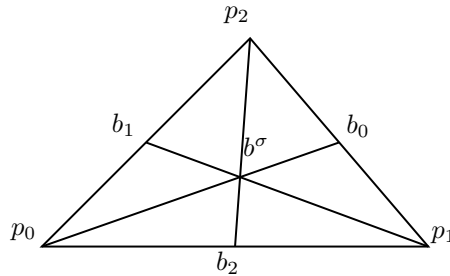


Figure 4: first order barycentric subdivision of the simplex σ

Definition 10 (Simplicial Approximation Theorem). Given simplicial complexes K and L . A smooth manifold \mathbf{M} represented by continuous map $\mathbf{M} : |K| \rightarrow |L|$ must have a simplicial approximation, and could be found its simplest kind after some barycentric subdivision of K by

$$\mathbf{M}_\Delta : \text{Sd}^q K \rightarrow L \quad \text{where } q \geq 1$$

Simplicial approximation theorem was the foundation of modern movie industry and game industry, since it provides a theoretical guarantee that the simplest discrete digital approximation of any smooth-shaped object exists.

Example 11 (Simplicial Approximation Theorem). See figure 1, one cannot construct a simplicial approximation to \mathbf{M} by its “parameter” K , but one can do so by K_2 , the first-order barycentric subdivision of “parameter”, as shown in figure 3

2 Chern-Gauss-Bonnet Theorem

Definition 12 (Induced Maps). Algebraic topology constructs functor

$$\mathfrak{C}_1 \rightarrow \mathfrak{C}_2$$

between $\mathfrak{C}_1 = \{\text{Topological Spaces, Homeomorphisms}\}$ and $\mathfrak{C}_2 = \{\text{Groups, Homomorphisms}\}$. Therefore, a continuous map $f : \mathbf{M} \rightarrow \mathbf{N}$ naturally induces homomorphism. there basically two kinds of *induced map*:

- *push-forward map*, denoted by $f_{\#}$ if on homotopy, and denoted by f_* if on homology.

$f_{\#}$ maps between fundamental groups² :

$$f_{\#} : \pi_1(\mathbf{M}) \rightarrow \pi_1(\mathbf{N})$$

f_* maps between p^{th} -homology groups, where p should be clear in the context:

$$f_* : H_p(\mathbf{M}) \rightarrow H_p(\mathbf{N})$$

- *pull-back map*, denoted as f^* if it maps between p^{th} -cohomology groups³, where p should be clear in the context:

$$f^* : H^p(\mathbf{N}) \rightarrow H^p(\mathbf{M})$$

Example 13 (Induced Maps of Surface). Suppose \mathbf{M} and \mathbf{N} are two closed surfaces, a continuous map:

$$f : \mathbf{M} \rightarrow \mathbf{N}$$

induces a push-forward map on first homology:

$$f_* : H_1(\mathbf{M}) \rightarrow H_1(\mathbf{N})$$

and a pull-back map on first cohomology:

$$f^* : H^1(\mathbf{N}) \rightarrow H^1(\mathbf{M})$$

² $f_{\#}$ takes “curves” to “curves”:

$$f_{\#} : C_p(\mathbf{M}) \rightarrow C_p(\mathbf{N})$$

$f_{\#}$ takes “cycles” to “cycles”:

$$f_{\#} : Z_p(\mathbf{M}) \rightarrow Z_p(\mathbf{N})$$

$f_{\#}$ takes “boundaries” to “boundaries”:

$$f_{\#} : B_p(\mathbf{M}) \rightarrow B_p(\mathbf{N})$$

³here we explain why push-forward and pull-back are opposite direction in a natural way.

– Points are sent forward. Given $p \in \mathbf{M}$ we have $f(p) \in \mathbf{N}$

– Functions are sent back, i.e. pull back from \mathbf{N} to \mathbf{M} . If we have a function $\omega : \mathbf{N} \rightarrow \mathbb{R}$ then we get the composition $\omega \circ f : \mathbf{M} \rightarrow \mathbb{R}$. The pull back can be considered a *functional* map, which maps from function on \mathbf{N} to function on \mathbf{M}

Suppose a curve $\sigma \in C_1(\mathbf{M}) \subset H_1(\mathbf{M})$ and a vector field $\omega \in C^1(\mathbf{N}) \subset H^1(\mathbf{M})$, then

$$\omega[f_*(\sigma)] = [f^*(\omega)](\sigma)$$

Definition 14 (Degree of a Map). Suppose \mathbf{M} and \mathbf{N} are two closed surfaces, the *degree of map* of a continuous map $f : \mathbf{M} \rightarrow \mathbf{N}$ is the algebraic number⁴ of pre-images $f^{-1}(q)$ for arbitrary point $q \in \mathbf{N}$, denoted by $\deg(f)$, which is independent of the choice of the point q . A quick example see figure 5

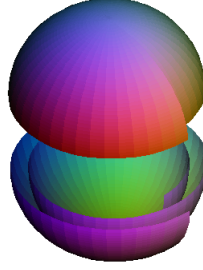


Figure 5: a continuous map $f : \mathbf{M} \rightarrow \mathbf{M}$ from a sphere to itself but in 2x speed. For every point $q \in \mathbf{M}$, there are two pre-images, so $\deg(f) = 2$

Example 15 (Degree of a Map). Suppose \mathbf{M} and \mathbf{N} are two closed surfaces, a continuous map:

$$f : \mathbf{M} \rightarrow \mathbf{N}$$

induces a push-forward map on second homology:

$$f_* : H_2(\mathbf{M}) \rightarrow H_2(\mathbf{N})$$

since $H_2(\mathbf{M}, \mathbb{Z}) \cong \mathbb{Z} \cong H_2(\mathbf{N}, \mathbb{Z})$, we also write its isomorphism:

$$\tilde{f}_* : \mathbb{Z} \rightarrow \mathbb{Z}$$

and it must have the form

$$\tilde{f}_*(z) = \deg(f) \cdot z$$

Definition 16 (Euler-Poincaré Characteristic). let g be the genus of a closed surface \mathbf{S} , then *Euler characteristic*, denoted as $\chi(\mathbf{S})$, is

$$\chi(\mathbf{S}) = 2(1 - g)$$

the discrete version, if \mathbf{S} triangulated in \mathbf{S}_Δ , is

$$\chi(\mathbf{S}_\Delta) = |\text{Faces}| - |\text{Edges}| + |\text{Vertices}|$$

Definition 17 (Gaussian Curvature). At any point on a surface, we can find a normal vector that is at right angles to the surface; planes containing the normal vector are called normal planes. The intersection of a normal plane and the surface will form a curve called a normal section and the curvature of this curve is the normal curvature. For most points on most surfaces, different normal sections will have different curvatures; the maximum and minimum values of these are called the principal curvatures, call these κ_1, κ_2 . The *Gaussian curvature* is the product of the two principal curvatures $K = \kappa_1 \cdot \kappa_2$, as shown in figure 6

⁴if Jacobian at that point q is positive then count +1, else then count -1, then degree of map is sum of total count

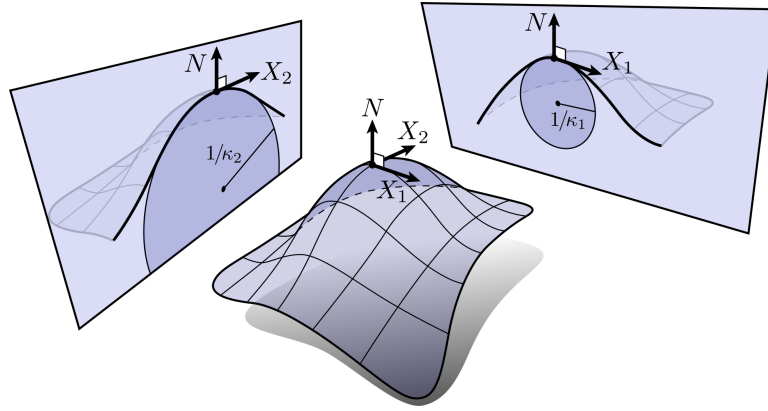


Figure 6: Gaussian curvature is the product of principal curvature κ_1 and κ_2 . The curvature of a circle is reciprocal of radius: $1/r$. So if the curvature is κ , the radius is $1/\kappa$, as noted in the figure

Theorem 18 (Chern-Gauss-Bonnet Theorem). *Let \mathbf{S} be a closed surface, $K(p)$ the Gaussian curvature at point p on surface, and $dA(p)$ the element area at point p on surface, then its total Gaussian curvature*

$$\int_{\mathbf{S}} K(p) dA(p) = 2\pi\chi(\mathbf{S})$$

Shiing-Shen Chern provided simple intrinsic proof of Gauss-Bonnet Theorem, which added his name to Gauss-Bonnet. We illustrate his beautiful proof by applying degree of Gauss map and homotopy between surfaces.

proof. consider the Gauss Map $G : \mathbf{S}^* \rightarrow \mathbb{S}^2$ from a canonical closed surface \mathbf{S}^* to unit sphere \mathbb{S}^2 . Whenever a point p on surface with normal $\mathbf{n}(p)$, the Gauss Map maps it to a point $G(p)$ on unit sphere with the same normal $\mathbf{n}(p)$.

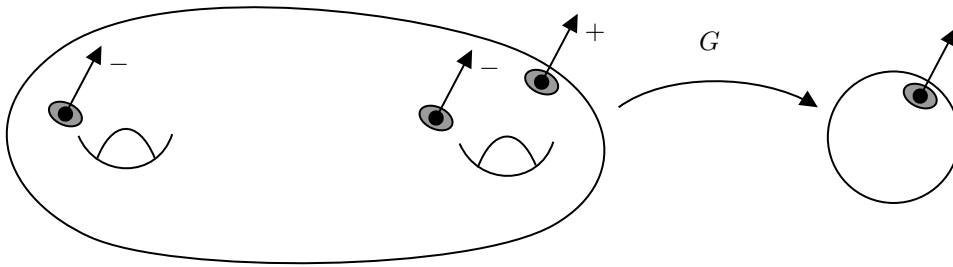


Figure 7: The canonical shape of genus g closed surface \mathbf{S}^* can guarantee $\deg(G) = 1 - g$ since the count of pre-image is strictly negative whenever a hole appears.

“canonical” here means it guarantees

$$\deg(G) = 1 - g$$

so the total area of the image of \mathbf{S}^* on unit sphere \mathbb{S}^2 is

$$\text{Area}(\mathbb{S}^2) \times \deg(G) = 4\pi \deg(G) = 4\pi(1 - g) = 2\pi\chi(\mathbf{S}^*)$$

Note that the total area of the image of \mathbf{S}^* on unit sphere \mathbb{S}^2 also equals to

$$\int_{\mathbf{S}^*} \frac{\text{Area}(G(p))}{\text{Area}(p)} dA(p)$$

which equals to, since it is a Gauss Map⁵:

$$\int_{\mathbf{S}^*} \frac{\text{Area}(G(p))}{\text{Area}(p)} dA(p) = \int_{\mathbf{S}^*} K(p) dA(p)$$

thus we get, for a canonical closed surface \mathbf{S}^* , the identity

$$\int_{\mathbf{S}^*} K(p) dA(p) = 2\pi\chi(\mathbf{S}^*)$$

Now consider the quantity

$$\frac{\int_{\mathbf{S}^*} K(p) dA(p)}{2\pi} = \chi(\mathbf{S}^*) \in \mathbb{Z}$$

is an integer, which should not change under continuous deformation from canonical shaped \mathbf{S}^* to any closed surface \mathbf{S} with same genus g , thus we get for any closed surface \mathbf{S}

$$\int_{\mathbf{S}} K(p) dA(p) = 2\pi\chi(\mathbf{S})$$

□

Example 19 (Chern-Gauss-Bonnet Theorem). let \mathbf{S} be a sphere with radius R , its genus $g = 0$, then $\chi(\mathbf{S}) = 2 \times (1 - 0) = 2$, its Gaussian curvature is constant $\frac{1}{R^2}$, according to Chern-Gauss-Bonnet formula

$$\int_{\mathbf{S}} \frac{1}{R^2} dA(p) = \frac{1}{R^2} \int_{\mathbf{S}} dA(p) = \frac{1}{R^2} \times \text{Area}(\mathbf{S}) = 2\pi\chi(\mathbf{S}) = 4\pi$$

indeed $\text{Area}(\mathbf{S}) = 4\pi R^2$.

3 Fixed Point Theorem

Definition 20 (Inclusion Map). an *inclusion map* i from A to B , where $A \subset B$, satisfies that for any element $x \in A$ we have $i(x) = x$, denoted as

$$i : A \hookrightarrow B$$

Theorem 21 (Brouwer's Fixed Point Theorem). Suppose $\Omega \subset \mathbb{R}^n$ is a compact convex set, $f : \Omega \rightarrow \Omega$ is a continuous map, then there exists a point $p \in \Omega$ such that

$$f(p) = p$$

⁵for Gauss Map, when shrinking a patch around point p , its limit is Gaussian curvature:

$$\lim_{\Omega_p \rightarrow 0} \frac{\text{Area}(G(\Omega_p))}{\text{Area}(\Omega_p)} = K(p)$$

proof. Assume $f : \Omega \rightarrow \Omega$ has no fixed point, namely

$$\forall p \in \Omega, \quad f(p) \neq p$$

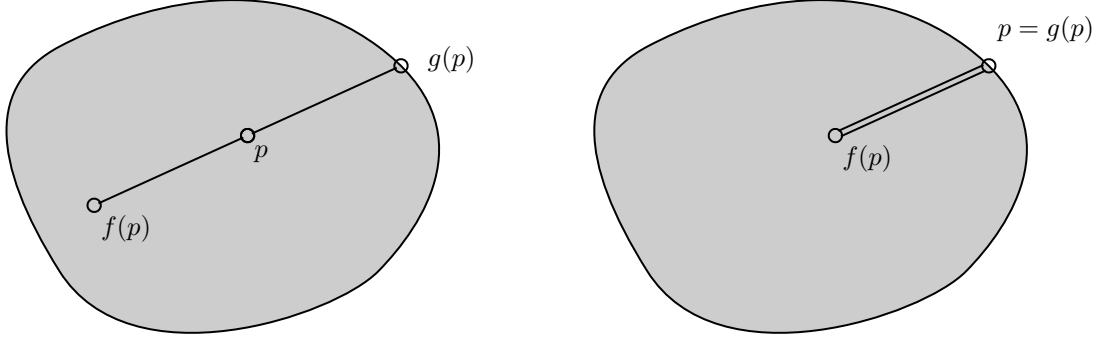


Figure 8: diagram of $g : \Omega \rightarrow \partial\Omega$ (left) and $(g \circ i) : \partial\Omega \rightarrow \partial\Omega$ (right)

We can construct $g : \Omega \rightarrow \partial\Omega$, a ray starting from $f(p)$ through p and intersect $\partial\Omega$ at $g(p)$. Because our assumption $f(p) \neq p$ and Ω is convex, g is well-defined. Note that if point $p \in \partial\Omega$ then $g(p) = p$, as shown in figure 8. We construct an inclusion map $i : \partial\Omega \hookrightarrow \Omega$, which maps a point $p \in \partial\Omega$ to itself. Then we compose it with g ,

$$\partial\Omega \xrightarrow{i} \Omega \xrightarrow{g} \partial\Omega$$

we get an identity map:

$$(g \circ i) : \partial\Omega \rightarrow \partial\Omega$$

which induces a push-forward map on $(n-1)^{th}$ homology:

$$(g \circ i)_* : H_{n-1}(\partial\Omega, \mathbb{Z}) \rightarrow H_{n-1}(\partial\Omega, \mathbb{Z})$$

since it is identity map,

$$(g \circ i)_* : z \mapsto z$$

$g : \Omega \rightarrow \partial\Omega$ induces a push-forward map on $(n-1)^{th}$ homology:

$$g_* : H_{n-1}(\Omega, \mathbb{Z}) \rightarrow H_{n-1}(\partial\Omega, \mathbb{Z})$$

however, since Ω is compact convex set

$$H_{n-1}(\Omega, \mathbb{Z}) = 0, \quad g_* = 0$$

so

$$(g \circ i)_* = g_* \circ i_* = 0$$

contradiction! $f : \Omega \rightarrow \Omega$ has fixed point. □

In 1910, Luitzen Egbertus Jan Brouwer proved his fixed point theorem, which ensured the existence of fixed point of a continuous self-map of convex compact space. Often, it can be stated as follow:

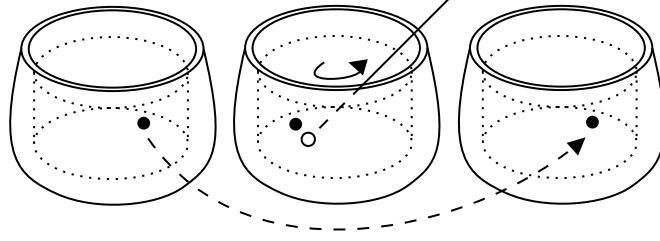


Figure 9: “swirling coffee” theorem: at least one molecule, “doesn’t move” before and after coffee swirling

Theorem 22 (“Swirling Coffee” Theorem). *Use a stick (volume can be ignored) to swirl a cup of coffee without making any bubble. In the end, there is a molecule with final position the same as initial position in your coffee.*

In 1926, Solomon Lefschetz gave a formula that relates the number of fixed points of a map to the induced push-forward maps on homology.

Definition 23 (Index of Fixed Point). Suppose \mathbf{M} is an n -dimensional topological space, p is a fixed point of self-map $f : \mathbf{M} \rightarrow \mathbf{M}$. Choose a neighborhood \mathbf{U} such that $p \in \mathbf{U} \subset \mathbf{M}$, consider the boundary of \mathbf{U} , which is a $(n - 1)$ -dimensional $\partial\mathbf{U}$. Similar to the concept “degree of a map” (see example 15), the induced push-forward map on $(n - 1)^{th}$ homology:

$$f_* : H_{n-1}(\partial\mathbf{U}, \mathbb{Z}) \rightarrow H_{n-1}(\partial\mathbf{U}, \mathbb{Z})$$

is $f_* : \mathbb{Z} \rightarrow \mathbb{Z}$ and must have the form $f_* : z \mapsto \lambda z$, where λ is an integer called the *algebraic index of fixed point* p of map f , denoted as

$$\text{Ind}(f, p) = \lambda$$

Definition 24 (Trace of Self-map). Let \mathbf{A} be a matrix representing a self-map $f : \mathbf{M} \rightarrow \mathbf{M}$ under any basis, then the *trace* of f , denoted as

$$\text{Tr}(f)$$

is $\text{Tr}(\mathbf{A})$, the trace of \mathbf{A} , which is independent of choice of basis.

Definition 25 (Lefschetz-Hopf Fixed Point Formula). Given compact topological space \mathbf{M} . The sum of indexes of all fixed points of a self-map $f : \mathbf{M} \rightarrow \mathbf{M}$ equals to the alternating sum of trace of push-forward map on k^{th} homology $f_{*k} : H_k(\mathbf{M}, \mathbb{Z}) \rightarrow H_k(\mathbf{M}, \mathbb{Z})$ induced by the self-map f

$$\sum_{p \in \text{Fix}(f)} \text{Ind}(f, p) = \sum_k (-1)^k \text{Tr}(f_{*k}) =: \Lambda(f)$$

where $\Lambda(f)$ is called *Lefschetz number*

Example 26 (Lefschetz-Hopf Fixed Point Formula). consider a simple self-map $f : [0, 1] \rightarrow [0, 1]$. we have

$$\Lambda(f) = \underbrace{\text{Tr}(f_{*0} : \mathbb{Z} \rightarrow \mathbb{Z})}_1 - \underbrace{\text{Tr}(f_{*1} : 0 \rightarrow 0)}_0 = 1 = \sum_{p \in \text{Fix}(f)} \text{Ind}(f, p)$$

as shown in figure 10

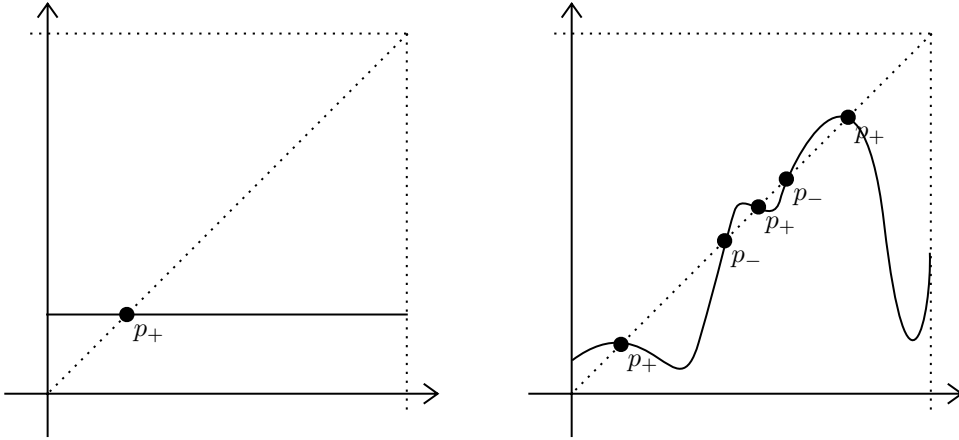


Figure 10: a self-map $f_1 : [0, 1] \rightarrow \text{const}$ (left) versus its homotopic self-map f_2 with same Lefschetz number $\Lambda(f_1) = \Lambda(f_2) = 1$

Theorem 27 (Lefschetz's Fixed Point Theorem). *Given a continuous self-map of a compact topological space $f : M \rightarrow M$, if its Lefschetz number $\Lambda(f) \neq 0$, then there is a point $p \in M$ such that*

$$f(p) = p$$

Proof (Advanced). Notation update:

- f_k : the induced push-forward map on k -dimensional space
- $f_k \mid C_k$: the induced map on k -chain group
- $f_k \mid H_k$: the induced map on k -homology
- \oplus : direct sum between groups, e.g. $A \oplus B = \{(a + b) \mid a \in A, b \in B\}$

According to simplicial approximation theorem, there must be approximated maps up to any precision. So we triangulate M first, and assume its induced map f can be both embedded in chain space and smooth space, as shown in the commutative diagram as figure 11:

$$\begin{array}{ccc}
 \frac{C_k}{Z_k} & \xrightarrow{f_k} & \frac{C_k}{Z_k} \\
 \partial_k \downarrow \quad \uparrow \partial_k^{-1} & & \downarrow \partial_k \\
 B_{k-1} & \xrightarrow{f_{k-1}} & B_{k-1}
 \end{array}$$

Figure 11: commutative diagram of induced map and boundary operator

we have

$$(f_{k-1} \mid B_{k-1}) = \partial_k \circ (f_k \mid \frac{C_k}{Z_k}) \circ \partial_k^{-1}$$

and thus

$$\begin{aligned}\mathrm{Tr}(f_{k-1} \mid B_{k-1}) &= \mathrm{Tr}([\partial_k][f_k \mid \frac{C_k}{Z_k}][\partial_k^{-1}]) \\ &= \mathrm{Tr}([f_k \mid \frac{C_k}{Z_k}][\partial_k^{-1}][\partial_k]) \\ &= \mathrm{Tr}(f_k \mid \frac{C_k}{Z_k})\end{aligned}$$

according to property of trace.

Let C_k be k -chain group, Z_k closed chain group, B_k exact chain group, H_k homology group. We have

$$C_k \cong \frac{C_k}{Z_k} \oplus Z_k \quad \text{and} \quad Z_k \cong B_k \oplus H_k$$

thus

$$\begin{aligned}\mathrm{Tr}(f_k \mid C_k) &= \mathrm{Tr}(f_k \mid \frac{C_k}{Z_k} \oplus Z_k) \\ &= \mathrm{Tr}(f_k \mid \frac{C_k}{Z_k}) + \mathrm{Tr}(f_k \mid Z_k) \\ &= \mathrm{Tr}(f_{k-1} \mid B_{k-1}) + \mathrm{Tr}(f_k \mid B_k) + \mathrm{Tr}(f_k \mid H_k)\end{aligned}$$

thus

$$\begin{aligned}\sum_k (-1)^k \mathrm{Tr}(f_k \mid C_k) &= \sum_k (-1)^k \overbrace{[\mathrm{Tr}(f_{k-1} \mid B_{k-1}) + \mathrm{Tr}(f_k \mid B_k) + \mathrm{Tr}(f_k \mid H_k)]}^{\text{cancel out in sequence}} \quad (1) \\ &= \sum_k (-1)^k \mathrm{Tr}(f_k \mid H_k) \quad (2) \\ &= \Lambda(f) \quad (3)\end{aligned}$$

according to Lefschetz-Hopf fixed point formula. Whenever $\Lambda(f) \neq 0$, there is an entry in a matrix such that $\mathrm{Tr}(f_k \mid C_k) \neq 0$, which means there is simplex $\sigma \in C_k$ such that $f_k(\sigma) \subset \sigma$, for any point in $|\sigma|$, the continuous map $f_k : |\sigma| \rightarrow |\sigma|$ must have a Brouwer's fixed point such that $f_k(p) = p$, which means

$$f(p) = p$$

□

Example 28 (Lefschetz Number, Betti Number and Euler-Poincaré Characteristic). Consider an identity map of a closed surface

$$\mathrm{id} : \mathbf{S} \rightarrow \mathbf{S}$$

the identity map is, of course, a self-map. According to equations 1, 2 and 3, we have

$$\begin{aligned}\Lambda(\mathrm{id}) &= \sum_k (-1)^k \mathrm{Tr}(\mathrm{id}_k \mid C_k) = \underbrace{\mathrm{Tr}(\mathrm{id}_2 \mid C_2)}_{|\text{Faces}|} - \underbrace{\mathrm{Tr}(\mathrm{id}_1 \mid C_1)}_{|\text{Edges}|} + \underbrace{\mathrm{Tr}(\mathrm{id}_0 \mid C_0)}_{|\text{Vertices}|} \\ &= \sum_k (-1)^k \mathrm{Tr}(\mathrm{id}_k \mid H_k) = \underbrace{\mathrm{Tr}(\mathrm{id}_2 \mid H_2)}_{b_2} - \underbrace{\mathrm{Tr}(\mathrm{id}_1 \mid H_1)}_{b_1} + \underbrace{\mathrm{Tr}(\mathrm{id}_0 \mid H_0)}_{b_0} \\ &= \chi(\mathbf{S})\end{aligned}$$

Here we show that for an identity map, the connection between its Lefschetz number and Euler-Poincaré characteristic, and where Euler (number of triangulation elements) and Poincaré (Betti number) coincide.

Geometrically, Betti number of surface can be understood as:

- b_0 is the number of connected components
- b_1 is the number of one-dimensional or “circular” holes
- b_2 is the number of two-dimensional “voids” or “cavities”

e.g. for a torus, $b_0 = 1$, $b_1 = 2$ and $b_2 = 1$