Computational Conformal Geometry	**授课日期: 2020年11月**

Lecture Note 1: Fundamental Groups and Covering Spaces

讲师: *Xianfeng (David) Gu* 撰写: 洪楠方
助教: 洪楠方 最后更新: *November 8, 2020*

This lecture is about algebraic surface topology. The key idea is to build a bridge between topology, which is abstract and hard to imagine, and algebraic structure, which is tangible and can be computed. In a categorical sense, we construct a functor

$$\mathfrak{C}_1 \mapsto \mathfrak{C}_2$$

between two categories[1] with structural information preserved, namely

$$\begin{aligned} \mathfrak{C}_1 &= \{\text{Topological Spaces, Homeomorphisms}\} \\ \mathfrak{C}_2 &= \{\text{Groups, Homomorphisms}\} \end{aligned}$$

Definition 1 (Topological Type)**.** All oriented compact surfaces can be classified by their genus g and number of boundaries b. Therefore, we use

$$(g, b)$$

to represent the topological type of an oriented surface $\mathbf{S}$.

Definition 2 (Homeomorphism)**.** A *homeomorphism* is a continuous function between topological spaces of the same topological type.

Definition 3 (Homomorphism)**.** A *homomorphism* is a structure-preserving map between two algebraic structures of the same type.

We now introduce *first homotopy group*, denoted as $\pi_1(\mathbf{S})$ (or $\pi_1(\mathbf{S}, q)$, if base point q is clear in context). The group structure of $\pi_1(\mathbf{S}, q)$ determines the topology of $\mathbf{S}$.

1 Surface Fundamental Group

Let $\mathbf{S}$ be a two-manifold with a base point $p \in \mathbf{S}$.

Definition 4 (Curve)**.** A *curve* is a continuous mapping $\gamma : [0,1] \mapsto \mathbf{S}$

Definition 5 (Loop)**.** A *closed curve* or *loop* through p is a curve s.t. $\gamma(0) = \gamma(1) = p$

Definition 6 (Homotopy)**.** Let $\gamma_0, \gamma_1 : [0,1] \mapsto \mathbf{S}$ be two curves. A *homotopy* connecting γ_0 and γ_1 is a continuous mapping

$$f : [0,1] \times [0,1] \mapsto \mathbf{S}$$

[1] The concepts of category and functor were covered in previous lectures

s.t.

$$f(0, t) = \gamma_0(t)$$

$$f(1, t) = \gamma_1(t)$$

We say γ_0 is homotopic to γ_1, if there exists a homotopy between them.

Definition 7 (Loop Product)**.** $\gamma_1 \cdot \gamma_2$ is

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq 0.5 \\ \gamma_2(2t-1) & \text{for } 0.5 \leq t \leq 1 \end{cases}$$

Definition 8 (Loop Inverse)**.** $\gamma^{-1}(t) := \gamma(1-t)$

Definition 9 (Fundamental Group)**.** Given a topological space $\mathbf{S}$, fix a base point $p \in \mathbf{S}$. Homotopy relation is an equivalence relation[2]. The set of all the loops through the base point p is Γ, which can be classified by homotopy relation and form a set of all the homotopy classes, denoted as $\Gamma/\sim$.

- The homotopy class of a loop γ is denoted by $[\gamma]$.
- The binary operation is defined as
  $$[\gamma_1][\gamma_2] := [\gamma_1 \cdot \gamma_2]$$
  .
- The unit element is defined as $[e]$, which is as trivial as a point.
- The inverse element is defined as
  $$[\gamma]^{-1} = [\gamma^{-1}]$$

then $\Gamma/\sim$ forms a group, so-called *fundamental group* of $\mathbf{S}$, or the first homotopy group, denoted as $\pi_1(\mathbf{S}, p)$.

[2]needs to be reflexive, symmetric and transitive