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| **Computational Conformal Geometry** | **授课日期: 2020年11月** |

# Lecture Note 4: Maps between Topological Spaces

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**Definition 1** (Continuous Map)**.** A *continuous map* $f$ is a continuous function between two topological spaces $\mathbf{M}$ and $\mathbf{N}$, denoted as,

$$f : \mathbf{M} \to \mathbf{N}$$

**Definition 2** (Simplicial Map)**.** A *simplicial map* $\varphi$ between simplicial complexes $K$ and $L$ is a function

$$\varphi : \mathrm{Vert}(K) \to \mathrm{Vert}(L)$$

from the vertex set of $K$, to that of $L$ such that whenever $v_0, v_1, ..., v_q$ span a $q$-simplex of $K$, $\varphi(v_0), \varphi(v_1), ..., \varphi(v_q)$ span a $p$-simplex ($p \le q$) of $L$. Of course, repetitions among $\varphi(v_0), \varphi(v_1), ..., \varphi(v_q)$ are allowed.

Note that the simplicial map $\varphi$ can be regarded as a function from $K$ to $L$: this function sends a simplex $\sigma$ of $K$ with vertices $v_0, v_1, ..., v_q$ to the simplex $\varphi(\sigma)$ of $L$ spanned by vertices $\varphi(v_0), \varphi(v_1), ..., \varphi(v_q)$, so we also write $\varphi$ as

$$\varphi : K \to L$$

Note that the simplicial map $\varphi$ also induces a continuous map

$$\varphi : |K| \to |L|$$

between $|K|$ and $|L|$ (the polyhedra[1] of $K$ and $L$), where a point inside $|K|$ spanned by vertex $v_0, v_1, ..., v_q$ is sent to a point inside $|L|$ continuously by

$$\varphi\left(\sum_{j=0}^{q} t_j v_j\right) = \sum_{j=0}^{q} t_j \varphi(v_j) \quad \text{whenever} \quad 0 \le t_j \le 1 \quad \text{for} \quad j = 0, 1, ..., q \quad \text{and} \quad \sum_{j=0}^{q} t_j = 1$$

As a closing remark, there are thus three equivalent ways of describing a simplicial map:

1. as a function between the vertex sets of two simplicial complexes, e.g. $\varphi : \mathrm{Vert}(K) \to \mathrm{Vert}(L)$
2. as a function from one simplicial complex to another, e.g. $\varphi : K \to L$
3. as a continuous map between the polyhedra of two simplicial complexes, e.g. $\varphi : |K| \to |L|$

We shall describe a simplicial map using the representation that is most appropriate in the given context.

[1] $|K|$ always denotes the polyhedra of simplicial complex $K$.

- for 0-simplex $\sigma_0$, $|\sigma_0|$ is itself
- for 1-simplex $\sigma_1$, $|\sigma_1|$ is itself
- for 2-simplex $\sigma_2$, $|\sigma_2|$ is the triangle it contains
- for 3-simplex $\sigma_3$, $|\sigma_3|$ is the tetrahedron it contains

# 1 Simplicial Approximation Theorem

One may have experience with **Minecraft** game or **Lego** toy. Any real world object can be discretized in lattice. The mathematical theorem behind is *simplicial approximation theorem*, which guarantees that a continuous manifold can be (by a slight deformation) approximated by a simplicial complex of the simplest kind given its embedded simplicial complex space.

**Example 3** (Manifold Embedded in Simplicial Complex)**.** See figure 1, The manifold $\mathbf{M}$ embedded in a given simplicial complex $L$ described by a continuous map from $|K|$, the "parameter", to $|L|$:

$$\mathbf{M} : |K| \to |L|$$

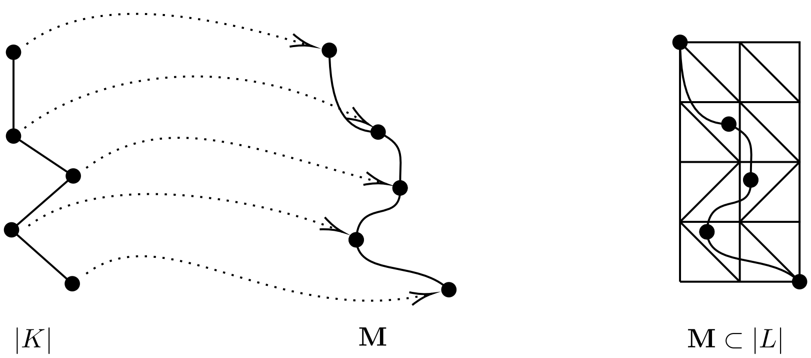


Figure 1: manifold $\mathbf{M}$ was represented by a continuous map: $\mathbf{M} : |K| \to |L|$

**Definition 4** (Star of a Vertex)**.** Let $K$ be a simplicial complex and let $p \in \mathrm{Vert}(K)$. Then the *star* of $p$, the "discrete version of the neighbor of a point", denoted by $\mathrm{st}(p)$, is defined by

$$\mathrm{st}(p) = \bigcup s^\circ \subset |K| \quad \text{where simplex} \quad s \in K \quad \text{such that} \quad p \in \mathrm{Vert}(s)$$

**Example 5** (Star of a Vertex)**.** As shown in figure 2, $\mathrm{st}(p)$ consists of the open shaded region, all the open simplices of which $p$ is a neighbor.

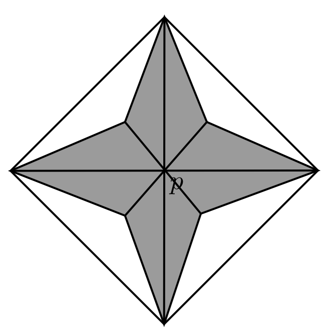


Figure 2: star of $p$, denoted by $\mathrm{st}(p)$

**Definition 6** (Simplicial Approximation)**.** Let $\mathbf{M}$ be a manifold represented by a continuous map $\mathbf{M}$ : $|K| \to |L|$. Its approximating candidate $\mathbf{M}_\Delta$ , represented by a simplicial map $\mathbf{M}_\Delta : K \to L$, is *simplicial approximation* to $\mathbf{M}$ if, for every vertex $p$ of $K$,

$$\mathbf{M}(\mathrm{st}(p)) \subset \mathrm{st}(\mathbf{M}_\Delta(p))$$

which means $\mathbf{M}$ carries neighboring simplices of $p$ inside the union of the simplices near $\mathbf{M}_\Delta(p)$. $\mathbf{M}_\Delta$ and $\mathbf{M}$ are close up to a meshing unit.

**Example 7** (Simplicial Approximation)**.** See figure 3, $\mathbf{M}_\Delta$ is an simplicial approximation to $\mathbf{M}$. $\mathbf{M}_\Delta$ (red) is the simplest approximation to $\mathbf{M}$, achieved by $\mathrm{Sd}^1 K$, the *first-order barycentric subdivision* of $K$. Barycentric subdivision and simplicial approximation theorem will be explained right away.

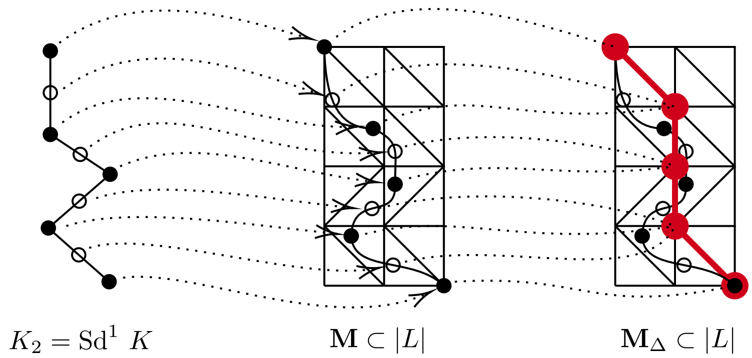


Figure 3: The simplicial approximation theorem guarantees that a simplest approximation in a given embedding mesh will be achieved by sufficient iterations of barycentric subdivision of "parameter".

**Definition 8** (Barycentric Subdivision)**.** If $s$ is a simplex, let $b^s$ denote its barycenter. If $K$ is a simplicial complex, define Sd $K$, the *barycentric subdivision* of $K$, to be the simplicial complex with

$$\mathrm{Vert}(\mathrm{Sd}\ K) = \{b^s : s \in K\}$$

note that here $s \in K$ are simplex of all dimensions in $K$. Recall that if $s$ is a 0-simplex then trivially $b^s = s$; if $s$ is a 1-simplex then $b^s$ is the central point of two vertices; and so on. The $q$ times iteration of barycentric subdivision is denoted by

$$\mathrm{Sd}^q K$$

**Example 9** (Barycentric Subdivision)**.** See figure 4, if simplex $\sigma = [p_0, p_1, p_2]$, then $\mathrm{Vert}(\mathrm{Sd}\ \sigma) = \{p_0, p_1, p_2, b_0, b_1, b_2, b^\sigma\}$.

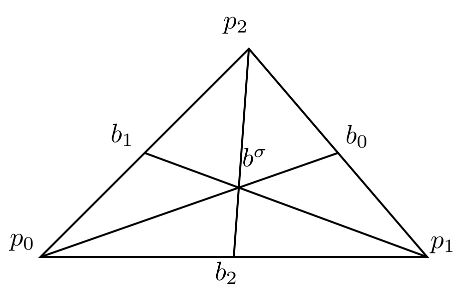


Figure 4: first order barycentric subdivision of the simplex $\sigma$

**Definition 10** (Simplicial Approximation Theorem)**.** Given simplicial complexes $K$ and $L$. A smooth manifold $\mathbf{M}$ represented by continuous map $\mathbf{M} : |K| \to |L|$ must have a simplicial approximation, and could be found its simplest kind after some barycentric subdivision of $K$ by

$$\mathbf{M}_\Delta : \mathrm{Sd}^q K \to L \quad \text{where} \quad q \geq 1$$

Simplicial approximation theorem was the foundation of modern movie industry and game industry, since it provides a theoretical guarantee that the simplest discrete digital approximation of any smooth-shaped object exists.

**Example 11** (Simplicial Approximation Theorem)**.** See figure 1, one cannot construct a simplicial approximation to $\mathbf{M}$ by its "parameter" $K$, but one can do so by $K_2$, the first-order barycentric subdivision of "parameter", as shown in figure 3

## 2 Chern-Gauss-Bonnet Theorem

**Definition 12** (Induced Maps)**.** Algebraic topology constructs functor

$$\mathfrak{C}_1 \to \mathfrak{C}_2$$

between $\mathfrak{C}_1 = \{\text{Topological Spaces, Homeomorphisms}\}$ and $\mathfrak{C}_2 = \{\text{Groups, Homomorphisms}\}$. Therefore, a continuous map $f : \mathbf{M} \to \mathbf{N}$ naturally induces homomorphism. there basically two kinds of *induced map*:

- *push-forward map*, denoted by $f_\#$ if on homotopy, and denoted by $f_*$ if on homology.

  $f_\#$ maps between fundamental groups[2] :

  $$f_\# : \pi_1(\mathbf{M}) \to \pi_1(\mathbf{N})$$

  $f_*$ maps between $p^{th}$-homology groups, where $p$ should be clear in the context:

  $$f_* : H_p(\mathbf{M}) \to H_p(\mathbf{N})$$

- *pull-back map*, denoted as $f^*$ if it maps between $p^{th}$-cohomology groups[3], where $p$ should be clear in the context:

  $$f^* : H^p(\mathbf{N}) \to H^p(\mathbf{M})$$

**Example 13** (Induced Maps of Surface)**.** Suppose $\mathbf{M}$ and $\mathbf{N}$ are two closed surfaces, a continuous map:

$$f : \mathbf{M} \to \mathbf{N}$$

induces a push-forward map on first homology:

$$f_* : H_1(\mathbf{M}) \to H_1(\mathbf{N})$$

and a pull-back map on first cohomology:

$$f^* : H^1(\mathbf{N}) \to H^1(\mathbf{M})$$

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[2] $f_\#$ takes "curves" to "curves":

$$f_\# : C_p(\mathbf{M}) \to C_p(\mathbf{N})$$

$f_\#$ takes "cycles" to "cycles":

$$f_\# : Z_p(\mathbf{M}) \to Z_p(\mathbf{N})$$

$f_\#$ takes "boundaries" to "boundaries":

$$f_\# : B_p(\mathbf{M}) \to B_p(\mathbf{N})$$

[3] here we explain why push-forward and pull-back are opposite direction in a natural way.

- Points are sent forward. Given $p \in \mathbf{M}$ we have $f(p) \in \mathbf{N}$
- Functions are sent back, i.e. pull back from $\mathbf{N}$ to $\mathbf{M}$. If we have a function $\omega : \mathbf{N} \to \mathbb{R}$ then we get the composition $\omega \circ f : \mathbf{M} \to \mathbb{R}$. The pull back can be considered a *functional* map, which maps from function on $\mathbf{N}$ to function on $\mathbf{M}$

Suppose a curve $\sigma \in C_1(\mathbf{M}) \subset H_1(\mathbf{M})$ and a vector field $\omega \in C^1(\mathbf{N}) \subset H^1(\mathbf{M})$, then

$$\omega[f_*(\sigma)] = [f^*(\omega)](\sigma)$$

**Definition 14** (Degree of a Map)**.** Suppose $\mathbf{M}$ and $\mathbf{N}$ are two closed surfaces, the *degree of map* of a continuous map $f : \mathbf{M} \to \mathbf{N}$ is the algebraic number[4] of pre-images $f^{-1}(q)$ for arbitrary point $q \in \mathbf{N}$, denoted by $\deg(f)$, which is independent of the choice of the point $q$. A quick example see figure 5

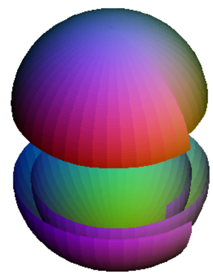

Figure 5: a continuous map $f : \mathbf{M} \to \mathbf{M}$ from a sphere to itself but in 2x speed. For every point $q \in \mathbf{M}$, there are two pre-images, so $\deg(f) = 2$

**Example 15** (Degree of a Map)**.** Suppose $\mathbf{M}$ and $\mathbf{N}$ are two closed surfaces, a continuous map:

$$f : \mathbf{M} \to \mathbf{N}$$

induces a push-forward map on second homology:

$$f_* : H_2(\mathbf{M}) \to H_2(\mathbf{N})$$

since $H_2(\mathbf{M}, \mathbb{Z}) \cong \mathbb{Z} \cong H_2(\mathbf{N}, \mathbb{Z})$, we also write its isomorphism:

$$\tilde{f}_* : \mathbb{Z} \to \mathbb{Z}$$

and it must have the form

$$\tilde{f}_*(z) = \deg(f) \cdot z$$

**Definition 16** (Euler-Poincaré Characteristic)**.** let $g$ be the genus of a closed surface $\mathbf{S}$, then *Euler characteristic*, denoted as $\chi(\mathbf{S})$, is

$$\chi(\mathbf{S}) = 2(1 - g)$$

the discrete version, if $\mathbf{S}$ triangulated in $\mathbf{S}_\Delta$, is

$$\chi(\mathbf{S}_\Delta) = |\text{Faces}| - |\text{Edges}| + |\text{Vertices}|$$

**Definition 17** (Gaussian Curvature)**.** At any point on a surface, we can find a normal vector that is at right angles to the surface; planes containing the normal vector are called normal planes. The intersection of a normal plane and the surface will form a curve called a normal section and the curvature of this curve is the normal curvature. For most points on most surfaces, different normal sections will have different curvatures; the maximum and minimum values of these are called the principal curvatures, call these $\kappa_1, \kappa_2$. The *Gaussian curvature* is the product of the two principal curvatures $K = \kappa_1 \cdot \kappa_2$, as shown in figure 6

[4] if Jacobian at that point $q$ is positive then count +1, else then count -1, then degree of map is sum of total count

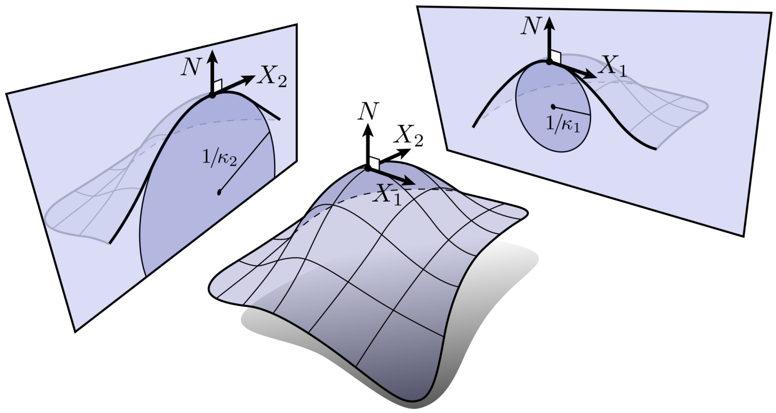


Figure 6: Gaussian curvature is the product of principal curvature $\kappa_1$ and $\kappa_2$. The curvature of a circle is reciprocal of radius: $1/r$. So if the curvature is $\kappa$, the radius is $1/\kappa$, as noted in the figure

**Theorem 18** (Chern-Gauss-Bonnet Theorem)**.** *Let $\boldsymbol{S}$ be a closed surface, $K(p)$ the Gaussian curvature at point $p$ on surface, and $dA(p)$ the element area at point $p$ on surface, then its total Gaussian curvature*

$$\int_{\boldsymbol{S}} K(p)dA(p) = 2\pi\chi(\boldsymbol{S})$$

Shiing-Shen Chern provided simple intrinsic proof of Gauss-Bonnet Theorem, which added his name to Gauss-Bonnet. We illustrate his beautiful proof by applying degree of Gauss map and homotopy between surfaces.

*proof.* consider the Gauss Map $G : \mathbf{S}^* \to \mathbb{S}^2$ from a canonical closed surface $\mathbf{S}^*$ to unit sphere $\mathbb{S}^2$. Whenever a point $p$ on surface with normal $\mathbf{n}(p)$, the Gauss Map maps it to a point $G(p)$ on unit sphere with the same normal $\mathbf{n}(p)$.

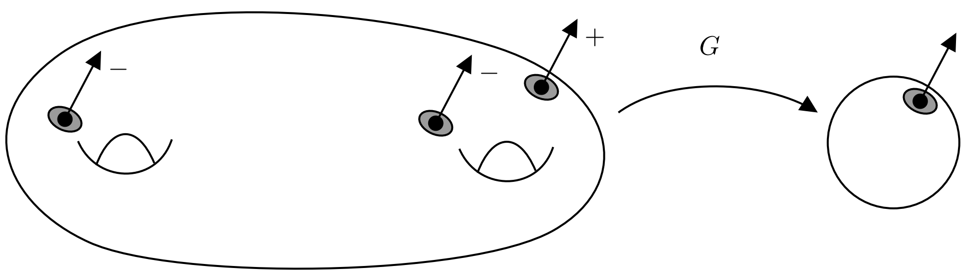


Figure 7: The canonical shape of genus $g$ closed surface $\mathbf{S}^*$ can guarantee $\deg(G) = 1 - g$ since the count of pre-image is strictly negative whenever a hole appears.

"canonical" here means it guarantees

$$\deg(G) = 1 - g$$

so the total area of the image of $\mathbf{S}^*$ on unit sphere $\mathbb{S}^2$ is

$$\mathrm{Area}(\mathbb{S}^2) \times \deg(G) = 4\pi \deg(G) = 4\pi(1 - g) = 2\pi\chi(\mathbf{S}^*)$$

Note that the total area of the image of $\mathbf{S}^*$ on unit sphere $\mathbb{S}^2$ also equals to

$$\int_{\mathbf{S}^*} \frac{\text{Area}(G(p))}{\text{Area}(p)} dA(p)$$

which equals to, since it is a Gauss Map[5]:

$$\int_{\mathbf{S}^*} \frac{\text{Area}(G(p))}{\text{Area}(p)} dA(p) = \int_{\mathbf{S}^*} K(p) dA(p)$$

thus we get, for a canonical closed surface $\mathbf{S}^*$, the identity

$$\int_{\mathbf{S}^*} K(p) dA(p) = 2\pi\chi(\mathbf{S}^*)$$

Now consider the quantity

$$\frac{\int_{\mathbf{S}^*} K(p) dA(p)}{2\pi} = \chi(\mathbf{S}^*) \in \mathbb{Z}$$

is an integer, which should not change under continuous deformation from canonical shaped $\mathbf{S}^*$ to any closed surface $\mathbf{S}$ with same genus $g$, thus we get for any closed surface $\mathbf{S}$

$$\int_{\mathbf{S}} K(p) dA(p) = 2\pi\chi(\mathbf{S})$$

□

**Example 19** (Chern-Gauss-Bonnet Theorem)**.** let $\mathbf{S}$ be a sphere with radius $R$, its genus $g = 0$, then $\chi(\mathbf{S}) = 2 \times (1 - 0) = 2$, its Gaussian curvature is constant $\frac{1}{R^2}$, according to Chern-Gauss-Bonnet formula

$$\int_{\mathbf{S}} \frac{1}{R^2} dA(p) = \frac{1}{R^2} \int_{\mathbf{S}} dA(p) = \frac{1}{R^2} \times \text{Area}(\mathbf{S}) = 2\pi\chi(\mathbf{S}) = 4\pi$$

indeed $\text{Area}(\mathbf{S}) = 4\pi R^2$.

## 3 Fixed Point Theorem

**Definition 20** (Inclusion Map)**.** an *inclusion map* $i$ from $A$ to $B$, where $A \subset B$, satisfies that for any element $x \in A$ we have $i(x) = x$, denoted as

$$i : A \hookrightarrow B$$

**Theorem 21** (Brouwer's Fixed Point Theorem)**.** *Suppose* $\Omega \subset \mathbb{R}^n$ *is a compact convex set,* $f : \Omega \to \Omega$ *is a continuous map, then there exists a point* $p \in \Omega$ *such that*

$$f(p) = p$$

[5]for Gauss Map, when shrinking a patch around point $p$, its limit is Gaussian curvature:

$$\lim_{\Omega_p \to 0} \frac{\text{Area}(G(\Omega_p))}{\text{Area}(\Omega_p)} = K(p)$$

*proof.* Assume $f : \Omega \to \Omega$ has no fixed point, namely

$$\forall p \in \Omega, \quad f(p) \neq p$$

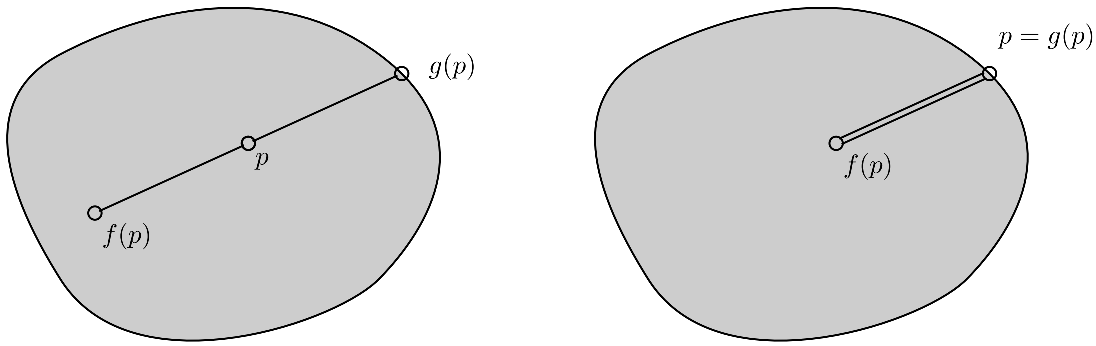


Figure 8: diagram of $g : \Omega \to \partial\Omega$ (left) and $(g \circ i) : \partial\Omega \to \partial\Omega$ (right)

We can construct $g : \Omega \to \partial\Omega$ , a ray starting from $f(p)$ through $p$ and intersect $\partial\Omega$ at $g(p)$. Because our assumption $f(p) \neq p$ and $\Omega$ is convex, $g$ is well-defined. Note that if point $p \in \partial\Omega$ then $g(p) = p$, as shown in figure 8. We construct an inclusion map $i : \partial\Omega \hookrightarrow \Omega$, which maps a point $p \in \partial\Omega$ to itself. Then we compose it with $g$,

$$\partial\Omega \overset{i}{\hookrightarrow} \Omega \xrightarrow{g} \partial\Omega$$

we get an identity map:

$$(g \circ i) : \partial\Omega \to \partial\Omega$$

which induces a push-forward map on $(n-1)^{th}$ homology:

$$(g \circ i)_* : H_{n-1}(\partial\Omega, \mathbb{Z}) \to H_{n-1}(\partial\Omega, \mathbb{Z})$$

since it is identity map,

$$(g \circ i)_* : z \mapsto z$$

$g : \Omega \to \partial\Omega$ induces a push-forward map on $(n-1)^{th}$ homology:

$$g_* : H_{n-1}(\Omega, \mathbb{Z}) \to H_{n-1}(\partial\Omega, \mathbb{Z})$$

however, since $\Omega$ is compact convex set

$$H_{n-1}(\Omega, \mathbb{Z}) = 0, \quad g_* = 0$$

so

$$(g \circ i)_* = g_* \circ i_* = 0$$

contradiction! $f : \Omega \to \Omega$ has fixed point. □

In 1910, Luitzen Egbertus Jan Brouwer proved his fixed point theorem, which ensured the existence of fixed point of a continuous self-map of convex compact space. Often, it can be stated as follow:

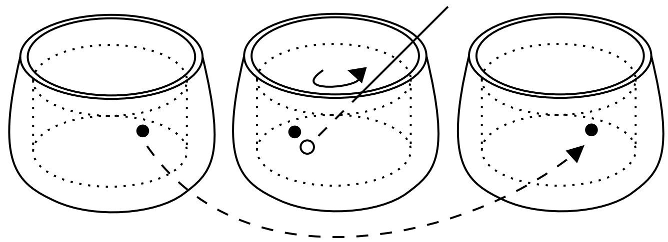

Figure 9: "swirling coffee" theorem: at least one molecule, "doesn't move" before and after coffee swirling

**Theorem 22** ("Swirling Coffee" Theorem)**.** *Use a stick (volume can be ignored) to swirl a cup of coffee without making any bubble. In the end, there is a molecule with final position the same as initial position in your coffee.*

In 1926, Solomon Lefschetz gave a formula that relates the number of fixed points of a map to the induced push-forward maps on homology.

**Definition 23** (Index of Fixed Point)**.** Suppose $\mathbf{M}$ is an $n$-dimensional topological space, $p$ is a fixed point of self-map $f : \mathbf{M} \to \mathbf{M}$. Choose a neighborhood $\mathbf{U}$ such that $p \in \mathbf{U} \subset \mathbf{M}$, consider the boundary of $\mathbf{U}$, which is a $(n-1)$-dimensional $\partial\mathbf{U}$. Similar to the concept "degree of a map" (see example 15), the induced push-forward map on $(n-1)^{th}$ homology:

$$f_* : H_{n-1}(\partial\mathbf{U}, \mathbb{Z}) \to H_{n-1}(\partial\mathbf{U}, \mathbb{Z})$$

is $f_* : \mathbb{Z} \to \mathbb{Z}$ and must have the form $f_* : z \mapsto \lambda z$, where $\lambda$ is an integer called the *algebraic index of fixed point* $p$ of map $f$, denoted as

$$\mathrm{Ind}(f, p) = \lambda$$

**Definition 24** (Trace of Self-map)**.** Let $\mathbf{A}$ be a matrix representing a self-map $f : \mathbf{M} \to \mathbf{M}$ under any basis, then the *trace* of $f$, denoted as

$$\mathrm{Tr}(f)$$

is $\mathrm{Tr}(\mathbf{A})$, the trace of $\mathbf{A}$, which is independent of choice of basis.

**Definition 25** (Lefschetz-Hopf Fixed Point Formula)**.** Given compact topological space $\mathbf{M}$. The sum of indexes of all fixed points of a self-map $f : \mathbf{M} \to \mathbf{M}$ equals to the alternating sum of trace of push-forward map on $k^{th}$ homology $f_{*k} : H_k(\mathbf{M}, \mathbb{Z}) \to H_k(\mathbf{M}, \mathbb{Z})$ induced by the self-map $f$

$$\sum_{p \in \mathrm{Fix}(f)} \mathrm{Ind}(f, p) = \sum_k (-1)^k \mathrm{Tr}(f_{*k}) =: \Lambda(f)$$

where $\Lambda(f)$ is called *Lefschetz number*

**Example 26** (Lefschetz-Hopf Fixed Point Formula)**.** consider a simple self-map $f : [0, 1] \to [0, 1]$. we have

$$\Lambda(f) = \underbrace{\mathrm{Tr}(f_{*0} : \mathbb{Z} \to \mathbb{Z})}_{1} - \underbrace{\mathrm{Tr}(f_{*1} : 0 \to 0)}_{0} = 1 = \sum_{p \in \mathrm{Fix}(f)} \mathrm{Ind}(f, p)$$

as shown in figure 10

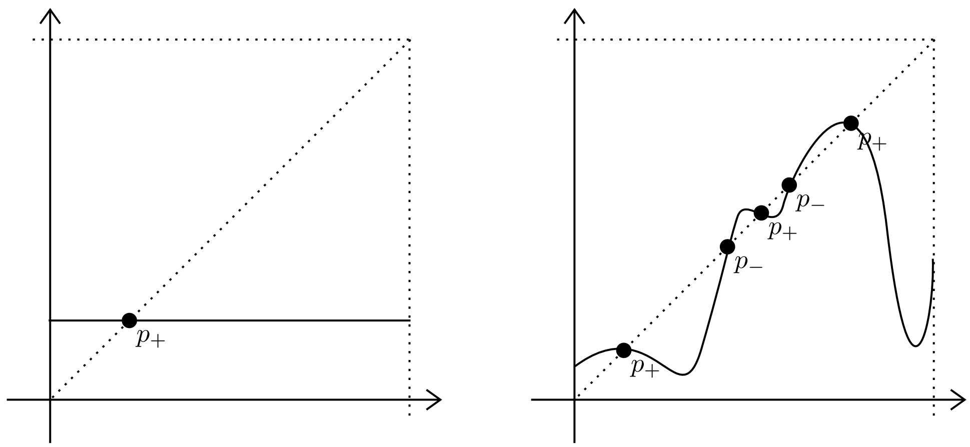


Figure 10: a self-map $f_1 : [0,1] \to$ const (left) versus its homotopic self-map $f_2$ with same Lefshcetz number $\Lambda(f_1) = \Lambda(f_2) = 1$

**Theorem 27** (Lefschetz's Fixed Point Theorem)**.** *Given a continuous self-map of a compact topological space $f : \boldsymbol{M} \to \boldsymbol{M}$, if its Lefschetz number $\Lambda(f) \neq 0$, then there is a point $p \in \boldsymbol{M}$ such that*

$$f(p) = p$$

*Proof (Advanced).* Notation update:

- $f_k$: the induced push-forward map on $k$-dimensional space
- $f_k \mid C_k$: the induced map on $k$-chain group
- $f_k \mid H_k$: the induced map on $k$-homology
- $\oplus$: direct sum between groups, e.g. $A \oplus B = \{(a+b) \mid a \in A, b \in B\}$

According to simplicial approximation theorem, there must be approximated maps up to any precision. So we triangulate $\mathbf{M}$ first, and assume its induced map $f$ can be both embedded in chain space and smooth space, as shown in the commutative diagram as figure 11:

$$\begin{tikzcd}
\frac{C_k}{Z_k} \arrow[r, "f_k"] \arrow[d, shift right, "\partial_k"'] & \frac{C_k}{Z_k} \arrow[d, "\partial_k"'] \\
B_{k-1} \arrow[r, "f_{k-1}"] \arrow[u, dashed, shift right, "\partial_k^{-1}"'] & B_{k-1}
\end{tikzcd}$$

Figure 11: commutative diagram of induced map and boundary operator

we have

$$(f_{k-1} \mid B_{k-1}) = \partial_k \circ (f_k \mid \frac{C_k}{Z_k}) \circ \partial_k^{-1}$$

and thus

$$\begin{aligned}
\mathrm{Tr}(f_{k-1} \mid B_{k-1}) &= \mathrm{Tr}([\partial_k][f_k \mid \frac{C_k}{Z_k}][\partial_k^{-1}]) \\
&= \mathrm{Tr}([f_k \mid \frac{C_k}{Z_k}][\partial_k^{-1}][\partial_k]) \\
&= \mathrm{Tr}(f_k \mid \frac{C_k}{Z_k})
\end{aligned}$$

according to property of trace.

Let $C_k$ be $k$-chain group, $Z_k$ closed chain group, $B_k$ exact chain group, $H_k$ homology group. We have

$$C_k \cong \frac{C_k}{Z_k} \oplus Z_k \qquad \text{and} \qquad Z_k \cong B_k \oplus H_k$$

thus

$$\begin{aligned}
\mathrm{Tr}(f_k \mid C_k) &= \mathrm{Tr}(f_k \mid \frac{C_k}{Z_k} \oplus Z_k) \\
&= \mathrm{Tr}(f_k \mid \frac{C_k}{Z_k}) + \mathrm{Tr}(f_k \mid Z_k) \\
&= \mathrm{Tr}(f_{k-1} \mid B_{k-1}) + \mathrm{Tr}(f_k \mid B_k) + \mathrm{Tr}(f_k \mid H_k)
\end{aligned}$$

thus

$$\begin{aligned}
\sum_k (-1)^k \mathrm{Tr}(f_k \mid C_k) &= \sum_k (-1)^k [\overbrace{(\mathrm{Tr} f_{k-1} \mid B_{k-1}) + \mathrm{Tr}(f_k \mid B_k)}^{\text{cancel out in sequence}} + \mathrm{Tr}(f_k \mid H_k)] \qquad (1) \\
&= \sum_k (-1)^k \mathrm{Tr}(f_k \mid H_k) \qquad (2) \\
&= \Lambda(f) \qquad (3)
\end{aligned}$$

according to Lefschetz-Hopf fixed point formula. Whenever $\Lambda(f) \neq 0$, there is an entry in a matrix such that $\mathrm{Tr}(f_k \mid C_k) \neq 0$, which means there is simplex $\sigma \in C_k$ such that $f_k(\sigma) \subset \sigma$, for any point in $|\sigma|$, the continuous map $f_k : |\sigma| \to |\sigma|$ must have a Brouwer's fixed point such that $f_k(p) = p$, which means

$$f(p) = p$$

□

**Example 28** (Lefschetz Number, Betti Number and Euler-Poincaré Characteristic)**.** Consider an identity map of a closed surface

$$\mathrm{id} : \mathbf{S} \to \mathbf{S}$$

the identity map is, of course, a self-map. According to equations 1, 2 and 3, we have

$$\begin{aligned}
\Lambda(\mathrm{id}) = \sum_k (-1)^k \mathrm{Tr}(\mathrm{id}_k \mid C_k) &= \underbrace{\mathrm{Tr}(\mathrm{id}_2 \mid C_2)}_{|\text{Faces}|} - \underbrace{\mathrm{Tr}(\mathrm{id}_1 \mid C_1)}_{|\text{Edges}|} + \underbrace{\mathrm{Tr}(\mathrm{id}_0 \mid C_0)}_{|\text{Vertices}|} \\
= \sum_k (-1)^k \mathrm{Tr}(\mathrm{id}_k \mid H_k) &= \underbrace{\mathrm{Tr}(\mathrm{id}_2 \mid H_2)}_{b_2} - \underbrace{\mathrm{Tr}(\mathrm{id}_1 \mid H_1)}_{b_1} + \underbrace{\mathrm{Tr}(\mathrm{id}_0 \mid H_0)}_{b_0} \\
= \chi(\mathbf{S})
\end{aligned}$$

Here we show that for an identity map, the connection between its Lefschetz number and Euler-Poincaré characteristic, and where Euler (number of triangulation elements) and Poincaré (Betti number) coincide.

Geometrically, Betti number of surface can be understood as:

- $b_0$ is the number of connected components
- $b_1$ is the number of one-dimensional or "circular" holes
- $b_2$ is the number of two-dimensional "voids" or "cavities"

e.g. for a tours, $b_0 = 1$, $b_1 = 2$ and $b_2 = 1$