Computational Conformal Geometry **授课日期: 2020年11月**

Lecture Note 4: Maps between Topological Spaces

讲师: *Xianfeng (David) Gu* 撰写: 洪楠方

助教: 洪楠方 最后更新: *January 12, 2021*

Definition 1 (Continuous Map). A *continuous map* f is a continuous function between two topological spaces $\mathbf{M}$ and $\mathbf{N}$, denoted as,

$$f : \mathbf{M} \to \mathbf{N}$$

Definition 2 (Simplicial Map). A *simplicial map* φ between simplicial complexes K and L is a function

$$\varphi : \mathrm{Vert}(K) \to \mathrm{Vert}(L)$$

from the vertex set of K, to that of L such that whenever $v_0, v_1, ..., v_q$ span a q-simplex of K, $\varphi(v_0), \varphi(v_1), ..., \varphi(v_q)$ span a p-simplex ($p \leq q$) of L. Of course, repetitions among $\varphi(v_0), \varphi(v_1), ..., \varphi(v_q)$ are allowed.

Note that the simplicial map φ can be regarded as a function from K to L: this function sends a simplex σ of K with vertices $v_0, v_1, ..., v_q$ to the simplex $\varphi(\sigma)$ of L spanned by vertices $\varphi(v_0), \varphi(v_1), ..., \varphi(v_q)$, so we also write φ as

$$\varphi : K \to L$$

Note that the simplicial map φ also induces a continuous map

$$\varphi : |K| \to |L|$$

between $|K|$ and $|L|$ (the polyhedra[1] of K and L), where a point inside $|K|$ spanned by vertex $v_0, v_1, ..., v_q$ is sent to a point inside $|L|$ continuously by

$$\varphi\left(\sum_{j=0}^{q} t_j v_j\right) = \sum_{j=0}^{q} t_j \varphi(v_j) \quad \text{whenever} \quad 0 \leq t_j \leq 1 \quad \text{for} \quad j = 0, 1, ..., q \quad \text{and} \quad \sum_{j=0}^{q} t_j = 1$$

As a closing remark, there are thus three equivalent ways of describing a simplicial map:

1. as a function between the vertex sets of two simplicial complexes, e.g. $\varphi : \mathrm{Vert}(K) \to \mathrm{Vert}(L)$
2. as a function from one simplicial complex to another, e.g. $\varphi : K \to L$
3. as a continuous map between the polyhedra of two simplicial complexes, e.g. $\varphi : |K| \to |L|$

We shall describe a simplicial map using the representation that is most appropriate in the given context.

[1] $|K|$ always denotes the polyhedra of simplicial complex K.

- for 0-simplex σ_0, $|\sigma_0|$ is itself
- for 1-simplex σ_1, $|\sigma_1|$ is itself
- for 2-simplex σ_2, $|\sigma_2|$ is the triangle it contains
- for 3-simplex σ_3, $|\sigma_3|$ is the tetrahedron it contains

1 Simplicial Approximation Theorem

One may have experience with **Minecraft** game or **Lego** toy. Any real world object can be discretized in lattice. The mathematical theorem behind is *simplicial approximation theorem*, which guarantees that a continuous manifold can be (by a slight deformation) approximated by a simplicial complex of the simplest kind given its embedded simplicial complex space.

Example 3 (Manifold Embedded in Simplicial Complex)**.** See figure 1, The manifold $\mathbf{M}$ embedded in a given simplicial complex L described by a continuous map from $|K|$, the "parameter", to $|L|$:

$$\mathbf{M} : |K| \to |L|$$

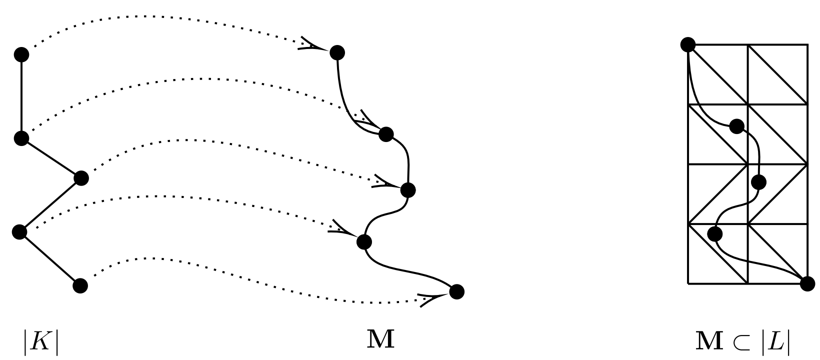

Figure 1: manifold $\mathbf{M}$ was represented by a continuous map: $\mathbf{M} : |K| \to |L|$

Definition 4 (Star of a Vertex)**.** Let K be a simplicial complex and let $p \in \mathrm{Vert}(K)$. Then the *star* of p, the "discrete version of the neighbor of a point", denoted by $\mathrm{st}(p)$, is defined by

$$\mathrm{st}(p) = \bigcup s^\circ \subset |K| \quad \text{where simplex} \quad s \in K \quad \text{such that} \quad p \in \mathrm{Vert}(s)$$

Example 5 (Star of a Vertex)**.** As shown in figure 2, $\mathrm{st}(p)$ consists of the open shaded region, all the open simplices of which p is a neighbor.

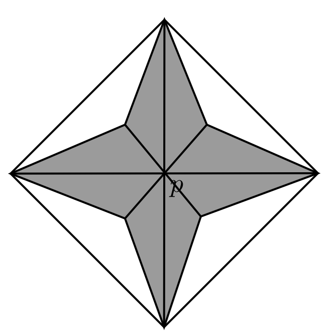

Figure 2: star of p, denoted by $\mathrm{st}(p)$

Definition 6 (Simplicial Approximation)**.** Let $\mathbf{M}$ be a manifold represented by a continuous map $\mathbf{M}$: $|K| \to |L|$. Its approximating candidate $\mathbf{M}_\Delta$, represented by a simplicial map $\mathbf{M}_\Delta : K \to L$, is *simplicial approximation* to $\mathbf{M}$ if, for every vertex p of K,

$$\mathbf{M}(\mathrm{st}(p)) \subset \mathrm{st}(\mathbf{M}_\Delta(p))$$

which means $\mathbf{M}$ carries neighboring simplices of p inside the union of the simplices near $\mathbf{M}_\Delta(p)$. $\mathbf{M}_\Delta$ and $\mathbf{M}$ are close up to a meshing unit.

Example 7 (Simplicial Approximation)**.** See figure 3, $\mathbf{M}_\Delta$ is an simplicial approximation to $\mathbf{M}$. $\mathbf{M}_\Delta$ (red) is the simplest approximation to $\mathbf{M}$, achieved by $\mathrm{Sd}^1 K$, the *first-order barycentric subdivision* of K. Barycentric subdivision and simplicial approximation theorem will be explained right away.

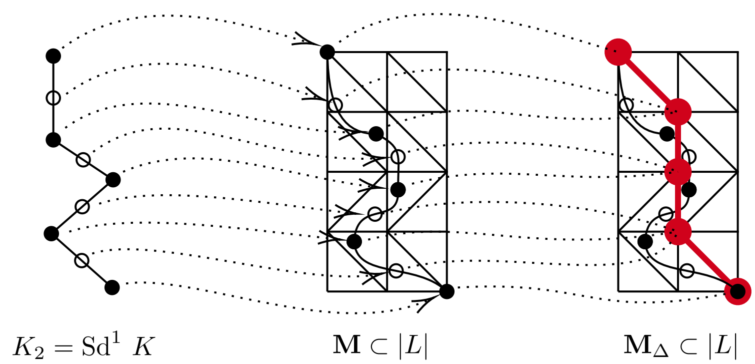

Figure 3: The simplicial approximation theorem guarantees that a simplest approximation in a given embedding mesh will be achieved by sufficient iterations of barycentric subdivision of "parameter".

Definition 8 (Barycentric Subdivision)**.** If s is a simplex, let b^s denote its barycenter. If K is a simplicial complex, define Sd K, the *barycentric subdivision* of K, to be the simplicial complex with

$$\mathrm{Vert}(\mathrm{Sd}\ K) = \{b^s : s \in K\}$$

note that here $s \in K$ are simplex of all dimensions in K. Recall that if s is a 0-simplex then trivially $b^s = s$; if s is a 1-simplex then b^s is the central point of two vertices; and so on. The q times iteration of barycentric subdivision is denoted by

$$\mathrm{Sd}^q K$$

Example 9 (Barycentric Subdivision)**.** See figure 4, if simplex $\sigma = [p_0, p_1, p_2]$, then $\mathrm{Vert}(\mathrm{Sd}\ \sigma) = \{p_0, p_1, p_2, b_0, b_1, b_2, b^\sigma\}$.

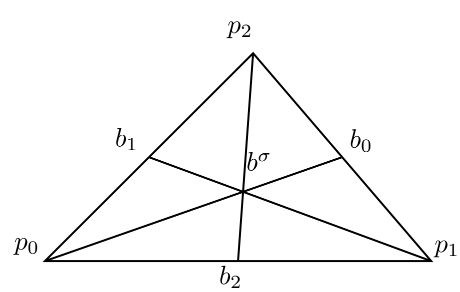

Figure 4: first order barycentric subdivision of the simplex σ

Definition 10 (Simplicial Approximation Theorem)**.** Given simplicial complexes K and L. A smooth manifold $\mathbf{M}$ represented by continuous map $\mathbf{M} : |K| \to |L|$ must have a simplicial approximation, and could be found its simplest kind after some barycentric subdivision of K by

$$\mathbf{M}_\Delta : \mathrm{Sd}^q K \to L \quad \text{where} \quad q \geq 1$$

Simplicial approximation theorem was the foundation of modern movie industry and game industry, since it provides a theoretical guarantee that the simplest discrete digital approximation of any smooth-shaped object exists.

Example 11 (Simplicial Approximation Theorem)**.** See figure 1, one cannot construct a simplicial approximation to $\mathbf{M}$ by its "parameter" K, but one can do so by K_2, the first-order barycentric subdivision of "parameter", as shown in figure 3

2 Chern-Gauss-Bonnet Theorem

Definition 12 (Induced Maps)**.** Algebraic topology constructs functor

$$\mathfrak{C}_1 \to \mathfrak{C}_2$$

between $\mathfrak{C}_1 = \{\text{Topological Spaces, Homeomorphisms}\}$ and $\mathfrak{C}_2 = \{\text{Groups, Homomorphisms}\}$. Therefore, a continuous map $f : \mathbf{M} \to \mathbf{N}$ naturally induces homomorphism. there basically two kinds of *induced map*:

- *push-forward map*, denoted by $f_\#$ if on homotopy, and denoted by f_* if on homology.

  $f_\#$ maps between fundamental groups[2] :

  $$f_\# : \pi_1(\mathbf{M}) \to \pi_1(\mathbf{N})$$

  f_* maps between p^{th}-homology groups, where p should be clear in the context:

  $$f_* : H_p(\mathbf{M}) \to H_p(\mathbf{N})$$

- *pull-back map*, denoted as f^* if it maps between p^{th}-cohomology groups[3], where p should be clear in the context:

  $$f^* : H^p(\mathbf{N}) \to H^p(\mathbf{M})$$

Example 13 (Induced Maps of Surface)**.** Suppose $\mathbf{M}$ and $\mathbf{N}$ are two closed surfaces, a continuous map:

$$f : \mathbf{M} \to \mathbf{N}$$

induces a push-forward map on first homology:

$$f_* : H_1(\mathbf{M}) \to H_1(\mathbf{N})$$

and a pull-back map on first cohomology:

$$f^* : H^1(\mathbf{N}) \to H^1(\mathbf{M})$$

[2] $f_\#$ takes "curves" to "curves":

$$f_\# : C_p(\mathbf{M}) \to C_p(\mathbf{N})$$

$f_\#$ takes "cycles" to "cycles":

$$f_\# : Z_p(\mathbf{M}) \to Z_p(\mathbf{N})$$

$f_\#$ takes "boundaries" to "boundaries":

$$f_\# : B_p(\mathbf{M}) \to B_p(\mathbf{N})$$

[3] here we explain why push-forward and pull-back are opposite direction in a natural way.

- Points are sent forward. Given $p \in \mathbf{M}$ we have $f(p) \in \mathbf{N}$
- Functions are sent back, i.e. pull back from $\mathbf{N}$ to $\mathbf{M}$. If we have a function $\omega : \mathbf{N} \to \mathbb{R}$ then we get the composition $\omega \circ f : \mathbf{M} \to \mathbb{R}$. The pull back can be considered a *functional* map, which maps from function on $\mathbf{N}$ to function on $\mathbf{M}$

Suppose a curve $\sigma \in C_1(\mathbf{M}) \subset H_1(\mathbf{M})$ and a vector field $\omega \in C^1(\mathbf{N}) \subset H^1(\mathbf{M})$, then

$$\omega[f_*(\sigma)] = [f^*(\omega)](\sigma)$$

Definition 14 (Degree of a Map)**.** Suppose $\mathbf{M}$ and $\mathbf{N}$ are two closed surfaces, the *degree of map* of a continuous map $f : \mathbf{M} \to \mathbf{N}$ is the algebraic number[4] of pre-images $f^{-1}(q)$ for arbitrary point $q \in \mathbf{N}$, denoted by $\deg(f)$, which is independent of the choice of the point q. A quick example see figure 5

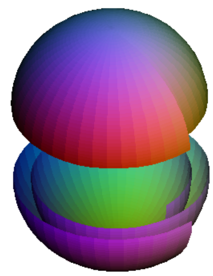

Figure 5: a continuous map $f : \mathbf{M} \to \mathbf{M}$ from a sphere to itself but in 2x speed. For every point $q \in \mathbf{M}$, there are two pre-images, so $\deg(f) = 2$

Example 15 (Degree of a Map)**.** Suppose $\mathbf{M}$ and $\mathbf{N}$ are two closed surfaces, a continuous map:

$$f : \mathbf{M} \to \mathbf{N}$$

induces a push-forward map on second homology:

$$f_* : H_2(\mathbf{M}) \to H_2(\mathbf{N})$$

since $H_2(\mathbf{M}, \mathbb{Z}) \cong \mathbb{Z} \cong H_2(\mathbf{N}, \mathbb{Z})$, we also write its isomorphism:

$$\tilde{f}_* : \mathbb{Z} \to \mathbb{Z}$$

and it must have the form

$$\tilde{f}_*(z) = \deg(f) \cdot z$$

Definition 16 (Euler-Poincaré Characteristic)**.** let g be the genus of a closed surface $\mathbf{S}$, then *Euler characteristic*, denoted as $\chi(\mathbf{S})$, is

$$\chi(\mathbf{S}) = 2(1 - g)$$

the discrete version, if $\mathbf{S}$ triangulated in $\mathbf{S}_\Delta$, is

$$\chi(\mathbf{S}_\Delta) = |\text{Faces}| - |\text{Edges}| + |\text{Vertices}|$$

Definition 17 (Gaussian Curvature)**.** At any point on a surface, we can find a normal vector that is at right angles to the surface; planes containing the normal vector are called normal planes. The intersection of a normal plane and the surface will form a curve called a normal section and the curvature of this curve is the normal curvature. For most points on most surfaces, different normal sections will have different curvatures; the maximum and minimum values of these are called the principal curvatures, call these κ_1, κ_2. The *Gaussian curvature* is the product of the two principal curvatures $K = \kappa_1 \cdot \kappa_2$, as shown in figure 6

[4] if Jacobian at that point q is positive then count +1, else then count -1, then degree of map is sum of total count

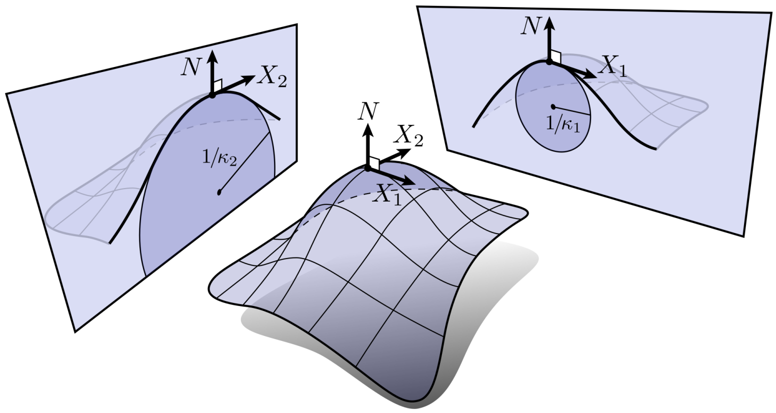

Figure 6: Gaussian curvature is the product of principal curvature κ_1 and κ_2. The curvature of a circle is reciprocal of radius: $1/r$. So if the curvature is κ, the radius is $1/\kappa$, as noted in the figure

Theorem 18 (Chern-Gauss-Bonnet Theorem)**.** *Let $\boldsymbol{S}$ be a closed surface, $K(p)$ the Gaussian curvature at point p on surface, and $dA(p)$ the element area at point p on surface, then its total Gaussian curvature*

$$\int_{\boldsymbol{S}} K(p)dA(p) = 2\pi\chi(\boldsymbol{S})$$

Shiing-Shen Chern provided simple intrinsic proof of Gauss-Bonnet Theorem, which added his name to Gauss-Bonnet. We illustrate his beautiful proof by applying degree of Gauss map and homotopy between surfaces.

proof. consider the Gauss Map $G : \mathbf{S}^* \to \mathbb{S}^2$ from a canonical closed surface $\mathbf{S}^*$ to unit sphere $\mathbb{S}^2$. Whenever a point p on surface with normal $\mathbf{n}(p)$, the Gauss Map maps it to a point $G(p)$ on unit sphere with the same normal $\mathbf{n}(p)$.

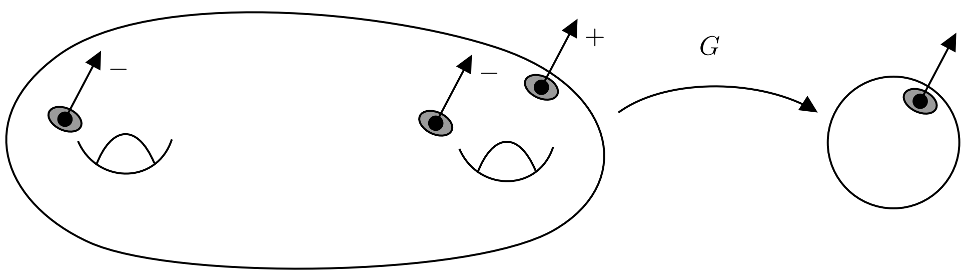

Figure 7: The canonical shape of genus g closed surface $\mathbf{S}^*$ can guarantee $\deg(G) = 1 - g$ since the count of pre-image is strictly negative whenever a hole appears.

"canonical" here means it guarantees

$$\deg(G) = 1 - g$$

so the total area of the image of $\mathbf{S}^*$ on unit sphere $\mathbb{S}^2$ is

$$\text{Area}(\mathbb{S}^2) \times \deg(G) = 4\pi \deg(G) = 4\pi(1 - g) = 2\pi\chi(\mathbf{S}^*)$$

Note that the total area of the image of $\mathbf{S}^*$ on unit sphere $\mathbb{S}^2$ also equals to

$$\int_{\mathbf{S}^*} \frac{\text{Area}(G(p))}{\text{Area}(p)} dA(p)$$

which equals to, since it is a Gauss Map[5]:

$$\int_{\mathbf{S}^*} \frac{\text{Area}(G(p))}{\text{Area}(p)} dA(p) = \int_{\mathbf{S}^*} K(p)dA(p)$$

thus we get, for a canonical closed surface $\mathbf{S}^*$, the identity

$$\int_{\mathbf{S}^*} K(p)dA(p) = 2\pi\chi(\mathbf{S}^*)$$

Now consider the quantity

$$\frac{\int_{\mathbf{S}^*} K(p)dA(p)}{2\pi} = \chi(\mathbf{S}^*) \in \mathbb{Z}$$

is an integer, which should not change under continuous deformation from canonical shaped $\mathbf{S}^*$ to any closed surface $\mathbf{S}$ with same genus g, thus we get for any closed surface $\mathbf{S}$

$$\int_{\mathbf{S}} K(p)dA(p) = 2\pi\chi(\mathbf{S})$$

□

Example 19 (Chern-Gauss-Bonnet Theorem)**.** let $\mathbf{S}$ be a sphere with radius R, its genus $g = 0$, then $\chi(\mathbf{S}) = 2 \times (1 - 0) = 2$, its Gaussian curvature is constant $\frac{1}{R^2}$, according to Chern-Gauss-Bonnet formula

$$\int_{\mathbf{S}} \frac{1}{R^2} dA(p) = \frac{1}{R^2} \int_{\mathbf{S}} dA(p) = \frac{1}{R^2} \times \text{Area}(\mathbf{S}) = 2\pi\chi(\mathbf{S}) = 4\pi$$

indeed $\text{Area}(\mathbf{S}) = 4\pi R^2$.

3 Fixed Point Theorem

Definition 20 (Inclusion Map)**.** an *inclusion map* i from A to B, where $A \subset B$, satisfies that for any element $x \in A$ we have $i(x) = x$, denoted as

$$i : A \hookrightarrow B$$

Theorem 21 (Brouwer's Fixed Point Theorem)**.** *Suppose $\Omega \subset \mathbb{R}^n$ is a compact convex set, $f : \Omega \to \Omega$ is a continuous map, then there exists a point $p \in \Omega$ such that*

$$f(p) = p$$

[5]for Gauss Map, when shrinking a patch around point p, its limit is Gaussian curvature:

$$\lim_{\Omega_p \to 0} \frac{\text{Area}(G(\Omega_p))}{\text{Area}(\Omega_p)} = K(p)$$

proof. Assume $f : \Omega \to \Omega$ has no fixed point, namely

$$\forall p \in \Omega, \quad f(p) \neq p$$

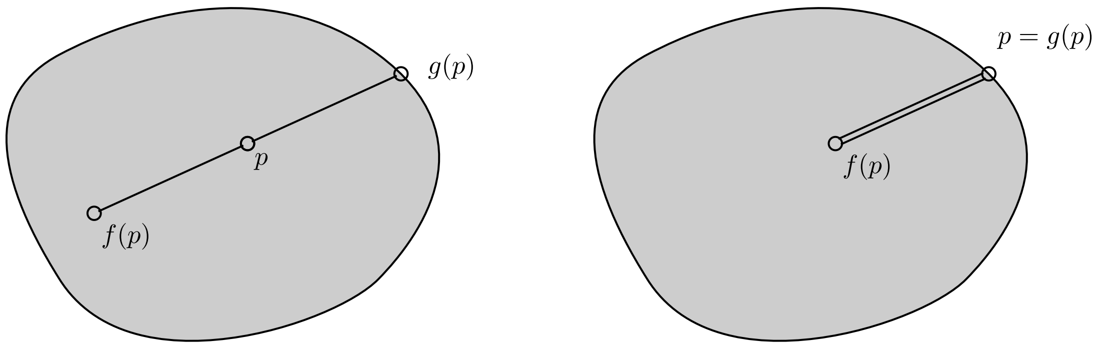

Figure 8: diagram of $g : \Omega \to \partial\Omega$ (left) and $(g \circ i) : \partial\Omega \to \partial\Omega$ (right)

We can construct $g : \Omega \to \partial\Omega$, a ray starting from $f(p)$ through p and intersect $\partial\Omega$ at $g(p)$. Because our assumption $f(p) \neq p$ and Ω is convex, g is well-defined. Note that if point $p \in \partial\Omega$ then $g(p) = p$, as shown in figure 8. We construct an inclusion map $i : \partial\Omega \hookrightarrow \Omega$, which maps a point $p \in \partial\Omega$ to itself. Then we compose it with g,

$$\partial\Omega \overset{i}{\hookrightarrow} \Omega \xrightarrow{g} \partial\Omega$$

we get an identity map:

$$(g \circ i) : \partial\Omega \to \partial\Omega$$

which induces a push-forward map on $(n-1)^{th}$ homology:

$$(g \circ i)_* : H_{n-1}(\partial\Omega, \mathbb{Z}) \to H_{n-1}(\partial\Omega, \mathbb{Z})$$

since it is identity map,

$$(g \circ i)_* : z \mapsto z$$

$g : \Omega \to \partial\Omega$ induces a push-forward map on $(n-1)^{th}$ homology:

$$g_* : H_{n-1}(\Omega, \mathbb{Z}) \to H_{n-1}(\partial\Omega, \mathbb{Z})$$

however, since Ω is compact convex set

$$H_{n-1}(\Omega, \mathbb{Z}) = 0, \quad g_* = 0$$

so

$$(g \circ i)_* = g_* \circ i_* = 0$$

contradiction! $f : \Omega \to \Omega$ has fixed point. $\square$

In 1910, Luitzen Egbertus Jan Brouwer proved his fixed point theorem, which ensured the existence of fixed point of a continuous self-map of convex compact space. Often, it can be stated as follow:

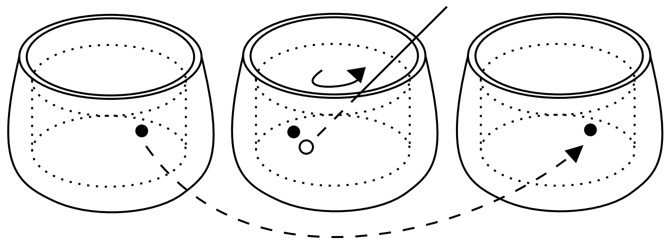

Figure 9: "swirling coffee" theorem: at least one molecule, "doesn't move" before and after coffee swirling

Example 22 ("Swirling Coffee" Theorem)**.** Use a stick (volume can be ignored) to swirl a cup of coffee without making any bubble. In the end, there is a molecule with final position the same as initial position in your coffee.

In 1926, Solomon Lefschetz gave a formula that relates the number of fixed points of a map to the induced push-forward maps on homology.

Definition 23 (Index of Fixed Point)**.** Suppose $\mathbf{M}$ is an n-dimensional topological space, p is a fixed point of self-map $f : \mathbf{M} \to \mathbf{M}$. Choose a neighborhood $\mathbf{U}$ such that $p \in \mathbf{U} \subset \mathbf{M}$, consider the boundary of $\mathbf{U}$, which is a $(n-1)$-dimensional $\partial\mathbf{U}$. Similar to the concept "degree of a map" (see example 15), the induced push-forward map on $(n-1)^{th}$ homology:

$$f_* : H_{n-1}(\partial\mathbf{U}, \mathbb{Z}) \to H_{n-1}(\partial\mathbf{U}, \mathbb{Z})$$

is $f_* : \mathbb{Z} \to \mathbb{Z}$ and must have the form $f_* : z \mapsto \lambda z$, where λ is an integer called the *algebraic index of fixed point* p of map f, denoted as

$$\mathrm{Ind}(f, p) = \lambda$$

Definition 24 (Trace of Self-map)**.** Let $\mathbf{A}$ be a matrix representing a self-map $f : \mathbf{M} \to \mathbf{M}$ under any basis, then the *trace* of f, denoted as

$$\mathrm{Tr}(f)$$

is $\mathrm{Tr}(\mathbf{A})$, the trace of $\mathbf{A}$, which is independent of choice of basis.

Definition 25 (Lefschetz-Hopf Fixed Point Formula)**.** Given compact topological space $\mathbf{M}$. The sum of indices of all fixed points of a self-map $f : \mathbf{M} \to \mathbf{M}$ equals to the alternating sum of trace of push-forward map on k^{th} homology $f_{*k} : H_k(\mathbf{M}, \mathbb{Z}) \to H_k(\mathbf{M}, \mathbb{Z})$ induced by the self-map f

$$\sum_{p \in \mathrm{Fix}(f)} \mathrm{Ind}(f, p) = \sum_k (-1)^k \mathrm{Tr}(f_{*k}) =: \Lambda(f)$$

where $\Lambda(f)$ is called *Lefschetz number*, and $\mathrm{Fix}(f)$ denotes the set of all fixed points of f

Example 26 (Lefschetz-Hopf Fixed Point Formula)**.** consider a simple self-map $f : [0, 1] \to [0, 1]$. we have

$$\Lambda(f) = \underbrace{\mathrm{Tr}(f_{*0} : \mathbb{Z} \to \mathbb{Z})}_{1} - \underbrace{\mathrm{Tr}(f_{*1} : 0 \to 0)}_{0} = 1 = \sum_{p \in \mathrm{Fix}(f)} \mathrm{Ind}(f, p)$$

as shown in figure 10

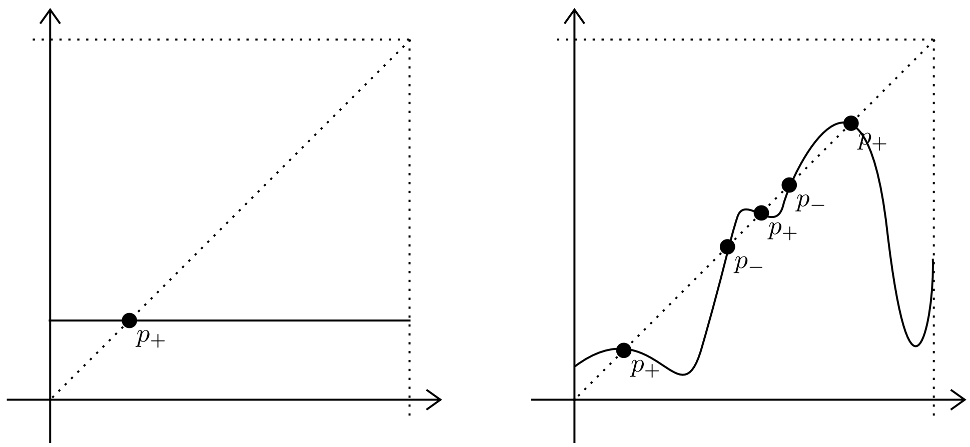

Figure 10: a self-map $f_1 : [0,1] \to$ const (left) versus its homotopic self-map f_2 with same Lefshcetz number $\Lambda(f_1) = \Lambda(f_2) = 1$

Theorem 27 (Lefschetz's Fixed Point Theorem)**.** *Given a continuous self-map of a compact topological space $f : \boldsymbol{M} \to \boldsymbol{M}$, if its Lefschetz number $\Lambda(f) \neq 0$, then there is a point $p \in \boldsymbol{M}$ such that*

$$f(p) = p$$

Proof (Advanced). Notation update:

- f_k: the induced push-forward map on k-dimensional space
- $f_k \mid C_k$: the induced map on k-chain group
- $f_k \mid H_k$: the induced map on k-homology
- $\oplus$: direct sum between groups, e.g. $A \oplus B = \{(a+b) \mid a \in A, b \in B\}$

According to simplicial approximation theorem, there must be approximated maps up to any precision. So we triangulate **M** first, and assume its induced map f can be both embedded in chain space and smooth space, as shown in the commutative diagram as figure 11:

$$\begin{tikzcd}
\frac{C_k}{Z_k} \arrow[r, "f_k"] \arrow[d, shift right, "\partial_k"'] & \frac{C_k}{Z_k} \arrow[d, "\partial_k"] \\
B_{k-1} \arrow[r, "f_{k-1}"] \arrow[u, dashed, shift right, "\partial_k^{-1}"'] & B_{k-1}
\end{tikzcd}$$

Figure 11: commutative diagram of induced map and boundary operator

we have

$$(f_{k-1} \mid B_{k-1}) = \partial_k \circ (f_k \mid \frac{C_k}{Z_k}) \circ \partial_k^{-1}$$

and thus

$$\begin{aligned}
\mathrm{Tr}(f_{k-1} \mid B_{k-1}) &= \mathrm{Tr}([\partial_k][f_k \mid \frac{C_k}{Z_k}][\partial_k^{-1}]) \\
&= \mathrm{Tr}([f_k \mid \frac{C_k}{Z_k}][\partial_k^{-1}][\partial_k]) \\
&= \mathrm{Tr}(f_k \mid \frac{C_k}{Z_k})
\end{aligned}$$

according to property of trace.

Let C_k be k-chain group, Z_k closed chain group, B_k exact chain group, H_k homology group. We have

$$C_k \cong \frac{C_k}{Z_k} \oplus Z_k \qquad \text{and} \qquad Z_k \cong B_k \oplus H_k$$

thus

$$\begin{aligned}
\mathrm{Tr}(f_k \mid C_k) &= \mathrm{Tr}(f_k \mid \frac{C_k}{Z_k} \oplus Z_k) \\
&= \mathrm{Tr}(f_k \mid \frac{C_k}{Z_k}) + \mathrm{Tr}(f_k \mid Z_k) \\
&= \mathrm{Tr}(f_{k-1} \mid B_{k-1}) + \mathrm{Tr}(f_k \mid B_k) + \mathrm{Tr}(f_k \mid H_k)
\end{aligned}$$

thus

$$\begin{aligned}
\sum_k (-1)^k \mathrm{Tr}(f_k \mid C_k) &= \sum_k (-1)^k [\overbrace{\mathrm{Tr}(f_{k-1} \mid B_{k-1}) + \mathrm{Tr}(f_k \mid B_k)}^{\text{cancel out in sequence}} + \mathrm{Tr}(f_k \mid H_k)] &(1)\\
&= \sum_k (-1)^k \mathrm{Tr}(f_k \mid H_k) &(2)\\
&= \Lambda(f) &(3)
\end{aligned}$$

according to Lefschetz-Hopf fixed point formula. Whenever $\Lambda(f) \neq 0$, there is an entry in a matrix such that $\mathrm{Tr}(f_k \mid C_k) \neq 0$, which means there is simplex $\sigma \in C_k$ such that $f_k(\sigma) \subset \sigma$, for any point in $|\sigma|$, the continuous map $f_k : |\sigma| \to |\sigma|$ must have a Brouwer's fixed point such that $f_k(p) = p$, which means

$$f(p) = p$$

□

Example 28 (Lefschetz Number, Betti Number and Euler-Poincaré Characteristic)**.** Consider an identity map of a closed surface

$$\mathrm{id} : \mathbf{S} \to \mathbf{S}$$

the identity map is, of course, a self-map. According to equations 1, 2 and 3, we have

$$\begin{aligned}
\Lambda(\mathrm{id}) &= \sum_k (-1)^k \mathrm{Tr}(\mathrm{id}_k \mid C_k) = \underbrace{\mathrm{Tr}(\mathrm{id}_2 \mid C_2)}_{|\text{Faces}|} - \underbrace{\mathrm{Tr}(\mathrm{id}_1 \mid C_1)}_{|\text{Edges}|} + \underbrace{\mathrm{Tr}(\mathrm{id}_0 \mid C_0)}_{|\text{Vertices}|} \\
&= \sum_k (-1)^k \mathrm{Tr}(\mathrm{id}_k \mid H_k) = \underbrace{\mathrm{Tr}(\mathrm{id}_2 \mid H_2)}_{b_2} - \underbrace{\mathrm{Tr}(\mathrm{id}_1 \mid H_1)}_{b_1} + \underbrace{\mathrm{Tr}(\mathrm{id}_0 \mid H_0)}_{b_0} \\
&= \chi(\mathbf{S})
\end{aligned}$$

Here we show that for an identity map, the connection between its Lefschetz number and Euler-Poincaré characteristic, and where Euler (number of triangulation elements) and Poincaré (Betti number) coincide.

Geometrically, Betti number of surface can be understood as:

- b_0 is the number of connected components
- b_1 is the number of one-dimensional or "circular" holes
- b_2 is the number of two-dimensional "voids" or "cavities"

e.g. for a tours, $b_0 = 1$, $b_1 = 2$ and $b_2 = 1$

4 Poincaré-Hopf Theorem

Lefschetz-Hopf fixed point formula directly leads to Poincaré-Hopf index theorem, which relates the number of zeros of a vector field to the topological invariant of space where the vector field is embedded.

Definition 29 (Isolated Zero Point)**.** Given a smooth vector field on a surface $\mathbf{S}$

$$v_{\mathbf{S}} : \mathbf{S} \to T\mathbf{S}$$

assigning each point $p \in \mathbf{S}$ a tangent vector $v_{\mathbf{S}}(p) \in T\mathbf{S}$, then $p \in \mathbf{S}$ is called a *zero point* if

$$v_{\mathbf{S}}(p) = \mathbf{0}$$

If there is a neighborhood $U(p)$ such that p is the unique zero in $U(p)$, then p is an *isolated zero point.*

We use

$$\mathrm{Zero}(v_{\mathbf{S}})$$

to denote the set of all zero points of a vector field $v_{\mathbf{S}}$

Definition 30 (Index of Zero Point)**.** Given a zero $p \in \mathrm{Zero}(v_{\mathbf{S}})$ of a vector field $v_{\mathbf{S}}$, choose a small disk $B(p, \varepsilon)$ and define a map φ from ∂B to unit circle $\mathbb{S}^1$:

$$\varphi : \partial B \to \mathbb{S}^1$$

where a point $q \in \partial B$ maps to $\varphi(q) \in \mathbb{S}^1$ with the same vector direction $\dfrac{v_{\mathbf{S}}(q)}{|v_{\mathbf{S}}(q)|}$, which induces a homomorphism:

$$\varphi_{\#} : \pi_1(\partial B) \to \pi_1(\mathbb{S}^1)$$

and must have the form

$$\varphi_{\#}(z) = kz$$

where k is called the *index of zero point*, denoted as

$$k =: \mathrm{Ind}(v_{\mathbf{S}}, p)$$

comment: very similar to "degree of map" and "index of fixed point"

Example 31 (Index of Zero Point)**.** see figure 12, if the mapping is by same orientation then we count as positive, else then count as negative. Index of zero point of sink field, source field and saddle field is $+1, +1$ and -1, respectively

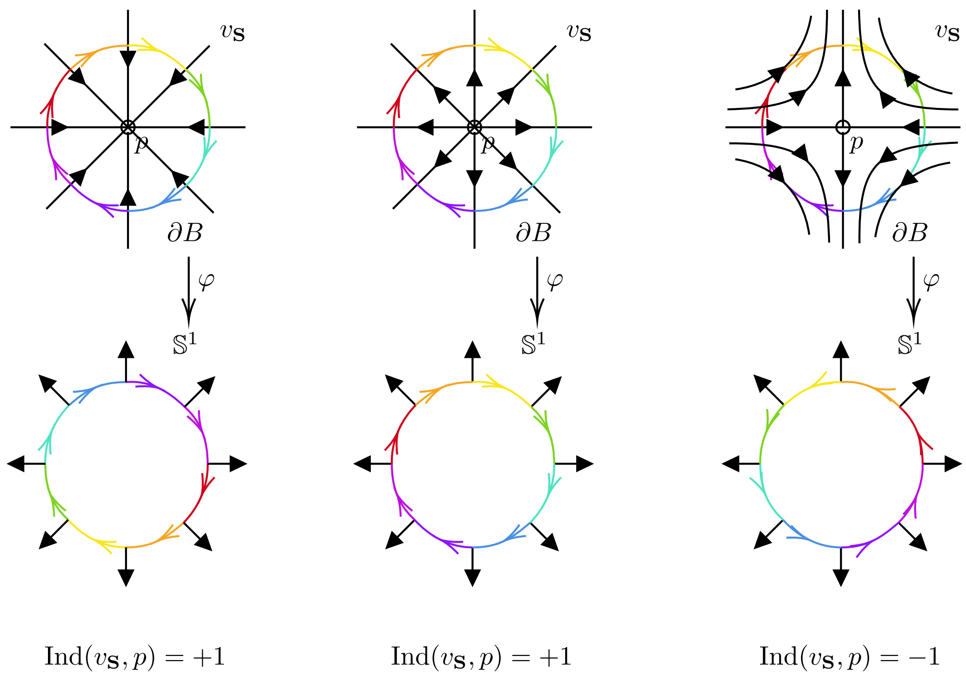

Figure 12: index of sink (left), source (middle) and saddle (right) is $+1, +1, -1$, respectively

Definition 32 (Poincaré-Hopf Index Theorem)**.** Assume $\mathbf{S}$ is a compact, oriented smooth surface, $v_{\mathbf{S}}$ is a smooth tangent vector field with isolated zeros. If $\mathbf{S}$ has boundaries[6], then $v_{\mathbf{S}}$ point along the exterior normal direction[7], then we have

$$\sum_{p \in \mathrm{Zero}(v_{\mathbf{S}})} \mathrm{Ind}(v_{\mathbf{S}}, p) = \chi(\mathbf{S})$$

where $\chi(\mathbf{S})$ is the Euler-Poincaré characteristic.

Proof. we construct a continuous self-map $f_\varepsilon : \mathbf{S} \to \mathbf{S}$ in following way:

$$f_\varepsilon(p) = p + \varepsilon v_{\mathbf{S}}(p)$$

we know the identity map (where $\varepsilon = 0$) is homotopic to f_ε:

$$f_0 \sim f_\varepsilon$$

thus we have (see example 28)

$$\Lambda(f_\varepsilon) = \Lambda(\mathrm{id}) = \chi(\mathbf{S})$$

notice that

$$f_\varepsilon(p) = p \qquad \text{if and only if} \qquad v_{\mathbf{S}} = \mathbf{0}$$

thus $\Lambda(f_\varepsilon)$, the summation of indices of fixed points of f_ε, equals to the summation of indices of zero points of $v_{\mathbf{S}}$, equals to Euler-Poincaré characteristic

[6] for a surface $\mathbf{S}$ with genus g and number of boundaries b, the Euler-Poincaré characteristic is

$$\chi(\mathbf{S}) = 2(1 - g) - b$$

[7] for a point $p \in \partial \mathbf{S}$, $v_{\mathbf{S}}(p) \cdot \mathbf{n}(p) > 0$

$$\sum_{p \in \mathrm{Fix}(f_\varepsilon)} \mathrm{Ind}(f_\varepsilon, p) = \sum_{p \in \mathrm{Zero}(v_{\mathbf{S}})} \mathrm{Ind}(v_{\mathbf{S}}, p) = \chi(\mathbf{S})$$

□

Example 33 (Poincaré-Hopf Index Theorem)**.** For a torus with $\chi(\mathbf{T}^2) = 0$, one can construct a tangent vector field without zero point. For a sphere with $\chi(\mathbb{S}^2) = 2$ and a bi-torus $\chi(\mathbf{T}^2 \oplus \mathbf{T}^2) = -2$, however, one cannot construct a tangent vector field without zero point. As shown in figure 13

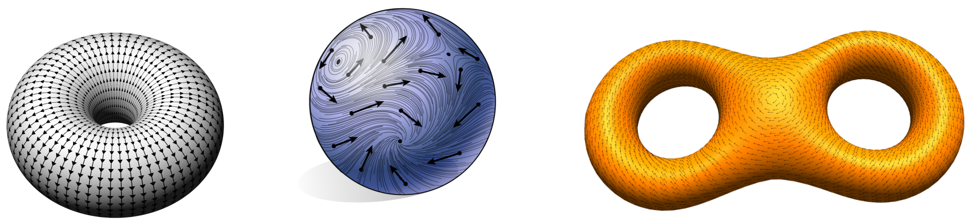

Figure 13: tangent vector field of torus, sphere and bi-torus

Example 34 ("Parietal Whorl" theorem)**.** We now show that parietal whorl in your head is guaranteed by Poincaré-Hopf index theorem.

- The hair region of human scalp can be considered as a smooth, compact, oriented surface with a boundary, so its Euler-Poincaré characteristic is 1
- The hair, with its direction and length, can be considered as a tangent vector field
- The parietal whorl, can be considered the zero point of the vector field

Given the fact that, the hairs on boundary always point along the exterior normal direction, as shown in figure 14 for currently-unknown developmental biological reason, Poincaré-Hopf index theorem guarantees that everyone has at least one parietal whorl

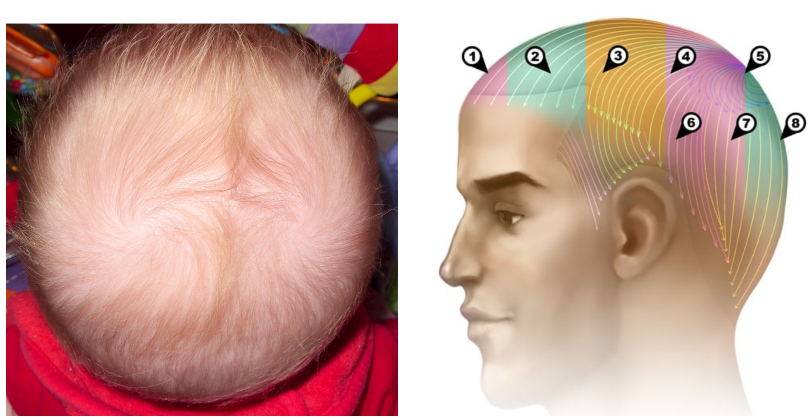

Figure 14: A baby with possibly sink, saddle and source on his head at the same time (left). Hairs on boundary of hair region on your scalp always point outward (right)