

Lecture Note 3: Topological Obstruction

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Poincaré-Hopf index theorem tells us that one cannot construct a smooth vector field over a sphere without zero point. Today we see this conclusion from another view.

1 Tangent Vector in Coordinate Chart

Historically, geometric techniques were developed mostly for Euclidean space. To study curved space, e.g. a manifold, we can construct local maps of open covers between manifold and Euclidean space.

Definition 1 (Smooth Manifold with Charts and Atlas). A manifold is a topological space \mathbf{M} covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ maps U_α to the Euclidean space \mathbb{R}^n . $(U_\alpha, \varphi_\alpha)$ is called a *coordinate chart* of \mathbf{M} . The set of all charts $\{(U_\alpha, \varphi_\alpha)\}$ form the *atlas* of \mathbf{M} . Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

is a *transition map*. If all transition maps are smooth, namely

$$\varphi_{\alpha\beta} \in C^\infty(\mathbb{R}^n)$$

then the manifold is a differentiable (or differential) manifold or a *smooth manifold*, as shown in figure 1

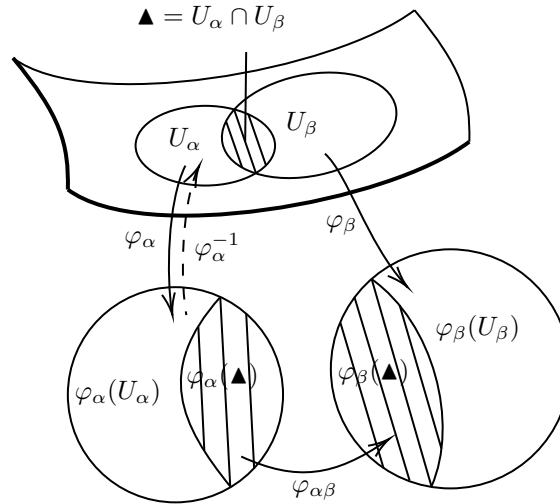


Figure 1: definition of smooth manifold was achieved by mapping it to Euclidean space patch by patch smoothly.

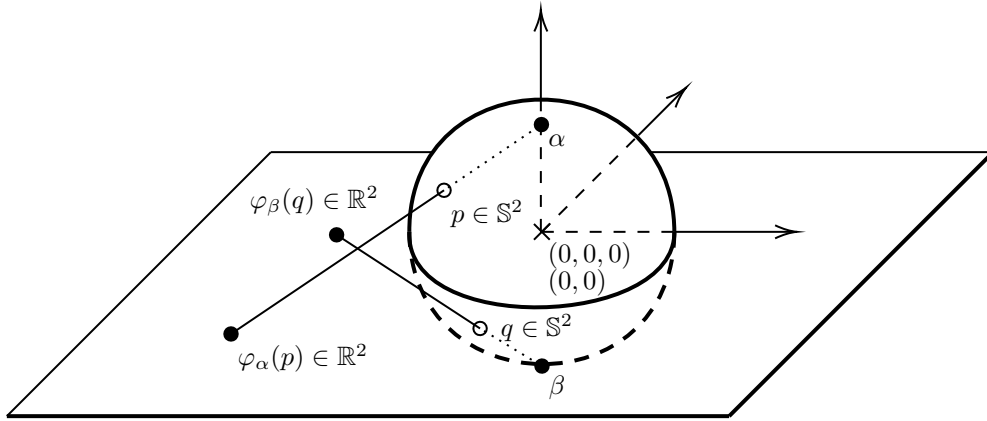


Figure 2: A unit sphere $\mathbb{S}^2 \in \mathbb{R}^3$ cannot be covered by only one chart, but can be covered by two charts, so-called stereo-graphic projection. The center of the sphere is $(0,0,0)$. We take its xy -plane as the image plane, containing the equator of sphere. The north pole α projects a point $p \in \mathbb{S}^2$ to the plane $\varphi_\alpha(p) \in \mathbb{R}^2$, and the south pole β projects a point $q \in \mathbb{S}^2$ to the plane $\varphi_\beta(q) \in \mathbb{R}^2$.

Example 2 (Stereo-graphic Projection). A manifold can hardly be covered by only one coordinate chart, thus it usually needs to be covered by multiple charts. A basic example is so-called *stereo-graphic projection*.

As shown in figure 2, north pole $\alpha = (0,0,1)$, south pole $\beta = (0,0,-1)$, let $p = (x_1, x_2, x_3)$, $\varphi_\alpha(p) = (x, y)$, $\varphi_\alpha(q) = (u, v)$

$$\begin{aligned}\varphi_\alpha : (x_1, x_2, x_3) &\mapsto \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3} \right) \\ \varphi_\alpha^{-1} : (x, y) &\mapsto \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2} \right) \\ \varphi_\beta : (x_1, x_2, x_3) &\mapsto \left(\frac{x_1}{1+x_3}, \frac{-x_2}{1+x_3} \right) \\ \varphi_\beta^{-1} : (u, v) &\mapsto \left(\frac{2u}{1+u^2+v^2}, \frac{-2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2} \right)\end{aligned}$$

Note that indeed $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} \in C^\infty$, the unit sphere is a smooth manifold. Notice that φ_α cannot cover α and φ_β cannot cover β . You need both to cover the whole sphere.

As shown in figure 3, let $p \in \mathbb{S}^2$. Any vector $d\mathbf{r} \in T_p\mathbb{S}^2$ through φ_α can be represented by $d\mathbf{r} = \partial_x dx + \partial_y dy$, where

$$\begin{aligned}\partial_x &= \frac{\partial \mathbf{r}}{\partial x} = \frac{\partial \varphi_\alpha^{-1}(x, y)}{\partial x} = \frac{2}{(1+x^2+y^2)^2} \begin{bmatrix} 1-x^2+y^2 \\ -2xy \\ 2x \end{bmatrix} \\ \partial_y &= \frac{\partial \mathbf{r}}{\partial y} = \frac{\partial \varphi_\alpha^{-1}(x, y)}{\partial y} = \frac{2}{(1+x^2+y^2)^2} \begin{bmatrix} -2xy \\ 1+x^2-y^2 \\ 2y \end{bmatrix}\end{aligned}$$

and the inner product

$$\begin{aligned}\langle \partial_x, \partial_x \rangle &= \langle \partial_y, \partial_y \rangle = \frac{4}{(1+x^2+y^2)^2} \\ \langle \partial_x, \partial_y \rangle &= 0\end{aligned}$$

so interestingly the bases of $T_p\mathbb{S}^2$ derived from partial derivative are orthogonal with equal length.

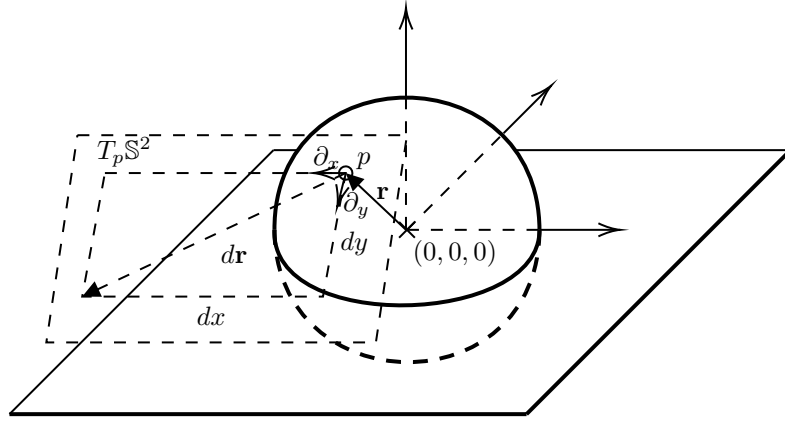


Figure 3: Let $p \in \mathbb{S}^2$. Let vector $\mathbf{r} = \varphi_\alpha^{-1}(x, y)$ parametrized by $(x, y) \in \mathbb{R}^2$, then any vector $d\mathbf{r}$ on $T_p\mathbb{S}^2$, the tangent plane at point p on \mathbb{S}^2 , can be represented by $d\mathbf{r} = \partial_x dx + \partial_y dy$.

Definition 3 (Riemannian Metric and Riemannian Manifold). Let \mathbf{M} be a smooth manifold, a *Riemannian metric* g on \mathbf{M} is a smooth family of inner products on the tangent spaces of \mathbf{M} . Namely, g associates to each point $p \in \mathbf{M}$ a positive definite symmetric bi-linear form on $T_p\mathbf{M}$:

$$g_p : T_p\mathbf{M} \times T_p\mathbf{M} \rightarrow \mathbb{R}$$

along with which comes a norm

$$|\cdot|_{g_p} : T_p\mathbf{M} \rightarrow \mathbb{R} \quad \text{defined by} \quad |\mathbf{v}|_{g_p} = \sqrt{g_p(\mathbf{v}, \mathbf{v})}$$

The smooth manifold \mathbf{M} endowed with this metric g is a *Riemannian manifold*, denoted by (\mathbf{M}, g) . Every smooth manifold has a Riemannian metric.

Continue example 2. All the coordinates are in \mathbb{R}^3 . For any tangent vector $d\mathbf{r} = \partial_x dx + \partial_y dy \in T_p\mathbb{S}^2$ at point p , we need (x, y) to parameterize the position of point $p \in \mathbb{S}^2$ and (dx, dy) to parameterize the direction and length of tangent vector $d\mathbf{r}$. We can use (x, y, dx, dy) to parameterize tangent vector.

We now introduce g_p^{can} , the *canonical Euclidean metric*, as a case of Riemannian metric¹ to measure the “distance” of two tangent vector at point p

$$g_p^{\text{can}} : T_p\mathbb{S}^2 \times T_p\mathbb{S}^2 \rightarrow \mathbb{R} \quad \text{is defined by} \quad (\partial_x dx_1 + \partial_y dy_1, \partial_x dx_2 + \partial_y dy_2) \mapsto dx_1 dx_2 + dy_1 dy_2$$

if we are only interested in unit tangent vector (“unit” in the sense of g_p^{can}) and denote $UT_p\mathbb{S}^2$ as unit tangent space, then we only need

$$|d\mathbf{r}|_{g_p^{\text{can}}} = \sqrt{g_p(d\mathbf{r}, d\mathbf{r})} = \sqrt{(dx)^2 + (dy)^2} = 1$$

then we can re-parameterize (dx, dy) as $(\cos \tau, \sin \tau)$, reducing four parameters to three:

$$(x, y, \tau)$$

if we are further only interested in unit tangent vector on equator of unit sphere, we can re-parameterize (x, y) as $(\cos \theta, \sin \theta)$, reducing three parameters to two:

$$(\theta, \tau)$$

¹Let x^1, \dots, x^n denote the standard coordinates on \mathbb{R}^n . Then define $g_p^{\text{can}} : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\left(\sum_i a_i \frac{\partial}{\partial x^i}, \sum_j b_j \frac{\partial}{\partial x^j} \right) \mapsto \sum_i a_i b_i$$

2 Shape of Smooth Non-zero Tangent Vector Field

We now consider a Riemann surface (\mathbf{M}, g) with non-zero unit tangent vector everywhere (“unit” is in the sense of g). All the possible unit tangent vector fields, which of course is non-zero, form a unit tangent bundle, denoted by $UT\mathbf{M}$:

$$UT\mathbf{M} := \bigcup_{p \in \mathbf{M}} \{p\} \times UT_p\mathbf{M} = \bigcup_{p \in \mathbf{M}} \{(p, d\mathbf{r}) \mid d\mathbf{r} \in T_p\mathbf{M}, |d\mathbf{r}|_g = 1\} = \{(p, d\mathbf{r}) \mid p \in \mathbf{M}, d\mathbf{r} \in T_p\mathbf{M}, |d\mathbf{r}|_g = 1\}$$

The unit tangent bundle of a surface is a 3-dimensional manifold. Then we consider a Riemann surface of simplest kind: a unit sphere with canonical Euclidean metric $(\mathbb{S}^2, g_p^{\text{can}})$:

$$UT\mathbb{S}^2 = \{(p, d\mathbf{r}) \mid p \in \mathbb{S}^2, d\mathbf{r} \in T_p\mathbb{S}^2, |d\mathbf{r}|_{g_p^{\text{can}}} = 1\}$$

Poincaré-Hopf theorem tells us that it is **impossible** to construct a **smooth** $v_{\mathbb{S}^2} \in UT\mathbb{S}^2$. We demonstrate such impossibility by topological obstruction.

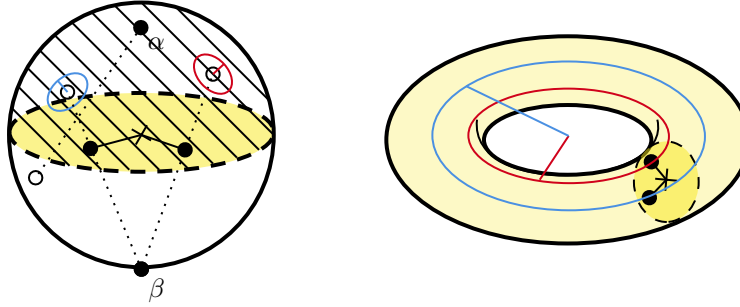


Figure 4: The topological space of unit tangent bundle of unit hemisphere is $\mathbb{S}^1 \times \mathbb{D}^2$, a solid torus. The sectioning disk in solid torus (deep yellow) corresponds to the image plane of φ_β , for example

We know that coordinate chart φ_α cannot cover point α and coordinate chart φ_β cannot cover point β . So we use φ_α for lower hemisphere and φ_β for upper hemisphere (figure 4), and glue them together through **equator**. We will show that to construct a smooth $v_{\mathbb{D}^2} \in UT\mathbb{D}^2$, the unit tangent vector field over unit hemisphere², is **okay**. But when we glue them together with constraint of smooth transition from φ_α to φ_β on equator, two hemispheres **cannot allow smooth vector fields over them at the same time**.

Firstly, we see that the shape of $UT\mathbb{D}^2$ is a **solid** torus

$$UT\mathbb{D}^2 \sim \mathbb{S}^1 \times \mathbb{D}^2$$

as shown in figure 4. Because $UT_p\mathbb{D}^2$, the set of all possible directions of unit tangent vector at a point on unit hemisphere, corresponds to a **fiber** that goes through a point on sectioning disk in solid torus (e.g. the red and blue curves in figure 4)

$$UT_p\mathbb{D}^2 \sim \mathbb{S}^1$$

If we cut the torus (to remove its genus), then the sectioning surface represents a particular $v_{\mathbb{D}^2}$. The sectioning surface, through which every fiber goes only once, is called **global section**. The smoothness of $v_{\mathbb{D}^2}$ is guaranteed by the smoothness of that global section. All the possible smooth $v_{\mathbb{D}^2}$ corresponds to all the possible global sections that can be smoothly deformed from the sectioning disk in solid torus

$$v_{\mathbb{D}^2} \sim \mathbb{D}^2$$

²we use \mathbb{D}^2 to denote a unit disk, which is homotopic to unit hemisphere, so we also use \mathbb{D}^2 to denote unit hemisphere

Secondly, notice that $UT(\partial\mathbb{D}^2)$, the unit tangent bundle of unit hemisphere on equator (the boundary of hemisphere) corresponds to a torus, the surface of that solid torus (figure 4)

$$UT(\partial\mathbb{D}^2) = UTS^1 = \partial(\mathbb{S}^1 \times \mathbb{D}^2) = \mathbb{S}^1 \times (\partial\mathbb{D}^2) = \mathbb{S}^1 \times \mathbb{S}^1 = \mathbf{T}^2$$

thus gluing two smooth $v_{\mathbb{D}^2}$ on equator smoothly, let's say $v_{\mathbb{D}_L^2}$ ("L" for lower hemisphere) and $v_{\mathbb{D}_U^2}$ ("U" for upper hemisphere), is very much of gluing two solid tori with homeomorphism of two tori such that two global sections, let's say \mathbb{D}_L^2 and \mathbb{D}_U^2 , forming a larger global section of $UT(\mathbb{S}^2)$

$$v_{\mathbb{D}_L^2} \bigcup_{UT(\partial\mathbb{D}_L^2) \sim UT(\partial\mathbb{D}_U^2)} v_{\mathbb{D}_U^2} \sim \mathbb{D}_L^2 \bigcup_{\mathbf{T}_L^2 \sim \mathbf{T}_U^2} \mathbb{D}_U^2$$

The topological obstruction means that one cannot find a global section of $UT(\mathbb{S}^2)$. Or in other words, with constraint of $\mathbf{T}_L^2 \sim \mathbf{T}_U^2$, by setting a global section \mathbb{D}_L^2 of lower solid torus freely, one cannot find a global section of upper solid torus, as we show later.

3 Topological Obstruction

The homeomorphism of two tori was guaranteed by smooth transition of charts on equator from φ_α to φ_β , namely, from (x, y, dx, dy) to (u, v, du, dv) . We check how different φ_β from φ_α , continue example 2

$$\begin{aligned} \partial_u &= \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial \varphi_\beta^{-1}(u, v)}{\partial u} = \frac{2}{(1 + u^2 + v^2)^2} \begin{bmatrix} 1 - u^2 + v^2 \\ 2uv \\ -2u \end{bmatrix} \\ \partial_v &= \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial \varphi_\beta^{-1}(u, v)}{\partial v} = \frac{2}{(1 + u^2 + v^2)^2} \begin{bmatrix} -2uv \\ -1 - u^2 + v^2 \\ -2v \end{bmatrix} \\ \langle \partial_u, \partial_u \rangle &= \langle \partial_v, \partial_v \rangle = \frac{4}{(1 + u^2 + v^2)^2} \\ \langle \partial_u, \partial_v \rangle &= 0 \end{aligned}$$

smooth transition from (dx, dy) to (du, dv) is guaranteed by differentiable Jacobian $\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$:

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

To compute $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $v_x = \frac{\partial v}{\partial x}$ and $v_y = \frac{\partial v}{\partial y}$, the most convenient way is by complex variable. If we parameterize (x, y) by complex number $z = x + iy$ and (u, v) by $w = u + iv$, notice that

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{\left(\frac{x_1}{1 - x_3}\right) - i\left(\frac{x_2}{1 - x_3}\right)}{\left(\frac{x_1}{1 - x_3}\right)^2 + \left(\frac{x_2}{1 - x_3}\right)^2} = \frac{x_1(1 - x_3) - ix_2(1 - x_3)}{\underbrace{(x_1^2 + x_2^2 + x_3^2)}_1 - x_3^2} = \frac{x_1 - ix_2}{1 + x_3} = u + iv = w$$

with $\frac{1}{z} = w$, we have $dw = -\frac{1}{z^2}dz$, we write

$$du + idv = -\frac{1}{z^2}(dx + idy) = -\frac{1}{(x + iy)^2}(dx + idy) = \frac{1}{(x^2 + y^2)^2} \begin{bmatrix} dx(y^2 - x^2) - dy(2xy) & \leftarrow \\ +i[dy(y^2 - x^2) + dx(2xy)] \end{bmatrix}$$

then by technique of complex variable:

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \frac{1}{(x^2 + y^2)^2} \begin{bmatrix} y^2 - x^2 & -2xy \\ 2xy & y^2 - x^2 \end{bmatrix}$$

is indeed differentiable near by $x^2 + y^2 = 1$, the equator.

Moreover, the transition of charts $\varphi : (z, dz) \mapsto (w, dw)$ is

$$\varphi : (z, dz) \mapsto \left(\frac{1}{z}, -\frac{1}{z^2}dz\right)$$

On equator, if parametrized by (θ, τ) , as $z = e^{i\theta}$ and $dz = e^{i\tau}$, we have

$$\varphi : (\theta, \tau) \mapsto (-\theta, \pi - 2\theta + \tau)$$

We use canonical representation of $\pi_1(\mathbf{T}_L^2)$ and $\pi_1(\mathbf{T}_U^2)$:

$$\pi_1(\mathbf{T}_L^2) = \langle a_L, b_L | [a_L, b_L] \rangle$$

$$\pi_1(\mathbf{T}_U^2) = \langle a_U, b_U | [a_U, b_U] \rangle$$

then φ induces a push-forward map on homotopy group³:

$$\varphi_\# : \pi_1(\mathbf{T}_L^2) \rightarrow \pi_1(\mathbf{T}_U^2)$$

by

$$\begin{aligned} a_L &\mapsto a_U \\ b_L &\mapsto a_U^{-2}b_U^{-1} \end{aligned}$$

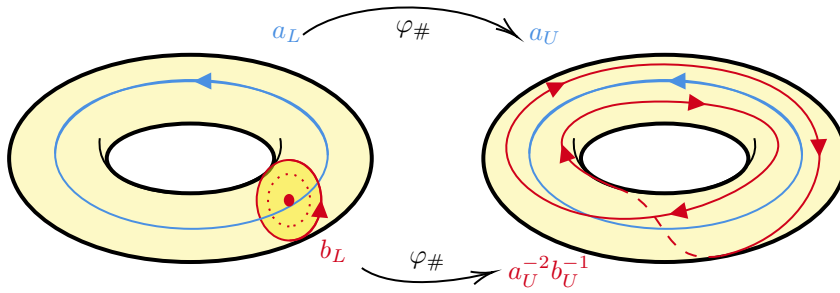


Figure 5: With constraint $\mathbf{T}_L^2 \sim \mathbf{T}_U^2$, by setting the global section \mathbb{D}_L^2 freely in lower solid torus, its boundary $\partial\mathbb{D}_L^2 = b_L$ maps to $a_U^{-2}b_U^{-1}$. While b_L can shrink to a point, $a_U^{-2}b_U^{-1}$ cannot, thus one cannot find a global section in upper solid torus with $a_U^{-2}b_U^{-1}$ as its boundary, which leads to a topological obstruction

As shown in figure 5, we finish construction of a topological obstruction to show that one cannot construct a smooth vector field over a sphere without zero point.

³check by corner points: e.g. $A = (0, 0) \mapsto \varphi(A) = (0, \pi), B = (0, 2\pi) \mapsto \varphi(B) = (0, 3\pi), C = (2\pi, 2\pi) \mapsto \varphi(C) = (-2\pi, -\pi), D = (2\pi, 0) \mapsto \varphi(D) = (-2\pi, -3\pi)$

4 Shape of Unit Tangent Bundle of Unit Sphere

We can derive the fundamental group of $UT\mathbb{S}^2$ using Van Kampen theorem.

Theorem 4 (Van Kampen (-Seifert) Theorem). *Topological space \mathbf{M} is decomposed into the union of \mathbf{U} and \mathbf{V} , the intersection of \mathbf{U} and \mathbf{V} is \mathbf{W} ,*

$$\mathbf{M} = \mathbf{U} \cup \mathbf{V}$$

$$\mathbf{W} = \mathbf{U} \cap \mathbf{V}$$

where \mathbf{U} , \mathbf{V} and \mathbf{W} are path connected.

$$i : \mathbf{W} \hookrightarrow \mathbf{U}$$

$$j : \mathbf{W} \hookrightarrow \mathbf{V}$$

are the inclusion maps. Pick a base point $p \in \mathbf{W}$, the fundamental groups

$$\pi_1(\mathbf{U}, p) = \langle u_1, \dots, u_k | \alpha_1, \dots, \alpha_l \rangle$$

$$\pi_1(\mathbf{V}, p) = \langle v_1, \dots, v_m | \beta_1, \dots, \beta_n \rangle$$

$$\pi_1(\mathbf{W}, p) = \langle w_1, \dots, w_p | \gamma_1, \dots, \gamma_q \rangle$$

then $\pi_1(\mathbf{M}, p)$ is given by

$$\pi_1(\mathbf{M}, p) = \langle u_1, \dots, u_k, v_1, \dots, v_m | \alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_n, i(w_1)j(w_1)^{-1}, \dots, i(w_p)j(w_p)^{-1} \rangle$$

One can use Van Kampen's theorem to compute fundamental groups for topological spaces that can be decomposed into simpler spaces.

Example 5 (Fundamental Group of Unit Tangent Bundle of Unit Sphere). We glue $UT\mathbb{D}_L^2$ and $UT\mathbb{D}_U^2$ with homomorphism:

$$\varphi_\#(a_L) = a_U$$

$$\varphi_\#(b_L) = a_U^{-2}b_U^{-1}$$

thus the set up:

$$UT\mathbb{S}^2 = UT\mathbb{D}_L^2 \bigcup_{\mathbf{T}_L^2 \sim \mathbf{T}_U^2} UT\mathbb{D}_U^2$$

$$\mathbf{T}^2 = UT\mathbb{D}_L^2 \bigcap UT\mathbb{D}_U^2$$

where $UT\mathbb{D}_L^2$, $UT\mathbb{D}_U^2$ and \mathbf{T}^2 are path connected.

$$i : \mathbf{T}^2 \hookrightarrow UT\mathbb{D}_L^2$$

$$j : \mathbf{T}^2 \hookrightarrow UT\mathbb{D}_U^2$$

are the inclusion maps. Pick a base point $p \in \mathbf{T}^2$, the fundamental groups

$$\pi_1(UT\mathbb{D}_L^2, p) = \langle a_L \rangle \quad \pi_1(\mathbf{T}_L^2, p) = \langle a_L, b_L | [a_L, b_L] \rangle$$

$$\pi_1(UT\mathbb{D}_U^2, p) = \langle a_U \rangle \quad \pi_1(\mathbf{T}_U^2, p) = \langle a_U, b_U | [a_U, b_U] \rangle$$

$$\pi_1(\mathbf{T}^2, p) = \langle a, b | [a, b] \rangle$$

the inclusion maps

$$i(a) = a_L, j(a) = a_U^{-1}$$

$$i(b) = b_L = \emptyset, j(b) = (a_U^{-2}b_U^{-1})^{-1}$$

then $\pi_1(UT\mathbb{S}^2, p)$ is given by

$$\pi_1(UT\mathbb{S}^2, p) = \langle a_L, a_U | a_L a_U, a_U^{-2}b_U^{-1} \rangle \cong \mathbb{Z}_2$$

5 Obstruction Class

The information of topological obstruction can be encoded in a 2-form Ω of smooth surface \mathbf{M} . The 2-form Ω is computed by random generated smooth vector field over \mathbf{M} . Surprisingly, all Ω generated in this way are cohomological to each other, thus they form an equivalence class, which we call it **obstruction class**. We denote $[\Omega]$ as obstruction class of $H^2(\mathbf{M}, \mathbb{R})$.

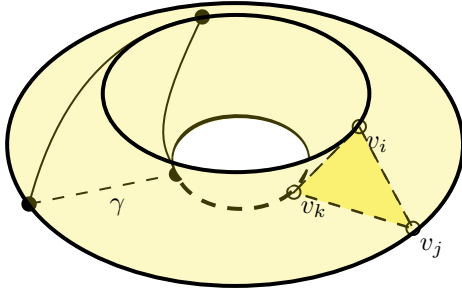
Definition 6 (Obstruction Class). Let \mathbf{M} be a smooth manifold. According to simplicial approximation theorem, there exists \mathbf{M}_Δ , the triangulation of \mathbf{M} , which is refined enough up to any precision. Given \mathbf{M}_Δ , we proceed without loss of generality. The unit tangent bundle of every 2-simplex $[v_i, v_j, v_k]$, is direct product of fiber and the 2-simplex:

$$UT[v_i, v_j, v_k] = \mathbb{S}^1 \times [v_i, v_j, v_k]$$

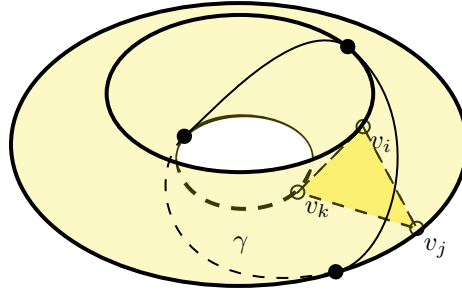
then we generate random tangent vector for each vertex, which means we generate three random points on the torus $UT(\partial[v_i, v_j, v_k])$, the surface (or boundary) of solid torus $UT[v_i, v_j, v_k]$. We see if the loop $\gamma \in UT(\partial[v_i, v_j, v_k])$ that goes through those three points can shrink to a point

Since $\pi_1(UT[v_i, v_j, v_k]) \cong \mathbb{Z}$, should $\gamma \in \pi_1(UT[v_i, v_j, v_k])$ give a number, which we assign it to the 2-form $\Omega([v_i, v_j, v_k])$, either zero if γ can shrink to a point, or non-zero if a local topological obstruction occurs. In the end, we will have a 2-form Ω which represents the *obstruction class*

$$[\Omega] \in H^2(\mathbf{M}, \mathbb{R})$$



$$\Omega(\gamma) \leftarrow 0$$



$$\Omega(\gamma) \leftarrow a \neq 0$$