Computational Conformal Geometry **授课日期: 2020年11月**

Lecture Note 2: Homology and Cohomology

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1 First Homotopy Group vs. First Homology Group

Homotopy relation does fully capture the topological spaces, but it is hard to compute: the homotopy group is non-Abelian (see figure 1). If we can instead represent the topological space by an Abelian group, which can be computed using linear algebra, it would be highly encouraged, even if some loss of information.

By a looser definition of equivalence relation, called homology relation, an Abelian group was formed. The loop product became commutative and therefore was replaced by notation +, called loop formal sum, or just formal sum when generalized to any dimension.

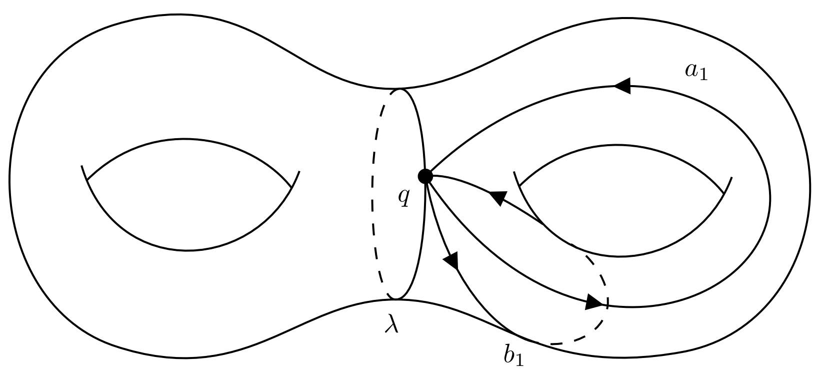

Figure 1: In first homotopy group $\pi_1(\mathbf{S}, q)$, we have $[\gamma] = [a_1 b_1 a_1^{-1} b_1^{-1}]$, but $[\gamma] \neq [e]$, so $[a_1][b_1] \neq [b_1][a_1]$, but in first homology group $H_1(\mathbf{S}, \mathbb{Z})$, $[a_1] + [b_1] = [b_1] + [a_1]$, and also we have $[\lambda] = \mathbf{0}$, see example 9

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Recall definition of *loop product*, which emphasizes the order of concatenation, and therefore it is not commutative. Why don't we just formally sum each parts up, and keep their orientation in record?

Definition 1 (Formal Sum)**.** If an oriented manifold $\mathbf{M}$ can be decomposed into finite simpler submanifolds $\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_n$ with the same orientation, then we write:

$$\mathbf{M} = \mathbf{m}_1 + \mathbf{m}_2 + ... + \mathbf{m}_n$$

where + denotes *formal sum.* Formal sum is commutative.

Definition 2 (Inverse of Formal Sum)**.** The *inverse of formal sum* of an oriented manifold $\mathbf{M}$ is the sum of inverse of submanifolds $\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_n$, denoted by "−":

$$-\mathbf{M} = -\mathbf{m}_1 - \mathbf{m}_2 - ... - \mathbf{m}_n$$

Definition 3 (Closure)**.** The *closure* of a subset $\mathbf{S}$ of points in a topological space consists of all points in $\mathbf{S}$ together with all limit points of $\mathbf{S}$, denoted by

$$\bar{\mathbf{S}}$$

Definition 4 (Interior)**.** The *interior* of a subset $\mathbf{S}$ of a topological space $\mathbf{X}$ is the union of all subsets of $\mathbf{S}$ that are open in $\mathbf{X}$, denoted by

$$\mathbf{S}^{\circ}$$

Definition 5 (Boundary and Boundary Operator)**.** The boundary of a subset $\mathbf{S}$ of a topological space $\mathbf{X}$ is the *closure* of $\mathbf{S}$ minus the interior of $\mathbf{S}$:

$$\partial \mathbf{S} := \bar{\mathbf{S}} \setminus \mathbf{S}^{\circ}$$

We also use $\partial_k \mathbf{\Sigma}$ to indicate that the boundary operator actions on a k-manifold $\mathbf{\Sigma}$.

Example 6 (Boundary of Surface or Loop)**.** For oriented surface, the boundary (loop) is positively oriented "as one walks along boundary on outside surface while cliff on your right". For oriented curve, the boundary are two end-points that the target point is positively oriented and the source point is negatively oriented. We use $\mathbf{0}$ to denote "nothing in space", e.g. the boundary of a sphere or a loop.

Up to this point, to form a group of k-manifolds, we already have:

- commutative binary operation, the *formal sum* "+"
- identity unit element, $\mathbf{0}$, "nothing in space"
- inverse of an element, which is its negatively oriented version.

We need one more thing, the equivalence class, to reveal topological invariant.

Definition 7 (Homology)**.** Let $\mathbf{S}$ be a k-manifold. Let γ_0 and γ_1 be two $(k-1)$-manifolds. A *homology relation* connecting γ_0 and γ_1 is a k-submanifold $\mathbf{\Sigma}$ such that:

$$\partial_k \mathbf{\Sigma} = \gamma_0 - \gamma_1$$

We say γ_0 is homological to γ_1 if there exists homology between them, denoted as $\gamma_0 \sim \gamma_1$ [1].

Definition 8 (First Homology Group)**.** Given a surface topological space $\mathbf{S}$. Homology relation is an equivalence relation. The set of all the loops and finite formal sum of them is Γ, which can be classified by homology relation and form a set of all the homology classes, denoted as $\Gamma/\sim$. To define a group:

- The homology class of a loop γ, denoted by $[\gamma]$, becomes group generator.

[1] Notice that Homotopy $\stackrel{\nLeftarrow}{\Rightarrow}$ Homology. To illustrate homology relation is an equivalence relation:

- (reflexive) $\gamma \sim \gamma$ since $\mathbf{0} = \gamma - \gamma$ trivially holds
- (symmetric) if $\gamma_0 \sim \gamma_1$, then $\partial_2 \mathbf{\Sigma} = \gamma_0 - \gamma_1$, then $\partial_2(\mathbf{S} \setminus \mathbf{\Sigma}) = \gamma_1 - \gamma_0$, then $\gamma_1 \sim \gamma_0$
- (transitive) if $\gamma_0 \sim \gamma_1$ and $\gamma_1 \sim \gamma_2$, suppose $\partial_2 \mathbf{\Sigma}_1 = \gamma_0 - \gamma_1$ and $\partial_2 \mathbf{\Sigma}_2 = \gamma_1 - \gamma_2$, then $\partial_2 \mathbf{\Sigma}_1 + \partial_2 \mathbf{\Sigma}_2 = \partial_2(\mathbf{\Sigma}_1 + \mathbf{\Sigma}_2) = \gamma_0 - \gamma_2$ then $\gamma_0 \sim \gamma_2$.

- The group binary operation is defined as

$$[\gamma_1] + [\gamma_2] := [\gamma_1 + \gamma_2]$$

  which is commutative

- The group unit element is defined as $\mathbf{0}$, which is "nothing in space".

- The group inverse element is defined as

$$[\gamma]^{-1} = -[\gamma] := [-\gamma]$$

then $\Gamma/\sim$ forms a group, so-called the first homology group, denoted as $H_1(\mathbf{S}, \mathbb{Z})$, if formal sum is over $\mathbb{Z}$, see foot note if over otherwise field[2]

Example 9 (Homology of loops)**.** See figure 2, $\mathbf{S}$ is a closed orientable surface with genus $g = 3$. We have

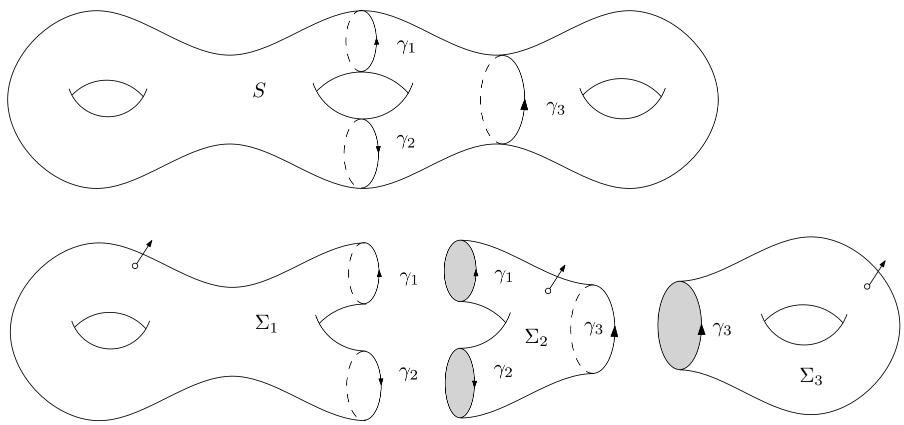

Figure 2: an example of homology relation

formal sum:

$$\mathbf{S} = \mathbf{\Sigma}_1 + \mathbf{\Sigma}_2 + \mathbf{\Sigma}_3$$

What can we say about the homological class about γ_1, γ_2 and γ_3? We can see that

$$\gamma_1 \sim \gamma_2 \sim (\gamma_2 + \gamma_3) \sim (\gamma_1 - \gamma_3)$$
$$\gamma_3 \sim (\gamma_1 - \gamma_2) \sim \mathbf{0}$$

since

$$\partial_2 \mathbf{\Sigma}_1 = \gamma_1 - \gamma_2$$
$$\partial_2 \mathbf{\Sigma}_2 = (\gamma_2 + \gamma_3) - \gamma_1 = \gamma_3 - (\gamma_1 - \gamma_2) = \gamma_2 - (\gamma_1 - \gamma_3)$$
$$\partial_2 \mathbf{\Sigma}_3 = \mathbf{0} - \gamma_3$$

[2]if formal sum over $\mathbb{Z}_2$, for example, then $[\gamma] + [\gamma] = \mathbf{0}$, we denoted first homology group as $H_1(\mathbf{S}, \mathbb{Z}_2)$. If formal sum over $\mathbb{R}$, for example, we allow $0.4[\gamma] - 1.6[\gamma] + \sqrt{2}[\gamma] = (\sqrt{2} - 1.2)[\gamma]$, then denote first homology group as $H_1(\mathbf{S}, \mathbb{R})$

2 Homology Group

Kernel is the generalization of zeros of a function. Image is the generalization of range of a function.

Definition 10 (Kernel)**.** Let G and H be groups and let $f : G \mapsto H$ be a homomorphism. Let e_H denote the identity unit element in H. The *kernel* of f is defined as

$$\ker f = \{g \in G \mid f(g) = e_H\}$$

Definition 11 (Loop Group)**.** Any closed loop (or finite formal sum of loops) γ on a closed oriented surface $\mathbf{S}$ will satisfy:

$$\partial_1 \gamma = \mathbf{0}$$

We denote *loop group* $Z_1(\mathbf{S})$ as

$$Z_1(\mathbf{S}) = \ker \partial_1 = \{\gamma \in \mathbf{S} \mid \partial_1(\gamma) = \mathbf{0}\}$$

Definition 12 (Image)**.** Let G and H be groups and let $f : G \mapsto H$ be a homomorphism. The *image* of f is defined as

$$\text{img } f = \{h \in H \mid \exists g \in G \ s.t. \ f(g) = h\}$$

Definition 13 (Boundary Group)**.** Any submanifold $\mathbf{\Sigma}$ on a closed oriented surface $\mathbf{S}$ will induce a closed boundary γ:

$$\partial_2 \mathbf{\Sigma} = \gamma$$

We denote *boundary group* $B_1(\mathbf{S})$ as

$$B_1(\mathbf{S}) = \text{img } \partial_2 = \{\gamma \in \mathbf{S} \mid \exists \mathbf{\Sigma} \in \mathbf{S} \ s.t. \ \partial_2 \mathbf{\Sigma} = \gamma\}$$

Definition 14 (Homology Group Structure)**.** The first homology group of $\mathbf{S}$ is the quotient group

$$H_1(\mathbf{S}, \mathbb{Z}) = \frac{Z_1(\mathbf{S})}{B_1(\mathbf{S})} = \frac{\ker \partial_1}{\text{img } \partial_2}$$

Which is consistent with homology relation, as we collapse $B_1(\mathbf{S})$ as identity (all $\partial_2 \mathbf{\Sigma}$ now become $\mathbf{0}$)

Generally, given $(k+1)$-manifold $\mathbf{M}$, k homology group is

$$H_k(\mathbf{M}, \mathbb{Z}) = \frac{Z_k(\mathbf{M})}{B_k(\mathbf{M})} = \frac{\ker \partial_k}{\text{img } \partial_{k+1}}$$

Figure 3 illustrates the relationship between groups:

- C_k, which is group of all k-submanifold
- Z_k, which is group of kernel of ∂_k on k-submanifold
- B_k, which is group of image of ∂_{k+1} on $(k+1)$-submanifold

$$B_k(\mathbf{M}) \subset Z_k(\mathbf{M}) \subset C_k(\mathbf{M})$$

and

$$\partial_k \circ \partial_{k+1} = \mathbf{0}$$

Theorem 15. *Suppose* $\mathbf{S}$ *is a path-connected genus* g *closed surface, then*

$$H_0(\mathbf{S}, \mathbb{Z}) \cong \mathbb{Z} \cong H_2(\mathbf{S}, \mathbb{Z})$$

$$H_1(\mathbf{S}, \mathbb{Z}) \cong \mathbb{Z}^{2g}$$

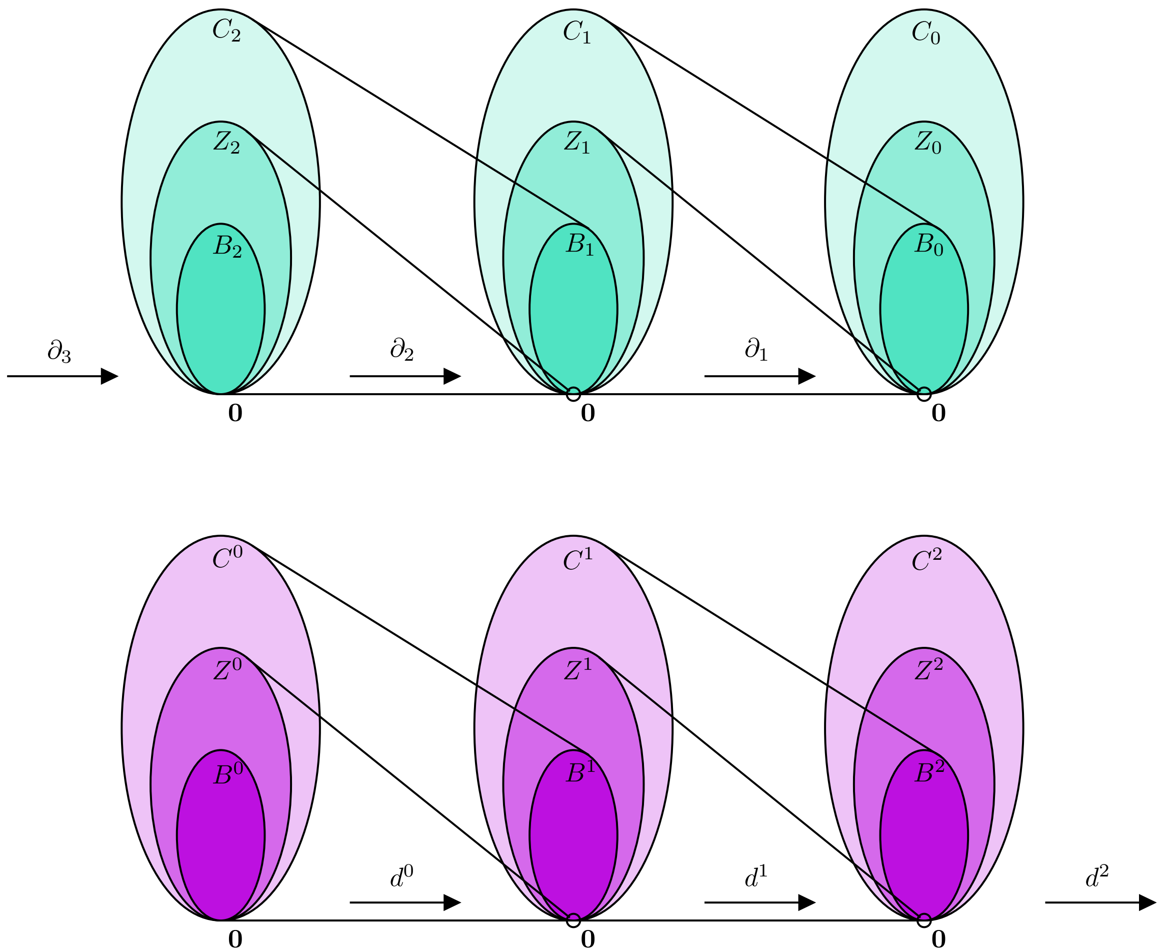

Figure 3: The relation of groups C_k, Z_k and B_k, and their duality, C^k, Z^k and B^k

3 First Homology Group vs. First Cohomology Group

We have another Abelian group that can encode the same topological information of homology group but can be computed even faster. However, few people understand. We would like to point out *k-form* is the generalization of function, and *coboundary operator* is the generalization of gradient. Before we get there, we now introduce some concept in differential geometry, which would be frequently used.

Definition 16 (Tangent Space)**.** Given a point x on closed surface $\mathbf{M}$, a *tangent space* of $\mathbf{M}$ through x, denoted as

$$T_x\mathbf{M}$$

is a vector space of plane that contains the possible directions in which one can tangentially pass through x. The elements of the *tangent space* $T_x\mathbf{M}$ at x are called the *tangent vectors* v at x, see figure 4:

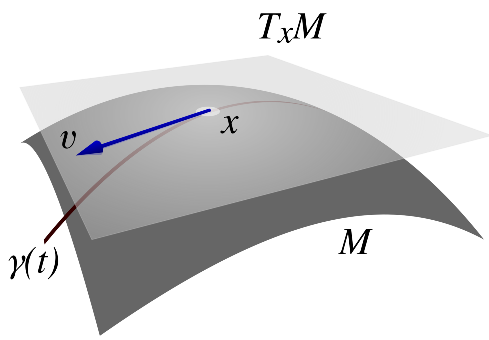

Figure 4: The tangent space $T_x\mathbf{M}$ and a tangent vector v on $T_x\mathbf{M}$, along a curve $\gamma(t)$ traveling through $x \in \mathbf{M}$

Definition 17 (Tangent Bundle)**.** The *tangent bundle* of a differentiable manifold $\mathbf{M}$ is a manifold $T\mathbf{M}$ which assembles all the tangent vectors in $\mathbf{M}$, given by the disjoint union of the tangent spaces of $\mathbf{M}$:

$$T\mathbf{M} := \bigcup_{x\in\mathbf{M}} \{x\} \times T_x\mathbf{M} = \bigcup_{x\in\mathbf{M}} \{(x,v) \mid v \in T_x\mathbf{M}\} = \{(x,v) \mid x \in \mathbf{M}, v \in T_x\mathbf{M}\}$$

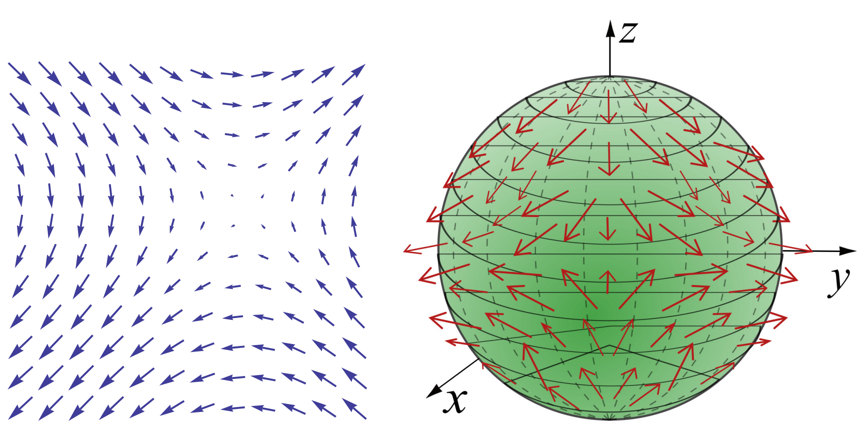

Figure 5: Vector field $f : \mathbf{D} \mapsto \mathbb{R}^2$ (left) and $g : \mathbf{S} \mapsto T\mathbf{S}$ (right)

Definition 18 (Vector Field)**.**

A *vector field* on the region $\mathbf{D} \subset \mathbb{R}^2$ is a vector-valued function

$$f : \mathbf{D} \mapsto \mathbb{R}^2$$

A *vector field* on a surface $\mathbf{S}$ is an assignment of a tangent vector to each point in $\mathbf{S}$. More precisely, a mapping from $\mathbf{S}$ to tangent bundle of $\mathbf{S}$:

$$g : \mathbf{S} \mapsto T\mathbf{S}$$

See figure 5

Definition 19 (Line Integral)**.** Integration of vector field F along a curve is called *line integral.*

$$\int_\gamma F(v) \cdot dv$$

or simply denoted as

$$\langle F, \gamma \rangle$$

Definition 20 (Curl Free Vector Field)**.** A *curl free field* F of is a vector field such that the line integral along any **loop** γ equals zero:

$$\oint_\gamma F(v) \cdot dv = 0$$

or written as

$$\text{curl } F = 0$$

Definition 21 (Gradient Vector Field)**.** A *gradient field* (or *conservative field*) F is a vector field such that the line integral along any **boundary** γ equals zero:

$$\oint_\gamma F(v) \cdot dv = 0$$

or equivalently we say that there exist a scalar field φ that its gradient is F:

$$F = \nabla\varphi \quad \text{or} \quad F = \text{grad } \varphi$$

Notice that just like boundary $\overset{\nLeftarrow}{\Rightarrow}$ loop

$$\text{gradient field} \overset{\nLeftarrow}{\Rightarrow} \text{curl free field}$$

Definition 22 (1-form)**.** Given an oriented closed surface $\mathbf{S}$, a *1-form* f of $\mathbf{S}$ encoded by a vector field F on $\mathbf{S}$ is a linear mapping from any curve $\gamma \in C_1(\mathbf{S})$ to line integral $\langle F, \gamma \rangle \in \mathbb{R}$:

$$f : C_1(\mathbf{S}) \mapsto \mathbb{R}$$

$$f(\gamma) := \langle F, \gamma \rangle$$

Since the *1-form* of $\mathbf{S}$ is very much of the vector filed F, so people use *vector field* and *1-form* interchangeably. Recall that group of any curve $C_1(\mathbf{S})$, group of loops $Z_1(\mathbf{S})$ and group of boundaries $B_1(\mathbf{S})$ with

$$B_1(\mathbf{S}) \subset Z_1(\mathbf{S}) \subset C_1(\mathbf{S})$$

since

$$\text{boundary} \overset{\nLeftarrow}{\Rightarrow} \text{loop}$$

Now we consider another way to describe such topological information.

Definition 23 (First Cohomology Group)**.** Given a surface topological space $\mathbf{S}$. We define three groups of 1-form (vector field) with binary operation "+" over $\mathbb{Z}$, unit element 0 and inverse as "-" prefix:

- group of all vector field over $\mathbf{S}$, denoted as
$$C^1(\mathbf{S})$$
- curl free field group, denoted as $Z^1(\mathbf{S})$
$$Z^1(\mathbf{S}) = \{f \in C^1(\mathbf{S}) \mid \langle f, \gamma \rangle = 0,\ \gamma \in Z_1(\mathbf{S})\}$$
- gradient field group, denoted as $B^1(\mathbf{S})$
$$B^1(\mathbf{S}) = \{f \in C^1(\mathbf{S}) \mid \langle f, \gamma \rangle = 0,\ \gamma \in B_1(\mathbf{S})\}$$

Since
$$\text{boundary} \overset{\nLeftarrow}{\Rightarrow} \text{loop}$$
We have
$$B^1(\mathbf{S}) \subset Z^1(\mathbf{S}) \subset C^1(\mathbf{S})$$
The *first cohomology group* $H^1(\mathbf{S}, \mathbb{Z})$ is achieved by collapsing $B^1(\mathbf{S})$ into identity:

$$H^1(\mathbf{S}, \mathbb{Z}) = \frac{Z^1(\mathbf{S})}{B^1(\mathbf{S})}$$

and notice that

$$H^1(\mathbf{S}, \mathbb{Z}) \cong H_1(\mathbf{S}, \mathbb{Z})$$

Until now, we may not have proper language to describe what is a cohomology relation, although we derive cohomology group. What does it mean if f_0 is cohomologous to f_1?

A scalar field φ on $\mathbf{S}$ is a 0-form. By gradient operator, it becomes a vector field F, the 1-form. Imagine any of tiny oriented curve γ on $\mathbf{S}$. The gradient can be thought of the difference of the scalar values of two end points, which is the summation of 0-form of boundary of the curve (because boundary will give one positive and one negative value):

$$\underbrace{F}_{\text{1-form}} = \underbrace{\text{grad} \overbrace{\varphi}^{\text{0-form}}}_{\text{1-form}} = \underbrace{\varphi \circ \partial}_{\text{1-form}}$$

Now the generalization of gradient by relating boundary operator is coboundary operator

Definition 24 (Coboundary and Coboundary Operator)**.** k-dimensional *Coboundary operator* d^k actions on a k-form f:
$$d^k f(\cdot) := f \circ \partial_{k+1}(\cdot)$$
we say $d^k f$, a $(k+1)$-form, is the coboundary of f, the k-form.

Notice that
$$d^k \circ d^{k-1}(\cdot) = 0$$
holds for any input of $(k-1)$-manifold. In the case of $d^1 \circ d^0(\cdot)$, namely, the curl of gradient is zero.

We also derive

Theorem 25 (Stokes Theorem)**.**

$$\langle dw, \sigma \rangle = \langle w, \partial\sigma \rangle$$

Definition 26 (Cohomology)**.** Let $\mathbf{S}$ be a surface topological space. Let f_0 and f_1 be two k-forms. A *cohomology relation* connecting f_0 and f_1 is a $(k-1)$-form φ such that:

$$d^{k-1}\varphi = f_0 - f_1$$

We say that f_0 is cohomologous to f_1 if there exists such $(k-1)$-form φ, denoted as $f_0 \sim f_1$, and cohomology class $[f_0] = [f_1]$.

Definition 27 (Cohomology Group Structure)**.** We omit further details, see figure 3:

$$H^k(\mathbf{S}, \mathbb{Z}) = \frac{Z^k(\mathbf{S})}{B^k(\mathbf{S})} = \frac{\ker d^k}{\text{img } d^{k-1}}$$

Theorem 28 (Poincaré Duality)**.** *Given n dimensional topological space $\mathbf{S}$:*

$$H^k(\mathbf{S}, \mathbb{Z}) \cong H_{n-k}(\mathbf{S}, \mathbb{Z})$$

When $n = 2$ as in the case of surface topology, given path-connected oriented closed genus g surface $\mathbf{S}$, we have

$$H^2(\mathbf{S}, \mathbb{Z}) \cong H_0(\mathbf{S}, \mathbb{Z}) \cong \mathbb{Z} \cong H_2(\mathbf{S}, \mathbb{Z}) \cong H^0(\mathbf{S}, \mathbb{Z})$$
$$H^1(\mathbf{S}, \mathbb{Z}) \cong H_1(\mathbf{S}, \mathbb{Z}) \cong \mathbb{Z}^{2g}$$

Definition 29 (Dual Cohomology Basis)**.** suppose a homology basis of $H_1(\mathbf{S})$ is $\{\gamma_1, \gamma_2, ..., \gamma_n\}$, the *dual cohomology basis* is $\{w_1, w_2, ..., w_n\}$, satisfying:

$$\langle w_i, \gamma_j \rangle = 1_{i=j}$$

where

$$1_{\mathcal{A}} := \begin{cases} 1 & \text{if } \mathcal{A} \text{ is true} \\ 0 & \text{otherwise} \end{cases}$$