Computational Conformal Geometry **授课日期: 2020年11月**

Lecture Note 1: Fundamental Group, Covering Space and Homology

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This lecture is about surface algebraic topology. The key idea is to build a bridge between topology, which is abstract and hard to imagine, and algebraic structure, which is tangible and can be computed. In a categorical sense, we construct a functor

$$\mathfrak{C}_1 \mapsto \mathfrak{C}_2$$

between two categories[1] with structural information preserved, namely

$$\begin{aligned}\mathfrak{C}_1 &= \{\text{Topological Spaces, Homeomorphisms}\}\\ \mathfrak{C}_2 &= \{\text{Groups, Homomorphisms}\}\end{aligned}$$

Definition 1 (Topological Type)**.** All oriented compact surfaces can be classified by their genus g and number of boundaries b. Therefore, we use

$$(g, b)$$

to represent the topological type of an oriented surface $\mathbf{S}$.

Definition 2 (Homeomorphism)**.** A *homeomorphism* is a continuous function between topological spaces of the same topological type.

Definition 3 (Homomorphism)**.** A *homomorphism* is a structure-preserving map between two algebraic structures of the same type.

We now introduce *first homotopy group*, denoted[2] as $\pi_1(\mathbf{S}, q)$. The group structure of $\pi_1(\mathbf{S}, q)$ determines the topology of $\mathbf{S}$.

1 Fundamental Group

Let $\mathbf{S}$ be a two-manifold with a base point $p \in \mathbf{S}$.

Definition 4 (Curve)**.** A *curve* is a continuous mapping $\gamma : [0, 1] \mapsto \mathbf{S}$

Definition 5 (Loop)**.** A *closed curve* or *loop* through p is a curve s.t. $\gamma(0) = \gamma(1) = p$

[1] The concepts of category and functor were covered in previous lectures

[2] Although the fundamental group in general depends on the choice of base point, it turns out that, up to isomorphism (actually, even up to inner isomorphism), this choice makes no difference as long as the space $\mathbf{S}$ is path-connected. For path-connected spaces, therefore, many authors therefore write $\pi_1(\mathbf{S})$ instead of $\pi_1(\mathbf{S}, q)$

Definition 6 (Homotopy)**.** Let $\gamma_0, \gamma_1 : [0,1] \mapsto \mathbf{S}$ be two curves. A *homotopy* connecting γ_0 and γ_1 is a continuous mapping

$$f : [0,1] \times [0,1] \mapsto \mathbf{S}$$

s.t.

$$f(0,t) = \gamma_0(t)$$
$$f(1,t) = \gamma_1(t)$$

We say γ_0 is homotopic to γ_1, if there exists a homotopy between them, denoted as $\gamma_0 \sim \gamma_1$.

Definition 7 (Loop Product)**.** $\gamma_1 \cdot \gamma_2$ is

$$\gamma_1 \cdot \gamma_2(t) := \begin{cases} \gamma_1(2t) & \text{for } 0 \le t \le 0.5 \\ \gamma_2(2t-1) & \text{for } 0.5 \le t \le 1 \end{cases}$$

Definition 8 (Loop Inverse)**.** $\gamma^{-1}(t) := \gamma(1-t)$

Definition 9 (Fundamental Group)**.** Given a surface topological space $\mathbf{S}$, fix a base point $p \in \mathbf{S}$. Homotopy relation is an equivalence relation[3]. The set of all the loops through the base point p is Γ, which can be classified by homotopy relation and form a set of all the homotopy classes, denoted as $\Gamma/\sim$. To define a group:

- The homotopy class of a loop γ, denoted by $[\gamma]$, becomes group generator.
- The group binary operation is defined as

$$[\gamma_1][\gamma_2] := [\gamma_1 \cdot \gamma_2]$$

  .

- The group unit element is defined as $[e]$, which is as trivial as a point.
- The group inverse element is defined as

$$[\gamma]^{-1} = [\gamma^{-1}]$$

then $\Gamma/\sim$ forms a group, so-called *fundamental group* of $\mathbf{S}$, or the first homotopy group, denoted as $\pi_1(\mathbf{S}, p)$.

Definition 10 (Word Group Representation)**.** Let $G = \{g_1, g_2, ..., g_n\}$ be n distinct symbols. Words of finite length generated by those symbols form a group with equivalence relations

- $\{g_1, g_2, ..., g_n\}$ becomes group generator.
- The group binary operation is defined as the concatenation of two word.
- The group unit element is empty word $\emptyset$
- The group inverse element is defined as reverse composition of the word.
- Certain segment of words can be replaced by $\emptyset$, which forms equivalence relations, denoted by set $R = \{R_1, R_2, ..., R_m\}$.

[3]needs to be reflexive, symmetric and transitive

Given a set of generators G and a set of relations R, all the equivalence classes of the words generated by G form a group under the concatenation, called *word group*, denoted as

$$\langle g_1, g_2, ..., g_n | R_1, R_2, ..., R_m \rangle$$

Word group representation can be used to process fundamental group in computer.

Theorem 11 (Van Kampen (-Seifert) Theorem)**.** *If*

- $\mathbf{X}$ *is a topological space;*
- $\mathbf{U}$ *and* $\mathbf{V}$ *are open, path connected subspaces of* $\mathbf{X}$*;*
- $\mathbf{U} \cap \mathbf{V}$ *is nonempty and path-connected;*
- $w \in \mathbf{U} \cap \mathbf{V}$*;*
- *homomorphisms* $I : \pi_1(\mathbf{U} \cap \mathbf{V}, w) \mapsto \pi_1(\mathbf{U}, w)$ *and* $J : \pi_1(\mathbf{U} \cap \mathbf{V}, w) \mapsto \pi_1(\mathbf{U}, w)$
- *the fundamental group of* $\mathbf{U}, \mathbf{V}$ *and* $\mathbf{U} \cap \mathbf{V}$

$$\pi_1(\mathbf{U}, w) = \langle u_1, ..., u_k | \alpha_1, ..., \alpha_l \rangle$$

$$\pi_1(\mathbf{V}, w) = \langle v_1, ..., v_m | \beta_1, ..., \beta_n \rangle$$

$$\pi_1(\mathbf{U} \cap \mathbf{V}, w) = \langle w_1, ..., w_p | \gamma_1, ..., \gamma_q \rangle$$

then $\pi_1(\mathbf{X}, w)$ *is the free product with amalgamation of* $\pi_1(\mathbf{U}, w)$ *and* $\pi_1(\mathbf{V}, w)$

$$\pi_1(\mathbf{X}, w) = \langle u_1, ..., u_k, v_1, ..., v_m | \alpha_1, ..., \alpha_l, \beta_1, ..., \beta_n, I(w_1)J(w_1)^{-1}, ..., I(w_p)J(w_p)^{-1} \rangle$$

One can use Van Kampen's theorem to calculate fundamental groups for topological spaces that can be decomposed into simpler spaces.

Theorem 12 (Canonical Representation of Surface Fundamental Group)**.** *Suppose* $\mathbf{S}$ *is a compact, oriented surface,* $p \in \mathbf{S}$ *is a fixed point, the fundamental group has a canonical*[4] *representation*

$$\pi_1(\mathbf{S}, p) = \langle a_1, b_1, a_2, b_2, ..., a_g, b_g | \prod_{i=1}^{g} [a_i, b_i] \rangle$$

where

$$[a_i, b_i] := a_i b_i a_i^{-1} b_i^{-1}$$

and g *is the genus of the surface* $\mathbf{S}$ *and* a_i, b_i *are canonical bases*[5]

[4] The canonical representation of the fundamental group of the surface is not unique. It is NP hard to verify if two given representations are isomorphic.

[5] we omit the definition of canonical basis.

Theorem 13. *Topological Spaces Homeomorphism* $\Leftrightarrow$ *Fundamental Groups Isomorphism*

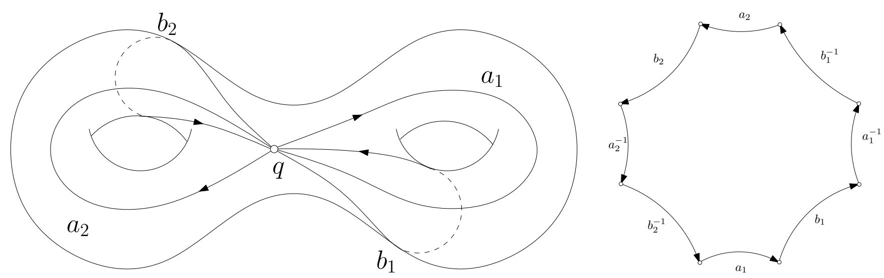

Figure 1: fundamental group canonical basis and fundamental domain

Proof. For each surface, find a canonical basis, slice the surface along the basis to get a $4g$ polygonal scheme, then construct a homeomorphism between the polygonal schema with consistent boundary condition. (e.g. bi-torus see figure 1) □

Definition 14 (Connected Sum)**.** The *connected sum* $\mathbf{S}_1 \oplus \mathbf{S}_2$ is formed by deleting the interior of disks $\mathbf{D}_i$ and attaching the resulting punctured surfaces $\mathbf{S}_i - \mathbf{D}_i$ to each other by a homeomorphism $h : \partial\mathbf{D}_1 \mapsto \partial\mathbf{D}_2$

$$\mathbf{S}_1 \oplus \mathbf{S}_2 := (\mathbf{S}_1 - \mathbf{D}_1) \cup_h (\mathbf{S}_2 - \mathbf{D}_2)$$

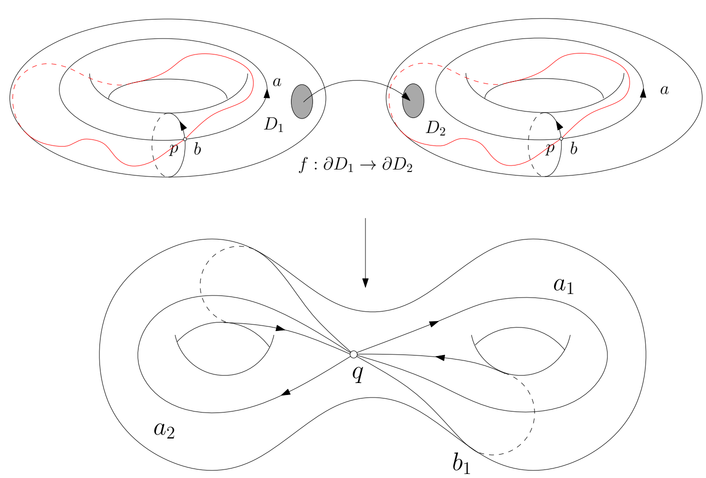

Figure 2: connected sum of two tori

Theorem 15 (Classification Theorem of Closed Surfaces)**.** *Any closed connected surface is homeomorphic to exactly one of the following surfaces:*

- *the sphere, a finite connected sum of tori,*
- *the connected sum of g tori for $g \geq 1$*

$$\underbrace{\mathbf{T}^2 \oplus \mathbf{T}^2 \oplus ... \oplus \mathbf{T}^2}_{g\ tori}$$

- *the connected sum of k real projective planes for $k \geq 1$.*

$$\mathbf{RP}^2 \oplus \mathbf{RP}^2 \oplus ... \oplus \mathbf{RP}^2$$

One can use theorem 11 to show that theorem 12 is true for

$$\mathbf{S} = \oplus_{i=1}^{g} \mathbf{T}^2$$

2 Quotient Group

Definition 16 (Coset)**.** Let H be a subgroup of the group G. Given an element g of G,

- the *left cosets* of H in G are the **sets** (not group!) obtained by multiplying each element of H by a fixed element g of G (where g is the left factor), denoted by

$$gH := \{gh : h \in H\}$$

- The *right cosets* are defined similarly, except that the element g is now a right factor, that is,

$$Hg := \{hg : h \in H\}$$

Definition 17 (Normal Subgroup)**.** Subgroup N of group G is *normal subgroup*, denoted as $N \triangleleft G$ if for all g in G, the left cosets gN and right cosets Ng are equal. Notice that any subgroup of an Abelian group is a normal subgroup.

Definition 18 (Quotient Group)**.** Let $N \triangleleft G$. To construct a *quotient group* G/N or $\dfrac{G}{N}$, N needs to be a normal subgroup of G:

- define the set G/N to be the set of all cosets[6] of N in G. That is, $G/N = \{\underset{a}{N} : a \in G\}$;
- for any two cosets $\underset{a}{N}$ and $\underset{b}{N}$, binary operation $*$ is defined as

$$\underset{a}{N} * \underset{b}{N} = \underset{ab}{N}$$

- the denominator N, the whole normal subgroup, collapsed into the unit[7] element

$$N = \{e\}$$

[6]since $N \triangleleft G$, left and right cosets coincide, we use $\underset{a}{N}$ to denote coset of N given $a \in G$

[7]In a quotient of a group, the equivalence class of the identity element is always a normal subgroup of the original group, and the other equivalence classes are precisely the cosets of that normal subgroup. We can alternatively think of quotient group as $G/\sim$, where $a \sim b$ if a and b are in the same coset of N

- the inverse is defined as

$$\underset{a}{N}^{-1} = \underset{a^{-1}}{N}$$

Notice that $G/e = G$ and $G/G = \{e\}$

The concepts of covering space and homology group was heavily built on quotient group.

3 Covering Space

Definition 19 (Covering Space)**.** Given topological spaces $\tilde{\mathbf{S}}$ and $\mathbf{S}$, a continuous map $f : \tilde{\mathbf{S}} \mapsto \mathbf{S}$ is surjective, such that

- for each point $q \in \mathbf{S}$, there is a neighborhood $\mathbf{U}$ of q;
- its preimage $f^{-1}(\mathbf{U}) = \cup_i \tilde{\mathbf{U}}_i$ is a disjoint union of open sets $\tilde{\mathbf{U}}_i$;
- f on each $\tilde{\mathbf{U}}_i$ is a local homeomorphism

then $(\tilde{\mathbf{S}}, f)$ is a *covering space* of *base space* $\mathbf{S}$, and f is called a projection map.

Definition 20 (Deck Transformation and Covering Group)**.** The automorphisms of $\tilde{\mathbf{S}}$, $g : \tilde{\mathbf{S}} \mapsto \tilde{\mathbf{S}}$, are called *deck transformations*, if they satisfy $f \circ g = f$. All the deck transformations form a group, the *covering group*, and denoted as

$$\mathrm{Deck}(\tilde{\mathbf{S}})$$

Theorem 21 (Covering Group Structure)**.** *Covering space* $\tilde{\mathbf{S}}$ *and base space* $\mathbf{S}$.

Suppose base points $\tilde{q} \in \tilde{\mathbf{S}}$, $f(\tilde{q}) = q \in \mathbf{S}$.

The projection map $f : \tilde{\mathbf{S}} \mapsto \mathbf{S}$ *induces a homomorphism between their fundamental groups*

$$f_* : \pi_1(\tilde{\mathbf{S}}, \tilde{q}) \mapsto \pi_1(\mathbf{S}, q)$$

If $f_*(\pi_1(\tilde{\mathbf{S}}, \tilde{q}))$ *is a normal subgroup of* $\pi_1(\mathbf{S}, q)$ *then the quotient group*

$$\frac{\pi_1(\mathbf{S}, q)}{f_*(\pi_1(\tilde{\mathbf{S}}, \tilde{q}))} \cong Deck(\tilde{\mathbf{S}})$$

Definition 22 (Universal Covering Space)**.** If a covering space $\tilde{\mathbf{S}}$ is simply connected (i.e. $\pi_1(\tilde{\mathbf{S}}) = \{e\}$), then $\tilde{\mathbf{S}}$ is called a *universal covering space* of $\mathbf{S}$.

$$\pi_1(\mathbf{S}) \cong \mathrm{Deck}(\tilde{\mathbf{S}})$$

Namely, the fundamental group of the base space is isomorphic to the deck transformation group of the universal covering space (see figure 3)

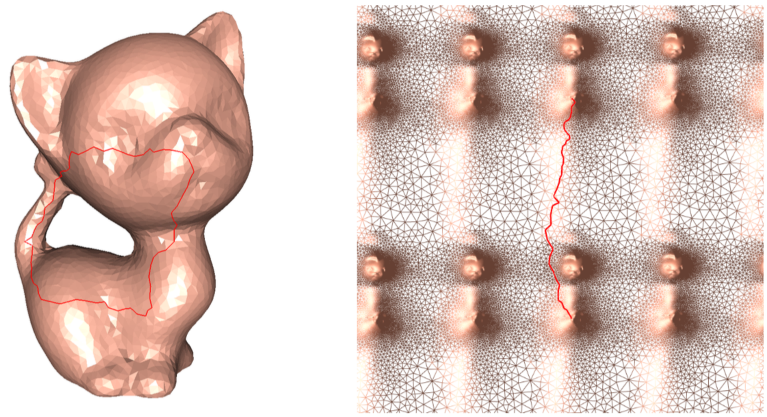

Figure 3: base space $\mathbf{S}$ on the left and universal covering space $\tilde{\mathbf{S}}$ on the right

4 First Homotopy Group vs. First Homology Group

Homotopy relation does fully capture the topological spaces, but it is hard to compute: the homotopy group is non-Abelian (see figure 4). If we can instead represent the topological space by an Abelian group, which can be computed using linear algebra, it would be highly encouraged, even if some loss of information.

By a looser definition of equivalence relation, called homology relation, an Abelian group was formed. The loop product became commutative and therefore was replaced by notation $+$, called loop formal sum, or just formal sum when generalized to any dimension.

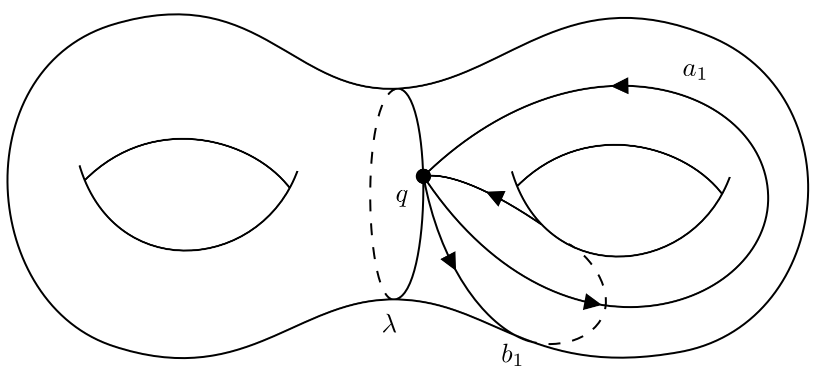

Figure 4: In first homotopy group $\pi_1(\mathbf{S}, q)$, we have $[\gamma] = [a_1 b_1 a_1^{-1} b_1^{-1}]$, but $[\gamma] \neq [e]$, so $[a_1][b_1] \neq [b_1][a_1]$, but in first homology group $H_1(\mathbf{S}, \mathbb{Z})$, $[a_1] + [b_1] = [b_1] + [a_1]$, and also we have $[\lambda] = \mathbf{0}$, see example 31

.

Recall definition 7 of *loop product*, which emphasizes the order of concatenation, and therefore it is not commutative. Why don't we just formally sum each parts up, and keep their orientation in record?

Definition 23 (Formal Sum)**.** If an oriented manifold $\mathbf{M}$ can be decomposed into finite simpler submanifolds

$\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_n$ with the same orientation, then we write:

$$\mathbf{M} = \mathbf{m}_1 + \mathbf{m}_2 + ... + \mathbf{m}_n$$

where $+$ denotes *formal sum*. Formal sum is commutative.

Definition 24 (Inverse of Formal Sum)**.** The *inverse of formal sum* of an oriented manifold $\mathbf{M}$ is the sum of inverse of submanifolds $\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_n$, denoted by "$-$":

$$-\mathbf{M} = -\mathbf{m}_1 - \mathbf{m}_2 - ... - \mathbf{m}_n$$

Definition 25 (Closure)**.** The *closure* of a subset $\mathbf{S}$ of points in a topological space consists of all points in $\mathbf{S}$ together with all limit points of $\mathbf{S}$, denoted by

$$\bar{\mathbf{S}}$$

Definition 26 (Interior)**.** The *interior* of a subset $\mathbf{S}$ of a topological space $\mathbf{X}$ is the union of all subsets of $\mathbf{S}$ that are open in $\mathbf{X}$, denoted by

$$\mathbf{S}^\circ$$

Definition 27 (Boundary and Boundary Operator)**.** The boundary of a subset $\mathbf{S}$ of a topological space $\mathbf{X}$ is the *closure* of $\mathbf{S}$ minus the interior of $\mathbf{S}$:

$$\partial\mathbf{S} := \bar{\mathbf{S}} \setminus \mathbf{S}^\circ$$

We also use $\partial_k\mathbf{\Sigma}$ to indicate that the boundary operator actions on a k-manifold $\mathbf{\Sigma}$.

Example 28 (Boundary of Surface or Loop)**.** For oriented surface, the boundary (loop) is positively oriented "as one walks along boundary on outside surface while cliff on your right". For oriented curve, the boundary are two end-points that the target point is positively oriented and the source point is negatively oriented. We use $\mathbf{0}$ to denote "nothing in space", e.g. the boundary of a sphere or a loop.

Up to this point, to form a group of k-manifolds, we already have:

- commutative binary operation, the *formal sum* "+"
- identity unit element, $\mathbf{0}$, "nothing in space"
- inverse of an element, which is its negatively oriented version.

We need one more thing, the equivalence class, to reveal topological invariant.

Definition 29 (Homology)**.** Let $\mathbf{S}$ be a k-manifold. Let γ_0 and γ_1 be two $(k-1)$-manifolds. A *homology relation* connecting γ_0 and γ_1 is a k-submanifold $\mathbf{\Sigma}$ such that:

$$\partial_k\mathbf{\Sigma} = \gamma_0 - \gamma_1$$

We say γ_0 is homological to γ_1 if there exists homology between them, denoted as $\gamma_0 \sim \gamma_1$ [8].

[8] Notice that Homotopy $\overset{\nLeftarrow}{\Rightarrow}$ Homology. To illustrate homology relation is an equivalence relation:

- (reflexive) $\gamma \sim \gamma$ since $\mathbf{0} = \gamma - \gamma$ trivially holds
- (symmetric) if $\gamma_0 \sim \gamma_1$, then $\partial_2\mathbf{\Sigma} = \gamma_0 - \gamma_1$, then $\partial_2(\mathbf{S} \setminus \mathbf{\Sigma}) = \gamma_1 - \gamma_0$, then $\gamma_1 \sim \gamma_0$
- (transitive) if $\gamma_0 \sim \gamma_1$ and $\gamma_1 \sim \gamma_2$, suppose $\partial_2\mathbf{\Sigma}_1 = \gamma_0 - \gamma_1$ and $\partial_2\mathbf{\Sigma}_2 = \gamma_1 - \gamma_2$, then $\partial_2\mathbf{\Sigma}_1 + \partial_2\mathbf{\Sigma}_2 = \partial_2(\mathbf{\Sigma}_1 + \mathbf{\Sigma}_2) = \gamma_0 - \gamma_2$ then $\gamma_0 \sim \gamma_2$.

Definition 30 (First Homology Group)**.** Given a surface topological space $\mathbf{S}$. Homology relation is an equivalence relation. The set of all the loops and finite formal sum of them is Γ, which can be classified by homology relation and form a set of all the homology classes, denoted as $\Gamma/\sim$. To define a group:

- The homology class of a loop γ, denoted by $[\gamma]$, becomes group generator.
- The group binary operation is defined as

$$[\gamma_1] + [\gamma_2] := [\gamma_1 + \gamma_2]$$

  which is commutative

- The group unit element is defined as $\mathbf{0}$, which is "nothing in space".
- The group inverse element is defined as

$$[\gamma]^{-1} = -[\gamma] := [-\gamma]$$

then $\Gamma/\sim$ forms a group, so-called the first homology group, denoted as $H_1(\mathbf{S}, \mathbb{Z})$, if formal sum is over $\mathbb{Z}$[9]

Example 31 (Homology of loops)**.** See figure 5, $\mathbf{S}$ is a closed orientable surface with genus $g = 3$. We have

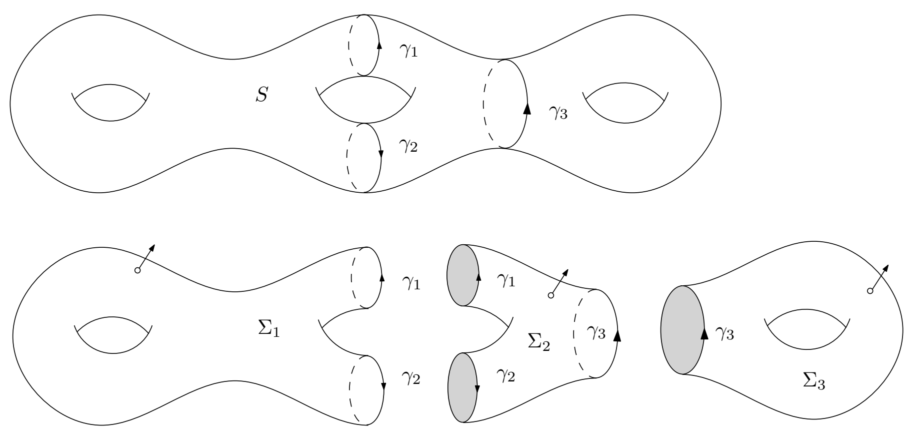

Figure 5: an example of homology relation

formal sum:

$$\mathbf{S} = \mathbf{\Sigma}_1 + \mathbf{\Sigma}_2 + \mathbf{\Sigma}_3$$

What can we say about the homological class about γ_1, γ_2 and γ_3? We can see that

$$\gamma_1 \sim \gamma_2 \sim (\gamma_2 + \gamma_3) \sim (\gamma_1 - \gamma_3)$$

$$\gamma_3 \sim (\gamma_1 - \gamma_2) \sim \mathbf{0}$$

[9]if formal sum over $\mathbb{Z}_2$, for example, then $[\gamma] + [\gamma] = \mathbf{0}$, we denoted first homology group as $H_1(\mathbf{S}, \mathbb{Z}_2)$

since

$$\begin{gathered}
\partial_2 \mathbf{\Sigma}_1 = \gamma_1 - \gamma_2 \\
\partial_2 \mathbf{\Sigma}_2 = (\gamma_2 + \gamma_3) - \gamma_1 = \gamma_3 - (\gamma_1 - \gamma_2) = \gamma_2 - (\gamma_1 - \gamma_3) \\
\partial_2 \mathbf{\Sigma}_3 = \mathbf{0} - \gamma_3
\end{gathered}$$

5 Homology Group

Kernel is the generalization of zeros of a function. Image is the generalization of range of a function.

Definition 32 (Kernel)**.** Let G and H be groups and let $f : G \mapsto H$ be a homomorphism. Let e_H denote the identity unit element in H. The *kernel* of f is defined as

$$\ker f = \{g \in G \mid f(g) = e_H\}$$

Definition 33 (Cycle Group)**.** Any closed cycle (or finite formal sum of cycles) γ on a closed oriented surface $\mathbf{S}$ will satisfy:

$$\partial_1 \gamma = \mathbf{0}$$

We denote *cycle group* $Z_1(\mathbf{S})$ as

$$Z_1(\mathbf{S}) = \ker \partial_1 = \{\gamma \in \mathbf{S} \mid \partial_1(\gamma) = \mathbf{0}\}$$

Definition 34 (Image)**.** Let G and H be groups and let $f : G \mapsto H$ be a homomorphism. The *image* of f is defined as

$$\text{img } f = \{h \in H \mid \exists g \in G \; s.t. \; f(g) = h\}$$

Definition 35 (Boundary Group)**.** Any submanifold $\mathbf{\Sigma}$ on a closed oriented surface $\mathbf{S}$ will induce a closed boundary γ:

$$\partial_2 \mathbf{\Sigma} = \gamma$$

We denote *boundary group* $B_1(\mathbf{S})$ as

$$B_1(\mathbf{S}) = \text{img } \partial_2 = \{\gamma \in \mathbf{S} \mid \exists \mathbf{\Sigma} \in \mathbf{S} \; s.t. \; \partial_2 \mathbf{\Sigma} = \gamma\}$$

Definition 36 (Homology Group Structure)**.** The first homology group of $\mathbf{S}$ is the quotient group

$$H_1(\mathbf{S}, \mathbb{Z}) = \frac{Z_1(\mathbf{S})}{B_1(\mathbf{S})} = \frac{\ker \partial_1}{\text{img } \partial_2}$$

Which is consistent with homology relation, as we collapse $B_1(\mathbf{S})$ as identity (all $\partial_2 \mathbf{\Sigma}$ now become $\mathbf{0}$)

Generally, given $(k+1)$-manifold $\mathbf{M}$, k homology group is

$$H_k(\mathbf{M}, \mathbb{Z}) = \frac{Z_k(\mathbf{M})}{B_k(\mathbf{M})} = \frac{\ker \partial_k}{\text{img } \partial_{k+1}}$$

Figure 6 illustrates the relationship between groups:

- C_k, which is group of all k-submanifold
- Z_k, which is group of kernel of ∂_k on k-submanifold
- B_k, which is group of image of ∂_{k+1} on $(k+1)$-submanifold

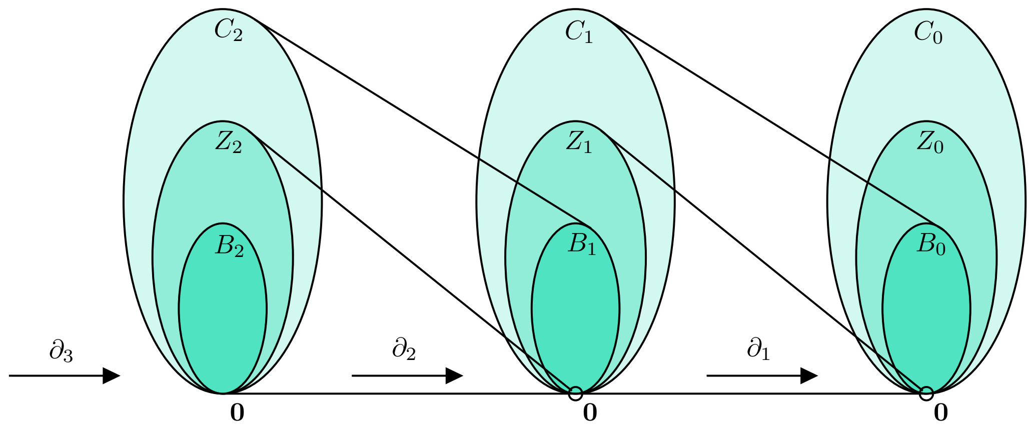

Figure 6: The relation of groups C_k, Z_k and B_k

Theorem 37. *Suppose* $\mathbf{S}$ *is a genus* g *closed surface, then*

$$H_0(\mathbf{S},\mathbb{Z}) \cong \mathbb{Z} \cong H_2(\mathbf{S},\mathbb{Z})$$

$$H_1(\mathbf{S},\mathbb{Z}) \cong \mathbb{Z}^{2g}$$

and generally for a n *dimensional closed manifold* $\mathbf{M}$*,*

$$H_k(\mathbf{M},\mathbb{Z}) \cong H_{n-k}(\mathbf{M},\mathbb{Z})$$

which implied by **Poincaré Duality**, that we will cover in next lecture.