Computational Conformal Geometry	**授课日期: 2020年11月**

Lecture Note 4: Maps between Topological Spaces

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Definition 1 (Continuous Map)**.** a *continuous map* is a continuous function between two topological spaces, denoted as, e.g.

$$f : \mathbf{M} \to \mathbf{N}$$

Definition 2 (Simplicial Map)**.** A *simplicial map* between simplicial complexes K and L is a function

$$\varphi : \mathrm{Vert}(K) \to \mathrm{Vert}(L)$$

from the vertex set of K to that of L sucht that whenever $v_0, v_1, ..., v_q$ span a q-simplex of K, $\varphi(v_0), \varphi(v_1), ..., \varphi(v_q)$ span a p-simplex ($p \leq q$) of L. Of course, repetition among $\varphi(v_0), \varphi(v_1), ..., \varphi(v_q)$ are allowed.

Note that the simplicial map φ can be regarded as a function from K to L: this function sends a simplex σ of K with vertices $v_0, v_1, ..., v_q$ to the simplex $\varphi(\sigma)$ of L spanned by vertices $\varphi(v_0), \varphi(v_1), ..., \varphi(v_q)$, so we also write φ as

$$\varphi : K \to L$$

Note that the simplicial map φ also induces a continuous map

$$\varphi : |K| \to |L|$$

between the polyhedra of K and L, denoted as $|K|$ and $|L|$, where a point inside polyhedron $|K|$ spanned by vertex $v_0, v_1, ..., v_q$ is sent to a point inside polyhedron $|L|$ continuously by

$$\varphi\left(\sum_{j=0}^{q} t_j v_j\right) = \sum_{j=0}^{q} t_j \varphi(v_j) \quad \text{whenever} \quad 0 \leq t_j \leq 1 \quad \text{for} \quad j = 0, 1, ..., q \quad \text{and} \quad \sum_{j=0}^{q} t_j = 1$$

As a closing remark, there are thus three equivalent ways of describing a simplicial map:

1. as a function between the vertex sets of two simplicial complexes, e.g. $\varphi : \mathrm{Vert}(K) \to \mathrm{Vert}(L)$
2. as a function from one simplicial complex to another, e.g. $\varphi : K \to L$
3. as a continuous map between the polyhedra of two simplicial complexes, e.g. $\varphi : |K| \to |L|$

We shall describe a simplicial map using the representation that is most appropriate in the given context.

1 Simplicial Approximation Theorem

You may have experience with *Minecraft* game or *Lego* toy. It seems obvious that any real world object can be represented by a labeled discretized lattice. The mathematical theorem to behind is *simplicial approximation* *theorem*, which is a foundational result for algebraic topology, guaranteeing that given an embedded mesh, a continuous manifold can be (by a slight deformation) approximated by a simplicial complex of the simplest kind.

The manifold $\mathbf{M}$ embedded by a given simplicial complex L, was described by a continuous map from a parametric simplicial complex K, with its space of parameter denoted by $|K|$, to the space of polyhedron of simplicial complex L, denoted by $|L|$:

$$\mathbf{M} : |K| \to |L|$$

Example 3 (manifold embedded by simplicial complex)**.** See figure 1, manifold $\mathbf{M}$ was parametrized by a continuous map from space of a simplicial complex K, denoted by $|K|$, to the polyhedron of another simplicial complex L, the embedded space of manifold $\mathbf{M}$, denoted by $|L|$

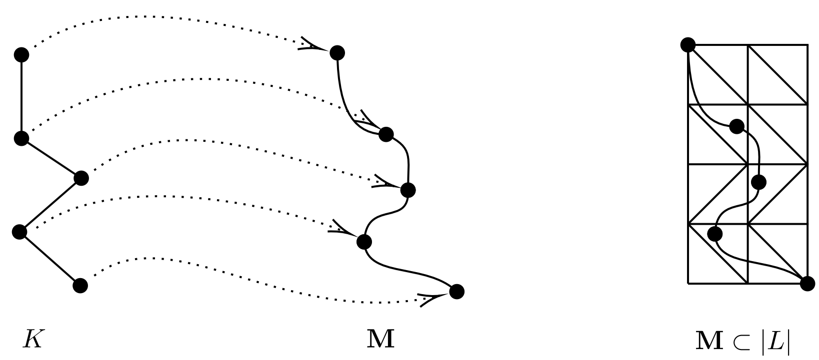

Figure 1: manifold $\mathbf{M}$ was represented by a continuous map: $\mathbf{M} : |K| \to |L|$

Definition 4 (Star of a Vertex, "discrete version of neighbor of a point")**.** Let K be a simplicial complex and let $p \in \mathrm{Vert}(K)$. Then the *star* of p, denoted by $\mathrm{st}(p)$, is defined by

$$\mathrm{st}(p) = \bigcup s^\circ \subset |K| \quad \text{where simplex} \quad s \in K \quad \text{such that} \quad p \in \mathrm{Vert}(s)$$

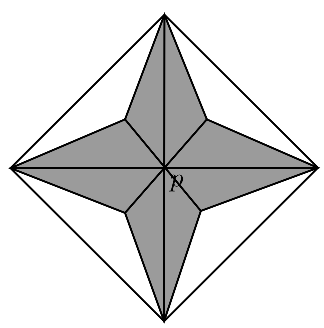

Figure 2: star of p, denoted by $\mathrm{st}(p)$

Example 5 (Star of a Vertex)**.** as shown in figure 2, $\mathrm{st}(p)$ consists of the open shaded region, all the open simplexes of which p is a neighbor.

Definition 6 (Simplicial Approximation)**.** a manifold $\mathbf{M}$, parametrized by simplicial complex K, embedded in simplicial complex L, represented by a continuous map

$$\mathbf{M} : |K| \to |L|$$

Its meshing approximation candidate $\mathbf{M}_\Delta$, parametrized by simplicial complex K, embedded in simplicial complex L, represented by a simplicial map

$$\mathbf{M}_\Delta : K \to L$$

Then $\mathbf{M}_\Delta$ is *simplicial approximation* to $\mathbf{M}$ if, for every vertex p of K,

$$\mathbf{M}(\mathrm{st}(p)) \subset \mathrm{st}(\mathbf{M}_\Delta(p))$$

which means $\mathbf{M}$ carries neighboring simplexes of p inside the union of the simplexes near $\mathbf{M}_\Delta(p)$. $\mathbf{M}_\Delta$ and $\mathbf{M}$ are close up to a meshing unit.

Example 7 (Simplicial Approximation)**.** See figure 3, $\mathbf{M}_\Delta$ is an simplicial approximation to $\mathbf{M}$.

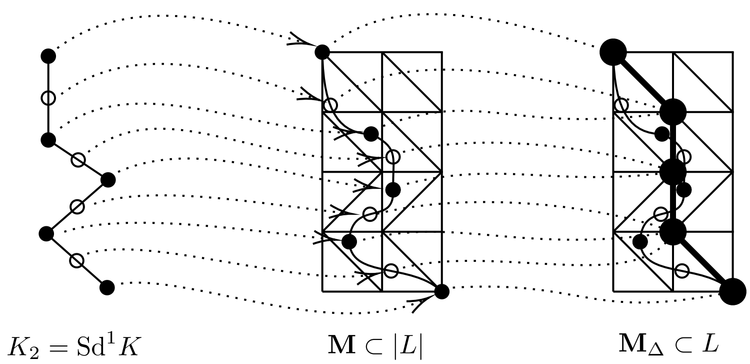

Figure 3: $\mathbf{M}_\Delta$, illustrated by biggest nodes and bolded edges on right side, is the simplest approximation to $\mathbf{M}$, achieved by first-order barycentric subdivision of K, denoted by $\mathrm{Sd}^1 K$. The simplicial approximation theorem guarantees that a simplest approximation in a given embedding mesh will be achieved by sufficient iterations of barycentric subdivision. (barycentric subdivision and simplicial approximation theorem will be explained right away)

Definition 8 (Barycentric Subdivision)**.** If s is a simplex, let b^s denote its barycenter. If K is a simplicial complex, define Sd K, the *barycentric subdivision* of K, to be the simplicial complex with

$$\mathrm{Vert}(\mathrm{Sd}\ K) = \{b^s : s \in K\}$$

note that here $s \in K$ are simplex of all dimensions in K. Recall that if s is a 0-simplex then trivially $b^s = s$; if s is a 1-simplex then b^s is the central point of two vertices; and so on. The q times iteration of barycentric subdivision is denoted by

$$\mathrm{Sd}^q K$$

Example 9 (Barycentric Subdivision)**.** If simplex $\sigma = [p_0, p_1, p_2]$, then $\mathrm{Vert}(\mathrm{Sd}\ \sigma) = \{p_0, p_1, p_2, b_0, b_1, b_2, b^\sigma\}$. See figure 4

Theorem 10 (Simplicial Approximation Theorem)**.** *Given simplicial complexes K and L. A smooth manifold $\boldsymbol{M}$, parametrized by K, embedded in $|L|$ (the polyhedron of L), represented by*

$$\boldsymbol{M} : |K| \to |L|$$

must have a simplicial approximation, and could be found its simplest kind after some barycentric subdivision by

$$\boldsymbol{M}_\Delta : Sd^q K \to L \quad \textit{where} \quad q \geq 1$$

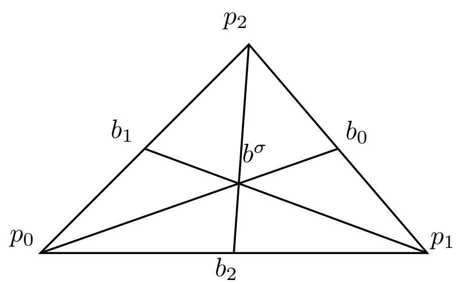

Figure 4: first order barycentric subdivision of the simplex σ

Simplicial approximation theorem was the foundation of modern movie industry and game industry, since it provide a theoretical guarantee that the simplest discrete digital approximation of any smooth-shaped object exists.

Example 11 (Simplicial Approximation Theorem)**.** See figure 1, one cannot construct a simplicial approximation to $\mathbf{M}$ by its mapping parameter K, but one can do so by its first-order barycentric subdivision K_2, as shown in figure 3

2 Chern-Gauss-Bonnet Theorem

Definition 12 (Induced Maps)**.** Algebraic topology constructs functor

$$\mathfrak{C}_1 \to \mathfrak{C}_2$$

between $\mathfrak{C}_1 = \{\text{Topological Spaces, Homeomorphisms}\}$ and $\mathfrak{C}_2 = \{\text{Groups, Homomorphisms}\}$. Therefore, a continuous map $f : \mathbf{M} \to \mathbf{N}$, where $\mathbf{M}$ and $\mathbf{N}$ are two manifolds, naturally induces homomorphism. there basically two kinds of *induced map*:

- *push-forward map*, denoted as $f_\#$ if on homotopy, and denoted as f_* if on homology.

  $f_\#$ maps between fundamental groups[1]:

  $$f_\# : \pi_1(\mathbf{M}) \to \pi_1(\mathbf{N})$$

  f_* maps between homology groups:

  $$f_* : H_p(\mathbf{M}) \to H_p(\mathbf{N})$$

- *push-forward map*, denoted as f^* if it maps between cohomology groups[2]:

  $$f^* : H^p(\mathbf{N}) \to H^p(\mathbf{M})$$

[1] $f_\#$ takes "curves" to "curves":

$$f_\# : C_p(\mathbf{M}) \to C_p(\mathbf{N})$$

$f_\#$ takes "cycles" to "cycles":

$$f_\# : Z_p(\mathbf{M}) \to Z_p(\mathbf{N})$$

$f_\#$ takes "boundaries" to "boundaries":

$$f_\# : B_p(\mathbf{M}) \to B_p(\mathbf{N})$$

[2] here we explain why push forward and pull back are opposite direction but both natural. Points are sent forward. Given $p \in \mathbf{M}$ we have $f(p) \in \mathbf{N}$. Functions are sent back, i.e. pull back from $\mathbf{N}$ to $\mathbf{M}$. If we have a function $\omega : \mathbf{N} \to \mathbb{R}$ then we get the composition $\omega \circ f : \mathbf{M} \to \mathbb{R}$. So the pull back can be consider a functional map, which maps from function on $\mathbf{N}$ to function on $\mathbf{M}$

Example 13 (Induced Maps of Surface)**.** Suppose $\mathbf{M}$ and $\mathbf{N}$ are two closed surfaces, a continuous map:

$$f : \mathbf{M} \to \mathbf{N}$$

induces a push-forward map on first homology:

$$f_* : H_1(\mathbf{M}) \to H_1(\mathbf{N})$$

and a pull-back map on first cohomology:

$$f^* : H^1(\mathbf{N}) \to H^1(\mathbf{M})$$

Suppose a curve $\sigma \in C_1(\mathbf{M}) \subset H_1(\mathbf{M})$ and a vector field $\omega \in C^1(\mathbf{N}) \subset H^1(\mathbf{M})$, then

$$\omega[f_*(\sigma)] = [f^*(\omega)](\sigma)$$

Definition 14 (Degree of a Map)**.** Suppose $\mathbf{M}$ and $\mathbf{N}$ are two closed surfaces, a continuous map:

$$f : \mathbf{M} \to \mathbf{N}$$

then the *degree of map* is the algebraic number[3] of pre-image $f^{-1}(q)$ for arbitrary point $q \in \mathbf{N}$, denoted as $\deg(f)$, which is independent of the choice of the point q. An example see figure 5

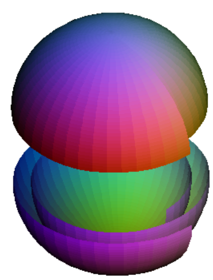

Figure 5: a continuous map $f : \mathbf{M} \to \mathbf{M}$ from a sphere to itself but in 2x speed. For every point $q \in \mathbf{M}$, there are two pre-image, so $\deg(f) = 2$

Example 15 (Degree of a Map)**.** Suppose $\mathbf{M}$ and $\mathbf{N}$ are two closed surfaces, a continuous map:

$$f : \mathbf{M} \to \mathbf{N}$$

induces a push-forward map on second homology:

$$f_* : H_2(\mathbf{M}) \to H_2(\mathbf{N})$$

since $H_2(\mathbf{M}, \mathbb{Z}) \cong \mathbb{Z} \cong H_2(\mathbf{N}, \mathbb{Z})$, we also write its isomorphism:

$$\tilde{f}_* : \mathbb{Z} \to \mathbb{Z}$$

and it must have the form

$$\tilde{f}_*(z) = \deg(f) \cdot z$$

[3]if Jacobian at that point q is positive then count +1, else then count -1, then degree of map is sum of total count

Definition 16 (Euler-Poincaré Characteristic)**.** let g be the genus of a closed surface $\mathbf{S}$, then *Euler characteristic*, denoted as $\chi(\mathbf{S})$, is

$$\chi(\mathbf{S}) = 2(1-g)$$

the discrete version, if $\mathbf{S}$ triangulated in $\mathbf{S}_\Delta$, is

$$\chi(\mathbf{S}_\Delta) = |\text{Faces}| - |\text{Edges}| + |\text{Vertices}|$$

Definition 17 (Gaussian Curvature)**.** At any point on a surface, we can find a normal vector that is at right angles to the surface; planes containing the normal vector are called normal planes. The intersection of a normal plane and the surface will form a curve called a normal section and the curvature of this curve is the normal curvature. For most points on most surfaces, different normal sections will have different curvatures; the maximum and minimum values of these are called the principal curvatures, call these κ_1, κ_2. The *Gaussian curvature* is the product of the two principal curvatures $K = \kappa_1 \cdot \kappa_2$, as shown in figure 6

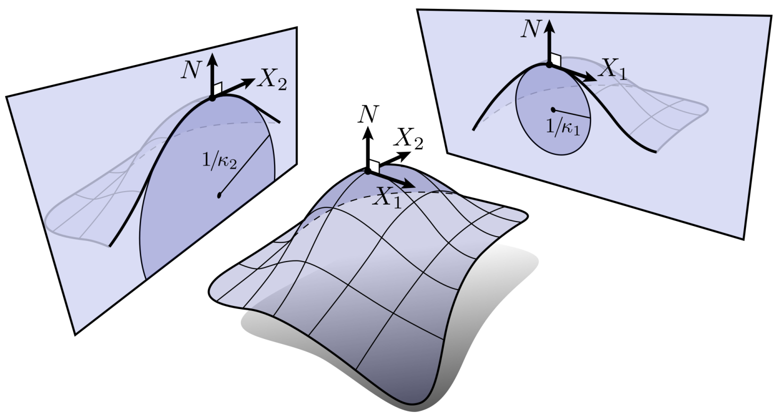

Figure 6: Gaussian curvature is the product of κ_1, κ_2

Theorem 18 (Chern-Gauss-Bonnet Theorem)**.** *Let $\boldsymbol{S}$ be a closed surface, $K(p)$ the Gaussian curvature at point p on surface, and $dA(p)$ the element area at point p on surface, then its total Gaussian curvature*

$$\int_{\boldsymbol{S}} K(p) dA(p) = 2\pi\chi(\boldsymbol{S})$$

Shiing-Shen Chern provided simple intrinsic proof of Gauss-Bonnet Theorem, which added his name to Gauss-Bonnet. We illustrate his beautiful proof by applying degree of Gauss map and homotopy between surfaces.

proof. consider the Gauss Map $G : \mathbf{S}^* \to \mathbb{S}^2$ from a canonical closed surface $\mathbf{S}^*$ to unit sphere $\mathbb{S}^2$. Whenever a point p on surface with normal $\mathbf{n}(p)$, the Gauss Map maps it to a point $G(p)$ on unit sphere with the same normal $\mathbf{n}(p)$.

Note that

$$\deg(G) = 1 - g$$

so the total area of the image of $\mathbf{S}^*$ on unit sphere $\mathbb{S}^2$ is

$$\text{Area}(\mathbb{S}^2) \times \deg(G) = 4\pi \deg(G) = 4\pi(1-g) = 2\pi\chi(\mathbf{S}^*)$$

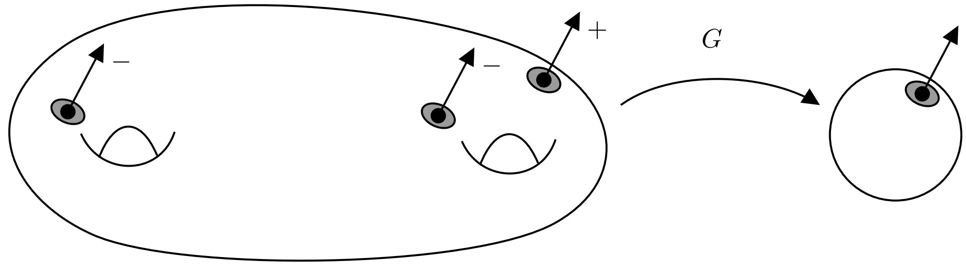

Figure 7: The canonical shape of genus g closed surface $\mathbf{S}^*$ can guarantee $\deg(G) = 1 - g$ since the count of pre-image is strictly negative whenever a hole appears.

Note that the total area of the image of $\mathbf{S}^*$ on unit sphere $\mathbb{S}^2$ also equals to

$$\int_{\mathbf{S}^*} \frac{\text{Area}(G(p))}{\text{Area}(p)} dA(p)$$

which equals to, since it is a Gauss Map[4]:

$$\int_{\mathbf{S}^*} \frac{\text{Area}(G(p))}{\text{Area}(p)} dA(p) = \int_{\mathbf{S}^*} K(p) dA(p)$$

thus we get, for a canonical closed surface $\mathbf{S}^*$, the identity

$$\int_{\mathbf{S}^*} K(p) dA(p) = 2\pi\chi(\mathbf{S}^*)$$

Now consider the quantity

$$\frac{\int_{\mathbf{S}^*} K(p) dA(p)}{2\pi} = \chi(\mathbf{S}^*) \in \mathbb{Z}$$

is an integer, which should not change under continuous deformation from canonical shaped $\mathbf{S}^*$ to any closed surface $\mathbf{S}$ with same genus g, thus we get for any closed surface $\mathbf{S}$

$$\int_{\mathbf{S}} K(p) dA(p) = 2\pi\chi(\mathbf{S})$$

□

Example 19 (Chern-Gauss-Bonnet Theorem)**.** let $\mathbf{S}$ be a sphere with radius R, its genus $g = 0$, then $\chi(\mathbf{S}) = 2 \times (1 - 0) = 2$, its Gaussian curvature is constant $\frac{1}{R^2}$, according to Chern-Gauss-Bonnet formula

$$\int_{\mathbf{S}} \frac{1}{R^2} dA(p) = \frac{1}{R^2} \int_{\mathbf{S}} dA(p) = \frac{1}{R^2} \times \text{Area}(\mathbf{S}) = 2\pi\chi(\mathbf{S}) = 4\pi$$

[4] for Gauss Map, when shrinking a patch around point p, its limit is Gaussian curvature:

$$\lim_{\Omega_p \to 0} \frac{\text{Area}(G(\Omega_p))}{\text{Area}(\Omega_p)} = K(p)$$

indeed Area($\mathbf{S}$) $= 4\pi R^2$.

3 Fixed Point Theorem

Definition 20 (Inclusion Map)**.** an *inclusion map* i from A to B, where $A \subset B$, satisfies that for any element $x \in A$ we have $i(x) = x$, denoted as

$$i : A \hookrightarrow B$$

Theorem 21 (Brouwer's Fixed Point Theorem)**.** *Suppose* $\Omega \subset \mathbb{R}^n$ *is a compact convex set,* $f : \Omega \to \Omega$ *is a continuous map, then there exists a point* $p \in \Omega$ *such that*

$$f(p) = p$$

proof. Assume $f : \Omega \to \Omega$ has no fixed point, namely

$$\forall p \in \Omega, \quad f(p) \neq p$$

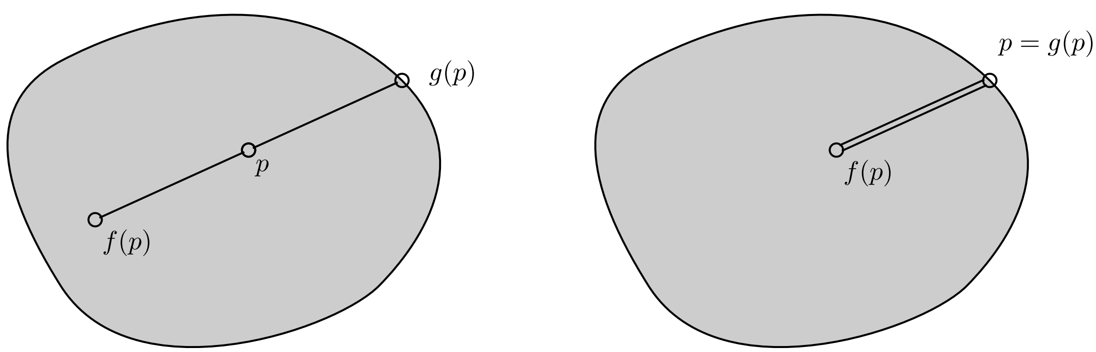

Figure 8: diagram of $g : \Omega \to \partial\Omega$ (left) and $(g \circ i) : \partial\Omega \to \partial\Omega$ (right)

We can construct $g : \Omega \to \partial\Omega$, a ray starting from $f(p)$ through p and intersect $\partial\Omega$ at $g(p)$. Because our assumption $f(p) \neq p$ and Ω is convex, g is well-defined. Note that if point $p \in \partial\Omega$ then $g(p) = p$, as shown in figure 8. We construct an inclusion map $i : \partial\Omega \hookrightarrow \Omega$, which maps a point $p \in \partial\Omega$ to itself. Then we compose it with g,

$$\partial\Omega \xhookrightarrow{i} \Omega \xrightarrow{g} \partial\Omega$$

we get an identity map:

$$(g \circ i) : \partial\Omega \to \partial\Omega$$

which induces a push-forward map on $(n-1)^{th}$ homology:

$$(g \circ i)_* : H_{n-1}(\partial\Omega, \mathbb{Z}) \to H_{n-1}(\partial\Omega, \mathbb{Z})$$

since it is identity map,

$$(g \circ i)_* : z \mapsto z$$

$g : \Omega \to \partial\Omega$ induces a push-forward map on $(n-1)^{th}$ homology:

$$g_* : H_{n-1}(\Omega, \mathbb{Z}) \to H_{n-1}(\partial\Omega, \mathbb{Z})$$

however, since Ω is compact convex set

$$H_{n-1}(\Omega, \mathbb{Z}) = 0, \quad g_* = 0$$

so

$$(g \circ i)_* = g_* \circ i_* = 0$$

contradiction! $f : \Omega \to \Omega$ has fixed point. $\square$

In 1910, Luitzen Egbertus Jan Brouwer proved his fixed point theorem, which ensured the existence of fixed point of a continuous self-map of convex compact space. Often, it can be stated as follow:

Theorem 22 ("Swirling Coffee" Theorem)**.** *Use a stick (volume can be ignored) to swirl a cup of coffee without making any bubble. In the end, there is a molecule with final position the same as initial position in your coffee.*

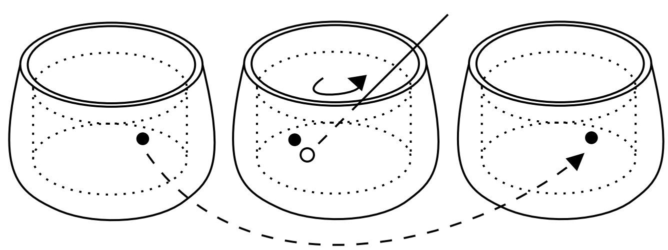

Figure 9: "swirling coffee" theorem: at least one molecule, "doesn't move" before and after coffee swirling

In 1926, Solomon Lefschetz gave a formula that relates the number of fixed points of a map to the induced push-forward maps on homology.

Definition 23 (Index of Fixed Point)**.** Suppose $\mathbf{M}$ is an n-dimensional topological space, p is a fixed point of self-map $f : \mathbf{M} \to \mathbf{M}$. Choose a neighborhood $\mathbf{U}$ such that $p \in \mathbf{U} \subset \mathbf{M}$, consider the boundary of $\mathbf{U}$, which is a $(n-1)$-dimensional $\partial\mathbf{U}$. Similar to the concept "degree of a map" (see example 15), the induced push-forward map on $(n-1)^{th}$ homology:

$$f_* : H_{n-1}(\partial\mathbf{U}, \mathbb{Z}) \to H_{n-1}(\partial\mathbf{U}, \mathbb{Z})$$

is $f_* : \mathbb{Z} \to \mathbb{Z}$ and must have the form $f_* : z \mapsto \lambda z$, where λ is an integer called the *algebraic index of fixed point* p of map f, denoted as

$$\mathrm{Ind}(f, p) = \lambda$$

Definition 24 (Trace of Self-map)**.** Let $\mathbf{A}$ be a matrix representing a self-map $f : \mathbf{M} \to \mathbf{M}$ under any basis, then the *trace* of f, denoted as

$$\mathrm{Tr}(f)$$

is $\mathrm{Tr}(\mathbf{A})$, the trace of $\mathbf{A}$, which is independent of choice of basis.

Definition 25 (Lefschetz-Hopf Fixed Point Formula)**.** Given compact topological space $\mathbf{M}$. The sum of indexes of all fixed points of a self-map $f : \mathbf{M} \to \mathbf{M}$ equals to the alternating sum of trace of push-forward map on k^{th} homology $f_{*k} : H_k(\mathbf{M}, \mathbb{Z}) \to H_k(\mathbf{M}, \mathbb{Z})$ induced by the self-map f

$$\sum_{p \in \mathrm{Fix}(f)} \mathrm{Ind}(f, p) = \sum_k (-1)^k \mathrm{Tr}(f_{*k}) =: \Lambda(f)$$

where $\Lambda(f)$ is called *Lefschetz number*

Example 26 (Lefschetz-Hopf Fixed Point Formula)**.** consider a simple self-map $f : [0, 1] \to [0, 1]$. we have

$$\Lambda(f) = \underbrace{\mathrm{Tr}(f_{*0} : \mathbb{Z} \to \mathbb{Z})}_{1} - \underbrace{\mathrm{Tr}(f_{*1} : 0 \to 0)}_{0} = 1 = \sum_{p \in \mathrm{Fix}(f)} \mathrm{Ind}(f, p)$$

as shown in figure 10

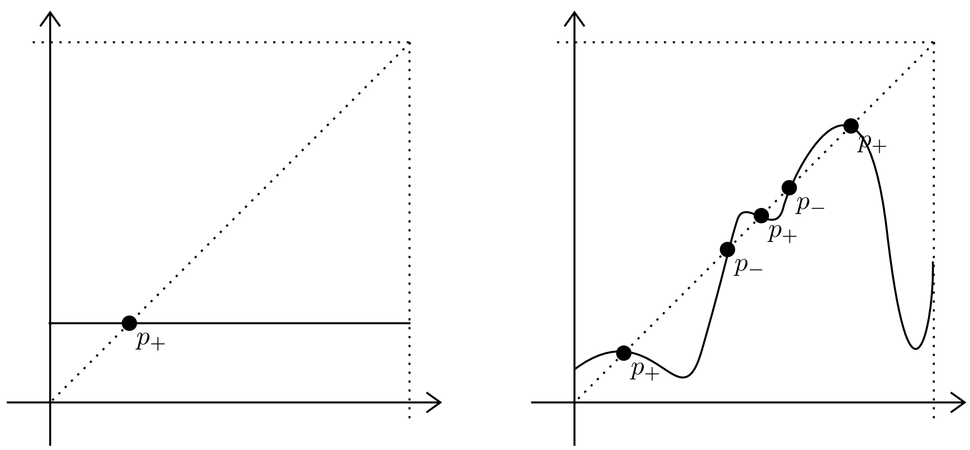

Figure 10: a self-map $f_1 : [0, 1] \to$ const (left) versus its homotopic self-map f_2 with same Lefshcetz number $\Lambda(f_1) = \Lambda(f_2) = 1$

Theorem 27 (Lefschetz's Fixed Point Theorem)**.** *Given a continuous self-map of a compact topological space $f : \boldsymbol{M} \to \boldsymbol{M}$, if its Lefschetz number $\Lambda(f) \neq 0$, then there is a point $p \in \boldsymbol{M}$ such that*

$$f(p) = p$$

Proof (Advanced). Notation update:

- f_k: the induced push-forward map on k-dimensional space
- $f_k \mid C_k$: the induced map on k-chain group
- $f_k \mid H_k$: the induced map on k-homology
- $\oplus$: direct sum between groups, e.g. $A \oplus B = \{(a + b) \mid a \in A, b \in B\}$

According to simplicial approximation theorem, there must be approximated maps up to any precision. So we triangulate $\mathbf{M}$ first, and assume its induced map f can be both embedded in chain space and smooth space, as shown in the commutative diagram as figure 11:

$$\begin{tikzcd}
\frac{C_k}{Z_k} \arrow[r, "f_k"] \arrow[d, shift right, "\partial_k"'] & \frac{C_k}{Z_k} \arrow[d, "\partial_k"'] \\
B_{k-1} \arrow[r, "f_{k-1}"] \arrow[u, dashed, shift right, "\partial_k^{-1}"'] & B_{k-1}
\end{tikzcd}$$

Figure 11: commutative diagram of induced map and boundary operator

we have

$$(f_{k-1} \mid B_{k-1}) = \partial_k \circ (f_k \mid \frac{C_k}{Z_k}) \circ \partial_k^{-1}$$

and thus

$$\begin{aligned}
\mathrm{Tr}(f_{k-1} \mid B_{k-1}) &= \mathrm{Tr}([\partial_k][f_k \mid \frac{C_k}{Z_k}][\partial_k^{-1}]) \\
&= \mathrm{Tr}([f_k \mid \frac{C_k}{Z_k}][\partial_k^{-1}][\partial_k]) \\
&= \mathrm{Tr}(f_k \mid \frac{C_k}{Z_k})
\end{aligned}$$

according to property of trace.

Let C_k be k-chain group, Z_k closed chain group, B_k exact chain group, H_k homology group. We have

$$C_k \cong \frac{C_k}{Z_k} \oplus Z_k \qquad \text{and} \qquad Z_k \cong B_k \oplus H_k$$

thus

$$\begin{aligned}
\mathrm{Tr}(f_k \mid C_k) &= \mathrm{Tr}(f_k \mid \frac{C_k}{Z_k} \oplus Z_k) \\
&= \mathrm{Tr}(f_k \mid \frac{C_k}{Z_k}) + \mathrm{Tr}(f_k \mid Z_k) \\
&= \mathrm{Tr}(f_{k-1} \mid B_{k-1}) + \mathrm{Tr}(f_k \mid B_k) + \mathrm{Tr}(f_k \mid H_k)
\end{aligned}$$

thus

$$\begin{aligned}
\sum_k (-1)^k \mathrm{Tr}(f_k \mid C_k) &= \sum_k (-1)^k \mathrm{Tr}(f_{k-1} \mid B_{k-1}) + \mathrm{Tr}(f_k \mid B_k) + \mathrm{Tr}(f_k \mid H_k) && (1) \\
&= \sum_k (-1)^k \mathrm{Tr}(f_k \mid H_k) && (2) \\
&= \Lambda(f) && (3)
\end{aligned}$$

according to Lefschetz-Hopf fixed point formula. Whenever $\Lambda(f) \neq 0$, there is an entry in a matrix such that $\mathrm{Tr}(f_k \mid C_k) \neq 0$, which means there is simplex $\sigma \in C_k$ such that $f_k(\sigma) \subset \sigma$, for any point in $|\sigma|$, the continuous map $f_k : |\sigma| \to |\sigma|$ must have a Brouwer's fixed point such that $f_k(p) = p$, which means

$$f(p) = p$$

□

Example 28 (Lefschetz Number, Betti Number and Euler-Poincaré Characteristic)**.** Consider an identity map of a closed surface

$$\mathrm{id} : \mathbf{S} \to \mathbf{S}$$

the identity map is, of course, a self-map. According to equations 1, 2 and 3, we have

$$\begin{aligned}
\Lambda(\mathrm{id}) &= \sum_k (-1)^k \mathrm{Tr}(\mathrm{id}_k \mid C_k) = \underbrace{\mathrm{Tr}(\mathrm{id}_2 \mid C_2)}_{|\text{Faces}|} - \underbrace{\mathrm{Tr}(\mathrm{id}_1 \mid C_1)}_{|\text{Edges}|} + \underbrace{\mathrm{Tr}(\mathrm{id}_0 \mid C_0)}_{|\text{Vertices}|} \\
&= \sum_k (-1)^k \mathrm{Tr}(\mathrm{id}_k \mid H_k) = \underbrace{\mathrm{Tr}(\mathrm{id}_2 \mid H_2)}_{b_2} - \underbrace{\mathrm{Tr}(\mathrm{id}_1 \mid H_1)}_{b_1} + \underbrace{\mathrm{Tr}(\mathrm{id}_0 \mid H_0)}_{b_0} \\
&= \chi(\mathbf{S})
\end{aligned}$$

Here we show that for an identity map, the connection between its Lefschetz number and Euler-Poincaré characteristic, and where Euler (number of triangulation element)and Poincaré (Betti number[5])coincide.

[5]geometrically, Betti number of surface can be understood as:

- b_0 is the number of connected components
- b_1 is the number of one-dimensional or "circular" holes
- b_2 is the number of two-dimensional "voids" or "cavities"