Category Theory **授课日期: 2020年10月**

Lecture Note 1: Shortest Path Revisit

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This new method to the Shortest Path problem serves two purposes:

- Apply the concept of order, preorder, and join learned in the previous chapter, to solve real life problems.
- What prerequisite a quantale have to meet, to form a V-category relation with a problem?

1 Finding the Shortest Path with Matrix Multiplication

We've all been through the struggling process of finding the shortest path on a weighted graph with numerous computing algorithms. The most important three of them, are Dijkstra's algorithm, Bellman-Ford algorithm, and A^* search algorithm.

Today, I will introduce a new method to find the shortest path. It is a operation on a relation called **V-categories**, by using a method that we are familiar with in linear algebra — **matrix multiplication**. Hopefully, It will inspire you to think about linear algebra from a different perspective, and whether linear algebra is potentially a footstone to many common problem we've confronted.

Now, here is an example :

Example 1 Given the weighted graph Y. We have two matrix derived from Y, M_Y and d_Y. M_Y is recording the length of paths that traverse either 0 or 1 edges. d_Y is the result matrix that displays the shortest path.

$Y :=$

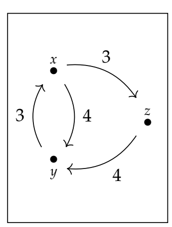


Figure 1: a weighted graph

Notice: If we can get from x to y in only one edge, we record its length in M_Y, else we use ∞.

To find all the shortest paths from every points to other points, we "multiply" the matrix by itself until the matrix stabilize such that the elements' value no longer change after multiplications. But be careful, the reason I use double quotation mark here is because, the multiplication rule is different from the usual matrix multiplication we've seen in linear algebra.

M_Y	x	y	z
x	0	4	3
y	3	0	∞
z	∞	4	0

d_Y	x	y	z
x	0	4	3
y	3	0	6
z	7	4	0

Table 1: matrix

In this case, the multiplication rule is based on a operation formula on a symmetric monoidal pre-order (SMP). The SMP instance here is **Cost** $= ([0, \infty], \geq, 0, +)$, which interpret (elements range, order, identity element, operation) respectively. So it means, the weight on the graph must be always between 0 and ∞. And the operation between elements in matrix multiplication is $+$.

If we draw it out, it looks like a order from 0 to infinity

$$0 \leq 2 \leq 3 \leq 4 \leq 5 \leq \text{.........} \leq \infty$$

Think of this order as a river. The water flows from the upstream(∞),to the downstream(0). And the multiplication rule is finding the downstream (aka. *Join*) of elements. For example, the join of (7 , 4) is 4, the join of (3 , ∞) is 3, the joint of (4 , 4) is 4.

For example, given two matrix multiplication:

$$\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix} \times \begin{bmatrix} \text{c} & \text{d} \\ \text{e} & \text{f} \end{bmatrix}$$

The result by conventional linear algebra method is shown below:

$$\begin{bmatrix} \text{ae + bg} & \text{af + bh} \\ \text{ce + dg} & \text{cf + dh} \end{bmatrix}$$

With the presented V-Category rule, we have:

$$\begin{bmatrix} \text{join(a + e , b + g)} & \text{join(a + f , b + h)} \\ \text{join(c + e , d + g)} & \text{join(c + f , d + h)} \end{bmatrix}$$

Applying the V-Category rule to M_Y in Table 1 until the result stabilized, we can get D_Y.

Notice: M_Y^n records the lengths of the shortest path traverse n edges or fewer. For example, M_Y^2 tells use the length of the shortest path traversing 2 edges or fewer. Similarly M_Y^3 tells us about the shortest path traversing 3 edges or fewer.

FYI, the rule for V-category multiplication is formally written as,$M : X \times Y \to V$ and N : Y × Z $\to V$. Their product is defined to be the matrix $(M * N) : X \times Z \to V$. we can use it as a general reference:

$$(M * N)(x, z) := \bigvee_{y \in Y} M(x, y) \otimes N(y, z)$$

Example 2 In the last example, we find the shortest path on a weighted graph with matrix multiplication, where the multiplication rule varies. We then add a monoidal preorder **Cost** with the matrix, to form a V-category. Lastly, we may use the V-category multiplication equation to conduct the matrix multiplication to find the shortest path.
Now, here is another example based on another symmetric monoidal pre-order. The SMPinstance here is **Bool** $(B, \leq, true, \wedge)$ t means, operation must include only either True or False. And the operation between elements in matrix multiplication is $\wedge$.
The order of Boolean is shown below, where F is the upstream and T is the downstream. The join of (F, T) is T, the join of (F, F) is F, and the join of (T, T) is T.

$$\mathrm{F} \longrightarrow \mathrm{T}$$

Given two boolean matrix, find their product with multiplication rule based on **Bool**:

$$\begin{bmatrix} \text{false} & \text{false} \\ \text{false} & \text{true} \\ \text{true} & \text{true} \end{bmatrix} \times \begin{bmatrix} \text{true} & \text{true} & \text{false} \\ \text{true} & \text{false} & \text{true} \end{bmatrix}$$

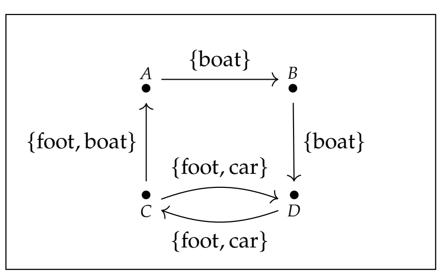
With the presented V-category rule, we have:

$$\begin{bmatrix} \text{join}(\mathrm{F} \wedge \mathrm{T}, \mathrm{F} \wedge \mathrm{T}) & \text{join}(\mathrm{F} \wedge \mathrm{T}, \mathrm{F} \wedge \mathrm{F}) & \text{join}(\mathrm{F} \wedge \mathrm{F}, \mathrm{F} \wedge \mathrm{T}) \\ \text{join}(\mathrm{F} \wedge \mathrm{T}, \mathrm{T} \wedge \mathrm{T}) & \text{join}(\mathrm{F} \wedge \mathrm{T}, \mathrm{T} \wedge \mathrm{F}) & \text{join}(\mathrm{F} \wedge \mathrm{F}, \mathrm{T} \wedge \mathrm{T}) \\ \text{join}(\mathrm{T} \wedge \mathrm{T}, \mathrm{T} \wedge \mathrm{T}) & \text{join}(\mathrm{T} \wedge \mathrm{T}, \mathrm{F} \wedge \mathrm{F}) & \text{join}(\mathrm{T} \wedge \mathrm{T}, \mathrm{T} \wedge \mathrm{T}) \end{bmatrix}$$

The result:

$$\begin{bmatrix} \text{false} & \text{false} & \text{false} \\ \text{true} & \text{false} & \text{true} \\ \text{true} & \text{true} & \text{true} \end{bmatrix}$$

Example 3 Let M be a set and let $\mathbf{M} = ([0, \infty], \geq, 0, +)$ be the monoidal preorder whose elements are subsets of M. X contains all the vertices. Draw a graph with four vertices and four or five edges, each labeled with a subset of $M =$ (car, boat, foot). The M-category X is all the modes of transportation that will get you from a to b, where a, b $\in X$.



Write out the corresponding four-by-four matrix of hom-objects, where hom-objects is defined as "for each path p from x to y take the intersection of the sets labelling the edges in p. Then, take the union of these sets over all paths p from x to y.

$\mathfrak{X}(\nearrow)$	A	B	C	D
A	M	$\{\text{boat}\}$	$\varnothing$	$\{\text{boat}\}$
B	$\varnothing$	M	$\varnothing$	$\{\text{boat}\}$
C	$\{\text{foot}, \text{boat}\}$	$\{\text{boat}\}$	M	M
D	$\{\text{foot}\}$	$\varnothing$	$\{\text{foot}, \text{car}\}$	M