Computational Conformal Geometry	**授课日期: 2020年11月**

Lecture Note 3: Topological Obstruction

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Poincaré-Hopf index theorem tells us that one cannot construct a smooth vector field over a sphere without zero point. Today we see this conclusion from another view.

1 Tangent Vector in Coordinate Chart

Historically, geometric techniques were developed mostly for Euclidean space. To study curved space, e.g. a manifold, we can construct local maps of open covers between manifold and Euclidean space.

Definition 1 (Smooth Manifold with Charts and Atlas)**.** A manifold is a topological space $\mathbf{M}$ covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ maps U_α to the Euclidean space $\mathbb{R}^n$. $(U_\alpha, \varphi_\alpha)$ is called a coordinate *chart* of $\mathbf{M}$. The set of all charts $\{(U_\alpha, \varphi_\alpha)\}$ form the *atlas* of $\mathbf{M}$. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a *transition map*. If all transition maps are smooth, namely

$$\varphi_{\alpha\beta} \in C^\infty(\mathbb{R}^n)$$

then the manifold is a differentiable (or differential) manifold or a *smooth manifold*, as shown in figure 1

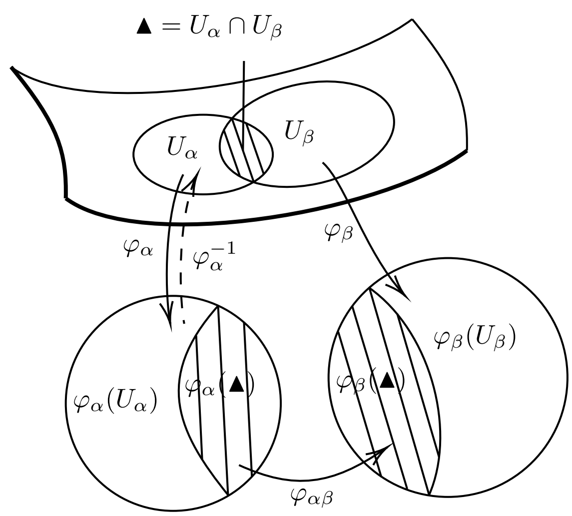

Figure 1: definition of smooth manifold was achieved by mapping it to Euclidean space patch by patch smoothly.

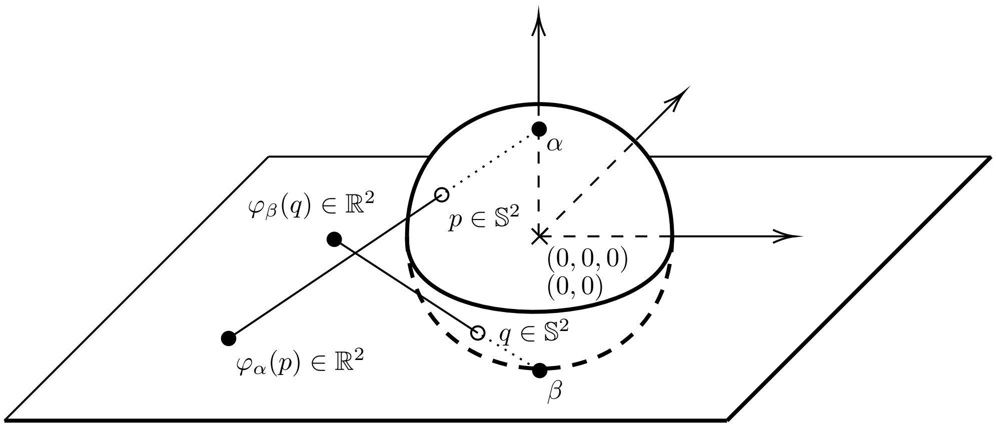

Figure 2: A unit sphere $\mathbb{S}^2 \in \mathbb{R}^3$ cannot be covered by only one chart, but can be covered by two charts, so-called stereo-graphic projection. The center of the sphere is $(0,0,0)$. We take its xy-plane as the image plane, containing the equator of sphere. The north pole α projects a point $p \in \mathbb{S}^2$ to the plane $\varphi_\alpha(p) \in \mathbb{R}^2$, and the south pole β projects a point $q \in \mathbb{S}^2$ to the plane $\varphi_\beta(q) \in \mathbb{R}^2$.

Example 2 (Stereo-graphic Projection)**.** A manifold can hardly be covered by only one coordinate chart, thus it usually needs to be covered by multiple charts. A basic example is so-called *stereo-graphic projection*.

As shown in figure 2, north pole $\alpha = (0,0,1)$, south pole $\beta = (0,0,-1)$, let $p = (x_1, x_2, x_3)$, $\varphi_\alpha(p) = (x, y)$, $\varphi_\alpha(q) = (u, v)$

$$
\begin{aligned}
\varphi_\alpha : (x_1, x_2, x_3) &\mapsto (\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}) \\
\varphi_\alpha^{-1} : (x, y) &\mapsto (\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2}) \\
\varphi_\beta : (x_1, x_2, x_3) &\mapsto (\frac{x_1}{1+x_3}, \frac{-x_2}{1+x_3}) \\
\varphi_\beta^{-1} : (u, v) &\mapsto (\frac{2u}{1+u^2+v^2}, \frac{-2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2})
\end{aligned}
$$

Note that indeed $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} \in C^\infty$, the unit sphere is a smooth manifold. Notice that φ_α cannot cover α and φ_β cannot cover β. You need both to cover the whole sphere.

As shown in figure 3, let $p \in \mathbb{S}^2$. Any vector $d\mathbf{r} \in T_p\mathbb{S}^2$ through φ_α can be represented by $d\mathbf{r} = \partial_x dx + \partial_y dy$, where

$$
\begin{aligned}
\partial_x &= \frac{\partial \mathbf{r}}{\partial x} = \frac{\partial \varphi_\alpha^{-1}(x,y)}{\partial x} = \frac{2}{(1+x^2+y^2)^2} \begin{bmatrix} 1-x^2+y^2 \\ -2xy \\ 2x \end{bmatrix} \\
\partial_y &= \frac{\partial \mathbf{r}}{\partial y} = \frac{\partial \varphi_\alpha^{-1}(x,y)}{\partial y} = \frac{2}{(1+x^2+y^2)^2} \begin{bmatrix} -2xy \\ 1+x^2-y^2 \\ 2y \end{bmatrix}
\end{aligned}
$$

and the inner product

$$
\begin{aligned}
\langle \partial_x, \partial_x \rangle = \langle \partial_y, \partial_y \rangle &= \frac{4}{(1+x^2+y^2)^2} \\
\langle \partial_x, \partial_y \rangle &= 0
\end{aligned}
$$

so interestingly the bases of $T_p\mathbb{S}^2$ derived from partial derivative are orthogonal with equal length.

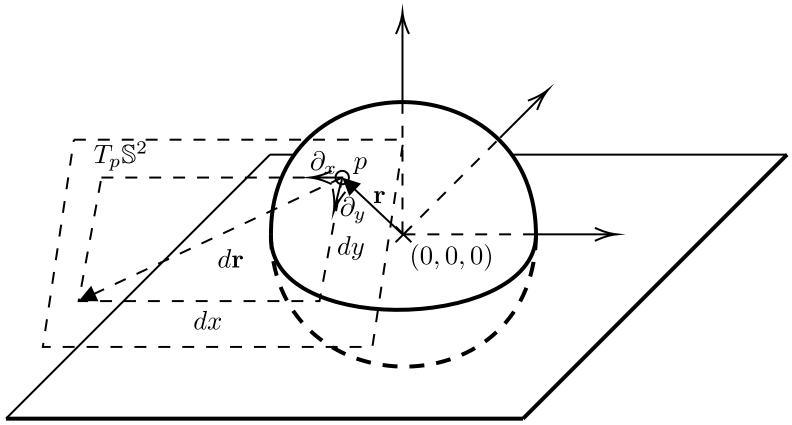

Figure 3: Let $p \in \mathbb{S}^2$. Let vector $\mathbf{r} = \varphi_\alpha^{-1}(x, y)$ parametrized by $(x, y) \in \mathbb{R}^2$, then any vector $d\mathbf{r}$ on $T_p\mathbb{S}^2$, the tangent plane at point p on $\mathbb{S}^2$, can be represented by $d\mathbf{r} = \partial_x dx + \partial_y dy$.

Definition 3 (Riemannian Metric and Riemannian Manifold)**.** Let $\mathbf{M}$ be a smooth manifold, a *Riemannian metric* g on $\mathbf{M}$ is a smooth family of inner products on the tangent spaces of $\mathbf{M}$. Namely, g associates to each point $p \in \mathbf{M}$ a positive definite symmetric bi-linear form on $T_p\mathbf{M}$:

$$g_p : T_p\mathbf{M} \times T_p\mathbf{M} \to \mathbb{R}$$

along with which comes a norm

$$|\cdot|_{g_p} : T_p\mathbf{M} \to \mathbb{R} \qquad \text{defined by} \qquad |\mathbf{v}|_{g_p} = \sqrt{g_p(\mathbf{v}, \mathbf{v})}$$

The smooth manifold $\mathbf{M}$ endowed with this metric g is a *Riemannian manifold*, denoted by $(\mathbf{M}, g)$. Every smooth manifold has a Riemannian metric.

Continue example 2. All the coordinates are in $\mathbb{R}^3$. For any tangent vector $d\mathbf{r} = \partial_x dx + \partial_y dy \in T_p\mathbb{S}^2$ at point p, we need (x, y) to parameterize the position of point $p \in \mathbb{S}^2$ and (dx, dy) to parameterize the direction and length of tangent vector $d\mathbf{r}$. We can use (x, y, dx, dy) to parameterize tangent vector.

We now introduce g_p^{can}, the *canonical Euclidean metric*, as a case of Riemannian metric[1] to measure the "distance" of two tangent vector at point p

$$g_p^{\text{can}} : T_p\mathbb{S}^2 \times T_p\mathbb{S}^2 \to \mathbb{R} \qquad \text{is defined by} \qquad (\partial_x dx_1 + \partial_y dy_1, \partial_x dx_2 + \partial_y dy_2) \mapsto dx_1 dx_2 + dy_1 dy_2$$

if we are only interested in unit tangent vector ("unit" in the sense of g_p^{can}) and denote $UT_p\mathbb{S}^2$ as unit tangent space, then we only need

$$|d\mathbf{r}|_{g_p^{\text{can}}} = \sqrt{g_p(d\mathbf{r}, d\mathbf{r})} = \sqrt{(dx)^2 + (dy)^2} = 1$$

then we can re-parameterize (dx, dy) as $(\cos\tau, \sin\tau)$, reducing four parameters to three:

$$(x, y, \tau)$$

if we are further only interested in unit tangent vector on equator of unit sphere, we can re-parameterize (x, y) as $(\cos\theta, \sin\theta)$, reducing three parameters to two:

$$(\theta, \tau)$$

[1] Let $x^1, ..., x^n$ denote the standard coordinates on $\mathbb{R}^n$. Then define $g_p^{\text{can}} : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$ by

$$\left(\sum_i a_i \frac{\partial}{\partial x^i}, \sum_j b_j \frac{\partial}{\partial x^j} \right) \mapsto \sum_i a_i b_i$$

2 Shape of Smooth Non-zero Tangent Vector Field

We now consider a Riemann surface $(\mathbf{M}, g)$ with non-zero unit tangent vector everywhere ("unit" is in the sense of g). All the possible unit tangent vector fields, which of course is non-zero, form a unit tangent bundle, denoted by $UT\mathbf{M}$:

$$UT\mathbf{M} := \bigcup_{p\in\mathbf{M}} \{p\} \times UT_p\mathbf{M} = \bigcup_{p\in\mathbf{M}} \{(p, d\mathbf{r}) \mid d\mathbf{r} \in T_p\mathbf{M}, |d\mathbf{r}|_g = 1\} = \{(p, d\mathbf{r}) \mid p \in \mathbf{M}, d\mathbf{r} \in T_p\mathbf{M}, |d\mathbf{r}|_g = 1\}$$

The unit tangent bundle of a surface is a 3-dimensional manifold. Then we consider a Riemann surface of simplest kind: a unit sphere with canonical Euclidean metric $(\mathbb{S}^2, g_p^{\mathrm{can}})$:

$$UT\mathbb{S}^2 = \{(p, d\mathbf{r}) \mid p \in \mathbb{S}^2, d\mathbf{r} \in T_p\mathbb{S}^2, |d\mathbf{r}|_{g_p^{\mathrm{can}}} = 1\}$$

Poincaré-Hopf theorem tells us that it is **impossible** to construct a **smooth** $v_{\mathbb{S}^2} \in UT\mathbb{S}^2$. We demonstrate such impossibility by topological obstruction.

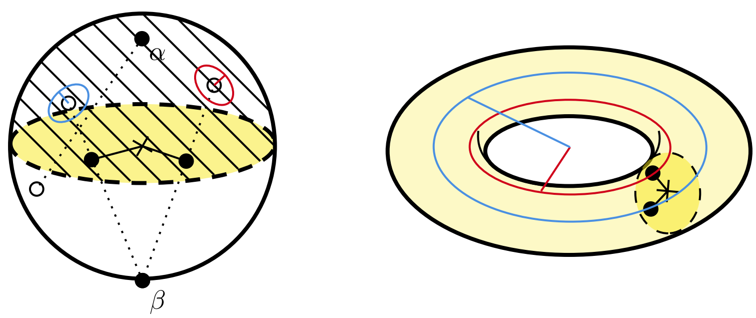

Figure 4: The topological space of unit tangent bundle of unit hemisphere is $\mathbb{S}^1 \times \mathbb{D}^2$, a solid torus. The sectioning disk in solid torus (deep yellow) corresponds to the image plane of φ_β, for example

We know that coordinate chart φ_α cannot cover point α and coordinate chart φ_α cannot cover point β. So we use φ_α for lower hemisphere and φ_β for upper hemisphere (figure 4), and glue them together through **equator**. We will show that to construct a smooth $v_{\mathbb{D}^2} \in UT\mathbb{D}^2$, the unit tangent vector field of unit hemisphere[2], **is okay**. But when we glue them together with constraint of smooth transition from φ_α to φ_β on equator, two hemispheres **cannot allow smooth vector fields over them at the same time**.

Firstly, we see that the shape of $UT\mathbb{D}^2$ is a **solid** torus

$$UT\mathbb{D}^2 \sim \mathbb{S}^1 \times \mathbb{D}^2$$

as shown in figure 4. Because $UT_p\mathbb{D}^2$, the set of all possible directions of unit tangent vector at a point on unit hemisphere, corresponds to a **fiber** that goes through a point on sectioning disk in solid torus (e.g. the red and blue curves in figure 4)

$$UT_p\mathbb{D}^2 \sim \mathbb{S}^1$$

If we cut the torus (to remove its genus), then the sectioning surface represents a particular $v_{\mathbb{D}^2}$. The sectioning surface, through which every fiber goes only once, is called **global section**. The smoothness of $v_{\mathbb{D}^2}$ is guaranteed by the smoothness of that global section. All the possible smooth $v_{\mathbb{D}^2}$ corresponds to all the possible global sections that can be smoothly deformed from the sectioning disk in solid torus

$$v_{\mathbb{D}^2} \sim \mathbb{D}^2$$

[2] we use $\mathbb{D}^2$ to denote a unit disk, which is homotopic to unit hemisphere, so we also use $\mathbb{D}^2$ to denote unit hemisphere

Secondly, notice that $UT(\partial\mathbb{D}^2)$, the unit tangent bundle of unit hemisphere on equator (the boundary of hemisphere) corresponds to a torus, the surface of that solid torus (figure 4)

$$UT(\partial\mathbb{D}^2) = UT\mathbb{S}^1 = \partial(\mathbb{S}^1 \times \mathbb{D}^2) = \mathbb{S}^1 \times (\partial\mathbb{D}^2) = \mathbb{S}^1 \times \mathbb{S}^1 = \mathbf{T}^2$$

thus gluing two smooth $v_{\mathbb{D}^2}$ on equator smoothly, let's say $v_{\mathbb{D}^2_L}$ ("L" for lower hemisphere) and $v_{\mathbb{D}^2_U}$ ("U" for upper hemisphere), is very much of gluing two solid tori with homeomorphism of two tori such that two global sections, let's say $\mathbb{D}^2_L$ and $\mathbb{D}^2_U$, forming a larger global section of $UT(\mathbb{S}^2)$

$$v_{\mathbb{D}^2_L} \bigcup_{UT(\partial\mathbb{D}^2_L)\sim UT(\partial\mathbb{D}^2_U)} v_{\mathbb{D}^2_U} \sim \mathbb{D}^2_L \bigcup_{\mathbf{T}^2_L\sim\mathbf{T}^2_U} \mathbb{D}^2_U$$

The topological obstruction means that one cannot find a global section of $UT(\mathbb{S}^2)$. Or in other words, with constraint of $\mathbf{T}^2_L \sim \mathbf{T}^2_U$, by setting a global section $\mathbb{D}^2_L$ of lower solid torus freely, one cannot find a global section of upper solid torus, as we show later.

3 Topological Obstruction

The homeomorphism of two tori was guaranteed by smooth transition of charts on equator from φ_α to φ_β, namely, from (x, y, dx, dy) to (u, v, du, dv). We check how different φ_β from φ_α, continue example 2

$$\partial_u = \frac{\partial\mathbf{r}}{\partial u} = \frac{\partial\varphi_\beta^{-1}(u,v)}{\partial u} = \frac{2}{(1+u^2+v^2)^2}\begin{bmatrix} 1-u^2+v^2 \\ 2uv \\ -2u \end{bmatrix}$$

$$\partial_v = \frac{\partial\mathbf{r}}{\partial v} = \frac{\partial\varphi_\beta^{-1}(u,v)}{\partial v} = \frac{2}{(1+u^2+v^2)^2}\begin{bmatrix} -2uv \\ -1-u^2+v^2 \\ -2v \end{bmatrix}$$

$$\langle\partial_u, \partial_u\rangle = \langle\partial_v, \partial_v\rangle = \frac{4}{(1+u^2+v^2)^2}$$

$$\langle\partial_u, \partial_v\rangle = 0$$

smooth transition from (dx, dy) to (du, dv) is guaranteed by differentiable Jacobian $\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$:

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix}$$

To compute $u_x = \dfrac{\partial u}{\partial x}, u_y = \dfrac{\partial u}{\partial y}, v_x = \dfrac{\partial v}{\partial x}$ and $v_y = \dfrac{\partial v}{\partial y}$, the most convenient way is by complex variable. If we parameterize (x, y) by complex number $z = x + iy$ and (u, v) by $w = u = iv$, notice that

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{\left(\dfrac{x_1}{1-x_3}\right) - i\left(\dfrac{x_2}{1-x_3}\right)}{\left(\dfrac{x_1}{1-x_3}\right)^2 + \left(\dfrac{x_2}{1-x_3}\right)^2} = \frac{x_1(1-x_3) - ix_2(1-x_3)}{\underbrace{(x_1^2+x_2^2+x_3^2)}_{1} - x_3^2} = \frac{x_1 - ix_2}{1+x_3} = u + iv = w$$

with $\frac{1}{z} = w$, we have $dw = -\frac{1}{z^2}dz$, we write

$$du + idv = -\frac{1}{z^2}(dx + idy) = -\frac{1}{(x+iy)^2}(dx + idy) = \frac{1}{(x^2+y^2)^2}\left[\begin{array}{l} dx(y^2-x^2) - dy(2xy) \quad \hookleftarrow \\ +i[dy(y^2-x^2) + dx(2xy)] \end{array}\right]$$

then by technique of complex variable:

$$\left[\begin{array}{cc} u_x & u_y \\ v_x & v_y \end{array}\right] = \frac{1}{(x^2+y^2)^2}\left[\begin{array}{cc} y^2-x^2 & -2xy \\ 2xy & y^2-x^2 \end{array}\right]$$

is indeed differentiable near by $x^2+y^2=1$, the equator.

Moreover, the transition of charts $\varphi : (z, dz) \mapsto (w, dw)$ is

$$\varphi : (z, dz) \mapsto (\frac{1}{z}, -\frac{1}{z^2}dz)$$

On equator, if parametrized by (θ, τ), as $z = e^{i\theta}$ and $dz = e^{i\tau}$, we have

$$\varphi : (\theta, \tau) \mapsto (-\theta, \pi - 2\theta + \tau)$$

We use canonical representation of $\pi_1(\mathbf{T}^2_L)$ and $\pi_1(\mathbf{T}^2_U)$:

$$\pi_1(\mathbf{T}^2_L) = \langle a_L, b_L | [a_L, b_L] \rangle$$

$$\pi_1(\mathbf{T}^2_U) = \langle a_U, b_U | [a_U, b_U] \rangle$$

then φ induces a push-forward map on homotopy group[3]:

$$\varphi_{\#} : \pi_1(\mathbf{T}^2_L) \to \pi_1(\mathbf{T}^2_U)$$

by

$$a_L \mapsto a_U$$

$$b_L \mapsto a_U^{-2} b_U^{-1}$$

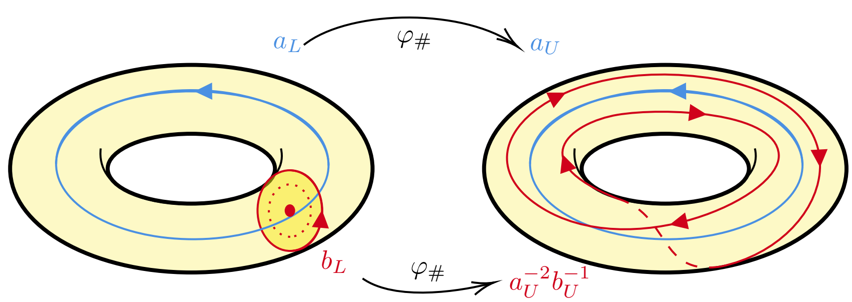

Figure 5: With constraint $\mathbf{T}^2_L \sim \mathbf{T}^2_U$, by setting the global section $\mathbb{D}^2_L$ freely in lower solid torus, its boundary $\partial\mathbb{D}^2_L = b_L$ maps to $a_U^{-2}b_U^{-1}$. While b_L can shrink to a point, $a_U^{-2}b_U^{-1}$ cannot, thus one cannot find a global section in upper solid torus with $a_U^{-2}b_U^{-1}$ as its boundary, which leads to a topological obstruction

As shown in figure 5, we finish construction of a topological obstruction to show that one cannot construct a smooth vector field over a sphere without zero point.

[3]check by corner points: e.g. $A = (0,0) \mapsto \varphi(A) = (0,\pi), B = (0,2\pi) \mapsto \varphi(B) = (0,3\pi), C = (2\pi,2\pi) \mapsto \varphi(C) = (-2\pi,-\pi), D = (2\pi,0) \mapsto \varphi(D) = (-2\pi,-3\pi)$

4 Shape of Unit Tangent Bundle of Unit Sphere

We can derive the fundamental group of $UT\mathbb{S}^2$ using Van Kampen theorem.

Theorem 4 (Van Kampen (-Seifert) Theorem)**.** *Topological space* $\boldsymbol{M}$ *is decomposed into the union of* $\mathbf{U}$ *and* $\mathbf{V}$*, the intersection of* $\mathbf{U}$ *and* $\mathbf{V}$ *is* $\boldsymbol{W}$*,*

$$\boldsymbol{M} = \boldsymbol{U} \cup \boldsymbol{V}$$

$$\boldsymbol{W} = \boldsymbol{U} \cap \boldsymbol{V}$$

where $\mathbf{U}$*,* $\mathbf{V}$ *and* $\boldsymbol{W}$ *are path connected.*

$$i : \boldsymbol{W} \hookrightarrow \mathbf{U}$$

$$j : \boldsymbol{W} \hookrightarrow \mathbf{V}$$

are the inclusion maps. Pick a base point $p \in \boldsymbol{W}$*, the fundamental groups*

$$\pi_1(\mathbf{U}, p) = \langle u_1, ..., u_k | \alpha_1, ..., \alpha_l \rangle$$

$$\pi_1(\mathbf{V}, p) = \langle v_1, ..., v_m | \beta_1, ..., \beta_n \rangle$$

$$\pi_1(\mathbf{W}, p) = \langle w_1, ..., w_p | \gamma_1, ..., \gamma_q \rangle$$

then $\pi_1(\mathbf{M}, p)$ *is given by*

$$\pi_1(\mathbf{M}, p) = \langle u_1, ..., u_k, v_1, ..., v_m | \alpha_1, ..., \alpha_l, \beta_1, ..., \beta_n, i(w_1)j(w_1)^{-1}, ..., i(w_p)j(w_p)^{-1} \rangle$$

One can use Van Kampen's theorem to compute fundamental groups for topological spaces that can be decomposed into simpler spaces.

Example 5 (Fundamental Group of Unit Tangent Bundle of Unit Sphere)**.** We glue $UT\mathbb{D}^2_L$ and $UT\mathbb{D}^2_U$ with homomorphism:

$$\varphi_\#(a_L) = a_U$$

$$\varphi_\#(b_L) = a_U^{-2} b_U^{-1}$$

thus the set up:

$$UT\mathbb{S}^2 = UT\mathbb{D}^2_L \bigcup_{\mathbf{T}^2_L \sim \mathbf{T}^2_U} UT\mathbb{D}^2_U$$

$$\mathbf{T}^2 = UT\mathbb{D}^2_L \bigcap UT\mathbb{D}^2_U$$

where $UT\mathbb{D}^2_L$, $UT\mathbb{D}^2_U$ and $\mathbf{T}^2$ are path connected.

$$i : \mathbf{T}^2 \hookrightarrow UT\mathbb{D}^2_L$$

$$j : \mathbf{T}^2 \hookrightarrow UT\mathbb{D}^2_U$$

are the inclusion maps. Pick a base point $p \in \mathbf{T}^2$, the fundamental groups

$$\pi_1(UT\mathbb{D}^2_L, p) = \langle a_L \rangle \qquad \pi_1(\mathbf{T}^2_L, p) = \langle a_L, b_L | [a_L, b_L] \rangle$$

$$\pi_1(UT\mathbb{D}^2_U, p) = \langle a_U \rangle \qquad \pi_1(\mathbf{T}^2_U, p) = \langle a_U, b_U | [a_U, b_U] \rangle$$

$$\pi_1(\mathbf{T}^2, p) = \langle a, b | [a, b] \rangle$$

the inclusion maps

$$i(a) = a_L, j(a) = a_U^{-1}$$

$$i(b) = b_L = \emptyset, j(b) = (a_U^{-2} b_U^{-1})^{-1}$$

then $\pi_1(UT\mathbb{S}^2, p)$ is given by

$$\pi_1(UT\mathbb{S}^2, p) = \langle a_L, a_U | a_L a_U, a_U^{-2} b_U^{-1} \rangle \cong \mathbb{Z}_2$$