Lecture Note 1: Fundamental Group and Covering Space

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This lecture is about surface algebraic topology. The key idea is to build a bridge between topology, which is abstract and hard to imagine, and algebraic structure, which is tangible and can be computed. In a categorical sense, we construct a functor

$$\mathfrak{C}_1 \mapsto \mathfrak{C}_2$$

between two categories[1] with structural information preserved, namely

$$\mathfrak{C}_1 = \{\text{Topological Spaces, Homeomorphisms}\}$$
$$\mathfrak{C}_2 = \{\text{Groups, Homomorphisms}\}$$

Definition 1 (Topological Type). All oriented compact surfaces can be classified by their genus g and number of boundaries b. Therefore, we use

$$(g, b)$$

to represent the topological type of an oriented surface $\mathbf{S}$.

Definition 2 (Homeomorphism). A *homeomorphism* is a continuous function between topological spaces of the same topological type.

Definition 3 (Homomorphism). A *homomorphism* is a structure-preserving map between two algebraic structures of the same type.

We now introduce *first homotopy group*, denoted[2] as $\pi_1(\mathbf{S}, q)$. The group structure of $\pi_1(\mathbf{S}, q)$ determines the topology of $\mathbf{S}$.

1   Fundamental Group

Let $\mathbf{S}$ be a two-manifold with a base point $p \in \mathbf{S}$.

Definition 4 (Curve). A *curve* is a continuous mapping $\gamma : [0, 1] \mapsto \mathbf{S}$

Definition 5 (Loop). A *closed curve* or *loop* through p is a curve s.t. $\gamma(0) = \gamma(1) = p$

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[1]The concepts of category and functor were covered in previous lectures

[2]Although the fundamental group in general depends on the choice of base point, it turns out that, up to isomorphism (actually, even up to inner isomorphism), this choice makes no difference as long as the space $\mathbf{S}$ is path-connected. For path-connected spaces, therefore, many authors therefore write $\pi_1(\mathbf{S})$ instead of $\pi_1(\mathbf{S}, q)$

Definition 6 (Homotopy). Let $\gamma_0, \gamma_1 : [0,1] \mapsto \mathbf{S}$ be two curves. A *homotopy* connecting γ_0 and γ_1 is a continuous mapping

$$f : [0,1] \times [0,1] \mapsto \mathbf{S}$$

s.t.

$$f(0,t) = \gamma_0(t)$$
$$f(1,t) = \gamma_1(t)$$

We say γ_0 is homotopic to γ_1, if there exists a homotopy between them.

Definition 7 (Loop Product). $\gamma_1 \cdot \gamma_2$ is

$$\gamma_1 \cdot \gamma_2(t) := \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq 0.5 \\ \gamma_2(2t-1) & \text{for } 0.5 \leq t \leq 1 \end{cases}$$

Definition 8 (Loop Inverse). $\gamma^{-1}(t) := \gamma(1-t)$

Definition 9 (Fundamental Group). Given a topological space $\mathbf{S}$, fix a base point $p \in \mathbf{S}$. Homotopy relation is an equivalence relation[3]. The set of all the loops through the base point p is Γ, which can be classified by homotopy relation and form a set of all the homotopy classes, denoted as $\Gamma/\sim$. To define a group:

- The homotopy class of a loop γ, denoted by $[\gamma]$, becomes group generator.

- The group binary operation is defined as

$$[\gamma_1][\gamma_2] := [\gamma_1 \cdot \gamma_2]$$

.

- The group unit element is defined as $[e]$, which is as trivial as a point.

- The group inverse element is defined as
$$[\gamma]^{-1} = [\gamma^{-1}]$$

then $\Gamma/\sim$ forms a group, so-called *fundamental group* of $\mathbf{S}$, or the first homotopy group, denoted as $\pi_1(\mathbf{S}, p)$.

Definition 10 (Word Group Representation). Let $G = \{g_1, g_2, ..., g_n\}$ be n distinct symbols. Words of finite length generated by those symbols form a group with equivalence relations

- $\{g_1, g_2, ..., g_n\}$ becomes group generator.

- The group binary operation is defined as the concatenation of two word.

- The group unit element is empty word $\emptyset$

- The group inverse element is defined as reverse composition of the word.

- Certain segment of words can be replaced by $\emptyset$, which forms equivalence relations, denoted by set $R = \{R_1, R_2, ..., R_m\}$.

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[3]needs to be reflexive, symmetric and transitive

Given a set of generators G and a set of relations R, all the equivalence classes of the words generated by G form a group under the concatenation, called *word group*, denoted as

$$\langle g_1, g_2, ..., g_n | R_1, R_2, ..., R_m \rangle$$

Word group representation can be used to process fundamental group in computer.

Theorem 11 (Van Kampen (-Seifert) Theorem)**.** *If*

- **X** *is a topological space;*

- **U** *and* **V** *are open, path connected subspaces of* **X***;*

- **U** ∩ **V** *is nonempty and path-connected;*

- $w \in$ **U** ∩ **V***;*

- *homomorphisms* $I : \pi_1(\mathbf{U} \cap \mathbf{V}, w) \mapsto \pi_1(\mathbf{U}, w)$ *and* $J : \pi_1(\mathbf{U} \cap \mathbf{V}, w) \mapsto \pi_1(\mathbf{U}, w)$

- *the fundamental group of* **U**, **V** *and* **U** ∩ **V**

$$\pi_1(\mathbf{U}, w) = \langle u_1, ..., u_k | \alpha_1, ..., \alpha_l \rangle$$

$$\pi_1(\mathbf{V}, w) = \langle v_1, ..., v_m | \beta_1, ..., \beta_n \rangle$$

$$\pi_1(\mathbf{U} \cap \mathbf{V}, w) = \langle w_1, ..., w_p | \gamma_1, ..., \gamma_q \rangle$$

then $\pi_1(\mathbf{X}, w)$ *is the free product with amalgamation of* $\pi_1(\mathbf{U}, w)$ *and* $\pi_1(\mathbf{V}, w)$

$$\pi_1(\mathbf{X}, w) = \langle u_1, ..., u_k, v_1, ..., v_m | \alpha_1, ..., \alpha_l, \beta_1, ..., \beta_n, I(w_1)J(w_1)^{-1}, ..., I(w_p)J(w_p)^{-1} \rangle$$

One can use Van Kampen's theorem to calculate fundamental groups for topological spaces that can be decomposed into simpler spaces.

Theorem 12 (Canonical Representation of Surface Fundamental Group)**.** *Suppose* **S** *is a compact, oriented surface,* $p \in$ **S** *is a fixed point, the fundamental group has a canonical[4] representation*

$$\pi_1(\mathbf{S}, p) = \langle a_1, b_1, a_2, b_2, ..., a_g, b_g | \prod_{i=1}^{g} [a_i, b_i] \rangle$$

where

$$[a_i, b_i] := a_i b_i a_i^{-1} b_i^{-1}$$

and g is the genus of the surface and a_i, b_i *are canonical bases[5]*

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[4]The canonical representation of the fundamental group of the surface is not unique. It is NP hard to verify if two given representations are isomorphic.

[5]we omit the definition of canonical basis.

Theorem 13. *Topological Spaces Homeomorphism ⇔ Fundamental Groups Isomorphism*
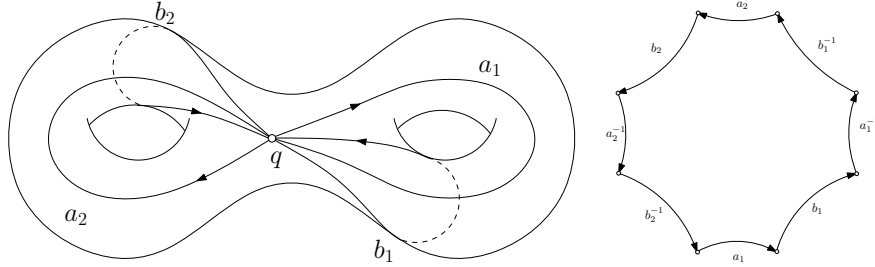

Figure 1: fundamental group canonical basis and fundamental domain

Proof. For each surface, find a canonical basis, slice the surface along the basis to get a 4g polygonal scheme, then construct a homeomorphism between the polygonal schema with consistent boundary condition. (e.g. bi-torus see figure 1) □

Definition 14 (Connected Sum). The *connected sum* $\mathbf{S}_1 \oplus \mathbf{S}_2$ is formed by deleting the interior of disks $\mathbf{D}_i$ and attaching the resulting punctured surfaces $\mathbf{S}_i - \mathbf{D}_i$ to each other by a homeomorphism $h : \partial \mathbf{D}_1 \mapsto \partial \mathbf{D}_2$

$$\mathbf{S}_1 \oplus \mathbf{S}_2 := (\mathbf{S}_1 - \mathbf{D}_1) \cup (\mathbf{S}_2 - \mathbf{D}_2)$$
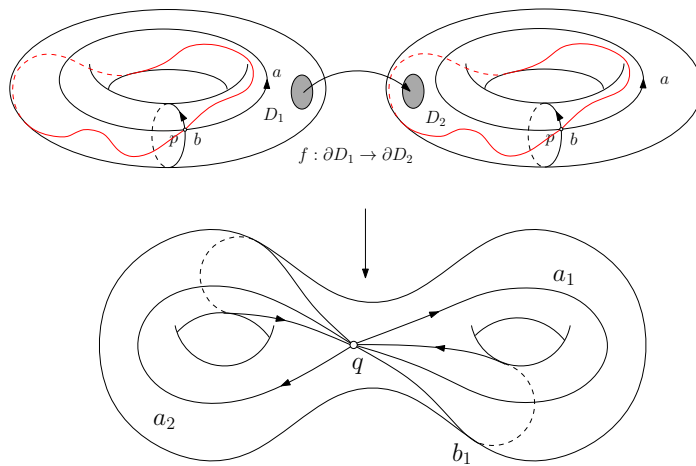

Figure 2: connected sum of two tori

Theorem 15 (Classification Theorem of Closed Surfaces). *Any closed connected surface is homeomorphic to exactly one of the following surfaces:*

- *the sphere, a finite connected sum of tori,*

- *the connected sum of g tori for $g \geq 1$*

$$\underbrace{\mathbf{T}^2 \oplus \mathbf{T}^2 \oplus ... \oplus \mathbf{T}^2}_{g \ tori}$$

- *the connected sum of k real projective planes for $k \geq 1$.*

$$\mathbf{RP}^2 \oplus \mathbf{RP}^2 \oplus ... \oplus \mathbf{RP}^2$$

One can use theorem 11 to show that theorem 12 is true for

$$\mathbf{S} = \oplus_{i=1}^g \mathbf{T}^2$$

2 Covering Space

Definition 16 (Covering Space). Given topological spaces $\tilde{\mathbf{S}}$ and $\mathbf{S}$, a continuous map $f : \tilde{\mathbf{S}} \mapsto \mathbf{S}$ is surjective, such that for each point $q \in \mathbf{S}$, there is a neighborhood $\mathbf{U}$ of q, its preimage $p^{-1}(\mathbf{U}) = \cup_i \tilde{\mathbf{U}}_i$ is a disjoint union of open sets $\tilde{\mathbf{U}}_i$, and the restriction of p on each $\tilde{\mathbf{U}}_i$ is a local homeomorphism, then $(\tilde{\mathbf{S}}, f)$ is a *covering space* of *base space* $\mathbf{S}$, and f is called a projection map.

Definition 17 (Deck Transformation). The automorphisms of $\tilde{\mathbf{S}}$, $g : \tilde{\mathbf{S}} \mapsto \tilde{\mathbf{S}}$, are called *deck transformations*, if they satisfy $f \circ g = f$. All the deck transformations form a group, the covering group, and denoted as

$$\mathrm{Deck}(\tilde{\mathbf{S}})$$

Theorem 18 (Covering Space Group Structure). *Covering space $\tilde{\mathbf{S}}$ and base space $\mathbf{S}$.*

Suppose base points $\tilde{q} \in \tilde{\mathbf{S}}$, $f(\tilde{q}) = q \in \mathbf{S}$.

The projection map $f : \tilde{\mathbf{S}} \mapsto \mathbf{S}$ induces a homomorphism between their fundamental groups

$$f_* : \pi_1(\tilde{\mathbf{S}}, \tilde{q}) \mapsto \pi_1(\mathbf{S}, q)$$

If $f_*(\pi_1(\tilde{\mathbf{S}}, \tilde{q}))$ is a normal subgroup of $\pi_1(\mathbf{S}, q)$ then

$$\frac{\pi_1(\mathbf{S}, q)}{f_*(\pi_1(\tilde{\mathbf{S}}, \tilde{q}))} \cong Deck(\tilde{\mathbf{S}})$$

Definition 19 (Universal Covering Space). If a covering space $\tilde{\mathbf{S}}$ is simply connected (i.e. $\pi_1(\tilde{\mathbf{S}}) = \{e\}$), then $\tilde{\mathbf{S}}$ is called a *universal covering space* of $\mathbf{S}$.

$$\pi_1(\mathbf{S}) \cong \mathrm{Deck}(\tilde{\mathbf{S}})$$

Namely, the fundamental group of the base space is isomorphic to the deck transformation group of the universal covering space.
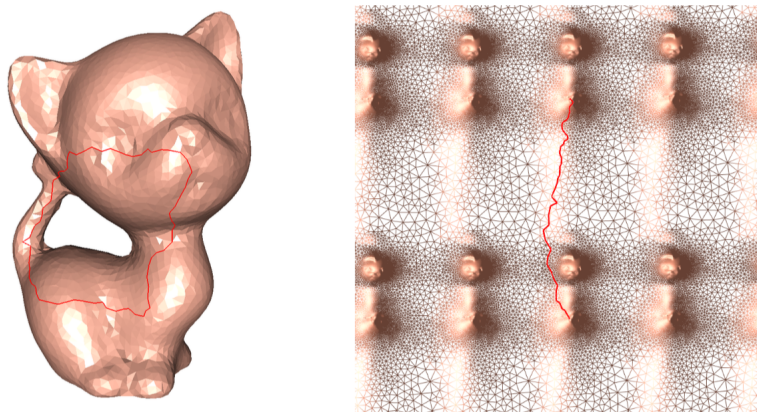
Figure 3: base space $\mathbf{S}$ on the left and universal covering space $\tilde{\mathbf{S}}$ on the right