Computational Conformal Geometry **授课日期: 2020年11月**

Lecture Note 1: Basic Surface Algebraic Topology

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This lecture is about surface algebraic topology. The key idea is to build a bridge between topology, which is abstract and hard to imagine, and algebraic structure, which is tangible and can be computed. In a categorical sense, we construct a functor

$$\mathfrak{C}_1 \to \mathfrak{C}_2$$

between two categories[1] with structural information preserved, namely

$$\mathfrak{C}_1 = \{\text{Topological Spaces, Homeomorphisms}\}$$
$$\mathfrak{C}_2 = \{\text{Groups, Homomorphisms}\}$$

Definition 1 (Topological Type)**.** All oriented compact surfaces can be classified by their genus g and number of boundaries b. Therefore, we use

$$(g, b)$$

to represent the topological type of an oriented surface $\mathbf{S}$.

Definition 2 (Homeomorphism)**.** A *homeomorphism* is a continuous function between topological spaces of the same topological type.

Definition 3 (Homomorphism)**.** A *homomorphism* is a structure-preserving map between two algebraic structures of the same type.

We now introduce *first homotopy group*, denoted[2] as $\pi_1(\mathbf{S}, q)$. The group structure of $\pi_1(\mathbf{S}, q)$ determines the topology of $\mathbf{S}$.

1 Fundamental Group

Let $\mathbf{S}$ be a two-manifold with a base point $p \in \mathbf{S}$.

Definition 4 (Curve)**.** A *curve* is a continuous mapping $\gamma : [0, 1] \to \mathbf{S}$

Definition 5 (Loop)**.** A *closed curve* or *loop* through p is a curve s.t. $\gamma(0) = \gamma(1) = p$

[1] The concepts of category and functor were covered in previous lectures

[2] Although the fundamental group in general depends on the choice of base point, it turns out that, up to isomorphism (actually, even up to inner isomorphism), this choice makes no difference as long as the space $\mathbf{S}$ is path-connected. For path-connected spaces, therefore, many authors therefore write $\pi_1(\mathbf{S})$ instead of $\pi_1(\mathbf{S}, q)$

Definition 6 (Homotopy)**.** Let $\gamma_0, \gamma_1 : [0,1] \to \mathbf{S}$ be two curves. A *homotopy* connecting γ_0 and γ_1 is a continuous mapping

$$f : [0,1] \times [0,1] \to \mathbf{S}$$

s.t.

$$f(0,t) = \gamma_0(t)$$
$$f(1,t) = \gamma_1(t)$$

We say γ_0 is homotopic to γ_1, if there exists a homotopy between them, denoted as $\gamma_0 \sim \gamma_1$.

Definition 7 (Loop Product)**.** $\gamma_1 \cdot \gamma_2$ is

$$\gamma_1 \cdot \gamma_2(t) := \begin{cases} \gamma_1(2t) & \text{for } 0 \le t \le 0.5 \\ \gamma_2(2t-1) & \text{for } 0.5 \le t \le 1 \end{cases}$$

Definition 8 (Loop Inverse)**.** $\gamma^{-1}(t) := \gamma(1-t)$

Definition 9 (Fundamental Group)**.** Given a surface topological space $\mathbf{S}$, fix a base point $p \in \mathbf{S}$. Homotopy relation is an equivalence relation[3]. The set of all the loops through the base point p is Γ, which can be classified by homotopy relation and form a set of all the homotopy classes, denoted as $\Gamma/\sim$. To define a group:

- The homotopy class of a loop γ, denoted by $[\gamma]$, becomes group generator.
- The group binary operation is defined as

  $$[\gamma_1][\gamma_2] := [\gamma_1 \cdot \gamma_2]$$

  .
- The group unit element is defined as $[e]$, which is as trivial as a point.
- The group inverse element is defined as

  $$[\gamma]^{-1} = [\gamma^{-1}]$$

then $\Gamma/\sim$ forms a group, so-called *fundamental group* of $\mathbf{S}$, or the first homotopy group, denoted as $\pi_1(\mathbf{S}, p)$.

Definition 10 (Word Group Representation)**.** Let $G = \{g_1, g_2, ..., g_n\}$ be n distinct symbols. Words of finite length generated by those symbols form a group with equivalence relations

- $\{g_1, g_2, ..., g_n\}$ becomes group generator.
- The group binary operation is defined as the concatenation of two words.
- The group unit element is empty word $\emptyset$
- The group inverse element is defined as.

  $$(g_1 g_2 ... g_k)^{-1} = g_k^{-1} ... g_2^{-1} g_1^{-1}$$

[3] needs to be reflexive, symmetric and transitive

- Certain segments of words can be replaced by $\emptyset$, which form equivalence relations, denoted by set $R = \{R_1, R_2, ..., R_m\}$.

Given a set of generators G and a set of relations R, all the equivalence classes of the words generated by G form a group under the concatenation, called *word group*, denoted as

$$\langle g_1, g_2, ..., g_n | R_1, R_2, ..., R_m \rangle$$

Word group representation can be used to process fundamental group in computer.

Theorem 11 (Canonical Representation of Surface Fundamental Group)**.** *Suppose* $\mathbf{S}$ *is a compact, oriented surface,* $p \in \mathbf{S}$ *is a fixed point, the fundamental group has a canonical*[4] *representation*

$$\pi_1(\mathbf{S}, p) = \langle a_1, b_1, a_2, b_2, ..., a_g, b_g | \prod_{i=1}^{g} [a_i, b_i] \rangle$$

where

$$[a_i, b_i] := a_i b_i a_i^{-1} b_i^{-1}$$

and g *is the genus of the surface* $\mathbf{S}$ *and* a_i, b_i *are canonical bases*[5]

Theorem 12. *Topological Spaces Homeomorphism* $\Leftrightarrow$ *Fundamental Groups Isomorphism*

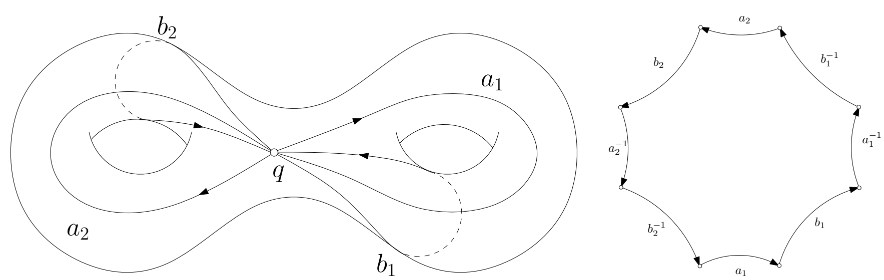

Figure 1: fundamental group canonical basis and fundamental domain

Proof. For each surface, find a canonical basis, slice the surface along the basis to get a $4g$ polygonal scheme, then construct a homeomorphism between the polygonal schema with consistent boundary condition. (e.g. bi-torus see figure 1) □

Definition 13 (Connected Sum)**.** The *connected sum* $\mathbf{S}_1 \oplus \mathbf{S}_2$ is formed by deleting the interior of disks $\mathbf{D}_i$ and attaching the resulting punctured surfaces $\mathbf{S}_i - \mathbf{D}_i$ to each other by a homeomorphism $h : \partial \mathbf{D}_1 \to \partial \mathbf{D}_2$

$$\mathbf{S}_1 \oplus \mathbf{S}_2 := (\mathbf{S}_1 - \mathbf{D}_1) \cup_h (\mathbf{S}_2 - \mathbf{D}_2)$$

[4]The canonical representation of the fundamental group of the surface is not unique. It is NP hard to verify if two given representations are isomorphic.

[5]we omit the definition of canonical basis.

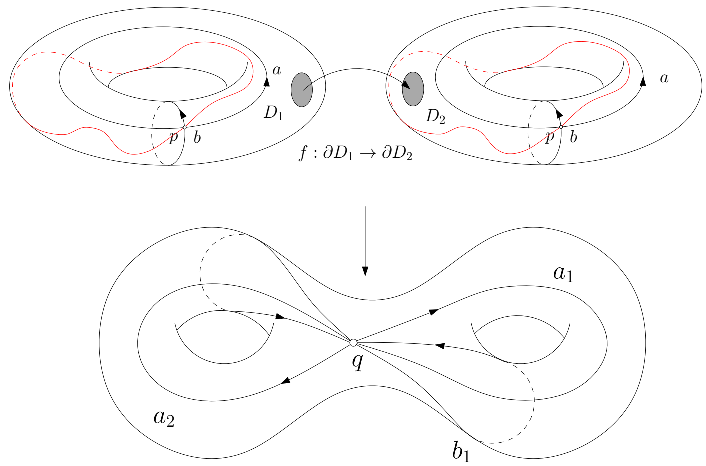

Figure 2: connected sum of two tori

Theorem 14 (Classification Theorem of Closed Surfaces)**.** *Any closed connected surface is homeomorphic to exactly one of the following surfaces:*

- *the sphere, a finite connected sum of tori,*
- *the connected sum of g tori for $g \geq 1$*

$$\underbrace{\mathbf{T}^2 \oplus \mathbf{T}^2 \oplus ... \oplus \mathbf{T}^2}_{g\ tori}$$

- *the connected sum of k real projective planes for $k \geq 1$.*

$$\mathbf{RP}^2 \oplus \mathbf{RP}^2 \oplus ... \oplus \mathbf{RP}^2$$

One can use Van Kampen theorem (will be discussed in later course) to show that theorem 11 is true for

$$\mathbf{S} = \oplus_{i=1}^{g} \mathbf{T}^2$$

2 Quotient Group

Definition 15 (Coset)**.** Let H be a subgroup of the group G. Given an element g of G,

- the *left cosets* of H in G are the **sets** (not group!) obtained by multiplying each element of H by a fixed element g of G (where g is the left factor), denoted by

$$gH := \{gh : h \in H\}$$

- The *right cosets* are defined similarly, except that the element g is now a right factor, that is,

$$Hg := \{hg : h \in H\}$$

Definition 16 (Normal Subgroup)**.** Subgroup N of group G is *normal subgroup*, denoted as $N \triangleleft G$ if for all g in G, the left cosets gN and right cosets Ng are equal. Notice that any subgroup of an Abelian group is a normal subgroup.

Definition 17 (Quotient Group)**.** Let $N \triangleleft G$. To construct a *quotient group* G/N or $\dfrac{G}{N}$, N needs to be a normal subgroup of G:

- define the set G/N to be the set of all cosets[6] of N in G. That is, $G/N = \{\underset{a}{N} : a \in G\}$;
- for any two cosets $\underset{a}{N}$ and $\underset{b}{N}$, binary operation $*$ is defined as

$$\underset{a}{N} * \underset{b}{N} = \underset{ab}{N}$$

- the denominator N, the whole normal subgroup, collapsed into the unit[7] element

$$N = \{e\}$$

- the inverse is defined as

$$\underset{a}{N}^{-1} = \underset{a^{-1}}{N}$$

Notice that $G/e = G$ and $G/G = \{e\}$

The concepts of quotient group will be frequently used in following chapters.

3 Covering Space

Definition 18 (Covering Space)**.** Given topological spaces $\tilde{\mathbf{S}}$ and $\mathbf{S}$, a continuous map $f : \tilde{\mathbf{S}} \to \mathbf{S}$ is surjective, such that

- for each point $q \in \mathbf{S}$, there is a neighborhood $\mathbf{U}$ of q;
- its preimage $f^{-1}(\mathbf{U}) = \cup_i \tilde{\mathbf{U}}_i$ is a disjoint union of open sets $\tilde{\mathbf{U}}_i$;
- f on each $\tilde{\mathbf{U}}_i$ is a local homeomorphism

then $(\tilde{\mathbf{S}}, f)$ is a *covering space* of *base space* $\mathbf{S}$, and f is called a projection map.

Definition 19 (Deck Transformation and Covering Group)**.** The automorphisms of $\tilde{\mathbf{S}}$, $g : \tilde{\mathbf{S}} \to \tilde{\mathbf{S}}$, are called *deck transformations*, if they satisfy $f \circ g = f$. All the deck transformations form a group, the *covering group*, and denoted as

$$\mathrm{Deck}(\tilde{\mathbf{S}})$$

[6]since $N \triangleleft G$, left and right cosets coincide, we use $\underset{a}{N}$ to denote coset of N given $a \in G$

[7]In a quotient of a group, the equivalence class of the identity element is always a normal subgroup of the original group, and the other equivalence classes are precisely the cosets of that normal subgroup. We can alternatively think of quotient group as $G/\sim$, where $a \sim b$ if a and b are in the same coset of N

Theorem 20 (Covering Group Structure)**.** *Covering space* $\tilde{\mathbf{S}}$ *and base space* $\mathbf{S}$*.*

Suppose base points $\tilde{q} \in \tilde{\mathbf{S}}$*,* $f(\tilde{q}) = q \in \mathbf{S}$*.*

The projection map $f : \tilde{\mathbf{S}} \to \mathbf{S}$ *induces a homomorphism between their fundamental groups*

$$f_* : \pi_1(\tilde{\mathbf{S}}, \tilde{q}) \to \pi_1(\mathbf{S}, q)$$

If $f_*(\pi_1(\tilde{\mathbf{S}}, \tilde{q}))$ *is a normal subgroup of* $\pi_1(\mathbf{S}, q)$ *then the quotient group*

$$\frac{\pi_1(\mathbf{S}, q)}{f_*(\pi_1(\tilde{\mathbf{S}}, \tilde{q}))} \cong Deck(\tilde{\mathbf{S}})$$

Definition 21 (Universal Covering Space)**.** If a covering space $\tilde{\mathbf{S}}$ is simply connected (i.e. $\pi_1(\tilde{\mathbf{S}}) = \{e\}$), then $\tilde{\mathbf{S}}$ is called a *universal covering space* of $\mathbf{S}$.

$$\pi_1(\mathbf{S}) \cong \mathrm{Deck}(\tilde{\mathbf{S}})$$

Namely, the fundamental group of the base space is isomorphic to the deck transformation group of the universal covering space (see figure 3)

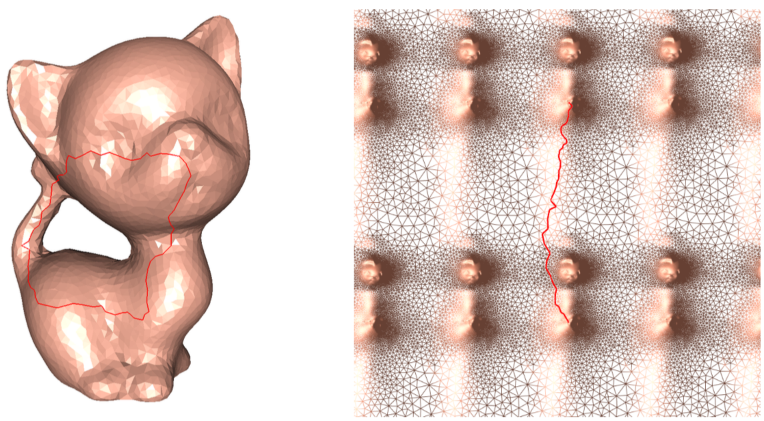

Figure 3: base space $\mathbf{S}$ on the left and universal covering space $\tilde{\mathbf{S}}$ on the right

4 First Homotopy Group vs. First Homology Group

Homotopy relation does fully capture the topological spaces, but it is hard to compute: the homotopy group is non-Abelian (see figure 4). If we can instead represent the topological space by an Abelian group, which can be computed using linear algebra, it would be highly encouraged, even if some loss of information.

By a looser definition of equivalence relation, called homology relation, an Abelian group was formed. The loop product became commutative and therefore was replaced by notation +, called loop formal sum, or just formal sum when generalized to any dimension.

Recall definition of *loop product*, which emphasizes the order of concatenation, and therefore it is not commutative. Why don't we just formally sum each parts up, and keep their orientation in record?

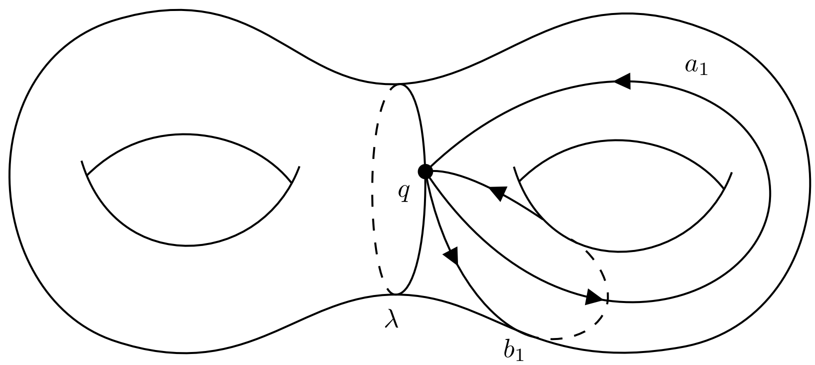

Figure 4: In first homotopy group $\pi_1(\mathbf{S}, q)$, we have $[\gamma] = [a_1 b_1 a_1^{-1} b_1^{-1}]$, but $[\gamma] \neq [e]$, so $[a_1][b_1] \neq [b_1][a_1]$, but in first homology group $H_1(\mathbf{S}, \mathbb{Z})$, $[a_1] + [b_1] = [b_1] + [a_1]$, and also we have $[\lambda] = \mathbf{0}$, see example 30

.

Definition 22 (Formal Sum)**.** If an oriented manifold $\mathbf{M}$ can be decomposed into finite simpler submanifolds $\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_n$ with the same orientation, then we write:

$$\mathbf{M} = \mathbf{m}_1 + \mathbf{m}_2 + ... + \mathbf{m}_n$$

where + denotes *formal sum.* Formal sum is commutative.

Definition 23 (Inverse of Formal Sum)**.** The *inverse of formal sum* of an oriented manifold $\mathbf{M}$ is the sum of inverse of submanifolds $\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_n$, denoted by "−":

$$-\mathbf{M} = -\mathbf{m}_1 - \mathbf{m}_2 - ... - \mathbf{m}_n$$

Definition 24 (Closure)**.** The *closure* of a subset $\mathbf{S}$ of points in a topological space consists of all points in $\mathbf{S}$ together with all limit points of $\mathbf{S}$, denoted by

$$\bar{\mathbf{S}}$$

Definition 25 (Interior)**.** The *interior* of a subset $\mathbf{S}$ of a topological space $\mathbf{X}$ is the union of all subsets of $\mathbf{S}$ that are open in $\mathbf{X}$, denoted by

$$\mathbf{S}^\circ$$

Definition 26 (Boundary and Boundary Operator)**.** The boundary of a subset $\mathbf{S}$ of a topological space $\mathbf{X}$ is the *closure* of $\mathbf{S}$ minus the interior of $\mathbf{S}$:

$$\partial \mathbf{S} := \bar{\mathbf{S}} \setminus \mathbf{S}^\circ$$

We also use $\partial_k \boldsymbol{\Sigma}$ to indicate that the boundary operator actions on a k-manifold $\boldsymbol{\Sigma}$.

Example 27 (Boundary of Surface or Loop)**.** For oriented surface, the boundary (loop) is positively oriented "as one walks along boundary on outside surface while cliff on your right". For oriented curve, the boundary are two end-points that the target point is positively oriented and the source point is negatively oriented. We use $\mathbf{0}$ to denote "nothing in space", e.g. the boundary of a sphere or a loop.

Up to this point, to form a group of k-manifolds, we already have:

- commutative binary operation, the *formal sum* "+"
- identity unit element, $\mathbf{0}$, "nothing in space"
- inverse of an element, which is its negatively oriented version.

We need one more thing, the equivalence class, to reveal topological invariant.

Definition 28 (Homology)**.** Let $\mathbf{S}$ be a k-manifold. Let γ_0 and γ_1 be two $(k-1)$-manifolds. A *homology relation* connecting γ_0 and γ_1 is a k-submanifold $\mathbf{\Sigma}$ such that:

$$\partial_k \mathbf{\Sigma} = \gamma_0 - \gamma_1$$

We say γ_0 is homological to γ_1 if there exists homology between them, denoted as $\gamma_0 \sim \gamma_1$ [8].

Definition 29 (First Homology Group)**.** Given a surface topological space $\mathbf{S}$. Homology relation is an equivalence relation. The set of all the loops and finite formal sum of them is Γ, which can be classified by homology relation and form a set of all the homology classes, denoted as $\Gamma/\sim$. To define a group:

- The homology class of a loop γ, denoted by $[\gamma]$, becomes group generator.
- The group binary operation is defined as

  $$[\gamma_1] + [\gamma_2] := [\gamma_1 + \gamma_2]$$

  which is commutative
- The group unit element is defined as $\mathbf{0}$, which is "nothing in space".
- The group inverse element is defined as

  $$[\gamma]^{-1} = -[\gamma] := [-\gamma]$$

then $\Gamma/\sim$ forms a group, so-called the first homology group, denoted as $H_1(\mathbf{S}, \mathbb{Z})$, if formal sum is over $\mathbb{Z}$, see foot note if over otherwise field[9]

Example 30 (Homology of loops)**.** See figure 5, $\mathbf{S}$ is a closed orientable surface with genus $g = 3$. We have formal sum:

$$\mathbf{S} = \mathbf{\Sigma}_1 + \mathbf{\Sigma}_2 + \mathbf{\Sigma}_3$$

What can we say about the homological class about γ_1, γ_2 and γ_3? We can see that

$$\gamma_1 \sim \gamma_2 \sim (\gamma_2 + \gamma_3) \sim (\gamma_1 - \gamma_3)$$

$$\gamma_3 \sim (\gamma_1 - \gamma_2) \sim \mathbf{0}$$

[8]Notice that Homotopy $\overset{\nLeftarrow}{\Rightarrow}$ Homology. To illustrate homology relation is an equivalence relation:

- (reflexive) $\gamma \sim \gamma$ since $\mathbf{0} = \gamma - \gamma$ trivially holds
- (symmetric) if $\gamma_0 \sim \gamma_1$, then $\partial_2 \mathbf{\Sigma} = \gamma_0 - \gamma_1$, then $\partial_2(\mathbf{S} \setminus \mathbf{\Sigma}) = \gamma_1 - \gamma_0$, then $\gamma_1 \sim \gamma_0$
- (transitive) if $\gamma_0 \sim \gamma_1$ and $\gamma_1 \sim \gamma_2$, suppose $\partial_2 \mathbf{\Sigma}_1 = \gamma_0 - \gamma_1$ and $\partial_2 \mathbf{\Sigma}_2 = \gamma_1 - \gamma_2$, then $\partial_2 \mathbf{\Sigma}_1 + \partial_2 \mathbf{\Sigma}_2 = \partial_2(\mathbf{\Sigma}_1 + \mathbf{\Sigma}_2) = \gamma_0 - \gamma_2$ then $\gamma_0 \sim \gamma_2$.

[9]if formal sum over $\mathbb{Z}_2$, for example, then $[\gamma] + [\gamma] = \mathbf{0}$, we denoted first homology group as $H_1(\mathbf{S}, \mathbb{Z}_2)$. If formal sum over $\mathbb{R}$, for example, we allow $0.4[\gamma] - 1.6[\gamma] + \sqrt{2}[\gamma] = (\sqrt{2} - 1.2)[\gamma]$, then denote first homology group as $H_1(\mathbf{S}, \mathbb{R})$

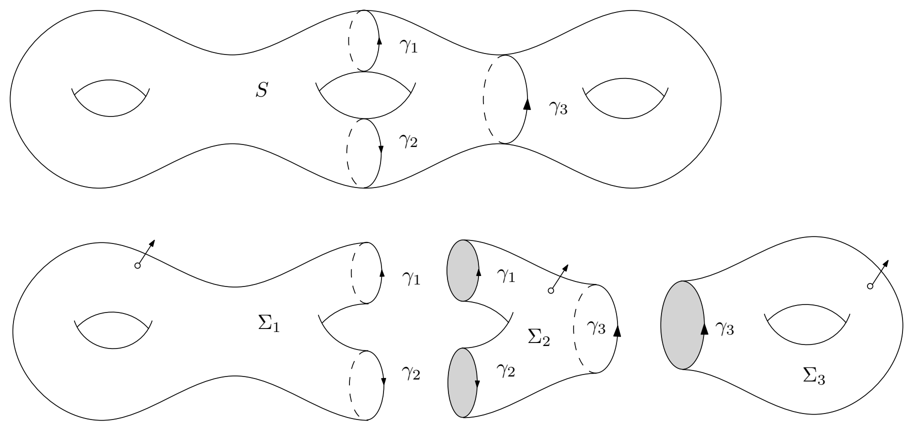

Figure 5: an example of homology relation

since

$$\partial_2 \mathbf{\Sigma}_1 = \gamma_1 - \gamma_2$$
$$\partial_2 \mathbf{\Sigma}_2 = (\gamma_2 + \gamma_3) - \gamma_1 = \gamma_3 - (\gamma_1 - \gamma_2) = \gamma_2 - (\gamma_1 - \gamma_3)$$
$$\partial_2 \mathbf{\Sigma}_3 = \mathbf{0} - \gamma_3$$

5 Homology Group

Kernel is the generalization of zeros of a function. Image is the generalization of range of a function.

Definition 31 (Kernel)**.** Let G and H be groups and let $f : G \to H$ be a homomorphism. Let e_H denote the identity unit element in H. The *kernel* of f is defined as

$$\ker f = \{g \in G \mid f(g) = e_H\}$$

Definition 32 (Loop Group)**.** Any closed loop (or finite formal sum of loops) γ on a closed oriented surface $\mathbf{S}$ will satisfy:

$$\partial_1 \gamma = \mathbf{0}$$

We denote *loop group* $Z_1(\mathbf{S})$ as

$$Z_1(\mathbf{S}) = \ker \partial_1 = \{\gamma \in \mathbf{S} \mid \partial_1(\gamma) = \mathbf{0}\}$$

Definition 33 (Image)**.** Let G and H be groups and let $f : G \to H$ be a homomorphism. The *image* of f is defined as

$$\text{img } f = \{h \in H \mid \exists g \in G \ s.t. \ f(g) = h\}$$

Definition 34 (Boundary Group)**.** Any submanifold $\mathbf{\Sigma}$ on a closed oriented surface $\mathbf{S}$ will induce a closed boundary γ:

$$\partial_2 \mathbf{\Sigma} = \gamma$$

We denote *boundary group* $B_1(\mathbf{S})$ as

$$B_1(\mathbf{S}) = \text{img } \partial_2 = \{\gamma \in \mathbf{S} \mid \exists \mathbf{\Sigma} \in \mathbf{S} \ s.t. \ \partial_2 \mathbf{\Sigma} = \gamma\}$$

Definition 35 (Homology Group Structure)**.** The first homology group of $\mathbf{S}$ is the quotient group

$$H_1(\mathbf{S}, \mathbb{Z}) = \frac{Z_1(\mathbf{S})}{B_1(\mathbf{S})} = \frac{\ker \partial_1}{\text{img } \partial_2}$$

Which is consistent with homology relation, as we collapse $B_1(\mathbf{S})$ as identity (all $\partial_2 \mathbf{\Sigma}$ now become $\mathbf{0}$)

Generally, given $(k+1)$-manifold $\mathbf{M}$, k homology group is

$$H_k(\mathbf{M}, \mathbb{Z}) = \frac{Z_k(\mathbf{M})}{B_k(\mathbf{M})} = \frac{\ker \partial_k}{\text{img } \partial_{k+1}}$$

Figure 6 illustrates the relationship between groups:

- C_k, which is group of all k-submanifold
- Z_k, which is group of kernel of ∂_k on k-submanifold
- B_k, which is group of image of ∂_{k+1} on $(k+1)$-submanifold

$$B_k(\mathbf{M}) \subset Z_k(\mathbf{M}) \subset C_k(\mathbf{M})$$

and

$$\partial_k \circ \partial_{k+1} = \mathbf{0}$$

Theorem 36. *Suppose* $\mathbf{S}$ *is a path-connected genus* g *closed surface, then*

$$H_0(\mathbf{S}, \mathbb{Z}) \cong \mathbb{Z} \cong H_2(\mathbf{S}, \mathbb{Z})$$

$$H_1(\mathbf{S}, \mathbb{Z}) \cong \mathbb{Z}^{2g}$$

6 First Homology Group vs. First Cohomology Group

We have another Abelian group that can encode the same topological information of homology group but can be computed even faster. However, few people understand. We would like to point out *k-form* is the generalization of function, and *coboundary operator* is the generalization of gradient. Before we get there, we now introduce some concept in differential geometry, which would be frequently used.

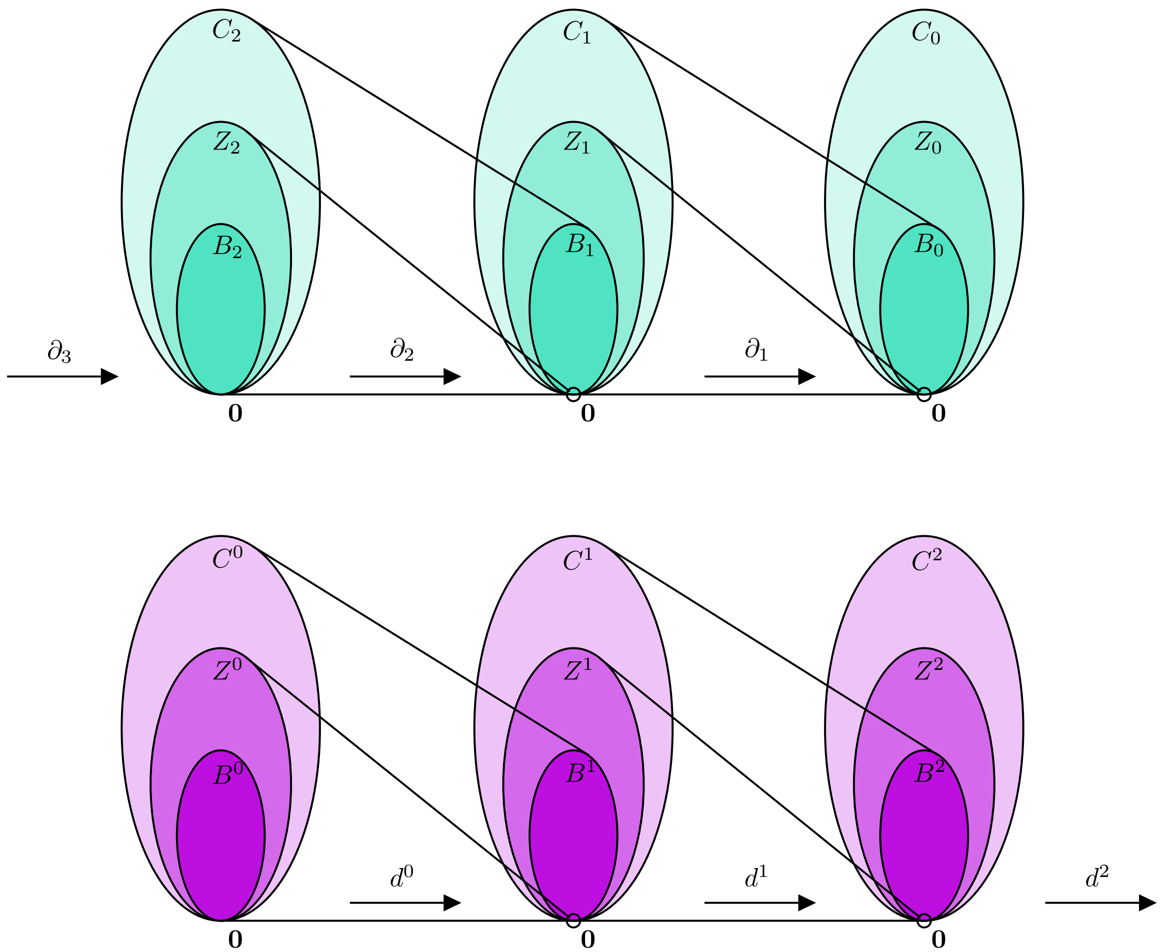

Figure 6: The relation of groups C_k, Z_k and B_k, and their duality, C^k, Z^k and B^k

Definition 37 (Tangent Space)**.** Given a point x on closed surface $\mathbf{M}$, a *tangent space* of $\mathbf{M}$ through x, denoted as

$$T_x\mathbf{M}$$

is a vector space of plane that contains the possible directions in which one can tangentially pass through x. The elements of the *tangent space* $T_x\mathbf{M}$ at x are called the *tangent vectors* v at x, see figure 7:

Definition 38 (Tangent Bundle)**.** The *tangent bundle* of a differentiable manifold $\mathbf{M}$ is a manifold $T\mathbf{M}$ which assembles all the tangent vectors in $\mathbf{M}$, given by the disjoint union of the tangent spaces of $\mathbf{M}$:

$$T\mathbf{M} := \bigcup_{x\in\mathbf{M}} \{x\} \times T_x\mathbf{M} = \bigcup_{x\in\mathbf{M}} \{(x,v) \mid v \in T_x\mathbf{M}\} = \{(x,v) \mid x \in \mathbf{M}, v \in T_x\mathbf{M}\}$$

Definition 39 (Vector Field)**.**

A *vector field* on the region $\mathbf{D} \subset \mathbb{R}^2$ is a vector-valued function

$$f : \mathbf{D} \to \mathbb{R}^2$$

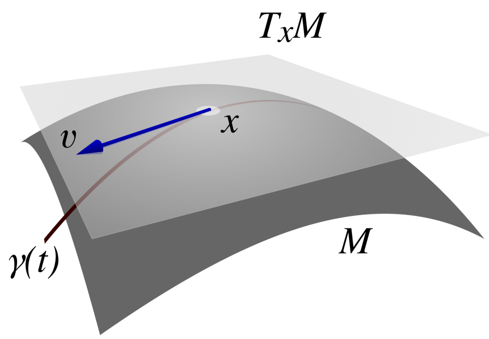

Figure 7: The tangent space $T_x\mathbf{M}$ and a tangent vector v on $T_x\mathbf{M}$, along a curve $\gamma(t)$ traveling through $x \in \mathbf{M}$

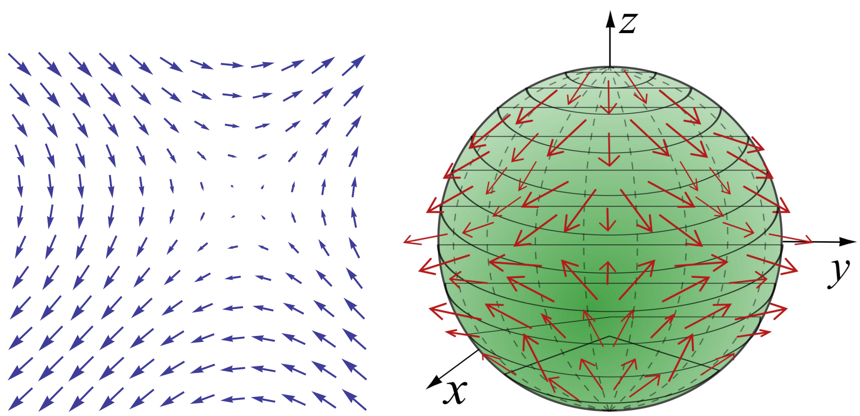

Figure 8: Vector field $f : \mathbf{D} \to \mathbb{R}^2$ (left) and $g : \mathbf{S} \to T\mathbf{S}$ (right)

A *vector field* on a surface $\mathbf{S}$ is an assignment of a tangent vector to each point in $\mathbf{S}$. More precisely, a mapping from $\mathbf{S}$ to tangent bundle of $\mathbf{S}$:

$$g : \mathbf{S} \to T\mathbf{S}$$

See figure 8

Definition 40 (Line Integral)**.** Integration of vector field F along a curve is called *line integral*.

$$\int_\gamma F(v) \cdot dv$$

or simply denoted as

$$\langle F, \gamma \rangle$$

Definition 41 (Curl Free Vector Field)**.** A *curl free field* F of is a vector field such that the line integral along any **loop** γ equals zero:

$$\oint_\gamma F(v) \cdot dv = 0$$

or written as

$$\text{curl } F = 0$$

Definition 42 (Gradient Vector Field)**.** A *gradient field* (or *conservative field*) F is a vector field such that the line integral along any **boundary** γ equals zero:

$$\oint_{\gamma} F(v) \cdot dv = 0$$

or equivalently we say that there exist a scalar field φ that its gradient is F:

$$F = \nabla\varphi \ \text{ or } \ F = \text{grad } \varphi$$

Notice that just like boundary $\overset{\not\Leftarrow}{\Rightarrow}$ loop

$$\text{gradient field} \overset{\not\Leftarrow}{\Rightarrow} \text{curl free field}$$

Definition 43 (1-form)**.** Given an oriented closed surface $\mathbf{S}$, a *1-form* f of $\mathbf{S}$ encoded by a vector field F on $\mathbf{S}$ is a linear mapping from any curve $\gamma \in C_1(\mathbf{S})$ to line integral $\langle F, \gamma \rangle \in \mathbb{R}$:

$$f : C_1(\mathbf{S}) \to \mathbb{R}$$

we write:

$$f(\gamma) = \langle f, \gamma \rangle := \langle F, \gamma \rangle$$

Since the *1-form* of $\mathbf{S}$ is very much of the vector filed F, so people use *vector field* and *1-form* interchangeably. Recall that group of any curve $C_1(\mathbf{S})$, group of loops $Z_1(\mathbf{S})$ and group of boundaries $B_1(\mathbf{S})$ with

$$B_1(\mathbf{S}) \subset Z_1(\mathbf{S}) \subset C_1(\mathbf{S})$$

since

$$\text{boundary} \overset{\not\Leftarrow}{\Rightarrow} \text{loop}$$

Now we consider another way to describe such topological information.

Definition 44 (First Cohomology Group)**.** Given a surface topological space $\mathbf{S}$. We define three groups of 1-form (vector field) with binary operation "+" over $\mathbb{Z}$, unit element 0 and inverse as "-" prefix:

- group of all vector field over $\mathbf{S}$, denoted as

$$C^1(\mathbf{S})$$

- curl free field group, denoted as $Z^1(\mathbf{S})$

$$Z^1(\mathbf{S}) = \{f \in C^1(\mathbf{S}) \mid \langle f, \gamma \rangle = 0, \ \gamma \in Z_1(\mathbf{S})\}$$

- gradient field group, denoted as $B^1(\mathbf{S})$

$$B^1(\mathbf{S}) = \{f \in C^1(\mathbf{S}) \mid \langle f, \gamma \rangle = 0, \ \gamma \in B_1(\mathbf{S})\}$$

Since

$$\text{boundary} \overset{\not\Leftarrow}{\Rightarrow} \text{loop}$$

We have

$$B^1(\mathbf{S}) \subset Z^1(\mathbf{S}) \subset C^1(\mathbf{S})$$

The *first cohomology group* $H^1(\mathbf{S}, \mathbb{Z})$ is achieved by collapsing $B^1(\mathbf{S})$ into identity:

$$H^1(\mathbf{S}, \mathbb{Z}) = \frac{Z^1(\mathbf{S})}{B^1(\mathbf{S})}$$

and notice that

$$H^1(\mathbf{S}, \mathbb{Z}) \cong H_1(\mathbf{S}, \mathbb{Z})$$

Until now, we may not have proper language to describe what is a cohomology relation, although we derive cohomology group. What does it mean if f_0 is cohomologous to f_1?

A scalar field φ on $\mathbf{S}$ is a 0-form. By gradient operator, it becomes a vector field F, the 1-form. Imagine any of tiny oriented curve γ on $\mathbf{S}$. The gradient can be thought of the difference of the scalar values of two end points, which is the summation of 0-form of boundary of the curve (because boundary will give one positive and one negative value):

$$\underbrace{F}_{\text{1-form}} = \underbrace{\text{grad } \overbrace{\varphi}^{\text{0-form}}}_{\text{1-form}} = \underbrace{\varphi \circ \partial}_{\text{1-form}}$$

Now the generalization of gradient by relating boundary operator is coboundary operator

Definition 45 (Coboundary and Coboundary Operator)**.** k-dimensional *Coboundary operator* d^k actions on a k-form f:

$$d^k f(\cdot) := f \circ \partial_{k+1}(\cdot)$$

we say $d^k f$, a $(k+1)$-form, is the coboundary of f, the k-form.

Notice that

$$d^k \circ d^{k-1}(\cdot) = 0$$

holds for any input of $(k-1)$-manifold. In the case of $d^1 \circ d^0(\cdot)$, namely, the curl of gradient is zero.

We also derive

Theorem 46 (Stokes Theorem)**.**

$$\langle dw, \sigma \rangle = \langle w, \partial\sigma \rangle$$

Definition 47 (Cohomology)**.** Let $\mathbf{S}$ be a surface topological space. Let f_0 and f_1 be two k-forms. A *cohomology relation* connecting f_0 and f_1 is a $(k-1)$-form φ such that:

$$d^{k-1}\varphi = f_0 - f_1$$

We say that f_0 is cohomologous to f_1 if there exists such $(k-1)$-form φ, denoted as $f_0 \sim f_1$, and cohomology class $[f_0] = [f_1]$.

Definition 48 (Cohomology Group Structure)**.** We omit further details, see figure 6:

$$H^k(\mathbf{S}, \mathbb{Z}) = \frac{Z^k(\mathbf{S})}{B^k(\mathbf{S})} = \frac{\ker d^k}{\text{img } d^{k-1}}$$

Theorem 49 (Poincaré Duality)**.** *Given n dimensional topological space* $\mathbf{S}$*:*

$$H^k(\mathbf{S}, \mathbb{Z}) \cong H_{n-k}(\mathbf{S}, \mathbb{Z})$$

When $n = 2$ as in the case of surface topology, given path-connected oriented closed genus g surface $\mathbf{S}$, we have

$$H^2(\mathbf{S}, \mathbb{Z}) \cong H_0(\mathbf{S}, \mathbb{Z}) \cong \mathbb{Z} \cong H_2(\mathbf{S}, \mathbb{Z}) \cong H^0(\mathbf{S}, \mathbb{Z})$$
$$H^1(\mathbf{S}, \mathbb{Z}) \cong H_1(\mathbf{S}, \mathbb{Z}) \cong \mathbb{Z}^{2g}$$

Definition 50 (Dual Cohomology Basis)**.** suppose a homology basis of $H_1(\mathbf{S})$ is $\{\gamma_1, \gamma_2, ..., \gamma_n\}$, the *dual cohomology basis* is $\{w_1, w_2, ..., w_n\}$, satisfying:

$$\langle w_i, \gamma_j \rangle = 1_{i=j}$$

where

$$1_{\mathcal{A}} := \begin{cases} 1 & \text{if } \mathcal{A} \text{ is true} \\ 0 & \text{otherwise} \end{cases}$$