Paper Review of Combining Dynamic Decoupling with Decoherence Free Subspace

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1 Introduction

This section gives the summary of the article *Comprehensive Encoding and Decoupling Solution to Problems of Decoherence and Design in Solid-State Quantum Computing*[1], but this review only focuses on DFS part of the paper.

The article[1] presents a solution to the problems of decoherence and design in solid-state quantum computing devices based on exchange interactions between qubits. The authors propose to use a simple encoding of logical qubits into two physical qubits, which protects against collective dephasing and eliminates the need for single-qubit operations. They also introduce efficient pulse sequences that can suppress leakage errors and other sources of decoherence by applying bang-bang controls on the encoded qubits. They illustrate their method with quantum dots and show how it can be generalized to other encodings and systems. They also suggest an empirical approach to determine the optimal bang-bang operations from experimental data. Afterwards, they show how to combine the encoded BB control with the quantum error correcting code so that a trade-off between qubits number and gate operation intensity is achieved. The article claims that their method offers a realistic and comprehensive solution to some of the major difficulties in quantum dot and other exchange-based quantum computers.

2 Theory

2.1 Dynamical Decoupling

Dynamical decoupling is a technique used in the field of quantum information and quantum computing to protect quantum systems from the adverse effects of decoherence and noise caused by their interaction with the environment. The basic idea behind dynamical decoupling is to apply a series of carefully designed control pulses (often called "bang-bang" pulses) to the quantum system. These control pulses aim to counteract the effects of environmental noise and other unwanted interactions. By applying these pulses, the effective coupling between the quantum system and its environment can be periodically reversed or modulated, such that the total effect of the coupling over time is minimized or even eliminated[2].

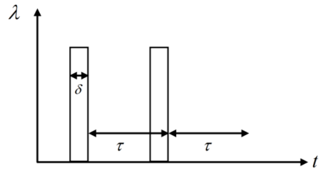

Figure 1: $\frac{\pi}{2}$ pulse

2.2 The Symmetrization of Dynamical Decoupling

Dynamical decoupling can be considered as a time-reversal process where all the operators that anti-commute with the pulse operator pick a minus sign in the evolution. Here is an example illustrating this[3]. Consider a single qubit system with the pure dephasing system-bath coupling Hamiltonian $H_{SB} = \sigma^z \otimes B^z$, and assume the system Hamiltonian commutes with σ^x. By applying the X-pulse in Figure 1, where $\delta\lambda = \frac{\pi}{2}$, and assuming the case of an ideal pulse ($\delta \to 0, \lambda \to \infty$) there is no system-bath interaction during the time the pulse is turned on, the joint system-bath evolution operator at time $t = 2\tau$ is (dropping overall factors of i and minus signs)

$$X f_\tau X f_\tau = \sigma^x e^{-i\tau H_{SB}} \sigma^x e^{-i\tau H_{SB}} = e^{-i\tau\sigma^x H_{SB}\sigma^x} e^{-i\tau H_{SB}} \quad (1)$$

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Since σ^x anti-commutes with the interaction Hamiltonian, we have

$$\begin{aligned}\sigma^x H_{SB} \sigma^x &= \sigma^x \sigma^z \sigma^x \otimes B^z \\ &= -\sigma^z \otimes B^z \\ &= -H_{SB}.\end{aligned} \tag{2}$$

Hence, $X f_\tau X f_\tau = e^{+i\tau H_{SB}} e^{-i\tau H_{SB}} = I$. This means, at time $t = 2\tau$, there is no effect of interaction Hamiltonian. The system is completely decoupled from the bath every 2τ.

We can generalize the aforementioned example by defining general evolution $f_\tau = e^{-i\tau(H_{SB}+H_B)}$. The system Hamiltonian is ignored here due to the assumption that the pulse operator will always commute with system Hamiltonian. Consider a group $\mathcal{G} = \{g_0, \cdots, g_K\}$ (with $g_0 \equiv I$) of unitary transformations g_j acting purely on the system. If we apply the following symmetrization sequence, after a total time of $t = (K+1)\tau$, we would have

$$\begin{aligned}U(T) &= \prod_{j=0}^{K} g_j^\dagger f_\tau g_j \\ &= \prod_{j=0}^{K} e^{-i\tau\left(g_j^\dagger H_{SB} g_j + H_B\right)} \\ &= e^{-i\tau\left(\sum_{j=0}^{K} g_j^\dagger H_{SB} g_j + (K+1) H_B\right)} + \mathcal{O}\left(T^2\right) \\ &= e^{-iT\left(H'_{SB} + H_B\right)} + \mathcal{O}\left(T^2\right)\end{aligned} \tag{3}$$

We therefore define the average Hamiltonian $H'_{SB} = \frac{1}{K+1}\sum_{j=0}^{K} g_j^\dagger H_{SB} g_j$. If we want to decouple the system with the bath, the group $\mathcal{G}$ should be carefully chosen according to one of the following rules:

- Pick a group G such that $H'_{SB} = 0$.
- According to Schur's Lemma, if we choose a group $\mathcal{G}$ so that its matrix representation over the relevant system Hilbert space is irreducible, then we will have $H'_{SB} \propto I_S$, which implies the system-bath decoupling.

A good example of applying the second rule is the n-qubit tensor product of the Pauli group which suffices to decouple the most general system-bath Hamiltonian in the case of n qubits: $H_{SB} = \sum_\alpha \sigma_1^{\alpha_1} \otimes \cdots \otimes \sigma_n^{\alpha_n} \otimes B^\alpha$ where $\alpha = \{\alpha_1, \ldots, \alpha_n\}$, and $\alpha_i \in \{0, x, y, z\}$.

The above illustrations all come from Lidar's book[3].

2.3 Combining with DFS - Example

With the context developed above, we can now discuss what is the advantage of combining decoherence-free subspace with dynamical decoupling.

The simplest decoherence-free subspace is to use two qubits, and to encode logical qubit as $|\bar{0}\rangle = |01\rangle$ and $|\bar{1}\rangle = |10\rangle$. We consider a system consisting of two qubits that are coupled to a bath by the independent dephasing interaction: $H_{SB} = \sigma_1^z \otimes B_1^z + \sigma_2^z \otimes B_2^z$. We can rewrite the interaction Hamiltonian in terms of collective decoherence and other decoherence: $H_{SB} = \frac{(\sigma_1^z+\sigma_2^z)}{2} \otimes B_+^z + \frac{(\sigma_1^z-\sigma_2^z)}{2} \otimes B_-^z$ where $B_\pm^z = B_1^z \pm B_2^z$. Here, $\frac{(\sigma_1^z+\sigma_2^z)}{2}$ is the collective decoherence to which encoding in the decoherence-free subspace is immune. Thus, we only care about differential dephasing $\frac{(\sigma_1^z-\sigma_2^z)}{2}$. This dephasing operator happens to be the logical Z operator operating on the DFS and can be decoupled by pulses that implemented the logic X or Y operators of DFS.

$$\bar{\sigma}^x = \frac{\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y}{2} \quad \bar{\sigma}^y = \frac{\sigma_1^y \sigma_2^x - \sigma_1^x \sigma_2^y}{2} \tag{4}$$

where $\bar{\sigma}^x$ and $\bar{\sigma}^y$ is the logical x and logical y of the the encoded system, respectively. Then, the $\frac{\pi}{2}$-pulse of logical X is:

$$\begin{aligned}e^{-i\frac{\pi}{2}\bar{\sigma}^x} &= e^{-i\frac{\pi}{4}\left(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y\right)} \\ &= e^{-i\frac{\pi}{4}\sigma_1^x\sigma_2^x} e^{-i\frac{\pi}{4}\sigma_1^y\sigma_2^y} \\ &= \frac{1}{\sqrt{2}}\left[I - i\sigma_1^x\sigma_2^x\right] \frac{1}{\sqrt{2}}\left[I - i\sigma_1^y\sigma_2^y\right] \\ &= -\frac{i}{2}\left(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y\right) = -i\bar{X}\end{aligned} \tag{5}$$

The decoupling resembles the unencoded process: $\bar{X} f_\tau \bar{X} f_\tau\Big|_{\text{DFS}} = \bar{I} \otimes \exp(-2i\tau\tilde{B}) + \mathcal{O}\left[(2\tau)^2\right]$. $\tilde{B}$ is the 'logical' bath Hamiltonian we don't care about.

2.4 Combining with DFS - General decoherence

The paper describes the general interaction Hamiltonian on two qubits:

$$H_{SB} = \sum_{\alpha_1,\alpha_2} (\sigma_1^{\alpha_1} \otimes \sigma_2^{\alpha_2}) \otimes B^{\alpha_1\alpha_2} \tag{6}$$

where $\alpha_i \in \{0, x, y, z\}$. It then lists all the possible errors than can happen on the two-qubit sys-

Effect on DFS states	Operators
unchanged	$I, \sigma_1^z + \sigma_2^z, \sigma_1^z\sigma_2^z, \sigma_1^x\sigma_2^x - \sigma_1^y\sigma_2^y, \sigma_1^x\sigma_2^y + \sigma_1^y\sigma_2^x$
logical op.	$\bar{\sigma}^z, \bar{\sigma}^y, \bar{\sigma}^x$
leakage	$\sigma_1^y, \sigma_2^y, \sigma_1^x\sigma_2^z, \sigma_1^z\sigma_2^x, \sigma_1^y\sigma_2^z, \sigma_1^z\sigma_2^y$

Table 1: Classification of all two-qubit error operators

tem and divides them into three groups based on the effect on DFS states. Table 1 shows this classification. The first type of error is protected by DFS and will not change the encoded state. The second type of error is the logical operator which can be deemed as logical Pauli errors. The third type of error is the error that transit from the DFS to outside states or vice versa ("leakage error").

We can now apply 'logical' dynamical decoupling to fight against the second type and the third type of error. Similarly, we look for two logical operators that anti-commutes with all the leakage errors and anti-commutes with all the logical Pauli errors, respectively. The former one happens to be $\exp(-i\pi\bar{\sigma}^x) = Z_1Z_2$. The latter one is a combination of logical X $\frac{\pi}{2}$-pulse $e^{-i\frac{\pi}{2}\bar{\sigma}^x} = -i\bar{X}$ and logical Z $\frac{\pi}{2}$-pulse $e^{-i\frac{\pi}{2}\bar{\sigma}^z} = -i\bar{Z}$. Firstly, we apply Z_1Z_2 to eliminate all leakage errors:

$$\begin{aligned} U_1(2\tau) &= ZZ \cdot f_\tau \cdot ZZ \cdot f_\tau \\ &= \exp\left[-2i\tau \sum_{\alpha\in\{x,y,z\}} \bar{\sigma}^\alpha \otimes \bar{B}^\alpha\right] + \mathcal{O}\left[(4\tau)^2\right] \end{aligned} \tag{7}$$

Afterwards, we apply $e^{-i\frac{\pi}{2}\bar{\sigma}^x} = -i\bar{X}$ to eliminate logical Y and logical Z errors

$$\begin{aligned} U_2(4\tau) &= \bar{X}U_1(2\tau)\bar{X}U_1(2\tau) \\ &= \exp\left[-4i\tau\left(\bar{\sigma}^x \otimes \bar{B}^x\right) \otimes B^{(j)}\right] \\ &+ \mathcal{O}\left[(2\tau)^2\right], \end{aligned} \tag{8}$$

Finally, we apply $e^{-i\frac{\pi}{2}\bar{\sigma}^z} = -i\bar{Z}$to get rid of all the errors:

$$\begin{aligned} U_3(8\tau) &= \bar{Z}U_2(4\tau)\bar{Z}U_2(4\tau) \\ &= \bar{I} \otimes e^{-8i\tau B'} + \mathcal{O}\left[(8\tau)^2\right]. \end{aligned} \tag{9}$$

Note the aforementioned examples and descriptions all come from the paper[1].

3 Discussion and Conclusion

Note that if we simply want to use dynamical decoupling to eliminate all the errors in Table 1. The selecting rules of group $\mathcal{G}$ tell us we might need $4^2 = 16$ different pulses in the worst case to achieve this goal, which requires at least a time of 32τ to accomplish this. However, if we can combine dynamical decoupling with DFS, we can used a pulse sequence that has length 8τ to achieve the same goal, shorter by a factor of 2 compared to the sequence we would have had to use without the DFS encoding. This demonstrates the space-time trade-off between using full DD without DFS encoding and using both methods[3].

I chose this paper because it reminds me of the principle behind Pauli Sandwiching Check[4]. Both of these two techniques decompose the error into Pauli matrices and try to find a group of Pauli elements that anti-commutes with all the error operators. I think they essentially follow the same principle. This also reflects some high-level common features of error-mitigation techniques which, I think, is worth exploring.

References

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[4] Alvin Gonzales, Ruslan Shaydulin, Zain H Saleem, and Martin Suchara. "Quantum error mitigation by pauli check sandwiching". Scientific Reports **13**, 2122 (2023). arXiv:2206.00215. code: alvinquantum/noise_mitigation_symmetry.