Review of Constructing Smaller Pauli Twirling Sets

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1 Introduction

1.1 Pauli Twirling

The technique of Pauli twirling is used to transform arbitrary noise channels into Pauli channels in error threshold estimations of quantum error correction codes. It is also a well-known technique in quantum information literature and has been used in various applications such as entanglement purification[1][2], randomized benchmarking[3][4], quantum process tomography[5][6], and error mitigation[7][8][9][10].
The essential step of Twirling is to conjugate the noise with a gate selected at random(random twirling) or in a predefined order(exact twirling) from a collection of gates known as the twirling set each time the circuit is executed. By using the complete set of Pauli operators as the twirling set, we can transform any noise channel into a Pauli channel, with its noise components matching the original noise's Pauli basis.

1.2 Constructing Smaller Twirling Set

The Gottesman-Knill theorem[11][12] states that quantum circuits involving only Clifford gates can be simulated efficiently on classical computers. However, when non-Clifford noise is introduced, simulating these circuits becomes intractable. Twirling can solve this issue. Using the full set of Pauli operators as the twirling set, any noise channel can be converted into a Pauli channel, which is effective in logical error estimation and error threshold estimation[13][14][15].
In the paper *Constructing Smaller Pauli Twirling Sets for Arbitrary Error Channels*[16], twirling is discussed as a technique for simulating noise and the impact of the noise on the performance of quantum error correction codes. Conventional twirling uses the full set of Pauli gates as the set of twirling gates. This paper presents a method to create a twirling set with a size comparable to the Pauli basis of the given error channel, which is often much smaller than the full set of Pauli gates. One advantage of minimizing the size of the twirling gate set is to increases the efficiency of simulations. In experiments, the advantage also includes reducing both the number of runs required and the circuit depth.

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2 Theory

2.1 Definitions

The paper defines a new Pauli multiplication - the * operation - which is just usual Pauli multiplication but ignoring all the ±1 and ± factors[16]. Under the new multiplication Z * X = X * Z = Y, which means all the multiplications are commutative now.
Commutator function ζ is defined as follow: For $g_i, g_j \in \mathcal{P}_n$, their commutator function $\zeta\left(g_i, g_j\right)$ is defined to be $g_i g_j = \zeta\left(g_i, g_j\right) g_j g_i$. We have the following relation

$$\zeta\left(g_i, g_j\right) = \begin{cases} 1 & \text{for } \left[g_i, g_j\right] = 0 \\ -1 & \text{for } \left\{g_i, g_j\right\} = 0 \end{cases}$$

Super-operators and error channels are written as $\overline{A}$, and follow the property $(\overline{A} + \overline{B})\rho = A\rho A^\dagger + B\rho B^\dagger$. Error channel $\mathscr{E}$ is written as: $\mathscr{E}(\rho) = \sum_M \overline{M}\rho \quad \text{with } \sum_M M^\dagger M = I$.
We can think of twirling as a super-super-operator that turns one super-operator into another[16]. In this paper, only random twirling is studied. Each run we choose a random element w_n from the twirling set W. Applying random twirling $\mathscr{T}_{W,N}^{random}$ using the twirling set W on the noise operator M the number of runs(twirling layers) N is defined as

$$\mathscr{T}_{W,N}^{random}(\overline{M}) = \frac{1}{N}\sum_{n=1}^{N} \overline{w_n M w_n^\dagger}$$

2.2 One-gate twirling

The paper defines an equivalent twirling method, called one-gate twirling, using only the generator(group theory concept) of the whole twirling set. Consider the special case where $W = \{I, w\}$, for which W only contains one extra gate other than the identity. Denote one-gate twirling as $\mathscr{T}_{\{I,w\}}$. The paper states that doing nested one-gate twirling with $\mathscr{T}_{\{I,w_1\}}$ on top of $\mathscr{T}_{\{I,w_2\}}$ on top of $\mathscr{T}_{\{I,w_3\}}$, etc, is equivalent to twirling with $W = \langle w_1, w_2, \ldots, w_n\rangle$, where $\langle w_1, w_2, \ldots, w_n\rangle$ denotes the full set of gates that can be generated from $\{w_1, w_2, \ldots, w_n\}$ using operation $*$.

$$\mathscr{T}_{\{I,w_1\}} \cdot \mathscr{T}_{\{I,w_2\}} \cdots = \mathscr{T}_{\langle w_1,w_2,\cdots\rangle}$$

2.3 The necessary conditions of twirling set

Break n-qubit noise operator M into its Pauli basis:

$$\begin{aligned} M &= \frac{1}{2^n} \sum_{g \in G} \operatorname{Tr}(gM) g \\ &= \frac{1}{2^n} \sum_{v \in V} \operatorname{Tr}(vM) v \end{aligned}$$

where V is the non-zero Pauli basis of M :$V = \{g \in G \mid \operatorname{Tr}(gM) \neq 0\}$. The intermediate process are ignored here since the derivation is long and tedious. Finally, we can have:

$$\begin{aligned} &\mathscr{T}_W(\overline{M})\rho = \\ &\frac{1}{|W|}\frac{1}{2^{2n}} \sum_{v,v' \in V} \operatorname{Tr}(vM) \operatorname{Tr}\left(v'M^\dagger\right) v\rho v' \sum_{w \in W} \zeta\left(w, vv'\right) \\ &\qquad = \underbrace{\frac{1}{2^{2n}} \sum_{v \in V} |\operatorname{Tr}(vM)|^2 \overline{v}\rho}_{v=v'} + \\ &\underbrace{\frac{1}{|W|}\frac{1}{2^{2n}} \sum_{\substack{v,v' \in V \\ v \neq v'}} \operatorname{Tr}(vM) \operatorname{Tr}\left(v'M^\dagger\right) v\rho v' \sum_{w \in W} \zeta\left(w, vv'\right)}_{v \neq v'} \end{aligned}$$

If we want to turn our error channel into a Pauli noise channel, the $v \neq v'$ term must vanish. This gives us the necessary condition of choosing a smaller twirling set W which must satisfy the following constraint:

$$\sum_{w \in W} \zeta\left(w, vv'\right) = 0 \quad \forall v, v' \in V \text{ and } v \neq v'$$

In this case, the result of twirling the noise operator M gives a Pauli channel which can be corrected with quantum error correcting code.

$$\mathscr{T}_W(\overline{M}) = \frac{1}{2^{2n}} \sum_{v \in V} |\operatorname{Tr}(vM)|^2 \overline{v} \tag{1}$$

2.4 Steps to construct smaller twirling set

It is not easy to understand the steps given in the paper. Discussing in detail the knowledge involved in those steps is beyond the scope of this review. Therefore, I will briefly summarize the idea behind these steps based on my understanding.

In general, the paper wants to find the group properties between generator set of twirling set and the Pauli basis set the elements of which span the basis for entire error channel. After finding such group properties, the paper constructs the Homomorphic mapping between the two set. With the help of Commutator tables and Generator tables which are defined in the paper, the desired Pauli twirling generator set W that satisfies our necessary condition (1) can be found.

3 The Main Result

This paper gives two example of how to do their steps in detail. One was to apply their methods on the [[7,1,3]] Steane code.

3.1 Steane code

Suppose the implementation of Steane code using spin qubits suffers from a small global field causing a global rotation of a small angle θ in the Z direction, leading to the following coherent noise:

$$M = \exp\left\{-i\theta \sum_{i=1}^{7} Z_i\right\} = I - i\theta \sum_{i=1}^{7} Z_i + O\left(\theta^2\right)$$

The higher order term $O\left(\theta^2\right)$ was ignored in the noise channel for the purpose of obtaining the reduced twirling set. The steps of obtaining the reduced twirling set are:

1. The Pauli basis of M is: $V = \{I\} + \{Z_n \mid n \in \mathbb{N}, 1 \leq n \leq 7\}$
2. No composition relations other than those involving the identity exists Within V, We

	$I(I)$	$\widetilde{h}_1(Z_1)$	$\widetilde{h}_2(Z_2)$	$\widetilde{h}_3(Z_3)$	$\widetilde{h}_1*\widetilde{h}_2(Z_4)$	$\widetilde{h}_1*\widetilde{h}_3(Z_5)$	$\widetilde{h}_2*\widetilde{h}_3(Z_6)$	$\widetilde{h}_1*\widetilde{h}_2*\widetilde{h}_3(Z_7)$
$\widetilde{q}_1$	1	-1	1	1	-1	-1	1	-1
$\widetilde{q}_2$	1	1	-1	1	-1	1	-1	-1
$\widetilde{q}_3$	1	1	1	-1	1	-1	-1	-1

Figure 1: Commutator table in the Steane Code example

	I	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7
$X_1X_4X_5X_7$	1	-1	1	1	-1	-1	1	-1
$X_2X_4X_6X_7$	1	1	-1	1	-1	1	-1	-1
$X_3X_5X_6X_7$	1	1	1	-1	1	-1	-1	-1

Figure 2: Commutator table of Twirling Generator Set and Pauli Error in the Steane Code

have:

$$\widetilde{V} = \{Z_n \mid n \in \mathbb{N}, 1 \leq n \leq 7\}$$
$$\widetilde{V}_S = \{Z_1\}$$

3. The smallest integer N that satisfies both $N \geq \log_2(|V|) = 3$ and $N \geq \left|\widetilde{V}_S\right| = 1$ is $N = 3$. Hence, we will define a generating set $\widetilde{H}$ of size 3.

4. Using the fact that $\widetilde{V}_S = \{Z_1\}$, the following mapping $\widetilde{V} \mapsto H_{\widetilde{V}} \subseteq H = \langle \widetilde{H} \rangle$ can be found:

$$\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\} \mapsto \left\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \tilde{h}_1 * \tilde{h}_2, \tilde{h}_1 * \tilde{h}_3, \tilde{h}_2 * \tilde{h}_3, \tilde{h}_1 * \tilde{h}_2 * \tilde{h}_3\right\}$$

5. Now starting with the generator table of $|\widetilde{H}| = 3$, we can construct the commutator table $\zeta\left(\tilde{q}_i, h_{\tilde{v},j}\right)$ shown in Fig 1. Note that the Pauli operators in the brackets are the elements in $\widetilde{V}$ that the elements in $H_{\widetilde{V}}$ map to.

6. To find $\widetilde{W}$ such that $\zeta(\tilde{w}_i, \tilde{v}_j) = \zeta(\tilde{q}_i, h_{\tilde{v},j})$. One possible choice is to have $\widetilde{W} = \{X_1X_4X_5X_7, X_2X_4X_6X_7, X_3X_5X_6X_7\}$. Here, $\tilde{w}_i$ is one element in the twirling generator set. $\tilde{v}_j$ is one element in the Pauli basis of M. $\tilde{q}_i$ is the row element of the commutator table. $h_{\tilde{v},j}$ is the column element of the commutator table. This gives us the table in Fig 2

7. Figure 3 shows the simulation results of this example.

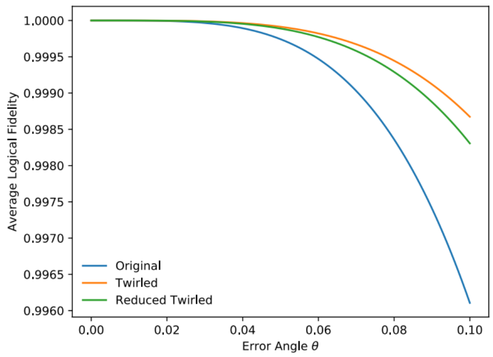

Figure 3: Simulation result of constructing smaller twirling set in the Steane Code example

3.2 Combining with error correcting code

The paper also proves that doing one-gate twirling with $W = \{I, s\}$ is equivalent to performing a s stabilizer check and throwing away the result. Here, s is a certain stabilizer operator. In the previous section, the paper proves that a given noise operator M can be twirled by doing nested one-gate twirling using the elements in the twirling generating set $\widetilde{W}$. The s stabilizer check can substitute the same element s in the $\widetilde{W}$. By doing stabilizer check s and discarding the result, we remove the element s from $\widetilde{W}$ and therefore further reduce the size of whole twirling set W.

4 Discussion

4.1 Strength

The contributions of the paper include outlining the necessary and sufficient conditions for a set of twirling gates to transform a given noise operator into a Pauli channel form, formalizing the method of constructing a smaller set as well as providing an example of how to combine twirling with stabilizer measurements which enables us to further reduce the size of the twirling gate set.

4.2 Limitation

The twirling set obtained from this paper's method is not the smallest possible set. The research about how to find the smallest possible set is necessary to further improve the efficiency of Pauli twirling.

4.3 Questions and Confusions

The paper concentrates on Pauli Twirling. Can the method proposed in this paper generalize to Clifford Twirling? If not, can we construct smaller twirling set for Clifford Twirling?

4.4 Possible Improvement

The paper gives the steps of their method and also includes two example illustrating how it works. In both example, the author gets the Pauli operators in twirling generating set directly from the commutator table. One aspect that can be improved is to detail the process of finding such Pauli operators.

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