Review of Pauli Sandwiching Check Technique

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1 Introduction

We are currently in the so-called noisy intermediate-scale quantum (NISQ) era[1]. The error mitigation technique plays an important role in improving state fidelity as well as reducing readout error. There are quite a lot error mitigation techniques already in use, such as probabilistic error cancellation[2] and dynamical decoupling[3]. One important characteristics of these techniques is that they don't introduce extra ancilla qubit for computation. In the paper *Quantum error mitigation by Pauli check sandwiching*[4], the authors claimed that, as qubit count and quality improves, new error mitigation methods can now utilize more qubits for reducing errors. A novel error mitigation technique working with ancilla qubits was therefore proposed in the paper. This technique is able to completely recover the noiseless state on the premise that noise does not affecting the checks state. To find check pairs for an arbitrary circuit, an efficient algorithm was designed. The results of numerical simulations on 1850 random input circuits demonstrates the efficacy of this design.

2 Design

2.1 Theory

The theory of Pauli Sandwiching Check(PSC) is mainly based on the fact that the Pauli group element $\hat{\sigma}_j \in \mathcal{P}_n$ either commute or anti-commute with the other element in the group and n-qubit Pauli group constitute a complete basis for any n-qubit operators. We know that we can treat errors as quantum maps and use Kraus operator

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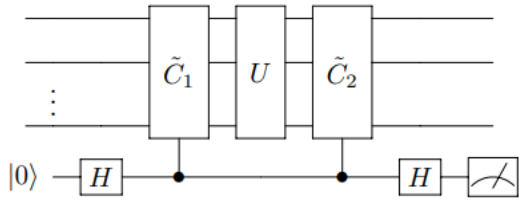


Figure 1: Schema of the one-layer Pauli Sandwiching Check

to model the quantum channel[5].

$$\begin{aligned}\epsilon(\rho_s) &= tr_E(U_{SE}\rho_s \otimes \rho_E U_{SE}^\dagger) \\ &= \sum_i^n \eta_i E_i \rho_s E_i^\dagger \\ \sum_i^n E_i E_i^\dagger &= \mathbb{I}\end{aligned}$$

where $\eta_i = 1$ and for completely positive (CP) map; $\eta_i = \pm 1$ and and there is at least one $\eta_i = -1$ for not completely positive map[4]. Note the Kraus operators E_i can be expanded in the Pauli basis.

$$E_i = \sum_{\hat{\sigma}_j \in \mathcal{P}_n} \alpha_{ij} \hat{\sigma}_j$$

where $\alpha_{ij} = tr(E_i \hat{\sigma}_j)/(2^n)$ is a complex constant. We now have all the theoretical tools to help us understand how PSC works. Figure 1 shows the schema of single layer PSC. We can assume $C_1 = \tilde{C}_1 \otimes |1\rangle\langle 1| + \mathbb{I} \otimes |0\rangle\langle 0|$ and $C_2 = \tilde{C}_2 \otimes |1\rangle\langle 1| + \mathbb{I} \otimes |0\rangle\langle 0|$. The constraints for selecting C_1 and C_2 are that they must be a n-qubit Pauli group elements and they satisfy the requirement $\tilde{C}_2 U \tilde{C}_1 = U$. The steps of doing single-layer protocol is as follows[4]:

1. Initialize the ancilla to $|0\rangle$.
2. Apply a Hadamard gate on ancilla qubit. Perform C_1 with the control on the ancilla qubit.Perform U on the compute qubits. Perform C_2 with the control on the ancilla qubit and target on the compute qubits.

3. Apply a Hadamard gate to the ancilla. Measure the ancilla in the Z basis, discard the results where the outcome is 1.

If $\epsilon(\rho_s) = \sum_i^n E_i \rho E_i^\dagger$We can write the density operator of the post-selected output state:

$$\rho_m = \frac{\sum_i \left[\left(\tilde{C}_2 E_i \tilde{C}_2^\dagger + E_i \right) U \rho U^\dagger \left(\tilde{C}_2 E_i^\dagger \tilde{C}_2^\dagger + E_i^\dagger \right) \right]}{tr \left(\sum_i \left[\left(\tilde{C}_2 E_i \tilde{C}_2^\dagger + E_i \right) U \rho U^\dagger \left(\tilde{C}_2 E_i^\dagger \tilde{C}_2^\dagger + E_i^\dagger \right) \right] \right)}$$

We can write it in terms of the new quantum map ϵ'

$$\rho_m = \frac{\epsilon'(U \rho U^\dagger)}{tr\left(\epsilon'(U \rho U^\dagger)\right)}$$

We then have the new Kraus operator $E_i' = \frac{\tilde{C}_2 E_i \tilde{C}_2^\dagger + E_i}{2}$. Recall that our $\tilde{C}_2 \in \mathcal{P}_n$ and we can expand E_i in Pauli basis. Since each Pauli basis either commutes or anti-commutes with $\tilde{C}_2$, those anti-commutes with $\tilde{C}_2$ will pick a minus sign after switching the order of $\tilde{C}_2$ and E_i in $\tilde{C}_2 E_i \tilde{C}_2^\dagger$.

$$\begin{aligned}
\frac{\tilde{C}_2 E_i \tilde{C}_2^\dagger + E_i}{2} &= \frac{\sum_{\hat{\sigma}_j \in \mathcal{P}_n} \alpha_{ij} \tilde{C}_2 \hat{\sigma}_j \tilde{C}_2^\dagger + \sum_{\hat{\sigma}_j \in \mathcal{P}_n} \alpha_{ij} \hat{\sigma}_j}{2} \\
&= \frac{\sum_{\hat{\sigma}_j \in \mathcal{P}_n} (-1)^m \alpha_{ij} \hat{\sigma}_j \tilde{C}_2 \tilde{C}_2^\dagger + \sum_{\hat{\sigma}_j \in \mathcal{P}_n} \alpha_{ij} \hat{\sigma}_j}{2} \\
&= \frac{\sum_{\hat{\sigma}_j \in \mathcal{P}_n} (1 + (-1)^m) \alpha_{ij} \hat{\sigma}_j}{2}
\end{aligned}$$

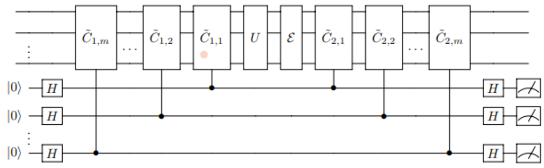
Therefore, the Pauli decomposition of the new Kraus operator E_i' only contains terms that commute with $\tilde{C}_2$.

$$E_i = \sum_{\hat{\sigma}_j \in \mathcal{P'}_n} \alpha_{ij} \hat{\sigma}_j$$

where $\mathcal{P'}_n$ is the Pauli group excluding he elements that anti-commute th $\tilde{C}_2$.
Similarly, if we use a multiple-layer PSC protocol as shown in Figure 2 and keep the result where all the measurement outcomes are 0 in step 3, we can eliminate all the Pauli terms that anti-commute with at least one of the $\tilde{C}_{2,k}$ checks.

$$E_i^{(k)} = \sum_{\hat{\sigma}_j \in \mathcal{G}_n^k} \alpha_{ij} \hat{\sigma}_j$$

where $E_i^{(k)}$ is the k-layer post-selected Kraus operator and $\mathcal{G}_n^k$ is the Pauli group excluding the elements that anti-commute with $\{\tilde{C}_{2,1}, \tilde{C}_{2,2}, \cdot, \tilde{C}_{2,k}\}$. If k equal the size of $\mathcal{P}_n$ (excluding global phases) 4^n, we get $E_i^{(k)} = \alpha_i I$. The



(a) $\mathcal{E}$ is an arbitrary noise map.

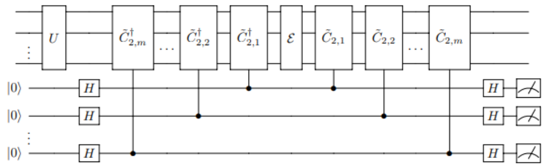


Figure 2: Schema of the multi-layer Pauli Sandwiching Check

constant will disappear after re-normalization. This means the error map is just identity operator which has no effect on our state. This means the upper bound of PCS gives us a fidelity of 1 **if errors are restricted to act only on the compute qubits**.

2.2 Technical Problem

To achieve the result of theoretical upper bound, we need solve one technical problem. If we have already found a set of $\tilde{C}_2$ checks, can we find the correspond set of $\tilde{C}_1$ checks such that $\tilde{C}_2 U \tilde{C}_1 = U$. The solution of the paper[4] is to construct a commutation relation table of the universal gate set. Since all the quantum circuits can be transpiled into circuits that only have universal gates, we can find $\tilde{C}_1$ checks by moving the $\tilde{C}_2$ to the front of the circuit through the commutation relations. The paper gives the pseudo code for doing this, if the algorithm fails to 'pushing' a certain $\tilde{C}_2$ check to the front, this check is simply discarded. Note that, in theory, we only need at most $2n$ $\tilde{C}_2$ weight-one checks to achieve unit fidelity[4]. Since $\tilde{C}_2 \in \mathcal{P}_n$, we have 4^n choices for $\tilde{C}_2$. Hence, discarding some choices for $\tilde{C}_2$ does not affect the implementation of the PSC if n is large.

2.3 Experiment Design

The paper uses numerical simulation to demonstrate the efficacy of PSC. The simulation is in a more realistic setting where all the gates and qubit states, including those involved in the parity checks, are noisy[4]. Measurements and the

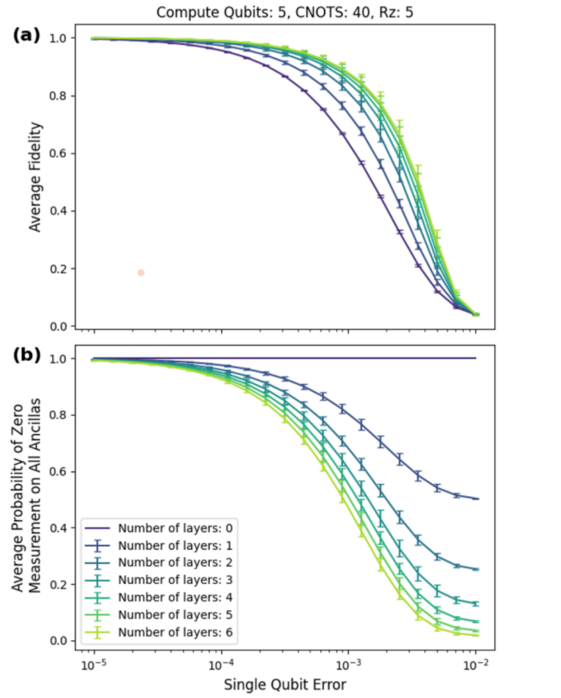


Figure 3: result of one experiment

circuit that generates the random input are exceptions to this setting. The simulation was running on Clifford circuits since the Gottesman–Knill theorem states that a Clifford circuit prepared and measured in computational basis can be simulated in polynomial time on a probabilistic classical computer[6]. The simulation was running with different layers of Pauli checks or different error rates with either 5 or 10 qubits.

3 Result

There are ten sets of experiment. Due to the limitation of the space, I cannot put all the simulation results here. Figure 3 is the result of one experiment with 5 qubits, 40 CNOT gates and 5 Rz gates. The figure below shows how the probability of measuring all 0 on ancilla qubits changes with single qubit error and number of layers of checks. In general, we can have a higher fidelity with more layers of checks. However, as the single qubit error rate increases, Pauli checks are progressively becoming useless. After exceeding a certain threshold(10^{-2} in this figure), the exist of Pauli checks does not contribute to higher fidelity any longer. All the simulation results show that fidelity is positively correlated with the number of layers up to some value. Since they don't take the noise of checks gates and check states into account when analyzing the effects of Pauli checks, the results are consistent with the theoretical analyses.

4 Discussion

4.1 Strength

I think one advantage of Pauli Sandwiching Check over other error mitigation techniques is that it guarantee a unit fidelity in theory. i.e. It has a quite high upper bound performance. The paper states that the method can also be easily combined with other error mitigation techniques since it is applicable to any circuit with arbitrary input state.

4.2 Limitation

The probability of measuring all 0 of ancilla qubit state $P(\overline{0}) = tr\left(\epsilon^{(m)}\left(U\rho U^{\dagger}\right)\right)$ tells us that we will have lower post-selecting probability if we simply use more layer of Pauli checks. Similarly, the cost of unit fidelity is a low probability of measuring 0 of all the checks, which means we have to discard most of the experimental results. This operation is expensive.

4.3 Questions

The design of PSC is reminiscent of Pauli Twirling and Parity Checks(Quantum Zeno effect). I was wondering that, compared with Pauli Twirling and Parity Check, which mitigation technique could give a better fidelity gain under some specific environment settings, such as qubit number, single qubit error rate. It is also worth exploring the performance of PSC and Pauli Twirling with different layers of checks and twirling gates respectively. Compare the results of these experiments will give us a deeper understanding of how to utilizing these error mitigation techniques.

4.4 Possible Improvement

According to the paper[4], the main limitation is the need to obtain the checks $\tilde{C}_1$ and $\tilde{C}_2$, with cost exponential in the number of qubits in the sub-circuit. This cost can be reduced to exponential in the number of non-Cliford gates by leveraging the extended stabilize formalism[7]. Besides, only Clifford circuits experiment has been conduct in this paper due to the problem of efficient simulation. I think it's also meaningful to test the efficacy of PSC on non-Clifford circuit. Experiments related to this are needed.

References

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