Reviews of Decoherence Free Subspace and Its Application in Quantum Error Correction

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I. Introduction

Error-mitigation techniques play an important role in the field of quantum computation. The effectiveness of quantum technologies in the near future is restricted by hardware errors or noise that result from flaws in qubits, such as unintended interactions with the surrounding environment. As these devices do not possess the requisite number of qubits or error rates to execute quantum error correction [8] [1] [6], the fidelity of computations must be enhanced by employing error mitigation techniques. There are various techniques to mitigate errors that have been developed. Zero-noise extrapolation [16] [4] uses different error rates to decrease the error when measuring an observable. Probabilistic error cancellation [17] uses an assortment of known noisy circuits to approach the expected value. Dynamical decoupling [18] utilizes timed control sequences to suppress interactions between the target quantum system and its surrounding environment. Compared to quantum correcting code, these techniques do not require a large number of additional qubits and intensive operations, and are therefore advantageous in the current NISQ era [13], as NISQ devices have limited qubits and gate operations due to fast decoherence times.

Decoherence-free subspace(DFS) is one of the most promising techniques among them. The basic idea of decoherence-free subspace is to find a subspace of the whole Hilbert space that is free of decoherence errors, and our state information should only reside in this subspace. In this paper review, we will focus on the theoretical foundation and applications of decoherence-free subspace. It was found that the majority of the foundational theories were established between 1995 and 2000. Recent research has focused primarily on the applications of DFS. Therefore, the theoretical content of this literature review is based on articles from that time period, while the content on applications includes both early and recent research. The content of this review comes from quite a lot literature, but the major articles being reviewed are [3] [20] [10] [18] [21] [7] [9].

II. Theory

The theory section concentrates on the theoretical foundation of the DFS. The formal definition of DFS is given by D. A. Lidar and K. B. Whaley [12]:

Definition 1 Consider a system with Hilbert space $\mathcal{H}_S$. A subspace $\tilde{\mathcal{H}}_S \subset \mathcal{H}_S$ is called decoherence free if any state $\rho_S(0)$ of the system initially prepared in this subspace is unitarily related to the final state $\rho_S(t)$ of the system, i.e.,

$$\rho_S(0) = \tilde{P}\rho_S(0)\tilde{P} \Rightarrow \rho_S(t) = U_S\rho_S(0)U_S^\dagger \tag{1}$$

where $U_S : \mathcal{H} \mapsto \mathcal{H}$ is unitary and $\tilde{P}$ is a projection operator onto $\tilde{\mathcal{H}}_S \subset \mathcal{H}_S$.

Noted that, in the early theoretical research, the analyses of DFS were under the assumption that only collective decoherence exists in the system. If the system features independent decoherence, which is the a fundamental premise in the field of quantum error correcting code, a DFS does not exist [10]. Hence, it is also worth the effort to give the definition of collective decoherence here.

Definition 2 Collective decoherence is the process that all the quantum states experience the same type of error, such as dephasing or depolarizing, at the same time.

Collective decoherence can arise naturally when the qubits are transmitted through a medium that varies slowly compared to the qubit dynamics, or when they are subject to a common magnetic field [14].

A. Paolo's work

Paolo's research [20] was the first to provide a formal theoretical explanation of how to locate a decoherence-free subspace (DFS). They utilized the spin-boson model to demonstrate their theory. In their work, N identical two-level systems (N qubits) were coupled with a single thermal bath made up of a group of non-interacting linear oscillators. The system Hamiltonian and the bath Hamiltonian is $\epsilon\sum_{i=1}^{N}\sigma_i^z$ and $\sum_k \omega_k b_k b_k^\dagger$, respectively. The interaction Hamiltonian is as follows

$$H_I = \sum_k \sum_{i=1}^{N} (g_{ki}\sigma_i^+ b_k + f_{ki}\sigma_i^+ b_k^\dagger + h_{ki}\sigma_i^z b_k + \text{H.c}) \tag{2}$$

The explanation for utilizing this type of interaction Hamiltonian can be found in the publication cited as [20], and we will assume it to be accurate. To satisfy the collective decoherence constraint, it is necessary for the coupling coefficients g_{ki}, f_{ki}, h_{ki} to remain independent of the qubit/two-level system index i. The interaction Hamiltonian that adheres to this constraint is as follows:

$$H_I = \sum_k (g_k S^+ b_k + f_k S^+ b_k^\dagger + h_k S^z b_k + \text{H.c}) \tag{3}$$

In this context, the global operators $S^\alpha = \sum_{i=1}^{N}\sigma_i^\alpha (\alpha \in \pm, z)$ act on all the two-level systems and consequently

produce collective evolution. The article then discovered that for the N=2 scenario, the singlet state $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ possesses the attribute $S^\alpha |\psi\rangle = 0 \ \forall \ \alpha \in \pm, z$. Therefore, for any state $|\psi_B\rangle \in \mathcal{H}_B$, the interaction Hamiltonian will destroy the state $|\psi\rangle \otimes |\psi_B\rangle$. If $|\psi_B\rangle$ is an eigenstate of H_B, then the state $|\psi\rangle \otimes |\psi_B\rangle$ becomes an eigenstate of the whole Hamiltonian H_{SB}, implying that the state of the system evolves unitarily. Through elementary sl(2) representation theory, one can discover the corresponding states for multi-qubit systems that are generalizations of the singlet state for N=2 [20]. By using a (reducible) representation $\mathcal{D}$ of sl(2) and performing the Clebsch-Gordan (CG) decomposition in terms of the $\mathcal{D}_j$'s, we have

$$\mathcal{D}^{\otimes N} = \bigotimes_{j \in J} n_j \mathcal{D}_j \tag{4}$$

The spin-boson model employs J to represent the potential total spin of the system, and $\mathcal{D}_j$ to indicate the representation of the system with a total spin of j. The integer n_j denotes the multiplicity with which $\mathcal{D}_j$ appears in the resolution of $\mathcal{D}$. Below are examples of spin-boson model representations for N=2,4,6:

$$\mathcal{D}_{1/2}^{\otimes 2} = \mathcal{D}_1 \oplus \mathcal{D}_0, \quad \mathcal{D}_{1/2}^{\otimes 4} = \mathcal{D}_2 \oplus 3\mathcal{D}_1 \oplus 2\mathcal{D}_0$$
$$\mathcal{D}_{1/2}^{\otimes 6} = \mathcal{D}_3 \oplus 5\mathcal{D}_2 \oplus 9D_1 \oplus 5\mathcal{D}_0$$

Our objective is to locate an n-qubit state that has a similar annihilation property as the singlet for N=2. Interestingly, the systems with a total spin of 0 possess this property. As a result, let $n(N)$ be the multiplicity of the $j = 0$ representation. In the aforementioned examples, we have $n(2) = 1, n(4) = 2, n(6) = 5$. Let C_N represent the $n(N)$-dimensional space that is spanned by the singlets. It is evident that if $|\psi^{(N)}\rangle \in C_N$, then $\forall\, |\psi_B\rangle \in \mathcal{H}_B$, $H_I |\psi^{(N)}\rangle \otimes |\psi_B\rangle = 0$. The remainder of Paolo's article establishes two theorems that demonstrate that if we initiate our system state in the decoherence-free subspace, the interaction between the spin and bath system will not cause decoherence in our system states. Due to the fact that the math involved is beyond the scope of this review, we will not go over the description and derivation of the theorems.

Paolo's subsequent work [21] shed light on the principle behind DFS by analyzing the group-theoretic structure of the system. The paper indicated that the reason why DFS exists comes from symmetry of evolution, and the working principle of some other error-mitigation techniques, such as dynamic decoupling, can be explained with the symmetric evolution as well. This work reveals that symmetrization can be accomplished through purely unitary methods, without the need for additional spatial resources. The analyses includes intensive use of representation theory and operator algebra which is beyond my comprehension. Therefore, in this review, I only summarize the major steps from the paper and ignore most of detail.

Invariant subspaces can be described in an abstract manner using the data $(\mathscr{H}, H, \mathscr{G}, \rho)$. This involves: (i) a finite-dimensional Hilbert space $\mathscr{H}$, (ii) a Hermitian operator H (Hamiltonian) over $\mathscr{H}$, (iii) a finite group $\mathscr{G}$ with order $|\mathscr{G}|$, (iv) a unitary representation ρ of $\mathscr{G}$ in $\mathscr{H}$, with $\rho : g \in \mathscr{G} \mapsto \rho_g = \exp\left(ih_g^\rho\right)$, and the properties $\rho_{gh} = \rho_g \rho_h$ and $\rho_{g-1} = \rho_g^\dagger$. The representation ρ is considered irreducible (irrep) if there are no non-trivial invariant subspaces in $\mathscr{H}$. The space $\mathscr{H}$ is split according to the $\mathscr{G}$-irreps, so that $\mathscr{H} = \oplus_J n_J \mathscr{H}_J$(Note that this expression is a generalized version of Equation (4)), where n_J is the multiplicity of the invariant subspace $\mathscr{H}_J$ associated with the Jth irrep of $\mathscr{G}$. We want to find invariant subspace with $J = 0$ so that we can eliminate our interaction Hamiltonian. The sector corresponding to the identity irrep is therefore what we need here. It is a subspace spanned by the vectors in $\mathscr{H}$ invariant under the action of $\mathscr{G}$: $\mathscr{H}_{\text{inv}}^\rho =: \{|\psi\rangle \in \mathscr{H} : \rho_g |\psi\rangle = |\psi\rangle, \forall g \in \mathscr{G}\}$.

According to the paper, $\pi_\rho =: |\mathscr{G}|^{-1} \sum_{g \in \mathscr{G}} \rho_g$ is the projector onto $\mathscr{H}_{\text{inv}}^\rho$. $\{|g\rangle\}$ is an orthonormal basis that is in a one-to-one correspondence with the elements of $\mathscr{G}$. $|0\rangle =: |\mathscr{G}|^{-1/2} \sum_{g \in \mathscr{G}} |g\rangle$ is the initial state in other Hilbert space(environment). $W_\rho =: \sum_{g \in \mathscr{G}} \rho_g \otimes \Pi_g$ is an unitary operator over system space and other space, where $\Pi_g =: |g\rangle\langle g|$. Let $|\psi\rangle$ be an arbitrary element of $\mathcal{H}$. Apply W_ρ to the initial state $|\Psi_0\rangle = |\psi\rangle \otimes |0\rangle$:

$$W_\rho |\Psi_0\rangle = \frac{1}{\sqrt{|\mathscr{G}|}} \sum_{g \in \mathscr{G}} \rho_g |\psi\rangle \otimes |g\rangle \tag{5}$$

By projecting over $|0\rangle$ i.e., applying $\mathbb{I} \otimes \Pi_0$. We finally get

$$|\Psi_0\rangle \mapsto |\mathscr{G}|^{-1} \sum_{g \in \mathscr{G}} \rho_g |\psi\rangle \otimes |0\rangle = (\pi_\rho \otimes \mathbb{I}) |\Psi_0\rangle \tag{6}$$

By discarding the ancillary factor, the $\mathscr{G}$-invariant component of $|\psi\rangle$ can be obtained with probability of success given by $\| (\mathbb{I} \otimes \Pi_0) W_\rho |\Psi_0\rangle \|^2 = \| \pi_\rho |\psi\rangle \|^2$.

B. D. A. Lidar's work

D. A. Lidar and his colleagues [10] developed the DFS theory using a semigroup approach and came to the same conclusion as Paolo. They employed the master equation in the form of Kraus operator's basis decomposition:

$$\mathcal{L}_t[\hat{R}] = -\mathrm{i}[\hat{H}(t), \hat{R}] + \sum_{i,j>1} g'_{i,j}(t) \left(\hat{F}_i \hat{R} \hat{F}_j^\dagger - \frac{1}{2} \left\{ \hat{F}_j^\dagger \hat{F}_i, \hat{R} \right\} \right) \tag{7}$$

rewrite this into the form:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \mathrm{L}[\rho] \equiv -\frac{i}{\hbar}[\mathbf{H}, \rho] + \mathrm{L}_D[\rho] \\ \mathrm{L}_D[\rho] &= \frac{1}{2} \sum_{\alpha,\beta=1}^{M} a_{\alpha\beta} \mathrm{L}_{\mathbf{F}_\alpha, \mathbf{F}_\beta}[\rho] \\ \mathrm{L}_{\mathbf{F}_\alpha, \mathbf{F}_\beta}[\rho] &= \left[\mathbf{F}_\alpha, \rho \mathbf{F}_\beta^\dagger \right] + \left[\mathbf{F}_\alpha \rho, \mathbf{F}_\beta^\dagger \right] \end{aligned} \tag{8}$$

Assume that $\{|i\rangle\}_{i=1}^{N_0}$ represents the basis for an invariant decoherence-free (DF) subspace $\tilde{\mathcal{H}} \subseteq \mathcal{H}$ with dimensionality N_0. The density matrix takes the following form:

$$\tilde{\rho} = \sum_{i,j=1}^{N_0} \tilde{\rho}_{ij} |i\rangle\langle j| \tag{9}$$

Let us assume that the error bases act on the DF states as follows: $\hat{\mathbf{F}}_\alpha|i\rangle = \sum_{j=1}^{N_0} c_{ij}^\alpha |j\rangle$. The DF condition is expressed as $\mathrm{L}_D[\tilde{\rho}] = 0$, which implies that $\mathrm{L}_{\mathbf{F}_\alpha,\mathbf{F}_\beta}[\tilde{\rho}]$ is equal to 0 for all values of α and β. We start with the following equation:

$$\begin{aligned}\mathrm{L}_{\mathbf{F}_\alpha,\mathbf{F}_\beta}[\tilde{\boldsymbol{\rho}}] = \sum_{ij,mn=1}^{N_0} \tilde{\rho}_{ij} \Big(2c_{jm}^{\beta^*} c_{\mathrm{in}}^\alpha |n\rangle\langle m| \\ - c_{mn}^{\beta*} c_{\mathrm{in}}^\alpha |m\rangle\,|j\rangle - c_{jm}^{\beta^*} c_{nm}^\alpha\,|i\rangle\,\langle n|\Big).\end{aligned} \tag{10}$$

To ensure that all values of $\mathrm{L}_{\mathbf{F}_\alpha,\mathbf{F}_\beta}[\tilde{\rho}]$ are equal to 0, two conditions must be met. The first condition states that there must only be one projection operator $|n\rangle\langle m|$ in each term, which leads to the relationship $c_{\mathrm{in}}^\alpha = c_i^\alpha \delta_{\mathrm{in}}$. The second condition states that if $c_i^\alpha \neq 0$, then $\frac{c_{\alpha j}}{c_{\alpha i}} + \frac{c_{\beta i}^*}{c_{\beta j}^*} = 2$. By utilizing the first condition, the aforementioned equation can be simplified to:

$$\mathrm{L}_{\mathbf{F}_\alpha,\mathbf{F}_\beta}[\tilde{\rho}] = \sum_{ij=1}^{N_0} \tilde{\rho}_{ij}|i\rangle\langle j| \left(2c_j^{\beta^*} c_i^\alpha - c_i^{\beta^*} c_i^\alpha - c_j^{\beta^*} c_j^\alpha\right) \tag{11}$$

Since condition (2) holds for all α and β, we have $\frac{c_{\alpha j}}{c_{\alpha i}} + (\frac{c_{\alpha i}}{c_{\alpha j}})^* = 2$ when $\alpha = \beta$. This implies that $c_{\alpha j}/c_{\alpha i} = 1\ \forall i, j$, which further implies that $c_{\alpha i}$ is independent of i, i.e., $\hat{\mathbf{F}}_\alpha|i\rangle = c_\alpha|i\rangle, \forall\alpha$. We can derive the relation $\left[\hat{\mathbf{F}}_\alpha, \hat{\mathbf{F}}_\beta\right]|i\rangle = 0$. If $\mathcal{L}$ is semisimple, we can represent the commutator with non-vanishing structure constants $f_{\alpha,\beta}^\gamma$ of the Lie algebra as follows: $\left[\hat{\mathbf{F}}_\alpha, \hat{\mathbf{F}}_\beta\right] = \sum_{\gamma=1}^M f_{\alpha,\beta}^\gamma \hat{\mathbf{F}}_\gamma$. This implies: $\sum_{\gamma=1}^M f_{\alpha,\beta}^\gamma c_\gamma = 0 \quad \forall\alpha,\beta$.

As the structure constants define the M-dimensional "adjoint" matrix representation $\mathcal{L}$: $[\mathrm{ad}(\hat{\mathbf{F}}_\alpha)]_{\gamma,\beta} = f_{\alpha,\beta}^\gamma$, and the generators of the Lie algebra are linearly independent, the only situation where the equation $\sum_{\gamma=1}^M f_{\alpha,\beta}^\gamma c_\gamma = 0 \quad \forall\alpha,\beta$ can be satisfied is when all $c_\gamma = 0$. This proves the following theorem:

Theorem 1 A necessary and sufficient condition for generic decoherence-free dynamics ($\mathrm{L}_D[\tilde{\rho}] = 0$) in a subspace $\tilde{\mathcal{H}} = \mathrm{Span}\left[\{|i\rangle\}_{i=1}^{N_0}\right]$ of the register Hilbert space, is that all basis states $|i\rangle$ are degenerate eigenstates of all the error generators $\left\{\hat{\mathbf{F}}_\alpha\right\} : \hat{\mathbf{F}}_\alpha|i\rangle = c_\alpha|i\rangle, \forall\alpha$; or, if $\mathcal{L}$ is semisimple, that all $|i\rangle$ are annihilated by all $\left\{\hat{\mathbf{F}}_\alpha\right\}$:

$$\hat{\mathbf{F}}_\alpha|i\rangle = 0 \quad \forall\alpha, i$$

This theorem is in agreement with what was observed in Paolo's work. In the spin-boson model mentioned earlier, the error generators $\{\hat{\mathbf{F}}_\alpha\}$ are the basis for the interaction Hamiltonian. As a reminder, the interaction Hamiltonian must annihilate all DF states, i.e., $H_I\,|\psi^{(N)}\rangle \otimes |\psi_B\rangle = 0$, which implies that $\hat{\mathbf{F}}_\alpha|i\rangle = 0 \quad \forall\alpha, i$.

D. A. Lidar et al.'s work [10] had other significant contributions, including demonstrating the stability of DFS under first-order symmetry-breaking perturbations, which has important implications for quantum computation. They also showed that the size of DFS (i.e., the dimension of the sub-Hilbert space) scales exponentially with the number of qubits, N, for $N \gg 1$. Their work also discussed universal quantum computation in DFS, which addresses the feasibility of encoding quantum information in the DFS and applying universal logical operators to the encoded quantum state.

D. A. Lidar et cl. [7] [12]'s subsequent work connected the methods of DFS and quantum error correcting code with the theory of noiseless subsystem. In [7], they associated error correcting codes with irreducible representations of operator algebras, demonstrating that noiseless subsystems are equivalent to error-correcting codes with infinite distance. Since they also used operator algebra just as what Paolo did in his work [21], this review does not include the detail of their analyses to avoid repetition. Some major contributions are emphasized here

- They demonstrated that quantum error correction can be reformulated without limiting the statistical characteristics of the noise in the environment, by describing error process in a broad algebraic context.
- The notion of a noiseless subsystem can be a unifying framework for quantum information protection analyses.
- They defined the notion of the noiseless subsystem.

  Definition 3 A noiseless subsystem is a subsystem where information is intrinsically stabilized against the effects of the noise with no need for detective and corrective action.

D. A. Lidar et cl. included how to use the theory of noiseless subsystem to describe DFS in their later work [12]. The description is similar to what we can see from Paolo's work [21] and is not included in this review due to the words limitation.

III. Applications

In this section, we discuss three early applications and some recent applications.

A. Pairing Quantum Bits

Luming and Guangcan established a DFS through the association of each qubit with an ancillary qubit [3]. They employed a spin-boson model to explain the functionality of their approach and named it as "free-Hamiltonian elimination". The equation (12) represents the Hamiltonian resulting from the pairing of each qubit l with its corresponding ancillary qubit l'.

$$\begin{aligned}H_{2L} =& \omega_0 \sum_{l=1}^{L} (\sigma_l^z + \sigma_{l'}^z) + \sum_{\omega} \bigcup_{l=1}^{L} \left(\omega a_{\omega l}^+ a_{\omega l}\right) \\ &+ \sum_{l=1}^{L} \sum_{\omega} \Big\{ \left[\lambda^{(1)} (\sigma_l^x + \sigma_{l'}^x) + \lambda^{(2)} (\sigma_l^y + \sigma_{l'}^y)\right. \\ &\left. + \lambda^{(3)} (\sigma_l^z + \sigma_{l'}^z)\right] g_{\omega l} \left(a_{\omega l}^+ + a_{\omega l}\right) \Big\}.\end{aligned} \tag{12}$$

The first, second, and third terms in this statement represent the system Hamiltonian, the bath Hamiltonian, and the interaction Hamiltonian, respectively. The objective is to ensure that the system Hamiltonian is compatible with the

operators used on the system component of the interaction Hamiltonian. Additionally, an external classical electromagnetic field that affects all qubit pairs was employed during the driven process. This process can be described using the Hamiltonian $H_{\mathrm{drv}} = \sum_{l=1}^{L} \left[g_1 \left(\sigma_l^x + \sigma_{l'}^x \right) + g_2 \left(\sigma_l^y + \sigma_{l'}^y \right) \right]$. By adjusting the phase and intensity of the electromagnetic field, the conditions $g_1 : g_2 : \omega_0 = \lambda^{(1)} : \lambda^{(2)} : \lambda^{(3)}$ can be met. As a result, the new Hamiltonian can be expressed:

$$\begin{aligned} H =& H_{2L} + H_{\mathrm{drv}} \\ =& \sum_{l=1}^{L} \left\{ \left(S_l + S_{l'} \right) \left[\frac{\omega_0}{\lambda^{(3)}} + \sum_{\omega} g_{\omega l} \left(a_{\omega l}^{+} + a_{\omega l} \right) \right] \right\} \\ &+ \sum_{\omega} \bigcup_{l=1}^{L} \left(\omega a_{\omega l}^{+} a_{\omega l} \right) \end{aligned} \tag{13}$$

where $S_l = \lambda^{(1)} \sigma_l^x + \lambda^{(2)} \sigma_l^y + \lambda^{(3)} \sigma_l^z$. It can be demonstrated that when the given Hamiltonian is employed, the system state evolves unitarily only if the initial state of the qubit pairs is a co-eigenstate of all the operators $S_l + S_{l'}$. These eigenstates of the operators $S_l + S_{l'}$ are referred to as coherence-preserving states. There is a one-to-one relationship between the logical state $|\Psi_L\rangle = \sum_{\{i_l\}} c_{\{i_l\}} |\{i_l\}\rangle$ and the coherence-preserving state of L qubit pairs $|\Psi_L\rangle_{coh} = \sum_{\{i_l\}} c_{\{i_l\}} |\{i_l, -i_l\}\rangle$. Luming and Guangcan demonstrated in their work [3] how to initialize the pair-qubit from the eigenstates of σ^z. They achieved this by utilizing single-qubit rotation operation $R_l(\theta)$ and the CNOT gate $C_{i,j}$. The remainder of their work indicated that (1) the encoding efficiency approaches 1 if all the qubit pairs can interact with the same modes of the environment, and (2) the coherence can still be preserved during gate operations by replacing the qubit gates with those designed for the qubit pairs.

It should be noted that the coherence-preserving states is the eigenstates of all the operators $S_l + S_{l'}$, and their corresponding eigenvalues m_l for these states are all 0. This implies that the coherence-preserving state they engineered can also annihilate all the interaction Hamiltonian. This finding is consistent with the theory proposed by Paolo and D.A. Lidar et al.

B. Concatenating DFS with Quantum Error Correcting Codes

D.A. Lidar et cl. [9] proposed a method to concatenating DFS with QECC. They wrote the error matrix as follows:

$$\mathbf{A}_a = \tilde{\mathbf{A}}_a + \epsilon \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{Q}_3 & \mathbf{Q}_4 \end{pmatrix} \tag{14}$$

where $\tilde{\mathbf{A}}_a$ is collective decoherence error(dominant). The second matrix represents the symmetry breaking perturbation(Independent error on each qubits). Under collective decoherence, the smallest DFS which can encode one logical qubit is made up of four physical qubits. If each physical qubit in QECC is made of 4 qubits which encode one-qubit information in DFS. We can, for example, construct a "perfect" 5-qubit [[5,1,3]] code that is free of both decoherence errors and Pauli errors. To fulfill this design, a modified controlled-not gate is desired to detect and correct errors that maps encoded state out of DFS, and the total number of qubits used in this design is 4x5=20.

C. Dynamic Decoupling

The work of Lorenza Viola and Seth Lloyd [18] did not use the idea of DFS. In Paolo's work [21], he proved that the symmetric evolution also accounts for why the technique of dynamic decoupling works. This means the theory of DFS can be also applied to dynamic decoupling. I would like to know if there is a connection between the two techniques, and that's the reason I include it in this review. The scheme they proposed was reminiscent of spin echo [5]. Their approach involves using π-pulses to flip the direction of a spin-1/2 system continually. The fundamental concept behind this strategy is to completely eliminate the system-bath interaction Hamiltonian by canceling out its impact over a longer period. Due to the length of the paper, it is not feasible to give a comprehensive overview in this brief review. I have summarized some of the essential aspects of their study as follows.

1) They considered the process of applying two successive π-pulses as a fundamental operation and evaluated the transformation of the density matrix of the system-bath system throughout a single spin-flip cycle.
2) Their examination revealed that if the rate of applying the π-pulse is faster than the correlation time of the environment τ_c, decoherence can be significantly reduced. The correlation time τ_c of the environment is related to the inverse of the cutoff frequency of the spectral density of the bath.
3) In the limit of continuous flipping and suppression,i.e., the time interval to apply π-pulse $\Delta t \to \infty$, decoherence is entirely and perfectly eliminated, regardless of the temperature or spectral density function.

D. Recent Applications

X.B. Wang applied DFS in quantum key distribution [19]. Their model assumed collective rotation errors. The state is encoded in the DFS as follows

$$\begin{aligned} |H'\rangle &= \left|\phi^+\right\rangle = \frac{1}{\sqrt{2}}(|H\rangle|H\rangle + |V\rangle|V\rangle) \\ |V'\rangle &= \left|\psi^-\right\rangle = \frac{1}{\sqrt{2}}(|H\rangle|V\rangle - |V\rangle|H\rangle) \\ |+'\rangle &= \frac{1}{\sqrt{2}}\left(|H'\rangle + |V'\rangle\right) = \frac{1}{\sqrt{2}}(|H\rangle|+\rangle - |V\rangle|-\rangle) \\ |-'\rangle &= \frac{1}{\sqrt{2}}\left(|H'\rangle - |V'\rangle\right) = \frac{1}{\sqrt{2}}(|H\rangle|-\rangle + |V\rangle|+\rangle) \end{aligned} \tag{15}$$

Thus, it can be free of collective rotation errors. $+, -$ represent the diagonal state and anti-diagonal state respectively. Keys are encoded in the four states and the security bound is the same as BB84 protocol [19].

There is a requirement in field of state engineering that the dynamics during the state engineering process should remain adiabatic to achieve the desired state evolution in all kinds of platforms with high fidelity. Jarrod et cl. [15] designed a protocol combing the quantum control with DFS. Their

proposed method involves the adiabatic control of a many-body system in a highly dissipative cavity, utilizing a DFS while experiencing collective decoherence.

Lei et cl. [2] accomplished the conversion of entangled states in DFS. Their study proposes various methods of converting polarized-entangled Knill-Laflamme-Milburn (KLM) states into Greenberger-Horne-Zeilinger (GHZ) states in DFS, utilizing weak cross-Kerr nonlinearity. The proposed schemes were analyzed numerically and found to have high fidelity and success probability.

IV. Discussion and Conclusion

Paolo's research focuses on the theoretical development of DFS and symmetric evolution. What I find most remarkable about their work is how they demonstrate that the principles behind DFS and dynamic decoupling are both rooted in the symmetry of evolution and can be explained using group-theoretic structures. Despite their differences in technical details, these two methods have the same underlying principles. This discovery has inspired me to include the work of Lorenza Viola and Seth Lloyd [18] in this review, which is not directly related to DFS. Both techniques aim to entirely eliminate the interaction Hamiltonian. The difference between the two approaches is that DFS involves designing or discovering a set of states that serve as the 0-eigenvalue eigenvectors of the interaction Hamiltonian, as well as the non-zero eigenvalue eigenvectors of the system/bath Hamiltonian, while dynamic decoupling physically removes the interaction Hamiltonian.

The nice thing about D. A. Lidar's work, in my opinion, is that they introduced the concept of noiseless subsystem which is a "universal language" that can describe both quantum error correcting code and DFS. We can easily find differences and similarities of these two methods under this unified framework. Compared to error correcting codes which actively detect and correct errors in the state, DFS is a passive error prevention scheme [20]. Rather than fixing errors, the quantum information is hidden from the environment, and hence it never becomes corrupted [11]. It is clear that the task of finding such subspace is non-trivial, and we may probably never find it in some cases. Paolo [21] proved that the symmetry of evolution is a necessary condition for having such subspace. I also want to point out that DFS is a special application of noiseless subsystem. There are a lot research focusing on noiseless subsystem and decoherence. The applications of noiseless subsystem are a much broader topic and are also very interesting.

As for applications of DFS, D.A. Lidar et cl. [9]'s proposal of concatenating DFS with QECC seems to be a good design, but the strange thing is that I did not find more papers discussing the experimental details of how to do this. There is probably a reason that makes this design not worth using. Recent applications of DFS are most related to the state preparation. This might be the dominant trend in the use of DFS.

This review has demonstrated that symmetry can result in the complete protection of quantum information. This is evident through the identification of a decoherence-free subspace in the collective decoherence of spin-boson model, made possible by the favorable symmetry of system-bath evolution. However, the limitations of DFS are also apparent. DFS is only effective against collective errors, and the assumption of collective decoherence is reasonable but not always applicable. Cluster decoherence may be a more realistic assumption in some situations, and in the extreme case, we have independent decoherence. Therefore, the task of finding DFS is non-trivial, and in certain scenarios, such a subspace may not exist.

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