Pre-Semester Course in Statistics

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Overview

1. Hypotheses and test procedures
2. Tests for population means
   - Normal known variance
   - Normal unknown variance
   - Large sample tests or CLT to the rescue
3. P-values
4. Two sample tests
   - Normal known variance
   - Large sample tests
   - Normal unknown variance
   - Two population variances

Hypothesis

A hypothesis is an assertion about parameters.

- The population mean $\mu = 2$.
- The proportion of successes $p = .3$.
- The population mean $\mu > 2$.
- The difference between two population means $\mu_1 - \mu_2 = 0$.
- The difference between two population means $\mu_1 - \mu_2 > 4$.

The null and the alternate

Typically there are two types of hypotheses in the hypothesis testing framework:

1. **Null hypothesis**: what one believes prior to the test. For example

$$
\begin{aligned}
H_0 &: \mu_1 = \mu_2, \\
H_0 &: \mu_1 = 10, \\
H_0 &: \mu_1 - \mu_2 = 5.
\end{aligned}
$$

2. **Alternative hypothesis**: a hypothesis contradictory to the null

$$
\begin{aligned}
H_A &: \mu_1 \neq \mu_2 (2 - sided) \\
H_A &: \mu_1 > \mu_2 (1 - sided) \\
H_A &: \mu_1 < \mu_2 (1 - sided).
\end{aligned}
$$

Two main uses of hypothesis testing

The null hypothesis is the one people focus on in classical hypothesis testing and there are two logical constructions of hypothesis testing.

- **Confirming a theory**: In physics one may believe that force is equal to mass times acceleration

  $$F = ma$$

  one can measure for objects of various masses the force and acceleration and use as a null hypothesis and alternative hypothesis

  $$\begin{aligned} H_0 &: \quad F - ma = 0 \\ H_A &: \quad F - ma \neq 0. \end{aligned}$$

  This confirmation of a hypothesis is quite common and in this case one typically wishes that the null is not rejected or there is strong evidence for the null.

Two main uses of hypothesis testing

The null hypothesis is the one people focus on in classical hypothesis testing and there are two logical constructions of hypothesis testing.

- **Repudiating a control**: A more common use of hypothesis testing is to set the null up as a control and show that there is evidence to reject it.
  An example is that rocket fuel makes cars run faster. In this case one can take speeds of cars spiked with rocket fuel and compute the population mean, μ_1, and compare this to the population mean of cars without rocket fuel, μ_2, and use the following null and alternative hypothesis

$$\begin{aligned} H_0 &: \quad \mu_1 = \mu_2 \\ H_A &: \quad \mu_1 > \mu_2. \end{aligned}$$

  In this case we would like to reject the null or that there is strong evidence against the null.

Two types of hypotheses

Two types of hypotheses are simple and composite:

- **Simple**: A simple hypothesis is one where the distribution under the hypothesis is fully specified. For example $X_1, ..., X_n \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2 = 5)$ and the hypothesis is
$$H : \mu = 10.$$
- **Composite**: For a composite hypothesis the distribution under the hypothesis is not fully specified. For example $X_1, ..., X_n \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2 = 5)$ and the hypothesis is
$$H : \mu > 10.$$
There are a variety of distributions that will satisfy this, any normal with mean greater than 10.

Typically the null is simple and the alternative is composite.

Test procedures

A test procedure has the following components:

- Data: Samples drawn from some distribution

$$X_1, ..., X_n \overset{iid}{\sim} p(x|\theta)$$

- The null hypothesis: this determines what parameter one cares about in the distribution. For example

$$H_0 : \mu = 20.$$

- Test statistic: A test statistic that is used to estimate the parameter one cares about from the above distribution. For example

$$\bar{X} = \frac{1}{n} \sum_i X_i.$$

- Rejection region: Values of the test statistic for which H_0 will be rejected.

Test procedures: *Silver content of Byzantine coins*

Data A number of coins from the reign of King Manuel I, Comnemus (1143 - 80) were discovered in Cyprus. They arise from four different coinages at intervals throughout his reign. Here are the silver content of the coins:

Table 2.3 Silver content of coins

First	Second	Third	Fourth
5.9	6.9	4.9	5.3
6.8	9.0	5.5	5.6
6.4	6.6	4.6	5.5
7.0	8.1	4.5	5.1
6.6	9.3		6.2
7.7	9.2		5.8
7.2	8.6		5.8
6.9			
6.2			

We would like to know what is the standard silver content of the **first** coinage.

Test procedures: *Silver content of Byzantine coins*

The null hypothesis We could find that the silver content of first coinage is range from 5 to 7. On the face of it the suspicion we might expect the standard to be 6.5. But there is a need for firm statistical evidence if it is to be confirmed. Suppose the true percentage of silver in coinage first is μ_1. The null hypothesis would be

$$H_0 : \mu_1 = 6.5$$

versus the two-sides alternative hypothesis

$$H_1 : \mu_1 \neq 6.5.$$

If there is a historical record suggests that the silver content can be larger than 6.5, we could use one-side alternative hypothesis

$$H_1 : \mu_1 > 6.5.$$

Test procedures: *Silver content of Byzantine coins*

Test statistic The first moment is the unbiased, consistent and efficient sample estimator for population expectation. Thus we use $\bar{X}_1$ as the estimator for μ_1.

$$\bar{X}_1 = \frac{1}{n}\sum_i X_1 i = \frac{1}{9}(5.9+6.8+6.4+7.0+6.6+7.7+7.2+6.9+6.2) \approx 6.744$$

Rejection region We need rejection region to tell us when the null hypothesis will be rejected. For example, if we set the reject region as

$$\bar{X}_1 > 6.5$$

In our example, $\bar{X}_1 = 6.744 > 6.5$, thus we reject the null hypothesis $\mu_1 = 6.5$.
But if we set the reject region as

$$\bar{X}_1 > 1.05 \times 6.5$$

We cannot reject the null hypothesis because

$$6.744 < 1.05 \times 6.5 = 6.825$$

Errors in hypothesis testing

There are two types of errors in hypothesis testing.

- **Type I**: Rejecting the null hypothesis, H_0, when it is true. The probability of this type of error is designated as

$$\alpha = \text{Pr}(H_0 \text{ is rejected when true}).$$

- **Type II**: Not rejecting the null hypothesis, H_0 when it is false. The probability of this type of error is designated as

$$\beta = \text{Pr}(H_0 \text{ not is rejected when false}).$$

Errors in hypothesis testing: an example

$$X_1, ..., X_n \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2 = 10).$$

$$\begin{aligned} H_0 &: \mu = 10, \\ H_A &: \mu > 10. \end{aligned}$$

Let us say that μ is really 10 and we select a rejection region of $\bar{X} > 15$. This means that for any $\bar{X} > 15$ greater than 15 we reject the null.

We can compute α and β for this case as follows:

$$\begin{aligned} \alpha &= \Pr(\bar{X} > 15 | X \sim \mathrm{N}(\mu = 10, \sigma^2 = 10)), \\ \beta &= \Pr(\bar{X} \leq 15 | X \sim \mathrm{N}(\mu > 10, \sigma^2 = 10)). \end{aligned}$$

There is no rejection region that can make both α and β equal to zero. So we seek a rejection region that controls both simultaneously.

The structure of tests

- Get data from a specified distribution.
- State null and alternative hypotheses.
- State the test statistic.
- State the acceptable type I error, α. This is called the α critical value.
- Compute the rejection region based on the above.
- See if the test statistic falls in the rejection region and reject the null based on this.

Normal population known σ

Typically the parameter σ is not known but this will describe the ideas. If $X_1, ..., X_n \overset{iid}{\sim} \mathsf{N}(\mu, \sigma^2)$ with known σ. The null and alternative hypotheses are

$$\begin{aligned} H_0 &: \mu = \mu_1, \\ H_A &: \mu > \mu_1. \end{aligned}$$

We want to compute rejection regions for this test.
We first have to specify the the level of type I error or the critical α of the test, say $\alpha = .05$. We specify the test statistic as $\bar{X}$. We now need to compute the value ℓ for which

$$.05 = \Pr(\bar{X} > \ell | X_i \sim \mathsf{N}(\mu = \mu_1, \sigma^2)).$$

To do this we need the "distribution of the test statistic under the null hypothesis."

Normal population known σ

If the null hypothesis is true and the data comes from a normal with known σ then we know that the following test statistic is distributed as a standard normal

$$z = \frac{\bar{X} - \mu_1}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1).$$

From this we can compute ℓ in the following

$$.05 = \Pr(\bar{X} > \ell | X_i \sim \mathrm{N}(\mu = \mu_1, \sigma^2)),$$

as $z_{.05}$. So our rejection region is $z > z_{.05}$.

The intuition behind is that if indeed $\mu = \mu_1$, the probability for $z > z_{.05}$ is 0.05, which is very low and thus $z > z_{.05}$ is unlikely to happen. So if it happens, we cannot accept previous assumption $\mu = \mu_1$ at the 0.05 level.

Normal population known σ

If $X_1, ..., X_n \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2)$ with known σ.

Null hypothesis $H_0 : \mu = \mu_1$.

Test statistic $z = \frac{\bar{X} - \mu_1}{\sigma / \sqrt{n}}$.

- $H_A : \mu > \mu_1$: the rejection region is $z \geq z_\alpha$.
- $H_A : \mu < \mu_1$: the rejection region is $z \leq -z_\alpha$.
- $H_A : \mu \neq \mu_1$: the rejection region is $z \leq -z_{\alpha/2}$ or $z \geq z_{\alpha/2}$.

Recipe

- Identify the parameter of interest, mean or variance or rate parameter.
- Determine the null value and state the null and alternative hypotheses.
- Give the formula for the computed value of the test statistic and plug in the null value and other known parameters in this step, for example μ_1 and σ in the above.
- Based on the distribution of the above statistic compute the rejection region for a selected significance level α.
- Reject or accept H_0.

Normal population unknown σ

If $X_1, ..., X_n \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2)$ with unknown σ. The null and alternative hypotheses are

$$\begin{aligned} H_0 &: \mu = \mu_1, \\ H_A &: \mu > \mu_1. \end{aligned}$$

We want to compute rejection regions for this test.
We first have to specify the the level of type I error or the critical α of the test, say $\alpha = .05$. We specify the test statistic as $\bar{X}$. We now need to compute the value ℓ for which

$$.05 = \Pr(\bar{X} > \ell | X_i \sim \mathrm{N}(\mu = \mu_1, \sigma^2)).$$

To do this we need the "distribution of the test statistic under the null hpothesis."

Normal population unknown σ

If the null hypothesis is true and the data comes from a normal with unknown σ then we know that the following test statistic is distributed as a t-distribution with $n-1$ degrees of freedom

$$t = \frac{\bar{X} - \mu_1}{S/\sqrt{n}} \sim \text{t-dist}_{n-1},$$

where S^2 is the estimate of the sample variance.

We proceed exactly as before but use the t-distribution to compute the rejection region.

Normal population unknown σ

If $X_1, ..., X_n \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2)$ with unknown σ.

Null hypothesis $H_0 : \mu = \mu_1$.

Test statistic $t = \frac{\bar{X} - \mu_1}{s/\sqrt{n}}$.

- $H_A : \mu > \mu_1$: the rejection region is $z \geq t_{\alpha, n-1}$.
- $H_A : \mu < \mu_1$: the rejection region is $z \leq -t_{\alpha, n-1}$.
- $H_A : \mu \neq \mu_1$: the rejection region is $t \leq -t_{\alpha/2, n-1}$ or $t \geq t_{\alpha/2, n-1}$.

Normal population unknown σ

The t-distribution is valid independent of the size of the sample. It holds for small n.

Large sample tests

If $X_1, ..., X_n \overset{iid}{\sim} p(\theta)$ where the distribution has bounded mean $\mu < \infty$ and variance $\sigma^2 < \infty$.

The null and alternative hypotheses are

$$
\begin{aligned}
H_0 &: \mu = \mu_1, \\
H_A &: \mu > \mu_1.
\end{aligned}
$$

We want to compute rejection regions for this test.
We first have to specify the the level of type I error or the critical α of the test, say $\alpha = .05$. We specify the test statistic as $\bar{X}$. We now need to compute the value ℓ for which

$$.05 = \Pr(\bar{X} > \ell | X_i \sim p(\theta)).$$

To do this we need the "distribution of the test statistic under the null hypothesis."

Large sample tests

If the null hypothesis is true then by the central limit theorem test statistic is distributed as standard normal

$$z = \frac{\bar{X} - \mu_1}{S/\sqrt{n}} \sim \mathrm{N}(0, 1),$$

where S^2 is the estimate of the sample variance.

We proceed exactly as before using z_α values.

Large sample tests

If $X_1, ..., X_n \overset{iid}{\sim} p(\theta)$ with bounded mean and variance.

Null hypothesis $H_0 : \mu = \mu_1$.

Test statistic $z = \frac{\bar{X} - \mu_1}{s/\sqrt{n}}$.

- $H_A : \mu > \mu_1$: the rejection region is $z \geq z_\alpha$.
- $H_A : \mu < \mu_1$: the rejection region is $z \leq -z_\alpha$.
- $H_A : \mu \neq \mu_1$: the rejection region is $z \leq -z_{\alpha/2}$ or $z \geq z_{\alpha/2}$.

Large sample test

If $X_1, ..., X_n \overset{iid}{\sim} p(\theta)$ with bounded mean and variance.

Null hypothesis $H_0 : \theta = \theta_1$.

Ideal test statistic

$$z = \frac{\hat{\theta} - \theta_1}{\sigma_{\hat{\theta}}},$$

but $\sigma_{\hat{\theta}}$ is unknown so assume $s_{\hat{\theta}} \approx \sigma_{\hat{\theta}}$ and use

$$z = \frac{\hat{\theta} - \theta_1}{s_{\hat{\theta}}}.$$

Example: population proportion

Given random variable $X \sim \text{binomial}(p, n)$ with known $n = 500$ we want to apply the hypothesis testing framework to $H_0 : p = p_1$.

Our estimate is $\hat{p} = \frac{X}{n}$ and we know p under the null hypothesis so $\sigma_{\hat{p}} = \sqrt{p_1(1 - p_1)/n}$ which results in the test statistic

$$z = \frac{\hat{p} - p_1}{\sqrt{p_1(1 - p_1)/n}},$$

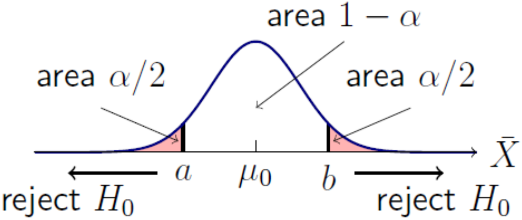
which if $\min(np_1, n(1 - p_1)) > 10$ is standard normal.

- $H_A : p > p_1$: the rejection region is $z \geq z_\alpha$.
- $H_A : p < p_1$: the rejection region is $z \leq -z_\alpha$.
- $H_A : p \neq p_1$: the rejection region is $z \leq -z_{\alpha/2}$ or $z \geq z_{\alpha/2}$.

Confidence interval

Confidence interval is the residual set of reject region. Given a α, we can calculate the 2-sides confidence interval that

$$Pr(-z_{\alpha/2} < z < z_{\alpha/2}) = 1 - \alpha$$



Formal definitions

Definition

The **P-value** *is the probability, assuming that H_0 is true, of obtaining a test statistic value at least as contradictory to H_0 as the value obtained on the data. The smaller the P-value the more contradictory is the data to H_0.*

The **P-value** is the sample or observed significance level. Once the P-value is determined one can reject or accept the null hypothesis at a α level by comparing the P-value to α

- Reject H_0: P-value $\leq \alpha$
- Do not reject H_0: P-value $> \alpha$.

Example

If $X_1, ..., X_n \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2)$ with known σ. The null and alternative hypotheses are

$$\begin{aligned} H_0 &: \mu = \mu_1, \\ H_A &: \mu > \mu_1. \end{aligned}$$

For a particular data set we can compute the following

$$z_{\text{data}} = \frac{\bar{x} - \mu_1}{\sigma/\sqrt{n}},$$

we can then compute

$$\text{p-value} = \Pr(z > z_{\text{data}})$$

which is the type I error if we set the critical region to z_{data}.

P-values for z-tests

For hypothesis tests using the z-statistic.

- Upper-tail test: $P = 1 - \Phi(z_{\text{data}})$.
- Lower-tail test: $P = \Phi(z_{\text{data}})$.
- Two-tailed test: $P = 2[1 - \Phi(|z_{\text{data}}|)]$.

Two sample tests

A very common situation is the following setup

1. $$X_1, \ldots, X_m \overset{iid}{\sim} p_1(\theta_1).$$
2. $$Y_1, \ldots, Y_n \overset{iid}{\sim} p_2(\theta_2).$$
3. X and Y are independent.

Questions:

$$\begin{aligned} \mu_1 - \mu_2 &= \Delta_0 \\ \mu_1 - \mu_2 &> \Delta_0 \\ \mu_1 - \mu_2 &< \Delta_0 \\ \mu_1 - \mu_2 &\neq \Delta_0. \end{aligned}$$

Normal population known σ

$$X_1, ..., X_m \overset{iid}{\sim} \mathrm{N}(\mu_1, \sigma_1^2)$$

$$Y_1, ..., Y_n \overset{iid}{\sim} \mathrm{N}(\mu_2, \sigma_2^2)$$

The null and alternative hypotheses are

$$\begin{aligned} H_0 &: \mu_1 - \mu_2 = \Delta_0, \\ H_A &: \mu_1 - \mu_2 > \Delta_0. \end{aligned}$$

We want to compute rejection regions for this test.
We first have to specify the the level of type I error or the critical α of the test, say $\alpha = .05$. We specify the test statistic as $\bar{X} - \bar{Y}$. We now need to compute the value ℓ for which

$$.05 = \Pr(\bar{X} - \bar{Y} > \ell | X_i \sim \mathrm{No}(\mu = \mu_1, \sigma_1^2) \text{ and } Y_i \sim \mathrm{N}(\mu = \mu_2, \sigma_2^2)).$$

To do this we need the "distribution of the test statistic under the null hypothesis."

Normal population known σ

If the null hypothesis is true and the data comes from a normal with known σ then we know that the following test statistic is distributed as a standard normal

$$z = \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} \sim \mathrm{N}(0, 1).$$

As we did in the case of the one sample test we use z_α values.

Normal population known σ

$$X_1, ..., X_m \overset{iid}{\sim} \mathrm{N}(\mu_1, \sigma_1^2)$$

$$Y_1, ..., Y_n \overset{iid}{\sim} \mathrm{N}(\mu_2, \sigma_2^2)$$

Null hypothesis $H_0 : \mu_1 - \mu_2 = \Delta_0$.

Test statistic

$$z = \frac{\bar{x} - \bar{y} - \Delta_0}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} \sim \mathrm{N}(0, 1).$$

- $H_A : \mu_1 - \mu_2 > \Delta_0$: the rejection region is $z \geq z_\alpha$.
- $H_A : \mu_1 - \mu_2 < \Delta_0$: the rejection region is $z \leq -z_\alpha$.
- $H_A : \mu_1 - \mu_2 \neq \Delta_0$: the rejection region is $z \leq -z_{\alpha/2}$ or $z \geq z_{\alpha/2}$.

Large sample tests

$$X_1, \ldots, X_m \overset{iid}{\sim} p_1(\theta_1).$$

$$Y_1, \ldots, Y_n \overset{iid}{\sim} p_2(\theta_2).$$

with mean and variance bounded for both distributions.

Null hypothesis $H_0 : \mu_1 - \mu_2 = \Delta_0$.

Test statistic

$$z = \frac{\bar{x} - \bar{y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} \sim \mathrm{N}(0, 1).$$

- $H_A : \mu_1 - \mu_2 > \Delta_0$: the rejection region is $z \geq z_\alpha$.
- $H_A : \mu_1 - \mu_2 < \Delta_0$: the rejection region is $z \leq -z_\alpha$.
- $H_A : \mu_1 - \mu_2 \neq \Delta_0$: the rejection region is $z \leq -z_{\alpha/2}$ or $z \geq z_{\alpha/2}$.

Normal population unknown σ

$$X_1, ..., X_m \overset{iid}{\sim} \mathrm{N}(\mu_1, \sigma_1^2)$$
$$Y_1, ..., Y_n \overset{iid}{\sim} \mathrm{N}(\mu_2, \sigma_2^2)$$

but σ_1 and σ_2 are unknown. The null and alternative hypotheses are

$$\begin{aligned} H_0 &: \mu_1 - \mu_2 = \Delta_0, \\ H_A &: \mu_1 - \mu_2 > \Delta_0. \end{aligned}$$

We want to compute rejection regions for this test.
We first have to specify the the level of type I error or the critical α of the test, say $\alpha = .05$. We specify the test statistic as $\bar{X} - \bar{Y}$. We now need to compute the value ℓ for which

$$.05 = \Pr(\bar{X} - \bar{Y} > \ell | X_i \sim \mathrm{No}(\mu = \mu_1, \sigma_1^2) \text{ and } Y_i \sim \mathrm{N}(\mu = \mu_2, \sigma_2^2)).$$

To do this we need the "distribution of the test statistic under the null hpothesis."

Normal population unknown σ

If the null hypothesis is true and the data comes from a normal with unknown σ then we know that the following test statistic is distributed as a t-distribution with a very complicated formula for the degrees of freedom

$$t = \frac{\bar{x} - \bar{y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}},$$

where S_1^2 and S_2^2 are the estimates of the sample variances.

The degrees of freedom ν is

$$\nu = \frac{\left(\frac{s_1^2}{m} + \frac{s_2^2}{n}\right)^2}{\frac{(s_1^2/m)^2}{m-1} + \frac{(s_2^2/n)^2}{n-1}}.$$

Note for $s_1 \approx s_2$ and $m \approx n$ then $\nu \approx 2n$.

Normal population unknown σ

$$X_1, ..., X_m \overset{iid}{\sim} \mathrm{N}(\mu_1, \sigma_1^2)$$

$$Y_1, ..., Y_n \overset{iid}{\sim} \mathrm{N}(\mu_2, \sigma_2^2)$$

but σ_1 and σ_2 are unknown.

Null hypothesis $H_0 : \mu_1 - \mu_2 = \Delta_0$.

Test statistic

$$t = \frac{\bar{x} - \bar{y} - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}},$$

- $H_A : \mu_1 - \mu_2 > \Delta_0$: the rejection region is $t \geq t_{\alpha,\nu}$.
- $H_A : \mu_1 - \mu_2 < \Delta_0$: the rejection region is $t \leq -t_{\alpha,\nu}$.
- $H_A : \mu_1 - \mu_2 \neq \Delta_0$: the rejection region is $t \leq -t_{\alpha/2,\nu}$ or $t \geq t_{\alpha/2,\nu}$.

Example: *Silver content of Byzantine coins*

Data The data give the silver content (%Ag) of the coins. They arise from four different coinages at intervals throughout the king's reign.

Table 2.3 Silver content of coins

First	Second	Third	Fourth
5.9	6.9	4.9	5.3
6.8	9.0	5.5	5.6
6.4	6.6	4.6	5.5
7.0	8.1	4.5	5.1
6.6	9.3		6.2
7.7	9.2		5.8
7.2	8.6		5.8
6.9			
6.2			

The question of interest is whether there is any significant difference in their silver content with the passage of time; there is a suspicion that it was deliberately and steadily reduced.

Example: *Silver content of Byzantine coins*

The null hypothesis On the face of it the suspicion could be correct in that the fourth coinage would seem to have lower silver content than, say, the first coinage, but there is a need for firm statistical evidence if it is to be confirmed. Suppose the true percentage of silver in coinage i is μ_i. The null hypothesis would be

$$H_0 : \mu_1 = \mu_4$$

versus the alternative hypothesis

$$H_1 : \mu_1 \neq \mu_4.$$

If we believed, right from the start, that no monarch of the period would ever *increase* the silver content, and that the only possibility of its changing would be in the direction of reduction, then the alternative hypothesis would be

$$H_1 : \mu_1 > \mu_4.$$

Example: *Silver content of Byzantine coins*

Test statistic The first moment is the unbiased, consistent and efficient sample estimator for population expectation. Thus we use $\bar{X}_1$ as the estimator for μ_1, and $\bar{X}_4$ for μ_4.

$$\bar{X}_1 = \frac{1}{m}\sum_i X_1 i = \frac{1}{9}(5.9+6.8+6.4+7.0+6.6+7.7+7.2+6.9+6.2) \approx 6.744$$

versus the alternative hypothesis

$$\bar{X}_4 = \frac{1}{n}\sum_i X_4 i. = \frac{1}{7}(5.3+5.6+5.5+5.1+6.2+5.8+5.8) \approx 5.614$$

$$s_1^2 \approx 0.295, s_4^2 \approx 0.131,$$

Example: *Silver content of Byzantine coins*

Rejection region

Null hypothesis $H_0 : \mu_1 - \mu_2 = 0$.

Test statistic

$$t = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} = 4.72,$$

- $H_A : \mu_1 - \mu_2 > \Delta_0$: the rejection region is $t \geq t_{\alpha,\nu}$.
- $H_A : \mu_1 - \mu_2 < \Delta_0$: the rejection region is $t \leq -t_{\alpha,\nu}$.
- $H_A : \mu_1 - \mu_2 \neq \Delta_0$: the rejection region is $t \leq -t_{\alpha/2,\nu}$ or $t \geq t_{\alpha/2,\nu}$.

Example: *Silver content of Byzantine coins*

```
. ztest x1=6.5

One-sample z test
------------------------------------------------------------------------------
Variable |     Obs        Mean    Std. Err.   Std. Dev.   [95% Conf. Interval]
---------+--------------------------------------------------------------------
      x1 |       9    6.744444    .3333333           1    6.091123    7.397766
------------------------------------------------------------------------------
    mean = mean(x1)                                               z =   0.7333
Ho: mean = 6.5

   Ha: mean < 6.5              Ha: mean != 6.5               Ha: mean > 6.5
 Pr(Z < z) = 0.7683         Pr(|Z| > |z|) = 0.4634          Pr(Z > z) = 0.2317
```

Example: *Silver content of Byzantine coins*

```
. ttest x1=x4, unpaired

Two-sample t test with equal variances
------------------------------------------------------------------------------
Variable |     Obs        Mean    Std. Err.   Std. Dev.   [95% Conf. Interval]
---------+--------------------------------------------------------------------
      x1 |       9    6.744444    .1811315    .5433946    6.326754    7.162135
      x4 |       7    5.614286    .1370237    .3625308    5.279001    5.949571
---------+--------------------------------------------------------------------
combined |      16        6.25    .1846167    .7384668    5.856499    6.643501
---------+--------------------------------------------------------------------
    diff |            1.130159    .2390758                .6173921    1.642925
------------------------------------------------------------------------------
    diff = mean(x1) - mean(x4)                                    t =   4.7272
Ho: diff = 0                                     degrees of freedom =       14

    Ha: diff < 0                 Ha: diff != 0                 Ha: diff > 0
 Pr(T < t) = 0.9998         Pr(|T| > |t|) = 0.0003          Pr(T > t) = 0.0002

.
end of do-file
```