# Amath 351 Summer 2019 Homework 1

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Due: 1:10 pm, July 5, 2019

Homework would not have a lot of partial credits. For one, problems are already separated into many parts. Secondly, you can check your answers with your classmates or even ask us questions on them! Remember to submit your homework on Canvas in a scanned (or typed-up), letter-size, portrait PDF file. Show all your work clearly and box your answer. Check your submission after you submitted it! If your submission is not clear, the grader reserves the right to not grade your homework. In such case, you will get a zero.

## ODE Classification

Being able to classify a given ODE allows us to choose a suitable solution method!

**Exercise 1. [20 pts]**
Classify the following ODEs based on their (1) order, (2) dimension, and whether they are (3) autonomous, (4) linear, (5) homogeneous (if linear), (6) constant-coefficient (if linear) or not. You answer should be like

a first order, 1-D, autonomous, linear, homogeneous, constant-coefficient ODE

or

a first order, 1-D, autonomous, nonlinear, NA , NA ODE

where NA stands for not-applicable.

**(a)** (5 pts) Newton's cooling law:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k\left(T_{\mathrm{env}} - T\right)$$

where $k$ and $T_{\mathrm{env}}$ does not depend on $t$. We already knew how to solve this, right?

**(b)** (5 pts) FitzHugh-Nagumo model for neuronal dynamics:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v - \frac{v^3}{3} - w + I(t), \frac{\mathrm{d}w}{\mathrm{d}t} = \frac{v + a - bw}{\tau}$$

where $v(t)$ is the membrane potential, $w(t)$ is the "strong ion current", $I(t)$ is the external current and $a, b, \tau$ are system parameters. We will learn to analyze this in Chapter 6.

**(c)** (5 pts) Cauchy-Euler Equation:

$$a_n x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1} x^{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + ... + a_0 y = 0$$

where all $a_0, a_1, ..., a_n$ are constants. We will learn how to solve this in Chapter 5.

**(d)** (5 pts) Bernoulli differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p(x)y(x) + q(x)y^r(x)$$

where $r \in \mathbb{R}\backslash\{0, 1\}$ (this means all real number except 0 and 1). You will learn how to solve this in Chapter 2!

**Exercise 2. [10 pts, no partial credit]**
Write down a (system of) ODE satisfying the following criteria.

**(a)** (5 pts) 1-D, 10th order autonomous nonlinear ODE

**(b)** (5 pts) 2-D 1st order constant-coefficient homogeneous linear ODE

## Superposition Principle

The principle shows why linear, homogeneous ODEs are so nice: we can easily construct the general solution once we have enough independent, particular solutions!

**Exercise 3. [15 pts] Superposition Principle for $n$th order linear, homogeneous ODE**
We have done the $n$-D, 1st order linear, homogeneous ODE in class. Since $n$th order linear, homogeneous ODE can always to rewritten into $n$-D, 1st order linear, homogeneous ODE, we already know the principle holds. Here we prove the principle from a different route: by direct using the linearity of differentiation.

**(a)** (3 pts) From the definition of derivative,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h},$$

show the linearity of the differentiation (assuming all derivatives exist),

$$\frac{\mathrm{d}}{\mathrm{d}t}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}t} + \frac{\mathrm{d}g}{\mathrm{d}t}$$

**(b)** (5 pts) Using the results in (a), argue that the same holds for the $n$th derivative (assuming the $n$th derivative of $f$ and $g$ both exist)

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}(f+g) = \frac{\mathrm{d}^n f}{\mathrm{d}t^n} + \frac{\mathrm{d}^n g}{\mathrm{d}t^n}.$$

**(c)** (7 pts) Using both (a) and (b), show that for a linear, homogeneous $n$th order ODE

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_1(x)y(x) = 0,$$

if both $\eta_1(x)$ and $\eta_2(x)$ are solutions, then so is the linear combination of them, $c_1\eta_1(x) + c_2\eta_2(x)$.

**Exercise 4. [13 pts] Applicability of the superposition principle**

**(a)** (5 pts) Given that $x_1(t) = 1$ and $x_2(t) = \ln(t)$ are solutions of the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2.$$

Show that their linear superposition, $z(t) = c_1 + c_2 \ln(t)$ is, in general (*i.e.*, for arbitrary $c_1$ and $c_2$), *not* a solution to this equation.

**(b)** (5 pts) Given that $x_1(t) = 1$ and $x_2(t) = 1 + e^{-2t}$ are solutions of the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = 2.$$

Show that their linear superposition, $z(t) = c_1 + c_2\left(1 + e^{-2t}\right)$ is, in general (*i.e.*, for arbitrary $c_1$ and $c_2$), *not* a solution to this equation.

**(c)** (3 pts) Explain why the superposition principle is not valid in (a) and (b). Now you know why you need to learn to classify ODEs.

**Exercise 5. [7 pts]**
Given that both

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} \text{ and } \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

are solutions of the system of ODEs:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 + x_2, \ \frac{\mathrm{d}x_2}{\mathrm{d}t} = 4x_1 + x_2$$

Show that the linear superposition of them,

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = c_1 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + c_2 \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} + c_2 e^{-t} \\ 2c_1 e^{3t} - 2c_2 e^{-t} \end{pmatrix}$$

is a solution of the system of ODEs for arbitrary $c_1$ and $c_2$. You could choose whether you want to use a matrix-vector notation to do this problem or not.

## 1-D Dynamical System

The geometrical perspective can help us check our answer for those ODEs we can solve and provide qualitative predictions for those ODEs we can not solve!

**Exercise 6. [18 pts] Logistic Growth**

The population density (total number/ occupying area) growth of a population, denoted as $\rho(t)$, can be modeled by the logistic growth equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = r\rho(1 - \frac{\rho}{K}) \tag{1}$$

where $r$ is the growth rate and $K$ is the environment capacity for reason that will be clear in our analysis. Compare to exponential growth model $\frac{\mathrm{d}\rho}{\mathrm{d}t} = r\rho$, the logistic growth model introduce the additional factor of limited food, which dominates when $\rho$ has value close to $K$.

**(a)** (4 pts) We can define a new time scale $\tau = at$ and a new population scale $M = b\rho$ to have a simpler equation

$$\frac{\mathrm{d}M}{\mathrm{d}\tau} = M(1 - M).$$

What is $a$ and $b$ in terms of $r$ and $K$? This process is called *non-dimensionalization*. We are rescaling the variable to make the remaining variables dimensionless. In many analysis, this makes the problem less tedious to solve.

**(b)** (4 pts) Solve $M(\tau)$ and then solve $\rho(t)$ given the initial condition of $\rho$ is $\rho(0) = \rho_0$.

**(c)** (7 pts) Find all the fixed points of Equation (1). Determine their stability by drawing the phase line plot with the phase portrait. Which fixed point do you expect $\rho(t)$ converge to as $t \to \infty$?

**(d)** (3 pts) By taking $t \to \infty$ of your solution $\rho(t)$ in (b), verify your prediction in (c). Now, it should be clear why $K$ is called environment capacity.

## Separable ODEs

Separable ODEs are simple integration practices. Review the integration technique you have learned if necessary. Also, it should be clear on the distinction between explicit solution and implicit solution from these exercises.

**Exercise 7. [17 pts, no partial credits]**

Solve the following separable ODEs (*i.e.* find the *general solution* with one undetermined constants from integration). You may want to check your answer by directly taking the derivative of your answer.

**(a)** (5 pts) Find the *explicit solution* of $\frac{\mathrm{d}x}{\mathrm{d}t} = \ln(t) + x^2 \ln(t)$.

**(b)** (5 pts) Find the *explicit solution* of $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{e^{1/t}}{t^2} \sec^2(x)$.

**(c)** (7 pts) Find the *implicit solution* of $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\left(a^2 - x^2\right)^{3/2}}{t^3 + 3t^2}$.