Introduction to Ordinary Differential Equations with Applications

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Preface

This is a lecture note for Amath 351 at University of Washington, Seattle (UW), a first course in differential equation after a one-year Calculus sequence. In this note, we aim to motivate most of our theoretical discussions from their applications. Unlike usual introduction of ordinary differential equation, we emphasize logic behind solution methods, pay extra emphasize on system of ODEs, and blend in the modern perspective of ODE from dynamical systems. The ability to manipulate algebraic equations, evaluate limits, derivatives and integrals on a freshman Calculus level (Math 124&125 at UW) are assumed. Linear algebra is heavily used in this note but not assumed as prerequisite. Having exposure to linear algebra, Taylor series, and complex variable would be helpful.

Main Reference The main references of this note are:

1. William E. Boyce and Richard C. Diprima, Elementary Differential Equations and Boundary Value Problems, 9th edition, Chap $1 \sim 5$, $7 \sim 9$. [B&D]
2. Steven H. Strogatz, Nonlinear Dynamics and Chaos, Chap 1, 2, 5, and 6. [S]
3. Professor Bernard Deconick's lecture note on Amath 351, 2009, UW
4. Professor Hong Qian's lecture note on Amath 568, Winter 2018, UW
5. Carl M. Bender and Steven A. Orszag, Advanced Mathematical Methods for Scientists and Engineers. [B&O]

Plenty of the exercises in this note are modified from them. Thank you! The last two are actually from the graduate level ODE class, Amath 568 at UW, I took from Professor Hong Qian.

A different emphasis In this note, we de-emphasize Laplace Transform and instead emphasize more on Systems of ODEs and Dynamical Systems due to the following reasons. On one hand, to compute the inverse of Laplace Transform in an undergraduate level (without Bromwich integral) is not general and full of manipulations that require "insights". The computation is much more straightforward after learning complex analysis and Contour integral. Moreover, after learning complex analysis, we can learn Laplace Transform together with Fourier Transform and have a more comprehensive understanding on integral transform method as a whole! On the other hand, system of ODEs and Dynamical Systems allows us to solve more problems where Laplace Transform "only" gives us a more "efficient" way to solve linear, constant coefficient ODEs. Dynamical Systems perspective even gives us a modern, geometrical understanding on ODEs, and allows us to understand Nonlinear Dynamics, which is pervasive in mathematical models and theories in complex systems such as biological systems or chemical systems.

Main Structure The structure of this note is to start from a general overview (Chap. 1), and then gradually solve harder and harder problems. We solve first order ODE (Chap 2), demonstrate general theory for linear higher order and system of first order ODEs (Chap 3), solve 2-D constant-coefficient linear system of ODEs exclusively as an example of our theory (Chap 4), 2nd order variable-coefficient linear ODE (Chap 5), and then stop at a glimpse of 2-D nonlinear ODE (Chap 6). This exploration matches the central ideas of how we solve new problems in applied mathematics:

> *reduce the new problem into one you knew how to solve*[1].

Note that for Chap 3 and 4, since a second order ODE can always be rewritten as a two dimensional system of first order ODEs, we will introduce some linear algebra and solve the latter in general. The solution of the former then follow rather directly. Chapter 4 also has a thorough introduction to Simple Harmonic Oscillator in physics, which is arguably the most important example in this class, in physics and in many applications. We even use the results to classify various types of *fixed points* in 2-D ODE in Chap 4 which is applied to nonlinear dynamical system in Chap 6! After this note, one would be able to extend what we learned to 3-D or higher dimensional systems!

Acknowledgment I thanks my parents Dr. Tso-Chi Yang 楊佐琦 and Hui-Yin Ma 馬慧英 for their continuous support, my sister Ying-Ju Yang 楊穎如 and my girlfriend Chia Hsieh 謝嘉 for their continuous accompany, and my Ph. D. adviser Dr. Hong Qian 錢紘 for his visionary ideas, our inspiring discussions and his insightful lectures on graduate-level ODE that help me establish a more comprehensive understanding on the subject.

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[1] I actually knew this from my brilliant friend Matthew Farkas when we studied for the qualification exam together in the summer of 2018.

Some words from the author

This lecture note is not thoroughly revised since this is the first time it is used in a class. If you spot any typo/mistake or have any suggestion on how to better explain some concepts, please do not hesitate to email me or let me know in person! I will really appreciate it.

Contents

Chapter 1

Differential Equations and Dynamical Systems

Main references for this Chapter are Chap 1 and 2 of Steven H. Strogatz, Nonlinear Dynamics and Chaos [S Chap 1,2], Carl M. Bender and Steven A. Orszag, Advanced Mathematical Methods for Scientists and Engineers [B&O Chap 1], and Chap 1 of William E. Boyce and Richard C. Diprima, Elementary Dierential Equa- tions and Boundary Value Problems [B&D Chap 1]. You can also read the introduction of differential equation on Wikipedia.

1.1 Introduction

To quantitatively characterize a system of interest, we assign variables and see how they changes in time, in space, or with respect to some variables that we can tweak freely. The variables that we can manipulate and change independently for the system are called *independent variables*. Typical examples of them are space coordinate (x, y, z) and time t (mostly the latter in this course because we are interested in the dynamics of the system). The state of system is then characterized by the variable that depend on the independent variables, called *dependent variables*. Unknown that does not change with the independent variables are usually referred as the *parameters* of the system. We could also investigate what happens to the dependent variables if we change the parameters. When the parameters are changed, the evolution of the dependent variables is different, giving us a different dynamics. In some sense, in such cases, you are making the parameters independent variables.

In general, we may have multiple dependent variables $\mathbf{y}$ to multiple independent variables $\mathbf{x}$ and we can write both of them as a *vector*. Here we use boldface characters, for instance, $\mathbf{x}$ to represent a (column) vector that "collects" $n \in \mathbb{N}$ components,

$$\mathbf{x} := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

For now, you can just treat it as one efficient way to write n thing at the same time.

The ideal relation we seek for $\mathbf{x}$ and $\mathbf{y}$ is *explicitly* as a functional relations $\mathbf{y} = \mathbf{f}(\mathbf{x})$. Unfortunately, in many cases, the relation is too "complicated" for us to write down an *explicit* functional relations. Instead, we may be able to *implicitly* write down algebraic equations $\mathbf{g}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ to relate them. To be more specific on our vector notation, if $\mathbf{y}$ contains $m \in \mathbb{N}$ components, then a more detailed expression for $\mathbf{y} = \mathbf{f}(\mathbf{x})$ would be m functions each with n arguments:

$$y_1 = f_1(x_1, x_2, ..., x_n), y_2 = f_2(x_1, x_2, ..., x_n), \cdots, y_m = f_m(x_1, x_2, ..., x_n).$$

And for the latter, $\mathbf{g}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ if $\mathbf{g}$ contains $k \in \mathbb{N}$ components. Each is an algebraic equation with a function equals to zero with $(m + n)$ arguments:

$$g_1(x_1, ..., x_n, y_1, ..., y_m) = 0, \cdots, g_k(x_1, ..., x_n, y_1, ..., y_m) = 0.$$

You see the advantage of writing things in a vector form, It is way neater! That said, if such notation bothers you, you can just think of it as one dimensional case for now, *i.e.* $y = f(x)$ and $g(x, y) = 0$.

1.1.1 What is a Differential Equation?

On the relation between $\mathbf{x}$ and $\mathbf{y}$, if $\mathbf{y}$ never changes with $\mathbf{x}$, then such trivial system is fully understood by $\mathbf{y} = \mathbf{f}(\mathbf{x}) = \mathbf{c}$ where $\mathbf{c}$ is a constant vector (parameters) independent to $\mathbf{x}$. Since most of the changes in our physical worlds are continuous and deterministic (at least naively), the "complicated" equations usually contain how a component in $\mathbf{y}$ changes when we changes a component in $\mathbf{x}$ a bit. For both one dimensional case, that is the *derivative* of y with respect to (w.r.t.) x defined as a limit for an infinitesimal change δx in x:

$$\frac{\mathrm{d}y}{\mathrm{d}x} := \lim_{\delta x \to 0} \frac{y(x + \delta x) - y(x)}{(x + \delta x) - x}.$$

In this note, we also denote the nth order derivative (take derivative for n times) as $y^{(n)}$ with parenthesizes around to distinguish it with the power of y. Sometimes, we will also used Newton's notation y' for first derivative and y'' for second derivative. Equations with derivatives are called *differential equations*.

What happens if we have more than one independent variable? We then need to know how a dependent variable y changes when we independently change each of the independent variables a bit. That gives the notion of *partial derivative*. For example, if y depends on $x_1, x_2, ..., x_n$ and if we (only) change the jth independent variable a bit, we have

$$\frac{\partial y}{\partial x_j} := \lim_{\delta x_j \to 0} \frac{y(x_1, ..., x_{j-1}, x_j + \delta x_j, x_{j+1}, ..., x_n) - y(\mathbf{x})}{\delta x_j}$$

giving us the rate of such change. We note the label j can take any number from 1 to n and will give rate of change in y with respect to (w.r.t.) different independent variables.

Therefore, depends on how many independent variables there are, we have two different kinds of differential equations. If we only have one independent variable so that there is no partial derivative, our complicated equation is generally in the form of

$$F(x, y_1, y_2, ..., \frac{\mathrm{d}}{\mathrm{d}x}y_1, \frac{\mathrm{d}}{\mathrm{d}x}y_2, ..., ..., \frac{\mathrm{d}^n}{\mathrm{d}x^n}y_i, ...) = 0$$

where y_i is the ith component of $\mathbf{y}$. This kind of differential equations that have only one independent variable are called *ordinary differential equation* (ODE) and is the differential equations we are going to investigate in this course. If there are more than one independent variables, there would be (a lot of) partial derivatives, so we called them *partial differential equations* (PDE).

1.1.2 Conditions and Solutions

Newton's 2nd law on its own does not determine a trajectory of a particle, $x(t)$. It gives us a rule on how the position and velocity of particles evolve in time. Usually, we also have *initial conditions* for the initial position $x(0)$ and the initial velocity $x^{(1)}(0)$ that specify the conditions. Same for general differential equations. We need *initial conditions* and/or conditions on the boundary of the system, called *boundary conditions*, to determine a unique solution.

IVP vs. BVP A problem with ODEs and initial conditions $(y(0), \frac{\mathrm{d}y}{\mathrm{d}x}(0), ...)$ specified at one point is called an *initial value problem* (IVP). And a problem with ODEs and boundary conditions (conditions like $y(0)$, $y(1)$, $\frac{\mathrm{d}y}{\mathrm{d}x}(1)$,...) at more than one point specified is called a *boundary values problem* (BVP). An IVP is in general easier than BVP since all the condition is at one space/time point. One can imagine our solution will then grow out from it according to the ODEs. BVP, however, is *global*. We can imagine that we will need to grow our solution a lot of times to match conditions at different space/time points. In this note we will mostly be focusing on IVPs.

Example 1.1. IVP

A problem like "A ball is thrown up from the ground level with initial velocity v_0. Find the trajectory of it $x(t)$." is an IVP. It is composed of an ODE and conditions evaluated at one time point.

Example 1.2. BVP

Suppose we have a 1 dimensional particle under a force field $F(x)$, so that Newton's second law tells us

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) = F(x).$$

If we know that at $t = 0$, $x(0) = x_0$ and at $t = 1$, $x(1) = x_1$. Solving the trajectory $x(t)$, $0 \leq t \leq 1$, is a boundary value problem.

A solution of ODEs is a functional relation between independent variable t and dependent variables $\mathbf{y}$ that is consistent with the ODE. For example, $y(t) = at$ is a solution to $\frac{\mathrm{d}y}{\mathrm{d}t} = a$. To check whether the relation, *e.g.* $y = f(t)$, is a solution to the ODE or not, we can just **plug it in**.

Explicit Solution vs. Implicit Solution The functional relation between independent and dependent variables can be expressed implicitly and if luckily explicitly. This gives two different type of solutions: *explicit solution* or an *implicit solution.* An explicit solution is in the form of $\mathbf{y} = \mathbf{f}(\mathbf{x})$ where we have the explicit function separating the dependent variables $\mathbf{y}$ from the independent variables $\mathbf{x}$. An implicit solution, on the other hand, is in the form of $\mathbf{g}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$, where dependent and independent variables are not explicitly separate. We should always try to get an explicit solution from an implicit solution by algebraically isolating and solving $\mathbf{y}$ from $\mathbf{g}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$.

Example 1.3. Explicit Solution vs. Implicit Solution

One example is to solve $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$, $y(0) = 1$. By moving parts with y to one side and moving parts with x to the other (if we can do that, we will can the ODE *separable*), we get

$$\frac{\mathrm{d}y}{y^2} = \mathrm{d}x \Rightarrow \int \frac{\mathrm{d}y}{y^2} = \int \mathrm{d}x \Rightarrow -\frac{1}{y} = x + c$$
$$\Rightarrow 0 = xy + cy + 1.$$

With the initial condition $y(0) = 1$, we can solve the integration constant c by plugging $(x, y) = (0, 1)$ in to get $0 = c + 1 \Rightarrow c = -1$. We thus get a *implicit* solution $(x - 1)\, y + 1 = 0$. A explicit solution can be found by a simple algebra,

$$y = \frac{1}{1 - x}.$$

We shall always try to get a explicit solution.

General solution vs. Particular solution When we are given by an ODE or a system of ODEs without specific conditions, the solution to them are not unique. In nice cases we will encounter in this note (much like nice integrals that you have closed-form solutions in Calculus), we would be able to derive a general, closed-form solution to the ODE. Such general form is called the *general solution* to the ODE. They will possess undetermined constants/parameters originated from the integration constants in them. Therefore, the number of those undetermined constants is closely related to how many dependent variables we have and the order of derivatives in the ODE. In a IVP or a BVP, these undetermined constants are specified by the conditions we posed on the problem.

In some cases, we may be able the *guess* a solution from experience. The solution we guessed (without undetermined constant) is a particular case in the general solution, *i.e.* it is the general solution evaluate at a specific combination of the undetermined constants. We called this kind of solution a *particular solution* of the ODE. A simple check on whether a solution is general or particular is to just check whether there is undetermined constant or not.

Example 1.4. General Solution vs. Particular Solution

Suppose we want to solve
$$\frac{\mathrm{d}x}{\mathrm{d}t} = x$$
which means the derivative of $x(t)$ is itself. From our experience in Calculus, we immediately get a *particular solution* $x(t) = e^t$. The *general solution*, on the other hand, is $x(t) = Ae^t$. We see that the particular solution is a particular case of the general solution when $A = 1$.

Suppose we want to solve
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -x$$
which means the second derivative of $x(t)$ is itself. From our experience in Calculus, we guess $x(t) = \sin(t)$, which is a particular solution. The general solution is actually $x(t) = A\sin(t) + B\cos(t)$ or equivalently $C\sin(t + D)$ or $E\cos(t + F)$ by trigs identity. Note that there are two undetermined constant because we have second order derivative in the ODE. Solving it involves two integrals, giving us two integration constants.

1.2 Classification of ODEs

As we go along and develop various methods to solve ODEs, each method is in fact specific to certain type of ODE. Therefore, we would need to fix our terminology and classify the type of ODE like how we distinguish ODE from PDE and distinguish different types of solution.

Order and Dimension

From our last example, the number of undetermined constants is related to whether the derivative is taken once or twice. Recall that in Calculus, the nth derivative of a function is also called the derivative of *order* n. We therefore say the order of an ODE is the maximum order of all the derivatives in it. For example, Newton's second law is with highest derivative second order, thus it is a second order ODE.

A closely related concept is the *dimension* of the ODE. It is the number of dependent variables in the problem. If we have more than one dependent variables, we usually call it a system of ODEs. As we will see more, most of the nth order ODE can be rewritten as a n dimensional (n-D) system of 1st order ODE (but not the backward). In a general setting where we can have a n-D system of mth order ODE, the number of undetermined constants would be $n \times m$. The idea of rewriting ODE with order higher than 1 into the higher dimensional system of (1st order) ODE is already in classical mechanics.

Example 1.5. Classical Mechanics

The famous Newton's second law in classical mechanics for a single particle system is governed by 3 equations, each for three direction in space. The trajectory of the particle is governed by

$$m\frac{\mathrm{d}^2\mathbf{x}(t)}{\mathrm{d}t^2} = \mathbf{F} \tag{1.2.1}$$

where both the position $\mathbf{x}$ and force $\mathbf{F}$ have three components corresponding to the x, y, and z direction. With two time derivative, it is a *second order* ODE.

If we use the momentum of the particle $\mathbf{p}_i = \mathbf{v}_i/m_i$ as another new dependent variables and rewrite Equation (1.2.1) as the following system of ODEs:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathbf{p}}{m} \text{ and } \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}. \tag{1.2.2}$$

We get in stead a 6 dimensional (three for $\mathbf{x}$ and three for $\mathbf{p}$) system of first order ODEs. We have our first taste on the equivalence between higher order ODE and higher dimensional system of (first order) ODEs.

When we finish introducing how to solve 1st order ODE and moving on the higher order ODE, we will learn how to solve higher dimensional linear system of (1st order) ODE and higher order ODE together. That requires some more usage of vector, matrix, and linear algebra.

Degree and Linear vs. Nonlinear

The degree of a polynomial is the highest power of its individual terms with non-zero coefficient, *e.g.* x^2+3x+4 is degree 2. Similarly, the *degree* of a ODE is given by the highest power of the dependent variables, ignoring the derivative notion. For instance, $yy^{(1)} = x^3$ is degree 2 and order 1, $\left(1+\left(y^{(2)}\right)^2\right)^3 = x^{1/2}$ is degree 6 and order 2. The degree and the order of an ODE are rather easy to be confused with. Just remember the name of degree is inherited from polynomial whereas the name of the order is inherited from derivative.

If all terms with dependent variables in an ODE have degree one, we say the ODE is *linear*. Otherwise, the ODE is said to be *nonlinear*. For example, $\sqrt{x}\cdot y^{(2)}(x) + 2x^5\cdot y^{(1)}(x) = x^2$ is linear whereas $y^{(1)}(x) = 2x\sqrt{y}$ is nonlinear.

The notion of an ODE being linear or not is important. After this course, you will learn how to in principle solve all linear ODEs. For nonlinear case, unfortunately, we can only solve very special cases. We will learn a bit on how to analyze them and still make prediction on its behavior, that is the contribution of the modern viewpoint from dynamical system in Chap 6.

The general form for a linear nth order ODE is given by (don't worry we will basically only solve up to $n = 2$ in this class)

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + p_{n-1}(t)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}t^{n-1}} + \cdots + p_1(t)\frac{\mathrm{d}y}{\mathrm{d}t} + p_0(t)y = q(t). \tag{1.2.3}$$

A linear system of (1st order) ODE is, on the other hand, given by

$$\begin{aligned}
\frac{\mathrm{d}z_1}{\mathrm{d}t} &= a_{11}z_1 + a_{12}z_2 + \ldots + a_{1n}z_n + b_1 \\
\frac{\mathrm{d}z_1}{\mathrm{d}t} &= a_{21}z_1 + a_{22}z_2 + \ldots + a_{2n}z_n + b_2 \\
&\vdots \\
\frac{\mathrm{d}z_n}{\mathrm{d}t} &= a_{n1}z_1 + a_{n2}z_2 + \ldots + a_{nn}z_n + b_n
\end{aligned} \tag{1.2.4}$$

This can in fact be written as a more compact form as

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}\mathbf{z} + \mathbf{b} \tag{1.2.5}$$

by introducing vectors and matrices:

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

So what is so great about linear ODEs in Equation (1.2.3) and (1.2.4)? We first note that if $q(t) = 0$ or $\mathbf{b}(t) = \mathbf{0}$, the ODE is *linear* and *homogeneous*. In such case, we have the superposition principle to construct solutions!

Superposition Principle:
For a system of ODEs that is linear and homogeneous. If both $\mathbf{z}_1$ and $\mathbf{z}_2$ are solutions, then so is $c_1\mathbf{z}_1 + c_2\mathbf{z}_2$ where $c_1, c_2 \in \mathbb{R}$.

Proof. This is true because with

$$\mathbf{z}_1 = \begin{pmatrix} z_{11} \\ z_{21} \\ \vdots \\ z_{n1} \end{pmatrix}, \mathbf{z}_2 = \begin{pmatrix} z_{12} \\ z_{22} \\ \vdots \\ z_{n2} \end{pmatrix}$$

both $\frac{\mathrm{d}}{\mathrm{d}t}$ and $\mathbf{A}$ can be distributed, *i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(c_1\mathbf{z}_1 + c_2\mathbf{z}_2\right) = c_1\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_1 + c_2\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_2$$

and

$$\mathbf{A}\left(c_1\mathbf{z}_1 + c_2\mathbf{z}_2\right) = c_1\mathbf{A}\mathbf{z}_1 + c_2\mathbf{A}\mathbf{z}_2.$$

Thus, $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_1 = \mathbf{A}\mathbf{z}_1$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_2 = \mathbf{A}\mathbf{z}_2$ imply $\frac{\mathrm{d}}{\mathrm{d}t}\left(c_1\mathbf{z}_1 + c_2\mathbf{z}_2\right) = \mathbf{A}\frac{\mathrm{d}}{\mathrm{d}t}\left(c_1\mathbf{z}_1 + c_2\mathbf{z}_2\right)$. □

For n-th order ODE, the principle is the same. We know there will be n undetermined constants. So, if we can find n "distinct" solutions $\eta_1, \eta_2, ..., \eta_n$, then we get the *general solution* as $y = c_1\eta_1 + c_2\eta_2 + ...c_n\eta_n$! The fundamental reason for superposition principle to hold is because differentiation itself is also *linear*:

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}(\eta_1 + \eta_2) = \frac{\mathrm{d}^n\eta_1}{\mathrm{d}t^n} + \frac{\mathrm{d}^n\eta_2}{\mathrm{d}t^n}, \forall n \in \mathbb{N}.$$

The steps remaining are just plugging in and check. See Exercise 1.9.

This is a rather strong and important result. For n-D linear, homogeneous system of ODEs or nth order linear homogeneous ODEs, there are n integral that needs to be done. Therefore, if we can find n mutually "distinct" solutions, then we have found the general solution with our n undetermined constants. What is the "distinct" condition we need? For the n-D vector case, we want any two of these vector solutions, say $\mathbf{v}_1$ and $\mathbf{v}_2$, to be non-parallel, or equivalently not up to a constant multiplication. Because in such case, we can combine $\mathbf{v}_1$ and $\mathbf{v}_2$, and then we are one solution (one undetermined constant) short. It is important to note that if the ODE is either inhomogeneous or nonlinear, then the superposition principle fails. See Exercise 1.11 and 1.12.

Autonomous vs. Nonautonomous

If the ODE does not have explicit dependence on the independent variables, *i.e.* the independent variables only affect the system through dependent variables, then we say the ODE is *autonomous* because it can "evolve" in time based on the value of its own state (dependent variables). Any autonomous system can directly be transformed into a dynamical system (in this note, we would just treat autonomous ODE as the "definition" of dynamical system). Note that being willing to increase the dimension of a system, nonautonomous system can be transformed into a autonomous dynamical system by introducing another dependent variable $\tau'(t) = 1$ such that $\tau = t$.

If the ODE is linear, then whether it is autonomous or not is fully determined by the coefficients. If the linear ODE is autonomous, the coefficient must be constant. Therefore, if we say an ODE is *constant-coefficient* we mean that the ODE is linear and autonomous.

Homogeneous vs. Inhomogeneous

For linear ODEs, we also care whether $\mathbf{y}(\mathbf{x}) = \mathbf{0}$ is a solution or not (for reason that would become clear later). If $\mathbf{y}(\mathbf{x}) = \mathbf{0}$, which we called the trivial solution, is a solution to the ODE. Then, we say the ODE is *homogeneous*. If not, it means the system is some how "driven" by some $\mathbf{x}$-dependent (or t-dependent) "force" by another driving term, we then say the ODE is *inhomogeneous* (or *nonhomogeneous*).

When we have constant-coefficient case, we can solve the ODE easily as we will investigate in detail for 2nd order case and 2-D system of ODEs case in Chap 4. It is the most important example throughout the course and in physics as well. It is called simple harmonic oscillator (SHO) in physics and it is omnipresent! Two typical realizations of SHO (with/without friction, with/without driving force) are shown in the following example.

Example 1.6. Simple Harmonic Oscillator (SHO)

Frictionless: A classic realization of a simple harmonic oscillator is a box attached to a ideal spring with all surface frictionless. The governing equation is given by the ODE

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -kx(t) \tag{1.2.6}$$

which is a 2nd order, linear, constant-coefficient, homogeneous, and autonomous.

Another typical realization of a SHO is the *LC circuit* in electronics. It is an electric circuit consisting of a inductor (L) and a capacitor (C). It forms a SHO for the charge $Q(t)$ across the capacity in the circuit. The governing ODE is

$$L\frac{\mathrm{d}^2Q(t)}{\mathrm{d}t^2} + \frac{Q(t)}{C} = 0.$$

Linear friction: A classic realization of a SHO is a box attached to a ideal spring with friction proportional to the velocity. The governing ODE is given by

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -\eta\frac{\mathrm{d}x(t)}{\mathrm{d}t} - kx(t) \tag{1.2.7}$$

which is 2nd order, linear, constant-coefficient, homogeneous, and autonomous.

Another classic realization of a SHO is a *RLC circuit*. It is a electric circuit consisting a resistor (R), a inductor (L), and a capacitor (C). The governing equation for the charge $Q(t)$ across the capacity is given by

$$L\frac{\mathrm{d}^2Q(t)}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q(t)}{\mathrm{d}t} + \frac{Q(t)}{C} = 0.$$

As we will discussed, depends on the level of the damping (from comparing η and R to other parameters), the transient behavior can be oscillatory or not.

Linear friction+periodic driving force: With the same setting as before but with additional sinusoidal driving force. The governing ODE is given by

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = A\sin(\omega t+\theta) - \eta\frac{\mathrm{d}x(t)}{\mathrm{d}t} - kx(t) \tag{1.2.8}$$

which is now 2nd order, linear, constant-coefficient, inhomogeneous, and non-autonomous. Similarly, a RLC circuit driven by a sinusoidal power source $V(t) = A\sin(\omega t+\theta)$ also gives a realization to such mathematical object:

$$L\frac{\mathrm{d}^2Q(t)}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q(t)}{\mathrm{d}t} + \frac{Q(t)}{C} = A\sin(\omega t+\theta).$$

As we will discussed later. The choice of the driving frequency can lead to significant difference in response. If the driving frequency matches the natural frequency of the system, the response is maximized, called *resonance*.

Let us now summarize how to categorize an ODE with the following table:

order	the highest order of derivative
dimension	the number of independent variables
autonomous	whether the ODE has explicit dependence on the independent variable or not
degree	highest power of the dependent variables (ignoring $\frac{\mathrm{d}}{\mathrm{d}t}$)
linear	all terms with dependent variables have degree 1
constant-coefficient	coefficients of all terms in a linear ODE are constants
homogeneous	the trivial solution $\mathbf{y} = \mathbf{0}$ is a solution in a linear ODE or not

More Examples from Physics, Chemistry and Biology

Almost all the laws of physics are mathematically represented by differential equations. Here is one more common example.

Example 1.7. Falling body with friction

If the friction is proportional to velocity. The governing ODE for the velocity is giving by a 1-D, 1st order ODE

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = mg - \eta v(t)$$

which is autonomous, inhomogeneous, degree 1, linear, and constant-coefficient.

If the friction is proportional to the linear combination of v and v^2. The governing ODE for the velocity is giving by a 1st order, 1-D ODE

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = mg - \left[\alpha v(t) + \beta v^2(t)\right]$$

which is autonomous, inhomogeneous, degree 2, and nonlinear.

We will be able to find nice closed-form explicit solutions for all the ODE examples above after this note. Here are some more examples that we can't solve in closed-form but will be able to analyze them and solve them numerically.

Example 1.8. Lorenz System

A "simple" mathematical model for atmosphere convection related to it is the famous Lorenz system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y-x), \ \frac{\mathrm{d}y}{\mathrm{d}t} = x\left(\rho - z\right) - y, \ \frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z \tag{1.2.9}$$

which is a 1st order, 3 dimensional, autonomous, homogeneous, degree 2, nonlinear system of ODEs.

Moreover, mathematical models in biology and chemistry are also mostly differential equations. In chemistry, for instance, with the *law of mass action*, concentrations of reactants in a chemical reaction are also described by a system of nonlinear ODEs.

Example 1.9. Chemical Reaction

Consider a weird chemical reaction: $2X + 2Y \rightleftharpoons 3X + B$ with forward($\rightarrow$) and backward($\leftarrow$) rates r_+ and r_-. Suppose the concentration of a rapidly-stirred reaction tank with chemicals X, Y, and with so many B that under all practical purpose B can be treated as fixed concentration. The ODEs that govern the dynamics of the concentration of X and Y, denoted as x and y, with the assumption of the law of mass action are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = r_+ x^2 y^2 - r_- b x^3, \; \frac{\mathrm{d}y}{\mathrm{d}t} = 2r_- b x^3 - 2r_+ x^2 y^2.$$

This is a 1st order, 2 dimensional, autonomous, homogeneous, degree 4, nonlinear system of ODEs.

One more example from ecology for interacting populations of species is

Example 1.10. Lotka-Volterra equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \beta x y, \; \frac{\mathrm{d}y}{\mathrm{d}t} = \delta x y - \gamma y$$

which describe the populations changes of a predator $y(t)$ and a prey $x(t)$. The parameters α is the growth rate of the prey, β gives the death rate when a prey encounter a predator, δ gives the combination of successful rate of a predator capturing a prey and it's corresponding reproductive ability, and γ characterize the death rate of a predator.

This is a 1st order, 2-dimensional, autonomous, homogeneous, degree 2, nonlinear system of ODEs.

An example in theoretical neuroscience is the Nobel-prize-winning *Hodgkin-Huxley* model, which is a four dimensional system of *nonlinear* ODEs that describe the dynamics of a membrane potential of a point neuron and how the concentration ions changes. Check it out if you are interested in understanding neuronal dynamics and the dynamics of the brain!

1.3 1-D Dynamical Systems and Nonlinearity

Let us now focus on 1-D 1st order ODE (with time as the independent variable t and an unknown $x(t)$ as the dependent variable)with the general form of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t). \tag{1.3.1}$$

The function $f(x,t)$ gives a rate of change for $x(t)$ for every given x and t. If the function $f(x,t)$ is linear in x, *i.e.* $f(x,t) = p(t)x + q(t)$, then the ODE is linear, if it is not explicitly depending one t, *i.e.* $f(x)$, then the ODE is autonomous.

A geometrical understanding of Equation (1.3.1) is to sketch a plot on $f(x,t)$ for each point on the plane (t,x) with a small line segment passes through (t,x)

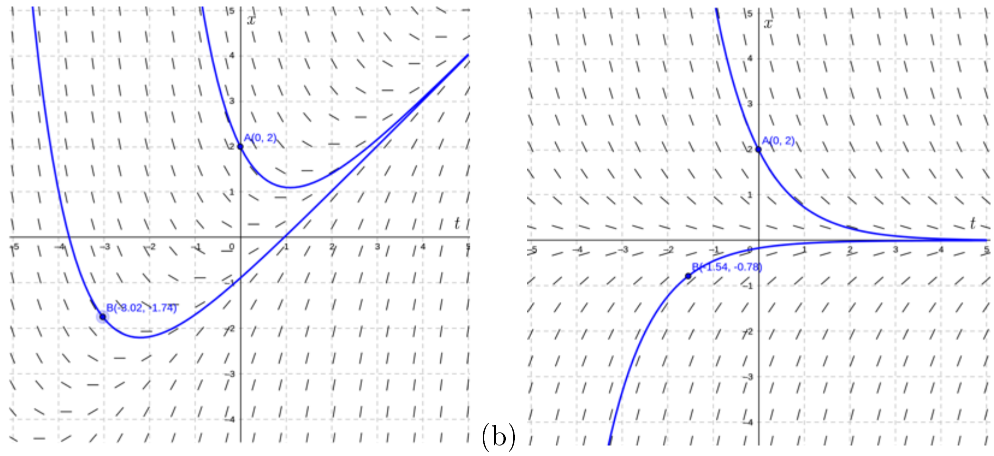
(a) (b)


Figure 1.3.1: (a) An example for non-autonomous ODE with $f(x,t) = t - x$. (b) An example for autonomous ODE with $f(x) = -x$. These plots are generated from the website of GeoGebra. Blue lines are solutions of the ODEs.

with slope $f(x,t)$ representing the local instantaneous speed of the particle, this is called a *Direction Field*[1]. It gives us a picture on how the solution looks like. Go to the website of GeoGebra to play with some functions $f(x,t)$. We show two examples in Figure 1.3.1. Note that on a direction field, if the ODE is autonomous, *i.e.* $f(x,t) = f(x)$, then the plot is the same along with time see Figure 1.3.1(b). In this kind of autonomous systems, we can have a better understanding on them from a modern dynamical system perspective.

1.3.1 Dynamical Systems Perspective

Systems with dynamics, *i.e.* whose state change with time, called *Dynamical systems*, can have many types. In this note, we focus on autonomous ODEs. In our one dimensional case, an general autonomous 1 dimensional first order ODE is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x). \tag{1.3.2}$$

Even if $f(x)$ is nonlinear, we can still study the system pretty well. The key is to understand the system geometrically.

Since the ODE is autonomous, we can understand the whole direction field by a slice of t. If we then plot $f(x)$ w.r.t. x, we then have the so called *phase line plot* of the system. From the *phase line plot*, we can sketch how a solution $x(t)$ evolves geometrically in the phase space (the space of all possible values of the dependent variable(s) x), giving us the *phase portrait*. We can know a lot of qualitative properties of the system from the phase portrait.

1. There may be zeros of $f(x)$. They plays significant roles on the dynamics

[1] A direction field can be treated as a 2-D phase portrait on the phase plane. When treating the system like that, we are extending the 1-D system to 2-D by introducing a second dimension τ with $\frac{\mathrm{d}\tau}{\mathrm{d}t} = 1$.

because they are points that are invariant in time, *i.e.* they are fixed in time. These zeros are called the *fixed points* of the dynamical system.

2. We can visualize which and how fast the system is going. Importantly, for points nearby a fixed point, it is getting closer to the fixed point or getting away! This means if we start with the fixed point and if the system got perturbed, this tells us whether the system going to stick around the same fixed point or run away. In more mathematically terms, whether the fixed point is (Lyapunov) *stable* or not! There are in fact different definitions for *stability.* We will not go into there. Take a look if you are interested.

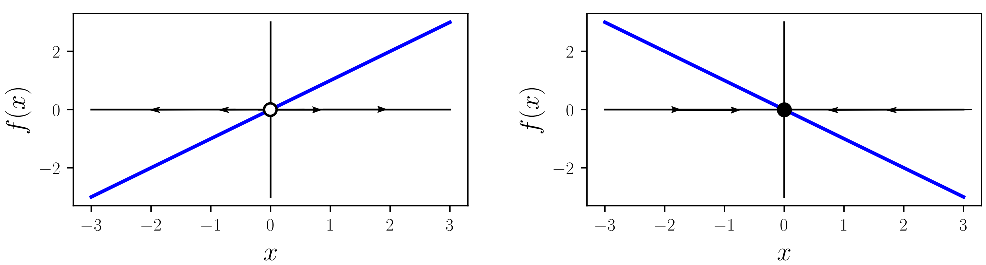
3. With different initial starting points ξ in the initial condition $x(0) = \xi$, we can see for a stable fixed point x^* as $t \to \infty$, what are the ξs that will approach to x^*. This set of points ξ is called the *basin of attraction* of a (asymptotic) stable fixed point.

In fact, this method can be generalized to higher dimensional system as we will see in the later Chapters! Here are some simple 1 dimensional examples.

Example 1.11. Linear cases

In Figure 1.3.1(b), we considered $\frac{\mathrm{d}x}{\mathrm{d}t} = -x$. The fixed point is at $x^* = 0$. It is (aymptotic) *stable* because from the phase portrait, we see that on the right of $x^* = 0$, $\frac{\mathrm{d}x}{\mathrm{d}t} < 0$; and on the left of $x^* = 0$, $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$. All the nearby points are "attracted" by the fixed point and they will go to x^* as $t \to \infty$. The basic of attraction is the whole real line $\mathbb{R}$.

Phase line plots and Phase portraits for $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) = \pm x$

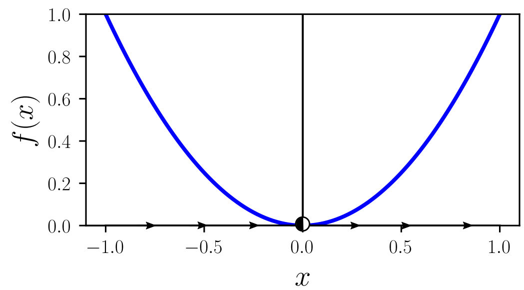


How about $\frac{\mathrm{d}x}{\mathrm{d}t} = x$? The fixed point is again at $x^* = 0$. But it is *unstable* because from the phase portrait, we see all the nearby points are "repelled" by the fixed point. Where do all the trajectories go as $t \to \infty$? Well, one can also consider ∞ and $-\infty$ as "fixed points". They are (asymptotically) "stable". One can say the basic of attraction for ∞ is the positive real line $\mathbb{R}^+$ whereas the basic of attraction for $-\infty$ is the negative real line $\mathbb{R}^-$. We also see that the two basic of attractions are separated by a unstable fixed point $x^* = 0$.

Other than stable fixed point and unstable fixed point, there is in fact a third type of fixed point with a semistability.

Example 1.12. Saddle point

Phase line plot and Phase portrait for $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) = x^2$



Consider $\frac{\mathrm{d}x}{\mathrm{d}t} = x^2$. We immediately see that the fixed point is at $x^* = 0$ (and $\pm\infty$ if you want to include them, if one say "find the fixed point on the real line $\mathbb{R}$", then we usually exclude $\pm\infty$ unless one consider the "extended real line".). However, we see that on both sides of the fixed point, $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$. The fixed point is stable when $x < 0$ but unstable for $x > 0$. We call this kind of semi-stable fixed point *saddle point.*

There would be a question on whether the "particle" go pass through the origin from left to right in finite time or not. We will investigate that in Example 1.14. The answer is no, if we start from the left, it takes infinite time for the "particle" to reach the origin.

With these simple examples, we know there are three kinds of fixed points in 1-D: stable, unstable, and saddle point. In the examples above, we can actually have a nice closed-form solution, so you may think this understanding is nothing. Well, let us try a harder one.

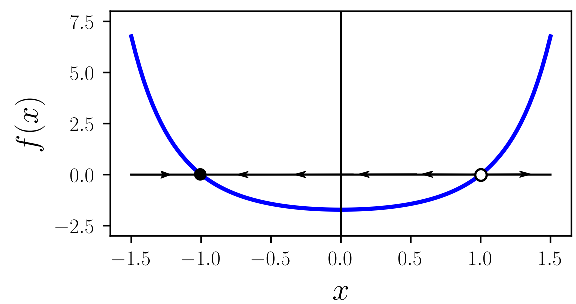
Example 1.13. A harder example

Consider $\frac{\mathrm{d}x}{\mathrm{d}t} = e^{x^2} - e$. This is still autonomous. Now, to solve where the fixed points are, we solve

$$e^{(x^*)^2} - e = 0 \Rightarrow e^{(x^*)^2} = e \Rightarrow (x^*)^2 = 1 \Rightarrow x^* = \pm 1.$$

So we have two fixed points. Now, even without using a calculator or computer, we can actually draw what $f(x) = e^{x^2} - e$ looks like. First of all, we see $f(x) \to \infty$ as $x \to \pm\infty$. Also, f is an even function, *i.e.* $f(x) = f(-x)$, so the phase line plot will be symmetric w.r.t. the y-axis. Now, the y intersect is at $f(0) = 1 - e < 0$ whereas the x intersects are the positions of the fixed points $x^* = \pm 1$. Last but not least, x^2 is monotonic for $x > 0$, so is e^{x^2}, and so is $f(x) = e^{x^2} - e$. There you have it. $f(x)$ is a concave up curve passing through $(0, 1 - e), (1, 0)$, and $(-1, 0)$.

Phase line plot and Phase portrait for $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) = e^{x^2} - e$



With that, we see that on the right of $x^* = 1$, $f(x) > 0$ whereas on the left of $x^* = 1$, $f(x) < 0$. Thus, $x^* = 1$ is unstable. With a similar procedure, we can see $x^* = -1$ is stable. The basic of attraction for $x^* = -1$ is then $(-\infty, 1)$. We know from this plot that if one start the dynamics with $x(0) < 1$, then $x \to -1$ as $t \to \infty$; if with $x(0) > 1$, then $x \to \infty$ as $t \to \infty$.

The geometrical understanding of dynamical system (autonomous ODE) can help us check our solution for ODEs that we can solve and even help us understand ODEs that we can not find explicit closed-form solution!

For nonlinear ODEs, stranger things can actually happen... Let us explore a bit more in the following subsection!

1.3.2 Nonlinearity, and the Existence and Uniqueness of solutions

Example 1.14. Blow up at finite time

One of the interesting properties of nonlinear ODE is that the solution can blow up to infinity at finite time whose value depends on the initial condition! Let us investigate more on the IVP:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2, x(0) = x_0.$$

Solving it, we move all the x on one side and t on the other and get

$$\frac{\mathrm{d}x}{x^2} = \mathrm{d}t \Rightarrow -\frac{1}{x} = t + c_1 \Rightarrow x(t) = \frac{1}{c_2 - t}.$$

With $x(0) = x_0$, we have that $c_2 = 1/x_0$.

Now, if the system starts at $x_0 < 0$ ($c_2 < 0$), we know from our previous example that it would approach to the saddle point at $x^* = 0$. But how long does it takes for this system to reach the saddle point? It takes infinite time since $x = \frac{1}{1/x_0 - t} \to 0$ only when $t \to \infty$! This tells us it took infinite time to get close to a saddle point. In other words, a saddle point is unreachable in finite time. Note that our discussion is general enough for any saddle point in arbitrary function that has non-vanish second derivative in 1-D. This is because for any smooth $f(x)$ on the RHS of the ODE. When we are close enough to the saddle point, $f(x)$ is well-approximated by a quadratic if it's second derivative at the saddle point is non-zero.

For $x_0 > 0$, we know $x(t) \to \infty$ as $t \to \infty$. Now, here comes the interesting things. With the solution

$$x(t) = \frac{1}{1/x_0 - t}$$

in hand, we see that x actually reaches infinity (can be considered as a fixed point) at a *finite* time $t = 1/x_0$! What's more, the finite time depends on the initial condition! This also shows us an existence of solution issue. It does not make sense to talk about the solution for this IVP with $x(0) = x_0$ after it blows up ($t > 1/x_0$) unless our "initial condition" is specified at different t.

Not only the existence of solutions could be challenged, the uniqueness of solutions may also be gone. See the following example which is actually from [S Exercise 2.5.6].

Example 1.15. A physics example for Non-uniqueness

Consider a leaky bucket. Let $h(\tau)$ be the height of the water remaining in the bucket at time τ; a be the area of a hole on the bottom; A be the cross-section area of the bucket; $v(\tau)$ be the velocity of the water passing through the hole.

The rate of the volume loss is $A\frac{\mathrm{d}h}{\mathrm{d}\tau} = av(\tau)$. Also from physics, we have $v = -\sqrt{2gh}$ (considered given, note that h is decreasing, v should be negative), this leads to $A\frac{\mathrm{d}h}{\mathrm{d}\tau} = -a\sqrt{2gh}$. By denoting $\frac{a\sqrt{2g}}{A} = b$, we have

$$\frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau} = -b\sqrt{h}.$$

Given a condition that the bucket is initially empty, $h(0) = 0$, can we solve the height of the bucket for some (negative) time before the bucket is empty $t = -\tau$? The answer is no! The bucket can stay empty for arbitrary long time! We want to solve $h(-\tau)$, rewriting the ODE in the backward time variable t, denoting $b = 1/\sqrt{r}$, and writing $x(t) = rh(t)$, we can reduce the ODE into

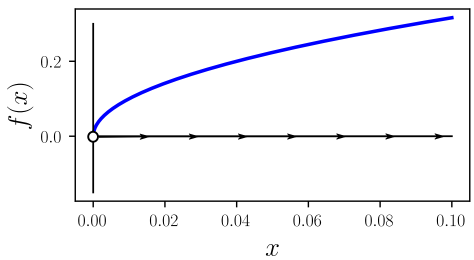
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{x}.$$

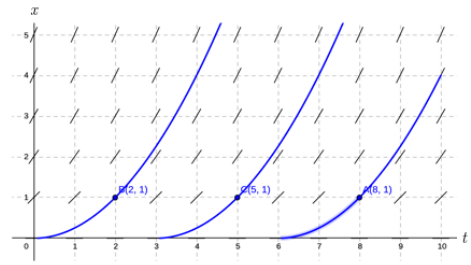
By blindly solving it, we have

$$x^{-1/2}\mathrm{d}x = \mathrm{d}t \Rightarrow 2x^{1/2} = t + C \Rightarrow x(t) = \left(\frac{t}{2} + c\right)^2.$$

By using the initial condition, we get $x(t) = t^2/4$ as a solution. However, $x(t) = 0$ is also a solution because we start at the fixed point. What happened? Which one is the solution?

Phase portrait and Direction field for $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) = x^{1/2}$





Direction field is from GeoGebra.

Algebraically, the solution $x(t) = \left(\frac{t}{2} + c\right)^2$ is for any $x \neq 0$. By "using" the initial condition, what we did is actually taking the $t \to 0$ limit and setting the limit to zero. Geometrically, by the phase portrait, we see that $f(x) = x^{1/2}$ has a infinite slope at the origin, $f^{(1)}(0) \to \infty$! In that case, a system at the origin doesn't know where to go! In fact, we can construct infinite number of solutions in this system. The system can stay at the origin for a arbitrary time interval T and then leave! The solution of that is

$$x(t) = \begin{cases} 0 & , 0 \leq t \leq T \\ \left(\frac{t-T}{2}\right)^2 & , t \geq T. \end{cases}$$

Since T is arbitrary, we have infinite number of choices, *i.e.* infinite number of solutions, in this IVP. The interval T corresponds to how long the bucket has stayed empty in our previous physics example.

Finally, as a "responsible mathematician", we shall at least state the Existence and Uniqueness Theorem for IVP in autonomous system. From the last example, one can already see that $f(x)$ has to be smooth enough. We cannot have infinite f' at the initial point.

Theorem. *Consider the IVP,* $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$ *and* $x(0) = x_0$. *Suppose* $f(x)$ *and* $f'(x)$ *are continuous on an open interval* R *of the* x*-axis, and suppose that* x_0 *is a point in* R. *Then, the IVP has a solution* $x(t)$ *on some time interval* $(-\tau, \tau)$ *about* $t = 0$ *and the solution is unique.*

With the theorem stated, can you pin down what goes wrong in the examples above so that the theorem does not apply at some time points?

One can further extend this Existence and Uniqueness Theorem to non-autonomous or higher dimensional IVPs. However, we will not go into there. We will only revise our work when there is something blowing up, having infinite slope, or other singular behavior.

1.4 Exercises

1.4.1 Checking a solution of a ODE: Derivative Practice

Exercise 1.1. Check that $y(x) = x^x$ is a solution to the nonlinear ODE,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + \frac{y \ln y}{x}.$$

Hint: We like the base to be e not x. How do we do that? The algebraic identity $a = e^{\ln(a)}$ will be helpful.

Exercise 1.2. Check that $x(t) = 2\arctan(\frac{e^t}{1+\sqrt{2}})$ solves

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sin(x).$$

Hint: you will need the half-angle tangent expression for sine: $\sin(x) = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$.

Exercise 1.3. Find the first and second derivative of

$$y(x) = 2c_1 \tan(c_1 \ln x + c_2) - 1$$

where c_1 and c_2 are arbitrary (integration) constants. And then verify that it is a general solution to the nonlinear ODE,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hint: Be careful on taking the second derivative. Also, to check, it is easier to check whether the ratio $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} / \frac{\mathrm{d}y}{\mathrm{d}x}$ is y/x or not!

Exercise 1.4. *Check that a function $u(x,t) = A(x-ct)+B(x+ct)$ is a solution to the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where c is the speed of wave. This solution is called the d'Alembert's formula.
Hint: Taking partial derivative is just treating all other variables as constants.

Exercise 1.5. *Check that the Gaussian distribution (or called Normal distribution) $u(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$ is a solution to the heat equation (or diffusion equation) $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$ where D is a parameter of the heat equation.
Hint: Taking partial derivative is just treating all other variables as constants.

1.4.2 DE Classification

Exercise 1.6. Classify the following ODEs based on their (1) order, (2) dimension, and whether they are (3) autonomous, (4) linear, (5) homogeneous (if linear), (6) constant-coefficient (if linear) or not. Example answers would be

a first order, 1-D, autonomous, linear, homogeneous, constant-coefficient ODE

or

a first order, 1-D, autonomous, nonlinear, – , – ODE.

(a) Newton's cooling law: $\frac{\mathrm{d}T}{\mathrm{d}t} = k\,(T_{\text{env}} - T)$

(b) FitzHugh-Nagumo model for neuronal dynamics:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v - \frac{v^3}{3} - w + I(t)$$
$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{v + a - bw}{\tau}$$

where $v(t)$ is the membrane potential, $w(t)$ is the "strong ion current", $I(t)$ is the external current and a, b, τ are system parameters.

(c) Logistic Growth of a population: $\frac{\mathrm{d}\rho}{\mathrm{d}t} = r\rho(1 - \frac{\rho}{K})$ where $\rho(t)$ is the population density, r and K are growth rate and the carrying capacity of the environment.

(d) Ideal Pendulum: $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\sin\theta$ where g is the gravitational constant, l is the length of the pendulum, and $\theta(t)$ is the angle difference from the vertical line, choosing counterclockwise as positive.

(e) Cauchy-Euler Equation:

$$a_n x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}x^{n-1}\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + \ldots + a_0 y = 0$$

where all $a_0, a_1, \ldots, a_n$ are constants.

(d) Bernoulli differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p(x)y(x) + q(x)y^r(x)$$

where $1 \in \mathbb{R}\setminus\{0, 1\}$.

(e) Ricatti equation: $\frac{\mathrm{d}y}{\mathrm{d}x} = q_0(x) + q_1(x)y(x) + q_2(x)y^2(x)$.

Exercise 1.7. Write down a (system of) ODE satisfying the following criteria.

(a) 1-D, 3rd order inhomogeneous non-autonomous linear ODE

(b) 2-D, 2nd order homogeneous nonlinear ODE with degree 4

(c) 1-D, 10th order autonomous nonlinear ODE with degree 2

(d) 2-D 1st order constant-coefficient homogeneous linear ODE

1.4.3 Linearity and Superposition Principle

Exercise 1.8. Verify that both $x_1(t) = t^{-2}$ and $x_2(t) = t^{-2}\ln(t)$ are solutions of the differential equation

$$t^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 5t\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = 0.$$

Show that the linear superposition of them, $x(t) = c_1 t^{-2} + c_2 t^{-2}\ln(t)$ is the general solution of the ODE.

Exercise 1.9.

(a) From the definition of derivative,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h},$$

show the linearity of the differentiation (assuming all derivative exists),

$$\frac{\mathrm{d}}{\mathrm{d}t}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}t} + \frac{\mathrm{d}g}{\mathrm{d}t}$$

(b) Using the results in (a), argue that the same holds for the nth derivative (assuming the nth derivative of f and g both exist)

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}(f+g) = \frac{\mathrm{d}^n f}{\mathrm{d}t^n} + \frac{\mathrm{d}^n g}{\mathrm{d}t^n}.$$

(c) Using both (a) and (b), show that for a linear, homogeneous nth order ODE

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_1(x)y(x) = 0,$$

if both $\eta_1(x)$ and $\eta_2(x)$ are solutions, then so is the linear combination of them, $c_1\eta_1(x) + c_2\eta_2(x)$.

Exercise 1.10. Both

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} \text{ and } \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

are solutions of the system of ODEs:

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1 + x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 4x_1 + x_2 \end{aligned}$$

Show that the linear superposition of them,

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = c_1 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + c_2 \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

is a solution of the system of ODEs.

Exercise 1.11. Verify that $x_1(t) = 1$ and $x_2(t) = \ln(t)$ are solutions of the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2.$$

Show that their linear superposition, $z(t) = c_1 + c_2 \ln t$ is, in general (say $c_2 \neq 0, 1$), *not* a solution to this equation. Why is the superposition principle not applicable here?

Exercise 1.12. Verify that $x_1(t) = 1$ and $x_2(t) = 1 + e^{-2t}$ are solutions of the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = 2.$$

Show that their linear superposition, $z(t) = c_1 + c_2\left(1 + e^{-2t}\right)$ is, in general (*i.e.*, for arbitrary c_1 and c_2), *not* a solution to this equation. Why is the superposition principle not applicable here?

1.4.4 Nonlinearity, Existence and Uniqueness of Solutions

Exercise 1.13. How long does it take to reach a fixed point?
In this problem [modified from S Exercise 2.5.1] we discussed

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x^a$$

for $a \in \mathbb{R}$ on a half line $x \geq 0$. Consider a particle with position $x(t)$ and instantaneous velocity $-(x(t))^a$ started at a initial condition $x(0) = 1 > 0$.

(a) Show that the particle is always moving toward the origin before hitting it.

(b) Solve the IVP for both $a = 1$ and $a \neq 1$.
Hint: if you are a bit confused on algebra rules. Do some examples! Think $a = 2$ for $a > 1$, $a = 1/2$ for $0 < a < 1$, for instances.

(c) Show that for $a \geq 1$, it takes infinite time for the particle to reach the origin. But for $a < 1$, it takes finite time! Compute the finite time in terms of a for the $a < 1$ case. Note that for $a = 1/2$, this is our leaky bucket example. Of course the time to reach the fixed point is finite. There is only finite amount of water!

1.4.5 1-D Dynamical System

Exercise 1.14. Terminal Velocity The velocity of a falling particle with linear friction is given by

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - \eta v.$$

Suppose the initial velocity is $v(0) = v_0$. Solve $v(t)$, find all fixed points, and determine the stability of them by drawing the phase portrait. Take $t \to \infty$ of $v(t)$ to get the terminal velocity.

Exercise 1.15. Logistic Growth The population density (total number/ occupying area) growth of a population can be modeled by the logistic growth equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = r\rho\left(1 - \frac{\rho}{K}\right) \tag{1.4.1}$$

where r is the growth rate and K is the environment capacity for reason that will be clear in our analysis. Compare to exponential growth model $\frac{\mathrm{d}\rho}{\mathrm{d}t} = r\rho$, the logistic growth model introduce the additional factor of limited food.

(a) We can define a new time scale $\tau = at$ and a new population scale $M = b\rho$ to have a simpler equation

$$\frac{\mathrm{d}M}{\mathrm{d}\tau} = M(1 - M).$$

What is a and b in terms of r and K? This process is called *nondimensionalization*. In many analysis, this makes the problem less tedious to solve.

(b) Solve $M(\tau)$ and then solve $N(t)$ given the initial condition of N is $N(0) = N_0$.

(c) Find all the fixed points of Equation (1.4.1). Determine their stability by drawing the phase portrait. Which fixed point do you expect $N(t)$ converge to as $t \to \infty$?

(d) By taking $t \to \infty$ of your solution $N(t)$ in (b), verify your prediction in (c). Now, it should be clear why K is called environment capacity.