CHAPTER 2

Consider the space of vectors

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : -\infty < x_i < \infty \text{ for all } i \right\}$$

$\mathbb{R}^n$ is what is called a <u>vector space</u>. This follows from all the properties of matrix addition / scalar multiplication that we gave in Chapter 1.

The other vector space that we will encounter is:

$$\mathbb{C}^n = \left\{ \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ \vdots \\ x_n + iy_n \end{pmatrix} : \begin{matrix} -\infty < x_i < \infty \\ -\infty < y_i < \infty \end{matrix} \text{ for all } i \right\}.$$

A subspace of a vector space is a subset with some special properties.

> Notation:
> $x \in S$: x is an element of S
> $S \subset V$: S is a subset of V

Suppose $S \subset \mathbb{R}^n$

- for $\vec{x}, \vec{y} \in S$, $\vec{x} + \vec{y} \in S$
- for $\vec{x} \in S$, $\alpha \in \mathbb{R}$ $\alpha \vec{x} \in S$

$\hookrightarrow$ $\alpha = 0$: $\vec{0} \in S$.

<u>Example</u>

$$S = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$$ is a subspace of $\mathbb{R}^3$

<u>Example</u>

$$S = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$$ is not a subspace of $\mathbb{R}^3$

Span and linear independence

<u>Definition</u> Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be elements of $\mathbb{R}^n$ (or $\mathbb{C}^n$)
A sum of the form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

where $c_j \in \mathbb{R}$ (or $c_j \in \mathbb{C}$) for all j, is called a linear combination of $\vec{v}_1, \ldots, \vec{v}_k$.
The span of $\vec{v}_1, \ldots, \vec{v}_k$, denoted $W = \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$, is the set of all possible linear combinations of $\vec{v}_1, \ldots, \vec{v}_k$.

<u>Proposition</u> The span of any finite collection of vectors is a subspace.

<u>Proof</u>: Suppose $W = \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$. We have to check the two properties of a subspace.

- For $\vec{x}, \vec{y} \in W$

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$
$$\vec{y} = d_1\vec{v}_1 + \cdots + d_k\vec{v}_k$$

$$\vec{x}+\vec{y} = (c_1+d_1)\vec{v}_1 + \cdots + (c_k+d_k)\vec{v}_k$$

A linear combination. Since W is the set of all possible linear combinations, $\vec{x}+\vec{y} \in W$.

- For $\alpha\vec{x} \in W$

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

$$\alpha\vec{x} = (\alpha c_1)\vec{v}_1 + \cdots + (\alpha c_k)\vec{v}_k$$

A linear combination. Since W is the set of all possible linear combinations, $\alpha\vec{x} \in W$. $\blacksquare$

<u>Definition</u> The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are called linearly dependent if there exists scalars, $c_1, \ldots, c_k$, not all zero, such that

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}.$$

Vectors that are not linearly dependent are called linearly independent.

Q: How do we test for dependence / independence?

<u>Theorem</u> Let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$ and let $A = (\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_k)$ be the $n \times k$ matrix whose columns are the vectors $\vec{v}_1, \ldots, \vec{v}_k$.

a) The vectors $\vec{v}_1, \ldots, \vec{v}_k$ are linearly dependent if and only if there is a non-zero (aka non-trivial) solution of $A\vec{x} = \vec{0}$.

b) The vectors are linearly independent if and only if the only solution of $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

c) A vector $\vec{b}$ lies in the span of $\vec{v}_1, \ldots, \vec{v}_k$ if and only if there is a solution of $A\vec{x} = \vec{b}$.

<u>Example</u> Are the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

linearly independent or dependent?

<u>Example</u> Is $\vec{b} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ in span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

How about $c = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$?

<u>Proposition</u> A set of k vectors in $\mathbb{R}^n$ is linearly independent if and only if the corresponding $n \times k$ matrix has rank k.

Q: $k > n$? No!

A set of k vectors in $\mathbb{R}^n$, $k > n$, must be

<u>Proposition</u> A set of k vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$ if and only if corresponding $n \times k$ matrix has rank n.

$$\Downarrow$$

$$k \geq n!$$

So, a collection of vectors in $\mathbb{R}^n$ that is both linearly independent and spans must be a collection of n vectors.

Basis and Dimension

<u>Definition</u> A <u>basis</u> of a subspace S of $\mathbb{R}^n$ is a finite collection of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in S$ that

a) spans S, and
b) is linearly independent.

What we just noticed:

<u>Theorem</u> Every basis of $\mathbb{R}^n$ consists of exactly n vectors.
A set of n vectors in $\mathbb{R}^n$ is a basis if and only if the $n \times n$ matrix $A = (\vec{v}_1 \dots \vec{v}_n)$ is nonsingular

- rank $A = n$.

<u>Theorem</u> Suppose S is a subspace of $\mathbb{R}^n$. Then every basis for S has the same number of vectors, say k. In this case $\dim S = k$.

Suppose S has a basis $\vec{v}_1, \vec{v}_2$
Now consider $\vec{w}_1, \vec{w}_2, \vec{w}_3 \in S$. A basis spans S, that is, every vector in S can be written as a linear combination of $\vec{v}_1, \vec{v}_2$:

$\vec{w}_1 = a_1\vec{v}_1 + a_2\vec{v}_2, \quad \vec{w}_2 = b_1\vec{v}_1 + b_2\vec{v}_2, \quad \vec{w}_3 = c_1\vec{v}_1 + c_2\vec{v}_2$

Then consider a linear combination:

$$d_1\vec{w}_1 + d_2\vec{w}_2 + d_3\vec{w}_3$$

$$\|$$

$$(a_1d_1)\vec{v}_1 + (a_2d_1)\vec{v}_2 + (b_1d_2)\vec{v}_1 + (b_2d_2)\vec{v}_2 + (c_1d_3)\vec{v}_1 + (c_2d_3)\vec{v}_2$$

$$= (a_1d_1 + b_1d_2 + c_1d_3)\vec{v}_1 + (a_2d_1 + b_2d_2 + c_2d_3)\vec{v}_2$$

Suppose $a_1 \neq 0$: Choose $d_1 = \dfrac{-b_1d_2 - c_1d_3}{a_1}$

Suppose $b_2 \neq 0$: Choose $d_2 = \dfrac{-a_2d_1 - c_2d_3}{b_2}$

Choose $d_3 \neq 0$.

$\Rightarrow \vec{w}_1, \vec{w}_2, \vec{w}_3$ are linearly dependent!

Now, consider a separate situation where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is a basis for S.

Can $\vec{w}_1, \vec{w}_2 \in S$ span S? Suppose $\vec{w}_1, \vec{w}_2$ are linearly independent. If they do span, then this is a basis, then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is dependent by the previous argument $\Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3$ is not a basis. A contradiction.

$\Rightarrow$ The do not span.

<u>Conclusion</u>: If S is a subspace of $\mathbb{R}^n$ and if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is a basis of S:

- Every collection of more than k vectors from S is linearly <u>dependent</u>
- Every collection of fewer than k vectors <u>does</u> <u>not</u> <u>span</u>.

$\left\{ \right.$
- A collection of k elements is a basis if and only if it spans
- A collection of k elements is a basis if and only if it is linearly independent.

The Fundamental Matrix Subspaces

<u>Definition</u>
- The image, $\operatorname{img} A$, of an $m \times n$ matrix A is the subspace of $\mathbb{R}^m$ spanned by the columns of A.
- The kernel (or null space) of A is given by

$$\ker A = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0} \}.$$

<u>Definition</u> The pivot columns of an $m \times n$ matrix A are the columns of A that correspond to columns of a reduced row echelon version of A containing pivots.

<u>Theorem</u> The pivot columns of A form a basis for $\operatorname{rng} A$.

<u>Ex</u> Find a basis for $\operatorname{img} A$ and $\ker A$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$A \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{pivot} & \text{pivot} & \text{free} \end{array}$$

x_3 free

$x_2 = x_3$

$x_1 = -x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\operatorname{img} A = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$\ker A = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Ex Do the same for:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

x_3 free
x_2 free
$x_1 = -x_2 - x_3$

$$\operatorname{img} A = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\ker A = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Every free variable corresponds to a basis vector!

Since the number of free variables and the number of pivots add up to be the number of columns

Theorem Let A be an $m \times n$ matrix. Then

$$\dim \ker A + \dim \operatorname{img} A = n.$$

Something we will be interested in later on is how can we understand solving

$$A\vec{x} = \vec{b}$$

when $\ker A \neq \{\vec{0}\}$ or $\vec{b} \notin \operatorname{img} A$.