AMATH 352

The textbook is Olver and Shakiban
Applied Linear Algebra

Motivation: Linear algebra is arguably the most algorithmic subject in mathematics. The goal of this course is to obtain a fundamental understanding of the subject so that if we can reduce a harder problem to linear algebra then we know how to solve the full problem.

Example: Find a quadratic polynomial $p(x)$ such that
$$p(-1)=0, \quad p(1)=0, \quad p'(0)=0$$

$$p(x) = ax^2 + bx + c$$

$$\left.\begin{aligned} a - b + c &= 0 \\ a + b + c &= 0 \\ 2a + b + 0 &= 0 \end{aligned}\right\} \quad p(x) = x^2 - 1 \quad \text{or} \quad p(x) = \alpha(x^2-1)$$

The solutions of this problem can be characterized by linear algebra

The essence of linear algebra is solving linear systems and characterizing when they can be solved.

CHAPTER 1 The solution of linear systems of equations

A linear system of equations (a linear system) is a set of equations of the form

$$\begin{cases} x + 2y + z = 2 \\ 2x + 6y + z = 7 \\ x + y + 4z = 3 \end{cases} \qquad (1)$$

where no variable is multiplying itself or another variable (no xz, x^2, xyz, etc.)

We can add and subtract equations from one another without changing the solution. For example, if we multiply row 1 by -2 and add it to row 2 we find

$$-2r_1 + r_2 \rightarrow r_2 \Rightarrow \begin{cases} x + 2y + z = 2 \\ 0 + 2y - z = 3 \\ x + y + 4z = 3 \end{cases}$$

$$-r_1 + r_3 \rightarrow r_3 \Rightarrow \begin{cases} x + 2y + z = 2 \\ 0 + 2y - z = 3 \\ 0 - y + 3z = 1 \end{cases}$$

then eliminate y from the last equation

$$\frac{1}{2}r_2 + r_3 \rightarrow r_3 \Rightarrow \begin{cases} x + 2y + z = 2 \\ 0 + 2y - z = 3 \\ 0 + 0 + \frac{5}{2}z = \frac{5}{2} \end{cases}$$

This linear system is now upper-triangular. It is solved with a process called backward substitution:

$$z = 1$$
$$2y - 1 = 3 \Rightarrow y = 2$$
$$x + 2(2) + 1 = 2 \Rightarrow x = -3.$$

Matrices and vectors

We use this section to fix notation.

A matrix is a rectangular array of numbers

$$\begin{pmatrix} 1.1 & 0 & 3 \\ 2 & 1.2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 4 & 1 \end{pmatrix}$$

In general,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

is an $m \times n$ matrix with m rows and n columns.

A row vector is a $1 \times n$ matrix and a column vector is an $m \times 1$ matrix. A general linear system is given by

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 \\ \quad \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m. \end{cases}$$

We call A the coefficient matrix.

<u>Matrix arithmetic</u> We have 3 basic operations with matrices.

1) matrix addition,
2) scalar multiplication, and
3) matrix multiplication

<u>Matrix addition</u>

Matrix addition is defined elementwise for matrices of the <u>same size</u>.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 4 & 4 \end{pmatrix}$$

In general, if A & B are $m \times n$ matrices then

$$A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}, \quad B = (b_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

and

$$A + B = (a_{ij} + b_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

<u>Scalar Mutiplication</u>

If $\alpha \in \mathbb{R}$ then

$$\alpha A = (\alpha a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

$$(-3) \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -3 & -6 \\ -9 & -12 \end{pmatrix}$$

Matrix Multiplication

We first define a vector product between a $1 \times n$ row vector and an $n \times 1$ column vector.

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n .$$

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -1 + 0 + 3 = 2$$

For general matrices, $C = AB$ where A is $m \times n$ and B is $n \times p$ is defined using the vector product of rows of A and columns of B

$$C = \left(a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj} \right)_{\substack{1 \le i \le m \\ 1 \le j \le p}}$$

$$\underset{3 \times 3}{\begin{pmatrix} 1 & 1 & 0 \\ 1\frac{1}{2} & 3 & 1 \\ 0 & 2 & 1 \end{pmatrix}} \underset{3 \times 2}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}} = \begin{pmatrix} 2 & 1 \\ 4\frac{1}{2} & 2\frac{1}{2} \\ 3 & 2 \end{pmatrix}$$

Other notation

zeros

$$I = I_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$n \times n$ identity matrix.

zeros

Basic Matrix Arithmetic

Matrix Addition:

$$A+B = B+C \qquad \text{Commutivity}$$

$$(A+B)+C = A+(B+C) \qquad \text{Associativity}$$

$$A+\underbrace{O}_{\text{zero matrix}} = O+A = A \qquad \text{Additive identity}$$

$$A+(-A) = O \qquad \text{Additive inverse}$$

Scalar Mutiplication:

$$c(dA) = d(cA) \qquad \text{Commutivity}$$

$$c(dA) = (cd)A \qquad \text{Associativity}$$

$$c(A+B) = cA + cB \qquad \text{Distributive}$$

$$1A = A \qquad \text{unit}$$

$$0 \cdot A = O \qquad \text{zero}$$

Matrix Multiplication:

$$(AB)C = A(BC) \qquad \text{Associativity}$$

$$A(B+C) = AB+AC \qquad \text{Distributive}$$

$$(A+B)C = AC+BC$$

$$IA = A = AI \qquad \text{Identity matrix}$$

$$AO = O = OA \qquad \text{zero matrix}$$

But

$AB \neq BA$ for most matrices

Exercise

Show that A, B must be square if AB and BA are both defined.

Regular Gaussian Elimination

We replace the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

first with a <u>matrix equation</u>

$$A\vec{x} = \vec{b}$$

$$A = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}, \quad \vec{x} = (x_j)_{1\le j\le n}, \quad \vec{b} = (b_i)_{1\le i\le m}.$$

This is then encoded using an <u>augmented matrix</u>

$$(A \mid \vec{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

<u>Example</u> For the system

$$\begin{aligned} 10x + 2y &= 3 \\ 4x + y &= 1 \end{aligned} \implies \left(\begin{array}{cc|c} 10 & 2 & 3 \\ 4 & 1 & 1 \end{array}\right)$$

<u>Example</u>

$$\begin{aligned} x + 2y + z &= 2 \\ 2x + 6y + z &= 7 \\ x + y + 4z &= 3 \end{aligned} \implies \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 6 & 1 & 7 \\ 1 & 1 & 4 & 3 \end{array}\right)$$

We now solve this last system using only the augmented matrix.

1st pivot

$$-2r_1 + r_2 \to r_2 \qquad \left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 1 & 1 & 4 & 8 \end{array}\right)$$

$$-r_1 + r_3 \to r_3 \qquad \left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 0 & -1 & 3 & 1 \end{array}\right)$$

$$\tfrac{1}{2}r_2 + r_3 \to r_3 \qquad \left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{2} & -1 & 3 \\ 0 & 0 & \boxed{5/2} & 5/2 \end{array}\right)$$

2nd pivot 3rd pivot

This algorithm (using just the addition of row) is called <u>regular Gaussian elimination</u> (RGE).

A matrix is called regular if RGE reduces it to upper-triangular form.

<u>Elementary matrices and the LU Factorization</u>

<u>Definition</u> The <u>elementary matrix</u> E associated with an elementary row operation for m-rowed matrices is the matrix obtained by applying the row operation to the $m \times m$ identity matrix I_m.

$$-2r_1 + r_2 \to r_2 \implies \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_1$$

$$E_1 \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix} \quad \checkmark$$

To encode the other row operations performed on this matrix:

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$$

$$-r_1 + r_3 \to r_3 \qquad\qquad \tfrac{1}{2} r_2 + r_3 \to r_3$$

Then

$$E_3 E_2 E_1 \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 5/2 \end{pmatrix}$$

<u>Claim:</u>

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{pmatrix}}_{M_3} E_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = I$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{M_2} E_2 = I$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{M_1} E_3 = I$$

Then

$$\underbrace{M_3 E_3}_{I} E_2 E_1 \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} = M_3 \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 5/2 \end{pmatrix}$$

$$\Downarrow$$

$$E_2 E_1 \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} = M_3 \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 5/2 \end{pmatrix}$$

$$\Downarrow$$

$$\underbrace{M_2 E_2}_{I} E_1 \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} = M_2 M_3 \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 5/2 \end{pmatrix}$$

$$\Downarrow$$

$$E_1 \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} = M_2 M_3 \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 5/2 \end{pmatrix}$$

$$\Downarrow$$

$$\underbrace{M_1 E_1}_{I} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} = M_1 M_2 M_3 \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 5/2 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} = M_1 M_2 M_3 \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 5/2 \end{pmatrix}$$

Some magic!

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1/2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 5/2 \end{pmatrix}$$

$$A \quad = \quad L \quad U$$

This is the LU factorization.

<u>Definition</u> A special lower- (upper-) triangular matrix is an $n \times n$ matrix that is lower- (upper-) triangular with ones on the diagonal.

The product of two (special) lower triangular matrices is also (special) lower triangular.

<u>Theorem</u> A matrix A is regular if and only if it can be factored

$$A = LU$$

where L is special lower triangular and U is upper triangular.

Now, we compute the LU factorization with simpler book keeping

<u>Example</u> Compute the LU factorization of

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 0 & -3 & 2 \end{pmatrix}$$

$-r_1 + r_2 \to r_2$, $\quad -2r_1 + r_3 \to r_3$

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix}$$

$-3/2\, r_2 + r_3 \to r_3$

The LU factorization is regular Gaussian Elimination with book keeping

Applications of the LU factorization

1) Forward and backward substitution

Solve $\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3/2 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\begin{pmatrix} u \\ v \\ w \end{pmatrix}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

First solve $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3/2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{cases} u = 1 \\ u + v = 0 \Rightarrow v = -1 \\ 2u + \frac{3}{2}v + w = 0 \\ \qquad \Rightarrow w = -\frac{1}{2} \end{cases}$$

Then solve

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1/2 \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$z = \frac{1}{8}$$

$$-2y + 4z = -1 \Rightarrow y = 3/4$$

$$x + 2y - z = 1 \Rightarrow x = -3/8$$

2) Determinants!

Permutations and Pivoting

Problem Compute the LU factorization of

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

We fail immediately.

To deal with this issue we add a new technique: the row interchange or row permutation.

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

Definition A square matrix is called non-singular if it can be reduced to an upper triangular matrix w/ non-zero diagonal entries – the pivots – by swapping rows and adding multiples of rows.

<u>Theorem</u> The system $A\vec{x}=\vec{b}$ has a unique solution $\vec{x}$ for every choice of $\vec{b}$ if and only if A is non-singular

<u>Partial Pivoting</u> In swapping rows in the previous example, we are left with some options:

$$r_1 \leftrightarrow r_2$$

or

$$r_1 \leftrightarrow r_3.$$

You can check that both choices lead to a solution. So why does it matter? Computers do not do exact arithmetic (generally) and rounding errors can accumulate.

It turns out that a good strategy to eliminate rounding errors is to make the pivots as large as possible.

<u>Example</u>

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

$r_1 \leftrightarrow r_3$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$-\frac{1}{2}r_1 + r_2 \to r_2$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1/2 & 3/2 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$r_2 \leftrightarrow r_3$
$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1/2 & 3/2 & 0 \end{array}\right)$$

$-\frac{1}{2}r_2 + r_3 \to r_3$
$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

Generalizing the LU factorization

An elementary permutation matrix is found by applying a row interchange to the identity matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad r_1 \leftrightarrow r_3 \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = P_1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad r_2 \leftrightarrow r_3 \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = P_2$$

$P_1 P_1 = I$
$P_2 P_2 = I$

So, let's find the full generalized LU factorization of

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

using partial pivoting.

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

Apply P_1:

$$P_1 A = \underbrace{P_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P_1}\ \underbrace{P_1 \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}}$$

$$= P_1 P_1 \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

$$P_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 1 & -1 \end{pmatrix}$$

$$P_2 P_1 A = P_2 \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P_2 P_2 \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= P_2 \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P_2 \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1/2 & 3/2 \end{pmatrix}$$

MAGIC!

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1/2 & 3/2 \end{pmatrix}$$

$$P_2 P_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

$P_2 P_1$ = Take identity matrix, swap 1, 3 then 2, 3

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

So,

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

Let's do this another way:

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

Interchange r_1 & r_3

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}$$

ignore the 1's and swap rows. Nothing happens because it's all zeros

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1/2 & 3/2 \\ 0 & 1 & -1 \end{pmatrix}$$

interchang r_2 & r_3

swap

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1/2 & 3/2 \end{pmatrix}$$

swap

ignore the 1's and swap rows.

$$\underset{P}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} A = \underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}}$$

<u>Transpose</u> Let $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$.

The <u>transpose</u> A^T of A is given by

$$A^T = (a_{ji})_{\substack{1 \le i \le n \\ 1 \le j \le m}}.$$

The rows of A become the colums of A^T:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

A key property:

$$(AB)^T = B^T A^T$$

Consider the matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

It follows that P is constructed by performing

$r_1 \leftrightarrow r_3$

then

$r_2 \leftrightarrow r_3$

on the identity matrix. So

$$P = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{P_1} \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{P_2}$$

both symmetric

Recall that $P_1 P_1 = I = P_2 P_2$. Then

$$P^T = P_2^T P_1^T = P_2 P_1$$

and

$$P^T P = P_2 \underbrace{P_1 P_1}_{I} P_2 = P_2 P_2 = I.$$

P^T is what is called the inverse of P.

The inverse of a matrix

We noted above that for

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ and by divine inspiration,

I found that for $M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$M_1 E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We now demonstrate that for every non-singular (square!) matrix A there exists A^{-1}, called the inverse matrix, such that

$$A A^{-1} = I = A^{-1} A.$$

Suppose A is 3×3, write

$$A^{-1} = (\vec{\alpha}_1 \ \vec{\alpha}_2 \ \vec{\alpha}_3)$$

then

$$A A^{-1} = (A\vec{\alpha}_1 \ A\vec{\alpha}_2 \ A\vec{\alpha}_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This implies

$$A\vec{\alpha}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A\vec{\alpha}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad A\vec{\alpha}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We need to solve 3 linear systems!

A good way to solve all three of these simultaneously is through a process called Gauss-Jordan elimination

<u>Example</u> Find the inverse of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

<u>Step 1</u> Augment

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$$

<u>Step 2</u> Gaussian elimination (w/ pivoting)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right)$$

<u>Step 3</u> A new row operation: Multiply rows to put 1's on the diagonal

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right)$$

<u>Step 4</u> Eliminate upper-triangular elements

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right) \Rightarrow A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -1/2 & 0 & 1/2 \end{pmatrix}.$$

General linear systems

Thus far we have focused on
square matrices $\longleftrightarrow$ n equations
n unknowns

and maybe augmented.

In many applications we have to consider
m equations and
n unknowns, $m \neq n$.

Example Balancing Chemical Equations

$$\overset{x_1}{(?)}\, C_3H_8 + \overset{x_2}{(?)}\, O_2 \longrightarrow \overset{x_3}{(?)}\, CO_2 + \overset{x_4}{(?)}\, H_2O$$

propane

Matter is neither created nor lost in this reaction.
We have 3 elements C, H, O so we can represent
each molecule as a vector

$$C_3H_8 \to \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}, \quad O_2 \to \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$CO_2 \to \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad H_2O \to \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Conservation tells us

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

<u>or</u>

$$\begin{bmatrix} 3 & 0 & -1 & 0 \\ 8 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is a <u>homogeneous</u> linear system

Let's do Gaussian elimination:

$$\begin{bmatrix} 3 & 0 & -1 & 0 \\ 8 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 0 & 8/3 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 8/3 & -2 \end{bmatrix}$$

Suppose we know x_4

$$\tfrac{8}{3} x_3 - 2 x_4 = 0$$

$$\Downarrow$$

$$x_3 = \tfrac{3}{4} x_4$$

$$2x_2 - 2x_3 - x_4 = 0 \Rightarrow 2x_2 = \tfrac{3}{2} x_4 + x_4$$

$$x_2 = \tfrac{5}{4} x_4$$

$8x_1 - x_3 = 0 \Rightarrow 3x_1 = \frac{3}{4}x_4 \Rightarrow x_1 = \frac{1}{4}x_4$

$$x_1 = \frac{1}{4}x_4$$

$$x_2 = \frac{5}{4}x_4$$

$$x_3 = \frac{3}{4}x_4$$

x_4 was never determined.

Choose $x_4 = 4$. Why?

$$\boxed{C_3H_8 + 5\,O_2 \rightarrow 3\,CO_2 + 4\,H_2O}$$

This matrix is in row echelon form.

pivots

pivot variables

free variables.

$$\begin{bmatrix} 3 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 8/3 & -2 \end{bmatrix}$$

$$x_1 \quad x_2 \quad x_3 \quad \boxed{x_4}$$

Example Put

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 2 & 2 & 0 & -1 \end{bmatrix}$$

in row echelon form

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 2 & 2 & -1 \\ 0 & -2 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 \quad x_2 \quad \boxed{x_3} \quad \boxed{x_4}$$

Row operation types:

1) A a multiple of one row to another.
2) Swap rows
3) Mutiply a row by a non-zero number

Fact Every matrix can be reduced to row echelon form using only operations of types 1 & 2.

Definition The rank of a matrix is the number of pivots.

Fact A square matrix is non-singular if and only if there is a pivot in every row.

Fact A system $A\vec{x} = \vec{b}$ of m linear equations in n unknows has either

(i) exactly one solution,
(ii) infinitely many solutions, or
(iii) no solution

Cholesky factorization

Consider the LU factorization of
$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

$A = A^T$!

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{pmatrix}$$

positive!

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix}}_{L}\underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{pmatrix}}_{D}\underbrace{\begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{pmatrix}}_{L^T}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{4/3} \end{pmatrix}\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{4/3} \end{pmatrix}\begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{2}/2 & \sqrt{3/2} & 0 \\ 0 & \sqrt{2/3} & \sqrt{4/3} \end{pmatrix}}_{\hat{L}}\underbrace{\begin{pmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{2}/2 & \sqrt{3/2} & 0 \\ 0 & \sqrt{2/3} & \sqrt{4/3} \end{pmatrix}^T}_{\hat{L}^T}$$

Cholesky decomposition.

Fact If $A = A^T$ and A has an LU factorization with $U_{ii} > 0$ for all i, then A has a Cholesky factorization.

Determinants

Fact Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{if } ad-bc = 0.$$

The scalar quantity $ad-bc$ is the determinant of A written

$$\det A = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc.$$

Theorem Associated with every square matrix, there exists a uniquely defined scalar quantity, know as its determinant, that obeys the following axioms.

(i) Adding a multiple of one row to another does not change the determinant (Type 1)

(ii) Interchanging rows changes the sign of the determinant. (Type 2)

(iii) Multiplying a row by any scalar (including zero) mutiplies the determinant by the same scalar (Type 3)

(iv) The determinant of an upper-triangular matrix U is the product of the diagonal entries.

$$\det U = u_{11} u_{22} \cdots u_{nn}.$$

Ex

- $\det I = 1$
- $\det \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = 3$
- $\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$
- $\det \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} = 0$ (why?)

Theorem If $A = LU$ is a regular matrix then

$$\det A = \det U = u_{11} u_{22} \cdots u_{nn}$$

More generally, if A is nonsingular $PA = LU$ then

$$\det A = (\det P)(\det U)$$
$$= (-1)^k \, u_{11} u_{22} \cdots u_{nn}.$$

k: # of row interchanges

Lastly, A is non-singular if and only if

$$\det A \neq 0.$$

Example

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & -3 & 4 \\ 0 & 2 & -2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Find det A.

$$\underset{\det A}{\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & -3 & 4 \\ 0 & 2 & -2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}} \to \underset{\det A}{\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 1 & 2 & -1 \end{pmatrix}} \to \underset{\det A}{\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & -1 \end{pmatrix}}$$

$$\to \underset{-\det A}{\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}}$$

$$\boxed{\begin{aligned} 9 &= -\det A \\ \det A &= -9. \end{aligned}}$$

Example

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 2 & -4 & 4 \\ 0 & 2 & -2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Find det A.

$$\underset{\det A}{\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 1 & 2 & -1 \end{pmatrix}} \to \underset{\frac{1}{2}\det A}{\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 1 & 2 & -1 \end{pmatrix}} \to \underset{\frac{1}{2}\det A}{\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & -1 \end{pmatrix}}$$

$$\to \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$-\frac{1}{2} \det A$$

$$\boxed{\begin{gathered} 9 = -\frac{1}{2} \det A \\ \det A = -18. \end{gathered}}$$