AMATH 383 HW 5

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Exercise 13.4

a

Assume $u(y,t) = \phi(y)e^{i\omega_0 t}$, then

$$\begin{aligned}\frac{\partial u}{\partial t} &= i\omega_0 \phi(y) e^{i\omega_0 t} \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}[\phi(y)' e^{i\omega_0 t}] \\ &= \phi''(y) e^{i\omega_0 t}\end{aligned}$$

Plug in partial differentiation of $u(y,t)$ into ODE,

$$\begin{aligned}\frac{\partial u}{\partial t} &= \alpha^2 \frac{\partial^2 u}{\partial y^2} \\ i\omega_0 \phi(y) e^{i\omega_0 t} &= \alpha^2 \phi''(y) e^{i\omega_0 t} \\ i\omega_0 \phi(y) &= \alpha^2 \phi''(y)\end{aligned}$$

Solve this second order ODE by assuming $\phi(y) = e^{\lambda y}$,

$$\begin{aligned}\phi''(y) - \frac{i\omega_0 \phi(y)}{\alpha^2} &= 0 \\ \lambda^2 e^{\lambda y} - \frac{i\omega_0}{\alpha^2} e^{\lambda y} &= 0 \\ \lambda &= \pm \frac{\sqrt{i\omega_0}}{\alpha} \\ \phi(y) &= c_1 e^{\frac{\sqrt{i\omega_0}}{\alpha} y} + c_2 e^{-\frac{\sqrt{i\omega_0}}{\alpha} y}\end{aligned}$$

Since $c_1 + c_2 = u_0$, $u(y,t)$ is bounded as $y \to \infty$, therefore $c_1 = 0$ since $\sqrt{\frac{i\omega_0}{\sigma^2}} > 0$, $e^{\frac{\sqrt{i\omega_0}}{\alpha} y}$ is unbounded as $y \to \infty$. Then, we get $c_2 = u_0$, plug in c_1, c_2 to function of $\phi(y)$:

$$\begin{aligned}\phi(y) &= c_1 e^{\frac{\sqrt{i\omega_0}}{\alpha}y} + c_2 e^{-\frac{\sqrt{i\omega_0}}{\alpha}y} \\ &= u_0 e^{-\frac{\sqrt{i\omega_0}}{\alpha}y} \\ u(y,t) &= u_0 e^{-\frac{\sqrt{i\omega_0}}{\alpha}y} e^{i\omega_0 t}\end{aligned}$$

b

The daily conductivity of soil is $0.01 \cdot 3600 \cdot 24 = 864 cm^2/day$.

Note that $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$, real part of $e^{i\theta} = cos(\theta)$. If we choose only real part, the function of $u(y,t)$ can be written as:

$$\begin{aligned}u(y,t) &= u_0 e^{-\sqrt{\frac{\omega_0}{2\alpha^2}}(1+i)y + i\omega_0 t} \\ &= u_0 e^{-\sqrt{\frac{\omega_0}{2\alpha^2}}(1)y} cos(-\sqrt{\frac{\omega_0}{2\alpha^2}}y + \omega_0 t)\end{aligned}$$

Note that for given y, and frequency $\omega_0 = 2\pi$, the period of $cos(-\sqrt{\frac{\omega_0}{2\alpha^2}}y + \omega_0 t)$ is 1. Let the unit of t be daily, then for any t, such that $a < t < a+1$, where a is natural number, $-1 \leq cos(-\sqrt{\frac{\omega_0}{2\alpha^2}}y + \omega_0 t) \leq 1$. Therefore, the daily variation of temperature ranges from $-u_0 e^{-\sqrt{\frac{\omega_0}{2\alpha^2}}(1)y}$ to $u_0 e^{-\sqrt{\frac{\omega_0}{2\alpha^2}}(1)y}$.

Plug the initial conditions $u_0 = 5, |u| = 2, \alpha^2 = 864$ into previous equation:

$$\begin{aligned}2 &= 5e^{-\sqrt{\frac{\omega_0}{2\alpha^2}}(1)y} \\ log(\frac{2}{5}) &= log(e^{-\sqrt{\frac{\omega_0}{2\alpha^2}}(1)y}) \\ y &= \frac{log(\frac{2}{5})}{-\sqrt{\frac{2\pi}{2\alpha^2}}} \approx 15.2\end{aligned}$$

Depth below 15.2 cm will control daily temperature variation within 2 degrees.

c

The yearly conductivity of soil is $0.01 \cdot 3600 \cdot 24 \cdot 365 = 315360 cm^2/year$.

Note that $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$, real part of $e^{i\theta} = cos(\theta)$. If we choose only real part, the function of $u(y,t)$ can be written as:

$$\begin{aligned} u(y,t) &= u_0 e^{-\sqrt{\frac{\omega_0}{2\alpha^2}}(1+i)y + i\omega_0 t} \\ &= u_0 e^{-\sqrt{\frac{\omega_0}{2\alpha^2}}(1)y} cos(-\sqrt{\frac{\omega_0}{2\alpha^2}}y + \omega_0 t) \end{aligned}$$

Note that for given y, and frequency $\omega_0 = 2\pi$, the period of $cos(-\sqrt{\frac{\omega_0}{2\alpha^2}}y + \omega_0 t)$ is 1. Let the unit of t be yearly, then for any t, such that $a < t < a+1$, where a is natural number, $-1 \leq cos(-\sqrt{\frac{\omega_0}{2\alpha^2}}y + \omega_0 t) \leq 1$. Therefore, the yearly variation of temperature ranges from $-u_0 e^{-\sqrt{\frac{\omega_0}{2\alpha^2}}(1)y}$ to $u_0 e^{-\sqrt{\frac{\omega_0}{2\alpha^2}}(1)y}$.

Plug the initial conditions $u_0 = 15, |u| = 2, \alpha^2 = 315360$ into previous equation:

$$\begin{aligned} 2 &= 15 e^{-\sqrt{\frac{\omega_0}{2\alpha^2}}(1)y} \\ log(\frac{2}{15}) &= log(e^{-\sqrt{\frac{\omega_0}{2\alpha^2}}(1)y}) \\ y &= \frac{log(\frac{2}{15})}{-\sqrt{\frac{2\pi}{2\alpha^2}}} \approx 638.55 \end{aligned}$$

Depth below 638.55 cm will control daily temperature variation within 2 degrees.

d

At $y = 0$, $u(y,t) = u_0 cos(\omega_0 t)$, for $y > 0$, $u(y,t) = u_0 e^{-\sqrt{\frac{\omega_0}{2\alpha^2}}(1)y} cos(-\sqrt{\frac{\omega_0}{2\alpha^2}}y + \omega_0 t)$.

We can see phase at $y = 0$ is $(\omega_0 t)$, phase at ideal depth is $(-\sqrt{\frac{\omega_0}{2\alpha^2}}y + \omega_0 t)$. The phase difference is $(-\sqrt{\frac{\omega_0}{2\alpha^2}}y)$.

Since we want the temperature at ideal depth to be inverse as to surface temperature, the phase shift should be $\frac{1}{2}$ or $-\frac{1}{2}$ (note the period is 1). Here, we choose phase shift $-\frac{1}{2}$ then we can come up with equation:

$$\begin{aligned} -(\sqrt{\frac{\omega_0}{2\alpha^2}}y) &= -\frac{1}{2} \cdot 2\pi \\ y &= \pi\sqrt{\frac{2\alpha^2}{\omega_0}} = \pi\sqrt{\frac{2 \cdot 315360}{2\pi}} \approx 995.1 \end{aligned}$$

At depth of 995.1 cm, the cellar will be perfectly out of phase than surface.

Exercise 2.7

a

Change in number of nodes that have k citations can be represented by

$$N_k(n+1) - N_k(n)$$

Such difference can also be explained by subtraction between two parts. The first part is number of nodes that previously had $(k-1)$ citation now being cited by new node. The second part is number of nodes that previously had k also begin cited by new node.

The first part can be represented by multiple probability for notes with $k-1$ citations of being cited with number of citations in new node:

$$\frac{(k-1)P_{k-1}(n)}{2m} \cdot m \to \frac{(k-1)P_{k-1}(n)}{2}$$

The second part can be represented by multiple probability for notes with k citations of being cited with number of citations in new node:

$$\frac{kP_k(n)}{2m} \cdot m \to \frac{kP_k(n)}{2}$$

Change in number of nodes that have k citations can be represented by part one minus part two:

$$N_k(n+1) - N_k(n) = \frac{(k-1)P_{k-1}(n)}{2} - \frac{kP_k(n)}{2}$$

b

The fraction of nodes with k citations P_k can be represented as:

$$P_k(n) = \frac{Nk(n)}{n}$$

When n is large enough, such fraction $P_k(n)$ is independent to n, therefore, we can write $P_k(n)$ as constant P_k

$$N_k(n) = n \cdot P_k$$
$$N_k(n+1) = (n+1)P_k$$

$$\begin{aligned}
N_k(n+1) - N_k(n) &= \frac{(k-1)P_{k-1}}{2} - \frac{kP_k}{2} \\
(n+1)P_k - n \cdot P_k &= \frac{(k-1)P_{k-1}}{2} - \frac{kP_k}{2} \\
\frac{P_k}{P_{k-1}} &= \frac{k-1}{k+2}
\end{aligned}$$

c

From previous part, we know the relationship between P_k and P_{k-1}.

$$\begin{aligned}
P_k &= \frac{k-1}{k+2}P_{k-1} \\
&= \frac{k-1}{k+2} \cdot \frac{k-2}{k+1} \cdot P_{k-2} \\
&= \frac{k-1}{k+2} \cdot \frac{k-2}{k+1} \cdot \frac{k-3}{k} \cdot P_{k-3} \\
&= \frac{(k-1)!}{(k+2)(k+1)k...5}P_2
\end{aligned}$$

By canceling the common factors, and consider P_2 as a constant, we get

$$P_k \propto \frac{1}{(k+2)(k+1)k}$$

When k goes large enough:

$$P_k \propto k^{-3}$$