AMATH 383 Homework 2

Nan Tang
University of Washington

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Exercise 9.4

a

Solve this set of differential equation through dividing $\frac{dy}{dt}$ by $\frac{dx}{dt}$.

$$\frac{dy}{dt}/\frac{dx}{dt} = \frac{-bx}{-axy}$$
$$\frac{dy}{dx} = \frac{b}{ay}$$
$$(ay)dy = (b)dx$$

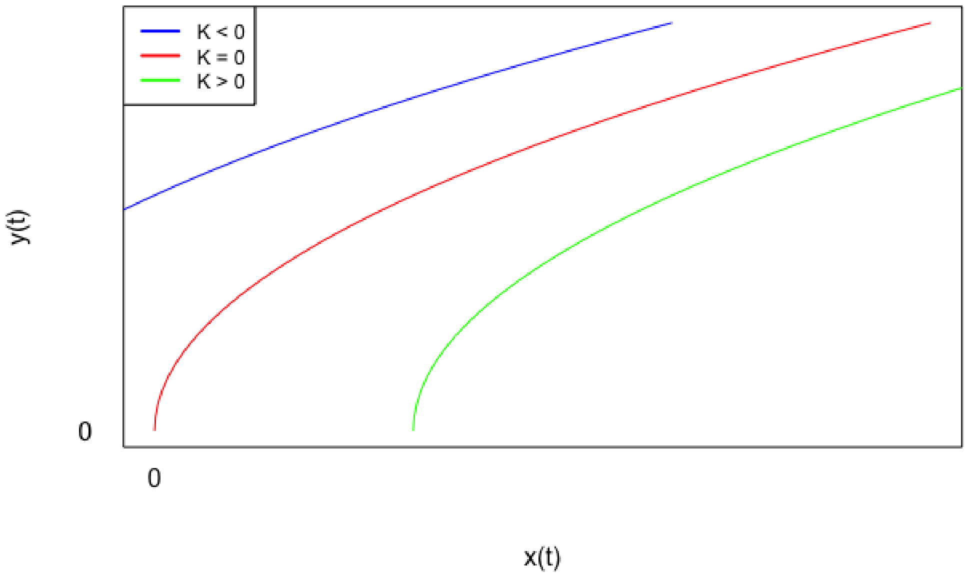
Integrate both sides of this equation, and let y_0, x_0 denote the initial value for $y(t), x(t)$.

$$\int_{y_0}^{y(t)} (ay)dy = \int_{x_0}^{x(t)} (b)dx$$
$$\frac{a}{2}(y(t)^2 - y_0^2) = b(x(t) - x_0)$$
$$bx(t) = \frac{a}{2}y(t)^2 - \frac{a}{2}y_0^2) + bx_0$$
$$x(t) = \frac{a}{2b}y(t)^2 + (x_0 - \frac{a}{2b}y_0^2)$$

The last equation shows the relation between $x(t)$ and $y(t)$ is a parabola equation. The intercept of this parabola on x and y axis is determined by the constant $(x_0 - \frac{a}{2b}y_0^2)$.

Since the derivatives of both $x(t)$ and $y(t)$ is negative, as t grows, $x(t)$ and $y(t)$ will go down to zero. In this model, the war ends until any of the side is eliminated. A stale means $y(t)$ and $x(t)$ arrive at zero simultaneous, i.e. no one left for both sides.

Let $K = (x_0 - \frac{a}{2b}y_0^2)$, and plot the $x(t)y(t)$ relationship.



Base on the graph, when $K < 0$, the solution ends on y-axis, indicating y wins; when $K > 0$, the solution line hits the x-axis, implying x wins. A stale will happen only if $x_0 - \frac{a}{2b}y_0^2 = 0$, since the solution goes into $y(t) = x(t) = 0$.

Note that $a = c_1\frac{A_g}{A_x}$ where $A_g = 2, A_x = 1000 \cdot x_0$; $b = c_2 p_x$ where $p_x = 0.1, c_1 = c_2$.

$$x_0 - \frac{a}{2b}y_0^2 = x_0 - \frac{c_1 A_g}{2A_x \cdot c_2 p_x}y_0^2 = x_0 - \frac{1}{100x_0}y_0^2$$

x_0 and y_0 denotes the initial force. When y_0 is ten times x_0, a stale will occur.

b

From the previous question, we know that when $x_0 - \frac{a}{2b}y_0^2 < 0$, the solution will ends up on y-axis as t is big enough. In this case, y wins since x loses all its force but y still has remaining power.

Plug in the value of a, b, c_1, c_2, we get

$$x_0 - \frac{a}{2b}y_0^2 < 0$$
$$x_0 < \frac{1}{100x_0}y)^2$$
$$x_0 < \frac{1}{10}y_0$$

When initial force of y is more than 10 times of x_0, y will win the battle.

The true initial values are $y_0^* \leq 6x_0^*$. Since y wins only if $y_0 \geq 10x_0$, US cannot prevail in the war.

Exercise 9.5

a

Given that $\frac{dN}{dt} = rN(1 - \frac{N}{K}) - \beta N$ and $N(t) = x(t) + y(t)$, we can replace $N(t)$ by $x(t), y(t)$.

$$\begin{aligned}\frac{dN}{dt} &= r(x+y)(1 - \frac{x+y}{K}) - \beta(x+y) \\ &= (x+y)(r - \frac{r(x+y)}{K}) - \beta(x+y)\end{aligned}$$

Note that $\frac{dN}{dt} = \frac{dy}{dt} + \frac{dx}{dt}$, therefore the only possible equation for $F(x,y)$ that satisfies $\frac{dx}{dt} = xF(x,y) + \beta x$ and $\frac{dy}{dt} = yF(x,y) + \beta y$ is $F(x,y) = r(1 - \frac{x+y}{K})$

b

The new model looks like

$$\begin{aligned}\frac{dx}{dt} &= x[r(1 - \frac{x+y}{K}) - \beta] \\ \frac{dy}{dt} &= y[r(1 - \frac{x+y}{K}) - \beta(1-\epsilon)]\end{aligned}$$

To find the equilibrium points, we want both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ to be zero.

There are four possible equilibrium solutions: $x = 0, y = 0$; $x = 0, r(1 - \frac{x+y}{K}) - \beta(1-\epsilon) = 0$; $r(1 - \frac{x+y}{K}) - \beta = 0, y = 0$ and $r(1 - \frac{x+y}{K}) - \beta = 0, r(1 - \frac{x+y}{K}) - \beta(1-\epsilon) = 0$

Equilibria 1 $(x^*, y^*) = (0, 0)$

when $x = 0, r(1 - \frac{x+y}{K}) - \beta(1-\epsilon) = 0$

$$\begin{aligned}r(1 - \frac{y}{K}) - \beta(1-\epsilon) &= 0 \\ y &= K - \frac{K}{r}\beta(1-\epsilon)\end{aligned}$$

Equilibria 2 $(x^*, y^*) = (0, K - \frac{K}{r}\beta(1-\epsilon))$

when $r(1-\frac{x+y}{K})-\beta=0, y=0$

$$r(1-\frac{x}{K})-\beta=0$$
$$x=K-\frac{K}{r}\beta$$

Equilibria 3 $(x^*,y^*)=(K-\frac{K}{r}\beta,0)$

when $r(1-\frac{x+y}{K})-\beta=0, r(1-\frac{x+y}{K})-\beta(1-\epsilon)=0$, there is not solution for x,y to satisfy this condition.

In summary, there are three equilibrium points for this model, they are $(0,0),(0,K-\frac{K}{r}\beta(1-\epsilon)),(K-\frac{K}{r}\beta,0)$

c

First get the matrix A for linear analysis.

- $a_{11}=\frac{\partial f}{\partial x}=r(1-\frac{x+y}{K})-\beta-\frac{r}{K}x$
- $a_{12}=\frac{\partial f}{\partial y}=-\frac{r}{K}x$
- $a_{21}=-\frac{r}{K}y$
- $a_{22}=r(1-\frac{x+y}{K})-\beta(1-\epsilon)-\frac{r}{K}y$

Equilibria 1 $(x^*,y^*)=(0,0)$

$a_{11}=r-\beta$, $a_{12}=0$ $a_{21}=0$, $a_{22}=r-\beta(1-\epsilon)$
$p=a_{11}+a_{22}=2r-\beta(2-\epsilon)$
$q=a_{11}a_{a22}-a_{12}a_{21}=(r-\beta)(r-\beta(1-\epsilon))$

Note that $\lambda_1=\frac{p}{2}+\frac{\sqrt{p^2-4q}}{2}$, $r>\beta>\beta(1-\epsilon)$, therefore $\lambda_1>\frac{p}{2}>0$. Point $(0,0)$ is unstable.

Equilibria 2 $(x^*,y^*)=(0,K-\frac{K}{r}\beta(1-\epsilon))$

$a_{11}=-\beta\epsilon$, $a_{12}=0$, $a_{21}=\beta(1-\epsilon)-r$, $a_{22}=\beta(1-\epsilon)-r$
$p=a_{11}+a_{22}=\beta-r-2\beta\epsilon$
$q=a_{11}a_{a22}-a_{12}a_{21}=r\beta\epsilon-\beta^2\epsilon+\beta^2\epsilon^2$

In this case, $p<0$ and $\sqrt{p^2-4q}<-p$. Therefore, both λ_1 and λ_2 are less than zero. The equilibrium point $(0,K-\frac{K}{r}\beta(1-\epsilon))$ is stable.

Equilibria 3 $(x^*, y^*) = (K - \frac{K}{r}\beta, 0)$

$a_{11} = \beta - r$, $a_{12} = \beta - r$, $a_{21} = 0$, $a_{22} = \beta\epsilon$
$p = a_{11} + a_{22} = \beta - r + \beta\epsilon$
$q = a_{11}a_{a22} - a_{12}a_{21} = (\beta - r)\beta\epsilon$

$\lambda_1 = \frac{1}{2}(p + \sqrt{p^2 - 4q}) = \frac{1}{2}(\beta - r + \epsilon + \beta - r - \epsilon) = \beta - r < 0$
$\lambda_2 = \frac{1}{2}(p - \sqrt{p^2 - 4q}) = \frac{1}{2}(\beta - r + \epsilon - \beta + r + \epsilon) = \epsilon > 0$

In this case, $\lambda_2 > 0$, thus the point $(K - \frac{K}{r}\beta, 0)$ is unstable.

d

From previous question, we can perceive that the only stable equilibrium point for this dynamic system is $x^* = 0$ and $y^* = K - \frac{K}{r}\beta(1 - \epsilon)$. As time goes on, x, representing population of Neanderthals, will finally go to zero, and the population of human y will converge to a stable point $K - \frac{K}{r}\beta(1 - \epsilon)$ which is determined by growth rate, mortality rate and carry capacity. Therefore, the extinction of Neanderthals is inevitable.

e

$$\begin{aligned}\frac{d}{dt}(\frac{x(t)}{y(t)}) &= \frac{1}{y}\frac{dx}{dt} - \frac{x}{y}\frac{1}{y}\frac{dy}{dt} \\ &= \frac{x}{y}(r(1 - \frac{x+y}{K}) - \beta) - \frac{x}{y}(r(1 - \frac{x+y}{K}) - \beta(1 - \epsilon)) \\ &= -\beta\epsilon(\frac{x(t)}{y(t)})\end{aligned}$$

Solution for this ODE is $\frac{x(t)}{y(t)} = (\frac{x(t_0)}{y(t_0)})e^{-\beta\epsilon t}$. Let A_0 denotes initial condition $\frac{x(t_0)}{y(t_0)}$, then the solution becomes $\frac{x(t)}{y(t)} = A_0 e^{-\epsilon\beta t}$.

Note that at $t = 10000$, $\frac{x(t)}{y(t)} = A_0/e$ and $\beta = \frac{1}{30}$, we can get equation:

$$\begin{aligned}A_0 e^{-\epsilon\beta t} &= A_0/e \\ e^{-\frac{10000}{30}\epsilon} &= e^{-1} \\ \epsilon &= \frac{30}{10000} = 0.003\end{aligned}$$

The mortality difference between humans and Neanderthals is 0.003.