AMATH 383 HW 3

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Exercise 6.6

a

Let $\frac{dp}{dt} = f(p) = 0$, then we get the equation

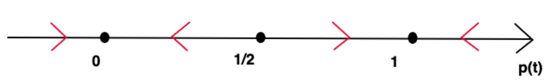
$$f(p) = \alpha p(1-p)(p-\frac{1}{2}) = 0$$

Since $\alpha \neq 0$, equilibrium points are $p = 0, p = \frac{1}{2}, p = 1$.

$$\begin{aligned}\frac{df(p)}{dp} &= \alpha(-(x-\frac{1}{2})x + (1-x)x + (1-x)(x-\frac{1}{2})) \\ &= \alpha(-\frac{6x^2-6x+1}{2})\end{aligned}$$

At $p = 0, f(p)' = -\frac{1}{2}\alpha$, at $p = \frac{1}{2}, f(p)' = \frac{1}{2}\alpha$, at $p = 1, f(p)' = -\frac{1}{2}\alpha$ Since in the past and in the present, right handed snail are always commoner than left handed, $p = 1$ where probability of right handed snail is approximately 100% should be a stable point. This indicates $f(p)'$ at $p = 1$ should have negative sign, i.e. $\alpha > 0$.

Thereby we get $p = 0, f(p)' < 0$, at $p = \frac{1}{2}, f(p)' > 0$, at $p = 1, f(p)' < 0$. $p = 0, 1$ are stable points, while $p = \frac{1}{2}$ is unstable point.



b

Note that $p = \frac{1}{2}$ is an unstable point. For any initial value of p that greater than $\frac{1}{2}$, as $t \to \infty$, $p(t)$ will asymptotically go to 1 (all right handed); for any initial value of p less than $\frac{1}{2}$, $p(t)$ will asymptotically go to 0 (all left handed). Therefore with initial condition $p(0) = \frac{1}{2}$, there is 50% chance all snails are left handed in hundred million years, and 50% chance all snails are right handed.

Exercise 6.8

Base on given information, the epidemic model should be

$$\begin{aligned}\frac{dI}{dt} &= \alpha I(t)N(t) \\ &= \alpha I(t)(1000 - I(t))\end{aligned}$$

At $I = 100$, $\frac{dI}{dt} = 90$,

$$\begin{aligned}90 &= \alpha \cdot 100(1000 - 100) \\ \alpha &= \frac{1}{1000} \\ \frac{dI}{dt} &= \frac{1}{1000} I(t)(1000 - I(t))\end{aligned}$$

This is a separable ODE, solve by separate integral, and partial fraction

$$\begin{aligned}\int \frac{1}{I(1000-I)} dI &= \int \frac{1}{1000} dt \\ \int \frac{1}{1000}(\frac{1}{I} + \frac{1}{1000-I}) dI &= \int \frac{1}{1000} dt \\ \int \frac{1}{I} + \frac{1}{1000-I} dI &= \int 1 dt \\ ln|I| - ln|1000 - I| &= t + c \\ \frac{I}{1000 - I} &= e^{t+c} \\ I &= \frac{1000}{1 + e^{-t-c}}\end{aligned}$$

Plug in initial condition $I(0) = 20$

$$\begin{aligned}20 &= \frac{1000}{1 + e^{-c}} \to e^{-c} = 49 \\ I(t) &= \frac{1000}{1 + 49e^{-t}}\end{aligned}$$

When $I = 900$

$$900 = \frac{1000}{1 + 49e^{-t}}$$
$$t = ln(49) + ln(9) \approx 6.09$$

After 6.09 days, 90% population will be infected.

Exercise 9.6

Plug in $P = pH - cE$ and $H = qEN$ into equations of $\frac{dE}{dt}$, $\frac{dN}{dt}$

$$g(E,N) = \frac{dE}{dt} = apgEn - acE$$
$$f(E,N) = \frac{dN}{dt} = rN(1 - \frac{N}{K}) - qEN$$

Equilibrium points are when both $g(E,N)$ and $f(E,N)$ are zero.

There are overall three equilibrium points $(E^*, N^*) = (0,0)$, $(E^*, N^*) = (0,K)$, $(E^*, N^*) = (\frac{r}{q}(1 - \frac{c}{Kpq}), \frac{c}{pq})$

$$a_{11} = \frac{\partial f}{\partial N} = r - \frac{2rN}{K} - qE$$
$$a_{12} = \frac{\partial f}{\partial E} = -qN$$
$$a_{21} = \frac{\partial g}{\partial N} = apqE$$
$$a_{22} = \frac{\partial g}{\partial E} = apqN - ac$$

Case 1: $(E^*, N^*) = (0,0)$

$a_{11} = r$, $a_{12} = 0$, $a_{21} = 0$, $a_{22} = -ac$

$p = a_{11} + a_{22} = r - ac$, $q = a_{11}a_{22} - a_{12}a_{21} = -acr$

$\lambda_1 = \frac{p}{2} + \frac{\sqrt{p^2-4q}}{2} = r$

$\lambda_2 = \frac{p}{2} - \frac{\sqrt{p^2-4q}}{2} = -ac$

since $r > 0$ therefore $\lambda_1 > 0$, it is an unstable point.

Case 2: $(E^*, N^*) = (0, K)$

$a_{11} = -r$, $a_{12} = -qK$, $a_{21} = 0$, $a_{22} = apqK - ac$

$p = a_{11} + a_{22} = -r + apqK - ac$, $q = a_{11}a_{22} - a_{12}a_{21} = -rapqK + rac$

$\lambda_1 = \frac{p}{2} + \frac{\sqrt{p^2-4q}}{2} = \frac{-r+apqK-ac}{2} + \frac{r+apqk-ac}{2} = apqK - ac$

Note that $\frac{c}{pqK} < 1 \to c < pqK$, therefore $\lambda_1 > 0$
$\lambda_2 = \frac{p}{2} - \frac{\sqrt{p^2-4q}}{2} = -r < 0$

Since $\lambda_1 > 0$, it is an unstable point.

Case 3: $(E^*, N^*) = (\frac{r}{q}(1 - \frac{c}{Kpq}), \frac{c}{pq})$

$a_{11} = -\frac{rc}{Kpq}$, $a_{12} = -\frac{c}{p}$, $a_{21} = apr - \frac{acr}{Kq}$, $a_{22} = 0$

$p = a_{11} + a_{22} = -\frac{rc}{Kpq}$, $q = a_{11}a_{22} - a_{12}a_{21} = acr - \frac{arc^2}{Kpq}$

Note that $q = acr(1 - \frac{c}{Kpq}) > 0$ since $\frac{c}{Kpq} < 1$. Then $p^2 - 4q < p^2 \to |\frac{p}{2}| > |\frac{\sqrt{p^2-4q}}{2}|$.

$\lambda_1 = \frac{p}{2} + \frac{\sqrt{p^2-4q}}{2} < 0$ since $p < 0$ and $|\frac{p}{2}| > |\frac{\sqrt{p^2-4q}}{2}|$

$\lambda_2 = \frac{p}{2} - \frac{\sqrt{p^2-4q}}{2} < 0$ since $p < 0, -\frac{\sqrt{p^2-4q}}{2} < 0$

Both λ_1 and $\lambda_2 < 0$, this point $(E^*, N^*) = (\frac{r}{q}(1 - \frac{c}{Kpq}), \frac{c}{pq})$ is a stable point.