7

Discrete Time Logistic Map, Periodic and Chaotic Solutions

Mathematics introduced:

overshoot instability; nonlinear map; linear and nonlinear stability; periodic and aperiodic solutions; deterministic chaos; sensitivity to initial conditions

7.1 Introduction

For population dynamics governed by a continuous first-order differential equation of the form studied in the previous chapter,

$$\frac{d}{dt}N = F(N),$$

the behavior of the solution is understood qualitatively by looking at the equilibria (also called the *fixed points*) of the equation. The long-term behavior of the solution is given by a monotonic approach to the stable equilibrium. There are no periodic oscillations and there is no erratic meandering. Initial conditions are unimportant in this picture.

Yet a richer time-dependent behavior is often seen in animal populations, including apparently random fluctuations in the size of their colonies. How should the simple models be modified so as to yield a richer, and more realistic, set of possible behaviors?

There are two approaches. One is to regard the fluctuations as being forced by external factors, such as by population immigration or environmental changes that affect the carrying capacity of the population. For example, if we are studying the logistic equation as a simple model for the population, i.e.,

$$\frac{d}{dt}N = rN\left(1 - \frac{N}{K}\right),$$

we may want to specify some fluctuations in the carrying capacity $K(t)$, or we may add an "immigration" term, $I(t)$, to the right-hand side of the above equation. If these externally specified terms have random fluctuations, so will the population $N(t)$ they induce.

Another approach is to inquire if the more interesting time behavior, including the seemingly random cases, can arise from a simple deterministic equation like this without needing outside influence. Robert May's (1974) paper showed that simple population dynamics can produce intrinsically interesting periodic or erratic solutions that are, on the surface, indistinguishable from those induced by external factors. There is an account of the discovery's impact on thinking in the field of theoretical ecology in the 1970s in the book *It Must Be Beautiful*, edited by Graham Farmelo.

Logistic Growth for Nonoverlapping Generations

May (1974) pointed out that while in some biological populations (such as humans) growth is a continuous process, generations overlap, and the appropriate mathematical description involves nonlinear differential equations, in other biological situations (such as 13-year periodical cicadas), population growth takes place at discrete time intervals and generations do not overlap. In these latter cases, nonlinear *difference* equations should be the more appropriate mathematical description.

Robert May considered the discrete version of the logistic equation

$$\frac{N((n+1)\Delta t) - N(n\Delta t)}{\Delta t} = r\,N(n\Delta t)\left[1 - \frac{N(n\Delta t)}{K}\right]. \tag{7.1}$$

It is more appropriate for populations in which births occur in well-defined breeding seasons, and the generations (Δt) are nonoverlapping, than the continuous version:

$$\frac{dN}{dt} = r\,N\left(1 - \frac{N}{K}\right), \tag{7.2}$$

which assumes that births are occurring continuously.

Equilibrium solutions are located at $N_1^* = 0$, $N_2^* = K$ for both discrete and continuous versions. (See Figure 7.1.)

For the continuous system, the stability of each of these two equilibria can be inferred easily from a plot of dN/dt vs. N. As can be seen in the left panel of Figure 7.1, the equilibrium $N^* = 0$ is unstable, while $N^* = K$ is stable, judging by the arrows, which show the direction of change of N. On the right panel of Figure 7.1, we make a similar plot of the rate of change of N in one generation: $(N(t+\Delta t) - N(t))/\Delta t$, vs. N. Again, the direction of the arrows indicates the direction of change of N. We see that the equilibrium $N^* = 0$ is unstable. The equilibrium $N^* = K$

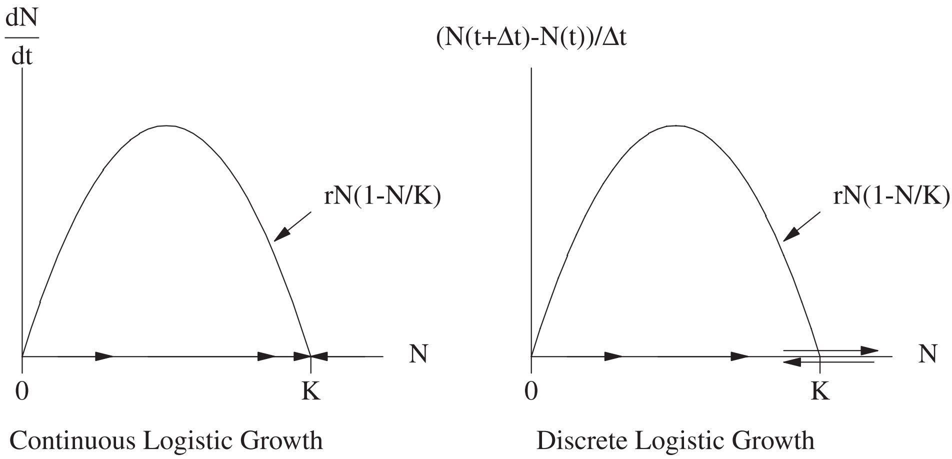

Figure 7.1. Continuous logistic growth (left) vs. discrete logistic growth (right). For the discrete version, there is the possibility that the change in population $N(t)$ in one generation (Δt) can be so large that it overshoots the equilibrium point (K), leading to an oscillatory instability.

is usually stable (as arrows point toward it), but an interesting phenomenon of overshoot may happen for large growths in one generation ($r\,\Delta t > 2$; see below). That is, if the growth rate is high, and the length of a generation is long, the change of the population from one generation to the next may be so large that N "overshoots" the equilibrium, with an ensuing oscillatory instability as a result of repeated back and forth overshoot. This instability is new and is not present in the continuous system. This new instability usually gives rise to periodic solutions, with the period depending on intrinsic parameters in the model without any external influence. The behavior of a system without even a single stable equilibrium present is more complicated but interesting.

7.2 Discrete Map

The difference equation (7.1) can be rewritten in the form of a mapping:

$$\boxed{N_{n+1} = f(N_n),} \tag{7.3}$$

where

$$f(N_n) = N_n + r\,\Delta t N_n\left(1 - \frac{N_n}{K}\right) \tag{7.4}$$

and $N_n \equiv N(n\Delta t)$.

The solution can be obtained by applying the mapping successively. That is, starting with a given N_0, Eq. (7.3) gives us N_1. Knowing N_1, it then yields N_2, etc.

In this formulation, the equilibrium N^* is determined from solving

$$N^* = f(N^*). \tag{7.5}$$

This equilibrium is stable to small perturbations if

$$|f'(N^*)| < 1. \tag{7.6}$$

This is because if we write

$$N_n = N^* + u_n$$

and assume u_n is small, Eq. (7.3) becomes approximately

$$u_{n+1} = f'(N^*)u_n.$$

Thus $f'(N^*)$ is the "amplifying factor" from u_n to u_{n+1}. For stability its magnitude should be less than 1.

There are two equilibria, satisfying Eq. (7.5):

$$N^* = 0 \text{ and } N^* = K.$$

Since

$$f'(N^*) = 1 + r\Delta t - 2\frac{r\Delta t}{K}N^*,$$

$$f'(0) = 1 + r\Delta t, \quad f'(K) = 1 - r\Delta t,$$

the equilibrium $N^* = 0$ is unstable. The equilibrium $N^* = K$ is usually stable, unless $r\Delta t > 2$. The three different behaviors of N_{n+1} vs. N_n near $N^* = K$ are shown in Figure 7.2 for different cases of $f'(K)$. For $0 > f'(K) > -1$, the solution is asymptotically stable as each iteration brings the solution closer and closer to the equilibrium. (From another perspective, the perturbation u_n from the equilibrium becomes smaller and smaller as n increases.) For $f'(K) = -1$, the solution approaches a distance from N^*. From there the perturbation neither grows nor decays. Instead the perturbation executes a periodic oscillation around the equilibrium, changing sign after each iteration but repeating after two iterations (i.e., $u_{n+2} = f'(K)u_{n+1} = f'(K)f'(K)u_n = u_n$). This is

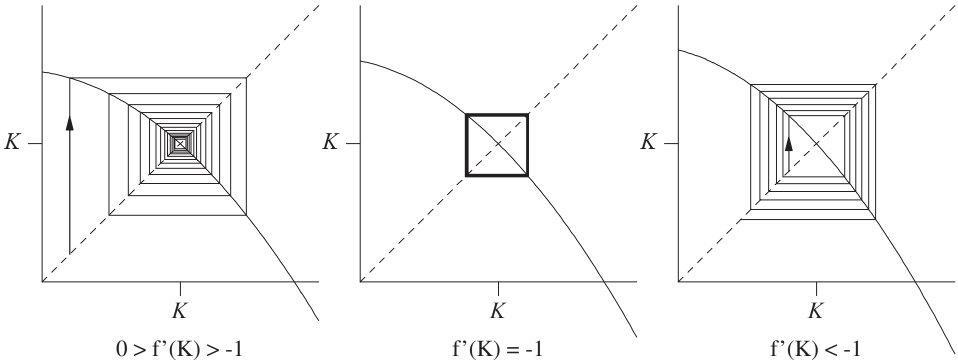

Figure 7.2. Solution of the discrete logistic equation near the equilibrium point. The horizontal axis is N_n and the vertical axis is N_{n+1}. The curve is $f(N_n) = N_n + r\,N_n(1 - N_n/K)$. The dashed line is $N_{n+1} = N_n$. Three cases of linear stability are shown: asymptotically stable, borderline stable (periodic oscillatory), and unstable.

called a 2-cycle; it is periodic with period $2\Delta t$. For $f'(K) < -1$, small perturbations grow. The solution is therefore linearly unstable. Nonlinearly it is not necessarily unstable, as we shall see.

7.3 Nonlinear Solution

We next solve the discrete logistic equation graphically for various values of the parameter $r\,\Delta t$. We construct a plot (see Figure 7.3) whose horizontal axis is N_n and whose vertical axis is N_{n+1}. On this plot we first draw $f(N_n)$ as a function of N_n. The graph is in the form of a parabola. Next we draw a straight line $N_{n+1} = N_n$ (a 45° line from the origin). The intersection of this straight line with the parabola then yields the equilibrium solution N^*.

The time-dependent solution to Eq. (7.1) can also be constructed from this plot. Given an N_0, we locate its value on the horizontal axis. We move up vertically until we hit the parabola, $f(N_0)$. This is then the value N_1 (from Eq. (7.3)), which we read off the vertical axis. Next we want to locate this N_1 on the horizontal axis; this is facilitated by the 45° line. That is, we use N_1 as the starting point on the vertical axis, we move horizontally until we hit the 45° line, and then we move down vertically until we hit the horizontal axis. This is the desired location for N_1.

We repeat this process to find N_2, N_3, etc. The resulting graph looks like a cobweb, and thus it is called a cobweb map. The graphical process we outlined above is called "cobwebbing."

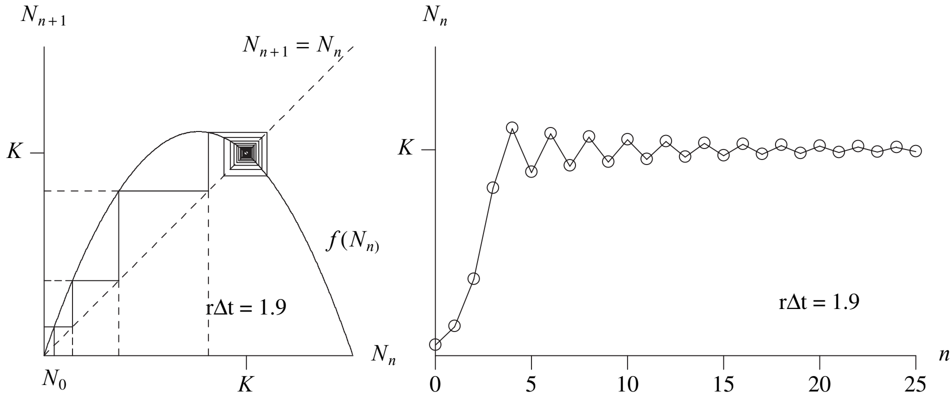

Figure 7.3. Graphical construction of the solution of the discrete logistic equation for the linearly stable case of $r\,\Delta t = 1.9$ (left panel). The curve is $f(N_n) = N_n + r\,N_n(1 - N_n/K)$. The diagonal dashed line is $N_{n+1} = N_n$. The right panel shows the solution $N(t)$ as a function of discrete $t = n\Delta t$. (Modified from original figure in Kot [2001], by permission.)

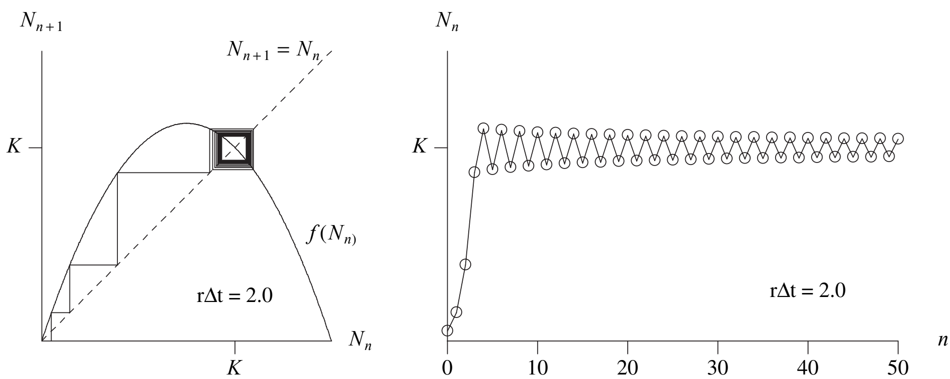

Figure 7.4. Same as Figure 7.3 except for the borderline case of $r\,\Delta t = 2.0$. Nonlinearly the solution remains the same as the linear solution: a 2-cycle periodic solution about (and close to) the equilibrium.

Figure 7.3 shows the cobwebbing construction and the solution to Eq. (7.3) for the stable case ($r\,\Delta t = 1.9$). The solution approaches asymptotically to the equilibrium $N^* \equiv K$. So the nonlinear stability of this equilibrium is the same as the linear determination.

Figure 7.4 is for the case of $r\,\Delta t = 2.0$. The nonlinear evolution is similar to the linear one. The solution is a 2-cycle.

Figure 7.5 is for the case of $r\,\Delta t = 2.2$. For this case, small perturbations from $N^* = K$ grow and therefore the equilibrium is linearly

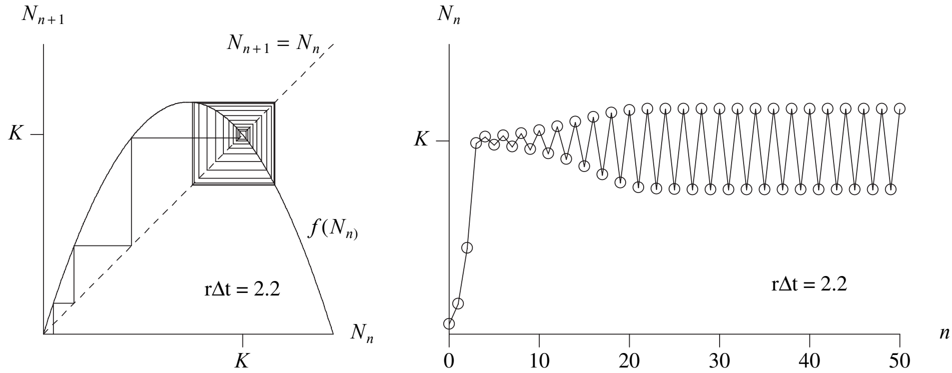

Figure 7.5. Same as Figure 7.3 except for the linearly unstable case of $r\,\Delta t = 2.2$. Nonlinearly the solution increasingly deviates from the equilibrium point until it settles down to a nonlinear 2-cycle periodic solution. (Modified from original figure in Kot [2001], by permission.)

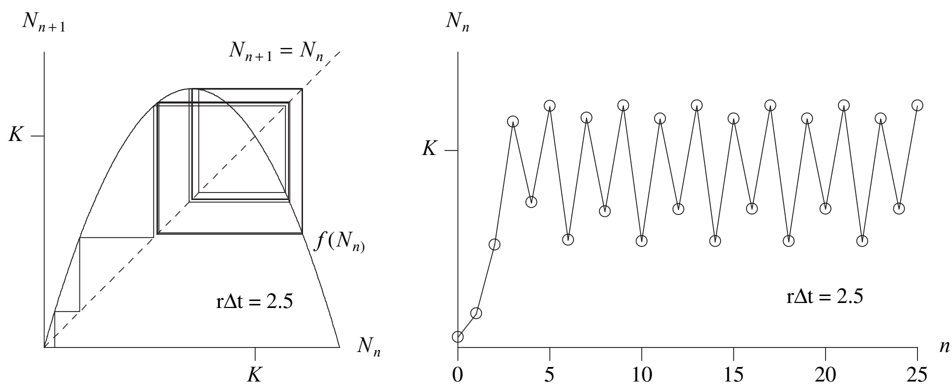

Figure 7.6. Same as Figure 7.3 except for $r\,\Delta t = 2.5$. The solution is a 4-cycle. (Modified from original figure in Kot [2001], by permission.)

unstable. However, our exact solution (by this graphical method) shows that the solution is nonlinearly periodic, with period $2\Delta t$. That is, the solution after a while repeats itself every two iterations.

Figure 7.6 is for the case of $r\,\Delta t = 2.5$. A 4-cycle periodic solution is found.

An 8-cycle solution is found for $r\,\Delta t = 2.55$ (see Figure 7.7).

There are more and more period doublings as $r\,\Delta t$ increases until at or above $r\,\Delta t = 2.5699456\ldots$ a period 3 appears. Then all periods are present. The solution does not approach N^* but is nevertheless

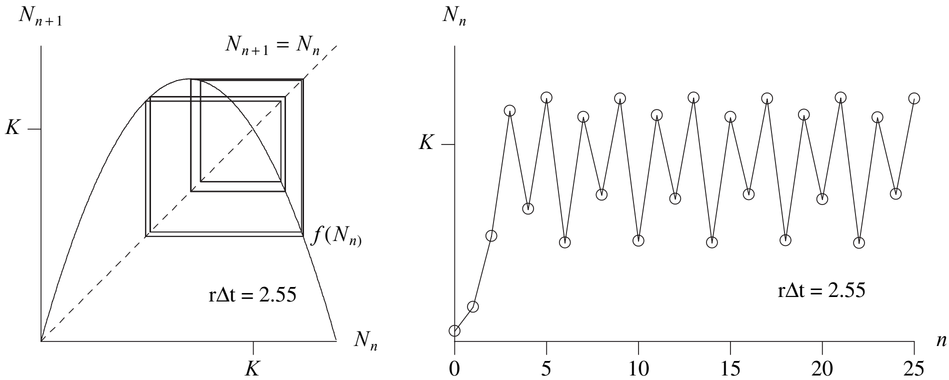

Figure 7.7. Same as Figure 7.3 except for $r\,\Delta t = 2.55$. The solution is an 8-cycle. (Modified from original figure in Kot [2001], by permission.)

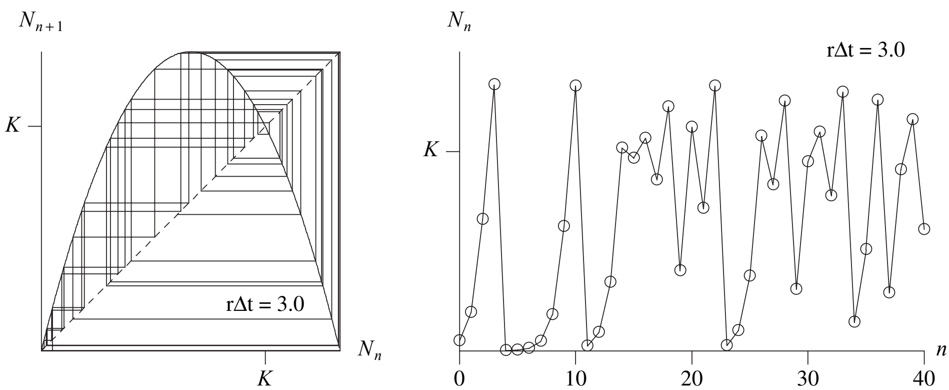

Figure 7.8. Same as Figure 7.3 except for $r\,\Delta t = 3.0$. The solution is aperiodic and bounded (chaos). (Modified from original figure in Kot [2001], by permission.)

bounded. It never repeats itself, and so is called *aperiodic*. Professor James Yorke at the University of Maryland coined the term "chaos" to describe this behavior in his paper with his student Tien-Yien Li entitled "Period Three Implies Chaos" (Yorke and Li, 1975).

The behavior for $r\,\Delta t = 3.0$ in Figure 7.8 is typical of this aperiodic behavior.

7.4 Sensitivity to Initial Conditions

A *chaotic* time series is one that is aperiodic. In addition, a chaotic solution is extremely sensitive to the initial condition.

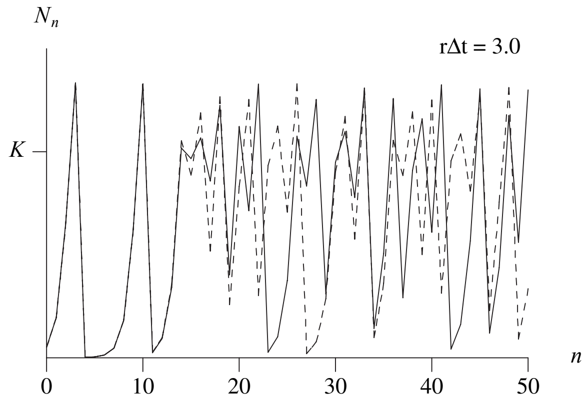

Figure 7.9. The dashed curve is the solution under the same condition as the solid curve, except with a slightly different initial condition. (Modified from original figure in Kot [2001], by permission.)

In Figure 7.9, the solution from Figure 7.8 is shown again in the solid line. The dashed line shows the solution starting with a slightly different initial condition ($N_0/K = 0.050001$ instead of 0.05). They are indistinguishable from each other for about 15 iterations. Then they diverge from each other dramatically. Loss of predictability is typical of chaotic systems.

7.5 Order Out of Chaos

Suppose we are given a time series of a variable (e.g., the left panel of Figure 7.10) and are told that it is all right to assume that there is only one degree of freedom. The behavior of this time series appears chaotic. We want to know if there is some underlying order to it. In particular, we want to find the equation (the map) that governs its evolution.

Looking back at earlier pages of this lecture we see that for $N_{n+1} = f(N_n)$, the functional form of f can be deduced by plotting the pairs (N_0, N_1), (N_1, N_2), (N_2, N_3), $\ldots$, etc. in an N_n, N_{n+1} plot. These points should lie on a more regular shape, a parabola in our previous example (see the right panel of Figure 7.10).

So, by plotting this way, we are discovering the form of $f(N)$ (the mapping) and hence the original deterministic equation that governs the apparently disorganized data. That is, we are uncovering the underlying order. Of course this will not work so easily if the chaotic time series is the result of a higher dimensional mapping.

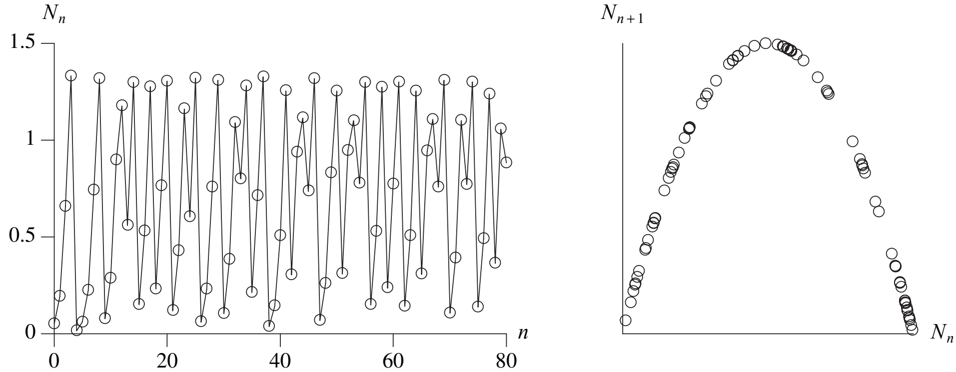

Figure 7.10. The left panel is a time series, which in this case appears random. The right panel is constructed by plotting the pair (N_n, N_{n+1}) from the data in the left panel.

7.6 Chaos Is Not Random

Before chaos was understood as an aperiodic solution to deterministic equations, with the property of hypersensitivity to initial conditions, chaotic behavior like that depicted in the left panel of Figure 7.10 was often thought of as random. In fact, in the late 1940s, the famous Hungarian-American mathematician and father of game theory, John von Neumann, even suggested using the logistic equation with $r\,\Delta t = 3$ as a random number generator (actually, $x_{n+1} = 4x_n(1 - x_n)$). Had the casinos adopted it, we would have been able to figure out, using the right panel of Figure 7.10, what the next outcome of a slot machine would be given its current state!

7.7 Exercises

1. Periodic solutions and their loss of stability

The logistic difference equation

$$(N_{n+1} - N_n)/\Delta t = r \cdot N_n \cdot (1 - N_n/K)$$

possesses chaotic solutions for $r\,\Delta t > 2.569946\ldots$. The above equation can be made dimensionless and rewritten in the simpler form:

$$X_{n+1} = f(X_n), \tag{7.7}$$

where

$$f(x) = ax(1 - x), \quad a \equiv 1 + r\Delta t, \quad a > 1$$

and

$$X_n \equiv N_n \cdot r\Delta t/((1 + r\Delta t)K).$$

We want to investigate the equilibrium solutions to this equation and their loss of stability as a increases.

a. Find the equilibria of Eq. (7.7) and determine their stability.

b. For $a > 3$, the solution to Eq. (7.7) includes a 2-cycle. That is, it obeys, along with the equilibria you just found in (a), this new equation:

$$X_{n+2} = X_n = X_2.$$

We know

$$X_{n+2} = f(X_{n+1}) = f(f(X_n)) \equiv g(X_n). \tag{7.8}$$

Find $g(x)$, and then solve for the equilibrium solution of Eq. (7.8), which satisfies $X^* = g(X^*)$. You should recover the equilibria in (a), plus two new ones if $a > 3$. These two new equilibria are not real for $a < 3$. In trying to solve for the equilibria you may need to solve a quartic equation, but you can take advantage of the fact that you already know two of the four solutions. After you factor these two factors out you are left with a quadratic equation to solve. To save you some algebra, it is factored for you here:

$$X^*[aX^* - (a - 1)][a^2X^{*2} - a(a + 1)X^* + (a + 1)] = 0.$$

c. What do you think is the nature of the two new equilibrium solutions you have found in (b)? (That is, are they true steady state solutions? You don't need to know their value to decide this. Just use the fact that they are the equilibria of (7.8) but not (7.7).)

d. Describe how you would go about determining the stability of the equilibria found in (b) (but do not actually do the stability calculation for each equilibrium; it is quite messy). Although you are not asked to do this, you may be interested to know that the 2-cycle solutions lose their stability when $a > 1 + \sqrt{6}$. Then 4-cycle solutions emerge, which can be obtained from

$X_{n+4} = f(X_{n+3}) = f(f(f(f(X_n))))$. The process can be repeated for n-cycle solutions as a increases further.

2. *Beverton–Holt model*

A goal of fishery scientists has been to determine the relationship between the size of a fish population and the number of offspring from it that survive to enter the fishery. Fishery scientists refer to a population of a fish as a stock, and the process of entering the fishery as recruitment. The relationship, called the stock-recruitment relationship, is often obtained by empirical fitting of the data to an analytic function. The Beverton–Holt stock recruitment is governed by the difference equation

$$N_{n+1} = \frac{RN_n}{1 + [(R-1)/K]N_n}, \quad R > 1.$$

a. Find the equilibria and determine their stability.

b. Find an exact, closed form solution. (*Hint*: Use the substitution $X_n = 1/N_n$. The equation for X_n turns out to be linear.) This is one of the rare cases where a nonlinear difference equation can be solved exactly.

3. *Ricker model*

Consider the following density-dependent difference equation for population density N_n at time n:

$$N_{n+1} = N_n e^{r[1-(N_n/K)]} \equiv f(N_n),$$

where $r > 0$ is the growth rate and $K > 0$ is the carrying capacity. This model is preferred by some over the discrete logistic model because N_t is always positive if N_0 is positive.

a. Find the equilibria N^* and discuss their linear stability as a function of r. Show that there is a change in stability at $r = 2$ ($r = 2$ is therefore called a bifurcation point).

b. Find the 2-cycle equilibria for $r > 2$ from

$$N_{n+2} = f(f(N_n)) \equiv g(N_n)$$

(i.e., find N^* such that $N^* = f(f(N^*))$). How would you interpret the nature of the two new "equilibria" you find here (which are not the equilibrium of $N_{n+1} = f(N_n)$)?

c. Show that, for large n, the maximum N_n is $f(K/r)$ and the minimum N_n is $f(f(K/r))$.

d. Use cobwebbing or another method to show that the solution becomes chaotic (bounded and aperiodic), for large r.