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Age of the Earth: Lord Kelvin's Model

Mathematics required:
solution of first-order ordinary differential equation by separation of variables; multivariate calculus; partial derivatives

Mathematics introduced:
the method of similarity solution in solving certain partial differential equations

13.1 Introduction

This subject is full of controversy, some of it scientific and some of it religious. Our interest here is not to settle the controversy or to take sides in the debate. Rather, we are interested in framing a mathematical problem in the context of the debate and seeing how modeling can be used to add quantitative information. The discovery of radioactivity and the development of radiometric dating methods are quite recent (in the early and mid 20th century, respectively). Imagine if you were born earlier, say in the 18th century, or that you don't believe in the validity of the current radiometric dating methods, as some people still don't. How do you arrive at an estimate of the age of the earth? This was the task facing William Thomson, Lord Kelvin (1824–1907) (Figure 13.1).

The prevailing views have swung pendulum-like between different extremes in the past few centuries. The current *scientific* view is that the earth is about 4.5 billion years old. Radiometric dating of rocks on earth found ancient rocks of about 3.5 billion years old, with the oldest rock found so far, in Northwestern Canada near Great Slave Lake, dating to 3.96 billion years. In Western Australia, geologists found a zircon crystal 4.4 billion years old trapped inside a rock dated 3.1 billion years old. (An article in the September 2001 issue of *National Geographic* has an interesting description of the zircon crystals, formed when magma cools, trapping a few uranium atoms in their lattices. The uranium atoms are protected from outside contaminants for billions of years: "Zircons are God's gift to geochemistry.") Moon rocks returned to Earth by the six Apollo and three Luna missions have been dated at 4.4–4.5 billion years. The meteorites have been dated at 4.58 billion years.

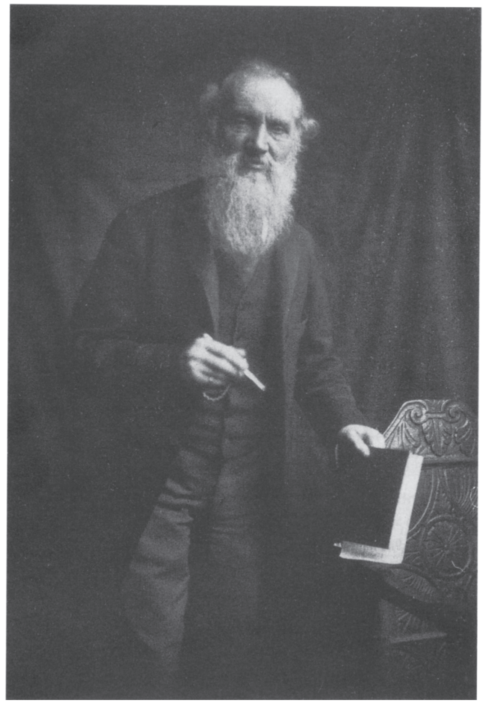

Figure 13.1. Lord Kelvin (William Thomson) (1824–1907).

Even today, Young Earth advocates, such as ICR (the Institute for Creation Research), believe in a literal reading of the *Genesis* account. Each day was a 24-hour day comparable to a modern day, and plants and animals were created in a mature functional state directly by God. By examining the various genealogies found in the Scriptures, it was estimated that Creation must have taken place sometime between six and fifteen thousand years ago. The most famous biblical chronology is due to Archbishop James Ussher, Primate of All Ireland, Vice Chancellor of Trinity College in Dublin, who in 1650 determined the date of Creation to be Sunday, October 23, 4004 BC, precisely! Modern creationists are willing to acknowledge the possibility of gaps in the genealogies, pushing this date back some. Even within the creationist camp there is currently a debate between the "Young Earth" creationists and the "Old Earth" creationists. The Old Earth creationists do not take the length of a day literally as 24 hours ("With the Lord a day is like a thousand years, and a thousand years are like a day," 2 Peter 3:8). However, as pointed out by the other camp, that Genesis intended a day to be 24 hours is made clear from the phrase following the description of each day: "There was evening, and there was morning." To make it even more explicit, the terms used are defined in *Genesis* 1:5, where "God called the light 'day' and the darkness He called 'night'." It is also pointed out by some in the Young Earth camp that the creation events are not in the expected order if long periods of time were involved: plants were created on day three, while the sun was not created until day four, and animals needed for the pollination of plants were not available until day five.

In Lord Kelvin's time (1850s) the debate was between the "Doctrine of Uniformity" and others such as Kelvin's. In *Proceedings of the Royal Society of Edinburgh* Lord Kelvin wrote:

> The Doctrine of Uniformity in Geology, as held by many of the most eminent of British geologists, assumes that the earth's surface and upper crust have been nearly as they are at present in temperature and other physical qualities during millions of millions of years. But the heat which we know, by observation, to be now conducted out of the earth yearly is so great, that if *this* action had been going on with any approach to uniformity for 20,000 million years, the amount of heat lost out of the earth would have been about as much as would heat, by 100° cent., a quantity of ordinary surface rock of 100 times the earth's bulk. This would be more than enough to melt a mass of surface rock equal to the bulk of the *whole earth*. (Thomson, 1866, pp. 512–513)

13.2 The Heat Conduction Problem

So, if not for 20,000 million years, how long? The problem from the perspective of heat conduction was solved by Thomson. At the age of 16 in 1840, at the University of Glasgow, Thomson read Fourier's *The Analytical Theory of Heat*. He later said: "I took *Fourier* out of the University Library, and in a fortnight I had mastered it—gone right through it." The subject of heat became a lifelong interest. The Kelvin temperature scale was later named after his work on absolute temperature in 1848.

In 1864, Thomson produced an estimate of the age of the earth by considering it as a warm, chemically inert sphere cooling through its surface. Thomson knew at the time that the temperature of the earth is hotter within and that the rocks were molten. Assuming that the hotter temperature was the temperature of the earth at an earlier time, he attempted to deduce the age of the earth. The mathematical formulation is as follows.

Let $t = 0$ be the time (the "beginning") when the earth's surface first solidified. The temperature of the earth at the time, $u_0(y)$, is taken to be the melting temperature of rock (which can be measured). y is the depth from the surface. He also assumed that the temperature at the surface of the earth for $t > 0$ has been more or less constant, i.e., $u(0, t) = u_s, t > 0$, and so u_s can be measured at present. This temperature is maintained against heat lost to space by heat conducted from below the surface. The equation of heat conduction (discovered by Joseph Fourier, 1800) was known to Thomson:

$$\boxed{\frac{\partial}{\partial t} u = \alpha^2 \frac{\partial^2}{\partial y^2} u, \quad y > 0, \quad t > 0,} \tag{13.1}$$

where $u(y, t)$ is the temperature at depth y and time t. α^2 is the conductivity of the earth and can be determined by measuring samples of surface rock. We have:

$$\begin{aligned} \text{partial differential equation (PDE):} &\quad \frac{\partial}{\partial t} u = \alpha^2 \frac{\partial^2}{\partial y^2} u, \ y > 0, \ t > 0, \\ \text{boundary condition (BC):} &\quad u(0, t) = u_s, \ t > 0, \\ \text{initial condition (IC):} &\quad u(y, 0) = u_0, \ y > 0. \end{aligned}$$

We can make the boundary condition homogeneous by letting

$$\psi(y, t) \equiv u(y, t) - u_s.$$

So in terms of ψ, the problem is:

$$\begin{aligned} \text{PDE}: &\quad \frac{\partial}{\partial t} \psi = \alpha^2 \frac{\partial^2}{\partial y^2} \psi, \ 0 < y < \infty, \ t > 0, \\ \text{BC}: &\quad \psi(0, t) = 0, \ t > 0, \\ \text{IC}: &\quad \psi(y, 0) = \psi_1 \equiv u_0 - u_s, \ \ y > 0. \end{aligned} \tag{13.2}$$

Lord Kelvin actually treated the earth as a sphere. We have simplified the problem to one dimension since we are interested only in the variation with depth. We will also artificially extend the domain to $0 < y < \infty$ for mathematical convenience. This does not cause any problem since the influence of the surface decays rapidly with depth. The fact that the mantle and the core of the earth are not solid also does not matter for the same reason.

Solution

In this problem, y measures depth and therefore has the dimension of length (in either meters or feet), t is time (in units of seconds), and the coefficient of conductivity α^2 is typically $0.012\ \text{cm}^2/\text{s}$. ψ is a temperature in either degrees Celsius or degrees Fahrenheit.

To solve this problem, we assume that $\psi(y,t)$ depends on y and t in the following combination:

$$\psi(y,t) = F\left(\frac{y}{\sqrt{\alpha^2 t}}\right). \tag{13.3}$$

This is a consequence of the fact that $y/\sqrt{\alpha^2 t}$ is a dimensionless quantity. That is, all units cancel in this combination (check this!). It turns out that it is the only possible combination of quantities in the problem that will make either y or t dimensionless. Since all mathematical functions must involve a dimensionless argument (e.g., $\sin(\omega t)$, $\exp(rt)$), it then follows that $\psi(y,t)$ must depend on y and t in the combination assumed. Had there been a spatial scale L and a time scale T, the solution could have been written as $F(y/L, t/T)$, and we would not have made much progress. In our case we let:

$$z \equiv \frac{y}{\sqrt{\alpha^2 t}};$$

then

$$\psi(y,t) = F(z).$$

This assumption is called a *similarity assumption* and works sometimes for problems without a typical length and a typical time scale. Assuming this is true, then ($F'(z) \equiv \frac{d}{dz}F(z)$)

$$\frac{\partial}{\partial t}\psi = \frac{\partial z}{\partial t}\cdot\frac{d}{dz}F(z) = \frac{-\frac{1}{2}y}{\sqrt{\alpha^2 t}}\frac{1}{t}F'(z) = -\frac{1}{2}z\frac{1}{t}F'(z),$$

$$\frac{\partial}{\partial y}\psi = \frac{\partial z}{\partial y}\frac{d}{dz}F(z) = \frac{1}{\sqrt{\alpha^2 t}}F'(z),$$

$$\frac{\partial^2}{\partial y^2}\psi = \frac{1}{\alpha^2 t}F''(z), \quad \alpha^2\frac{\partial^2}{\partial y^2}\psi = \frac{1}{t}F''(z).$$

Thus

$$\frac{\partial}{\partial t}\psi = \alpha^2\frac{\partial^2}{\partial y^2}\psi$$

becomes

$$\frac{1}{t}F''(z) = -\frac{1}{2}z\frac{1}{t}F'(z),$$

which is

$$F''(z) + \frac{1}{2}zF'(z) = 0.$$

This is actually a first-order ordinary differential equation for $W(z) \equiv F'(z)$:

$$\frac{d}{dz}W + \frac{z}{2}W = 0.$$

Separation of variables yields

$$\frac{dW}{W} = -\frac{z}{2}dz.$$

So

$$W(z) = W(0)e^{-z^2/4}$$

and

$$F(z) = \int_0^z W(z)dz = W(0)\int_0^z e^{-z^2/4}dz = 2W(0)\int_0^{\frac{z}{2}} e^{-z'^2}dz'.$$

We have used the fact $F(0) = 0$ because $z = \frac{y}{\sqrt{\alpha^2 t}}$, and the boundary condition $\psi(0,t) = 0$. The initial condition $\psi(y,0) = \psi_1$ implies that $F(\infty) = \psi_1$ since $z = \frac{y}{\sqrt{\alpha^2 t}} \to \infty$ for $t \to 0^+$, $y > 0$. Using the integral identity,

$$\int_{-\infty}^{\infty} e^{-z'^2}dz' = 2\int_0^{\infty} e^{-z'^2}dz' = \sqrt{\pi}, \quad \text{so} \int_0^{\infty} e^{-z'^2}dz' = \frac{\sqrt{\pi}}{2}.$$

The condition

$$F(\infty) = \psi_1$$

then leads to

$$2W(0)\int_0^{\infty} e^{-z'^2}dz' = \psi_1.$$

So $W(0) = \psi_1/\sqrt{\pi}$. Finally

$$\psi(y,t) = \frac{2\psi_1}{\sqrt{\pi}} \int_0^{\frac{y}{2\sqrt{\alpha^2 t}}} e^{-z'^2} dz',$$

$$u(y,t) = u_s + \frac{2(u_0 - u_s)}{\sqrt{\pi}} \int_0^{\frac{y}{2\sqrt{\alpha^2 t}}} e^{-z'^2} dz'.$$

The temperature gradient is obtained as

$$\frac{\partial u}{\partial y}(y,t) = \frac{(u_0 - u_s)}{\sqrt{\pi\alpha^2 t}} e^{-\frac{y^2}{4\alpha^2 t}}. \tag{13.4}$$

The temperature gradient, $\frac{\partial}{\partial y}u(y,t)$, can be measured near the surface (but not so close to the surface as to be affected by the seasonal change of weather).

Let

$$\Delta \equiv \frac{\partial}{\partial y}u(y,t), \quad y \sim 0;$$

then

$$\boxed{\Delta = \frac{(u_0 - u_s)}{\sqrt{\pi\alpha^2 t}}.} \tag{13.5}$$

Solve for t, the age of the Earth:

$$\boxed{t = (u_0 - u_s)^2/(\pi\alpha^2\Delta^2).} \tag{13.6}$$

This result can be anticipated from "back of envelope" dimensional arguments.

The temperature gradient is approximately equal to the temperature difference $(u_0 - u_s)$, divided by the length scale over which the difference occurs. There is no length scale in the problem other than the combination $X \equiv \sqrt{\alpha^2 t}$, which has the dimension of length. So

$$\Delta \sim \frac{(u_0 - u_s)}{\sqrt{\alpha^2 t}}.$$

Comparing this with the exact solution we see that we have missed only by a factor $\frac{1}{\sqrt{\pi}}$—not much for this problem.

13.3 Numbers

> When you cannot measure it, when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind.
>
> — *Lord Kelvin*

Estimate

$$t = \frac{1}{\pi\alpha^2}\left(\frac{u_0 - u_s}{\Delta}\right)^2.$$

Kelvin knew at the time that the temperature increases by about "1°F every 50 ft downward." The conductivity of the rock (Edinburgh rocks vs. Greenwich rocks) is about 0.012 cm^2/s. Kelvin gave wild guesses of the temperature to melt rocks: 10,000°F ("not realistic") and 7,000°F ("closer to the truth").

With $u_0 \sim 7{,}000°F$, Kelvin got $t = 98$ million years, with a possible range of 20 to 400 million years:

$$\begin{aligned} t &\sim \frac{1}{\pi 0.012\,\mathrm{cm}^2/\mathrm{s}}\left(\frac{7{,}000°\mathrm{F}}{1°\mathrm{F}/50\,\mathrm{ft}}\right)^2 \\ &\sim \frac{350{,}000^2\,\mathrm{ft}^2}{\pi 0.012\,\mathrm{cm}^2}\cdot \mathrm{s} \\ &= 98 \text{ million years.} \end{aligned}$$

Kelvin also estimated the age of the sun from the solar constant, which yields the rate at which the sun is losing energy:

$$S = 1 \times 10^6\ \mathrm{cal\,cm^{-2}\,year^{-1}},$$

which gives the total output of the sun, $\sim 3 \times 10^{33}$ cal/year. Kelvin was unaware of the radioactive heat source. He assumed that the sun produced its energy from the gravitational potential of matter falling into the sun, including meteorites and even planets (and later, the matter that composed the mass of the sun itself), and published a value of 50 million years for the age of the sun in 1853.

Thus there seemed to be some consistency in Kelvin's estimates of the ages of the earth and the sun. But both are too short for Darwin's theory of evolution. Kelvin became an opponent of Darwin's theory.

The discovery of radioactivity at the turn of the 20th century offered a solution to the problems of both the Earth and the sun. Lord Rayleigh (see Strutt 1906) worked on the radioactivity in igneous rocks and proposed a shallow distribution of heat sources that neatly removed the thermal conductivity problem that had led Kelvin to his erroneous conclusion.

13.4 Exercises

1. Fourier's wine cellar

Joseph Fourier (1768–1830) was an expert on heat conduction. To find the perfect depth to build his wine cellar, he solved the following problem on subsoil temperature variations:

$$\frac{\partial}{\partial t}u = \alpha^2 \frac{\partial^2}{\partial y^2}u, \quad 0 < y < \infty,$$

subject to the boundary conditions:

$$u(y, t) \text{ bounded as } y \to \infty,$$

$$u(0, t) = u_0 e^{i\omega_0 t}.$$

Here ω_0 is the frequency of the temperature variation at the surface, $\omega_0 = 2\pi/(1\,\text{day})$ for daily variations and $\omega_0 = 2\pi/(1\text{ year})$ for seasonal variations. u is the temperature of the soil. y is the depth underground, positive downward, with $y = 0$ being the surface. For soil, the thermal conductivity is $\alpha^2 \cong 0.01\,\text{cm}^2/\text{s}$. We use the complex exponential instead of $\cos \omega_0 t$ because it makes the algebra easier. One can always take the real part of the solution afterwards.

a. Solve the problem (i.e., find $u(y, t)$) by assuming that your solution can be written in the form

$$u(y, t) = \phi(y) e^{i\omega_0 t}.$$

b. For daily variations, $u_0 = 5°\text{C}$ and $\omega_0 = 2\pi/(1\text{ day})$. Find the depth below which the magnitude of temperature variation is less than $2°\text{C}$.

c. Do the same for seasonal variations, with $u_0 = 15°\text{C}$ and $\omega_0 = 2\pi/(1 \text{ year})$.

d. An ideal depth to locate a wine cellar is where the temperature variation is not only small but perfectly out of phase with the seasonal oscillation at the surface. At this depth it would be winter when it is summer at the surface. Find this depth. ($\sqrt{-1} = i$, $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$, $e^{i\theta} = \cos\theta + i\sin\theta$, real part of $e^{i\theta} = \cos\theta$.)

2. Preventing nuclear meltdown

When the temperature in the core of a nuclear reactor reaches its critical value u_c, a meltdown occurs as the temperature from nuclear fission increases rapidly. There is very little that can be done once this happens. To protect against meltdowns, supercooled cooling rods are activated at the first sign of trouble. These rods attempt to maintain the temperatures at the two ends of a reactor (at $x = -L$ and $x = L$) at a fixed temperature u_0, which is below u_c. A one-dimensional problem for the temperature $u(x, t)$ of the nuclear reactor is given by

$$\text{PDE:} \quad \frac{\partial}{\partial t}u = \alpha^2 \frac{\partial^2}{\partial x^2}u + ae^u,$$

$$\text{BC:} \quad u = u_0 \text{ at } x = -L \quad \text{and} \quad x = L, \quad t > 0.$$

a. Find the steady state solution. (*Hint*: Set $\frac{\partial}{\partial t}u = 0$. Solve $\alpha^2 \frac{d^2}{dx^2}u + ae^u = 0$ by multiplying it by $\frac{d}{dx}u$ and integrating it in x.)
Express the constant of integration in terms of the maximum temperature $u_{\max}$. (*Hint*: $u = u_{\max}$ where $\frac{d}{dx}u = 0$.)

b. Show that this maximum temperature occurs at the center ($x = 0$) of the reactor.

c. We want to keep the maximum temperature of the reactor safely below the critical temperature, i.e.,

$$u_{\max} = u_c - b,$$

by a given margin b. What should the temperature of the cooling rods be?

d. Suppose we are too late in inserting the cooling rods and we fail to maintain the maximum temperature below the critical value. Temperature increases rapidly in time, and heat conduction is ineffective.

Dropping the heat conduction term, we have

$$\frac{\partial}{\partial t}u = ae^{u}.$$

Solve this equation (which is an ordinary differential equation in time) using separation of variables for ordinary differential equations. Assume u at $t = 0$ to be u_c.