Exercise 4.1

a

$y(t)$ is the amount of ^{210}Pb at time t, λ is the decay rate of ^{210}Pb.

While in the ore, the amount of ^{210}Pb as r that replenished from ^{226}Ra decaying is the only source of increment. The only decrement for ^{210}Pb is decaying, which depends on current amount of ^{210}Pb. Therefore, the change in ^{210}Pb can be expressed as $\frac{dy}{dt} = -\lambda y + r$.

After manufacture, element ^{226}Ra are removed, from which ^{210}Pb replenishes its amount. Hence, the amount of ^{210}Pb no longer gets increment. Meanwhile, the decaying process of ^{210}Pb continues, which has same equation as before. Therefore, the change in ^{210}Pb can be written as $\frac{dy}{dt} = -\lambda y$

b

While in the ore, ^{210}Pb and ^{226}Ra are in a radioactive equilibrium , implying $\frac{dy}{dt} = 0$, or $-\lambda y + r = 0$. Let t_0 denotes the last minute before manufacture. Then, at the time t_0, equilibrium exists but no longer holds, i.e. $\lambda y(t_0) = r$

c

Solve for $\frac{dy}{dt} = -\lambda y(t) \Rightarrow$ general aolution: $y(t) = y(t_0)e^{-\lambda(t-t_0)}$

Plug in the initial condition: $y(t_0) = \frac{r}{\lambda} \Rightarrow y(t) = \frac{r}{\lambda}e^{-\lambda(t-t_0)}$

d

From previous step, we know that solution for $\frac{dy}{dt} = -\lambda y(t)$ is $y(t) = \frac{r}{\lambda}e^{-\lambda(t-t_0)}$

The rate of change $\frac{dy}{dt} = -\lambda\frac{r}{\lambda}e^{-\lambda(t-t_0)} = -re^{-\lambda(t-t_0)}$

Note that decay rate of ^{210}Pb λ is $\frac{ln2}{22}$, $r = 0 \to 200$ and $\frac{dy}{dt} = -8.5$

Then we can get the equation $re^{-\lambda(t-t_0)} = 8.5 \Rightarrow (0 \to 200)e^{-\frac{ln2}{22}(t-t_0)} = 8.5$

Since $(t - t_0)$ is positively correlated with value of r, we choose the maximum value of r to find the maximum value of $(t - t_0)$.

Solve for $200e^{-\frac{ln2}{22}(t-t_0)} = 8.5 \Rightarrow (t - t_0) \approx 100.24$

The maximum age of this painting is 100 years.

Exercise 4.4

Plug $k = 0$ into the equation for change of infected cells $\frac{dT^*}{dt} = kVT - \delta T$, we get $\frac{dT^*}{dt} = -\delta T^*$

Solve for this equation, we get equation for number of infected cells $T^*(t) = Ce^{-\delta t}$

Since at time $t = 0$, T^* is approximate constant as T_0^*. Plug this initial condition into T^*t, we get $T^*(t) = T_0^* e^{-\delta t}$

Plug equation of $T^*(t)$ into the equation for production of virus, we get $P(t) = N\delta T_0^* e^{-\delta t}$

Then the equation for change of virus becomes $\frac{dV}{dt} = N\delta T_0^* e^{-\delta t} - cV$ where T_0^*, N, δ are constant

$$\begin{aligned}
\text{let } u(t) &= e^{\int c dt} = e^{ct} \\
\int q(t)u(t)dt &= \int N\delta T_0^* e^{-\delta t} e^{ct} dt \\
&= N\delta T_0^* \int e^{(c-\delta)t} dt = \frac{N\delta T_0^*}{c-\delta}(e^{(c-\delta)t}) \\
V(t) &= (\int q(t)u(t)dt + C)/u(t) \\
&= e^{-ct}(\frac{N\delta T_0^*}{c-\delta} e^{c-\delta t} + C), \text{ where C is a constant} \\
&= \frac{N\delta T_0^*}{c-\delta} e^{-\delta t} + Ce^{-ct}
\end{aligned}$$

Let V_0 denotes amount of virus at $t = 0$ and assume that $\frac{dV}{dt} = 0$ at the beginning.

Then $P(0) - cV(0) = 0 \Rightarrow N\delta T_0^* = cV(0)$

Plug the initial condition that $V(t = 0) = V_0$ and $N\delta T_0^* = cV(0)$ into the general solution.

$$\begin{aligned}
V(t = 0) &= \frac{N\delta T_0^*}{c-\delta} e^0 + Ce^0 \text{where C is a constant} \\
V_0 &= \frac{cV_0}{c-\delta} + C \\
C &= -\frac{\delta V_0}{c-\delta}
\end{aligned}$$

The solution for the equation of virion is $V(t) = \frac{V_0}{c-\delta}(ce^{-\delta t} - \delta e^{-ct})$

Exercise 4.5

a

Solve for $\frac{dV_I}{dt} = -cV_I$, we get $V_I(t) = Ce^{-ct}$, where C is a constant.

Plug the initial condition $V_I(t=0) = V_0$ into general solution, we get $V_I(t) = V_0 e^{-ct}$

Substitute $V_I(t)$ into equation $\frac{dT^*}{dt} = kV_I T - \delta T^*$ and assuming $T = T_0$ is a constant.

$$\begin{aligned}
\frac{dT^*}{dt} &= kV_0T_0e^{-ct} - \delta T^* \\
u(t) &= e^{\int \delta dt} = e^{\delta t} \\
\int q(t)u(t)dt &= \int kT_0V_0e^{-ct} * e^{\delta t}dt \\
&= kT_0V_0 \int e^{\delta - c}dt = \frac{kT_0V_0}{\delta - c}e^{(\delta - c)t} \\
T^*(t) &= \frac{\int q(t)u(t)dt + C}{u(t)}, \text{ where C is a constant} \\
&= \frac{kT_0V_0}{\delta - c}e^{-ct} + Ce^{-\delta t}
\end{aligned}$$

Let T_0^* denotes $T^*(t=0)$. Since $\frac{dT^*}{dt} = 0$ at $t = 0$, $kV_0T_0 = \delta T_0^* \Rightarrow T_0^* = \frac{kV_0T_0}{\delta}$

Plug initial condition into general solution

$$\begin{aligned}
T_0^* &= \frac{kT_0V_0}{\delta - c} + C \\
\frac{kV_0T_0}{\delta} &= \frac{kT_0V_0}{\delta - c} + C \\
C &= -\frac{(kV_0T_0)c}{\delta(\delta - c)}
\end{aligned}$$

Therefore the particular solution for $T^*(t)$ can be written as

$$T^*(t) = kV_0T_0\frac{\delta e^{-ct} - ce^{-\delta t}}{\delta(\delta - c)} = kV_0T_0\frac{ce^{-\delta t} - \delta e^{-ct}}{\delta(c - \delta)}$$

b

Substitute $T^*(t)$ into equation of $\frac{dV_{NI}}{dt}$, we get

$$\begin{aligned}
\frac{dV_{NI}}{dt} &= \frac{N\delta kV_0T_0}{\delta(c - \delta)}(ce^{-\delta t} - \delta e^{-ct}) - cV_{NI} \\
&= \frac{NkV_0T_0}{c - \delta}(ce^{-\delta t} - \delta e^{-ct}) - cV_{NI}
\end{aligned}$$

Solve for this ODE

$$
\begin{aligned}
u(t) &= e^{ct} \\
\int q(t)u(t)dt &= \int \frac{NkV_0T_0}{c-\delta}(ce^{-\delta t} - \delta e^{-ct})e^{ct}dt \\
&= \frac{NkV_0T_0}{c-\delta}\int (ce^{(c-\delta)t} - \delta e^{(c-c)t}) \\
&= \frac{NkV_0T_0}{c-\delta}(\frac{c}{c-\delta}e^{(c-\delta)t} - \delta t) \\
V_{NI}(t) &= (\int q(t)u(t)dt + C)/u(t), \text{ where C is a constant} \\
&= (\frac{NkV_0T_0}{c-\delta}(\frac{c}{c-\delta}e^{(c-\delta)t} - \delta t) + C)/e^{ct} \\
&= \frac{NkV_0T_0}{c-\delta}(\frac{c}{c-\delta}e^{-\delta t} - \delta te^{-ct} + Ce^{-ct})
\end{aligned}
$$

At $t = 0$, the amount of non-infectious virions is zero, i.e. $V_{NI}(t = 0) = 0$. Plug the initial condition into general solution

$$
\begin{aligned}
\frac{NkV_0T_0}{c-\delta}(\frac{c}{c-\delta}e^{-\delta t_0} - \delta t_0 e^{-ct_0} + Ce^{-ct_0}) = 0 \\
C = -\frac{NkV_0T_0}{c-\delta}\frac{c}{c-\delta}
\end{aligned}
$$

Therefore, the solution can be written as $V_{NI}(t) = \frac{NkV_0T_0}{c-\delta}[\frac{c}{c-\delta}(e^{-\delta t} - e^{-ct}) - \delta te^{-ct}]$

c

Population of virion is consist of non-infectious and infectious ones.

At time t, the amount of non-infectious virion is $V_{NI}(t) = \frac{NkV_0T_0}{c-\delta}[\frac{c}{c-\delta}(e^{-\delta t} - e^{-ct}) - \delta te^{-ct}]$

while the amount of infectious virion is $V_I(t) = V_0e^{-ct}$

$V(t) = V_{NI}(t) + V_I(t) = \frac{NkV_0T_0}{c-\delta}[\frac{c}{c-\delta}(e^{-\delta t} - e^{-ct}) - \delta te^{-ct}] + V_0e^{-ct}$