Simple Linear Regression II

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We will look at hypothesis tests for the estimate of the slope. Hypothesis tests for the estimated intercept can be constructed analogously.

Example: Hypothesis testing

Example: Snowfall data `ftcollinssnow` from R package `alr4`.

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Data: The amounts of snowfall (in inches) for 93 years in Ft. Collins.

Let y denote the amount of late season snowfall, and x denote the amount of early season snowfall.
Given the regression model (and given that ϵ_i iid $\mathcal{N}(0, \sigma^2)$, $i = 1, \ldots, n$):

$$y = \beta_0 + \beta_1 x + \epsilon$$

the test of interest is:

$$H_0 : \beta_1 = \beta_1^* \text{ (and } H_A : \beta_1 \neq \beta_1^* \text{)}.$$

Example: Hypothesis testing

Assume ϵ_i iid $\mathcal{N}(0,\sigma^2)$, $i=1,\ldots,n$. For testing the null hypothesis:

$$H_0 : \beta_1 = \beta_1^* \text{ (and } H_A : \beta_1 \neq \beta_1^* \text{)}.$$

we can compute the t-statistic as

$$T = \frac{\hat{\beta}_1 - \beta_1^*}{SE(\hat{\beta}_1 | X = x)}$$

where β_1^* is the value from the null hypothesis.

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The t-statistic follows Student's t_{n-2} distribution under the null hypothesis:

$$T | X = x \sim t_{n-2}.$$

Example: Hypothesis testing

We make a test decision based on the p-value. Recall: a *p*-value is

"The probability, under the null hypothesis, of obtaining a result as or more extreme than the observed result."

If t is the observed value of our test statistic, then the p-value of the test is calculated as

$$P(|T| \geq |t| \, \big| H_0), T \sim t_{n-2}.$$

Example: Hypothesis testing

Let $\beta_1^* = 0$. Given the estimated slope and its standard error for Ft. Collins snowfall data over 93 years

$$\hat{\beta}_1 = 0.2035,\ SE(\hat{\beta}_1 | X = x) = 0.1310,$$

calculate the test statistic for testing $H_0 : \beta_1 = 0$.

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T? What distribution does the test statistic follow?

Under what assumptions? Do you reject H_0?

Example: Hypothesis testing

Since $\beta_1^* = 0$, and since the data was collected over 93 years (93 samples), we calculate the observed value of the test statistic as follows (denoted with lowercase t):

$$t = \frac{0.20335 - 0}{0.1310} = 1.553.$$

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Given that the two-sided p-value is:

$$P(|T| \geq |t| \,\big|\, H_0) = P(|T| \geq 1.553) = 0.124,$$

is there evidence against the null hypothesis that the early and late season snowfalls are independent?

Analysis of variance

An alternative way to address the hypothesis

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Model	RSS
$y_i = \beta_0 + \epsilon_i$	$\sum_{i=1}^{n}(y_i - \hat{\beta}_0)^2 = \sum_{i=1}^{n}(y_i - \overline{y})^2 = SYY$
$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$	$SYY - \frac{SXY^2}{SXX} = SYY - SS_{reg}$

Analysis of variance

Formally, the hypothesis test for comparing the two models is:

$$H_0 : \mathrm{E}[Y|X = x] = \beta_0,$$
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ANOVA Table

$RSS = SYY - SS_{reg}$

Source	df	SS	MS	F	p-value
Regression	1	SS_{reg}	$SS_{reg}/1$	$MS_{reg}/\hat{\sigma}^2$	
Residual	n-2	RSS	$\hat{\sigma}^2 = \frac{RSS}{n-2}$		
Total	n-1	SYY			

The mean square column is obtained by dividing the sum of squares (SS) by its corresponding degrees of freedom (df).

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$$SXX = 10954.069,$$

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Total		17572.408			

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Source	df	SS	MS	F	p-value
Regression	1	453.5759	453.5759	2.4111	0.1239
Residual	91	17188.83	118.1190		
Total	92	17572.408			

$$SS_{reg} = \frac{SXY^2}{SXX} = \frac{2229.014^2}{10954.069} = 453.5759$$

$$RSS = SYY - SS_{reg} = 17188.83$$

$$\frac{RSS}{91} = 188.1190 = \hat{\sigma}^2$$

$$F = \frac{453.5759}{188.1190} = 2.4111, P(F^* \geq 2.4111) = 0.1239, \text{ where } F^* \sim F_{1,91}.$$

Example: Analysis of variance

What do we conclude for testing the hypothesis

$$H_0 : \mathrm{E}[Y|X = x] = \beta_0,$$
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Is there evidence against the null?

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Note: the p-value for the F-statistic in this example is the same as the p-value for the t-statistic testing $H_0 : \beta_1 = 0$ ($H_A : \beta_1 \neq 0$) in the earlier example with the Ft. Collins snowfall data.

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$$F = \frac{SS_{reg}}{\hat{\sigma}^2} = \frac{SXY^2}{\hat{\sigma}^2 SXX} = \frac{\hat{\beta}_1^2}{SE(\hat{\beta}_1|X = x)^2} = T^2.$$

$T^2 = F$

$\hat{\beta}_1 = \frac{SXY}{SXX}$

$SE(\hat{\beta}_1 | X = x) = \frac{\hat{\sigma}}{\sqrt{SXX}}$

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Note on reporting p-values: It is better to report a p-value and let the reader decide whether the result is significant,rather than to simply report significance at some pre-determined level.

Recall: Confidence Intervals

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$$\hat{\beta}_0 | X = x \sim \mathcal{N}(\beta_0, \sigma^2(\frac{1}{n} + \frac{\overline{x}^2}{SXX})),$$

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given that $\epsilon_i | X = x$ iid $\mathcal{N}(0, \sigma^2)$, $i = 1, \ldots, n$. Then

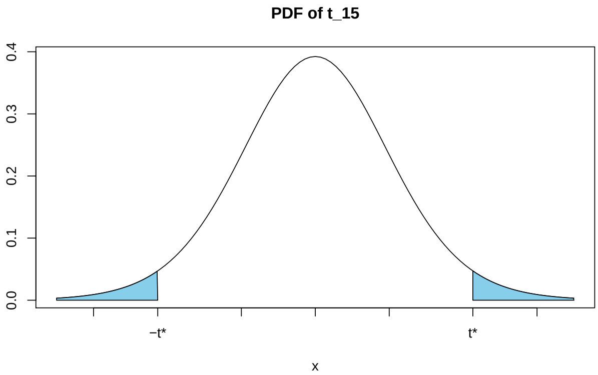
$$\frac{\hat{\beta}_0 - \beta_0}{\sigma^2(\frac{1}{n} + \frac{\overline{x}^2}{SXX})} | X = x \sim \mathcal{N}(0, 1).$$

Since σ^2 is usually not known and is instead estimated as $\hat{\sigma}^2$,

$$\frac{\hat{\beta}_0 - \beta_0}{\hat{\sigma}^2(\frac{1}{n} + \frac{\overline{x}^2}{SXX})} | X = x \sim t_{n-2}.$$

The t-distribution with $n-2$ degrees of freedom is the appropriate reference distribution for constructing the confidence intervals for $\hat{\beta}_0$ and $\hat{\beta}_1$.

$n = 17$ and we are interested in a 90% CI for β_0



$P(\frac{\hat{\beta}_0 - \beta_0}{\sqrt{\hat{\sigma}^2/SXX}} \leq |t^*| \Big| X = x) = 0.9$, so $t^* = t_{0.95,15}$. Then

$$P(-t^* \leq \frac{\hat{\beta}_0 - \beta_0}{SE(\hat{\beta}_0 | X = x)} \leq t^*) = 0.9.$$

Confidence Interval for $\hat{\beta}_0$

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a 90% confidence interval for $\hat{\beta}_0$ when $n = 17$ is:

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Estimate ± $(1-\alpha/2)$-quantile of the prob. dist. × SE of estimate.

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The interpretation of confidence intervals is based on repeated sampling. If samples of size n are drawn repeatedly and, say, 95% confidence intervals are estimated for the intercept, then 95% of those intervals (on average) would contain the true parameter β_0.

Duality: Confidence intervals and hypothesis testing

A $(1-\alpha)\times 100\%$ confidence interval for $\hat{\beta}_0$ is the set of points β_0^* such that

$$\hat{\beta}_0 - t_{1-\alpha/2,n-2}\cdot SE(\hat{\beta}_0) \leq \beta_0^* \leq \hat{\beta}_0 + t_{1-\alpha/2,n-2}\cdot SE(\hat{\beta}_0),$$

Any such β_0^* represents the null hypothesis that would not be rejected at the $100\times\alpha\%$:

$$H_0 : \beta_0 = \beta_0^* \text{ (and } H_A : \beta_0 \neq \beta_0^*).$$

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When σ^2 is estimated, we can construct a confidence region for $(\hat{\beta}_0, \hat{\beta}_1)$ using the Scheffé method. The reference distribution will be $F_{2,n-2}$. We will not discuss the details now.

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In R, we can use functions `confint(.)` and `confidenceEllipse(.)` to obtain the confidence intervals and regions.

Example: Snow geese

Example: Consider the regression of photographic count on observer's estimate (snow geese example). Obtain 95% confidence region for the slope and intercept estimates.

```
>summary(lm(photo~obs))
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   2.1712     3.9266   0.553    0.588
obs           0.9957     0.1380   7.214 2.07e-06
---
Residual standard error: 5.804 on 16 degrees of freedom
Multiple R-squared:  0.7648,Adjusted R-squared:  0.7501
F-statistic: 52.04 on 1 and 16 DF,  p-value: 2.066e-06

> qt(0.975,16)
[1] 2.119905
```

Example: Snow geese

The estimates of $(\hat{\beta}_0, \hat{\beta}_1)$ and their standard errors are

$$\hat{\beta}_0 = 2.1712, SE(\hat{\beta}_0|X = x) = 3.9266, t_{0.975,16} = 2.12$$
$$\hat{\beta}_1 = 0.9957, SE(\hat{\beta}_1|X = x) = 0.1380, t_{0.975,16} = 2.12$$

Let's construct:

- the 95% confidence interval for β_0,
- the 95% confidence interval for β_1,
- the 95% joint **confidence region** for (β_0, β_1).

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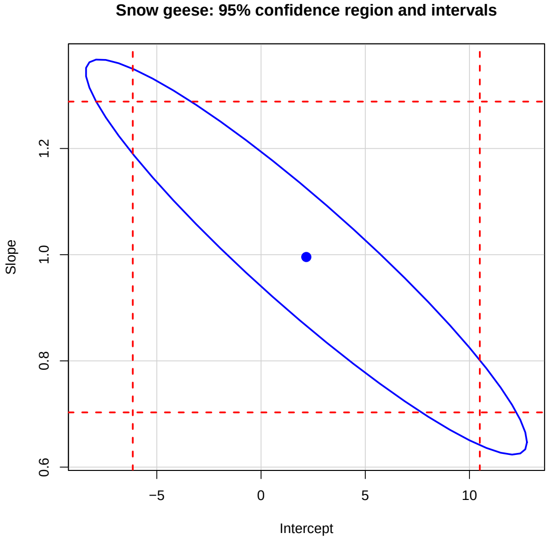

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Can you write down this null hypothesis formally?

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$$H_0 : (\beta_0, \beta_1) = (0, 1) \text{ versus } H_A : (\beta_0, \beta_1) \neq (0, 1)$$

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Let's plot the value of the null.

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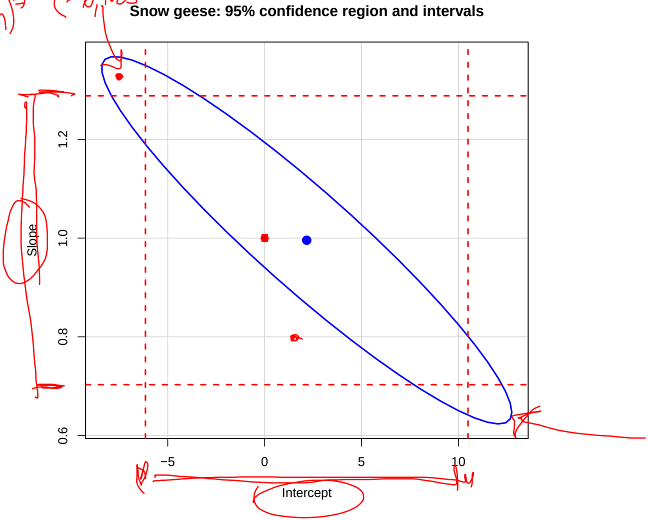
$(\beta_0, \beta_1) = (-6, 1.25)$



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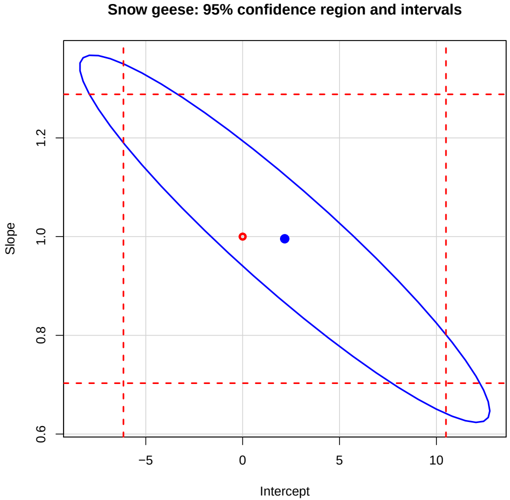

Can we reject the null hypothesis?

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Hence, we cannot reject the null hypothesis that observer's count is exact.

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This difference affects how we construct *fitted value confidence intervals* and *prediction confidence intervals*. (prediction intervals).

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The **fitted values** are values on the regression line

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The least squares fitted value is an **unbiased** estimate of the mean.

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The least squares fitted value is an **unbiased** estimate of the mean. The **variance** for the fitted least squares estimate is:

$$\begin{aligned}
\mathrm{Var}[\hat{y}_*|X = x_*] &= \mathrm{Var}[\hat{\beta}_0 + \hat{\beta}_1 x_*|X = x_*] \\
&= \mathrm{Var}[\hat{\beta}_0|X = x_*] + x_*^2 \mathrm{Var}[\hat{\beta}_1|X = x_*] + 2x_* \mathrm{Cov}[\hat{\beta}_0, \hat{\beta}_1|X = x_*] \\
&= \sigma^2\left(\frac{1}{n} + \frac{\overline{x}^2}{SXX}\right) + \sigma^2 x_*^2 \frac{1}{SXX} - 2\sigma^2 x_* \frac{\overline{x}}{SXX} = \sigma^2\left(\frac{1}{n} + \frac{(x_* - \overline{x})^2}{SXX}\right).
\end{aligned}$$

Fitted values

Then

$$SE(\hat{y}_* | X = x_*) = \hat{\sigma}\left(\frac{1}{n} + \frac{(x_* - \overline{x})^2}{SXX}\right)^{1/2}$$

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Note that the confidence interval for the fitted value is wider the further away we are from $\overline{x}$.

Predicted values

Since the true value of y_* according to our model is

$$y_* = \underbrace{\beta_0 + \beta_1 x_*} + \epsilon_*,$$

As before:

$$\mathrm{E}[y_* - \hat{y}_* | X = x_*] = 0.$$

Predicted values

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$\rightarrow \mathrm{Var}[\hat{y}_* | X = x_*]$

$$\mathrm{E}[y_* - \hat{y}_* | X = x_*] = 0.$$

What about $\mathrm{Var}[y_* - \hat{y}_* | X = x_*]$? How far away is our predicted (fitted) value from the actual value y_*? Using the formula for the variance of the sum of two uncorrelated variables, we obtain:

$$\mathrm{Var}[\hat{y}_* + \epsilon_*] = \mathrm{Var}[y_* - \hat{y}_* | X = x_*] = \mathrm{Var}[\beta_0 + \beta_1 x_* + \epsilon_* - \hat{y}_* | X = x_*]$$

$$= \mathrm{Var}[\epsilon_* | X = x_*] + \mathrm{Var}[\hat{y}_* | X = x_*] = \sigma^2 + \sigma^2 \left(\frac{1}{n} + \frac{(x_* - \overline{x})^2}{SXX} \right)$$

$$= \sigma^2 \left(1 + \frac{1}{n} + \frac{(x_* - \overline{x})^2}{SXX} \right).$$

Compare: Uncertainty in fitted and predicted values

Since $\mathrm{Var}[y_* - \hat{y}_* | X = x_*] = \sigma^2(1 + \frac{1}{n} + \frac{(x_* - \overline{x})^2}{SXX})$ The standard error is:

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The prediction interval for y_* is always wider than the confidence interval for $\hat{y}_*$.

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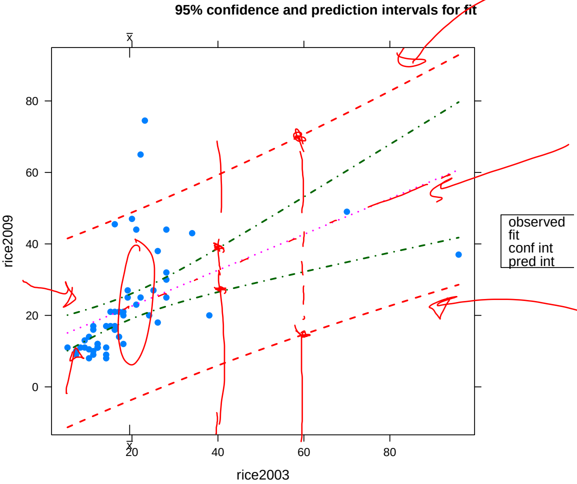

~~Example: snowgeese data.~~

R^2: the Coefficient of Determination

R^2 is the proportion of variability in the response that is explained by the regression.

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R^2 is a measure of goodness of fit of a linear regression.

Example: Snow geese

Calculate R^2 from:

```
> anova(lm(photo~obs))
Analysis of Variance Table

Response: photo
          Df  Sum Sq Mean Sq F value    Pr(>F)
obs        1 1752.70 1752.70  52.037 2.066e-06
Residuals 16  538.91   33.68
---
```

SSreg

RSS

summary(lm(y ~ x))

RSS + SSreg = Syy

$$R^2 = 1 - \frac{RSS}{Syy}$$

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```

```
> 1752.70/(1752.70+538.91)
[1] 0.7648335
```

Model Fit

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This can be both an advantage and a disadvantage: one must understand the practical significance of $\hat{\sigma}$ in order to interpret its value.

Mean Squared Error

Another way to asses model fit is using the **generalization error** (**mean squared error of the estimator**)
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It can be shown that:

$$MSE(\hat{y}) = \boxed{(\mathrm{E}[\hat{y}|X = x] - y)^2} + \mathrm{Var}[\hat{y}|X = x] = \boxed{Bias(\hat{y})^2} + \mathrm{Var}[\hat{y}|X = x].$$

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This is known as the bias-variance trade-off.

For least squares estimates:

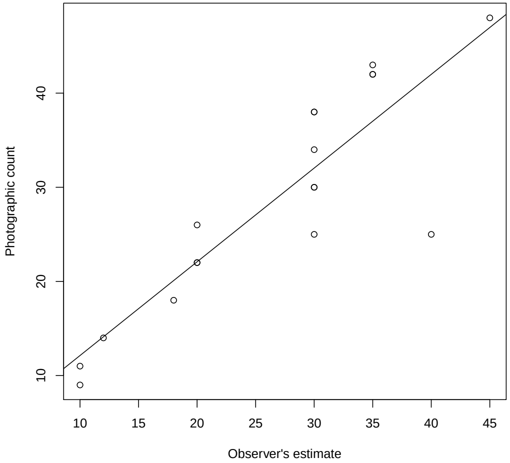
$$MSE(\hat{y}) = \sigma^2\left(\frac{1}{n} + \frac{(x-\overline{x})^2}{SXX}\right).$$

The **estimated (within sample) mean squared error** computed as

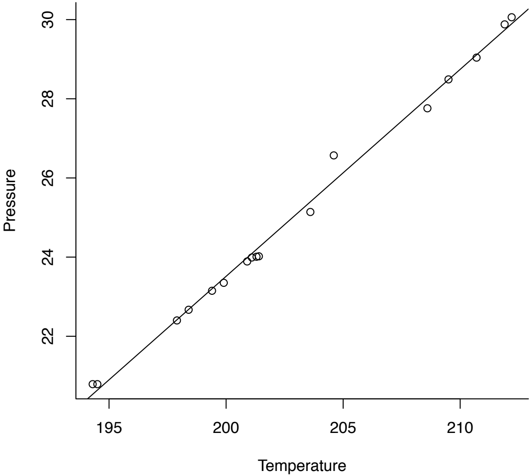
$$\boxed{MSE = \frac{1}{n}\sum_{i=1}^{n}(y_i-\hat{y}_i)^2.}$$

We will revisit the MSE later in the course.

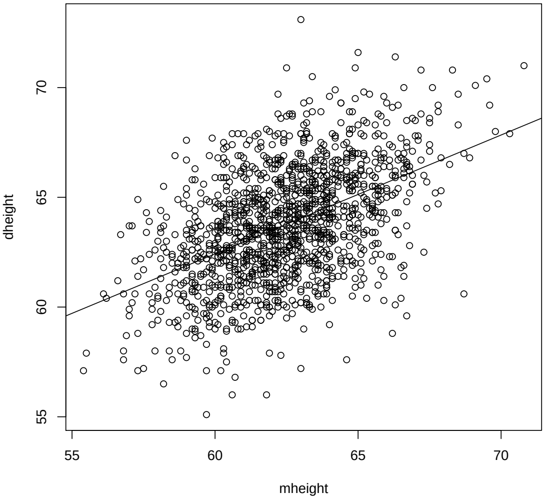
Snow geese: R-squared=0.7648, $\hat{\sigma}$ = 5.804, MSE=29.94



Forbes data: R-squared=0.9944, $\hat{\sigma}$ = 0.2328, MSE = 0.048



Heights: R-squared=0.2408, $\hat{\sigma}$ = 2.266, MSE = 5.129



Example: Interpretation of the slope

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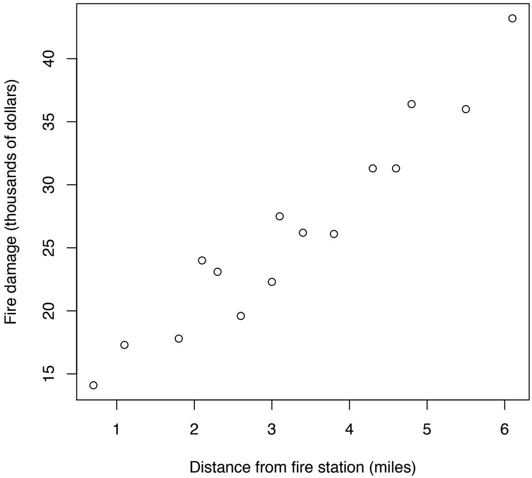
Is the amount of fire damage related to the proximity of the nearest fire station?

Let y be the amount of fire damage in thousands of dollars and x be the distance to the nearest fire station in miles.

A sample of 15 recent residential fires was selected.

Data: `fire.df` in R package `s20x`.

Example: Fire damage



Example: Fire damage

Fitting the regression

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \text{ with } \epsilon_i \sim \mathcal{N}(0, \sigma^2), \epsilon_i \text{ iid.}$$

we obtain the following (partial) R output:

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  10.2779     1.4203   7.237 6.59e-06
distance      4.9193     0.3927  12.525 1.25e-08
---
Residual standard error: 2.316 on 13 degrees of freedom
Multiple R-squared:  0.9235,Adjusted R-squared:  0.9176
F-statistic: 156.9 on 1 and 13 DF,  p-value: 1.248e-08
```

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We can provide the usual interpretation for $\hat{\beta}_1$:

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False!

Note: Observational studies cannot be used to infer causal relationship without additional information external to the study.