Final Exam 2015 Outline Solutions

1 a) $P(\text{eff} \mid X_1 = 1) = \dfrac{P(X_1 = 1 \mid E)\, P(E)}{P(X_1 = 1 \mid E)\, P(E) + P(X_1 = 1 \mid E^c)\, P(E^c)}$

$= 0.3 \times 0.2 / (0.06 + 0.16) = 3/11$

b) $P(\text{eff} \mid X_1 = 1, X_2 = 2) = \dfrac{0.2 \times 0.3 \times 0.2}{0.2 \times 0.3 \times 0.2 + 0.2 \times 0.2 \times 0.8} = 3/11$

c) As seen in (b), $X = 2$ does not change the probabilities that the drug is effective. He still has probability $3/11$ which is greater than the pop prob. 0.2.

2 a) $EX = a + b$, $EY = 6$. Equal for any real a, b s.t. $a + b = 6$

b) $\text{var}(X) = a^2 + 2b^2$, $\text{var}(Y) = 1 + 8 + 27 = 36 = 6^2$

so $|a| \le 6$, $|b| \le \sqrt{18}$ with $a^2 + 2b^2 = 6^2$

c) X and Y are both Normal, so we need (a) and (b)

$a + b = 6$, $a^2 + 2b^2 = 36 \Rightarrow b = 0$, $a = 6$ or ~~$a = \frac{18}{5}$, $b = \frac{12}{5}$~~

d) $\text{cov}(X, Y) = a\,\text{var}(Z_1) + 2b\,\text{var}(Z_2) = a + 4b$ $\quad a = 2,\ b = 4$

So any a, b s.t. $a + 4b = 0$

3. a) $X \sim \text{Bin}(10000, 0.15)$ approx by $N(1500, 1500 \times 0.85)$

b) $X \sim \text{Hg}(10000, 100, 1600)$, approx by $\text{Bin}\left(1600, \frac{100}{10000}\right)$

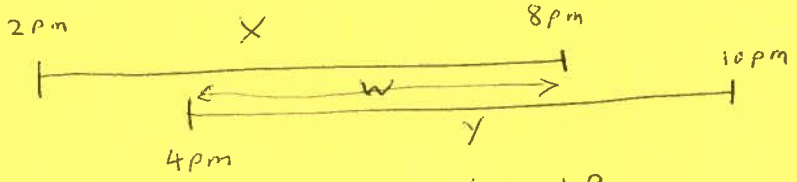
c) $X \sim \text{Bin}(10000, 0.001)$ approx by $\text{Poisson}(10)$

d) $X = \frac{1}{100}\sum_1^{100} Y_i$, $Y_i \sim u(0, 200)$. by CLT, approx by

$N\left(\frac{200}{2}, \frac{200^2}{12} \cdot \frac{1}{100}\right)$

$\sim N\left(100, \frac{100}{3}\right)$

4



(a) $\mathbb{E}X = \text{var}(X) = 3 \times 6 = 18$

$\text{cov}(X,Y) = \text{var}(W) = 3 \times 4 = 12$

(b) Given an event in $(4,10)$, $P(\text{event is in } (4,8)) = \frac{4}{6} = \frac{2}{3}$

Fred's total is $\underset{2-4\text{pm}}{\text{Poisson}(6)} + \underset{4-8\text{pm}}{\text{Bin}(27, \tfrac{2}{3})}$

$\mathbb{E}(\text{Fred}) = 6 + 18 = 24$

$\text{var}(\text{Fred}) = 6 + 27 \times \frac{2}{3} \times \frac{1}{3} = 12$

(c) $T = \sum_1^{18} V_k$ where $V_k \sim \mathcal{E}(3)$ time between events. independent

$\mathbb{E}T = 18/3 = 6$ (i.e. 10:00pm)

$\text{var}(T) = 18 \cdot \frac{1}{3^2} = 2$

$P(\text{later than 10pm}) = P(\text{less than 18 in 6 hours})$

$= P(\text{Poisson}(18) \leq 18) = \sum_0^{17} e^{-18} \frac{18^k}{k!}$

5 (a) $f_{X,Y}(x,y) = e^{-y}/y \quad 0 < x < y < \infty$

$f_Y(y) = \int_0^y \frac{e^{-y}}{y}\,dx = \frac{e^{-y}}{y}[x]_0^y = e^{-y} \quad 0 < y < \infty$

$f_{X|Y}(x|y) = \frac{e^{-y}}{y} / e^{-y} = \frac{1}{y} \quad 0 \leq x \leq y$ i.e. $U(0,y)$

(b) $X|Y \sim U(0,y)$ $\mathbb{E}(X|Y=y) = \frac{1}{2}y$

$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(\frac{1}{2}Y) = \frac{1}{2}\mathbb{E}Y = \frac{1}{2}$

$\mathbb{E}(XY) = \mathbb{E}(\mathbb{E}(XY|Y)) = \mathbb{E}(Y\,\mathbb{E}(X|Y)) = \frac{1}{2}\mathbb{E}Y^2$

$\text{cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}X \cdot \mathbb{E}Y = \frac{1}{2}\mathbb{E}Y^2 - \frac{1}{2}(\mathbb{E}Y)^2 = \frac{1}{2}\text{var}(Y) = \frac{1}{2}$

(c) $W = X/Y \quad P(W \leq w | Y=y) = P(X \leq wy | Y=y)$

$= \frac{wy}{y}$ as $(X|Y)$ is $U(0,y)$

$= w \quad 0 \leq w \leq 1$, not depending on Y.

So W is $U(0,1)$ independent of Y