1.(a) $f_{X,Y}(x,y) = \frac{1}{x^2} \cdot \frac{2}{y^3}$ $\quad 1 \le x < \infty \quad 1 \le y < \infty$

$f_{X,Y}(x,y)$ can be written as two functions that contain only x and y respectively

ranges are seperable . So X Y are independent

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$
$$= \int_1^{\infty} \frac{2}{x^2y^3}\,dx$$
$$= \frac{2}{y^3}(-x^{-1})\Big|_{x=1}^{x=\infty}$$
$$= \frac{2}{y^3}\left(-\frac{1}{x}\Big|_{x=1}^{x=\infty}\right)$$
$$= \frac{2}{y^3}(-0+1)$$
$$= \frac{2}{y^3}$$

$$\therefore f_Y(y) = \begin{cases} \frac{2}{y^3}, & 1 \le y < \infty \\ 0, & \text{otherwise} \end{cases}$$

(b) $f_{X,Y}(x,y) = \frac{3}{x^2y^3}$ $\quad 1 \le y \le x < \infty$

Notice that the ranges of x and y are dependent on each other. So X, Y are not independent

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$
$$= \int_y^{\infty} \frac{3}{x^2y^3}\,dx$$
$$= \frac{3}{y^3}\left(-\frac{1}{x}\right)\Big|_{x=y}^{x=\infty}$$
$$= -\frac{3}{y^3}\left(0-\frac{1}{y}\right)$$
$$= \frac{3}{y^4}$$

$$\therefore f_Y(y) = \begin{cases} \frac{3}{y^4}, & 1 \le y < \infty \\ 0, & \text{otherwise} \end{cases}$$

(c) $$P(Y<X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_1^{\infty}\left[\int_1^{x} \frac{2}{x^2y^3}\,dy\right]dx$$
$$= \int_1^{\infty} \frac{2}{x^2}\left(-\frac{1}{2y^2}\Big|_{y=1}^{y=x}\right)dx$$
$$= \int_1^{\infty} \frac{2}{x^2}\left(-\frac{1}{2x^2}+\frac{1}{2}\right)dx$$
$$= \int_1^{\infty} -\frac{1}{x^4}+\frac{1}{x^2}\,dx$$
$$= \int_1^{\infty} -x^{-4}+x^{-2}\,dx$$
$$= \frac{1}{3}x^{-3}\Big|_1^{\infty}+\left(-\frac{1}{x}\Big|_1^{\infty}\right)$$
$$= \left(0-\frac{1}{3}\right)+(-0+1)$$
$$= \frac{2}{3}$$

Midterm 2, 2016

Question # 1

from

Xuehan Zhao

Midterm 2, 2016

Questions # 2 and # 3

from Erica Chen

2. (a) let X be the # of times Fred stops Fred will be late if $X \geq 3$; $P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2)$

$X \sim \text{Bin}(8, \frac{1}{3})$

$$P(X \geq 3) = 1 - \binom{8}{0}\left(\frac{2}{3}\right)^8 - \binom{8}{1}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^7 - \binom{8}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^6$$

(b) Only one out of the first 6 lights was red. Fred will be late if the last two lights are all red

$$P(\text{last two all red}) = \left(\frac{1}{3}\right)^2 = \boxed{\frac{1}{9}}$$

(c) The 3rd red light was at stop 5, this is a $\text{NegBin}(3, \frac{1}{3})$

Let X be the stop that Fred met the 3rd red light

$$P(X=5) = \binom{4}{2}\left(\frac{1}{3}\right)^3 \cdot \left(\frac{2}{3}\right)^2 = 6 \cdot \left(\frac{1}{27}\right) \cdot \left(\frac{4}{9}\right) = \boxed{\frac{8}{81}}$$

(d) Fred has 3 stops left, and at each stop with 1/3 probability of getting red. this is $\text{Bin}(3, \frac{1}{3})$ The expected amount of red light in the next 3 stops is $3 \cdot \frac{1}{3} = 1$

The expected total amount of red lights that day is $3 + 1 = \boxed{4}$

3. (a) Let X be the # of days that Fred is late in 180 days

$X \sim Bin(180, 0.53)$ Since for large n. Binomial can be approximated by Normal Distribution. We can approximate X by

$Y \sim N(\underset{np}{\underset{\uparrow}{180 \cdot 0.53}}, \underset{np(1-p)}{\underset{\uparrow}{180 \cdot 0.53(1-0.53)}})$

We are interested in the probability where $X > 100$

$P(X > 100) \underset{\uparrow}{\approx} P(Y > 100.5) = 1 - P(Y \leq 100.5)$

continuity correction since X is discrete

$= 1 - P\left(\frac{Y - 180 \cdot 0.53}{\sqrt{180 \cdot 0.53(1-0.53)}} \leq \frac{100.5 - 180 \cdot 0.53}{\sqrt{180 \cdot 0.53(1-0.53)}}\right)$

why $+\frac{1}{2}$?

-1

work out Φ and use $Z \sim (0,1)$ to find the probability.

(b) (i) Let W be the event that the bus is late on a given day.

$W \sim Bern(0.53)$. $E[W] = 0.53$. $Var(W) = 0.53 \cdot 0.47$

$Y = 30 \cdot W$ (the sum of 30 independent identical distributed Bernoulli r.v)

$E[Y] = 30E[W] = 30 \cdot 0.53$

$Var(Y) = 30 \, Var(W) = 30 \cdot 0.53 \cdot 0.47$

(ii) $E[X - Y] = E[X] - E[Y]$

$E[X] = E[40W] = 40E[W] = 40 \cdot 0.53$

$E[X - Y] = 40 \cdot 0.53 - 30 \cdot 0.53 = \boxed{5.3}$

(iii) Since only 5 days Sam and Sarah both take the bus

Let Z be the number of days they are late in these 5 days

$Var(Z) = 5 \cdot 0.53 \cdot 0.47$

Z and X - Z and Y - Z are independent since they are not both on the ~~sa~~ bus

$Cov(X, Y) = Cov(X - Z + Z, Y - Z + Z)$

$= Cov(X-Z, Y-Z) + Cov(Z, Y-Z) + Cov(X-Z, Z)$

$+ Cov(Z, Z)$. since Z, X - Z, Y - Z are independent

$= Cov(Z, Z) = Var(Z) = \boxed{5 \cdot 0.53 \cdot 0.47}$