Lecture 5: Conditional probability and Bayes Theorem L&M 2.4; Wed, Oct 3, 2018

5.1 Conditional probability L & M, 2.4

(i) Idea: revise probabilities in light of restrictions or partial information.

(ii) Definition: probability of D given E is $P(D \mid E) \; = \; P(D \cap E)/P(E)$, provided $P(E) \; > \; 0$.

(iii) Example: Parents type AB and O have three children. (Note: just like tossing fair coin 3 times.)
$\Omega \; = \; \{AAA, AAB, ABA, ABB, BAA, BAB, BBA, BBB\}$ and these are equiprobable (1/8 each).
Let D be at least 2 A children: $P(D) \; = \; 4/8 \; = \; 1/2$. Note also $P(\text{first child is } A) = 4/8 = 1/2$.
$P(D \mid \text{first child is } A) \; = \; P(D \; \cap \; \text{first child is } A)/P(\text{first child is } A) \; = \; (3/8)/(4/8) \; = \; 3/4$.

5.2 Example: In a population suppose $P(A) = 0.41, \; P(B) = 0.16, \; P(AB) = 0.07, P(O) = 0.36$.

$P(\text{have antigen } A) = P(A) + P(AB) = 0.48, \; P(\text{have antigen } B) = P(B) + P(AB) = 0.23$.

$$\begin{aligned} P(\text{have antigen } A|\text{have antigen } B) &= P(AB)/(P(B) + P(AB)) \; = \; 0.07/0.23 \; = \; 0.304 \; < \; 0.48 = P(\text{have } A) \\ P(\text{have antigen } B|\text{have antigen } A) &= P(AB)/(P(A) + P(AB)) \; = \; 0.07/0.48 \; = \; 0.146 \; < \; 0.23 = P(\text{have } B) \end{aligned}$$

Note they are NOT equal, but in both cases having the one decreases the probability of also having the other.

5.3 The chain rule L&M P. 40-41 If $P(E_1 \cap E_2 \cap ... \cap E_n) \; > \; 0$,

$$P(E_1 \cap E_2 \cap ... \cap E_n) \; = \; P(E_1).P(E_2|E_1).P(E_3 \mid E_1 \cap E_2)...P(E_n \mid E_1 \cap E_2 \cap ... \cap E_n)$$

To prove it, write it out:

$$P(E_1).P(E_2|E_1).P(E_3 \mid E_1 \cap E_2)...P(E_n \mid E_1 \cap E_2 \cap ... \cap E_n) \; =$$
$$P(E_1)\frac{P(E_2 \cap E_1)}{P(E_1)} \frac{P(E_3 \cap E_2 \cap E_1)}{P(E_2 \cap E_1)} ... \qquad ... \frac{P(E_n \cap ... \cap E_3 \cap E_2 \cap E_1)}{P(E_{n-1} \cap .. \cap E_2 \cap E_1)}$$

Note: we could put E_1, E_2 ... in any order we choose.

Common sense example: Draw 3 cards from a 52-card pack without replacement:

the probability of no face card in 3 cards is (40/52).(39/51).(38/50).

5.4 Law of total probability: L&M P.43

Mutually exclusive and exhaustive events H_i ("hypotheses", or "states of nature") $i = 1, ..., k$
Suppose H_i has probability $P(H_i)$ and $\sum_{i=1}^{k} P(H_i) = 1$. (Note $H_i \cap H_j = \Phi$, $P(H_i \cap H_j) = 0$.

$$\begin{aligned} D &= (D \cap H_1) \cup (D \cap H_2) \cup ... \cup (D \cap H_k) \\ P(D) &= \left(\sum_{j=1}^{k} P(D \mid H_j)P(H_j) \right) \end{aligned}$$

5.5 Bayes' formula: L&M p.48

Assume $H_i, \; i = 1, ..., k$ as above, and D with $P(D) > 0$,

$$P(H_j|D) \; = \; \frac{P(H_i \cap D)}{P(D)} \; = \; \frac{P(D|H_i)P(H_i)}{\sum_{j=1}^{k} P(D|H_j)P(H_j)}$$

See example (i) on P.4: two biased coins C_1 and C_2 with $P_1(H) = 1/4$ and $P_2(H) = 3/4$.

A random coin is chosen ($P(C_1) = P(C_2) = 0.5$), and tossed. $P(H) = 0.5 * (1/4) + 0.5 * (3/4) = 0.5$.

The result is H: $\quad P^*(C_1) = P(C_1 \mid \text{head}) = (1/4) \times (1/2)/((1/4) \times (1/2) + (3/4) \times (1/2)) \; = \; 1/4$.

(a) Repeat the whole "experiment" choosing a random coin: $P(H) = 0.5$ as before.

(b) Toss the **same** coin again: $P(H) = P^*(C_1) * (1/4) + P^*(C_2) * 3/4 = (1/4) * (1/4) + (3/4) * 3/4) = 5/8$.

Lecture 6: Independence: L & M 2.5 Friday Oct 5 (part 1)

6.1 Independent events; L&M P.53-54

(i) Definition: E and F are independent if $P(E \cap F) \;=\; P(E) \times P(F)$.

(ii) Interpretation: Knowing F happens does not affect the probability of E: $P(E|F) \;=\; P(E)$.

(iii) If E and F are independent, $P(E \cup F) = P(E) + P(F) - P(E).P(F)$ so
$P(E^c \cap F^c) = 1 - P(E \cup F) = 1 - P(E) - P(F) + P(E).P(F) \;=\; (1 - P(E)).(1 - P(F)) \;=\; P(E^c).P(F^c)$.
That is E^c and F^c are independent. (So are E^c and F, and E and F^c).

6.2 Example from above: Checking independence

$P(A) = 0.41,\ P(B) = 0.16,\ P(AB) = 0.07, P(O) = 0.36$.

$P(\text{have antigen } A) = P(A) + P(AB) = 0.48,\ P(\text{have antigen } B) = P(B) + P(AB) = 0.23$.

$0.48 \times 0.23 \;=\; 0.1104 \;\neq 0.07 = P(AB)$: Having antigen A is NOT independent of having antigen B. Knowing a person has the B antigen decreases the probability they have the A antigen.

6.3: Independence of multiple events; L&M P58-59

Definition: E_1, E_2, ..., E_n are (jointly) independent if for every subset $E_{r_1}, E_{r_2}, ...$ with $r_1 < r_2 < .... \leq n$

$$P(E_{r_1} \cap E_{r_2} \cap .... \cap E_{r_k}) \;\;=\;\; P(E_{r_1}) \times P(E_{r_2}) \times .... \times P(E_{r_k}).$$

6.4 Pairwise and joint independence: examples

(i) Pairwise independence without joint independence: example.

Two independent rolls of a fair die. D_1 is first throw gives odd number.

D_2 is second throw gives odd number. D_3 is sum of two throws is odd number.

$P(D_1) \;=\; P(D_2) \;=\; P(D_3) \;=\; 1/2$. $\qquad P(D_1 \cap D_2) \;=\; P(D_1 \cap D_3) \;=\; P(D_2 \cap D_3) \;=\; 1/4$.

But $P(D_1 \cap D_2 \cap D_3) \;=\; 0$, not 1/8. These three events are pairwise independent but NOT jointly independent.

(ii) The three-way independence, without pairwise, is clearly also possible.

Let F, G, I be Swiss adults fluent in French, German and Italian.

Suppose $P(F) \;=\; P(G) \;=\; P(I) \;=\; 1/2$, and $P(F \cap G \cap I) \;=\; P(F) \times P(G) \times P(I) \;=\; 1/8$.

But this does not determine $P(F \cap G)$ etc.

For example, we could have as shown. Then

$P(F \cap G) \;=\; 3/8,\;\; P(F \cap I) \;=\; 1/8, P(G \cap I) = 1/8$.

	I			I^c		
	F	F^c		F	F^c	
G	1/8	0		1/4	1/8	1/2
G^c	0	3/8		1/8	0	1/2
	1/8	3/8		3/8	1/8	

6.5 Updating information from independent experiments

For the general formula, see the "extra notes" on P.4.

For the coins example; choose one coin at random and toss it twice:

What is the probability the second toss is heads given the first is heads?

Solution 1: $P(\text{2 nd. head} \mid \text{first head}) \;=\; P(\text{both heads})/P(\text{1 st head}) \;=$

$$((1/4) \times (1/4) \times (1/2) \;+\; (3/4) \times (3/4) \times (1/2))/((1/4) \times (1/2) \;+\; (3/4) \times (1/2)) \;\;=\;\; 5/8.$$

Solution 2: After first head, $P^*(C_1) = P(C_1 \mid \text{head}) = (1/4) \times (1/2)/((1/4) \times (1/2) + (3/4) \times (1/2)) \;=\; 1/4$.

So $P^*(C_2) = P(C_2 \mid \text{head}) \;=\; (1 - (1/4)) \;=\; 3/4$.

$P(\text{heads again}) \;=\; P^*(C_1)P(\text{head} \mid C_1) \;+\; P^*(C_2)P(\text{head} \mid C_2) \;=\; (1/4) \times (1/4) + (3/4) \times (3/4) \;=\; 5/8$.

The two head outcomes are independent *given the coin chosen*, but not overall.

Lecture 7: Conditional probability and independence Examples Oct 5: Part 2

7.1 Testing for a rare disease L&M P50-51

Suppose we have a quite effective test, so $P(+ \mid \text{disease}) = 0.99$, and test is quite accurate, so $P(+ \mid \text{no disease}) = 0.02$). Now suppose the frequency of the disease is 0.001.

$$\begin{aligned} P(+\text{ test result}) &= P(+ \mid \text{disease})P(\text{disease}) + P(+ \mid \text{no disease})P(\text{no disease}) \\ &= 0.99 \times 0.001 + 0.02 \times (1 - 0.001) = 0.00099 + 0.0198 = 0.02097 \\ P(\text{disease} \mid +) &= P(+ \mid \text{disease})P(\text{disease})/P(+) = 0.99 \times 0.001/0.02097 = 0.047 \end{aligned}$$

Less than 5% of people testing positive actually have the disease!

The probability of spots given measles is large: the probability of measles given spots is small.

The probability of 4 legs given elephant close to 1: the probability of elephant given 4 legs is close to 0.

7.2 Mendelian segregation when parents are known

Mendel discovered that every pea plant has two *factors* or *alleles* for flower color: R (red) and W (white). Each plant has RR or RW and is red, or WW and is white.

We cross two plants which we know are RW. The offspring plants are *independent* and each is RR, RW or WW with probabilities 1/4, 1/2, 1/4.

What is the probability an offspring is RR, given it is red.

Answer: $P(\text{red}) = 3/4$, $P(RR \cap \text{red}) = P(RR) = 1/4$, so $P(RR \mid \text{red}) = (1/4)/(3/4) = 1/3$.

7.3 Using population allele frequencies

Suppose in the population the R allele has frequency 0.3, and W 0.7, and the types of the two alleles in an individual are independent. What is the probability a red-flowered pea-plant is RR?

Solution: By independence, $P(RR) = 0.3 \times 0.3 = 0.09$, $P(WW) = 0.7 \times 0.7 = 0.49$,

$$P(RW) = 1 - 0.09 - 0.49 = 0.42.$$

Overall $P(\text{red}) = P(RR) + P(RW) = 0.09 + 0.42 = 0.51$,

so then $P(RR \mid \text{red}) = P(RR \cap \text{red})/P(\text{red}) = 0.09/0.51 = 0.176$.

7.4 Updating probabilities with new information

The red pea plant is crossed to a white one (WW). The first offspring has red flowers. What now is the probability the red parent plant is RR?

Solution: $P(\text{red offspring} \mid RR\text{ parent}) = 1$, $P(\text{red offspring} \mid RW\text{ parent}) = (1/2)$.

$$\begin{aligned} P(\text{red offspring}) &= P(\text{red offspring} \mid RR)P(RR) + P(\text{red offspring} \mid RW)P(RW) \\ &= 1 \times 0.176 + (1/2)(1 - 0.176) = 0.588. \end{aligned}$$

$$\begin{aligned} P(\text{red parent is } RR \mid \text{red offspring}) &= P(\text{red offspring} \mid \text{red parent is } RR) \times P(\text{red parent is } RR)/P(\text{red offspring}) \\ &= 1 \times 0.176/0.588 = 0.299 \approx 0.3. \end{aligned}$$

7.5 Independent events

We want to know if a red plant is RW or RR: we "self" it – cross it with iteself. If it is RR all the offspring will have red flowers. Mendel decided to classify the plant as RR if all of 10 offspring had red flowers.

If the parent plant is RW, each offspring is red with probability 3/4, and white with probability 1/4, and the colors of offspring pea plants are *independent*.

The probability all 10 offspring are red is $(3/4)^{10} = 0.0563$

Mendel would have misclassified over 5% of his RW parent plants.

Extra notes on updating information sequentially

(i) The probability of new data

For partition of Ω, $H_1, ..., H_k$ and two experiments D and E that are independent **given** the state H_j:

$$P(E \mid D) \;=\; P(E \cap D)/P(D) \;=\; \sum_{j=1}^{k} P(E \cap D \cap H_j)/P(D).$$

Now let $\quad P^*(H_i) \;=\; P(H_i \mid D) \;=\; P(D \cap H_i)/P(D)$, The probability of H_i after observing data D.

Then $\quad P(D \cap E \cap H_i) \;=\; P(E \mid D \cap H_i)P(H_i \mid D)P(D) \;=\; P(E \mid H_i)P^*(H_i)P(D)$ (why?)

So $\quad P(E \mid D) \;=\; \sum_{i=1}^{k} P(E \mid H_i)P^*(H_i). \;=\; \sum_{i=1}^{k} P(E \mid H_i)P(H_i \mid D).$

(ii) Updating the probability of H_i

In general (even if E and D are not independent given H_i), we can consider D and E together:

$$P(H_i \mid D \cap E) \;=\; P(D \cap E \mid H_i)P(H_i)/\left(\sum_{j=1}^{k} P(D \cap E \mid H_j)P(H_j)\right)$$

However, from above, provided D and E are independent given each H_i, $i = 1, ..., k$:

$$P(H_i \mid D \cap E) \;=\; P(E \mid H_i)P^*(H_i)/\left(\sum_{j=1}^{k} P(E \mid H_j)P^*(H_j)\right)$$

That is, we can first update from $P(H_i)$ to $P^*(H_i) = P(H_i \mid D)$ and then use these probabilities in updating to $P(H_i \mid D \cap E)$. And then so also for the next event, and the next, ...

(iii) Example 1: see the example of two coins already discussed.

(iv) Example 2: (a) In a population 25% of people are type bb, 50% are type bg, and the remaining 25% have grey eyes, gg.

If bb marries gg, all kids have brown eyes. If bg marries gg, kids are indep. 50/50, brown/grey eyes.

Let B_0 be the event Sarah has brown eyes, bb is event Sarah is type bb, and bg is event Sarah is type bg.

(b) Sarah marries Paul, who has grey eyes. Their first child has brown eyes: event B_1.

(c) Sarah and Paul's second child also has brown eyes: event B_2.

(d) Sarah and Paul's third child has grey eyes; event G_3.

	bb	bg	gg	$bb\|...$	$bg\|...$		bb	bg	$P^*(bb)$	$P^*(bg)$
-	0.25	0.5	0.25	1/4	1/2		0.25	0.5	1/4	1/2
$\cap B_0$	$\times 1 = 0.25$	$\times 1 = 0.5$	0	1/3	2/3		1	1	1/3	2/3
$\cap B_1$	$\times 1 = 0.25$	$\times 0.5 = 0.25$	-	1/2	1/2		1	0.5	1/2	1/2
$\cap B_2$	$\times 1 = 0.25$	$\times 0.5 = 0.125$	-	2/3	1/3		1	0.5	2/3	1/3
$\cap G_3$	$\times 0 = 0$	$\times 0.5$	-	0	1		0	0.5	0	1

The left side shows the cumulative way of looking at the problem.

The right side shows the conditional updating view– the second formula above.

Note it is exact same result and almost exact same computations; the idea is that using the conditional updating form we do not need to know what went before, only the current $P^*(bb)$ and $P^*(bg)$.

8: Combinatorial, Binomial and Hypergeometric probabilities, L&M 2.7, 3.2 Oct 10.

8.1 Combinatorial probabilities: L&M 2.7

If there are n possible equiprobable outcomes of an experiment, and m of them satisfy a condition specifying an event A, then the probability of the event A is $P(A) = m/n$.

Example 1: Draw 3 cards from a 52-card pack (without replacement): $\binom{52}{3}$ ways.

Event A: no face card (there are 40 non-face cards): $\binom{40}{3}$ ways.

$$P(A) = \binom{40}{3}/\binom{52}{3} = \frac{40.39.38}{52.51.50} = 0.447$$

(Recall we got this same answer earlier, by a different route.)

Example 2: The birthday problem; L&M P.94 Suppose we have k people.

Ignore Feb 29, and suppose each of the other 365 days has equal probability of being each person's birthday.

Number of ways birthdays can be = $365.365.365....365 = 365^k$.

Number of ways of having *different* birthdays = $365.364.....(365-k+1)$.

Probability all have different birthdays = $365.364.....(365-k+1)/(365^k)$

This turns out to be approx. $0.75,\ 0.5,\ 0.03$ for $k = 15,\ 23,\ 50$.

8.2 Binomial probabilities: independent trials: L & M 3.2

Example 1 above: but now replace the card each time.

On each draw there are 40 non-face cards in the 52 cards, and the trials are independent:

$$P(\text{no face card}) = (40/52).(40/52).(40/52) = 0.455.$$

General binomial sampling: N fish in a pond; k are red, $N-k$ are blue.

n times over, we catch a fish and put it back. Each time $P(\text{red}) = p = k/N$.

Probability of any sequence $RRBRBBBRRBB....$ is $p.p.(1-p).p.(1-p).(1-p).(1-p).p.p...$

Probability of any sequence with x red and $n-x$ blue is $p^x(1-p)^{n-x}$

There are $\binom{n}{x}$ such sequences (why?). The probability we sample x red fish is $\binom{n}{x}p^k(1-p)^{n-x}$.

This is **Sampling with replacement**: the outcomes are **independent**.

8.3 Hypergeometric probabilities; L&M 3.2

N fish in a pond; k are red, $N-k$ are blue.

Sample n fish **without replacement**. What is the probability x are red?

Number of ways of choosing the x red fish is $\binom{k}{x}$, and the $(n-x)$ blue fish is $\binom{N-k}{n-x}$.

The total number of ways of choosing n fish from N is $\binom{N}{n}$.

Probability is $\binom{k}{x} \times \binom{N-k}{n-x} / \binom{N}{n}$