Some notes about Poisson random variables and processes

1: Information on the crib sheet: can be quoted as/when needed

(a) The number of events N in time t is Poisson: $N \sim \mathcal{P}o(\lambda t)$

(b) Events occur independently. That is, numbers of events in disjoint time intervals are independent.

(c) The time T to the next event is Exponential: $T \sim \mathcal{E}(\lambda)$

(d) Given an event in time interval $(0, s)$, the time T it occurred is Uniform: $T \sim U(0, s)$.

(e) Multiple independent Poisson processes rates λ_i can be combined to a single process rate $\sum_i \lambda_i$.

(f) If each event of a Poisson process rate λ is independently detected with probability p, detected events form a Poisson process rate λp.

2. Sum of Poisson random variables

(a) If $X_i \sim \mathcal{P}o(\mu_i)$, independent, then $Y = \sum_i X_i \sim \mathcal{P}o(\sum_i \mu_i)$.

(b) The easiest way to see this is to think about a Poisson process rate λ and disjoint time intervals length t_i. So if $X_i \sim \mathcal{P}o(\lambda t_i)$, then $Y = \sum_i X_i$ is the total events in total time $\sum_i t_i$, so is Poisson.

(c) Now consider independent Poisson processes rates $lambda_i$ (as in 1(e)).
The total number of events (of all types) N in time t, is sum of independent Poissons means $\lambda_i t$.
So this is $\mathcal{P}o(\sum_i \lambda_i t)$.
Since 1(a) characterizes Poisson process, this shows 1(e).

3. Splitting up the Poisson process

(a) Conditioning in time:
From 1(d): given an event in $(0, t)$, the probability it is in $(0, s)$ is s/t. $(0 < s < t)$.
The events are all independent.
So given $N = n$ events in $(0, t)$, then number X in $(0, s)$ is Binomial: $(X \mid N = n) \sim Bin(n, s/t)$.

(b) Considering one type in the situation 1(e):
Given an event in the combined process, the probability it is type i is $\lambda_i / \sum_j \lambda_j$.
This is not too hard to show, but if needed you will be told you can assume it as in the Homework 9 example.
The events are all independent.
So given total $N = n$ events in $(0, t)$, then number X that are type i is Binomial: $(X \mid N = n) \sim Bin(n, \lambda_i / \sum_j \lambda_j)$.

(c) Counts of types are not independent, given $N = n$.
The different types start as independent Poisson processes in 1(e).
BUT Note that the number X of type i and number Y of type j, given total $N = n$, are **not** independent: the event cannot be both type i and type j!!
(It's like the multinomial; in fact is is multinomial, but do not worry about that.)

(d) Getting back the independent Poisson processes: see 1(f)
In 1(f) we have a combined process rate λ two "types" (here detected and not detected e.g. typos, but could be any two types type-1 and type-2).
Given each is independently type-1 with probability p,
then if X is the number of type 1 events, and N the total number of events $(X \mid N = n) \sim Bin(n, p)$.
In time $(0, t)$, $N \sim \mathcal{P}o(\lambda t)$.
1(f) then says $X \sim \mathcal{P}o(\lambda p t)$: a Poisson process rate λp.
It is on the cheat-sheet: you could quote it is asked!
Similarly the number Y of type-2 is Poisson: $Y \sim \mathcal{P}o(\lambda(1 - p))$.
BUT 1(f) does **not** say that these are two **independent** Poisson processes. To show that you have to consider the joint probablity:

$$P(X = x, Y = y) = P(X = x, N = x + y) = P(X = x | N = (x + y))P(N = (x + y))$$

You know the first term in the Binomial $(X \mid N = n)$ amd the second is overall Poisson.
Plugging these in we find that X and Y **are** independent Poisson.
This is in your notes (or 27.3 in the printed notes: notes8.pdf file).
It is worth checking your notes on this.