STAT 340 Final Exam: Fall 2018 Dec 12, 2018
Useful facts

1. Permutations and combinations

There are $n! = \prod_{i=1}^{n} i = 1.2.3.4.\ldots.n$ permutations of n objects.

There are $\binom{n}{k} = n!/(k!(n-k)!)$ ways of choosing a given k objects from n.

2. Joint and conditional probabilities

If C and D are any events: $\mathrm{P}(C \bigcup D) = \mathrm{P}(C) + \mathrm{P}(D) - \mathrm{P}(C \bigcap D)$.

The conditional probability of C given D is $\mathrm{P}(C \mid D) = \mathrm{P}(C \bigcap D) / \mathrm{P}(D)$.

C and D are independent if and only if $\mathrm{P}(C \bigcap D) = \mathrm{P}(C) \times \mathrm{P}(D)$.

3. Laws and theorems

Suppose $E_1, \ldots, E_k$ is a partition of Ω.

That is $E_i \bigcap E_j$ is empty, and $E_1 \bigcup E_2 \bigcup \ldots \bigcup E_k = \Omega$.

The law of total probability states that: $\mathrm{P}(D) = \sum_{j=1}^{k} \mathrm{P}(D \bigcap E_j) = \sum_{j=1}^{k} \mathrm{P}(D \mid E_j)\, \mathrm{P}(E_j)$

Bayes' Theorem states that: $\mathrm{P}(E_i \mid D) = \mathrm{P}(D \mid E_i)\, \mathrm{P}(E_i)/\mathrm{P}(D)$

4. Random variables and distributions

	discrete (mass)	continuous (density)
Probability mass/density function	pmf: $\mathrm{P}(X = x) = p_X(x)$	pdf: $f_X(x)$
Cumulative dist. func. $\mathrm{P}(X \le x)$	$F_X(x) = \sum_{w \le x} p_X(w)$	$F_X(x) = \int_{-\infty}^{x} f_X(w) dw$
Joint mass/density func. of (X, Y)	$p_{X,Y}(x, y) = \mathrm{P}(X = x, Y = y)$	$f_{X,Y}(x, y)$
Marginal mass/density of X	$p_X(x) = \sum_y p_{X,Y}(x, y)$	$f_X(x) = \int_{y=-\infty}^{\infty} f_{X,Y}(x, y) dy$
Conditional of X given $Y = y$	$p_{X\|Y}(x\|y) = p_{X,Y}(x, y)/p_Y(y)$	$f_{X\|Y}(x\|y) = f_{X,Y}(x, y)/f_Y(y)$
Independence of X and Y	$p_{X,Y}(x, y) = \mathrm{P}(X = x)\mathrm{P}(Y = y)$	$f_{X,Y}(x, y) = f_X(x) f_Y(y)$
or	$p_{X\|Y}(x\|y) = p_X(x)$	$f_{X\|Y}(x\|y) = f_X(x)$
	Must hold for all x and all y	

5. Moments of random variables:

(a) Expectation: E(X)	$\mathrm{E}(X) = \sum_x x\ \mathrm{P}(X = x)$	$\int_{-\infty}^{\infty} x f_X(x) dx$
and of $g(X)$	$\mathrm{E}(g(X)) = \sum_x g(x)\ \mathrm{P}(X = x)$	$\int_{-\infty}^{\infty} g(x)\ f_X(x)\ dx$
and of $g(X, Y)$	$\mathrm{E}(g(X, Y)) = \sum_x \sum_y g(x, y)\ \mathrm{P}_{X,Y}(x, y)$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\ f_{X,Y}(x, y)\ dx\ dy$

in each case **provided** the sum/integral converges absolutely.

(b) For any random variables X:

Variance: $\mathrm{var}(X) = \mathrm{E}((X - E(X))^2) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

Note also: $\mathrm{E}(aX + b) = a\, \mathrm{E}(X) + b$, $\quad \mathrm{var}(aX + b) = a^2\, \mathrm{var}(X)$.

(c) For any random variables X, Y, Z and W:

Covariance: $\mathrm{cov}(X, Y) = \mathrm{E}((X - E(X))(Y - E(Y))) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)$

Correlation: $\rho(X, Y) = \mathrm{cov}(X, Y)/\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}, \quad -1 \le \rho(X, Y) \le 1$

Note also:

$$\begin{aligned}
\mathrm{E}(X + Y) &= \mathrm{E}(X) + \mathrm{E}(Y), \\
\mathrm{var}(X + Y) &= \mathrm{var}(X) + \mathrm{var}(Y) + 2\, \mathrm{cov}(X, Y) \\
\mathrm{cov}(aX + b, cW + d) &= ac\, \mathrm{cov}(X, W), \\
\mathrm{cov}(X + Y, W + Z) &= \mathrm{cov}(X, W) + \mathrm{cov}(X, Z) + \mathrm{cov}(Y, W) + \mathrm{cov}(Y, Z)
\end{aligned}$$

6. Conditional Expectations of random variables:

(a) If $h(y) = \mathrm{E}(X \mid Y = y)$, we write $h(Y) = \mathrm{E}(X|Y)$. Note this is a random variable, a function of Y.

Result 1: $\mathrm{E}(X) \;=\; \mathrm{E}(h(Y)) = \mathrm{E}(\mathrm{E}(X \mid Y))$ and more generally $\mathrm{E}(g(X,Y)) = \mathrm{E}(\mathrm{E}(g(X,Y) \mid Y))$

Result 2: $\mathrm{var}(X) = \mathrm{E}(\mathrm{var}(X|Y)) \;+\; \mathrm{var}(\mathrm{E}(X|Y))$.

(b) If $X_1, X_2, ...$, are independent, with mean μ and variance σ^2, and independent of random variable N which takes values 1,2,3....,, and $W = \sum_{i=1}^{N} X_i$, then

$$\mathrm{E}(W) = \mathrm{E}(\mathrm{E}(W|N)) = \mu \mathrm{E}(N) \text{ and } \mathrm{var}(W) = \sigma^2 \mathrm{E}(N) + \mu^2 \mathrm{var}(N).$$

7. A note about Poisson Process rate λ

(a) The number of events N in time t is Poisson: $N \;\sim\; \mathcal{P}o(\lambda t)$

(b) Events occur independently. That is, numbers of events in disjoint time intervals are independent.

(c) The time T to the next event is Exponential: $T \;\sim \mathcal{E}(\lambda)$

(d) Given an event in time interval $(0,s)$, the time T it occurred is Uniform: $T \;\sim\; U(0,s)$.

(e) Multiple independent Poisson processes rates λ_i can be combined to a single process rate $\sum_i \lambda_i$.

(f) If each event of a Poisson process rate λ is independently detected with probability p, detected events form a Poisson process rate λp.

8. Standard distributions:

	pmf or pdf	mean	variance
(a) Binomial; $B(n,p)$	$\mathrm{P}(X=k) \;=\; \binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
index n, parameter p	$k = 0, 1, 2, ..., n$		
(b) Multinomial: $Mn(n,(p_1,..,p_k))$	$P(X_i = n_i, i = 1, ...k) =$	$X_i \;\sim\; Bin(n, p_i),\; i = 1, ..., k$	
$\sum_i p_i = 1 \quad \sum_i n_i \;=\; n$	$(n!/(n_1!)....(n_k!))\; p_1^{n_1}....p_k^{n_k}$		
(c) Hypergeometric (N,k,n);	$P(X=x) \;=\; \binom{k}{x}\binom{N-k}{n-x}/\binom{N}{n}$	$nk/N \equiv np$	$np(1-p)\frac{N-n}{N-1}$
sample n from N, $p = k/N$	$x = \min(0, k+n-N), ..., \max(n,k)$		
sample n from N	$k = 0, ..., n,\; k \leq m,\; k \geq m+n-N$		
(d) Geometric; $G(p)$;	$\mathrm{P}(X=k) \;=\; p(1-p)^{k-1}$	$1/p$	$(1-p)/p^2$
parameter p	$k = 1, 2, 3, 4, ...$		
(e) Neg. Binomial; $NegB(r,p)$;	$\mathrm{P}(X=k) \;=\; \binom{k-1}{r-1} p^r (1-p)^{k-r}$	r/p	$r(1-p)/p^2$
index r, parameter p	$k = r, r+1, r+2, ...$		
(f) Poisson; $\mathcal{P}o(\mu)$	$\mathrm{P}(X=k) \;=\; \exp(-\mu)\mu^k/k!, \quad k = 0, 1, 2, ...$	μ	μ
(g) Uniform on (a,b); $U(a,b)$;	$f_X(x) = 1/(b-a), \quad a < x < b$	$(b+a)/2$	$(b-a)^2/12$
(h) Normal, $N(\mu, \sigma^2)$	$f_X(x) = (1/\sqrt{2\pi\sigma^2})\exp(-(x-\mu)^2/2\sigma^2)$	μ	σ^2
(i) Exponential, $\mathcal{E}(\lambda)$	$f_X(x) = \lambda \exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$
rate parameter λ	$0 \leq x < \infty$		