STAT 340 Midterm-2: Autumn 2018 Nov 19, 2018

1. Permutations and combinations

There are $n! = \prod_{i=1}^{n} i = 1.2.3.4.\ldots.n$ permutations of n objects.

There are $\binom{n}{k} = n!/(k!(n-k)!)$ ways of choosing a given k objects from n.

2. Joint and conditional probabilities

If C and D are any events: $\mathrm{P}(C \bigcup D) = \mathrm{P}(C) + \mathrm{P}(D) - \mathrm{P}(C \bigcap D)$.

The conditional probability of C given D is $\mathrm{P}(C \mid D) = \mathrm{P}(C \bigcap D) / \mathrm{P}(D)$.

C and D are independent if $\mathrm{P}(C \bigcap D) = \mathrm{P}(C).\mathrm{P}(D)$.

3. Laws and theorems

Suppose $E_1, \ldots, E_k$ is a partition of Ω. That is $E_i \bigcap E_j$ is empty, and $E_1 \bigcup E_2 \bigcup \ldots \bigcup E_k = \Omega$.

The law of total probability states that: $\mathrm{P}(D) = \sum_{j=1}^{k} \mathrm{P}(D \bigcap E_j) = \sum_{j=1}^{k} \mathrm{P}(D \mid E_j)\ \mathrm{P}(E_j)$

Bayes' Theorem states that: $\mathrm{P}(E_i \mid D) = \mathrm{P}(D \mid E_i)\ \mathrm{P}(E_i)/\mathrm{P}(D)$

4. Random variables and distributions

	discrete (mass)	continuous (density)
Probability mass/density function	pmf: $\mathrm{P}(X = x) = p_X(x)$	pdf: $f_X(x)$
Probability $P(X \in B)$	$\sum_{x \in B} p_X(x)$	$\int_{x \in B} f_X(x)dx$
Cumulative dist. func. CDF, $\mathrm{P}(X \leq x)$	$F_X(x) = \sum_{w \leq x} p_X(w)$	$F_X(x) = \int_{-\infty}^{x} f_X(w)dw$
Joint mass/density func. of (X, Y)	$p_{X,Y}(x, y) = \mathrm{P}(X = x, Y = y)$	$f_{X,Y}(x, y)$
Marginal mass/density of X	$p_X(x) = \sum_y p_{X,Y}(x, y)$	$f_X(x) = \int_{y=-\infty}^{\infty} f_{X,Y}(x, y)dy$
Independence of X and Y	$p_{X,Y}(x, y) = \mathrm{P}(X = x)\mathrm{P}(Y = y)$	$f_{X,Y}(x, y) = f_X(x)f_Y(y)$

Must hold for all x and all y

5. Moments of random variables: (provided the relevant sums/integrals converge absolutely.)

Expectation:	$\mathrm{E}(X)$	$\sum_x x\ \mathrm{P}(X = x)$	$\int_{-\infty}^{\infty} x f_X(x)dx$
	$\mathrm{E}(g(X))$	$\sum_x g(x)\ \mathrm{P}(X = x)$	$\int_{-\infty}^{\infty} g(x)f_X(x)dx$

For any random variables X:

Variance: $\mathrm{var}(X) = \mathrm{E}((X - E(X))^2) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

Note also:

$$\begin{aligned} \mathrm{E}(aX + b) &= a\ \mathrm{E}(X) + b, \\ \mathrm{var}(aX + b) &= a^2\ \mathrm{var}(X). \end{aligned}$$

For any random variables X, Y, Z and W:

Covariance: $\mathrm{cov}(X, Y) = \mathrm{E}((X - E(X))(Y - E(Y))) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)$

Correlation: $\rho(X, Y) = \mathrm{cov}(X, Y)/\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}, \quad -1 \leq \rho(X, Y) \leq 1$

Note also:

$$\begin{aligned} \mathrm{E}(X + Y) &= \mathrm{E}(X) + \mathrm{E}(Y), \\ \mathrm{var}(X + Y) &= \mathrm{var}(X) + \mathrm{var}(Y) + 2\mathrm{cov}(X, Y) \\ \mathrm{cov}(aX + b, cW + d) &= ac\ \mathrm{cov}(X, W), \\ \mathrm{cov}(X + Y, W + Z) &= \mathrm{cov}(X, W) + \mathrm{cov}(X, Z) + \mathrm{cov}(Y, W) + \mathrm{cov}(Y, Z) \end{aligned}$$

6. Standard distributions:	pmf or pdf	mean	variance
(a) Binomial; $B(n,p)$	$\mathrm{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
index n, parameter p	$k = 0,1,2,...,n$		
(b) Hypergeometric (N,k,n);	$P(X=x) = \binom{k}{x}\binom{N-k}{n-x}/\binom{N}{n}$	$nk/N \equiv np$	$np(1-p)\frac{N-n}{N-1}$
sample n from N, $p=k/N$	$x = \min(0,k+n-N),...,\max(n,k)$		
(c) Poisson with parameter μ, $\mathcal{P}o(\mu)$	$P(X=x) = \exp(-\mu)\mu^x/x!$ $x = 0,1,2,3,4.....$	μ	μ
(d) Geometric; $Geom(p)$;	$\mathrm{P}(X=k) = p(1-p)^{k-1}$	$1/p$	$(1-p)/p^2$
parameter p	$k = 1,2,3,4,......$		
(e) Neg. Binomial; $NegB(r,p)$;	$\mathrm{P}(X=k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$	r/p	$r(1-p)/p^2$
index r, parameter p	$k = r, r+1, r+2, ...$		
(f) Multinomial: $Mn(n,(p_1,..,p_k))$	$P(X_i = n_i, i = 1,...k) =$	$X_i \sim Bin(n,p_i),\ i = 1,...,k$	
$\sum_i p_i = 1 \quad \sum_i n_i = n$	$(n!/(n_1!)....(n_k!))\ p_1^{n_1}....p_k^{n_k}$		
(g) Uniform on (a,b); $U(a,b)$;	$f_X(x) = 1/(b-a), \quad a < x < b$	$(b+a)/2$	$(b-a)^2/12$
(h) Exponential; $\mathcal{E}(\lambda)$	$f_X(x) = \lambda \exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$
rate parameter λ	$0 \leq x < \infty$		
(j) Normal; $N(\mu,\sigma^2)$	$f_X(x) = (1/\sqrt{2\pi\sigma^2})\exp(-(x-\mu)^2/2\sigma^2)$ $-\infty < x < \infty$	μ	σ^2