16 + 15 + 15 (46) well done

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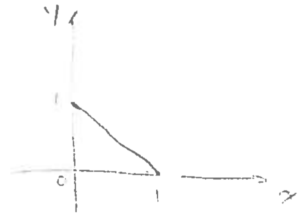
Q1.



(a) $(x+y) \le 1$

$y \le 1-x$

$\therefore 0 \le y \le 1-x$

also $x \le 1-y$

since $y \ge 0$

so $x \le 1$

$f_X(x) = \int_0^{1-x} 2\, dy$

$= [2y]_0^{1-x}$

$= 2(1-x)$

so $f_X(x) = \begin{cases} 2(1-x), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

(b) $E(X) = \int_0^1 x\, f_X(x)\, dx = \int_0^1 2x(1-x)\, dx = \int_0^1 2x - 2x^2\, dx = [x^2 - \frac{2}{3}x^3]_0^1$

$= 1 - \frac{2}{3}$

$= \frac{1}{3}$

$Var(X) = E(X^2) - (EX)^2 = \int_0^1 x^2 \cdot f_X(x)\, dx - (\frac{1}{3})^2$

$= \int_0^1 x^2 \cdot 2(1-x)\, dx - \frac{1}{9}$

$= [\frac{2}{3}x^3 - \frac{1}{2}x^4]_0^1 - \frac{1}{9}$

$= \frac{2}{3} - \frac{1}{2} - \frac{1}{9}$

$= \frac{1}{18}$

(c) $E(XY) = \iint_{\substack{0 \le x,\\ 0 \le y\\ x+y \le 1}} xy \cdot f_{XY}(x,y)\, dx\, dy = \int_0^1 y\left(\int_0^{1-y} x \cdot 2\, dx\right) dy$

* $x + y \le 1$
$0 \le x \le 1-y$

$= \int_0^1 y\left([x^2]_0^{1-y}\right) dy$

$= \int_0^1 y \cdot (1-y)^2\, dy$

$= \int_0^1 y - 2y^2 + y^3\, dy$

$= [\frac{1}{2}y^2 - \frac{2}{3}y^3 + \frac{1}{4}y^4]_0^1$

$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12}$

(d) $Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{E(XY) - E(X)E(Y)}{\sqrt{Var(X)\,Var(Y)}}$

By symmetry. $E(Y) = E(X) = \frac{1}{3}$, $Var(Y) = Var(X) = \frac{1}{18}$

so $Corr(X,Y) = \frac{\frac{1}{12} - \frac{1}{3} \times \frac{1}{3}}{\sqrt{\frac{1}{18} \times \frac{1}{18}}} = \frac{\frac{1}{12} - \frac{1}{9}}{\frac{1}{18}} = \frac{3-4}{2} = -\frac{1}{2}$

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Q2:

(a) $E(X) = E(3+3Z_1+Z_2) = 3+3E(Z_1)+E(Z_2) = 3+0+0=3$

$Var(X) = Var(3+3Z_1+Z_2) = 9\,Var(Z_1) + Var(Z_2) \Leftarrow$ since Z_1 and Z_2 are independent

$= 9+1 = 10$

$E(Y) = E(6-2Z_1+3Z_3) = 6-2E(Z_1)+3E(Z_3) = 6$

$Var(Y) = Var(6-2Z_1+3Z_3) = 4\,Var(Z_1)+9\,Var(Z_3) \Leftarrow$ since Z_1 and Z_3 are independent

$= 13$

(b) $Cov(X,Y) = Cov(3+3Z_1+Z_2,\ 6-2Z_1+3Z_3)$

$= Cov(3Z_1, -2Z_1)$ why -1

$= -6\,Cov(Z_1, Z_1)$

$= -6\,Var(Z_1)$

$= -6$

(c) $E(Y-3X) = E(Y) - 3E(X) = 6-3\times3 = 6-9 = -3$

$Var(Y-3X) = Var(Y) + 9\,Var(X) + 2\,Cov(Y,-3X)$

$= 13+9\times10 - 6\,Cov(X,Y)$

$= 13+90-6\times(-6)$

$= 13+90+36$

$= 139$

(d) because Z_1, Z_2 and Z_3 are independent normal variables, X and Y are also normal distributed, and thus $Y-3X$ is also normal distributed with mean (-3), variance (139) from part (c)

$P(Y+3>3X) = P(Y-3X>-3)$

$(Y-3X) \sim N(-3,\ 139)$

so -3 is actually the mean of $Y-3X$.

And because the normal distribution is symmetric.

$\Rightarrow P(Y+3>3X) = P(Y-3X>-3) = \frac{1}{2}$.

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Q3

(a) Let W be number of mugs unpacked up to and including the first blue mug from the question we have $W \sim Geom(\frac{1}{3})$

$$E(W) = \frac{1}{p} = \frac{1}{\frac{1}{3}} = 3$$

$$P(W=3) = \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^1 = \frac{4}{27}$$

(b) P(unpack only 12 mugs) = P(only unpack 8 white mug ∩ only unpack 4 blue mugs)

$$= \binom{12}{8} \left(\frac{2}{3}\right)^8 \left(\frac{1}{3}\right)^4$$

What distribution?

why

-1

(c) since

(d) number of ways that there are 4 white mugs and 4 blue mugs on each shelf.

$$\binom{8}{4}\binom{8}{4}$$

number of ways that 16 mugs can be displayed on shelves $\binom{16}{8}$

so $P(\text{4 white \& 4 blue on each shelf}) = \frac{\binom{8}{4}\binom{8}{4}}{\binom{16}{8}}$

(c) if last mugs ~~pick~~ unpacked is blue mugs, there will be 7 blue mugs and 8 whites mugs always, which satisfies (at least 8 white & at least 8 blue so the students will not unpack one more mugs so the last mug must be white

let W_r be number of mugs unpack ~~and~~ up to and including the 8th mug

$W_r \sim NegB(8, \frac{2}{3})$.

$$P(W_r = 16) = \binom{15}{7} \left(\frac{2}{3}\right)^7 \left(\frac{1}{3}\right)^8 \times \left(\frac{2}{3}\right) = \binom{15}{7}\left(\frac{2}{3}\right)^8\left(\frac{1}{3}\right)^8$$