Lecture 1: Sample spaces and events: Larson & Marx, 2.2; Sept 26, 2018

1.1 Sample spaces

The *sample space* Ω is the set of all possible outcomes of an experiment.

One and only one outcome can occur.

1.2 Examples

(i) Child is boy or girl: $\Omega = \{\text{boy, girl}\}$

(ii) Toss of one die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

(iii) Number of traffic accidents: $\Omega = \{0, 1, 2, 3, 4, ...\} = \{0\} \cup \mathcal{Z}_+$.

(iv) Time waiting for the bus: $\Omega = (0, \infty) = \Re_+$, the positive half line.

1.3 Events

Any subset E of Ω is a *event*.

(The book says this: it is OK for countable sample spaces, but an oversimplification for a space like $\Re_+$.)

1.4 Combining events

(i) If E is an event, not-E (the *complement* of E: written E^c) is an event.

(ii) If E is an event and F is an event, then "E and/or F" is an event. "E and/or F" is written $E \cup F$.

(iii) If E_1, E_2, ... are events then $E_1 \cup E_2 \cup E_3$.... is an event. (Countable unions.)

(iv) If E is an event and F is an event, then "E and F" is an event. "E and F" is written $E \cap F$.

(v) If E_1, E_2, ... are events then $E_1 \cap E_2 \cap E_3$.... is an event. (Countable intersections.)

1.5 More events:

(i) The empty set Φ is an event: so $\Omega = \Phi^c$ is an event.

If $E \cap F = \Phi$, E and F are *disjoint* also known as *mutually exclusive*.

(ii) E and E^c are *disjoint events*: $E \cup E^c = \Omega$, $E \cap E^c = \Phi$.

(iii) Events E_1, E_2 .....E_k are *mutually exclusive* if $E_i \cap E_j = \Phi$ for all pairs (i, j) $(i, j = 1, ..., k,\ i \neq j)$.

(iv) Events E_1, E_2 .....E_k are *exhaustive* if $E_1 \cup E_2 \cup .... \cup E_k = \Omega$.

(v) If Ω is discrete, the elements of Ω are a set of mutually exclusive and exhaustive events.

1.6 A genetic example: The ABO blood types.

We can be blood type A, B, AB or O.

Let our "experiment" be finding the blood types of two children in a family.

Then $\Omega = \{(i, j); i, j = A, B, AB, O\}$.

Let E_1 be the first child is type A. Let E_2 be the first child is type O.

Let E_3 be the second child is type B.

Let E_4 be at least one child is type A. Let E_5 be at most one child is type A.

Let E_6 be the two children have the same blood type.

Let E_7 be the two children have different blood types.

Which pairs of events are *complements*?

Which pairs of events are *disjoint*?

Which pair of events is *mutually exclusive and exhaustive*?

What is the intersection of E_4 and E_5?

Lecture 2: Probabilities of Events: Larson & Marx 2.3; Sept 28,2018 (a)

2.1 Probability axioms

For each event E we assume we can assign a number $P(E)$ which satisfies the following three axioms:

(i) $P(E) \geq 0$ for every event E.

(ii) $P(\Omega) = 1$

(iii) If $E_1, E_2, ...$ are *mutually exclusive* $P(E_1 \cup E_2 \cup E_3 \cup ....) \ = \ P(E_1) + P(E_2) + P(E_3) + ...$

Note: for a countable sample space, each outcome (element of Ω) has a probability, and each event is a union of outcomes, with probability the sum of the probabilities of the outcomes.

2.2 Probability interpretation as a limiting frequency

A useful *interpretation* of $P(E)$ is that it is the proportion of times an outcome in E occurs in a large number of repetitions of the same experiment with outcomes in the sample space Ω.

Example: Sampling an individual from a very large population.

$\Omega = \{A, B, AB, O\}$.

$P(A)$ can be interpreted as the proportion of A blood-type individuals in the population. If we repeat the sampling of an individual again, and again, the proportion of times we observe the individual to have blood type A is $P(A)$.

For the USA population, roughly, $P(A) = 0.41$, $P(B) = 0.16$, $P(AB) = 0.07$, and $P(O) = 0.36$.

$P(\text{antigen A on red blood cells}) \ = \ P(\{A\} \cup \{AB\}) \ = \ P(A) + P(AB) \ = \ 0.48$ for this example.

2.3 Basic probability formulae

(i) $\Omega \ = \ E \cup E^c$, $E \cap E^c \ = \ \Phi$, so $P(E^c) + P(E) \ = \ P(\Omega) \ = \ 1$, or $P(E^c) \ = \ 1 - P(E)$.

This also shows $P(E) \leq 1$, since all probabilities are non-negative.

(ii) If $E \subset F$, $F \ = \ E \cup (F \cap E^c)$; $P(F) \ = \ P(E) + P(F \cap E^c) \ \geq \ P(E)$.

(iii) $E \cup F \ = \ E \cup (E^c \cap F)$, so $P(E \cup F) \ = \ P(E) + P(E^c \cap F)$.

So $P(E \cup F) + P(E \cap F) \ = \ P(E) + P(E^c \cap F) + P(E \cap F) \ = \ P(E) + P(F)$,

or $P(E \cup F) \ = \ P(E) + P(F) - P(E \cap F)$. **Note:** $P(E \cup F) \ \leq \ P(E) + P(F)$ **always**.

2.4 Two important probability formulae

(i) Law of total probability

Suppose $E_1, E_2, ...,$ form a *partition* of Ω. That is, $E_1, E_2, ...$ are mutually exclusive and exhaustive.

That is, $E_i \cap E_j \ = \ \Phi$ (disjoint), and $\Omega = E_1 \cup E_2 \cup ....$

Then for any event F, $F \ = \ \bigcup_i (F \cap E_i)$, $P(F) \ = \ \sum_i P(F \cap E_i)$.

Special case: if e_i is ith outcome in a countable Ω, and $E_i = \{e_i\}$,

$$F \cap E_i \ = \ E_i \text{ or } F \cap E_i \ = \ \Phi, \text{ and } P(F) = \sum_{e_i \in F} P(E_i).$$

(ii) The inclusion and exclusion formula

$$\begin{aligned}
P(D \cup E) &= P(D) + P(E) - P(D \cap E). \\
P(C \cup D \cup E) &= P(C) + P(D) + P(E) - P(C \cap D) - P(D \cap E) - P(C \cap E) + P(C \cap D \cap E). \\
P(E_1 \cup E_2 \cup ... \cup E_k) &= P(E_1) + P(E_2) + ... + P(E_k) \\
&\quad -P(E_1 \cap E_2) - \text{all the other 2-way} \\
&\quad +P(E_1 \cap E_2 \cap E_3) + \text{all the other 3-way} \\
&\quad -P(E_1 \cap E_2 \cap E_3 \cap E_4) - \text{all the other 4-way} \ \ ...... \pm P(E_1 \cap E_2 \cap .... \cap E_k).
\end{aligned}$$

Note: This sum of positive and negative terms may not be well defined as $k \to \infty$: see 4.3

Lecture 3: Permutations and combinations: Larson & Marx 2.6; Sept 28,2018 (b)

3.1 Basic principle of counting

If an experiment has k steps, and if earlier choices do NOT limit later ones, then if step-1 can be done in n_1 ways, step-2 in n_2 ways, ... step-k in n_k ways,

then there are $n_1 \times n_2 \times ... \times n_k$ possible outcomes for (step-1, ..., step-k).

Corollary: There are 2^k subsets of a set size k.

Proof: Each element i, $i = 1, ..., k$ can be chosen, or not: $n_i = 2$, $i = 1, ..., k$.

So total possible is $2 \times 2 \times .... \times 2 = 2^k$.

Note: for proper (not Ω), non-empty (not Φ) subsets, there are $2^k - 2$.

3.2 Permutations and combinations

(i) The number of ways of ordering n distinct objects is $n(n-1)(n-2)....3.2.1 \ = \ n!$ (n-factorial).

(ii) The number of ways of choosing k distinct objects, in order, from n is $n(n-1)....(n-k+1) \ = \ n!/(n-k)!$.

(iii) If we do not care about the order in which the k objects are selected, there are $k!$ selections that give the same *combination*.

That is there are $n!/((n-k)!k!)$ distinct *combinations*: this is often written ${}_nC_k$ or $\binom{n}{k}$.

(iv) A useful formula: (L & M, P.88)

$$\binom{n}{k} \ = \ \binom{n-1}{k-1} + \binom{n-1}{k}$$

Consider the number of choices that do and do not contain the particular object "Fred".

3.3 The binomial theorem; L&M 2.6 P.87

$$(x+y)^n \ = \ \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Note in each bracket we choose x or y. There are 2^n sequences.

The number of sequences in which there are k choices of x is $\binom{n}{k}$, and each has value $x^k y^{n-k}$.

The case $x = y = 1$ gives $\binom{n}{k}$ as the number of sizs-k subsets of n objects.

The case of 4.4 (ii) (next page) is a special case with $x = y = \frac{1}{2}$.

3.4 Multinomial combinations; L & M Theorem 2.6.2, P.86

Number of ways of arranging n_1 objects type-1, n_2 objects type-2, ... n_k objects type-k,

where $n_1 + n_2 + .. + n_k \ = \ n$:

Choose the n_1 positions for type 1: $\binom{n}{n_1} \ = \ n!/(n_1!(n-n_1)!)$.

Now out of the remaining $(n - n_1)$ positions choose n_2 for type-2:

number of ways $= \binom{n-n_1}{n_2} \ = \ (n-n_1)!/(n_2!(n-n_1-n_2)!)$. etc. ...

Total number of ways is

$$\frac{n!}{n_1!(n-n_1)!} \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \frac{(n-n1-n_2)!}{n_3!(n-n_1-n_2-n_3)!} ... \frac{(n-n1-n_2-...-n_{k-1})!}{n_k!0!} \ = \ \frac{n!}{n_1! \ n_2!....n_k!}$$

Example: Twelve students go to donate blood: 5 are type A, 2 are type B, one is AB, and 4 are type O. How many different orderings of the types of blood in the 12 blood donation tubes are there?

Answer: $12!/(5! \times 2! \times 1! \times 4!) \ = \ (12.11.10.9.8.7.6)/(2.4.3.2) \ = \ 12.11.10.9.7 \ = \ 914,760$.

4: Some additional notes on sets, probabilities, and combinatorics

4.1 The collection of all events

For (finite or) countable Ω, events are all subsets of Ω, but this does not work for $\Omega = \Re$.

More generally, Ω is event, E an event $\Rightarrow E^c$ an event, and $E_1, E_2, ...$ events $\Rightarrow \bigcup_{i=1}^{\infty} E_i$ an event.

(Such collections, closed under complements and countable unions, are called σ-fields.)

Note $\Phi = \Omega^c$ is an event, and $\bigcap E_i = (\bigcup E_i^c)^c$ are then also events.

4.2 Increasing and decreasing sequences of events

(i) $A_1,\ A_2,\ , A_3, ...$ are nested increasing sets if $A_1 \subset A_2 \subset A_3 \subset .....$ Then $\bigcup_1^n A_i = A_n$ and $\bigcap_1^n A_i = A_1$.

(ii) $A_1,\ A_2,\ , A_3, ...$ are nested decreasing sets if $A_1 \supset A_2 \supset A_3 \supset .....$ Then $\bigcup_1^n A_i = A_1$ and $\bigcap_1^n A_i = A_n$.

Example: In a sequence of tries (maybe not independent), let A_n be event of no successes in n tries, a decreasing sequence. So $\lim_{n\to\infty} A_n$ is event of no success ever: $D_n = A_n^c$ (success by try n) is increasing.

4.3 Nested sets Theorem: Let $A_1, A_2,\ ....$ be any events in Ω.

(i) If $A_1 \subset A_2 \subset A_3 \subset ....$, $P(A_1 \bigcup A_2 \bigcup A_3 .....) = \lim_{n\to\infty} P(A_n)$.

(ii) If $A_1 \supset A_2 \supset A_3 \supset ....$, $P(A_1 \bigcap A_2 \bigcap A_3 .....) = \lim_{n\to\infty} P(A_n)$.

Proof: (i) Let $B_i = A_i \cap A_{i-1}^c$; Then B_i are disjoint and $B_1 \bigcup B_2 \bigcup ... \bigcup B_n = A_1 \bigcup A_2 \bigcup ... \bigcup A_n = A_n$,

so $P(A_1 \bigcup A_2 \bigcup A_3 .....) = P(B_1 \bigcup B_2 \bigcup B_3 .....) = \sum_1^{\infty} P(B_i) = \lim_{n\to\infty}(\sum_1^n P(B_i))$

$$= \lim_{n\to\infty} P(B_1 \bigcup B_2 \bigcup .... \bigcup B_n) = \lim_{n\to\infty} P(A_n).$$

(ii) Let $D_i = A_i^c$, so from (i) $P(D_1 \bigcup D_2 \bigcup D_3 .....) = \lim_{n\to\infty} P(D_n)$.

But $P(D_1 \bigcup D_2 \bigcup D_3 .....) = P((A_1 \bigcap A_2 \bigcap ...)^c) = 1 - P(A_1 \bigcap A_2 \bigcap ....)$.

So $P(A_1 \bigcap A_2 \bigcap ....) = 1 - P(D_1 \bigcup D_2 \bigcup D_3 .....) = \lim_{n\to\infty}(1 - P(D_n)) = \lim_{n\to\infty} P(A_n)$.

Continuing above example: In a sequence of trials, let $A_n = \{$ no success by trial $n\}$ and $D_n = A_n^c$.

(i) If trials are **independent**, with constant probability of success $p > 0$,

$P(A_n) = (1-p)^n \longrightarrow 0$. Eventually, with probability 1, a success occurs.

(ii) Suppose the probability of success on try k is p_k. Then $P(D_n) \le \sum_{k=1}^n p_k$.

If p_k decreases fast (e.g. $p_k = 0.1/k^2$) then $\lim P(D_n) < 1$; eventual success is not certain.

4.4 Binomial counts and Stirling's formula L&M 2.6, P. 76-77

(i) Suppose there are N equiprobable outcomes in Ω.

Suppose event E is true for R of these outcomes. Then $P(E) = R/N$.

(ii) An AB parent and an O parent can have an A child or a B child.

Suppose they have n children: there are 2^n possible sequences of A and B children.

Assume these are equiprobable. (In fact, they are.)

$\binom{n}{k}$ of these sequences have k A children. $\quad P(k\ A \text{ children out of } n) = \binom{n}{k}/2^n$.

(iii) $n!$ can be approximated for large n by $\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$. Also $\binom{n}{k}$ is largest when $k \approx n/2$.

Then, for large n,

$$\binom{n}{n/2} = \frac{n!}{(n/2)!(n/2)!} \approx \frac{\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}}{\sqrt{2\pi}(n/2)^{(n/2)+\frac{1}{2}} e^{-(n/2)} \times \sqrt{2\pi}(n/2)^{(n/2)+\frac{1}{2}} e^{-(n/2)}}$$

$$= (1/\sqrt{2\pi}) 2^{n+1} n^{n+\frac{1}{2}-(n/2)-\frac{1}{2}-(n/2)-\frac{1}{2}} = (1/\sqrt{2\pi})(2/\sqrt{n}) 2^n$$

$$\text{Or } P((n/2)\ A \text{ children out of } n) = \binom{n}{n/2}(\frac{1}{2})^n \approx 1/\sqrt{2\pi(n/4)}$$

This result will come back in approximating Binomial probabilities by the Normal probability distribution.