Explain/justify your answers: the explanation is more important than the numerical answer.
All numerical answers may be left as numerical expressions.
An information page (2-sided) is provided.
This exam has 5 questions; it continues on the back.

1. (*12 points: 4 each part*)
In a certain population, everyone is equally susceptible to colds. The number of colds suffered by each person during each winter season ranges from 0 to 4 with probability 0.2 for each value (see table).
A new cold prevention drug is introduced, which, *for people for whom the new drug is effective* changes the probabilities as shown. Unfortunately, the effects of the drug last **only** the duration of one winter season, and the drug is **only effective** in 20% of people.

	Probabilities for X colds				
X =	0	1	2	3	4
No drug or drug ineffective	0.2	0.2	0.2	0.2	0.2
Drug effective	0.4	0.3	0.2	0.1	0.0

(a) Fred decides to take the drug. Given that he gets 1 cold that winter, what is the conditional probability that the drug is effective for Fred?
(b) The next year he takes the drug again. Given that he gets 2 colds in this winter, what is the updated conditional probability that the drug is effective for Fred?
(c) The third winter he decides not to bother to take the drug, and he gets 2 colds. He therefore argues that the drug must not have been effective for him, since he got just as few colds the year he did not take it. Comment on his logic.

2. (*12 points; 3 each part*)
Suppose Z_1, Z_2, and Z_3 are independent Normal random variables. Each of Z_i $(i = 1, 2, 3)$ has expectation 1, and Z_i has variance i $(i = 1, 2, 3)$. That is, $Z_1 \sim N(1, 1), Z_2 \sim N(1, 2)$, and $Z_3 \sim N(1, 3)$.
Let $X = aZ_1 + bZ_2$, $Y = Z_1 + 2Z_2 + 3Z_3$.
(a) For what values of a and b (if any) do X and Y have the same expectation?
(b) For what values of a and b (if any) do X and Y have the same variance?
(c) For what values of a and b (if any) do X and Y have the same distribution?
(d) For what values of a and b (if any) does $\mathrm{cov}(X, Y) = 0$?
Hint: First show $E(Y) = 6$ *and* $var(Y) = 36$.

3. (*12 points: 3 each part*)
A common condition C in new-born babies occurs with probability $p = 0.15$, and a much rarer birth defect D occurs with probability $p^* = 1/1000 = 0.001$. Each of the conditions C and D occur apparently independently and at random in any baby.
In King County (K) in 2014, there are 10,000 births; 100 of these births take place at a small hospital H.
(a) What probability distribution provides the most accurate model for computing the probability that more than 1600 babies with condition C are born in the county K in 2014? What *other* probability distribution might be more computationally convenient, and would provide a good approximation for this probability?
(b) In fact, in 2014, it was observed that in the county K exactly 1500 babies were born with C. What probability distribution provides the most accurate model for computing the probability that exactly 12 of these 1500 babies with C were born at hospital H? What *other* probability distribution might be more computationally convenient, and would provide a good approximation for this probability?

Question 3 continued

(c) What probability distribution provides the most accurate model for computing the probability that exactly 5 babies with the more serious condition D are born in county K in 2014? What *other* probability distribution might be more computationally convenient, and would provide a good approximation for this probability?

(d) Hospital H notes that the profit they make on each birth is \$X, where X is uniformly distributed between 0 and 200. (Thanks to medical insurance, the profit does not depend on the condition of the baby.)

What would be a computationally convenient approximation for the probability distribution of the average profit per birth made by hospital H in 2014?

(Justify your answers, stating assumptions where necessary. Be sure to specify the parameters of the distributions you name; you may leave these parameter values as numerical expressions).

4. (*12 points; 4 each part*)

In a busy 24-hour copy center, malfunctions of the copy machine occur as a Poisson process, rate 3/hour. That is, the numbers of malfunctions in disjoint time intervals are independent, and the number in time interval t hours has a Poisson distribution with mean $3t$.

(These are minor malfunctions such as bad copies – not breakdowns causing an interruption in service.)

(a) On a certain day, Fred counts the machine malfunctions from 2 p.m. to 8 p.m. and Jon separately counts the malfunctions from 4 p.m. to 10 p.m. What is the expected number of malfunctions counted by Jon, and what is the variance? What is the covariance between the number counted by Fred, and the number counted by Jon?

(b) Given that a particular malfunction occurs during Jon's 4 p.m. to 10 p.m. shift, what is the probability it occurs before Fred goes home at 8 p.m.?

Given that Jon counts a total of 27 malfunctions during his shift, what are the mean and variance of the *total* number counted by Fred on that same day?

(c) Instead of ending his shift at the fixed time of 10.00 p.m., Jon agrees to stay at the copy center until 18 malfunctions have occurred during his shift. What are the mean and variance of the length of his shift? Write the probability he has to stay later than 10.00 p.m. as the probability a Poisson random variable with a mean you should specify is less than some number that you should also specify.

(**Do NOT attempt to evaluate this probability**).

Hint for (a) and (b); Split Fred's hours into the two independent parts; 2-4 p.m. and 4-8 p.m.

5. (*12 points: 4 each part*)

The jointly continuous random variables X and Y have joint density function (pdf)

$f_{X,Y}(x, y) \ = \ e^{-y}/y$ on the set $0 < x < y < \infty$ and $f_{X,Y}(x, y) \ = \ 0$ for all other (x, y).

(a) Show that the marginal pdf, $f_Y(y)$ of Y, is $f_Y(y) \ = \ e^{-y}$ on $0 < y < \infty$ and find the conditional pdf of X given that $Y = y$.

(b) Using this conditional pdf from (a), show that $\mathrm{E}(X \mid Y = y) = \frac{1}{2}y$.

Hence show that $\mathrm{E}(X) \ = \frac{1}{2}\mathrm{E}(Y) \ = \frac{1}{2}$, and $\mathrm{E}(XY) = \frac{1}{2}\mathrm{E}(Y^2)$ so that $\mathrm{cov}(X, Y) \ = \frac{1}{2}\mathrm{var}(Y) \ = \frac{1}{2}$.

(You may cite standard results about exponential random variables.)

(c) By considering the conditional probabilities for $W = X/Y$ given that $Y = y$, show that W and Y are independent. What is the distribution of W?