Rewrite.

Final Q1 (a). Let S_2 denotes getting first success on 2nd trial.

Want to know $P(A=G \mid S_2)$

$P(A=G \mid S_2) = P(S_2 \mid A=G) \cdot P(A=G) / P(S_2)$

$P(S_2 \mid A=G) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(S_2 \mid B=G) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$

Since $A=G$ and $B=G$ are all cases,

$P(S_2) = P(S_2 \mid A=G) \cdot P(A=G) + P(S_2 \mid B=G) \cdot P(B=G)$

Note that $P(A=G) = P(B=G) = \frac{1}{2}$

$P(S_2) = \frac{1}{4} \cdot \frac{1}{2} + \frac{2}{9} \cdot \frac{1}{2} = \frac{1}{8} + \frac{1}{9} = \frac{17}{72}$

$P(A=G \mid S_2) = P(S_2 \mid A=G) \cdot P(A=G) / P(S_2)$

$= \frac{1}{4} \cdot \frac{1}{2} / \frac{17}{72}$

$= \frac{9}{17}$

Q1 (b). Let S_3 denotes getting first success on 3rd trial.

Want to know $P(B=R \mid S_3)$

$P(B=R \mid S_3) = P(S_3 \mid B=R) \cdot P(B=R) / P(S_3)$

$P(S_3 \mid B=R) = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{27}$

$P(S_3 \mid A=R) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

$P(S_3) = P(S_3 \mid B=R) \cdot P(B=R) + P(S_3 \mid A=R) \cdot P(A=R)$

$= \frac{4}{27} \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{2} = \frac{2}{27} + \frac{1}{16} = \frac{59}{432}$

$P(B=R \mid S_3) = P(S_3 \mid B=R) \cdot P(B=R) / P(S_3)$.

$= \frac{4}{27} \cdot \frac{1}{2} / \frac{59}{432}$

$= \frac{32}{59}$

Rewrite

Q1 (c). Let R_1 denotes getting report that the facility had green chips getting first sucess on second trial.
Let R_2 denotes getting report that facility tested red one gets first sucess on third trial
Let M_1 denotes the match that $A=G$ $B=R$.
Let M_2 denotes the match $A=R$ and $B=G$.

Want to know $P(M_1 \mid R_1, R_2)$; (R_1, R_2) denotes getting both report.!

$P(M_1 \mid R_1, R_2) = P(R_1, R_2 \mid M_1) \cdot P(M_1) \,/\, P(R_1, R_2)$.

Since getting first success in these two facilities are independent.

$$
\begin{aligned}
P(R_1, R_2 \mid M_1) &= P(R_1 \cap R_2 \mid M_1) \\
&= P((R_1 \mid M_1) \cap (R_2 \mid M_1)) \\
&= P(R_1 \mid M_1) \cdot P(R_2 \mid M_1) \\
&= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{27}
\end{aligned}
$$

Similarly $P(R_1, R_2 \mid M_2) = P(R_1 \mid M_2) \cdot P(R_2 \mid M_2)$

$$= \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{36}$$

Note that $P(M_1) = P(M_2) = \frac{1}{2}$ and

$$
\begin{aligned}
P(R_1, R_2) &= P(R_1, R_2 \mid M_1) \cdot P(M_1) + P(R_1, R_2 \mid M_2) \cdot P(M_2) \\
&= \frac{1}{2}\left(\frac{1}{27} + \frac{1}{36}\right) = \frac{7}{216}
\end{aligned}
$$

$$
\begin{aligned}
P(M_1 \mid R_1, R_2) &= P(R_1, R_2 \mid M_1) \cdot P(M_1) \,/\, P(R_1, R_2) \\
&= \frac{1}{27} \cdot \frac{1}{2} \,/\, \frac{7}{216} \\
&= \frac{4}{7}.
\end{aligned}
$$

With combined reports, the probability that A is green and B is red equals $\frac{4}{7}$

Rewrite

Q2 (b). Known $X \sim \mathcal{E}(2)$ $Y \sim \mathcal{E}(1)$ X, Y independent.
Want to know distribution of $2X$ and $P(2X < Y)$

Since $X \sim \mathcal{E}(2)$, the cdf of X $F_X(x) = P(X < x) = 1 - e^{-2x} \ (x > 0)$
the cdf of $2X$ $F_{2X}(x) = P(2X < x)$

$$= P(X < \tfrac{x}{2})$$
$$= 1 - e^{-x}$$

$$f_{2X}(x) = \frac{d}{dx} F_{2X}(x) = e^{-x}$$

Thus $2X \sim \mathcal{E}(1)$ with pdf $f_{2X}(x) = e^{-x}$.

$$P(2X < Y) = \int_0^{\infty} \int_0^{y} e^{-x} \cdot e^{-y} \, dx \, dy$$
$$= \int_0^{\infty} \left(\int_0^{y} e^{-x} dx \right) e^{-y} \, dy$$
$$= \int_0^{\infty} (-e^{-2y} + e^{-y}) \, dy$$
$$= \left(\tfrac{1}{2} e^{-2y} - e^{-y} \right) \Big|_0^{\infty}$$
$$= \tfrac{1}{2}$$

Rewrite.

Q4 (a). Let C denotes number of C salmon in t hours
Let S denotes # S salmon in t hours.
Let N denotes total # salmon in t hours, $N = S + C$.

Known: $N \sim Po(5t)$, $P(C) = \frac{2}{5}$ $P(S) = \frac{3}{5}$.

For each fish, it's either C or S,
the event that getting x C salmon among $(x+y)$ fishes is binomial.
$C \sim Bin(\frac{2}{5}, N = C+S)$.

$P(C = x \mid N = C+S = x+y) = \binom{x+y}{x} \frac{2}{5}^{x} \cdot \frac{3}{5}^{y}$.

$P(C = x, S = y \mid N = C+S = x+y) = P(C = x \mid N = C+S = x+y) = \frac{(x+y)!}{x!\,y!} \frac{2}{5}^{x} \frac{3}{5}^{y}$

$N \sim Po(5t)$, $P(N = x+y) = e^{-5t} 5t^{x+y} / (x+y)!$

$$\begin{aligned}
P(C=x, S=y) &= P(C=x, S=y \mid N = x+y) \cdot P(N = x+y) \\
&= \left(\frac{(x+y)!}{x!\,y!}\left(\frac{2}{5}\right)^{x} \cdot \left(\frac{3}{5}\right)^{y}\right) \cdot \left(e^{-5t} (5t)^{x+y} / (x+y)!\right) \\
&= \left(\left(\frac{2}{5}\right)^{x} \cdot \left(\frac{3}{5}\right)^{y} \cdot e^{-5t} \cdot (5t)^{x+y}\right) / (x! \cdot y!) \\
&= \left((2t)^{x} \cdot (3t)^{y} \cdot e^{-2t} \cdot e^{-3t}\right) / (x! \cdot y!) \\
&= \left(e^{-2t} \cdot (2t)^{x} / x!\right) \cdot \left(e^{-3t} \cdot (3t)^{y}\right) / y! \\
&= P(Po(2t) = x) \cdot P(Po(3t) = y).
\end{aligned}$$

This proves C and S are independent, for $P(S \cap C) = P(S) \cdot P(C)$.

$P(\text{success} \mid A) = \frac{1}{2}$ $\quad P(\text{success} \mid B) = \frac{1}{3}$

in green

a. $P(\text{1st not success and 2nd success} \mid A) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ ✓

$P(\text{1st} \ldots \mid B) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$ ✓

$P(\text{1st not success, 2nd success}) = \frac{1}{2} \cdot \frac{2}{9} + \frac{1}{4} \cdot \frac{1}{2} = \frac{17}{36}$ ← forgot $P(A)$ & $P(B)$

$P(A \mid \text{1st not success, 2nd success}) = P(\text{2nd success} \mid A) \cdot P(A) / P(\text{2nd success})$

$= \frac{1}{4} \cdot \frac{1}{2} / \frac{17}{36}$

$= \frac{9}{34} \; / \frac{1}{2} = \frac{9}{17}$ $\quad$ −2

in red

b. $P(\text{1, 2 not success 3 success} \mid A) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$ ✓

$P(\text{1, 2 not success 3 success} \mid B) = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{27}$ ✓

$P(\text{3rd is success}) = \frac{1}{8} + \frac{4}{27} = \frac{5}{18}$ ✗ same error $\quad$ −1

$P(B \mid \text{3rd is success}) = P(\text{3rd is success} \mid B) \cdot P(B) / P(\text{3rd is success})$

$= \frac{4}{27} \cdot \frac{1}{2} / \frac{5}{18}$

$= \frac{2}{45}$

c. $P(\text{green is A and red is B}) = \frac{4}{17} \cdot \frac{2}{45}$.

✗ $\quad$ −4

2. $X \sim \mathcal{E}(2)$ $Y \sim \mathcal{E}(1)$

a. $P(X<Y) = \int_0^\infty \int_0^y 2e^{-2x} \cdot e^{-y} \, dx \, dy$

$= \int_0^\infty -e^{-2x} \cdot e^{-y} \Big|_0^y \, dy$

$= \int_0^\infty -e^{-3y} + e^{-y} \, dy$

$= \frac{1}{3} e^{-3y} - e^{-y} \Big|_0^\infty$

$= 1 - \frac{1}{3} = \frac{2}{3}$.

b. 2X still exponential distribution. why

$f(2x) = 2e^{-4x}$.

$P(2X<Y) = \int_0^\infty \int_0^{\frac{1}{2}y} 2e^{-4x} \cdot e^{-y} \, dx \, dy$.

$= \int_0^\infty -\frac{1}{2} e^{-4x} \cdot e^{-y} \Big|_0^{\frac{1}{2}y} \, dy$

$= \int_0^\infty -\frac{1}{2} e^{-3y} \cdot \quad + \frac{1}{2} e^{-y} \, dy$

$= \frac{1}{2} \int_0^\infty -e^{-3y} + e^{-y} \, dy$

$= \frac{1}{2} P(X<Y) = \frac{1}{3}$

-2

c. $W = \min(X, Y)$. if $W>w$ then both $X>w$ and $Y>w$

$P(W>w) = P(X>w) \cap P(Y>w)$ since X Y indep.

$= P(X>w) \cdot P(Y>w)$.

$= \int_w^\infty 2e^{-2x} dx \cdot \int_w^\infty e^{-y} dy$

$= (-e^{-2x} \Big|_w^\infty) \cdot (-e^{-y} \Big|_w^\infty)$

$= e^{-3w}$

so $W \sim \mathcal{E}(3)$ with mean $\frac{1}{3}$

d. $V = \max(X, Y)$ so if $V \le v$, $X<v$ and $Y<v$.

$P(V \le v) = P(X \le v) \cap P(Y \le v)$

indep

$= P(X \le v) \cdot P(Y \le v)$.

$\longrightarrow$

$= P(2A) \cup P(2B) \cup P(2AB) \cup P(2O)$

3. a. i $P(\text{first two have same}) = 0.4 \cdot 0.4 + 0.2 \cdot 0.2 + 0.1 \cdot 0.1 + 0.3 \cdot 0.3$

ii. $P(\text{first two have same} \mid \text{neither } O)$

$$= \frac{P(2A) \cup P(2B) \cup P(2AB)}{P(\text{neither } O)}$$

$$= \frac{0.4 \cdot 0.4 + 0.2 \cdot 0.2 + 0.1 \cdot 0.1}{(1-0.3)(1-0.3)}$$

b. i let X denotes # people have A.

mean / sd ?

$-\frac{1}{2}$

$X \sim N(1000, 0.4)$

$P(X \geq 370) = P(N(1000, 0.4) \geq 369.5)$ by correction.

ii. let Y denotes # people have A^*

$Y \sim Po(\lambda = 1000 \cdot 0.4 \cdot 0.001) = Po(0.4)$.

$P(Y=1) = P(Po(0.4) = 1)$.

c. i $P(2 \text{ have same}) = P(2A) \cap P(2B) \cap P(2AB) \cap P(2O)$

$$= \frac{4}{10} \cdot \frac{3}{9} + \frac{2}{10} \cdot \frac{1}{9} + \frac{1}{10} \cdot 0 + \frac{3}{10} \cdot \frac{2}{9}$$

$$= \frac{12+2+6}{90} = \frac{2}{9}$$

ii. $P(2 \text{ have same} \mid \text{neither } O) = P(2A) \cap P(2B)) / P(\text{not } 2O)$

x neither O

$$= \left(\frac{4}{10} \cdot \frac{3}{9} + \frac{2}{10} \cdot \frac{1}{9}\right) / \left(\frac{7}{10} \cdot \frac{6}{9}\right).$$

$-\frac{1}{2}$

$N = C + S. \quad N \sim Po(5t).$

4. (a). $P(C=x\ S=y) = P(C=x, S=y \mid N=(x+y)) \,/\, P(N=x+y)$

$P(N=x+y) = e^{-5t} \cdot (5t)^{x+y} / (x+y)!$

$(C=x, S=y \mid N=(x+y)) \sim U(x+y).$ -3

(b) i Since S, C are poisson distribution and indep.

$\#$ of $C \sim Bin(25, \frac{2}{5})$

$E(C) = 25 \cdot \frac{2}{5} = 10.$

$Var(C) = 25 \cdot \frac{2}{5} \cdot \frac{3}{5} = 6.$

ii $C \sim Po(2 \cdot 4) = Po(8).$ Since S, C indep.

$E(C) = 8$

$E(N) = 8.$

(c).

N=25

8 12

N=20

S=10

12 4

S=16.

4 8

In 8 am - 12 p.m. $C_1 \sim Bin(25, \frac{2}{5}).$ $E(C_1) = 10$ $Var(C_1) = 6.$

In 12 p.m - 4 p.m C_2 constant $E(C_2) = 20 - 10 = 10$ $Var(C_2) = 0.$

In 4 → 8. $C_3 \sim Po(2 \cdot 4)$ $E(C_3) = 8$ $Var(C_3) = 8.$

$E(C) = 10 + 10 + 8 = 28$

$Var(C) = 6 + 0 + 8 = 14.$

5. $f_{XY}(x,y) = e^{-y} \quad 0 \le x \le y \le \infty$. 0 other.

a. $f_X(x) = \int_x^\infty e^{-y}\, dy$

$= -e^{-y} \Big|_x^\infty$

$= e^{-x}. \quad 0 \le x \le \infty.$

$f_X(x) = 0$ otherwise since $f_{XY} = 0$.

$f_{(Y|X)}(y|x) = f_{XY}(x,y) / f_X(x)$

$= e^{-y} / e^{-x} = e^{-y+x}. \quad 0 \le x \le y < \infty.$

$f_{(Y|X)}(y|x) = 0$ other wise. since $f_{XY} = 0$.

b. $E(Y|X=x) = \int_x^\infty y\, f_{Y|X}(y|x)\, dy$

$= \int_x^\infty y e^{-y+x}\, dy$

$= -y e^{-y+x} - e^{-y+x} \Big|_x^\infty$

$= x+1$

$Var(Y|X=x) = E(Y^2|X=x) - E(Y|X=x)^2$

$= \int_x^\infty y^2 e^{-y+x}\, dy - (x+1)^2$

$= 1.$

why ??

$E(Y) = E(E(Y|X))$

$= E(X+1) = E(X)+1 = 2.$

$Var(Y) = E(Var(Y|X)) + Var(E(Y|X)).$

$= E(1) + Var(X+1)$

$= 1 + Var(X).$

$-1\frac{1}{2}$

$\rightarrow .= ?$

c. $W = Y - X$.

$$P(W \le w \mid X = x) = P(Y - X \le w \mid X = x)$$
$$= P(Y \le x + w \mid X = x).$$

$$f_{Y|X}(y|x) = e^{-y+x}.$$

$$F_{Y|X}(x+w) = \int_x^{x+w} e^{-y+x}\, dy \qquad \text{Note } Y \ge X.$$
$$= -e^{-y+x} \Big|_x^{x+w}$$
$$= -e^{-w} + e^0$$
$$= -e^{-w} + 1$$

this shows XW functionally independent. why?

$-\frac{1}{2}$

$$P(W \le w \mid X = x) = P(W \le w) = F_{Y|X}(x+w) = -e^{-w} + 1$$

$$f_W(w) = dF_{Y|X}/dw = e^{-w} \qquad 0 \le w \le \infty.$$

otherwise $f_W(w) = 0$.