Q1

Q1-a

```
n <- 100
mu <- 5
sample_dt <- rnorm(n, mu, 1)

> summary(sample_dt)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  2.757   4.368   4.955   5.085   5.875   7.455
```

Q1-b

Rewrite the prior as normal prior with density $\pi(\theta) \sim N(\theta_0, \tau_0^2)$ where $\tau_0^2 = \infty$.

The likelihood function is

$$L(\theta|x_1...x_n) = (\frac{1}{\sqrt{2\pi\sigma^2}})^n e^{\frac{-1}{2\sigma^2}\sum_0^n (x_i-\theta)^2}$$

$$\begin{aligned}
\sum_0^n (x_i - \theta)^2 &= \sum (x_i - \bar{x} + \bar{x} - \theta)^2 \\
&= \sum (x_i - \bar{x})^2 + (\bar{x} - \theta)^2
\end{aligned}$$

Since $\sum (x_i - \bar{x})(\bar{x} - \theta) = 0$

$$\text{define: } s^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$$

$$\sum_0^n (x_i - \theta)^2 = ns^2 + n(\bar{x} - \theta)^2$$

The posterior function is

$$\begin{aligned}
\pi(\theta|x_1...x_n) &= L(\theta|x_1...x_n)\pi(\theta) \\
&\propto e^{-\frac{1}{2\sigma^2}(ns^2 + n(\bar{x}-\theta))^2} e^{-\frac{1}{2\tau_0^2}(\theta-\theta_0)^2} \\
&\propto e^{-\frac{n}{2\sigma^2}(s^2 + (\bar{x}-\theta))^2} e^{-\frac{1}{2\tau_0^2}(\theta-\theta_0)^2}
\end{aligned}$$

$$\text{define: } \tau_1^2 = (\frac{1}{\tau_0^2} + \frac{n}{\sigma^2})^{-1}$$

$$w = \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}}$$

$$\theta_1 = w\bar{X} + (1 - w)\theta_0$$

Then the posterior function can be represented as

$$\pi(\theta|x_1...x_n) \propto e^{-\frac{1}{2\tau_1^2}(\theta-\theta_1)^2}$$

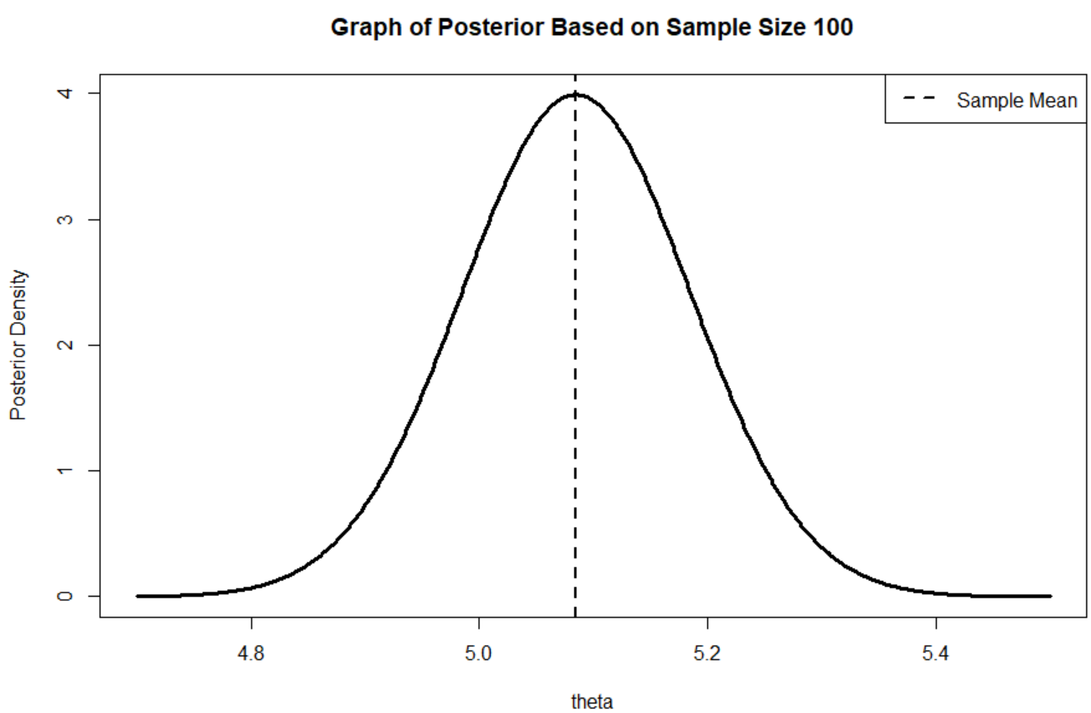
Right hand side of this formula is part of normal density function. Thus, the posterior function follows normal distribution $N(\theta_1, \tau_1^2)$

$$\begin{aligned}
w &= \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}},\ \sigma=1,\ \tau_0^2=\infty \\
w &= \frac{n}{n+0} = 1 \\
\theta_1 &= \bar{X} \\
\tau_1^2 &= (\frac{1}{\tau_0^2}+\frac{n}{\sigma^2})^{-1},\ \tau_0^2=\infty,\ \sigma=1 \\
\tau_1^2 &= \frac{1}{n}
\end{aligned}$$

In this case, we get mean and variance of the posterior are $\bar{X}$ and $\frac{1}{n}$. The exact density function of posterior can be written as

$$\begin{aligned}
\pi(\theta|x_1...x_n) &= \frac{1}{\sqrt{2\pi\tau_1^2}} e^{-\frac{1}{2\tau_1^2}(\theta-\theta_1)^2} \\
&= \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{n}{2}(\theta-\bar{X})^2}
\end{aligned}$$

Based on the sample data generated in previous question, where $n = 100$ and $\bar{X} = 5.085$. $\pi(\theta|x_1...x_n) \sim N(5.085, 0.01)$



Q1-c

The posterior distribution of θ follows normal distribution with mean $\bar{X} = 5.085$ and variance $\frac{1}{n} = 0.01$, therefore

$$\frac{\theta - \mathbb{E}(\theta|\bar{X})}{Var(\theta|\bar{X})} \sim N(0,1))$$

95% credible interval for θ can be represented by

$$\begin{aligned} 95\%CI &= P(L(\bar{X} < \theta < U(\bar{X}))) \\ &= [\mathbb{E}(\theta|\bar{X}) - Z_{\frac{0.05}{2}}\sqrt{Var(\theta|\bar{X})}, \mathbb{E}(\theta|\bar{X}) + Z_{\frac{0.05}{2}}\sqrt{Var(\theta|\bar{X})}] \\ &= [5.085 - 1.96 \cdot 0.1, 5.085 + 1.96 \cdot 0.1] \\ &= [4.889, 5.281] \end{aligned}$$

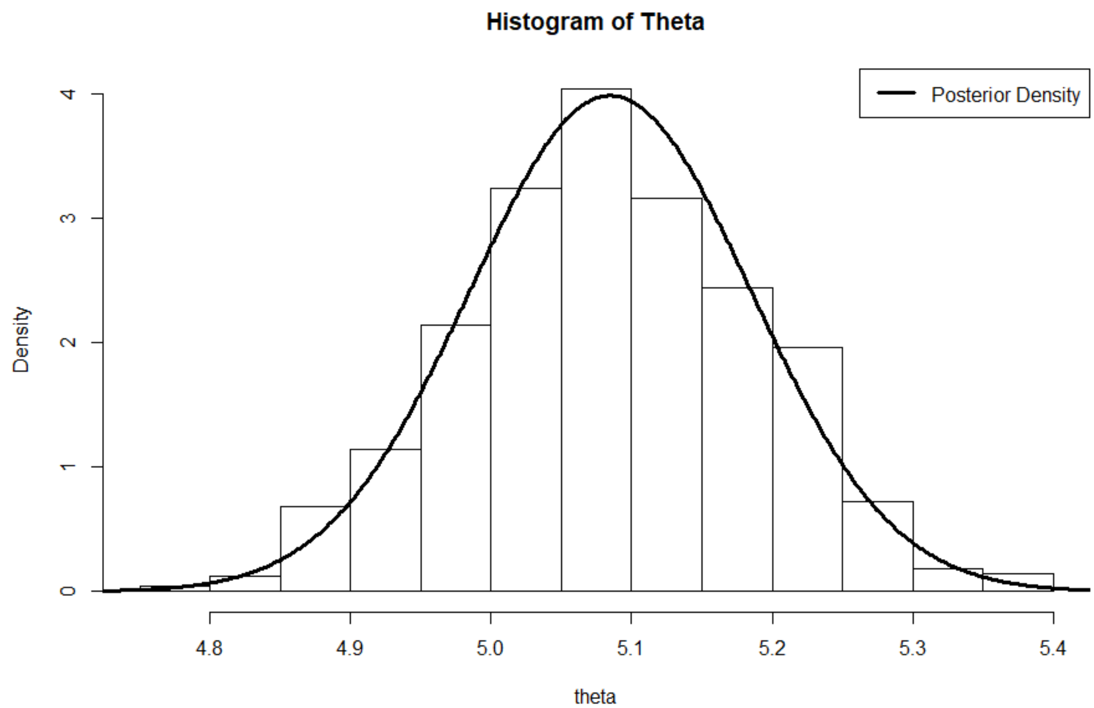
Given the sample data, the probability that true θ falls in between 4.889 and 5.281 is 0.95.

Q1-d

```
post_mu <- 5.085
post_sigma <- sqrt(1/100)
theta_dt <- rnorm(1000, post_mu, post_sigma)

theta_value <- seq(4.7, 5.5, 0.001)
post_density <- dnorm(theta_value, post_mu, post_sigma)

hist(theta_dt, breaks=20, probability=T, main='Histogram of Theta',
     xlab='theta', ylab='Density')
lines(theta_value, post_density, lwd=3)
legend('topright', 'Posterior Density', lty=1, lwd=3, cex=1)
```



Q1-e

```
> quantile(theta_dt, 0.025)
   2.5%
4.88424
> quantile(theta_dt, 0.975)
   97.5%
5.280327
```

The lower bound is 4.88424, and the upper bound is 5.280327. The interval captured from simulated data is approximately identical to calculated credible interval in previous question.