Homework 5

Instructions: This homework is due in class on Friday May 17.

Please read the following guidelines for presenting your work and follow them diligently.

- Write your full name clearly on the top right of the first page. **Staple** pages on the left hand corner. Write neatly in complete sentences.
- You are required to work all the problems, however, only 5 will be graded. The page numbers below refer to the fifth edition of the text.
- Answer the questions in the order in which they are posed. Clearly number the questions as I have.
- You must first work independently on the homework. Please post questions on the discussion board or come to office hours once you have tried the problems.
- Be sure to show/explain your work thoughtfully. How you write your answers is important.
- If you use R to make plots or as a calculator, it is enough to simply include the output (e.g., appropriately labeled plot) in the main part of your homework without the R code.

---

1. Recall that a Bayesian test of $H_0 : \theta \leq \theta^*$ versus $H_1 : \theta > \theta^*$ in the normal model

$$\begin{aligned} X &\sim N(\theta, \sigma^2), \\ \theta &\sim N(\theta_0, \tau_0^2). \end{aligned}$$

will reject H_0 iff

$$\bar{X} > \theta^* + \frac{\sigma^2(\theta^* - \theta_0)}{n\tau_0^2}$$

.

(a) Calculate the *size* of this test. In other words, treat X as random and θ as fixed and calculate

$$\alpha \;=\; sup\, P(reject\ H_0|H_0)$$

(Note : α will be an expression in terms of $\theta^*, \theta_0, \sigma^2, n$ and τ_0^2.)

(b) Make and include a plot of α as a function of $\delta = (\theta^* - \theta_0)$ by specifying $n = 1, \sigma^2 = 1$ and $\tau_0^2 = 1$. Comment on what you observe. (You can specify δ in the range -5 to 5 in increments of 0.01.)

2. Grace, the editor of the student newspaper, is going to conduct a survey of students to determine the level of support for the current President of the students' association. She needs to determine the prior distribution for θ, the probability that a student supports the President. She decides her prior mean is 0.5, and her prior standard deviation is 0.15.

   (a) Determine the $Beta(a, b)$ prior that matches her prior belief. (You will need to derive any formulas you use for the mean and variance of a Beta distribution.)

   (b) Out of the 68 students that she randomly polls, $x = 21$ support the current President. Determine the posterior distribution of θ. (Write the sampling model first before diving in)

   (c) Calculate and interpret a 95% equal tail credible interval for θ.

3. Repeat parts (b) and (c) from problem 2 for a Beta(1,1) prior. Graph both posterior distributions on the same graph. What do you notice about how they compare with each other?

4. Suppose X is a geometric random variable with PMF:

$$p_\theta(x) \quad = \quad (1-\theta)^{x-1}\theta, \quad x = 1, 2, \ldots,$$

   and we place a uniform prior on θ, the probability of a success.

   (a) Show that the posterior distribution of θ is a $Beta(2, x)$, that is,

$$\pi(\theta|x) \quad = \quad \frac{\Gamma(2+x)}{\Gamma(x)\Gamma(2)}(1-\theta)^{x-1}\theta, \; 0 < \theta < 1.$$

   (b) Suppose we want to test $H_0 : \theta = 3/4$ based on observing $x = 2$. Intuitively, do you find H_0 plausible? Why or why not? Now calculate the Bayesian p-value and make a conclusion at the 5% level.

5. This problem is here to help you go back and forth between a histogram of data and a normal probability plot. For each part, construct a histogram of the data and also the normal probability plot. Then write a few sentences explaining why the probability plot makes sense based on the histogram.

   (a) Generate a sample of size 25 from a standard normal distribution. (To generate a normal in R use rnorm)

   (b) Repeat part(a) for $Y = Z/U$ where $Z \sim N(0, 1)$ and U is an independent uniform on (0,1). (To generate a uniform in R use runif)