INSTRUCTOR'S SOLUTIONS MANUAL

AN INTRODUCTION TO MATHEMATICAL STATISTICS AND ITS APPLICATIONS

FIFTH EDITION

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Chapter 2: Probability

Section 2.2: Sample Spaces and the Algebra of Sets

2.2.1 $S = \{(s,s,s), (s,s,f), (s,f,s), (f,s,s), (s,f,f), (f,s,f), (f,f,s), (f,f,f)\}$
$A = \{(s,f,s), (f,s,s)\}$; $B = \{(f,f,f)\}$

2.2.2 Let (x, y, z) denote a red x, a blue y, and a green z. Then
$A = \{(2,2,1), (2,1,2), (1,2,2), (1,1,3), (1,3,1), (3,1,1)\}$

2.2.3 (1,3,4), (1,3,5), (1,3,6), (2,3,4), (2,3,5), (2,3,6)

2.2.4 There are 16 ways to get an ace and a 7, 16 ways to get a 2 and a 6, 16 ways to get a 3 and a 5, and 6 ways to get two 4's, giving 54 total.

2.2.5 The outcome sought is (4, 4). It is "harder" to obtain than the set {(5, 3), (3, 5), (6, 2), (2, 6)} of other outcomes making a total of 8.

2.2.6 The set N of five card hands in hearts that are not flushes are called *straight flushes*. These are five cards whose denominations are consecutive. Each one is characterized by the lowest value in the hand. The choices for the lowest value are A, 2, 3, …, 10. (Notice that an ace can be high or low). Thus, N has 10 elements.

2.2.7 P = {right triangles with sides $(5, a, b)$: $a^2 + b^2 = 25$}

2.2.8 A = {*SSBBBB*, *SBSBBB*, *SBBSBB*, *SBBBSB*, *BSSBBB*, *BSBSBB*, *BSBBSB*, *BBSSBB*, *BBSBSB*, *BBBSSB*}

2.2.9 **(a)** S = {(0, 0, 0, 0) (0, 0, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1), (0, 1, 0, 0), (0, 1, 0, 1), (0, 1, 1, 0), (0, 1, 1, 1), (1, 0, 0, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 0, 1, 1,), (1, 1, 0, 0), (1, 1, 0, 1), (1, 1, 1, 0), (1, 1, 1, 1,)}

(b) A = {(0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0,)}

(c) $1 + k$

2.2.10 **(a)** S = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 4), (4, 1), (4, 2), (4, 4)}

(b) {2, 3, 4, 5, 6, 8}

2.2.11 Let p_1 and p_2 denote the two perpetrators and i_1, i_2, and i_3, the three in the lineup who are innocent.
Then $S = \{(p_1,i_1), (p_1,i_2), (p_1,i_3), (p_2,i_1), (p_2,i_2), (p_2,i_3), (p_1,p_2), (i_1,i_2), (i_1,i_3), (i_2,i_3)\}$.
The event A contains every outcome in S except (p_1, p_2).

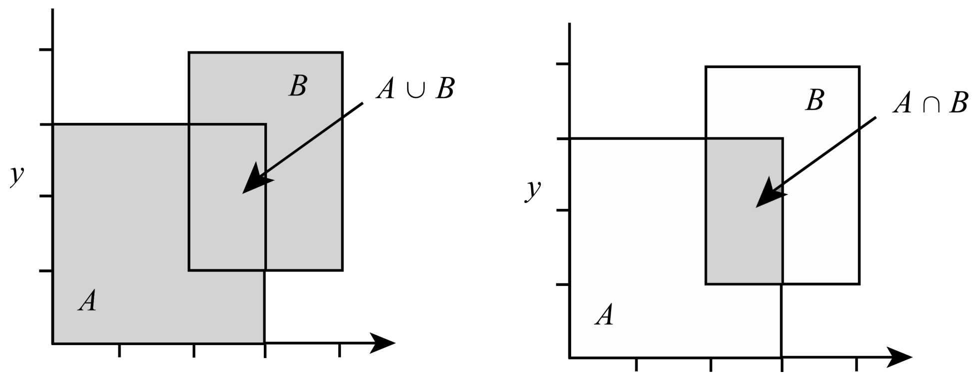
2.2.12 The quadratic equation will have complex roots—that is, the event A will occur—if $b^2 - 4ac < 0$.

2.2.13 In order for the shooter to win with a point of 9, one of the following (countably infinite) sequences of sums must be rolled: (9,9), (9, no 7 or no 9,9), (9, no 7 or no 9, no 7 or no 9,9), …

2.2.14 Let (x, y) denote the strategy of putting x white chips and y black chips in the first urn (which results in $10 - x$ white chips and $10 - y$ black chips being in the second urn). Then $S = \{(x, y): x = 0,1,...,10,\ \ y = 0,1,...,10, \text{ and } 1 \le x + y \le 19\}$. Intuitively, the optimal strategies are (1, 0) and (9, 10).

2.2.15 Let A_k be the set of chips put in the urn at $1/2^k$ minute until midnight. For example, $A_1 = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$. Then the set of chips in the urn at midnight is $\bigcup_{k=1}^{\infty}(A_k - \{k+1\}) = \varnothing$.

2.2.16

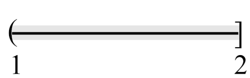


2.2.17 If $x^2 + 2x \le 8$, then $(x + 4)(x - 2) \le 0$ and $A = \{x:\ -4 \le x \le 2\}$. Similarly, if $x^2 + x \le 6$, then $(x + 3)(x - 2) \le 0$ and $B = \{x:\ -3 \le x \le 2)$. Therefore, $A \cap B = \{x:\ -3 \le x \le 2\}$ and $A \cup B = \{x:\ -4 \le x \le 2\}$.

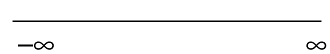
2.2.18 $A \cap B \cap C = \{x:\ x = 2, 3, 4\}$

2.2.19 The system fails if either the first pair fails or the second pair fails (or both pairs fail). For either pair to fail, though, both of its components must fail. Therefore, $A = (A_{11} \cap A_{21}) \cup (A_{12} \cap A_{22})$.

2.2.20 **(a)**

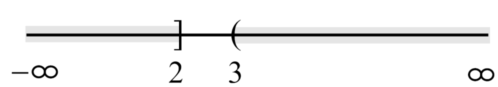


(b)



(c) empty set

(d)



2.2.21 40

2.2.22 **(a)** $\{E1, E2\}$ **(b)** $\{S1, S2, T1, T2\}$ **(c)** $\{A, I\}$

2.2.23 **(a)** If s is a member of $A \cup (B \cap C)$ then s belongs to A or to $B \cap C$. If it is a member of A or of $B \cap C$, then it belongs to $A \cup B$ and to $A \cup C$. Thus, it is a member of $(A \cup B) \cap (A \cup C)$. Conversely, choose s in $(A \cup B) \cap (A \cup C)$. If it belongs to A, then it belongs to $A \cup (B \cap C)$. If it does not belong to A, then it must be a member of $B \cap C$. In that case it also is a member of $A \cup (B \cap C)$.

(b) If s is a member of $A \cap (B \cup C)$ then s belongs to A and to $B \cup C$. If it is a member of B, then it belongs to $A \cap B$ and, hence, $(A \cap B) \cup (A \cap C)$. Similarly, if it belongs to C, it is a member of $(A \cap B) \cup (A \cap C)$. Conversely, choose s in $(A \cap B) \cup (A \cap C)$. Then it belongs to A. If it is a member of $A \cap B$ then it belongs to $A \cap (B \cup C)$. Similarly, if it belongs to $A \cap C$, then it must be a member of $A \cap (B \cup C)$.

2.2.24 Let $B = A_1 \cup A_2 \cup \ldots \cup A_k$. Then $A_1^C \cap A_2^C \cap \ldots \cap A_k^C = (A_1 \cup A_2 \cup \ldots \cup A_k)^C = B^C$. Then the expression is simply $B \cup B^C = S$.

2.2.25 **(a)** Let s be a member of $A \cup (B \cup C)$. Then s belongs to either A or $B \cup C$ (or both). If s belongs to A, it necessarily belongs to $(A \cup B) \cup C$. If s belongs to $B \cup C$, it belongs to B or C or both, so it must belong to $(A \cup B) \cup C$. Now, suppose s belongs to $(A \cup B) \cup C$. Then it belongs to either $A \cup B$ or C or both. If it belongs to C, it must belong to $A \cup (B \cup C)$. If it belongs to $A \cup B$, it must belong to either A or B or both, so it must belong to $A \cup (B \cup C)$.

(b) Suppose s belongs to $A \cap (B \cap C)$, so it is a member of A and also $B \cap C$. Then it is a member of A and of B and C. That makes it a member of $(A \cap B) \cap C$. Conversely, if s is a member of $(A \cap B) \cap C$, a similar argument shows it belongs to $A \cap (B \cap C)$.

2.2.26 **(a)** $A^C \cap B^C \cap C^C$
(b) $A \cap B \cap C$
(c) $A \cap B^C \cap C^C$
(d) $(A \cap B^C \cap C^C) \cup (A^C \cap B \cap C^C) \cup (A^C \cap B^C \cap C)$
(e) $(A \cap B \cap C^C) \cup (A \cap B^C \cap C) \cup (A^C \cap B \cap C)$

2.2.27 A is a subset of B.

2.2.28 **(a)** $\{0\} \cup \{x: 5 \le x \le 10\}$ **(b)** $\{x: 3 \le x < 5\}$ **(c)** $\{x: 0 < x \le 7\}$
(d) $\{x: 0 < x < 3\}$ **(e)** $\{0\} \cup \{x: 3 \le x \le 10\}$ **(f)** $\{0\} \cup \{x: 7 < x \le 10\}$

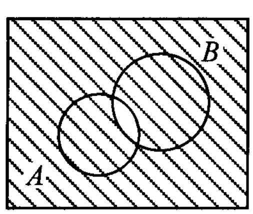
2.2.29 **(a)** B and C **(b)** B is a subset of A.

2.2.30 **(a)** $A_1 \cap A_2 \cap A_3$
(b) $A_1 \cup A_2 \cup A_3$

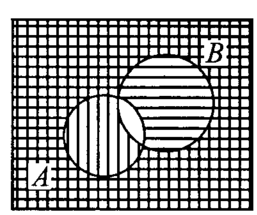
The second protocol would be better if speed of approval matters. For very important issues, the first protocol is superior.

2.2.31 Let A and B denote the students who saw the movie the first time and the second time, respectively. Then N(**a**) = 850, N(**b**) = 690, and $N[(A \cup B)^C] = 4700$ (implying that $N(A \cup B) = 1300$). Therefore, $N(A \cap B)$ = number who saw movie twice $= 850 + 690 - 1300 = 240$.

2.2.32 **(a)**



$(A \cap B)^C = A^C \cup B^C$



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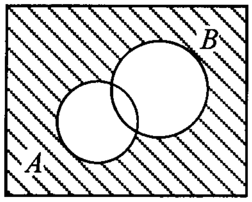
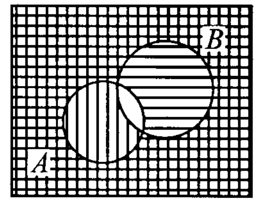
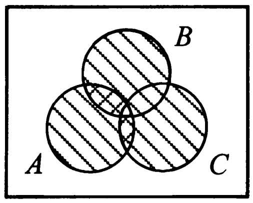


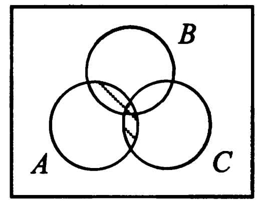
$(A \cup B)^C = A^C \cap B^C$



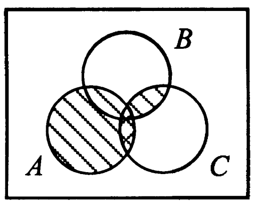
2.2.33 **(a)**



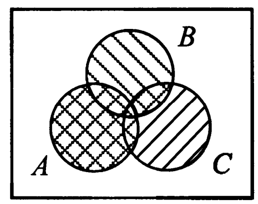
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



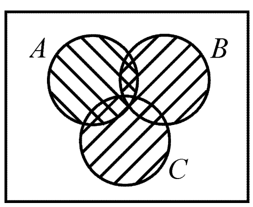
(b)



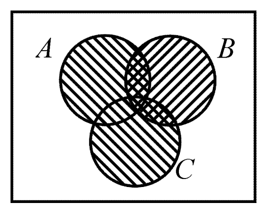
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



2.2.34 **(a)**

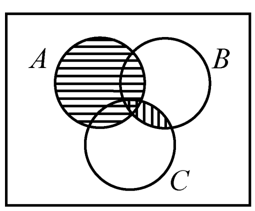


$A \cup (B \cup C)$

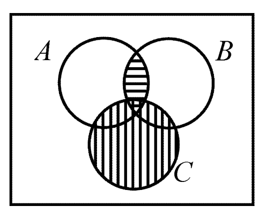


$(A \cup B) \cup C$

(b)



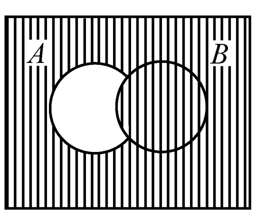
$A \cap (B \cap C)$



$(A \cap B) \cap C$

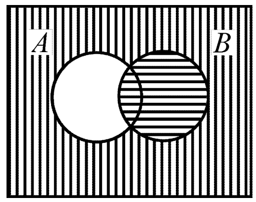
2.2.35 A and B are subsets of $A \cup B$.

2.2.36 **(a)**



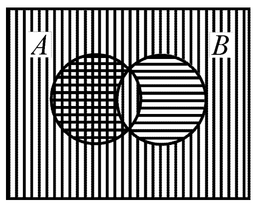
$(A \cap B^C)^C = A^C \cup B$

(b)



$B \cup (A \cup B)^C = A^C \cup B$

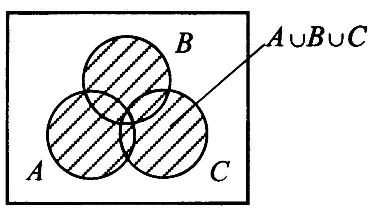
(c)



$A \cap (A \cap B)^C = A \cap B^C$

2.2.37 Let A be the set of those with MCAT scores ≥ 27 and B be the set of those with GPAs ≥ 3.5. We are given that N(a) = 1000, N(b) = 400, and $N(A \cap B) = 300$. Then $N(A^C \cap B^C) = N[(A \cup B)^C] = 1200 - N(A \cup B) = 1200 - [(N(a) + N(b) - N(A \cap B)]$ $= 1200 - [(1000 + 400 - 300] = 100$. The requested proportion is 100/1200.

2.2.38



$N(A \cup B \cup C) = N(a) + N(b) + N(c) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$

2.2.39 Let A be the set of those saying "yes" to the first question and B be the set of those saying "yes" to the second question. We are given that N(a) = 600, N(b) = 400, and $N(A^C \cap B) = 300$. Then $N(A \cap B) = N(b) - N(A^C \cap B) = 400 - 300 = 100$. $N(A \cap B^C) = N(a) - N(A \cap B) = 600 - 100$ $= 500$.

2.2.40 $N[(A \cap B)^C] = 120 - N(A \cup B) = 120 - [N(A^C \cap B) + N(A \cap B^C) + N(A \cap B)]$ $= 120 - [50 + 15 + 2] = 53$

Section 2.3: The Probability Function

2.3.1 Let L and V denote the sets of programs with offensive language and too much violence, respectively. Then $P(L) = 0.42$, $P(V) = 0.27$, and $P(L \cap V) = 0.10$. Therefore, P(program complies) $= P((L \cup V)^C) = 1 - [P(L) + P(V) - P(L \cap V)] = 0.41$.

2.3.2 $P(A$ or B but not both$) = P(A \cup B) - P(A \cap B) = P(a) + P(b) - P(A \cap B) - P(A \cap B)$ $= 0.4 + 0.5 - 0.1 - 0.1 = 0.7$

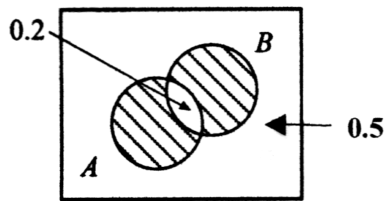
2.3.3 **(a)** $1 - P(A \cap B)$ **(b)** $P(\text{b}) - P(A \cap B)$

2.3.4 $P(A \cup B) = P(\text{a}) + P(\text{b}) - P(A \cap B) = 0.3$; $P(\text{a}) - P(A \cap B) = 0.1$. Therefore, $P(\text{b}) = 0.2$.

2.3.5 No. $P(A_1 \cup A_2 \cup A_3) = P(\text{at least one “6” appears}) = 1 - P(\text{no 6's appear}) = 1 - \left(\frac{5}{6}\right)^3 \neq \frac{1}{2}$.

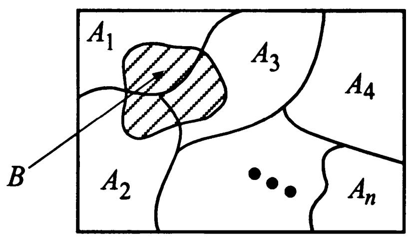
The A_i's are not mutually exclusive, so $P(A_1 \cup A_2 \cup A_3) \neq P(A_1) + P(A_2) + P(A_3)$.

2.3.6



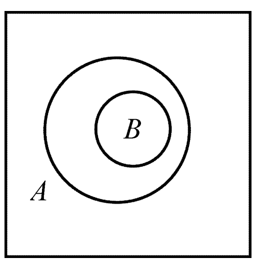
$P(A$ or B but not both$) = 0.5 - 0.2 = 0.3$

2.3.7

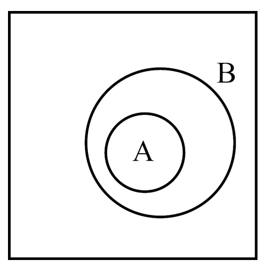


By inspection, $B = (B \cap A_1) \cup (B \cap A_2) \cup \ldots \cup (B \cap A_n)$.

2.3.8 **(a)**



(b)



2.3.9 $P(\text{odd man out}) = 1 - P(\text{no odd man out}) = 1 - P(HHH \text{ or } TTT) = 1 - \frac{2}{8} = \frac{3}{4}$

2.3.10 $A = \{2, 4, 6, \ldots, 24\}$; $B = \{3, 6, 9, \ldots, 24)$; $A \cap B = \{6, 12, 18, 24\}$.
Therefore, $P(A \cup B) = P(\text{a}) + P(\text{b}) - P(A \cap B) = \frac{12}{24} + \frac{8}{24} - \frac{4}{24} = \frac{16}{24}$.

2.3.11 Let A: State wins Saturday and B: State wins next Saturday. Then $P(\text{a}) = 0.10$, $P(\text{b}) = 0.30$, and $P(\text{lose both}) = 0.65 = 1 - P(A \cup B)$, which implies that $P(A \cup B) = 0.35$. Therefore, $P(A \cap B) = 0.10 + 0.30 - 0.35 = 0.05$, so $P(\text{State wins exactly once}) = P(A \cup B) - P(A \cap B) = 0.35 - 0.05 = 0.30$.

2.3.12 Since A_1 and A_2 are mutually exclusive and cover the entire sample space, $p_1 + p_2 = 1$.
But $3p_1 - p_2 = \frac{1}{2}$, so $p_2 = \frac{5}{8}$.

2.3.13 Let F: female is hired and T: minority is hired. Then $P(\text{f}) = 0.60$, $P(T) = 0.30$, and $P(F^C \cap T^C) = 0.25 = 1 - P(F \cup T)$. Since $P(F \cup T) = 0.75$, $P(F \cap T)$ $= 0.60 + 0.30 - 0.75 = 0.15$.

2.3.14 The smallest value of $P[(A \cup B \cup C)^C]$ occurs when $P(A \cup B \cup C)$ is as large as possible. This, in turn, occurs when A, B, and C are mutually disjoint. The largest value for $P(A \cup B \cup C)$ is $P(\text{a}) + P(\text{b}) + P(\text{c}) = 0.2 + 0.1 + 0.3 = 0.6$. Thus, the smallest value for $P[(A \cup B \cup C)^C]$ is 0.4.

2.3.15 **(a)** $X^C \cap Y = \{(H, T, T, H), (T, H, H, T)\}$, so $P(X^C \cap Y) = 2/16$
(b) $X \cap Y^C = \{(H, T, T, T), (T, T, T, H), (T, H, H, H), (H, H, H, T)\}$ so $P(X \cap Y^C) = 4/16$

2.3.16 $A = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$
$A \cap B^C = \{(1, 5), (3, 3), (5, 1)\}$, so $P(A \cap B^C) = 3/36 = 1/12$.

2.3.17 $A \cap B$, $(A \cap B) \cup (A \cap C)$, A, $A \cup B$, S

2.3.18 Let A be the event of getting arrested for the first scam; B, for the second. We are given $P(\mathbf{a}) = 1/10$, $P(\text{b}) = 1/30$, and $P(A \cap B) = 0.0025$. Her chances of not getting arrested are $P[(A \cup B)^C] = 1 - P(A \cup B) = 1 - [P(\text{a}) + P(\text{b}) - P(A \cap B)] = 1 - [1/10 + 1/30 - 0.0025] = 0.869$

Section 2.4: Conditional Probability

2.4.1 $P(\text{sum} = 10 | \text{sum exceeds } 8) = \dfrac{P(\text{sum} = 10 \text{ and sum exceeds } 8)}{P(\text{sum exceeds } 8)}$

$= \dfrac{P(\text{sum} = 10)}{P(\text{sum } = 9, 10, 11, \text{ or } 12)} = \dfrac{3/36}{4/36 + 3/36 + 2/36 + 1/36} = \dfrac{3}{10}$.

2.4.2 $P(A|B) + P(B|A) = 0.75 = \dfrac{P(A \cap B)}{P(B)} + \dfrac{P(A \cap B)}{P(A)} = \dfrac{10P(A \cap B)}{4} + 5P(A \cap B)$, which implies that

$P(A \cap B) = 0.1$.

2.4.3 If $P(A|B) = \dfrac{P(A \cap B)}{P(B)} < P(A)$, then $P(A \cap B) < P(\text{a}) \cdot P(\text{b})$. It follows that

$P(B|A) = \dfrac{P(A \cap B)}{P(A)} < \dfrac{P(A) \cdot P(B)}{P(A)} = P(\text{b})$.

2.4.4 $P(E|A \cup B) = \dfrac{P(E \cap (A \cup B))}{P(A \cup B)} = \dfrac{P(E)}{P(A \cup B)} = \dfrac{P(A \cup B) - P(A \cap B)}{P(A \cup B)} = \dfrac{0.4 - 0.1}{0.4} = \dfrac{3}{4}$.

2.4.5 The answer would remain the same. Distinguishing only three family types does not make them equally likely; (girl, boy) families will occur twice as often as either (boy, boy) or (girl, girl) families.

2.4.6 $P(A \cup B) = 0.8$ and $P(A \cup B) - P(A \cap B) = 0.6$, so $P(A \cap B) = 0.2$. Also, $P(A|B) = 0.6 = \dfrac{P(A \cap B)}{P(B)}$, so $P(b) = \dfrac{0.2}{0.6} = \dfrac{1}{3}$ and $P(a) = 0.8 + 0.2 - \dfrac{1}{3} = \dfrac{2}{3}$.

2.4.7 Let R_i be the event that a red chip is selected on the ith draw, $i = 1, 2$.
Then $P(\text{both are red}) = P(R_1 \cap R_2) = P(R_2 \mid R_1)P(R_1) = \dfrac{3}{4} \cdot \dfrac{1}{2} = \dfrac{3}{8}$.

2.4.8 $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A) + P(B) - P(A \cup B)}{P(B)} = \dfrac{a + b - P(A \cup B)}{b}$.
But $P(A \cup B) \leq 1$, so $P(A|B) \geq \dfrac{a + b - 1}{b}$.

2.4.9 Let W_i be the event that a white chip is selected on the ith draw, $i = 1,2$. Then
$P(W_2|W_1) = \dfrac{P(W_1 \cap W_2)}{P(W_1)}$. If both chips in the urn are white, $P(W_1) = 1$;
if one is white and one is black, $P(W_1) = \dfrac{1}{2}$. Since each chip distribution is equally likely,
$P(W_1) = 1 \cdot \dfrac{1}{2} + \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{3}{4}$. Similarly, $P(W_1 \cap W_2) = 1 \cdot \dfrac{1}{2} + \dfrac{1}{4} \cdot \dfrac{1}{2} = \dfrac{5}{8}$, so $P(W_2|W_1) = \dfrac{5/8}{3/4} = \dfrac{5}{6}$.

2.4.10 $P[(A \cap B) | (A \cup B)^C] = \dfrac{P[(A \cap B) \cap (A \cup B)^C]}{P[(A \cup B)^C]} = \dfrac{P(\varnothing)}{P[(A \cup B)^C]} = 0$

2.4.11 **(a)** $P(A^C \cap B^C) = 1 - P(A \cup B) = 1 - [P(a) + P(b) - P(A \cap B)] = 1 - [0.65 + 0.55 - 0.25] = 0.05$

(b) $P[(A^C \cap B) \cup (A \cap B^C)] = P(A^C \cap B) + P(A \cap B^C) = [P(a) - P(A \cap B)] + [P(b) - P(A \cap B)]$
$= [0.65 - 0.25] + [0.55 - 0.25] = 0.70$

(c) $P(A \cup B) = 0.95$

(d) $P[(A \cap B)^C] = 1 - P(A \cap B) = 1 - 0.25 = 0.75$

(e) $P\{[(A^C \cap B) \cup (A \cap B^C)] | A \cup B\} = \dfrac{P[(A^C \cap B) \cup (A \cap B^C)]}{P(A \cup B)} = 0.70/0.95 = 70/95$

(f) $P(A \cap B) | A \cup B) = P(A \cap B)/P(A \cup B) = 0.25/0.95 = 25/95$

(g) $P(B|A^C) = P(A^C \cap B)/P(A^C)] = [P(b) - P(A \cap B)]/[1 - P(a)] = [0.55 - 0.25]/[1 - 0.65]$
$= 30/35$

2.4.12 $P(\text{No. of heads} \geq 2 | \text{ No. of heads} \leq 2)$
$= P(\text{No. of heads} \geq 2 \text{ and No. of heads} \leq 2)/P(\text{No. of heads} \leq 2)$
$= P(\text{No. of heads} = 2)/P(\text{No. of heads} \leq 2) = (3/8)/(7/8) = 3/7$

2.4.13 $P(\text{first die} \geq 4|\text{sum} = 8) = P(\text{first die} \geq 4 \text{ and sum} = 8)/P(\text{sum} = 8)$
$= P(\{(4, 4), (5, 3), (6, 2)\}/P(\{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}) = 3/5$

2.4.14 There are 4 ways to choose three aces (count which one is left out). There are 48 ways to choose the card that is not an ace, so there are $4 \times 48 = 192$ sets of cards where exactly three are aces. That gives 193 sets where there are at least three aces. The conditional probability is (1/270,725)/(193/270,725) = 1/193.

2.4.15 First note that $P(A \cup B) = 1 - P[(A \cup B)^C] = 1 - 0.2 = 0.8$.
Then $P(\text{b}) = P(A \cup B) - P(A \cap B^C) - P(A \cap B) = 0.8 - 0.3 - 0.1 = 0.5$. Finally
$P(A|B) = P(A \cap B)/P(\text{b}) = 0.1/0.5 = 1/5$

2.4.16 $P(A|B) = 0.5$ implies $P(A \cap B) = 0.5P(\text{b})$. $P(B|A) = 0.4$ implies $P(A \cap B) = (0.4)P(\text{a})$.
Thus, $0.5P(\text{b}) = 0.4P(\text{a})$ or $P(\text{b}) = 0.8P(\text{a})$.
Then, $0.9 = P(\text{a}) + P(\text{b}) = P(\text{a}) + 0.8P(\text{a})$ or $P(\text{a}) = 0.9/1.8 = 0.5$.

2.4.17 $P[(A \cap B)^C] = P[(A \cup B)^C] + P(A \cap B^C) + P(A^C \cap B) = 0.2 + 0.1 + 0.3 = 0.6$
$P(A \cup B|(A \cap B)^C) = P[(A \cap B^C) \cup (A^C \cap B)]/P((A \cap B)^C) = [0.1 + 0.3]/0.6 = 2/3$

2.4.18 $P(\text{sum} \geq 8|\text{at least one die shows 5})$
$= P(\text{sum} \geq 8 \text{ and at least one die shows 5})/P(\text{at least one die shows 5})$
$= P(\{(5, 3), (5, 4), (5, 6), (3, 5), (4, 5), (6, 5), (5, 5)\})/(11/36) = 7/11$

2.4.19 P(Outandout wins|Australian Doll and Dusty Stake don't win)
$= P$(Outandout wins and Australian Doll and Dusty Stake don't win)/P(Australian Doll and Dusty Stake don't win) = 0.20/0.55 = 20/55

2.4.20 Suppose the guard will randomly choose to name Bob or Charley if they are the two to go free. Then the probability the guard will name Bob, for example, is
P(Andy, Bob) + (1/2)P(Bob, Charley) = 1/3 + (1/2)(1/3) = 1/2.
The probability Andy will go free given the guard names Bob is P(Andy, Bob)/P(Guard names Bob) = (1/3)/(1/2) = 2/3. A similar argument holds for the guard naming Charley. Andy's concern is not justified.

2.4.21 $P(BBRWW) = P(\text{b})P(B|B)P(R|BB)P(W|BBR)P(W|BBRW) = \frac{4}{15}\cdot\frac{3}{14}\cdot\frac{5}{13}\cdot\frac{6}{12}\cdot\frac{5}{11}$
$=.0050$. $P(2, 6, 4, 9, 13) = \frac{1}{15}\cdot\frac{1}{14}\cdot\frac{1}{13}\cdot\frac{1}{12}\cdot\frac{1}{11} = \frac{1}{360{,}360}$.

2.4.22 Let K_i be the event that the ith key tried opens the door, $i = 1, 2, \ldots, n$. Then P(door opens first time with 3rd key) = $P(K_1^C \cap K_2^C \cap K_3) = P(K_1^C)\cdot P(K_2^C \mid K_1^C)\cdot P(K_3 \mid K_1^C \cap K_2^C) =$
$\frac{n-1}{n}\cdot\frac{n-2}{n-1}\cdot\frac{1}{n-2} = \frac{1}{n}$.

2.4.23 (1/52)(1/51)(1/50)(1/49) = 1/6,497,400

2.4.24 (1/2)(1/2)(1/2)(2/3)(3/4) = 1/16

2.4.25 Let A_i be the event "Bearing came from supplier i", $i = 1, 2, 3$. Let B be the event "Bearing in toy manufacturer's inventory is defective."
Then $P(A_1) = 0.5$, $P(A_2) = 0.3$, $P(A_3) = 0.2$ and $P(B|A_1) = 0.02$, $P(B|A_2) = 0.03$, $P(B|A_3) = 0.04$
Combining these probabilities according to Theorem 2.4.1 gives
$P(b) = (0.02)(0.5) + (0.03)(0.3) + (0.04)(0.2) = 0.027$
meaning that the manufacturer can expect 2.7% of her ball-bearing stock to be defective.

2.4.26 Let B be the event that the face (or sum of faces) equals 6. Let A_1 be the event that a Head comes up and A_2, the event that a Tail comes up. Then $P(b) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2)$
$= \frac{1}{6}\cdot\frac{1}{2}+\frac{5}{36}\cdot\frac{1}{2} = 0.15$.

2.4.27 Let B be the event that the countries go to war. Let A be the event that terrorism increases. Then $P(b) = P(B|A)P(a) + P(B|A^C)P(A^C) = (0.65)(0.30) + (0.05)(0.70) = 0.23$.

2.4.28 Let B be the event that a donation is received; let A_1, A_2, and A_3 denote the events that the call is placed to Belle Meade, Oak Hill, and Antioch, respectively.
Then $P(b) = \sum_{i=1}^{3} P(B|A_i)P(A_i) = (0.60)\cdot\frac{1000}{4000}+(0.55)\cdot\frac{1000}{4000}+(0.35)\cdot\frac{2000}{4000} = 0.46$.

2.4.29 Let B denote the event that the person interviewed answers truthfully, and let A be the event that the person interviewed is a man.
Then $P(b) = P(B|A)P(a) + P(B|A^C)P(A^C) = (0.78)(0.47) + (0.63)(0.53) = 0.70$.

2.4.30 Let B be the event that a red chip is ultimately drawn from Urn I. Let A_{RW}, for example, denote the event that a red is transferred from Urn I and a white is transferred from Urn II. Then
$P(b) = P(B|A_{RR})P(A_{RR}) + P(B|A_{RW})P(A_{RW}) + P(B|A_{WR})P(A_{WR}) + P(B|A_{WW})P(A_{WW})$
$= \frac{3}{4}\left(\frac{3}{4}\cdot\frac{2}{4}\right)+\frac{2}{4}\left(\frac{3}{4}\cdot\frac{2}{4}\right)+1\left(\frac{1}{4}\cdot\frac{2}{4}\right)+\frac{3}{4}\left(\frac{1}{4}\cdot\frac{2}{4}\right)=\frac{11}{16}$.

2.4.31 Let B denote the event that someone will test positive, and let A denote the event that someone is infected. Then $P(b) = P(B|A)P(a) + P(B|A^C)P(A^C) = (0.999)(0.0001) + (0.0001)(0.9999)$
$= 0.00019989$.

2.4.32 The optimal allocation has 1 white chip in one urn and the other 19 chips (9 white and 10 black) in the other urn. Then $P(\text{white is drawn}) = 1\cdot\frac{1}{2}+\frac{9}{19}\cdot\frac{1}{2} = 0.74$.

2.4.33 If B is the event that Backwater wins and A is the event that their first-string quarterback plays, then $P(b) = P(B|A)P(a) + P(B|A^C)P(A^C) = (0.75)(0.70) + (0.40)(0.30) = 0.645$.

2.4.34 Since the identities of the six chips drawn are not known, their selection does not affect any probability associated with the seventh chip. Therefore,
$P(\text{seventh chip drawn is red}) = \text{P(first chip drawn is red)} = \frac{40}{100}$.

2.4.35 No. Let B denote the event that the person calling the toss is correct. Let A_H be the event that the coin comes up Heads and let A_T be the event that the coin comes up Tails.
Then $P(b) = P(B|A_H)P(A_H) + P(B|A_T)P(A_T) = (0.7)\left(\frac{1}{2}\right) + (0.3)\left(\frac{1}{2}\right) = \frac{1}{2}$.

2.4.36 Let B be the event of a guilty verdict; let A be the event that the defense can discredit the police. Then $P(b) = P(B|A)P(a) + P(B|A^C)P(A^C) = 0.15(0.70) + 0.80(0.30) = 0.345$

2.4.37 Let A_1 be the event of a 3.5-4.0 GPA; A_2, of a 3.0-3.5 GPA; and A_3, of a GPA less than 3.0. If B is the event of getting into medical school, then
$P(b) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) = (0.8)(0.25) + (0.5)(0.35) + (0.1)(0.40)$
$= 0.415$

2.4.38 Let B be the event of early release; let A be the event that the prisoner is related to someone on the governor's staff.
Then $P(b) = P(B|A)P(a) + P(B|A^C)P(A^C) = (0.90)(0.40) + (0.01)(0.60) = 0.366$

2.4.39 Let A_1 be the event of being a Humanities major; A_2, of being a Natural Science major; A_3, of being a History major; and A_4, of being a Social Science major. If B is the event of a male student, then $P(b) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) + P(B|A_4)P(A_4)$
$= (0.40)(0.4) + (0.85)(0.1) + (0.55)(0.3) + (0.25)(0.2) = 0.46$

2.4.40 Let B denote the event that the chip drawn from Urn II is red; let A_R and A_W denote the events that the chips transferred are red and white, respectively.

$$\text{Then } P(A_W \mid B) = \frac{P(B \mid A_W)P(A_W)}{P(B \mid A_R)P(A_R) + P(B \mid A_W)P(A_W)} = \frac{(2/4)(2/3)}{(3/4)(1/3) + (2/4)(2/3)} = \frac{4}{7}$$

2.4.41 Let A_i be the event that Urn i is chosen, i = I, II, III. Then, $P(A_i) = 1/3$, i = I, II, III. Suppose B is the event a red chip is drawn. Note that $P(B|A_1) = 3/8$, $P(B|A_2) = 1/2$ and $P(B|A_3) = 5/8$.

$$P(A_3 \mid B) = \frac{P(B \mid A_3)P(A_3)}{P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + P(B \mid A_3)P(A_3)}$$

$$= \frac{(5/8)(1/3)}{(3/8)(1/3) + (1/2)(1/3) + (5/8)(1/3)} = 5/12.$$

2.4.42 If B is the event that the warning light flashes and A is the event that the oil pressure is low, then

$$P(A|B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^C)P(A^C)} = \frac{(0.99)(0.10)}{(0.99)(0.10) + (0.02)(0.90)} = 0.85$$

2.4.43 Let B be the event that the basement leaks, and let A_T, A_W, and A_H denote the events that the house was built by Tara, Westview, and Hearthstone, respectively. Then $P(B|A_T) = 0.60$, $P(B|A_W) = 0.50$, and $P(B|A_H) = 0.40$. Also, $P(A_T) = 2/11$, $P(A_W) = 3/11$, and $P(A_H) = 6/11$. Applying Bayes' rule to each of the builders shows that $P(A_T|B) = 0.24$, $P(A_W|B) = 0.29$, and $P(A_H|B) = 0.47$, implying that Hearthstone is the most likely contractor.

2.4.44 Let B denote the event that Francesca passed, and let A_X and A_Y denote the events that she was enrolled in Professor X's section and Professor Y's section, respectively. Since $P(B|A_X) = 0.85$, $P(B|A_Y) = 0.60$, $P(A_X) = 0.4$, and $P(A_Y) = 0.6$,

$$P(A_X|B) = \frac{(0.85)(0.4)}{(0.85)(0.4)+(0.60)(0.6)} = 0.486$$

2.4.45 Let B denote the event that a check bounces, and let A be the event that a customer wears sunglasses. Then $P(B|A) = 0.50$, $P(B|A^C) = 1 - 0.98 = 0.02$, and $P(\mathbf{a}) = 0.10$, so

$$P(A|B) = \frac{(0.50)(0.10)}{(0.50)(0.10)+(0.02)(0.90)} = 0.74$$

2.4.46 Let B be the event that Basil dies, and define A_1, A_2, and A_3 to be the events that he ordered cherries flambe, chocolate mousse, or no dessert, respectively. Then $P(B|A_1) = 0.60$, $P(B|A_2) = 0.90$, $P(B|A_3) = 0$, $P(A_1) = 0.50$, $P(A_2) = 0.40$, and $P(A_3) = 0.10$. Comparing $P(A_1|B)$ and $P(A_2|B)$ suggests that Margo should be considered the prime suspect:

$$P(A_1|B) = \frac{(0.60)(0.50)}{(0.60)(0.50)+(0.90)(0.40)+(0)(0.10)} = 0.45$$

$$P(A_2|B) = \frac{(0.90)(0.40)}{(0.60)(0.50)+(0.90)(0.40)+(0)(0.10)} = 0.55$$

2.4.47 Define B to be the event that Josh answers a randomly selected question correctly, and let A_1 and A_2 denote the events that he was 1) unprepared for the question and 2) prepared for the question, respectively. Then $P(B|A_1) = 0.20$, $P(B|A_2) = 1$, $P(A_2) = p$, $P(A_1) = 1 - p$, and

$$P(A_2|B) = 0.92 = \frac{P(B\mid A_2)P(A_2)}{P(B\mid A_1)P(A_1)+P(B\mid A_2)P(A_2)} = \frac{1\cdot p}{(0.20)(1-p)+(1\cdot p)}$$

which implies that $p = 0.70$ (meaning that Josh was prepared for $(0.70)(20) = 14$ of the questions).

2.4.48 Let B denote the event that the program diagnoses the child as abused, and let A be the event that the child is abused. Then $P(\mathrm{a}) = 1/90$, $P(B|A) = 0.90$, and $P(B|A^C) = 0.03$, so

$$P(A|B) = \frac{(0.90)(1/90)}{(0.90)(1/90)+(0.03)(89/90)} = 0.25$$

If $P(\mathrm{a}) = 1/1000$, $P(A|B) = 0.029$; if $P(\mathrm{a}) = 1/50$, $P(A|B) = 0.38$.

2.4.49 Let A_1 be the event of being a Humanities major; A_2, of being a History and Culture major; and A_3, of being a Science major. If B is the event of being a woman, then

$$P(A_2|B) = \frac{(0.45)(0.5)}{(0.75)(0.3)+(0.45)(0.5)+(0.30)(0.2)} = 225/510$$

2.4.50 Let B be the event that a 1 is received. Let A be the event that a 1 was sent. Then

$$P(A^C|B) = \frac{(0.10)(0.3)}{(0.95)(0.7)+(0.10)(0.3)} = 30/695$$

2.4.51 Let B be the event that Zach's girlfriend responds promptly. Let A be the event that Zach sent an e-mail, so A^C is the event of leaving a message. Then

$$P(A|B) = \frac{(0.8)(2/3)}{(0.8)(2/3)+(0.9)(1/3)} = 16/25$$

2.4.52 Let A be the event that the shipment came from Warehouse A with events B and C defined similarly. Let D be the event of a complaint.

$$P(C|D) = \frac{P(D\mid C)P(C)}{P(D\mid A)P(A)+P(D\mid B)P(B)+P(D\mid C)P(C)}$$

$$= \frac{(0.02)(0.5)}{(0.03)(0.3)+(0.05)(0.2)+(0.02)(0.5)} = 10/29$$

2.4.53 Let A_i be the event that Drawer i is chosen, i, = 1, 2, 3. If B is the event a silver coin is selected, then $P(A_3|B) = \dfrac{(0.5)(1/3)}{(0)(1/3)+(1)(1/3)+(0.5)(1/3)} = 1/3$

Section 2.5: Independence

2.5.1 **(a)** No, because $P(A \cap B) > 0$.
(b) No, because $P(A \cap B) = 0.2 \neq P(\text{a}) \cdot P(\text{b}) = (0.6)(0.5) = 0.3$
(c) $P(A^C \cup B^C) = P((A \cap B)^C) = 1 - P(A \cap B) = 1 - 0.2 = 0.8$.

2.5.2 Let C and M be the events that Spike passes chemistry and mathematics, respectively. Since $P(C \cap M) = 0.12 \neq P(\text{c}) \cdot P(M) = (0.35)(0.40) = 0.14$, C and M are not independent.
$P(\text{Spike fails both}) = 1 - P(\text{Spike passes at least one}) = 1 - P(C \cup M)$
$= 1 - [P(\text{c}) + P(M) - P(C \cap M)] = 0.37$.

2.5.3 $P(\text{one face is twice the other face}) = P((1, 2), (2, 1), (2, 4), (4, 2), (3, 6), (6, 3)) = \dfrac{6}{36}$.

2.5.4 Let R_i, B_i, and W_i be the events that red, black, and white chips are drawn from urn i, i = 1, 2. Then $P(\text{both chips drawn are same color}) = P((R_1 \cap R_2) \cup (B_1 \cap B_2) \cup (W_1 \cap W_2))$
$= P(R_1) \cdot P(R_2) + P(B_1) \cdot P(B_2) + P(W_1) \cdot P(W_2)$ [because the intersections are mutually exclusive and the individual draws are independent]. But $P(R_1) \cdot P(R_2) + P(B_1) \cdot P(B_2) + P(W_1) \cdot P(W_2)$

$$= \left(\frac{3}{10}\right)\left(\frac{2}{9}\right)+\left(\frac{2}{10}\right)\left(\frac{4}{9}\right)+\left(\frac{5}{10}\right)\left(\frac{3}{9}\right) = 0.32.$$

2.5.5 $P(\text{Dana wins at least 1 game out of 2}) = 0.3$, which implies that $P(\text{Dana loses 2 games out of 2}) = 0.7$. Therefore, $P(\text{Dana wins at least 1 game out of 4}) = 1 - P(\text{Dana loses all 4 games})$
$= 1 - P(\text{Dana loses first 2 games \underline{and} Dana loses second 2 games}) = 1 - (0.7)(0.7) = 0.51$.

2.5.6 Six equally-likely orderings are possible for any set of three distinct random numbers: $x_1 < x_2 < x_3$, $x_1 < x_3 < x_2$, $x_2 < x_1 < x_3$, $x_2 < x_3 < x_1$, $x_3 < x_1 < x_2$, and $x_3 < x_2 < x_1$. By inspection, $P(\text{a}) = \frac{2}{6}$, and $P(\text{b}) = \frac{1}{6}$, so $P(A \cap B) = P(\text{a}) \cdot P(\text{b}) = \frac{1}{18}$.

2.5.7 **(a)** 1. $P(A \cup B) = P(\text{a}) + P(\text{b}) - P(A \cap B) = 1/4 + 1/8 + 0 = 3/8$
2. $P(A \cup B) = P(\text{a}) + P(\text{b}) - P(\text{a})P(\text{b}) = 1/4 + 1/8 - (1/4)(1/8) = 11/32$

(b) 1. $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0}{P(B)} = 0$

2. $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)P(B)}{P(B)} = P(A) = 1/4$

2.5.8 **(a)** $P(A \cup B \cup C) = P(\text{a}) + P(\text{b}) + P(\text{c}) - P(\text{a})P(\text{b}) - P(\text{a})P(\text{c}) - P(\text{b})P(\text{c}) + P(\text{a})P(\text{b})P(\text{c})$
(b) $P(A \cup B \cup C) = 1 - P[(A \cup B \cup C)^C] = 1 - P(A^C \cap B^C \cap C^C) = 1 - P(A^C)P(B^C)P(C^C)$

2.5.9 Let A_i be the event of i heads in the first two tosses, $i = 0, 1, 2$. Let B_i be the event of i heads in the last two tosses, $i = 0, 1, 2$. The A's and B's are independent. The event of interest is $(A_0 \cap B_0) \cup (A_1 \cap B_1) \cup (A_2 \cap B_2)$ and $P[(A_0 \cap B_0) \cup (A_1 \cap B_1) \cup (A_2 \cap B_2)]$
$= P(A_0)P(B_0) + P(A_1)P(B_1) + P(A_2)P(B_2) = (1/4)(1/4) + (1/2)(1/2) + (1/4)(1/4) = 6/16$

2.5.10 A and B are disjoint, so they cannot be independent.

2.5.11 Equation 2.5.3: $P(A \cap B \cap C) = P(\{1, 3)\}) = 1/36 = (2/6)(3/6)(6/36) = P(\text{a})P(\text{b})P(\text{c})$
Equation 2.5.4: $P(B \cap C) = P(\{1, 3), (5,6)\}) = 2/36 \neq (3/6)(6/36) = P(\text{b})P(\text{c})$

2.5.12 Equation 2.5 3: $P(A \cap B \cap C) = P(\{2, 4, 10, 12)\}) = 4/36 \neq (1/2)(1/2)(1/2) = P(\text{a})P(\text{b})P(\text{c})$
Equation 2.5.4: $P(A \cap B) = P(\{2, 4, 10, 12, 24, 26, 32, 34, 36)\}) = 9/36 = 1/4 = (1/2)(1/2)$
$= P(\text{a})P(\text{b})$
$P(A \cap C) = P(\{1, 2, 3, 4, 5, 10, 11, 12, 13)\}) = 9/36 = 1/4 = (1/2)(1/2) = P(\text{a})P(\text{c})$
$P(B \cap C) = P(\{2, 4, 6, 8, 10, 12, 14, 16, 18)\}) = 9/36 = 1/4 = (1/2)(1/2) = P(\text{a})P(\text{c})$

2.5.13 11 [= 6 verifications of the form $P(A_i \cap A_j) = P(A_i) \cdot P(A_j)$ + 4 verifications of the form $P(A_i \cap A_j \cap A_k) = P(A_i) \cdot P(A_j) \cdot P(A_k)$ + 1 verification that $P(A_1 \cap A_2 \cap A_3 \cap A_4)$
$= P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4)$].

2.5.14 $P(\text{a}) = \frac{3}{6}$, $P(\text{b}) = \frac{2}{6}$, $P(\text{c}) = \frac{6}{36}$, $P(A \cap B) = \frac{6}{36}$, $P(A \cap C) = \frac{3}{36}$, $P(B \cap C) = \frac{2}{36}$, and $P(A \cap B \cap C) = \frac{1}{36}$. It follows that A, B, and C are mutually independent because $P(A \cap B \cap C) = \frac{1}{36} = P(\text{a}) \cdot P(\text{b}) \cdot P(\text{c}) = \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{6}{36}$, $P(A \cap B) = \frac{6}{36} = P(\text{a}) \cdot P(\text{b}) = \frac{3}{6} \cdot \frac{2}{6}$, $P(A \cap C) = \frac{3}{36} = P(\text{a}) \cdot P(\text{c}) = \frac{3}{6} \cdot \frac{6}{36}$, and $P(B \cap C) = \frac{2}{36} = P(\text{b}) \cdot P(\text{c}) = \frac{2}{6} \cdot \frac{6}{36}$.

2.5.15 $P(A \cap B \cap C) = 0$ (since the sum of two odd numbers is necessarily even) $\neq P(\text{a}) \cdot P(\text{b}) \cdot P(\text{c})$ > 0, so A, B, and C are not mutually independent. However, $P(A \cap B) = \frac{9}{36}$ $= P(\text{a}) \cdot P(\text{b}) = \frac{3}{6} \cdot \frac{3}{6}$, $P(A \cap C) = \frac{9}{36} = P(\text{a}) \cdot P(\text{c}) = \frac{3}{6} \cdot \frac{18}{36}$, and $P(B \cap C) = \frac{9}{36} = P(\text{b}) \cdot P(\text{c}) =$ $\frac{3}{6} \cdot \frac{18}{36}$, so A, B, and C are pairwise independent.

2.5.16 Let R_i and G_i be the events that the ith light is red and green, respectively, $i = 1, 2, 3, 4$. Then $P(R_1) = P(R_2) = \frac{1}{3}$ and $P(R_3) = P(R_4) = \frac{1}{2}$. Because of the considerable distance between the intersections, what happens from light to light can be considered independent events. P(driver stops at least 3 times) = P(driver stops exactly 3 times) + P(driver stops all 4 times)
$= P((R_1 \cap R_2 \cap R_3 \cap G_4) \cup (R_1 \cap R_2 \cap G_3 \cap R_4) \cup (R_1 \cap G_2 \cap R_3 \cap R_4)$
$\cup (G_1 \cap R_2 \cap R_3 \cap R_4) \cup (R_1 \cap R_2 \cap R_3 \cap R_4)) = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$
$+ \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{7}{36}$.

2.5.17 Let M, L, and G be the events that a student passes the mathematics, language, and general knowledge tests, respectively. Then $P(M) = \frac{6175}{9500}$, $P(L) = \frac{7600}{9500}$, and $P(\text{g}) = \frac{8075}{9500}$.
P(student fails to qualify) = P(student fails at least one exam)
= 1 − P(student passes all three exams) $= 1 - P(M \cap L \cap G) = 1 - P(M) \cdot P(L) \cdot P(\text{g}) = 0.56$.

2.5.18 Let A_i denote the event that switch A_i closes, $i = 1, 2, 3, 4$. Since the A_i's are independent events, P(circuit is completed) $= P((A_1 \cap A_2) \cup (A_3 \cap A_4)) = P(A_1 \cap A_2) + P(A_3 \cap A_4)$
$- P((A_1 \cap A_2) \cap (A_3 \cap A_4)) = 2p^2 - p^4$.

2.5.19 Let p be the probability of having a winning game card.
Then 0.32 = P(winning at least once in 5 tries) = 1 − P(not winning in 5 tries) $= 1 - (1 - p)^5$,
so $p = 0.074$

2.5.20 Let A_H, A_T, B_H, B_T, C_H, and C_T denote the events that players A, B, and C throw heads and tails on individual tosses. Then $P(A \text{ throws first head}) = P(A_H \cup (A_T \cap B_T \cap C_T \cap A_H) \cup \cdots)$
$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{8}\right) + \frac{1}{2}\left(\frac{1}{8}\right)^2 + \cdots = \frac{1}{2}\left(\frac{1}{1 - 1/8}\right) = \frac{4}{7}$. Similarly, $P(B$ throws first head)
$= P((A_T \cap B_H) \cup (A_T \cap B_T \cap C_T \cap A_T \cap B_H) \cup \ldots) = \frac{1}{4} + \frac{1}{4}\left(\frac{1}{8}\right) + \frac{1}{4}\left(\frac{1}{8}\right)^2 + \ldots = \frac{1}{4}\left(\frac{1}{1 - 1/8}\right) = \frac{2}{7}$.
$P(C \text{ throws first head}) = 1 - \frac{4}{7} - \frac{2}{7} = \frac{1}{7}$.

2.5.21 P(at least one child becomes adult) = 1 − P(no child becomes adult) $= 1 - 0.2^n$.
Then $1 - 2^n \geq 0.75$ implies $n \geq \frac{\ln 0.25}{\ln 0.2}$ or $n \geq 0.86$, so take $n = 1$.

2.5.22 P(at least one viewer can name actor) = 1 − P(no viewer can name actor) = $1 - (0.85)^{10} = 0.80$. P(exactly one viewer can name actor) = $10\,(0.15)\,(0.85)^9 = 0.347$.

2.5.23 Let B be the event that no heads appear, and let A_i be the event that i coins are tossed, i = 1, 2, …, 6. Then $P(b) = \sum_{i=1}^{6} P(B\,|\,A_i)P(A_i) = \frac{1}{2}\left(\frac{1}{6}\right) + \left(\frac{1}{2}\right)^2\left(\frac{1}{6}\right) + \ldots + \left(\frac{1}{2}\right)^6\left(\frac{1}{6}\right) = \frac{63}{384}$.

2.5.24 P(at least one red chip is drawn from at least one urn) = 1 − P(all chips drawn are white)
$= 1 - \left(\frac{4}{7}\right)^r \cdot \left(\frac{4}{7}\right)^r \cdots \left(\frac{4}{7}\right)^r = 1 - \left(\frac{4}{7}\right)^{rm}$.

2.5.25 P(at least one double six in n throws) = 1 − P(no double sixes in n throws) = $1 - \left(\frac{35}{36}\right)^n$. By trial and error, the smallest n for which P(at least one double six in n throws) exceeds 0.50 is 25 $\left[1 - \left(\frac{35}{36}\right)^{24} = 0.49;\ 1 - \left(\frac{35}{36}\right)^{25} = 0.51\right]$.

2.5.26 Let A be the event that a sum of 8 appears before a sum of 7. Let B be the event that a sum of 8 appears on a given roll and let C be the event that the sum appearing on a given roll is neither 7 nor 8. Then $P(b) = \frac{5}{36}$, $P(c) = \frac{25}{36}$, and $P(a) = P(b) + P(c)P(b) + P(c)P(c)P(b)$
$+ \cdots = \frac{5}{36} + \frac{25}{36}\frac{5}{36} + \left(\frac{25}{36}\right)^2\frac{5}{36} + \cdots = \frac{5}{36}\sum_{k=0}^{\infty}\left(\frac{25}{36}\right)^k = \frac{5}{36}\left(\frac{1}{1 - 25/36}\right) = \frac{5}{11}$.

2.5.27 Let W, B, and R denote the events of getting a white, black and red chip, respectively, on a given draw. Then P(white appears before red) = $P(W \cup (B \cap W) \cup (B \cap B \cap W) \cup \cdots)$

$$= \frac{w}{w+b+r} + \frac{b}{w+b+r}\cdot\frac{w}{w+b+r} + \left(\frac{b}{w+b+r}\right)^2 \cdot \frac{w}{w+b+r} + \cdots$$

$$= \frac{w}{w+b+r}\cdot\left(\frac{1}{1 - b/(w+b+r)}\right) = \frac{w}{w+r}.$$

2.5.28 $P(B|A_1) = 1 - P(\text{all } m \text{ I-teams fail}) = 1 - (1-r)^m$; similarly, $P(B|A_2) = 1 - (1-r)^{n-m}$. From Theorem 2.4.1, $P(b) = [1 - (1-r)^m]p + [1 - (1-r)^{n-m}](1-p)$. Treating m as a continuous variable and differentiating $P(b)$ gives
$\frac{dP(B)}{dm} = -p(1-r)^m \cdot \ln(1-r) + (1-p)(1-r)^{n-m} \cdot \ln(1-r)$. Setting $\frac{dP(B)}{dm} = 0$ implies that
$m = \frac{n}{2} + \frac{\ln[(1-p)/p]}{2\ln(1-r)}$.

2.5.29 P(at least one four) = 1 − P(no fours) = $1 - (0.9)^n$. $1 - (0.9)^n \geq 0.7$ implies $n = 12$

2.5.30 Let B be the event that all n tosses come up heads. Let A_1 be the event that the coin has two heads, and let A_2 be the event the coin is fair. Then

$$P(A_2 \mid B) = \frac{(1/2)^n(8/9)}{1(1/9)+(1/2)^n(8/9)} = \frac{8(1/2)^n}{1+8(1/2)^n}$$

By inspection, the limit of $P(A_2 \mid B)$ as n goes to infinity is 0.

Section 2.6: Combinatorics

2.6.1 $2 \cdot 3 \cdot 2 \cdot 2 = 24$

2.6.2 $20 \cdot 9 \cdot 6 \cdot 20 = 21{,}600$

2.6.3 $3 \cdot 3 \cdot 5 = 45$. Included will be aeu and cdx.

2.6.4 **(a)** $26^2 \cdot 10^4 = 6{,}760{,}000$
(b) $26^2 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 3{,}407{,}040$
(c) The total number of plates *with* four zeros is $26 \cdot 26$, so the total number *not* having four zeros must be $26^2 \cdot 10^4 - 26^2 = 6{,}759{,}324$.

2.6.5 There are 9 choices for the first digit (1 through 9), 9 choices for the second digit (0 + whichever eight digits are not appearing in the hundreds place), and 8 choices for the last digit. The number of admissible integers, then, is $9 \cdot 9 \cdot 8 = 648$. For the integer to be odd, the last digit must be either 1, 3, 5, 7, or 9. That leaves 8 choices for the first digit and 8 choices for the second digit, making a total of 320 ($= 8 \cdot 8 \cdot 5$) odd integers.

2.6.6 For each topping, the customer has 2 choices: "add" or "do not add." The eight available toppings, then, can produce a total of $2^8 = 256$ different hamburgers.

2.6.7 The bases can be occupied in any of 2^7 ways (each of the seven can be either "empty" or "occupied"). Moreover, the batter can come to the plate facing any of five possible "out" situations (0 through 4). It follows that the number of base-out configurations is $5 \cdot 2^7$, or 640.

2.6.8 With 4 choices for the first digit, 1 for the third digit, 5 for the last digit, and 10 for each of the remaining six digits, the total number of admissible zip codes is 20,000,000($= 4 \cdot 10^6 \cdot 1 \cdot 5$).

2.6.9 $4 \cdot 14 \cdot 6 + 4 \cdot 6 \cdot 5 + 14 \cdot 6 \cdot 5 + 4 \cdot 14 \cdot 5 = 1156$

2.6.10 There are two mutually exclusive sets of ways for the black and white keys to alternate—the black keys can be 1^{st}, 3^{rd}, 5^{th}, and 7^{th} notes in the melody, or the 2^{nd}, 4^{th} 6^{th}, and 8^{th}. Since there are 5 black keys and 7 white keys, there are $5 \cdot 7 \cdot 5 \cdot 7 \cdot 5 \cdot 7 \cdot 5 \cdot 7$ variations in the first set and $7 \cdot 5 \cdot 7 \cdot 5 \cdot 7 \cdot 5 \cdot 7 \cdot 5$ in the second set. The total number of alternating melodies is the sum $5^4 7^4 + 7^4 5^4 = 3{,}001{,}250$.

2.6.11 The number of usable garage codes is $2^8 - 1 = 255$, because the "combination" where none of the buttons is pushed is inadmissible (recall Example 2.6.3). Five additional families can be added before the eight-button system becomes inadequate.

2.6.12 4, because $2^1 + 2^2 + 2^3 < 26$ but $2^1 + 2^2 + 2^3 + 2^4 \geq 26$.

2.6.13 In order to exceed 256, the binary sequence of coins must have a head in the ninth position and at least one head somewhere in the first eight tosses. The number of sequences satisfying those conditions is $2^8 - 1$, or 255. (The "1" corresponds to the sequences TTTTTTTTH, whose value would not exceed 256.)

2.6.14 There are 3 choices for the vowel and 4 choices for the consonant, so there are $3 \cdot 4 = 12$ choices, if order doesn't matter. If we are taking ordered arrangements, then there are 24 ways, since each unordered selection can be written vowel first or consonant first.

2.6.15 There are $1 \cdot 3$ ways if the ace of clubs is the first card and $12 \cdot 4$ ways if it is not. The total is then $3 + 12 \cdot 4 = 51$

2.6.16 Monica has $3 \cdot 5 \cdot 2 = 30$ routes from Nashville to Anchorage, so there are $30 \cdot 30 = 900$ choices of round trips.

2.6.17 ${}_6P_3 = 6 \cdot 5 \cdot 4 = 120$

2.6.18 ${}_4P_4 = 4! = 24$; ${}_2P_2 \cdot {}_2P_2 = 4$

2.6.19 $\log_{10}(30!) \doteq \log_{10}\left(\sqrt{2\pi}\right) + \left(30 + \frac{1}{2}\right)\log_{10}(30) - 30\log_{10}e = 32.42246$, which implies that $30! \doteq 10^{32.42246} = 2.645 \times 10^{32}$.

2.6.20 ${}_9P_9 = 9! = 362{,}880$

2.6.21 There are 2 choices for the first digit, 6 choices for the middle digit, and 5 choices for the last digit, so the number of admissible integers that can be formed from the digits 1 through 7 is 60 $(= 2 \cdot 6 \cdot 5)$.

2.6.22 **(a)** ${}_8P_8 = 8! = 40{,}320$

(b) The men can be arranged in, say, the odd-numbered chairs in ${}_4P_4$ ways; for each of those permutations, the women can be seated in the even-numbered chairs in ${}_4P_4$ ways. But the men could also be in the even-numbered chairs. It follows that the total number of alternating seating arrangements is ${}_4P_4 \cdot {}_4P_4 + {}_4P_4 \cdot {}_4P_4 = 1152$.

2.6.23 There are 4 different sets of three semesters in which the electives could be taken. For each of those sets, the electives can be selected and arranged in ${}_{10}P_3$ ways, which means that the number of possible schedules is $4 \cdot {}_{10}P_3$, or 2880.

2.6.24 ${}_6P_6 = 720$; ${}_6P_6 \cdot {}_6P_6 = 518{,}400$; $6!6!2^6$ is the number of ways six male/female cheerleading teams can be positioned along a sideline if each team has the option of putting the male in front or the female in front; $6!6!2^6 2^{12}$ is the number of arrangements subject to the conditions of the previous answer but with the additional option that each cheerleader can face either forwards or backwards.

2.6.25 The number of playing sequences where at least one side is out of order = total number of playing sequences − number of correct playing sequences = ${}_6P_6 - 1 = 719$.

2.6.26 Within each of the n families, members can be lined up in ${}_mP_m = m!$ ways. Since the n families can be permuted in ${}_nP_n = n!$ ways, the total number of admissible ways to arrange the nm people is $n! \cdot (m!)^n$.

2.6.27 There are ${}_2P_2 = 2$ ways for you and a friend to be arranged, ${}_8P_8$ ways for the other eight to be permuted, and six ways for you and a friend to be in consecutive positions in line. By the multiplication rule, the number of admissible arrangements is ${}_2P_2 \cdot {}_8P_8 \cdot 6 = 483{,}840$.

2.6.28 By inspection, ${}_nP_1 = n$. Assume that ${}_nP_k = n(n-1)\cdots(n-k+1)$ is the number of ways to arrange k distinct objects without repetition. Notice that $n - k$ options would be available for a $(k + 1)$st object added to the sequences. By the multiplication rule, the number of sequences of length $k + 1$ must be $n(n-1)\cdots(n-\mathrm{k}+1)(n-k)$. But the latter is the formula for ${}_nP_{k+1}$.

2.6.29 $(13!)^4$

2.6.30 By definition, $(n + 1)! = (n + 1) \cdot n!$; let $n = 0$.

2.6.31 ${}_9P_2 \cdot {}_4C_1 = 288$

2.6.32 Two people between them: $4 \cdot 2 \cdot 5! = 960$
Three people between them: $3 \cdot 2 \cdot 5! = 720$
Four people between them: $2 \cdot 2 \cdot 5! = 480$
Five people between them: $1 \cdot 2 \cdot 5! = 240$
Total number of ways: 2400

2.6.33 **(a)** $(4!)(5!) = 2880$ **(b)** $6(4!)(5!) = 17,280$
(c) $(4!)(5!) = 2880$ **(d)** $\binom{9}{4}(2)(5!) = 30,240$

2.6.34 TENNESSEE can be permuted in $\dfrac{9!}{4!2!2!1!} = 3780$ ways;
FLORIDA can be permuted in $7! = 5040$ ways.

2.6.35 If the first digit is a 4, the remaining six digits can be arranged in $\dfrac{6!}{3!(1!)^3} = 120$ ways; if the first digit is a 5, the remaining six digits can be arranged in $\dfrac{6!}{2!2!(1!)^2} = 180$ ways. The total number of admissible numbers, then, is $120 + 180 = 300$.

2.6.36 **(a)** $8!/3!3!2! = 560$ **(b)** $8! = 40{,}320$ **(c)** $8!/3!(1!)^5 = 6720$

2.6.37 **(a)** $4! \cdot 3! \cdot 3! = 864$
(b) $3! \cdot 4!3!3! = 5184$ (each of the 3! permutations of the three nationalities can generate 4!3!3! arrangements of the ten people in line)

(c) $10! = 3{,}628{,}800$
(d) $10!/4!3!3! = 4200$

2.6.38 Altogether, the letters in S L U M G U L L I O N can be permuted in $\dfrac{11!}{3!2!(1!)^6}$ ways. The seven consonants can be arranged in $7!/3!(1!)^4$ ways, of which $4!$ have the property that the three L's come first. By the reasoning used in Example 2.6.13, it follows that the number of admissible arrangements is $4!/(7!/3!) \cdot \dfrac{11!}{3!2!}$, or 95,040.

2.6.39 Imagine a field of 4 entrants (A, B, C, D) assigned to positions 1 through 4, where positions 1 and 2 correspond to the opponents for game 1 and positions 3 and 4 correspond to the opponents for game 2. Although the four players can be assigned to the four positions in $4!$ ways, not all of those permutations yield different tournaments. For example, $\dfrac{B}{1}\dfrac{C}{2}\dfrac{A}{3}\dfrac{D}{4}$ and $\dfrac{A}{1}\dfrac{D}{2}\dfrac{B}{3}\dfrac{C}{4}$ produce the same set of games, as do $\dfrac{B}{1}\dfrac{C}{2}\dfrac{A}{3}\dfrac{D}{4}$ and $\dfrac{C}{1}\dfrac{B}{2}\dfrac{A}{3}\dfrac{D}{4}$. In general, n games can be arranged in $n!$ ways, and the two players in each game can be permuted in $2!$ ways. Given a field of $2n$ entrants, then, the number of distinct pairings is $(2n)!/n!(2!)^n$, or $1 \cdot 3 \cdot 5 \cdots (2n - 1)$.

2.6.40 Since x^{12} can be the result of the factors $x^6 \cdot x^6 \cdot 1 \cdots 1$ or $x^3 \cdot x^3 \cdot x^3 \cdot x^3 \cdot 1 \cdots 1$ or $x^6 \cdot x^3 \cdot x^3 \cdot 1 \cdots 1$, the analysis described in Example 2.6.16 implies that the coefficient of x^{12} is $\dfrac{18!}{2!16!} + \dfrac{18!}{4!14!} + \dfrac{18!}{1!2!15!} = 5661$.

2.6.41 The letters in E L E E M O S Y N A R Y minus the pair S Y can be permuted in $10!/3!$ ways. Since S Y can be positioned in front of, within, or behind those ten letters in 11 ways, the number of admissible arrangements is $11 \cdot 10!/3! = 6{,}652{,}800$.

2.6.42 Each admissible spelling of ABRACADABRA can be viewed as a path consisting of 10 steps, five to the right (R) and five to the left (L). Thus, each spelling corresponds to a permutation of the five R's and five L's. There are $\dfrac{10!}{5!5!} = 252$ such permutations.

2.6.43 Six, because the first four pitches must include two balls and two strikes, which can occur in $4!/2!2! = 6$ ways.

2.6.44 $9!/2!3!1!3! = 5040$ (recall Example 2.6.16)

2.6.45 Think of the six points being numbered 1 through 6. Any permutation of three A's and three B's—for example, $\dfrac{A}{1}\dfrac{A}{2}\dfrac{B}{3}\dfrac{B}{4}\dfrac{A}{5}\dfrac{B}{6}$—corresponds to the three vertices chosen for triangle A and the three for triangle B. It follows that $6!/3!3! = 20$ different sets of two triangles can be drawn.

2.6.46 Consider $k!$ objects categorized into $(k - 1)!$ groups, each group being of size k. By Theorem 2.6.2, the number of ways to arrange the $k!$ objects is $(k!)!/(k!)^{(k-1)!}$, but the latter must be an integer.

2.6.47 There are $\frac{14!}{2!2!1!2!2!3!1!1!}$ total permutations of the letters. There are $\frac{5!}{2!2!1!} = 30$ arrangements of the vowels, only one of which leaves the vowels in their original position. Thus, there are $\frac{1}{30} \cdot \frac{14!}{2!2!1!2!1!3!1!1!} = 30{,}270{,}240$ arrangements of the word leaving the vowels in their original position.

2.6.48 $\frac{15!}{4!3!1!3!1!1!1!1!} = 1,513,512,000$

2.6.49 The three courses with A grades can be: emf, emp, emh, efp, efh, eph, mfp, mfh, mph, fph, or 10 possibilities. From the point of view of Theorem 2.6.2, the grade assignments correspond to the set of permutations of three A's and two B's, which equals $\frac{5!}{3!2!} = 10$.

2.6.50 Since every (unordered) set of two letters describes a different line, the number of possible lines is $\binom{5}{2} = 10$.

2.6.51 To achieve the two-to-one ratio, six pledges need to be chosen from the set of 10 and three from the set of 15, so the number of admissible classes is $\binom{10}{6} \cdot \binom{15}{3} = 95{,}550$.

2.6.52 Of the eight crew members, five need to be on a given side of the boat. Clearly, the remaining three can be assigned to the sides in 3 ways. Moreover, the rowers on each side can be permuted in 4! ways. By the multiplication rule, then, the number of ways to arrange the crew is 1728 $(= 3 \cdot 4! \cdot 4!)$.

2.6.53 **(a)** $\binom{9}{4} = 126$ **(b)** $\binom{5}{2}\binom{4}{2} = 60$ **(c)** $\binom{9}{4} - \binom{5}{4} - \binom{4}{4} = 120$

2.6.54 $\binom{7}{5} = 21$; order does not matter.

2.6.55 Consider a simpler problem: Two teams of two each are to be chosen from a set of four players—*A*, *B*, *C*, and *D*. Although a single team can be chosen in $\binom{4}{2}$ ways, the number of *pairs* of teams is only $\binom{4}{2}\Big/2$, because [(*A B*), (*C D*)] and [(*C D*), (*A B*)] would correspond to the same matchup. Applying that reasoning here means that the ten players can split up in $\binom{10}{5}\Big/2 = 126$ ways.

2.6.56 Number the spaces between the twenty pages from 1 to 19. Choosing any two of these spaces partitions the reading assignment into three non-zero, numbers, x_1, x_2, and x_3, corresponding to the numbers of pages read on Monday, Tuesday, and Wednesday, respectively. Therefore, the number of ways to complete the reading assignment is $\binom{19}{2} = 171$.

2.6.57 The four *I*'s need to occupy any of the $\binom{8}{4}$ sets of four spaces between and around the other seven letters. Since the latter can be permuted in $\frac{7!}{2!4!1!}$ ways, the total number of admissible arrangements is $\binom{8}{4}\cdot\frac{7!}{2!4!1!} = 7350$.

2.6.58 Let $x = y = 1$ in the expansion $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$. The total number of hamburgers referred to in Question 2.6.6 (= 2^8) must also be equal to the number of ways to choose k condiments, k = 0, 1, 2, …, 8—that is, $\binom{8}{0}+\binom{8}{1}+\ldots+\binom{8}{8}$.

2.6.59 Consider the problem of selecting an unordered sample of n objects from a set of $2n$ objects, where the $2n$ have been divided into two groups, each of size n. Clearly, we could choose n from the first group and 0 from the second group, or $n - 1$ from the first group and 1 from the second group, and so on. Altogether, $\binom{2n}{n}$ must equal $\binom{n}{n}\binom{n}{0}+\binom{n}{n-1}\binom{n}{1}+\ldots+\binom{n}{0}\binom{n}{n}$. But $\binom{n}{j}=\binom{n}{n-j}$, $j = 0, 1, \ldots, n$ so $\binom{2n}{n}=\sum_{j=0}^{n}\binom{n}{j}^2$.

2.6.60 Let $x = y = 1$ in the expansion $(x - y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k(-y)^{n-k}$. Then $x - y = 0$ and the sum reduces to $0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}$, or equivalently, $\binom{n}{1}+\binom{n}{3}+\ldots = \binom{n}{0}+\binom{n}{2}+\ldots$.

2.6.61 The ratio of two successive terms in the sequence is $\binom{n}{j+1}\Big/\binom{n}{j}=\frac{n-j}{j+1}$. For small j, $n - j > j + 1$, implying that the terms are increasing. For $j > \frac{n-1}{2}$, though, the ratio is less than 1, meaning the terms are decreasing.

2.6.62 Four months of daily performance create a need for roughly 120 different sets of jokes. If n denotes the number of different jokes that Mitch has to learn, the question is asking for the smallest n for which $\binom{n}{4}\geq 120$. By trial and error, $n = 9$.

2.6.63 Using Newton's binomial expansion, the equation $(1 + t)^d \cdot (1 + t)^e = (1 + t)^{d+e}$ can be written

$$\left(\sum_{j=0}^{d}\binom{d}{j}t^j\right)\cdot\left(\sum_{j=0}^{e}\binom{e}{j}t^j\right)=\sum_{j=0}^{d+e}\binom{d+e}{j}t^j$$

Since the exponent k can arise as $t^0 \cdot t^k$, $t^1 \cdot t^{k-1}$, … , or $t^k \cdot t^0$, it follows that $\binom{d}{0}\binom{e}{k}+\binom{d}{1}\binom{e}{k-1}+\ldots+\binom{d}{k}\binom{e}{0}=\binom{d+e}{k}$. That is, $\binom{d+e}{k}=\sum_{j=0}^{k}\binom{d}{j}\binom{e}{k-j}$.

Section 2.7: Combinatorial Probability

2.7.1 $\binom{7}{2}\binom{3}{2}\Big/\binom{10}{4}$

2.7.2 $P(\text{sum} = 5) = \dfrac{\text{Number of pairs that sum to 5}}{\text{Total number of pairs}} = 2\Big/\binom{6}{2}=\dfrac{2}{15}$.

2.7.3 P(numbers differ by more than 2) = 1 − P(numbers differ by one) − P(numbers differ by 2)
$= 1 - 19\Big/\binom{20}{2}-18\Big/\binom{20}{2}=\dfrac{153}{190} = 0.81$.

2.7.4 $P(A \cup B) = P(\text{a}) + P(\text{b}) - P(A \cap B) = \binom{4}{4}\binom{48}{9}\Big/\binom{52}{13}+\binom{4}{4}\binom{48}{9}\Big/\binom{52}{13}-\binom{4}{4}\binom{4}{4}\binom{44}{5}\Big/\binom{52}{13}$

2.7.5 Let A_1 be the event that an urn with $3W$ and $3R$ is sampled; let A_2 be the event that the urn with $5W$ and $1R$ is sampled. Let B be the event that the three chips drawn are white. By Bayes' rule,

$$P(A_2 \mid B)=\frac{P(B\mid A_2)P(A_2)}{P(B\mid A_1)P(A_1)+P(B\mid A_2)P(A_2)}$$

$$=\frac{\left[\binom{5}{3}\binom{1}{0}\Big/\binom{6}{3}\right]\cdot(1/10)}{\left[\binom{3}{3}\binom{3}{0}\Big/\binom{6}{3}\right]\cdot(9/10)+\left[\binom{5}{3}\binom{1}{0}\Big/\binom{6}{3}\right]\cdot(1/10)}=\frac{10}{19}$$

2.7.6 $\binom{2}{1}^{50}\Big/\binom{100}{50}$

2.7.7 $6/6^n = 1/6^{n-1}$

2.7.8 There are 6 faces that could be the "three-of-a-kind" and 5 faces that could be the "two-of-a-kind." Moreover, the five dice bearing those two numbers could occur in any of $5!/2!3! = \binom{5}{2}$ orders. It follows that P("full house") $= 6 \cdot 5 \cdot \binom{5}{2} \Big/ 6^5 = 50/6^4$

2.7.9 By Theorem, 2.6.2, the $2n$ grains of sand can be arranged in $(2n)!/n!n!$ ways. Two of those arrangements have the property that the colors will completely separate. Therefore, the probability of the latter is $2(n!)^2/(2n)!$

2.7.10 P(monkey spells CALCULUS) $= 1/[8!/(2!)^3(1!)^2] = 1/5040$;
P(monkey spells ALGEBRA) $= 1/[7!/2!(1!)^5] = 2/5040$.

2.7.11 P(different floors) $= 7!/7^7$; P(same floor) $= 7/7^7 = 1/7^6$. The assumption being made is that all possible departure patterns are equally likely, which is probably not true, since residents living on lower floors would be less inclined to wait for the elevator than would those living on the top floors.

2.7.12 The total number of distinguishable permutations of the phrase is $\dfrac{23!}{2!2!4!2!1!3!2!4!2!2!1!1!}$. The number of permutations where all of the S's are adjacent is counted by treating the S's as a single letter that appears once. The denominator above will have one of the 4! replaced by 1!. The number of such permutations, then, is $\dfrac{23!}{2!2!4!2!1!3!2!1!2!2!1!1!}$. The probability that the S's are adjacent is then the ratio of these two terms or $4!23!/26! = 1/650$. The requested probability is then the complement, 649/650.

2.7.13 The 10 short pieces and 10 long pieces can be lined up in a row in $20!/(10)!(10)!$ ways. Consider each of the 10 pairs of consecutive pieces as defining the reconstructed sticks. Each of those pairs could combine a short piece (S) and a long piece (L) in two ways: SL or LS. Therefore, the number of permutations that would produce 10 sticks, each having a short and a long component is 2^{10}, so the desired probability is $2^{10} \Big/ \binom{20}{10}$.

2.7.14 $6!/6^6$

2.7.15 Any of $\binom{k}{2}$ people could share any of 365 possible birthdays. The remaining $k - 2$ people can generate $364 \cdot 363 \cdots (365 - k + 2)$ sequences of distinct birthdays. Therefore, P(exactly one match) $= \binom{k}{2} \cdot 365 \cdot 364 \cdots (365 - k + 2)/365^k$.

2.7.16 The expression $\binom{12}{1}\binom{11}{1}\binom{10}{1}$ <u>orders</u> the denominations of the three single cards—in effect, each set of three denominations would be counted 3! times. The denominator (= $\binom{52}{5}$) in that particular probability calculation, though, does not consider the cards to be ordered. To be consistent, the denominations for the three single cards must be treated as a combination, meaning the number of choices is $\binom{12}{3}$.

2.7.17 To get a flush, Dana needs to draw any three of the remaining eleven diamonds. Since only forty-seven cards are effectively left in the deck (others may already have been dealt, but their identities are unknown), P(Dana draws to flush) = $\binom{11}{3}\Big/\binom{47}{3}$.

2.7.18 P(draws to full house or four-of-a-kind) = P(draws to full house) + P(draws to four-of-a-kind) $= \frac{3}{47} + \frac{1}{47} = \frac{4}{47}$.

2.7.19 There are two pairs of cards that would give Tim a straight flush (5 of clubs and 7 of clubs or 7 of clubs and 10 of clubs). Therefore, P(Tim draws to straight flush) = $2\Big/\binom{47}{2}$. A flush, by definition, consists of five cards in the same suit whose denominations are not all consecutive. It follows that P(Tim draws to flush) = $\left[\binom{10}{2} - 2\right]\Big/\binom{47}{2}$, where the "2" refers to the straight flushes cited earlier.

2.7.20 A sum of 48 requires four 10's and an 8 or three 10's and two 9's; a sum of 49 requires four 10's and a 9; no sums higher than 49 are possible. Therefore,
$P(\text{sum} \geq 48) = \left[\binom{4}{4}\binom{4}{1} + \binom{4}{3}\binom{4}{2} + \binom{4}{4}\binom{4}{1}\right]\Big/\binom{52}{5} = 32\Big/\binom{52}{5}$.

2.7.21 $\binom{5}{3}\binom{4}{2}^3\binom{3}{1}\binom{4}{2}\binom{2}{1}\binom{4}{1}\Big/\binom{52}{9}$

2.7.22 $\binom{32}{13}\Big/\binom{52}{13}$

2.7.23 $\left[\binom{2}{1}\binom{2}{1}\right]^4\binom{32}{4}\Big/\binom{48}{12}$

2.7.24 Any permutation of $\frac{n+r}{2}$ steps forward and $\frac{n-r}{2}$ steps backward will result in a net gain of r steps forward. Since the total number of (equally-likely) paths is 2^n,

$$P(\text{conventioneer ends up } r \text{ steps forward}) = \frac{n!\Big/\left(\frac{n+r}{2}\right)!\left(\frac{n-r}{2}\right)!}{2^n}.$$

Chapter 3: Random Variables

Section 3.2: Binomial and Hypergeometric Probabilities

3.2.1 The number of days, k, the stock rises is binomial with n = 4 and p = 0.25. The stock will be the same after four days if k = 2. The probability that k = 2 is $\binom{4}{2}(0.25)^2(0.75)^2 = 0.211$

3.2.2 Let k be the number of control rods properly inserted. The system fails if $k \le 4$. The probability of that occurrence is given by the binomial probability sum
$\sum_{k=0}^{4}\binom{10}{k}(0.8)^k(0.2)^{10-k} = 0.0064$

3.2.3 The probability of 12 female presidents is 0.23^{12}, which is approximately 1/50,000,000.

3.2.4 $1-\binom{6}{0}(0.153)^0(0.847)^6-\binom{6}{1}(0.153)^1(0.847)^5 = 0.231$

3.2.5 $1-\sum_{k=8}^{11}\binom{11}{k}(0.9)^k(0.1)^{11-k} = 0.0185$

3.2.6 The probability of k sightings is given by the binomial probability model with n = 10,000 and p = 1/100,000. The probability of at least one genuine sighting is the probability that $k \ge 1$. The probability of the complementary event, k = 0, is $(99{,}999/100{,}000)^{10{,}000} = 0.905$.
Thus, the probability that $k \ge 1$ is 1 − 0.905 = 0.095.

3.2.7 For the two-engine plane, P(Flight lands safely) = P(One or two engines work properly)
= $\binom{2}{1}(0.6)^1(0.4)^1+\binom{2}{2}(0.6)^2(0.4)^0 = 0.84$
For the four-engine plane, P(Flight lands safely) = P(Two or more engines work properly)
= $\binom{4}{2}(0.6)^2(0.4)^2+\binom{4}{3}(0.6)^3(0.4)^1+\binom{4}{4}(0.6)^4(0.4)^0 = 0.8208$
The two-engine plane is a better choice.

3.2.8 Probabilities for the first system are binomial with n = 50 and p = 0.05. The probability that $k \ge 1$ is $1-(0.95)^{50}$ = 1 − 0.077 = 0.923.
Probabilities for the second system are binomial with n = 100 and p = 0.02. The probability that $k \ge 1$ is $1-(0.98)^{100}$ = 1 − 0.133 = 0.867
System 2 is superior from a bulb replacement perspective.

3.2.9 The number of 6's obtained in n tosses is binomial with p = 1/6. The first probability in question has n = 6. The probability that $k \geq 1$ is $1-(5/6)^6$ = 1 – 0.33 = 0.67.
For the second situation, n = 12. The probability that $k \geq 2$ one minus the probability that k = 0 or 1, which is $1-(5/6)^{12}-12(1/6)(5/6)^{11}$ = 0.62. Finally, take n = 18. The probability that $k \geq 3$ is one minus the probability that k = 0, 1, or 2, which is
$1-(5/6)^{18}-18(1/6)(5/6)^{17}-153(1/6)^2(5/6)^{16}$ = 0.60.

3.2.10 The number of missile hits on the plane is binomial with n = 6 and p = 0.2. The probability that the plane will crash is the probability that $k \geq 2$. This event is the complement of the event that k = 0 or 1, so the probability is $1 - (0.8)^6 - 6(0.2)(0.8)^5 = 0.345$
The number of rocket hits on the plane is also binomial, but with n = 10 and p = 0.05.
The probability that the boat will be disabled is $P(k \geq 1)$, which is $1 - (0.95)^{10} = 0.401$

3.2.11 The number of girls is binomial with n = 4 and p = 1/2. The probability of two girls and two boys is $\binom{4}{2}(0.5)^4 = 0.375$. The probability of three and one is $2\binom{4}{3}(0.5)^4$ = 0.5, so the latter is more likely.

3.2.12 The number of recoveries if the drug is effective is binomial with n = 12 and p = 1/2. The drug will discredited if the number of recoveries is 6 or less. The probability of this is
$\sum_{k=0}^{6}\binom{12}{k}(0.5)^{12}$ = 0.613.

3.2.13 The probability it takes k calls to get four drivers is $\binom{k-1}{3}0.80^4 0.20^{k-4}$. We seek the smallest number n so that $\sum_{k=4}^{n}\binom{k-1}{3}0.80^4 0.20^{k-4} \geq 0.95$. By trial and error, n = 7.

3.2.14 The probability of any shell hitting the bunker is 30/500 = 0.06. The probability of exactly k shells hitting the bunker is $p(k) = \binom{25}{k}(0.06)^k(0.94)^{25-k}$. The probability the bunker is destroyed is $1 - p(0) - p(1) - p(2) = 0.187$.

3.2.15 (1) The probability that any one of the seven measurements will be in the interval (1/2, 1) is 0.50. The probability that exactly three will fall in the interval is
$\binom{7}{3}0.5^7$ = 0.273

(2) The probability that any one of the seven measurements will be in the interval (3/4, 1) is 0.25. The probability that fewer than 3 will fall in the interval is
$\sum_{k=0}^{2}\binom{7}{k}(0.25)^k(0.75)^{7-k}$ = 0.756

3.2.16 Use the methods of Example 3.2.3 for $p = 0.5$. Then the probabilities of Team A winning the series in 4, 5, 6, and 7 games are 0.0625, 0.125, 0.156, and 0.156, respectively. Since Team B has the same set of probabilities, the probability of the series ending in 4, 5, 6, and 7 games is double that for *A*, or 0.125, 0.250, 0.312, and 0.312, respectively. The "expected" frequencies are the number of years, 58, times the probability of each length. For example, we would "expect" 58(0.126) = 7.3 series of 4 games. The table below gives the comparison of observed and expected frequencies.

Number of games	Observed Number of years	Expected Number of years
4	12	58(0.125) = 7.3
5	10	58(0.250) = 14.5
6	12	58(0.312) = 18.1
7	24	58(0.312) = 18.1

Note that the model has equal expected frequencies for 6 and 7 length series, but the observed numbers are quite different. This model does not fit the data well.

3.2.17 By the binomial theorem, $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$. Let $x = p$ and $y = 1 - p$.

Then $1 = [p + (1 - p)]^n = \sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}$

3.2.18 Any particular sequence having k_1 of Outcome 1 and k_2 of Outcome 2, must have $n - k_1 - k_2$ of Outcome 3. The probability of such a sequence is $p_1^{k_1} p_2^{k_2}(1-p_2-p_2)^{n-k_1-k_2}$.

The number of such sequences depends on the number of ways to choose the k_1 positions in the sequence for Outcome 1 and the k_2 positions for Outcome 2. The k_1 positions can be chosen in $\binom{n}{k_1}$ ways. For each such choice, the k_2 positions can be chosen in $\binom{n-k_1}{k_2}$ ways. Thus, $P(k_1$ of Outcome 1 and k_2 of Outcome 2) $= \binom{n}{k_1}\binom{n-k_1}{k_2}p_1^{k_1}p_2^{k_2}(1-p_1-p_2)^{n-k_1-k_2}$

$$= \frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{k_2!(n-k_1-k_2)!}p_1^{k_1}p_2^{k_2}(1-p_1-p_2)^{n-k_1-k_2}$$

$$= \frac{n!}{k_1!k_2!(n-k_1-k_2)!}p_1^{k_1}p_2^{k_2}(1-p_1-p_2)^{n-k_1-k_2}$$

3.2.19 In the notation of Question 3.2.18, $p_1 = 0.5$ and $p_2 = 0.3$, with $n = 10$. Then the probability of 3 of Outcome 1 and 5 of Outcome 2 is $\frac{10!}{3!5!2!}(0.5)^3(0.3)^5(0.2)^2 = 0.031$

3.2.20 Use the hypergeometric model with $N = 12$, $n = 5$, $r = 4$, and $w = 12 - 4 = 8$. The probability that the committee will contain two accountants ($k = 2$) is $\dfrac{\binom{4}{2}\binom{8}{3}}{\binom{12}{5}} = 14/33$

3.2.21 "At least twice as many black bears as tan-colored" translates into spotting 4, 5, or 6 black bears. The probability is $\dfrac{\binom{6}{4}\binom{3}{2}}{\binom{9}{6}} + \dfrac{\binom{6}{5}\binom{3}{1}}{\binom{9}{6}} + \dfrac{\binom{6}{6}\binom{3}{0}}{\binom{9}{6}} = 64/84$

3.2.22 The probabilities are hypergeometric with $N = 4050$, $n = 65$, $r = 514$, and $w = 4050 - 514 = 3536$. The probability that k children have not been vaccinated is

$\dfrac{\binom{514}{k}\binom{3536}{65-k}}{\binom{4050}{65}}$, $k = 0, 1, 2, \ldots, 65$

3.2.23 The probability that k nuclear missiles will be destroyed by the anti-ballistic missiles is hypergeometric with $N = 10$, $n = 7$, $r = 6$, and $w = 10 - 6 = 4$. The probability the Country B will be hit by at least one nuclear missile is one minus the probability that $k = 6$, or

$1 - \dfrac{\binom{6}{6}\binom{4}{1}}{\binom{10}{7}} = 0.967$

3.2.24 Let k be the number of questions chosen that Anne has studied. Then the probabilities for k are hypergeometric with $N = 10$, $n = 5$, $r = 8$, and $w = 10 - 8 = 2$. The probability of her getting at least four correct is the probability that $k = 4$ or 5, which is

$\dfrac{\binom{8}{4}\binom{2}{1}}{\binom{10}{5}} + \dfrac{\binom{8}{5}\binom{2}{0}}{\binom{10}{5}} = \dfrac{140}{252} + \dfrac{56}{252} = 0.778$

3.2.25 The probabilities for the number of men chosen are hypergeometric with $N = 18$, $n = 5$, $r = 8$, and $w = 10$. The event that both men and women are represented is the complement of the event that 0 or 5 men will be chosen, or $1 - \dfrac{\binom{8}{0}\binom{10}{5}}{\binom{18}{5}} + \dfrac{\binom{8}{5}\binom{10}{0}}{\binom{18}{5}} = 1 - \dfrac{252}{8568} + \dfrac{56}{8568} = 0.964$

3.2.26 The probability is hypergeometric with $N = 80$, $n = 10$, $r = 20$, and $w = 60$, and equals

$$\frac{\binom{20}{6}\binom{60}{4}}{\binom{80}{10}} = 0.0115$$

3.2.27 First, calculate the probability that exactly one real diamond is taken during the first three grabs. There are three possible positions in the sequence for the real diamond, so this probability is

$$\frac{3(10)(25)(24)}{(35)(34)(33)}.$$

The probability of a real diamond being taken on the fourth removal is 9/32. Thus, the desired probability is $\frac{3(10)(25)(24)}{(35)(34)(33)} \times \frac{9}{32} = \frac{162{,}000}{1{,}256{,}640} = 0.129.$

3.2.28 $\dfrac{\binom{2}{2}\binom{8}{0} + \binom{6}{2}\binom{4}{0} + \binom{2}{2}\binom{8}{0}}{\binom{10}{2}} = \dfrac{1+15+1}{45} = \dfrac{17}{45} = 0.378$

3.2.29 The k-th term of $(1 + \mu)^N = \binom{N}{k}\mu^k$

$$(1 + \mu)^r\,(1 + \mu)^{N-r} = \left(\sum_{i=1}^{r}\binom{r}{i}\mu^i\right)\left(\sum_{j=1}^{N-r}\binom{N-r}{j}\mu^j\right)$$

The k-th term of this product is $\sum_{i=1}^{r}\binom{r}{i}\binom{N-r}{k-i}\mu^k$

Equating coefficients gives $\binom{N}{k} = \sum_{i=1}^{k}\binom{r}{i}\binom{N-r}{k-i}$.

Dividing through by $\binom{N}{k}$ shows that the hypergeometric terms sum to 1.

3.2.30 $$\frac{\binom{r}{k+1}\binom{w}{n-k-1}}{\binom{N}{n}} \div \frac{\binom{r}{k}\binom{w}{n-k}}{\binom{N}{n}} = \binom{r}{k+1}\binom{w}{n-k-1} \div \binom{r}{k}\binom{w}{n-k}$$

$$= \binom{r}{k+1}\binom{w}{n-k-1} \div \binom{r}{k}\binom{w}{n-k}$$

$$= \frac{r!}{(k+1)!(r-k-1)!} \cdot \frac{w!}{(n-k-1)!(w-n+k+1)!} \cdot \frac{k!(r-k)!}{r!} \cdot \frac{(n-k)!(w-n+k)!}{w!}$$

$$= \frac{n-k}{(k+1)} \cdot \frac{r-k}{(w-n+k+1)}.$$

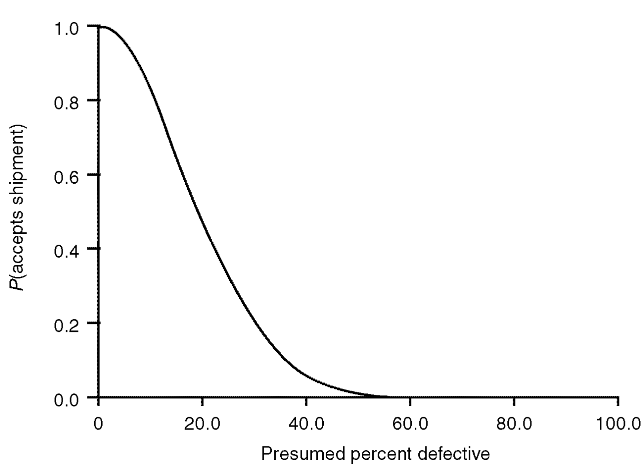
3.2.31 Let W_0, W_1 and W_2 be the events of drawing zero, one, or two white chips, respectively, from Urn I. Let A be the event of drawing a white chip from Urn II. Then
$P(\text{a}) = P(A|W_0)P(W_0) + P(A|W_1)P(W_1) + P(A|W_2)P(W_2)$

$$= \frac{5}{11}\frac{\binom{5}{2}\binom{4}{0}}{\binom{9}{2}} + \frac{6}{11}\frac{\binom{5}{1}\binom{4}{1}}{\binom{9}{2}} + \frac{7}{11}\frac{\binom{5}{0}\binom{4}{2}}{\binom{9}{2}} = 53/99$$

3.2.32 For any value of r = number of defective items, the probability of accepting the sample is

$$p_r = \frac{\binom{r}{0}\binom{100-r}{10}}{\binom{100}{10}} + \frac{\binom{r}{1}\binom{100-r}{9}}{\binom{100}{10}}$$

Then the operating characteristic curve is the plot of the presumed percent defective versus the probability of accepting the shipment, or $100(r/100) = r$ on the x-axis and p_r on the y-axis. If there are 16 defective, you will accept the shipment approximately 50% of the time.



3.2.33 There are $\frac{r!}{r_1!r_2!r_3!}$ ways to divide the red chips into three groups of the given sizes. There are $\frac{(N-r)!}{(n_1-r_1)!(n_2-r_2)!(n_3-r_3)!}$ ways to divide the white chips into the three groups of the required sizes. The total number of ways to divide the N objects into groups of n_1, n_2, and n_3 objects is $\frac{N!}{n_1!n_2!n_3!}$. Thus, the desired probability is $\dfrac{\frac{r!}{r_1!r_2!r_3!}\frac{(N-r)!}{(n_1-r_1)!(n_2-r_2)!(n_3-r_3)!}}{\frac{N!}{n_1!n_2!n_3!}} = \dfrac{\binom{n_1}{r_1}\binom{n_2}{r_2}\binom{n_3}{r_3}}{\binom{N}{r}}$.

3.2.34 First, calculate the probability that the first group contains two disease carriers and the others have one each. The probability of this, according to Question 3.2.33, is $\dfrac{\binom{7}{2}\binom{7}{1}\binom{7}{1}}{\binom{21}{4}}$ = 49/285.

The probability that either of the other two groups has 2 carriers and the others have one is the same. Thus, the probability that each group has at least one diseased member is $3\frac{49}{285} = \frac{49}{95}$ = 0.516. Then the probability that at least one group is disease free is $1 - 0.516 = 0.484$.

3.2.35 There are $\binom{N}{n}$ total ways to choose the sample. There are $\binom{n_i}{k_i}$ ways to arrange for k_i of the n_i objects to be chosen, for each i. Using the multiplication rule shows that the probability of getting k_1 objects of the first kind, k_2 objects of the second kind, …, k_t objects of the t-th kind is $\dfrac{\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_t}{k_t}}{\binom{N}{n}}$

3.2.36 In the notation of Question 3.2.33, let $n_1 = 5$, $n_2 = 4$, $n_3 = 4$, $n_4 = 3$, so $N = 16$. The sample size is given to be $n = 8$, and $k_1 = k_2 = k_3 = k_4 = 2$. Then the probability that each class has two representatives is $\dfrac{\binom{5}{2}\binom{4}{2}\binom{4}{2}\binom{3}{2}}{\binom{16}{8}} = \dfrac{(10)(6)(6)(3)}{12{,}870} = \dfrac{1080}{12{,}870}$ = 0.084.

Section 3.3: Discrete Random Variables

3.3.1 **(a)** Each outcome has probability 1/10

Outcome	X = larger no. drawn
1, 2	2
1, 3	3
1, 4	4
1, 5	5
2, 3	3
2, 4	4
2, 5	5
3, 4	4
3, 5	5
4, 5	5

Counting the number of each value of the larger of the two and multiplying by 1/10 gives the pdf:

k	$p_X(k)$
2	1/10
3	2/10
4	3/10
5	4/10

(b)

Outcome	X = larger no. drawn	V = sum of two nos.
1, 2	2	3
1, 3	3	4
1, 4	4	5
1, 5	5	6
2, 3	3	5
2, 4	4	6
2, 5	5	7
3, 4	4	7
3, 5	5	8
4, 5	5	9

k	$p_X(k)$
3	1/10
4	1/10
5	2/10
6	2/10
7	2/10
8	1/10
9	1/10

3.3.2 **(a)** There are $5 \times 5 = 25$ total outcomes. The set of outcomes leading to a maximum of k is $(X = k) = \{(j, k) | 1 \le j \le k - 1\} \cup \{(k, j) | 1 \le j \le k - 1\} \cup \{(k, k\}$, which has $2(k - 1) + 1 = 2k - 1$ elements. Thus, $p_X(k) = (2k - 1)/25$

(b)

Outcomes	V = sum of two nos.
(1, 1)	2
(1, 2) (2, 1)	3
(1, 3) (2, 2) (3, 1)	4
(1, 4) (2, 3) (3, 2) (4, 1)	5
(1, 5) (2, 4) (3, 3) (4, 2) (5, 1)	6
(2, 5) (3, 4) (4, ,3) (5, 2)	7
(3, 5) (4, 4), (5, 3)	8
(4, 5) (5, 4)	9
(5, 5)	10

$p_V(k) = (k - 1)/25$ for $k = 1, 2, 3, 4, 5, 6$ and $p_V(k) = (11 - k)/25$ for $k > 6$

3.3.3 $p_X(k) = P(X = k) = P(X \le k) - P(X \le k - 1)$. But the event $(X \le k)$ occurs when all three dice are $\le k$ and that can occur in k^3 ways. Thus $P(X \le k) = k^3/216$.
Similarly, $P(X \le k - 1) = (k - 1)^3/216$. Thus $p_X(k) = k^3/216 - (k - 1)^3/216$.

3.3.4 $p_X(1) = 6/6^3 = 6/216 = 1/36$
$p_X(2) = 3(6)(5)/6^3 = 90/216 = 15/36$
$p_X(3) = (6)(5)(4)/6^3 = 120/216 = 20/36$

3.3.5

Outcomes	V = no. heads – no. tails
(H, H, H)	3
(H, H, T) (H, T, H) (T, H, H)	1
(T, T, H) (T, H, T) (T, H, H)	−1
(T, T, T)	−3

$p_X(3) = 1/8$, $p_X(1) = 3/8$, $p_X(-1) = 3/8$, $p_X(-3) = 1/8$

3.3.6

Outcomes	k	$p_X(k)$
(1, 1)	2	(1/6)(1/6) = 1/36
(2, 1)	3	(2/6)(1/6) = 2/36
(1, 3) (3, 1)	4	(1/6)(1/6) + (2/6)(1/6) = 3/36
(1, 4) (2, 3) (4, 1)	5	(1/6)(1/6) + (2/6)(1/6) + (1/6)(1/6) = 4/36
(1, 5) (2, 4) (3, 3)	6	(1/6)(1/6) + (2/6)(1/6) + (2/6)(1/6) = 5/36
(1, 6) (2, 5) (3, 4) (4, 3)	7	(1/6)(1/6) +(2/6)(1/6)+ (2/6)(1/6)+ (1/6)(1/6) = 6/36
(2, 6) (3, 5) (4, 4)	8	(2/6)(1/6) + (2/6)(1/6) + (1/6)(1/6) = 5/36
(1, 8) (3, 6) (4, 5)	9	(1/6)(1/6) + (2/6)(1/6) + (1/6)(1/6) = 4/36
(2, 8) (4, 6)	10	(2/6)(1/6) + (1/6)(1/6) = 3/36
(3, 8)	11	(2/6)(1/6) = 2/36
(4, 8)	12	(1/6)(1/6) = 1/36

3.3.7 This is similar to Question 3.3.5. If there are k steps to the right (heads), then there are $4 - k$ steps to the left (tails). The final position X is number of heads $-$ number of tails $= k - (4 - k) = 2k - 4$.

The probability of this is the binomial of getting k heads in 4 tosses = $\binom{4}{k}\frac{1}{16}$.

Thus, $p_X(2k-4) = \binom{4}{k}\frac{1}{16}$, $k = 0, 1, 2, 3, 4$

3.3.8 $p_X(2k-4) = \binom{4}{k}\left(\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{4-k}$, $k = 0, 1, 2, 3, 4$

3.3.9 Consider the case $k = 0$ as an example. If you are on the left, with your friend on your immediate right, you can stand in positions 1, 2, 3, or 4. The remaining people can stand in 3! ways. Each of these must be multiplied by 2, since your friend could be the one on the left. The total number of permutations of the five people is 5!
Thus, $p_X(0) = (2)(4)(3!)/5! = 48/120 = 4/10$. In a similar manner
$p_X(1) = (2)(3)(3!)/5! = 36/120 = 3/10$
$p_X(2) = (2)(2)(3!)/5! = 24/120 = 2/10$
$p_X(3) = (2)(1)(3!)/5! = 12/120 = 1/10$

3.3.10 $p_{X_1}(k) = p_{X_2}(k) = \dfrac{\binom{2}{k}\binom{2}{2-k}}{\binom{4}{2}}$, $k = 0, 1, 2$. For $X_1 + X_2 = m$, let

$X_1 = k$ and $X_2 = m - k$, for $k = 0, 1, \ldots$, m. Then $p_{X_3}(m) = \sum_{k=0}^{m} p_{X_1}(k)p_{X_2}(m-k)$,

$m = 0, 1, 2, 3, 4$, or

m	$p_{X_3}(m)$
0	1/36
1	2/9
2	1/2
3	2/9
4	1/36

3.3.11 $P(2X+1=k) = P[X=(k-1)/2]$, so $p_{2X+1}(k) = p_X\left(\frac{k-1}{2}\right) = \binom{4}{\frac{k-1}{2}}\left(\frac{2}{3}\right)^{\frac{k-1}{2}}\left(\frac{1}{3}\right)^{4-\frac{k-1}{2}}$,

$k = 1, 3, 5, 7, 9$

3.3.12 $F_X(k) = P(X \le k) = k^3$, as explained in the solution to Question 3.3.3.

3.3.13 $F_X(k) = P(X \le k) = \sum_{j=0}^{k} P(X=j) = \sum_{j=0}^{k}\binom{4}{j}\left(\frac{1}{6}\right)^j\left(\frac{5}{6}\right)^{4-j}$

3.3.14 $p_x(k) = F_X(k) - F_X(k-1) = \dfrac{k(k+1)}{42} - \dfrac{(k-1)k}{42} = \dfrac{k}{21}$

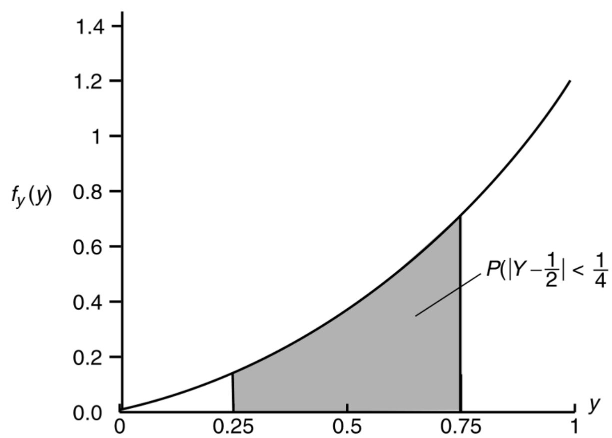
3.3.15 See the solution to Question 3.3.3.

Section 3.4: Continuous Random Variables

3.4.1 $P(0 \le Y \le 1/2) = \int_0^{1/2} 4y^3 dy = y^4\Big|_0^{1/2} = 1/16$

3.4.2 $P(3/4 \le Y \le 1) = \int_{3/4}^{1} \dfrac{2}{3} + \dfrac{2}{3}y\, dy = \dfrac{2y}{3} + \dfrac{y^2}{3}\Big|_{3/4}^{1} = 1 - \dfrac{11}{16} = \dfrac{5}{16}$

3.4.3 $P\left(\left|Y - 1/2\right| < 1/4\right) = P(1/4 < Y < 3/4) = \int_{1/4}^{3/4} \dfrac{3}{2}y^2\, dy = \dfrac{y^3}{2}\Big|_{1/4}^{3/4} = \dfrac{27}{128} - \dfrac{1}{128} = \dfrac{26}{128} = \dfrac{13}{64}$



3.4.4 $P(Y > 1) = \int_1^3 (1/9)y^2 dy = (1/27)\,y^3\Big|_1^3 = 1 - 1/27 = 26/27$

3.4.5 **(a)** $\int_{10}^{\infty} 0.2e^{-0.2y} dy = -e^{-0.2y}\Big|_{10}^{\infty} = e^{-2} = 0.135$

(b) If A = probability customer leaves on first trip, and B = probability customer leaves on second trip, then P(a) = P(b) = 0.135. In this notation,
$p_X(1) = P(a)P(B^C) + P(A^C)P(b) = 2(0.865)(0.135) = 0.23355$

3.4.6 Clearly the function given is non-negative and continuous. We must show that it integrates to 1.

$$\int_0^1 (n+2)(n+1)y^n(1-y)dy = \int_0^1 (n+2)(n+1)(y^n - y^{n+1})dy = (n+2)(n+1)\left(\frac{y^{n+1}}{n+1} - \frac{y^{n+2}}{n+2}\right)\Bigg|_0^1$$

$$= \left[(n+2)y^{n+1} - (n+1)y^{n+2}\right]\Big|_0^1 = 1$$

3.4.7 $F_Y(y) = P(Y \le y) = \int_0^y 4t^3 dt = t^4 \Big|_0^y = y^4$. Then $P(Y \le 1/2) = F_Y(1/2) = (1/2)^4 = 1/16$

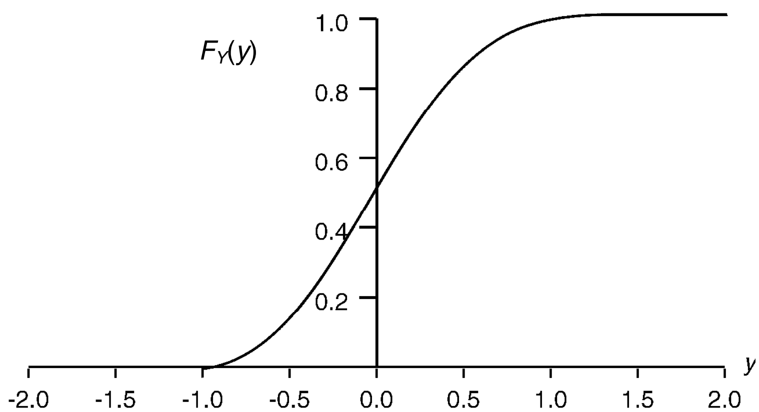
3.4.8 $F_Y(y) = P(Y \le y) = \int_0^y \lambda e^{-\lambda t} dt = -e^{-\lambda t} \Big|_0^y = 1 - e^{-\lambda y}$

3.4.9 For $y < -1$, $F_Y(y) = 0$

For $-1 \le y < 0$, $F_Y(y) = \int_{-1}^y (1+t)dt = \frac{1}{2} + y + \frac{1}{2}y^2$

For $0 \le y \le 1$, $F_Y(y) = \int_{-1}^y \left(1 - |t|\right)dt = \frac{1}{2} + \int_0^y (1-t)dt = \frac{1}{2} + y - \frac{1}{2}y^2$

For $y > 1$, $F_Y(y) = 1$



3.4.10 (1) $P(1/2 < Y \le 3/4) = F_Y(3/4) - F_Y(1/2) = (3/4)^2 - (1/2)^2 = 0.3125$

(2) $f_Y(y) = \frac{d}{dy} F_Y = \frac{d}{dy} y^2 = 2y$, $0 \le y < 1$

$P(1/2 < Y \le 3/4) = \int_{1/2}^{3/4} 2y\,dy = y^2 \Big|_{1/2}^{3/4} = 0.3125$

3.4.11 **(a)** $P(Y < 2) = F_Y(2)$, since F_Y is continuous over [0, 2]. Then $F_Y(2) = \ln 2 = 0.693$

(b) $P(2 < Y \le 2.5) = F_Y(2.5) - F_Y(2) = \ln 2.5 - \ln 2 = 0.223$

(c) The probability is the same as (b) since F_Y is continuous over $[0, e]$

(d) $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \ln y = \frac{1}{y}$, $1 \le y \le e$

3.4.12 First note that $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(4y^3 - 3y^4) = 12y^2 - 12y^3$, $0 \le y \le 1$.

Then $P(1/4 < Y \le 3/4) = \int_{1/4}^{3/4} (12y^2 - 12y^3)dy = (4y^3 - 3y^4)\Big|_{1/4}^{3/4} = 0.6875$.

3.4.13 $f_Y(y) = \dfrac{d}{dy}\dfrac{1}{12}(y^2 + y^3) = \dfrac{1}{6}y + \dfrac{1}{4}y^2, 0 \le y \le 2$

3.4.14 Integrating by parts, we find that $F_Y(y) = \int_0^y te^{-t}dt = -te^{-t} - e^{-t}\Big|_0^y = 1 - (1+y)e^{-y}$.

3.4.15 $F'(y) = -1(1+e^{-y})^{-2}(-e^{-y}) = \dfrac{e^{-y}}{(1+e^{-y})^2} > 0$, so $F(y)$ is increasing. The other two assertions follow from the facts that $\lim\limits_{y\to-\infty} e^{-y} = \infty$ and $\lim\limits_{y\to\infty} e^{-y} = 0$.

3.4.16 $F_W(w) = P(W \le w) = P(2Y \le w) = P(Y \le w/2) = F_Y(w/2)$

$f_W(w) = \dfrac{d}{dw}F(w) = \dfrac{d}{dw}F_Y(w/2) = f_W(w/2)\cdot(w/2)' = \dfrac{1}{2}4(w/2)^3 = \dfrac{1}{4}w^3$

where $2(0) \le w \le 2(1)$ or $0 \le w \le 2$

3.4.17 $P(-a < Y < a) = P(-a < Y \le 0) + P(0 < Y < a)$

$= \int_{-a}^{0} f_Y(y)dy + \int_0^a f_Y(y)dy = -\int_a^0 f_Y(-y)dy + \int_0^a f_Y(y)dy$

$= \int_0^a f_Y(y)dy + \int_0^a f_Y(y)dy = 2[F_Y(a) - F_Y(0)]$

But by the symmetry of f_Y, $F_Y(0) = 1/2$. Thus, $2[F_Y(a) - F_Y(0)] = 2[F_Y(a) - 1/2] = 2F_Y(a) - 1$

3.4.18 $F_Y(y) = \int_0^y (1/\lambda)e^{-t/\lambda}dt = 1 - e^{-t/\lambda}$, so

$h(y) = \dfrac{(1/\lambda)e^{-y/\lambda}}{1-(1-e^{-y/\lambda})} = 1/\lambda$

Since the hazard rate is constant, the item does not age. Its reliability does not decrease over time.

Section 3.5: Expected Values

3.5.1 $E(X) = -1(0.935) + 2(0.0514) + 18(0.0115) + 180(0.0016) + 1{,}300(1.35 \times 10^{-4})$
$+ 2{,}600(6.12 \times 10^{-6}) + 10{,}000(1.12 \times 10^{-7}) = -0.144668$

3.5.2 Let X be the winnings of betting on red in Monte Carlo. Then $E(X) = \dfrac{18}{37} - \dfrac{19}{37} = \dfrac{-1}{37}$.

Let X* be the winnings of betting on red in Las Vegas. Then $E(X) = \dfrac{18}{38} - \dfrac{20}{38} = \dfrac{-2}{38}$.

The amount bet, M, is the solution to the equation $M\left(\dfrac{-1}{37} - \dfrac{-2}{38}\right) = \$3{,}000$ or M is approximately equal to \$117, 167.

3.5.3 $E(X)$ = \$30,000(0.857375) + \$18,000(0.135375) + \$6,000(0.007125) + (–\$6,000)(0.000125)
= \$28,200.00

3.5.4 Rule A: Expected value = $-5 + 0 \cdot \frac{\binom{2}{0}\binom{4}{2}}{\binom{6}{2}} + 2 \cdot \frac{\binom{2}{1}\binom{4}{1}}{\binom{6}{2}} + 10 \cdot \frac{\binom{2}{2}\binom{4}{0}}{\binom{6}{2}} = -49/15$

Rule B: Expected value = $-5 + 0 \cdot \frac{\binom{2}{0}\binom{4}{2}}{\binom{6}{2}} + 1 \cdot \frac{\binom{2}{1}\binom{4}{1}}{\binom{6}{2}} + 20 \cdot \frac{\binom{2}{2}\binom{4}{0}}{\binom{6}{2}} = -47/15$

Neither game is fair to the player, but Rule B has the better payoff.

3.5.5 P is the solution to the equation $\sum_{k=1}^{5}[kP(1-p_k) - 50{,}000 p_k] = P\sum_{k=1}^{5} k(1-p_k) - 50{,}000\sum_{k=1}^{5} p_k$ $= 1000$, where p_k is the probability of death in year k, k = 1, 2, 3, 4, 5.
Since $\sum_{k=1}^{5} p_k = 0.00272$ and $\sum_{k=1}^{5} k(1-p_k) = 4.99164$, the equation becomes
4.99164P – 50,000(0.00272) = 1000, or P = \$227.58.

3.5.6 The random variable X is hypergeometric, where $r = 4$, $w = 96$, $n = 20$. Then $E(X) = \frac{4(20)}{4+96} = \frac{4}{5}$.

3.5.7 This is a hypergeometric problem where r = number of students needing vaccinations = 125 and w = number of students already vaccinated = 642 – 125 = 517. An absenteeism rate of 12% corresponds to a sample $n = (0.12)(642) \doteq 77$ missing students. The expected number of unvaccinated students who are absent when the physician visits is $\frac{125(77)}{125+517} \doteq 15$.

3.5.8 **(a)** $E(Y) = \int_0^1 y \cdot 3(1-y)^2\,dy = \int_0^1 3(y - 2y^2 + y^3)\,dy$

$= 3\left[\frac{1}{2}y^2 - \frac{2}{3}y^3 + \frac{1}{4}y^4\right]_0^1 = \frac{1}{4}$

(b) $E(Y) = \int_0^\infty y \cdot 4ye^{-2y}\,dy = 4\left[-\frac{1}{2}y^2e^{-2y} - \frac{1}{2}ye^{-2y} - \frac{1}{4}e^{-2y}\right]_0^\infty = 1$

(c) $E(Y) = \int_0^1 y \cdot \left(\frac{3}{4}\right)dy + \int_2^3 y \cdot \left(\frac{1}{4}\right)dy = \left.\frac{3y^2}{8}\right|_0^1 + \left.\frac{y^2}{8}\right|_2^3 = 1$

(d) $E(Y) = \int_0^{\pi/2} y \cdot \sin y\,dy = (-y\cos y + \sin y)\Big|_0^{\pi/2} = 1$

3.5.9 $E(Y) = \int_0^3 y\left(\frac{1}{9}y^2\right)dy = \frac{1}{9}\int_0^3 y^3\,dy = \left.\frac{y^4}{36}\right|_0^3 = \frac{9}{4}$ years

3.5.10 $E(Y) = \int_a^b y\frac{1}{b-a}dy = \frac{y^2}{2(b-a)}\bigg|_a^b = \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} = \frac{b+a}{2}$. This simply says that a uniform bar will balance at its middle.

3.5.11 $E(Y) = \int_0^\infty y \cdot \lambda e^{-\lambda y}dy = \left(-ye^{-\lambda y} - \frac{1}{\lambda}e^{-\lambda y}\right)\bigg|_0^\infty = \frac{1}{\lambda}$

3.5.12 Since $\frac{1}{y^2} \geq 0$, this function will be a pdf if its integral is 1, and $\int_1^\infty \frac{1}{y^2}dy = -\frac{1}{y}\bigg|_1^\infty = 1$. However, what would be its expected value is $\int_1^\infty y\frac{1}{y^2}dy = \int_1^\infty \frac{1}{y}dy = \ln y\big|_1^\infty$, but this last quantity is infinite.

3.5.13 Let X be the number of cars passing the emissions test. Then X is binomial with $n = 200$ and $p = 0.80$. Two formulas for $E(X)$ are:

(1) $E(X) = \sum_{k=1}^{n} k\binom{n}{k}p^k(1-p)^{n-k} = \sum_{k=1}^{200} k\binom{200}{k}(0.80)^k(0.20)^{200-k}$

(2) $E(X) = np = 200(0.80) = 160$

3.5.14 The probability that an observation of Y lies in the interval (1/2, 1) is $\int_{1/2}^{1} 3y^2dy = y^3\big|_{1/2}^{1} = \frac{7}{8}$

Then X is binomial with $n = 15$ and $p = 7/8$. $E(X) = 15(7/8) = 105/8$.

3.5.15 If birthdays are randomly distributed throughout the year, the city should expect revenue of (\$50)(74,806)(30/365) or \$307,421.92.

3.5.16 If we assume that the probability of bankruptcy due to fraud is 23/68, then we can expect 9(23/68) = 3.04, or roughly 3 of the 9 additional bankruptcies will be due to fraud.

3.5.17 For the experiment described, construct the table:

Sample	Larger of the two, k
1, 2	2
1, 3	3
1, 4	4
2, 3	3
2, 4	4
3, 4	4

Each of the six samples is equally likely to be drawn, so $p_X(2) = 1/6$, $p_X(3) = 2/6$, and $p_X(4) = 3/6$. Then $E(X) = 2(1/6) + 3(2/6) + 4(3/6) = 20/6 = 10/3$.

3.5.18

Outcome	X
HHH	6
HHT	2
HTH	4
HTT	1
THH	2
THT	0
TTH	1
TTT	0

From the table, we can calculate $p_X(0) = 1/4$, $p_X(1) = 1/4$, $p_X(2) = 1/4$, $p_X(4) = 1/8$, $p_X(6) = 1/8$. Then $E(X) = 0 \cdot (1/4) + 1 \cdot (1/4) + 2 \cdot (1/4) + 4 \cdot (1/8) + 6 \cdot (1/8) = 2$.

3.5.19 The "fair" ante is the expected value of X, which is

$$\sum_{k=1}^{9} 2^k\left(\frac{1}{2^k}\right) + \sum_{k=10}^{\infty} 1000\left(\frac{1}{2^k}\right) = 9 + \frac{1000}{2^{10}}\sum_{k=0}^{\infty}\left(\frac{1}{2^k}\right) = 9 + \frac{1000}{2^{10}}\frac{1}{1-\frac{1}{2}} = 9 + \frac{1000}{512} = \frac{5608}{512} = \$10.95$$

3.5.20 **(a)** $E(X) = \sum_{k=1}^{\infty} c^k\left(\frac{1}{2}\right)^k = \sum_{k=1}^{\infty}\left(\frac{c}{2}\right)^k = \frac{c}{2}\sum_{k=0}^{\infty}\left(\frac{c}{2}\right)^k = \frac{c}{2-c}$

(b) $\sum_{k=1}^{\infty} \log 2^k\left(\frac{1}{2}\right)^k = \log 2\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k$

To evaluate the sum requires a special technique: For a parameter t, $0 < t < 1$, note that

$\sum_{k=1}^{\infty} t^k = \frac{t}{1-t}$. Differentiate both sides of the equation with respect to t to obtain

$\sum_{k=1}^{\infty} kt^{k-1} = \frac{1}{(1-t)^2}$. Multiplying both sides by t gives the desired equation:

$\sum_{k=1}^{\infty} kt^k = \frac{t}{(1-t)^2}$. In the case of interest, $t = 1/2$, so $\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k = 2$, and $E(X) = 2 \cdot \log 2$.

3.5.21 $p_X(1) = \frac{6}{216} = \frac{1}{36}$

$p_X(2) = \frac{3(6)(5)}{216} = \frac{15}{36}$

$p_X(3) = \frac{6(5)(4)}{216} = \frac{20}{36}$

$E(X) = 1 \cdot \frac{1}{36} + 2 \cdot \frac{15}{36} + 3 \cdot \frac{20}{36} = \frac{91}{36}$

3.5.22 For the experiment described, construct the table

Sample	Absolute value of difference
1, 2	1
1, 3	2
1, 4	3
1, 5	4
2, 3	1
2, 4	2
2, 5	3
3, 4	1
3, 5	2
4, 5	1

If X denotes the absolute value of the difference, then from the table:
$p_X(1) = 4/10$, $p_X(2) = 3/10$, $p_X(3) = 2/10$, $p_X(4) = 1/10$
$E(X) = 1(4/10) + 2(3/10) + 3(2/10) + 4(1/10) = 2$

3.5.23 Let X be the length of the series. Then $p_X(k) = 2\binom{k-1}{3}\left(\frac{1}{2}\right)^k$, $k = 4, 5, 6, 7$.

$$E(X) = \sum_{k=4}^{7}(k)(2)\binom{k-1}{3}\left(\frac{1}{2}\right)^k = 4\left(\frac{2}{16}\right)+5\left(\frac{4}{16}\right)+6\left(\frac{5}{16}\right)+7\left(\frac{5}{16}\right)=\frac{93}{16} = 5.8125$$

3.5.24 Let X = number of drawings to obtain a white chip. Then

$p_X(k) = \frac{1}{k}\cdot\frac{1}{k+1}$, $k = 1, 2, \dots$

$$E(X) = \sum_{k=1}^{\infty} k\left(\frac{1}{k(k+1)}\right) = \sum_{k=1}^{\infty}\frac{1}{k+1}.$$

For each n, let $T_n = \sum_{i=2^n}^{i=2^{n+1}}\frac{1}{i}$. Then $T_n \geq \frac{2^n}{2^{n+1}} = \frac{1}{2}$.

$\sum_{k=1}^{\infty}\frac{1}{k+1} \geq \sum_{n=1}^{\infty}T_n = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$ This last sum is infinite, so $E(X)$ does not exist.

3.5.25 $$E(X) = \sum_{k=1}^{r} k\frac{\binom{r}{k}\binom{w}{n-k}}{\binom{r+w}{n}} = \sum_{k=1}^{r} k\frac{\frac{r!}{k!(r-k)!}\binom{w}{n-k}}{\frac{(r+w)!}{n!(r+w-n)!}}$$

Factor out the presumed value of $E(X) = rn/(r + w)$:

$$E(X) = \frac{rn}{r+w}\sum_{k=1}^{r}\frac{\frac{(r-1)!}{(k-1)!(r-k)!}\binom{w}{n-k}}{\frac{(r-1+w)!}{(n-1)!(r+w-n)!}} = \frac{rn}{r+w}\sum_{k=1}^{r}\frac{\binom{r-1}{k-1}\binom{w}{n-k}}{\binom{r-1+w}{n-1}}$$

Change the index of summation to begin at 0, which gives

$E(X) = \dfrac{rn}{r+w}\sum_{k=0}^{r-1}\dfrac{\binom{r-1}{k}\binom{w}{n-1-k}}{\binom{r-1+w}{n-1}}$. The terms of the summation are urn probabilities where

there are $r-1$ red balls, w white balls, and a sample size of $n-1$ is drawn. Since these are the probabilities of a hypergeometric pdf, the sum is one. This leaves us with the desired equality $E(X) = \dfrac{rn}{r+w}$.

3.5.26 $E(X) = \sum_{j=1}^{\infty} jp_X(j) = \sum_{j=1}^{\infty}\sum_{k=1}^{j} p_X(j) = \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} p_X(j) = \sum_{k=1}^{\infty} P(X \geq k)$

3.5.27 **(a)** $0.5 = \int_0^m (\theta+1)y^\theta\, dy = y^{\theta+1}\Big|_0^m = m^{\theta+1}$, so $m = (0.5)^{\frac{1}{\theta+1}}$

(b) $0.5 = \int_0^m \left(y+\frac{1}{2}\right)dy = \left(\frac{y^2}{2}+\frac{y}{2}\right)\Big|_0^m = \frac{m^2}{2}+\frac{m}{2}$. Solving the quadratic equation

$\frac{1}{2}(m^2+m-1) = 0$ gives $m = \dfrac{-1+\sqrt{5}}{2}$.

3.5.28 $E(3X-4) = 3E(X) - 4 = 3(10)(2/5) - 4 = 8$

3.5.29 \$100(12)(0.11) = \$132

3.5.30 $E(W) = \int_0^1 w\left(\frac{1}{\sqrt{w}}-1\right)dw = \left(\frac{2}{3}w^{3/2}-\frac{w^2}{2}\right)\Big|_0^1 = 1/6$

Also, $E(W) = E(Y^2) = \int_0^1 y^2[2(1-y)]\,dy = \left(\frac{2}{3}y^3 - \frac{2}{4}y^4\right)\Big|_0^1 = 1/6$.

3.5.31 $E(Q) = \int_0^\infty 2(1-e^{-2y})6e^{-6y}dy = 12\int_0^\infty (e^{-6y}-e^{-8y})dy$

$= 12\left[-\frac{1}{6}e^{-6y}+\frac{1}{8}e^{-8y}\right]_0^\infty = \frac{1}{2}$, or \$50,000

3.5.32 $E(\text{Volume}) = \int_0^1 5y^2 6y(1-y)dy = 30\int_0^1 (y^3-y^4)dy = 30\left[\frac{1}{4}y^4 - \frac{1}{5}y^5\right]_0^1 = 1.5 \text{ in}^3$

3.5.33 Class average = $E(g(Y)) = \int_0^{100} 10y^{1/2}\frac{1}{5000}(100-y)dy$

$= \frac{1}{500}\int_0^1 (100y^{1/2} - y^{3/2})dy = \frac{1}{500}\left[\frac{2}{3}100y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^{100}$

= 53.3, so the professor's "curve" did not work.

3.5.34 $E(W) = \int_0^1 \left(y - \frac{2}{3} \right)^2 (2y)\, dy = 2\left(\frac{1}{4} y^4 - \frac{4}{9} y^3 + \frac{2}{9} y^2 \right)\Bigg|_0^1 = 1/36$

3.5.35 The area of the triangle = $\frac{1}{4} y^2$, so E(Area) = $\int_6^{10} \frac{1}{4} y^2 \frac{1}{10-6} dy = \frac{1}{16} \frac{y^3}{3}\Bigg|_6^{10}$ = 16.33.

3.5.36 1 = $\sum_{i=1}^{n} ki = k\frac{n(n+1)}{2}$ implies $k = \frac{2}{n(n+1)}$

$E\left(\frac{1}{X} \right) = \sum_{i=1}^{n} \frac{1}{i} \frac{2}{n(n+1)} i$ = 2/(n + 1)

Section 3.6: The Variance

3.6.1 If sampling is done with replacement, X is binomial with n = 2 and p = 2/5. By Theorem 3.5.1, μ = 2(2/5) = 4/5.
$E(X^2)$ = 0 · (9/25) + 1· (12/25) + 4 · (4/25) = 28/25. Then Var(X) = 28/25 − (4/5)^2 = 12/25.

3.6.2 $\mu = \int_0^1 y\left(\frac{3}{4} \right) dy + \int_2^3 y\left(\frac{1}{4} \right) dy = 1$

$E(X^2) = \int_0^1 y^2\left(\frac{3}{4} \right) dy + \int_2^3 y^2\left(\frac{1}{4} \right) dy = \frac{11}{6}$

Var(X) = $\frac{11}{6} - 1 = \frac{5}{6}$

3.6.3 Since X is hypergeometric, $\mu = \frac{3(6)}{10} = \frac{9}{5}$

$E(X^2) = \sum_{k=0}^{3} k^2 \frac{\binom{6}{k}\binom{4}{3-k}}{\binom{10}{3}}$ = 0 · (4/120) + 1 · (36/120) + 4 · (60/12) + 9 · (20/120) =
456/120 = 38/10
Var(X) = 38/10 − (9/5)^2 = 28/50 = 0.56, and σ = 0.748

3.6.4 μ = 1/2. $E(Y^2) = \int_0^1 y^2 (1) dy$ = 1/3. Var(Y) = 1/3 − (1/2)^2 = 1/12

3.6.5 $\mu = \int_0^1 y 3(1-y)^2 dy = 3\int_0^1 (y - 2y^2 + y^3) dy$ = 1/4

$E(Y^2) = \int_0^1 y^2 3(1-y)^2 dy = 3\int_0^1 (y^2 - 2y^3 + y^4) dy$ = 1/10

Var(Y) = 1/10 − (1/4)^2 = 3/80

3.6.6 $\mu = \int_0^k y \frac{2y}{k^2} dy = \frac{2k}{3}$. $E(Y^2) = \int_0^k y^2 \frac{2y}{k^2} dy = \frac{k^2}{2}$

$\text{Var}(Y) = \frac{k^2}{2} - \left(\frac{2k}{3}\right)^2 = \frac{k^2}{18}$. Var($Y$) = 2 implies $\frac{k^2}{18} = 2$ or $k = 6$.

3.6.7 $f_Y(y) = \begin{cases} 1-y, & 0 \le y \le 1 \\ 1/2, & 2 \le y \le 3 \\ 0, & \text{elsewhere} \end{cases}$

$\mu = \int_0^1 y(1-y)dy + \int_2^3 y\left(\frac{1}{2}\right)dy = 17/12$

$E(Y^2) = \int_0^1 y^2(1-y)dy + \int_2^3 y^2\left(\frac{1}{2}\right)dy = 13/4$

$\sigma = \sqrt{13/4 - (17/12)^2} = \sqrt{179}/12 = 1.115$

3.6.8 **(a)** $\int_1^\infty \frac{2}{y^3} dy = \left.\frac{-1}{y^2}\right|_1^\infty = 1$

(b) $E(Y) = \int_1^\infty y \frac{2}{y^3} dy = \left.\frac{-2}{y}\right|_1^\infty = 2$

(c) $E(Y^2) = \int_1^\infty y^2 \frac{2}{y^3} dy = 2\ln y \,\Big|_1^\infty$, which is infinite.

3.6.9 Let Y = Frankie's selection. Johnny wants to choose k so that $E[(Y-k)^2]$ is minimized. The minimum occurs when $k = E(Y) = (a+b)/2$ (see Question 3.6.13).

3.6.10 $E(Y) = \int_0^1 y(5y^4)dy = 5\int_0^1 y^5 dy = \left.\frac{5}{6}y^6\right|_0^1 = \frac{5}{6}$

$E(Y^2) = \int_0^1 y^2(5y^4)dy = 5\int_0^1 y^6 dy = \left.\frac{5}{7}y^7\right|_0^1 = \frac{5}{7}$

$\text{Var}(Y) = E(Y^2) - E(Y)^2 = \frac{5}{7} - \left(\frac{5}{6}\right)^2 = \frac{5}{7} - \frac{25}{36} = \frac{5}{252}$

3.6.11 Using integration by parts, we find that

$E(Y^2) = \int_0^\infty y^2 \lambda e^{-\lambda y} dy = -y^2 e^{-\lambda y}\Big|_0^\infty + \int_0^\infty 2ye^{-\lambda y} dy = 0 + \int_0^\infty 2ye^{-\lambda y} dy,$

The right hand term is $2\int_0^\infty ye^{-\lambda y} dy = \frac{2}{\lambda}\int_0^\infty y\lambda e^{-\lambda y} dy = \frac{2}{\lambda}E(Y) = \frac{2}{\lambda}\frac{1}{\lambda} = \frac{2}{\lambda^2}$.

Then $\text{Var}(Y) = E(Y^2) - E(Y)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$.

3.6.12 For the given Y, $E(Y) = 1/2$ and $\text{Var}(Y) = 1/4$. Then

$$P\left(Y > E(Y) + 2\sqrt{\text{Var}(Y)}\right) = P\left(Y > \frac{1}{2} + 2\sqrt{\frac{1}{4}}\right) = P\left(Y > \frac{3}{2}\right)$$

$$= \int_{3/2}^{\infty} 2e^{-2y}dy = 1 - \int_{\infty}^{3/2} 2e^{-2y}dy = 1 - (1 - e^{-2(3/2)}) = e^{-3}$$

$= 0.0498$

3.6.13 $E[(X - a)^2] = E[((X - \mu) + (\mu - a))^2]$
$= E[(X - \mu)^2] + E[(\mu - a)^2] + 2(\mu - a)E(X - \mu)$
$= \text{Var}(X) + (\mu - a)^2$, since $E(X - \mu) = 0$. This is minimized when $a = \mu$, so the minimum of $g(\mathbf{a})$ $= \text{Var}(X)$.

3.6.14 $\text{Var}(-5Y + 12) = (-5)^2\text{Var}(Y) = 25(3/80) = 15/16$

3.6.15 $\text{Var}\left(\frac{5}{9}(Y - 32)\right) = \left(\frac{5}{9}\right)^2 \text{Var}(Y)$, by Theorem 3.6.2. So $\sigma\left(\frac{5}{9}(Y - 32)\right) = \left(\frac{5}{9}\right)\sigma(Y) = \frac{5}{9}(15.7) =$ 8.7°C.

3.6.16 (1) $E[(W - \mu)/\sigma] = (1/\sigma)E[(W - \mu)] = 0$
(2) $\text{Var}[(W - \mu)/\sigma] = (1/\sigma^2)\text{Var}[(W - \mu)] = (1/\sigma^2)\sigma^2 = 1$

3.6.17 **(a)** $f_Y(y) = \dfrac{1}{b-a} f_U\left(\dfrac{y-a}{b-a}\right) = \dfrac{1}{b-a}$, $(b - a)(0) + a \leq y \leq (b - a)(1) + a$, or

$f_Y(y) = \dfrac{1}{b-a}$, $a \leq y \leq b$, which is the uniform pdf over $[a, b]$

(b) $\text{Var}(Y) = \text{Var}[(b - a)U + a] = (b - a)^2\ \text{Var}(U) = (b - a)^2/12$

3.6.18 $E(Y) = 5.5$ and $\text{Var}(Y) = 0.75$.

$$E(W_1) = 0.2281 + (0.9948)E(Y) + E(E_1) = 0.2281 + (0.9948)(5.5) + 0 = 5.6995$$

$$E(W_2) = -0.0748 + (1.0024)E(Y) + E(E_2) = -0.0748 + (1.0024)(5.5) + 0 = 5.4384$$

$$\text{Var}(W_1) = (0.9948)^2\,\text{Var}(Y) + \text{Var}(E_1) = (0.9948)^2(0.75) + 0.0427 = 0.7849$$

$$\text{Var}(W_2) = (1.0024)^2\,\text{Var}(Y) + \text{Var}(E_2) = (1.0024)^2(0.75) + 0.0159 = 0.7695$$

The above two equalities follow from the second corollary to Theorem 3.9.5. So the second procedure is better, since the mean of W_2 is closer to the true mean, and it has smaller variance.

3.6.19 $E(Y^r) = \displaystyle\int_0^2 y^r \frac{1}{2}dy = \frac{1}{2}\frac{y^{r+1}}{r+1}\bigg|_0^2 = \frac{2^r}{r+1}$

$$E[(Y - 1)^6] = \sum_{j=0}^{6}\binom{6}{j}E(Y^j)(-1)^{6-j} = \sum_{j=0}^{6}\binom{6}{j}\frac{2^r}{r+1}(-1)^{6-j}$$

$= (1)(1) + (-6)(1) + 15(4/3) + (-20)(2) + (15)(16/5) + (-6)(32/6) + (1)(64/7) = 1/7$

3.6.20 For the given f_Y, $\mu = 1$ and $\sigma = 1$.

$$\gamma_1 = \frac{E\left[(Y-1)^3\right]}{1} = \sum_{j=0}^{3}\binom{3}{j}E(Y^j)(-1)^{3-j} = \sum_{j=0}^{3}\binom{3}{j}(j!)(-1)^{3-j}$$

(1)(1)(−1) + (3)(1)(1) + (3)(2)(−1) + (1)(6)(1) = 2

3.6.21 For the uniform random variable U, $E(U) = 1/2$ and $\text{Var}(U) = 1/12$. Also the k-th moment of U is $E(U^k) = \int_0^1 u^k\,du = 1/(k+1)$. Then the coefficient of kurtosis is

$$\gamma_2 = \frac{E[(U-\mu)^4]}{\sigma^4} = \frac{E[(U-\mu)^4]}{\text{Var}^2(U)} = \frac{E[(U-1/2)^4]}{(1/12)^2} = (144)\sum_{k=0}^{4}\binom{4}{k}\frac{1}{k+1}\left(-\frac{1}{2}\right)^{4-k}$$

$= (144)(1/80) = 9/5$

3.6.22 $10 = E\left[(W-2)^3\right] = \sum_{j=0}^{3}\binom{3}{j}E(W^j)(-2)^{3-j} = (1)(1)(-8) + (3)(2)(4) + (3)E(W^2)(-2) + (1)(4)(1)$

This would imply that $E(W^2) = 5/3$. In that case, $\text{Var}(W) = 5/3 - (2)^2 < 0$, which is not possible.

3.6.23 Let $E(X) = \mu$; let σ be the standard deviation of X. Then $E(aX + b) = a\mu + b$. Also, $\text{Var}(aX + b) = a^2\sigma^2$, so the standard deviation of $aX + b = a\sigma$. Then

$$\gamma_1 = \frac{E\left[\left((aX+b)-(a\mu+b)\right)^3\right]}{(a\sigma)^3}$$

$$= \frac{a^3E\left[(X-\mu)^3\right]}{a^3\sigma^3} = \frac{E\left[(X-\mu)^3\right]}{\sigma^3} = \gamma_1(X)$$

The demonstration for γ_2 is similar.

3.6.24 **(a)** Question 3.4.6 established that Y is a pdf for any positive integer n. As a corollary, we know that $1 = \int_0^1 (n+2)(n+1y^n(1-y)dy$ or equivalently, for any positive integer n,

$$\int_0^1 y^n(1-y)dy = \frac{1}{(n+2)(n+1)}$$

Then $E(Y^2) = \int_0^1 y^n(n+2)(n+1)y^n(1-y)dy = (n+2)(n+1)\int_0^1 y^{n+2}(1-y)dy$

$= \frac{(n+2)(n+1)}{(n+4)(n+3)}$. By a similar argument, $E(Y) = \frac{(n+2)(n+1)}{(n+4)(n+3)} = \frac{(n+1)}{(n+3)}$.

Thus, $\text{Var}(Y) = E(Y^2) - E(Y)^2 = \frac{(n+2)(n+1)}{(n+4)(n+3)} - \frac{(n+1)^2}{(n+3)^2} = \frac{2(n+1)}{(n+4)(n+3)^2}$

(b) $E(Y^k) = \int_0^1 y^k(n+2)(n+1)y^n(1-y)dy = (n+2)(n+1)\int_0^1 y^{n+k}(1-y)dy$

$$= \frac{(n+2)(n+1)}{(n+k+2)(n+k+1)}$$

3.6.25 **(a)** $1 = \int_1^{\infty} cy^{-6}dy = c\left[\frac{y^{-5}}{-5}\right]_1^{\infty} = c\frac{1}{5}$, so $c = 5$.

(b) $E(Y^r) = 5\int_1^{\infty} y^r y^{-6}dy = 5\left[\frac{y^{r-5}}{r-5}\right]_1^{\infty}$. For this last expression to be finite, r must be < 5.

The highest integral moment is $r = 4$.

Section 3.7: Joint Densities

3.7.1 $1 = \sum_{x,y} p(x,y) = c\sum_{x,y} xy = c[(1)(1) + (2)(1) + (2)(2) + (3)(1)] = 10c$, so $c = 1/10$

3.7.2 $1 = \int_0^1\int_0^1 c(x^2+y^2)dxdy = c\int_0^1\int_0^1 x^2dxdy + c\int_0^1\int_0^1 y^2dydx$

$= c\int_0^1\left[\frac{x^3}{3}\right]_0^1 dy + c\int_0^1\left[\frac{y^3}{3}\right]_0^1 dx = c\int_0^1\frac{1}{3}dy + c\int_0^1\frac{1}{3}dx = \frac{2}{3}c$, $c = 3/2$.

3.7.3 $1 = \int_0^1\int_0^y c(x+y)dxdy = c\int_0^1\left[\frac{x^2}{2}+xy\right]_0^y dy = c\int_0^1\frac{3y^2}{2}dy = c\left[\frac{y^3}{2}\right]_0^1 = \frac{c}{2}$, so $c = 2$.

3.7.4 $1 = c\int_0^1\left(\int_0^y xydx\right)dy = c\int_0^1\left(\left[\frac{x^2y}{2}\right]_0^y\right)dy = c\int_0^1\frac{y^3}{2}dy$

$c\left[\frac{y^4}{8}\right]_0^1 = c\left(\frac{1}{8}\right)$, so $c = 8$

3.7.5 $P(X = x, Y = y) = \dfrac{\binom{3}{x}\binom{2}{y}\binom{4}{3-x-y}}{\binom{9}{3}}$, $0 \le x \le 3$, $0 \le y \le 2$, $x + y \le 3$

3.7.6 $P(X = x, Y = y) = \dfrac{\binom{4}{x}\binom{4}{y}\binom{44}{4-x-y}}{\binom{52}{4}}$, $0 \le x \le 4$, $0 \le y \le 4$, $x + y \le 4$

3.7.7 $P(X > Y) = p_{X,Y}(1, 0) + p_{X,Y}(2, 0) + p_{X,Y}(2, 1)$
$= 6/50 + 4/50 + 3/50 = 13/50$

3.7.8

Outcome	X	Y
H H H	1	3
H H T	0	2
H T H	1	2
H T T	0	1
T H H	1	2
T H T	0	1
T T H	1	1
T T T	0	0

(x, y)	$p_{X,Y}(x, y)$
(0, 0)	1/8
(0, 1)	2/8
(0, 2)	1/8
(0, 3)	0
(1, 0)	0
(1, 1)	1/8
(1, 2)	2/8
(1, 3)	1/8

3.7.9

		Number of 2's, X		
		0	1	2
Number of 3's, Y	0	16/36	8/36	1/36
	1	8/36	2/36	0
	2	1/36	0	0

From the matrix above, we calculate

$p_Z(0) = p_{X,Y}(0, 0) = 16/36$
$p_Z(1) = p_{X,Y}(0, 1) + p_{X,Y}(1, 0) = 2(8/36) = 16/36$
$p_Z(2) = p_{X,Y}(0, 2) + p_{X,Y}(2, 0) + p_{X,Y}(1, 1) = 4/36$

3.7.10 **(a)** $1 = \int_0^1\int_0^1 c\, dxdy = \text{c}$, so $c = 1$ **(b)** $P(0 < X < 1/2, 0 < Y < 1/4) = \int_0^{1/4}\int_0^{1/2} 1\, dxdy = 1/8$

3.7.11 $P(Y< 3X) = \int_0^\infty\int_x^{3x} 2e^{-(x+y)}dydx = \int_0^\infty e^{-x}\left(\int_x^{3x} 2e^{-y}dy\right)dx$

$= 2\int_0^\infty e^{-x}\left(\left[-e^{-y}\right]_x^{3x}\right)dx = 2\int_0^\infty e^{-x}\left[e^{-x} - e^{-3x}\right]dx$

$= 2\int_0^\infty \left[e^{-2x} - e^{-4x}\right]dx = 2\left[-\frac{1}{2}e^{-2x} + \frac{1}{4}e^{-4x}\right]_0^\infty = \frac{1}{2}$

3.7.12 The density is the bivariate uniform over a circle of radius 2. The area of the circle is $\pi(2)^2 = 4\pi$. Thus, $f_{X,Y}(x, y) = 1/4\pi$ for $x^2 + y^2 \le 4$.

3.7.13 $P(X < 2Y) = \int_0^1 \int_{x/2}^1 (x+y)\,dy\,dx$

$$= \int_0^1 \int_{x/2}^1 x\,dy\,dx + \int_0^1 \int_{x/2}^1 y\,dy\,dx$$

$$= \int_0^1 \left[x - \frac{x^2}{2}\right]dx + \int_0^1 \left[\frac{1}{2} - \frac{x^2}{8}\right]dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{6}\right]_0^1 + \left[\frac{x}{2} - \frac{x^3}{24}\right]_0^1 = \frac{19}{24}$$

3.7.14 The probability of an observation falling into the interval (0, 1/3) is $\int_0^{1/3} 2t\,dt = 1/9$. The probability of an observation falling into the interval (1/3, 2/3) is $\int_{1/3}^{2/3} 2t\,dt = 1/3$. Assume without any loss of generality that the five observations are done in order. To calculate $p_{X,Y}(1, 2)$, note that there are $\binom{5}{1}$ places where the observation in (0, 1/3) could occur, and $\binom{4}{2}$ choices for the location of the observations in (1/3, 2/3). Then $p_{X,Y}(1, 2) = \binom{5}{1}\binom{4}{2}(1/9)^1(1/3)^2(5/9)^2$ = 750/6561.

3.7.15 The set where $y > h/2$ is a triangle with height $h/2$ and base $b/2$. Its area is $bh/8$. Thus the area of the set where $y < h/2$ is $bh/2 - bh/8 = 3bh/8$. The probability that a randomly chosen point will fall in the lower half of the triangle is $(3bh/8)/(bh/2) = 3/4$.

3.7.16 $p_X(x) = \dfrac{\binom{3}{x}}{\binom{9}{3}} \sum_{y=0}^{\min(2,3-x)} \binom{2}{y}\binom{4}{3-x-y} = \dfrac{\binom{3}{x}\binom{6}{3-x}}{\binom{9}{3}}, x = 0,1,2,3$

3.7.17 From the solution to Question 3.7.8, $p_X(x) = 1/8 + 2/8 + 1/8 = 1/2$, $x = 0, 1$,
$p_Y(0) = 1/8$, $p_Y(1) = 3/8$, $p_Y(2) = 3/8$, $p_Y(3) = 1/8$.

3.7.18 Let X_1 be the number in the upper quarter; X_2, the number in the middle half. From Question 3.2.18, we know that $P(X_1 = 2, X_2 = 2) = \dfrac{6!}{2!2!2!}(0.25)^2(0.50)^2(0.25)^2 = 0.088$. The simplest way to deal with the marginal probability is to recognize that the probability of belonging to the middle half is binomial with $n = 6$ and $p = .05$. This probability is $\binom{6}{2}(0.5)^2(0.5)^4 = 0.234$.

3.7.19 **(a)** $f_X(x) = \int_0^1 \frac{1}{2}dy = \frac{y}{2}\Big|_0^1 = \frac{1}{2}$, $0 \le x \le 2$

$f_Y(y) = \int_0^2 \frac{1}{2}dx = \frac{x}{2}\Big|_0^2 = 1$, $0 \le y \le 1$

(b) $f_X(x) = \int_0^1 \frac{3}{2}y^2\,dy = \frac{1}{2}y^3\Big|_0^1 = 1/2$, $0 \le x \le 2$

$f_Y(y) = \int_0^2 \frac{3}{2}y^2\,dx = \frac{3}{2}y^2x\Big|_0^2 = 3y^2$, $0 \le y \le 1$

(c) $f_X(x) = \int_0^1 \frac{2}{3}(x+2y)dy = \frac{2}{3}\left(xy + y^2\right)\Big|_0^1 = \frac{2}{3}(x+1)$, $0 \le x \le 1$

$f_Y(y) = \int_0^1 \frac{2}{3}(x+2y)dx = \frac{2}{3}\left(\frac{x^2}{2} + 2xy\right)\Bigg|_0^1 = \frac{4}{3}y + \frac{1}{3}$, $0 \le y \le 1$

(d) $f_X(x) = c\int_0^1 (x+y)dy = c\left(xy + \frac{y^2}{2}\right)\Bigg|_0^1 = c\left(x + \frac{1}{2}\right)$, $0 \le x \le 1$

In order for the above to be a density, $1 = \int_0^1 c\left(x + \frac{1}{2}\right)dx = c\left(\frac{x^2}{2} + \frac{x}{2}\right)\Bigg|_0^1 = c$, so

$f_X(x) = x + \frac{1}{2}$, $0 \le x \le 1$

$f_Y(y) = y + \frac{1}{2}$, $0 \le y \le 1$, by symmetry of the joint pdf

(e) $f_X(x) = \int_0^1 4xy\,dy = 2xy^2\Big|_0^1 = 2x$, $0 \le x \le 1$

$f_Y(y) = 2y$, $0 \le y \le 1$, by the symmetry of the joint pdf

(f) $f_X(x) = \int_0^\infty xye^{-(x+y)}dy = xe^{-x}\int_0^\infty ye^{-y}dy$

$= xe^{-x}(-ye^{-y} - e^{-y})\Big|_0^\infty = xe^{-x}$, $0 \le x$

$f_Y(y) = ye^{-y}$, $0 \le y$, by symmetry of the joint pdf

(g) $f_X(x) = \int_0^\infty ye^{-xy-y}dy = \int_0^\infty ye^{-(x+1)y}dy$

Integrating by parts gives

$$\left(-\frac{y}{x+1}e^{-(x+1)y} - \left(\frac{1}{x+1}\right)^2 e^{-(x+1)y}\right)\Bigg|_0^\infty = \left(\frac{1}{x+1}\right)^2, \ 0 < x$$

$f_Y(y) = \int_0^\infty ye^{-xy-y}dx = \int_0^\infty ye^{-y}e^{-xy}dx = ye^{-y}\left(-\frac{1}{y}\right)e^{-xy}\Big|_0^\infty = e^{-y}$, where $0 \le y$.

3.7.20 **(a)** $f_X(x) = \int_x^2 \frac{1}{2} dy = \left.\frac{y}{2}\right|_x^2 = 1 - \frac{x}{2}$, $0 \le x \le 2$

$f_Y(y) = \int_0^y \frac{1}{2}\, dx = \frac{1}{2} y$, $0 \le y \le 2$

(b) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy = \int_0^x \frac{1}{x} dy = \frac{1}{x} y \Big|_0^x = 1$, $0 \le x \le 1$

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx = \int_y^1 \frac{1}{x} dx = \ln x \Big|_y^1 = -\ln y$, $0 \le y \le 1$

(c) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy = \int_0^{1-x} 6x\, dy = 6xy \Big|_0^{1-x} = 6x(1 - x)$, $0 \le x \le 1$

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx = \int_0^{1-y} 6x\, dx = 3x^2\Big|_0^{1-y} = 3(1 - y)^2$, $0 \le y \le 1$

3.7.21 $f_X(x) = \int_0^{1-x} 6(1 - x - y) dy = 6\left(y - xy - \frac{y^2}{2}\right)\Bigg|_0^{1-x} = 6\left[(1 - x) - x(1 - x) - \frac{(1 - x)^2}{2}\right]$

$= 3 - 6x + 3x^2$, $0 \le x \le 1$

3.7.22 $f_Y(y) = \int_0^y 2e^{-x}e^{-y} dx = -2e^{-x}\, e^{-y}\Big|_0^y = 2e^{-y} - 2e^{-2y}$, $0 \le y$

3.7.23 $p_X(x) = \sum_{y=0}^{4-x} \frac{4!}{x!\,y!(4 - x - y)!}\left(\frac{1}{2}\right)^x \left(\frac{1}{3}\right)^y \left(\frac{1}{6}\right)^{4-x-y}$

$= \frac{4!}{x!(4 - x)!}\left(\frac{1}{2}\right)^x \sum_{y=0}^{4-x} \frac{(4 - x)!}{y![(4 - x) - y]!}\left(\frac{1}{3}\right)^y \left(\frac{1}{6}\right)^{4-x-y} = \frac{4!}{x!(4 - x)!}\left(\frac{1}{2}\right)^x \left(\frac{1}{3} + \frac{1}{6}\right)^{4-x}$

$= \binom{4}{x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$

Thus, X is binomial with $n = 4$ and $p = 1/2$. Similarly, Y is binomial with $n = 4$ and $p = 1/3$.

3.7.24 **(a)** Consider any outcome with x of the first kind, y of the second, and (necessarily) $n - x - y$ of the third kind. The probability of such an outcome is $p_1^x p_2^y \left(1 - p_1 - p_2\right)^{n-x-y}$. All we need to know now is how many outcomes there are with x of the first kind, y of the second, and $n - x - y$ of the third kind. This question is resolved by the number of ways to choose the places for these three kinds of outcomes, which by Theorem 2.6.2 is $\frac{n!}{x!\,y!(n - x - y)!}$.

(b) The proof is an adaptation of the solution to Question 3.7.23.

3.7.25 **(a)** $S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

(b) $F_{X,Y}(1, 2) = P(X \le 1, Y \le 2) = P(\{(H, 1), (H, 2), (T, 1), (T, 2)\}) = 4/12 = 1/3$

3.7.26 $F_{X,Y}(1, 2) = \sum_{i=0}^{1}\sum_{j=0}^{2} p_{X,Y}(i, j)$

$$= \frac{\binom{5}{0}\binom{4}{1}\binom{3}{3}}{\binom{12}{4}} + \frac{\binom{5}{0}\binom{4}{2}\binom{3}{2}}{\binom{12}{4}} + \frac{\binom{5}{1}\binom{4}{0}\binom{3}{3}}{\binom{12}{4}} + \frac{\binom{5}{1}\binom{4}{1}\binom{3}{2}}{\binom{12}{4}} + \frac{\binom{5}{1}\binom{4}{2}\binom{3}{1}}{\binom{12}{4}}$$

$$= \frac{4+18+5+60+90}{495} = \frac{177}{495} = 0.358$$

3.7.27 **(a)** $F_{X,Y}(u, v) = \int_0^u\int_0^v \frac{3}{2}y^2 dy\, dx = \int_0^u\left[\frac{1}{2}y^3\Big|_0^v\right]dx = \int_0^u \frac{1}{2}v^3 dx = \frac{1}{2}uv^3$

(b) $F_{X,Y}(u, v) = \int_0^u\int_0^v \frac{2}{3}(x+2y)dy\, dx = \int_0^u\left[\frac{2}{3}(xy+y^2)\Big|_0^v\right]dx = \int_0^u \frac{2}{3}(vx+v^2dx) = \frac{1}{3}u^2v + \frac{2}{3}uv^2$

(c) $F_{X,Y}(u, v) = \int_0^u\int_0^v 4xydy\, dx = \int_0^u x\left[2y^2\Big|_0^v\right]dx = 2v^2\int_0^u xdx = u^2v^2$

3.7.28 **(a)** For $0 \le u \le v \le 2$, $F_{X,Y}(u,v) = \int_0^u\int_x^v \frac{1}{2}dy\, dx = \int_0^u \frac{y}{2}\Big|_x^v dx = \frac{1}{2}\int_0^u (v-x)dx = \frac{1}{4}(2uv-u^2)$

(b) $F_{X,Y}(u, v) = \int_0^v\int_y^u \frac{1}{x}dy\, dx = \int_0^v\left[\ln x\Big|_y^u\right]dy = \int_0^v \ln u - \ln y dy = v\ln u - v\ln v + v$

(c) Case I: $v \le 1 - u$

$F_{X,Y}(u, v) = \int_0^v\int_0^u 6xdx\, dy = \int_0^v\left[3x^2\Big|_0^u\right]dy = \int_0^v 3u^2dy = 3u^2v$

Case II: $v > 1 - u$

$F_{X,Y}(u, v) = \int_0^u\int_0^v 6xdy\, dx = \int_{1-v}^u\int_{1-x}^v 6xdy\, dx$

$= 3u^2v - [3u^2v - 3u^2 + 2u^3 - 3(1-v)^2v + 3(1-v)^2 - 2(1-v)^3]$

$= 3u^2 - 2u^3 + 3(1-v)^2v - 3(1-v)^2 + 2(1-v)^3 = 3u^2 - 2u^3 - (1-v)^3$

3.7.29 By Theorem 3.7.3, $f_{X,Y} = \frac{\partial^2}{\partial x\partial y}F_{X,Y} = \frac{\partial^2}{\partial x\partial y}(xy) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}xy\right)$

$= \frac{\partial}{\partial x}(x) = 1,\ 0 \le x \le 1,\ 0 \le y \le 1.$

The graph of $f_{X,Y}$ is a square plate of height one over the unit square.

3.7.30 By Theorem, 3.7.3, $f_{X,Y} = \frac{\partial^2}{\partial x\partial y}F_{X,Y} = \frac{\partial^2}{\partial x\partial y}[(1-e^{-\lambda y})(1-e^{-\lambda x})]$

$= \frac{\partial}{\partial x}\frac{\partial}{\partial y}[(1-e^{-\lambda y})(1-e^{-\lambda x})] = \frac{\partial}{\partial x}[(\lambda e^{-\lambda y})(1-e^{-\lambda x})]$

$= \lambda e^{-\lambda y}\lambda e^{-\lambda x},\ x \ge 0,\ y \ge 0$

3.7.31 First note that $1 = F_{X,Y}(1, 1) = k[4(1^2)(1^2) + 5(1)(1^4)] = 9k$, so $k = 1/9$.

Then $f_{X,Y} = \dfrac{\partial^2}{\partial x \partial y} F_{X,Y} = \dfrac{\partial^2}{\partial x \partial y}\left(\dfrac{4}{9}x^2y^2 + \dfrac{5}{9}xy^4\right)$

$$= \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\frac{4}{9}x^2y^2 + \frac{5}{9}xy^4\right) = \frac{\partial}{\partial x}\left(\frac{8}{9}x^2y + \frac{20}{9}xy^3\right) = \frac{16}{9}xy + \frac{20}{9}y^3$$

$$P(0 < X < 1/2,\ 1/2 < Y < 1) = \int_0^{1/2}\int_{1/2}^{1}\left(\frac{16}{9}xy + \frac{20}{9}y^3\right)dy\,dx$$

$$= \int_0^{1/2} \frac{8}{9}xy^2 + \frac{5}{9}y^4\Big|_{1/2}^{1} dx = \int_0^{1/2}\left(\frac{2}{3}x + \frac{25}{48}\right)dx = \frac{1}{3}x^2 + \frac{25}{48}x\ \Big|_0^{1/2} = 11/32$$

3.7.32 $P(a < X \le b, Y \le d) = F_{X,Y}(b, d) - F_{X,Y}(a, d)$
$P(a < X \le b, Y \le c) = F_{X,Y}(b, c) - F_{X,Y}(a, c)$
$P(a < X \le b, c < Y \le d) = P(a < X \le b, Y \le d) - P(a < X \le b, Y \le c)$
$= (F_{X,Y}(b, d) - F_{X,Y}(a, d)) - (F_{X,Y}(b, c) - F_{X,Y}(a, c)) = F_{X,Y}(b, d) - F_{X,Y}(a, d) - F_{X,Y}(b, c) + F_{X,Y}(a, c)$

3.7.33 $P(X_1 \ge 1050, X_2 \ge 1050, X_3 \ge 1050, X_4 \ge 1050)$

$$= \int_{1050}^{\infty}\int_{1050}^{\infty}\int_{1050}^{\infty}\int_{1050}^{\infty}\prod_{i=1}^{4}\frac{1}{1000}e^{-x_i/1000}dx_1dx_2dx_3dx_4$$

$$= \left(\int_{1050}^{\infty}\frac{1}{1000}e^{-x/1000}dx\right)^4 = (e^{-1.05})^4 = 0.015$$

3.7.34 **(a)** $p_{X,Y,Z}(x, y, z) = \dfrac{\binom{4}{x}\binom{4}{y}\binom{4}{z}\binom{40}{6-x-y-z}}{\binom{52}{6}}$ where $0 \le x, y, z \le 4$, $x + y + z \le 6$

(b) $p_{X,Y}(x, y) = \dfrac{\binom{4}{x}\binom{4}{y}\binom{44}{6-x-y}}{\binom{52}{6}}$ where $0 \le x, y \le 4$, $x + y \le 6$

$p_{X,Z}(x, z) = \dfrac{\binom{4}{x}\binom{4}{z}\binom{44}{6-x-z}}{\binom{52}{6}}$ where $0 \le x, z \le 4$, $x + z \le 6$

3.7.35 $p_{X,Y}(0, 1) = \displaystyle\sum_{z=0}^{2} p_{X,Y,Z}(0,1,z) = \frac{3!}{0!1!}\left(\frac{1}{2}\right)^0\left(\frac{1}{12}\right)^1\sum_{z=0}^{2}\frac{1}{z!(2-z)!}\left(\frac{1}{6}\right)^z\left(\frac{1}{4}\right)^{2-z}$

$$= \frac{3!}{0!1!}\left(\frac{1}{2}\right)^0\left(\frac{1}{12}\right)^1\left(\frac{1}{2}\right)\sum_{z=0}^{2}\frac{2!}{z!(2-z)!}\left(\frac{1}{6}\right)^z\left(\frac{1}{4}\right)^{2-z} = \frac{3!}{0!1!}\left(\frac{1}{2}\right)^0\left(\frac{1}{12}\right)^1\left(\frac{1}{2}\right)\left(\frac{1}{6}+\frac{1}{4}\right)^2 = \frac{25}{576}$$

3.7.36 **(a)** $f_{X,Y}(x, y) = \int_0^\infty f_{X,Y,Z}(x,y,z)dz = \int_0^\infty (x+y)e^{-z}dz$

$= (x+y)\left[-e^{-z}\right]_0^\infty = (x + y),\ 0 \le x, y \le 1$

(b) $f_{Y,Z}(y, z) = \int_0^1 f_{X,Y,Z}(x,y,z)dx = \int_0^1 (x+y)e^{-z}dx = e^{-z}\left[\frac{x^2}{2}+xy\right]_0^1 = \left(\frac{1}{2}+y\right)e^{-z},$

$0 \le y \le 1,\ z \ge 0$

(c) $f_Z(z) = \int_0^1 f_{Y,Z}(y,z)dy = \int_0^1 \left(\frac{1}{2}+y\right)e^{-z}dy = e^{-z}\left[\frac{y}{2}+\frac{y^2}{2}\right]_0^1 = e^{-z},\ z \ge 0$

3.7.37 $f_{W,X}(w, x) = \int_0^1\int_0^1 f_{W,X,Y,Z}(w,x,y,z)dydz = \int_0^1\int_0^1 16wxyz\ dydz = \int_0^1\left[8wxy^2z\right]_0^1 dz = \int_0^1 [8wxz]dz$

$= \left[4wxz^2\right]_0^1 = 4wx,\ 0 < w, x < 1$

$P(0 < W < 1/2,\ 1/2 < X < 1) = \int_0^{1/2}\int_{1/2}^1 4wx\ dxdw =$

$\int_0^{1/2} 2w\left[x^2\right]_{1/2}^1 dx = \int_0^{1/2}\frac{3}{2}w\ dw = \frac{3}{4}w^2\Big|_0^{1/2} = \frac{3}{16}$

3.7.38 We must show that $p_{X,Y}(j, k) = p_X(j)p_Y(k)$. But for any pair (j, k), $p_{X,Y}(j, k) = 1/36 = (1/6)(1/6) = p_X(j)p_Y(k)$.

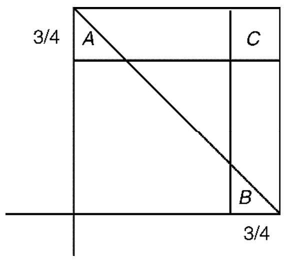
3.7.39 The marginal pdfs for $f_{X,Y}$ are $f_X(x) = \lambda e^{-\lambda x}$ and $f_Y(y) = \lambda e^{-\lambda y}$ (Hint: see the solution to 3.7.19**(f)**). Their product is $f_{X,Y}$, so X and Y are independent. The probability that one component fails to last 1000 hours is $1 - e^{-1000\lambda}$. Because of independence of the two components, the probability that two components both fail is the square of that, or $(1 - e^{-1000\lambda})^2$.

3.7.40 **(a)** $p_{X,Y}(x, y) = \begin{cases} \frac{1}{4}\cdot\frac{2}{5} = \frac{1}{10} & y = x \\ \frac{1}{4}\cdot\frac{1}{5} = \frac{1}{20} & y \ne x \end{cases}$

(b) $p_X(x) = 1/4$, since each ball in Urn I is equally likely to be drawn
$p_Y(y) = 1/10 + 3(1/20) = 1/4$

(c) $P(X = 1, Y = 1) = 1/10$, but $P(X = 1)P(Y = 1) = 1/16$

3.7.41 First, note $k = 2$. Then, 2 times area of $A = P(Y \ge 3/4)$. Also, 2 times area of $B = P(X \ge 3/4)$. The square C is the set $(X \ge 3/4) \cap (Y \ge 3/4)$. However, C is in the region where the density is 0. Thus, $P((X \ge 3/4) \cap (Y \ge 3/4))$ is zero, but the product $P(X \ge 3/4)P(Y \ge 3/4)$ is not zero.



3.7.42 $f_X(x) = \int_0^1 \frac{2}{3}(x+2y)dy = \frac{2}{3}(x+1)$

$f_Y(y) = \int_0^1 \frac{2}{3}(x+2y)dx = \frac{2}{3}\left(2y+\frac{1}{2}\right)$

But $\frac{2}{3}(x+1)\frac{2}{3}\left(2y+\frac{1}{2}\right) \neq \frac{2}{3}(x+2y)$.

3.7.43 $P(Y < X) = \int_0^1\int_0^x f_{X,Y}(x, y)\, dydx = \int_0^1\int_0^x (2x)(3y^2)\, dydx = \int_0^1 2x^4 dx = \frac{2}{5}$

3.7.44 $F_X(x) = \int_0^x \frac{t}{2}dt = \frac{x^2}{4}$. $F_Y(y) = \int_0^y 2t\, dt = y^2$

$F_{X,Y}(x, y) = F_X(x)F_Y(y) = \frac{x^2y^2}{4}$, $0 \le x \le 2$, $0 \le y \le 1$

3.7.45 $P\left(\frac{Y}{X} > 2\right) = P(Y > 2X) = \int_0^1\int_0^{y/2}(2x)(1)dxdy = \int_0^1\left[x^2\right]_0^{y/2}dy = \left.\frac{y^3}{12}\right|_0^1 = \frac{1}{12}$

3.7.46 $P(a < X < b, c < Y < d) = \int_a^b\int_c^d xye^{-(x+y)}dydx = \int_a^b xe^{-x}\left(\int_c^d ye^{-y}dy\right)dx$

$= \int_a^b xe^{-x}dx\int_c^d ye^{-y}dy = P(a < X < b)\, P(c < Y < d)$

3.7.47 Take $a = c = 0$, $b = d = 1/2$. Then $P(0 < X < 1/2, 0 < Y < 1/2) = \int_0^{1/2}\int_0^{1/2}(2x + y - 2xy)dydx$ $= 5/32$.

$f_X(x) = \int_0^1 (2x + y - 2xy)dy = x + 1/2$, so $P(0 < X < 1/2) = \int_0^{1/2}\left(x+\frac{1}{2}\right)dx = \frac{3}{8}$

$f_Y(y) = \int_0^1 (2x + y - 2xy)dx = 1$, so $P(0 < X < 1/2) = 1/2$. But, $5/32 \neq (3/8)(1/2)$

3.7.48 We proceed by showing that the events $g(X)\ \varepsilon\ A$ and $h(Y)\ \varepsilon B$ are independent, for sets of real numbers, A and B. Note that $P(g(X)\ \varepsilon A$ and $h(Y)\ \varepsilon B) = P(X\ \varepsilon\ g^{-1}(\text{a})$ and $Y\ \varepsilon\ g^{-1}(\text{b}))$. Since X and Y are independent, $P(X\ \varepsilon\ g^{-1}(\text{a})$ and $Y\ \varepsilon\ g^{-1}(\text{b})) = P(X\ \varepsilon\ g^{-1}(\text{a}))P(Y\ \varepsilon\ g^{-1}(\text{b})) =$ $P(g(X)\ \varepsilon\ A)P(h(Y)\ \varepsilon\ B)$

3.7.49 Let K be the region of the plane where $f_{X,Y} \neq 0$. If K is not a rectangle with sides parallel to the coordinate axes, there exists a rectangle $A = \{(x, y) | a \le x \le b, c \le y \le d\}$ with $A \cap K = \varnothing$, but for $A_1 = \{(x, y) | a \le x \le b, \text{all } y\}$ and $A_2 = \{(x, y) | \text{all } x, c \le y \le d\}$, $A_1 \cap K \neq \varnothing$ and $A_2 \cap K \neq \varnothing$. Then $P(\mathbf{a}) = 0$, but $P(A_1) \neq 0$ and $P(A_2) \neq 0$. However, $A = A_1 \cap A_2$, so $P(A_1 \cap A_2) \neq P(A_1)P(A_2)$.

3.7.50 $f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots x_n) = \prod_{j=1}^{n} (1/\lambda)e^{-x_j/\lambda} = (1/\lambda)^n e^{-\frac{1}{\lambda}\sum_{j=1}^{n} x_j}$

3.7.51 **(a)** $P(X_1 < 1/2) = \int_0^{1/2} 4x^3 dx = x^4 \Big|_0^{1/2} = 1/16$

(b) This asks for the probability of exactly one success in a binomial experiment with $n = 4$ and $p = 1/16$, so the probability is $\binom{4}{1}(1/16)^1(15/16)^3 = 0.206$.

(c) $f_{X_1,X_2,X_3,X_4}(x_1, x_2, x_3, x_4) = \prod_{j=1}^{4} 4x_j^3 = 256(x_1x_2x_3x_4)^3,\ 0 \le x_1, x_2, x_3, x_4 \le 1$

(d) $F_{X_2,X_3}(x_2, x_3) = \int_0^{x_3}\int_0^{x_2} (4s^3)(4t^3)dsdt = \int_0^{x_2} 4s^2 ds \int_0^{x_3} 4t^3 dt = x_2^4 x_3^4$, $0 \le x_2, x_3 \le 1$.

3.7.52 $P(X_1 < 1/2, X_2 > 1/2, X_3 < 1/2, X_4 > 1/2, \ldots, X_{2k} > 1/2)$
$= P(X_1 < 1/2)P(X_2 > 1/2)P(X_3 < 1/2)P(X_4 > 1/2), \ldots, P(X_{2k} > 1/2)$ because the X_i are independent. Since the X_i are uniform over the unit interval, $P(X_i < 1/2) = P(X_i > 1/2) = 1/2$. Thus the desired probability is $(1/2)^{2k}$.

Section 3.8: Transforming and Combining Random Variables

3.8.1 **(a)** $p_{X+Y}(w) = \sum_{\text{all } x} p_X(x)p_Y(w - x)$. Since $p_X(x) = 0$ for negative x, we can take the lower limit of the sum to be 0. Since $p_Y(w - x) = 0$ for $w - x < 0$, or $x > w$, we can take the upper limit of the sum to be w. Then we obtain

$$p_{X+Y}(w) = \sum_{k=0}^{w} e^{-\lambda}\frac{\lambda^k}{k!}e^{-\mu}\frac{\mu^{w-k}}{(w-k)!} = e^{-(\lambda+\mu)}\sum_{k=0}^{w}\frac{1}{k!(w-k)!}\lambda^k\mu^{w-k}$$

$$= e^{-(\lambda+\mu)}\frac{1}{w!}\sum_{k=0}^{w}\frac{w!}{k!(w-k)!}\lambda^k\mu^{w-k} = e^{-(\lambda+\mu)}\frac{1}{w!}(\lambda+\mu)^w,\ w = 0, 1, 2, \ldots$$

This pdf has the same form as the ones for X and Y, but with parameter $\lambda + \mu$.

(b) $p_{X+Y}(w) = \sum_{\text{all } x} p_X(x)p_Y(w - x)$. The lower limit of the sum is 1.

For this pdf, we must have $w - k \ge 1$ so the upper limit of the sum is $w - 1$. Then

$$p_{X+Y}(w) = \sum_{k=1}^{w-1}(1-p)^{k-1}p(1-p)^{w-k-1}p = (1-p)^{w-2}p^2\sum_{k=1}^{w-1}1 = (w-1)(1-p)^{w-2}p^2,\ w = 2, 3, 4, \ldots$$

The pdf for $X + Y$ does not have the same form as those for X and Y, but Section 4.5 will show that they all belong to the same family—the negative binomial.

3.8.2 $f_{X+Y}(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)dx = \int_0^w (xe^{-x})(e^{-(w-x)})dx = e^{-w}\int_0^w x\,dx = \frac{w^2}{2}e^{-w},\ w \geq 0$

3.8.3 First suppose that $0 \leq w \leq 1$. As in the previous problem the upper limit of the integral is w, and $f_{X+Y}(w) = \int_0^w (1)(1)dx = w$. Now consider the case $1 \leq w \leq 2$. Here, the first integrand vanishes unless x is ≤ 1. Also, the second pdf is 0 unless $w - x \leq 1$ or $x \geq w - 1$. Then $f_{X+Y}(w)$ $= \int_{w-1}^1 (1)(1)dx = 2 - w$.

In summary, $f_{X+Y}(w) = \begin{cases} w & 0 \leq w \leq 1 \\ 2-w & 1 \leq w \leq 2 \end{cases}$

3.8.4 Consider the continuous case. It suffices to show that $F_{V,X+Y} = F_V F_{X+Y}$. $F_{V,X+Y}(v,w)$ $= P(V \leq v, X + Y \leq w) =$

$$\int_{-\infty}^{v}\int_{-\infty}^{\infty}\int_{-\infty}^{w-x} f_V(v) f_X(x) f_Y(y)dydxdv = \int_{-\infty}^{v} f_V(v)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{w-x} f_X(x) f_Y(y)dydx\right)dv$$

$= F_V(v)F_{X+Y}(w)$

3.8.5 $F_W(w) = P(W \leq w) = P(Y^2 \leq w) = P\left(Y \leq \sqrt{w}\right) = f = F_Y\left(\sqrt{w}\right)$

Now differentiate both sides to obtain $f_W(w) = \frac{d}{dw}F_W(w) = \frac{d}{dw}F_Y\left(\sqrt{w}\right) = \frac{1}{2\sqrt{w}} f_Y\left(\sqrt{w}\right)$.

3.8.6 From Question 3.8.5, $f_W(w) = \frac{1}{2\sqrt{w}} f_Y\left(\sqrt{w}\right)$. Since $f_Y\left(\sqrt{w}\right) = 1$, $f_W(w) = \frac{1}{2\sqrt{w}}$, $0 \leq w \leq 1$.

3.8.7 From Question 3.8.5,

$f_W(w) = \frac{1}{2\sqrt{w}} f_Y\left(\sqrt{w}\right)$. Thus $f_W(w) = \frac{1}{2\sqrt{w}} 6\sqrt{w}\left(1-\sqrt{w}\right) = 3\left(1-\sqrt{w}\right)$ where $0 \leq w \leq 1$.

3.8.8 From Question 3.8.5

$$f_{Y^2}(u) = \frac{1}{2\sqrt{u}} f_Y\left(\sqrt{u}\right) = \frac{1}{2\sqrt{u}} a\left(\sqrt{u}\right)^2 e^{-b\left(\sqrt{u}\right)^2} = \frac{1}{2}a\sqrt{u}e^{-bu}.$$

Then $f_W(w) = \frac{2}{m} f_u\left(\frac{2}{m}w\right) = \frac{2}{m}\frac{1}{2}a\sqrt{\frac{2}{m}w}e^{-b(2w/m)} = \frac{\sqrt{2}}{m^{3/2}}a\sqrt{w}e^{-b(2w/m)}$, $0 \leq w$.

3.8.9 **(a)** Let $W = XY$. Then $f_W(w) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(x) f_Y(w/x)dx$

Since $f_Y(w/x) \neq 0$ when $0 \leq w/x \leq 1$, then we need only consider $w \leq x$.

Similarly, $f_X(x) \neq 0$ implies $x \leq 1$. Thus the integral becomes $\int_w^1 \frac{1}{x}dx = \ln x\Big|_w^1 = -\ln w$, $0 \leq w \leq 1$.

(b) Again let $W = XY$. Since the range of integration here is the same as in Part **(a)**, we can write $f_V(v) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(x) f_Y(w/x)dx = \int_w^1 \frac{1}{x}2(x)2(w/x)dx = 4w\int_w^1 \frac{1}{x}dx = -4w\ln w,\ 0 \leq w \leq 1$.

3.8.10 **(a)** Let $W = Y/X$. Then $f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(wx)dx$

Since $f_X(x) = 0$ for $x < 0$, the lower limit of the integral is 0. Since $f_Y(wx) = 0$ for $wx > 1$, we must have $wx \le 1$ or $x \le 1/w$.

Case I: $0 \le w \le 1$: In this case $1/w > 1$, so the upper limit of the integral is 1.

$F_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xw)dx = \int_0^1 x(1)(1)dx = 1/2$

Case II: $w > 1$: In this case $1/w \le 1$, so the upper limit of the integral is $1/w$.

$f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xw)dx = \int_0^{1/w} x(1)(1)dx = \dfrac{1}{2w^2}$

(b) Case I: $0 \le w \le 1$: The limits of the integral are 0 and 1.

$F_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xw)dx = \int_0^1 x(2x)(2wx)dx = w$

Case II: $w > 1$: The limits of the integral are 0 and $1/w$.

$f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xw)dx = \int_0^{1/w} x(2x)(2xw)dx = \dfrac{1}{w^3}$

3.8.11 Let $W = Y/X$. Then $f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(wx)dx = \int_0^{\infty} x(xe^{-x})e^{-wx}dx = \int_0^{\infty} x^2 e^{-(1+w)x}dx$

$$= \frac{1}{1+w}\left(\int_0^{\infty} x^2(1+w)e^{-(1+w)x}dx\right)$$

Let V be the exponential random variable with parameter $1 + w$. Then the quantity in parentheses above is $E(V^2)$.

But $E(V^2) = \text{Var}(V) + E^2(V) = \dfrac{1}{(1+w)^2} + \dfrac{1}{(1+w)^2} = \dfrac{2}{(1+w)^2}$ (See Question 3.6.11)

Thus, $f_W(w) = \dfrac{1}{1+w}\left(\dfrac{2}{(1+w)^2}\right) = \dfrac{2}{(1+w)^3}$, $0 \le w$.

Section 3.9: Further Properties of the Mean and Variance

3.9.1 Let X_i be the number from the i-th draw, $i = 1, \ldots, r$. Then for each i, $E(X_i) = \dfrac{1+2+\ldots+n}{n} = \dfrac{n+1}{2}$. The sum of the numbers drawn is $\sum_{i=1}^{r} X_i$, so the expected value of the sum is $\sum_{i=1}^{r} E(X_i) = \dfrac{r(n+1)}{2}$.

3.9.2 $f_{X,Y}(x, y) = \lambda^2 e^{-\lambda(x+y)} = \left(\lambda e^{-\lambda x}\right)\left(\lambda e^{-\lambda y}\right)$ implies that $f_X(x) = \lambda e^{-\lambda x}$ and $f_Y(y) = \lambda e^{-\lambda y}$.

Then $E(X + Y) = E(X) + E(Y) = \dfrac{1}{\lambda} + \dfrac{1}{\lambda} = \dfrac{2}{\lambda}$

3.9.3 From Question 3.7.19(c), $f_X(x) = \frac{2}{3}(x+1)$, $0 \le x \le 1$, so $E(X) = \int_0^1 x\frac{2}{3}(x+1)dx$

$= \frac{2}{3}\int_0^1 (x^2 + x)dx = \frac{5}{9}$

Also, $f_Y(y) = \frac{4}{3}y + \frac{1}{3}$, $0 \le y \le 1$, so $E(Y) = \int_0^1 y\left(\frac{4}{3}y + \frac{1}{3}\right)dy = \int_0^1\left(\frac{4}{3}y^2 + \frac{1}{3}y\right)dy = \frac{11}{8}$.

Then $E(X + Y) = E(X) + E(Y) = \frac{5}{9} + \frac{11}{18} = \frac{7}{6}$.

3.9.4 Let $X_i = 1$ if a shot with the first gun is a bull's eye and 0 otherwise, $i = 1, \ldots, 10$. $E(X_i) = 0.30$. Let $V_i = 1$ if a shot with the second gun is a bull's-eye and 0 otherwise, $i = 1, \ldots, 10$. $E(V_i) = 0.40$.

Cathie's score is $4\sum_{i=1}^{10} X_i + 6\sum_{i=1}^{10} V_i$, and her expected score is $E\left(4\sum_{i=1}^{10} X_i + 6\sum_{i=1}^{10} V_i\right)$

$= 4\sum_{i=1}^{10} E(X_i) + 6\sum_{i=1}^{10} E(V_i) = 4(10)(0.30) + 6(10)(0.40) = 36$.

3.9.5 $\mu = E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i) = \sum_{i=1}^{n} a_i \mu = \mu\sum_{i=1}^{n} a_i$, so the given equality occurs if and only if

$\sum_{i=1}^{n} a_i = 1$.

3.9.6 Let X_i be the daily closing price of the stock on day i. The daily expected gain is $E(X_i) = (1/8)p - (1/8)q = (1/8)(p - q)$. After n days the expected gain is $(n/8)(p - q)$.

3.9.7 **(a)** $E(X_i)$ is the probability that the i-th ball drawn is red, $1 \le i \le n$. Draw the balls in order without replacement, but do not note the colors. Then look at the i-th ball <u>first</u>. The probability that it is red is surely independent of when it is drawn. Thus, all of these expected values are the same and each equals $r/(r + w)$.

(b) Let X be the number of red balls drawn. Then $X = \sum_{i=1}^{n} X_i$ and $E(X) = \sum_{i=1}^{n} E(X_i) = nr/(r + w)$.

3.9.8 Let X_1 = number showing on face 1; X_2 = number showing on face 2. Since X_1 and X_2 are independent, $E(X_1X_2) = E(X_1)E(X_2) = (3.5)(3.5) = 12.25$.

3.9.9 First note that $1 = \int_{10}^{20}\int_{10}^{20} k(x + y)dydx = k \cdot 3000$, so $k = \frac{1}{3000}$.

If $\frac{1}{R} = \frac{1}{X} + \frac{1}{Y}$, then $R = \frac{XY}{X + Y}$.

$E(R) = \frac{1}{3000}\int_{10}^{20}\int_{10}^{20} \frac{xy}{x + y}(x + y)dydx = \frac{1}{3000}\int_{10}^{20}\int_{10}^{20} xydydx = 7.5$.

3.9.10 From Question 3.8.5, $f_{X^2}(w) = \frac{1}{2\sqrt{w}}$, so $E(X^2) = \int_0^1 w\frac{1}{2\sqrt{w}}dw = \frac{1}{2}\int_0^1 \sqrt{w}dw = \frac{1}{3}$, with a similar result holding for Y^2. Then $E(X^2 + Y^2) = 2/3$.

3.9.11 The area of the triangle is the random variable $W = \frac{1}{2}XY$. Then

$$E\left(\frac{1}{2}XY\right) = \frac{1}{2}E(XY) = \frac{1}{2}E(X)E(Y) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$$

3.9.12 The Y_i are independent for $i = 1, 2, \ldots, n$. Thus,

$$E\left(\sqrt[n]{Y_1 \cdot Y_2 \cdot \ldots \cdot Y_n}\right) = E\left(\sqrt[n]{Y_1}\right)\cdot E\left(\sqrt[n]{Y_2}\right)\cdot \ldots E\left(\sqrt[n]{Y_n}\right)$$

The Y_i all have the same uniform pdf, so it suffices to calculate $E\left(\sqrt[n]{Y_1}\right)$, which is

$\int_0^1 \sqrt[n]{y}\cdot 1\,dy = \frac{n}{n+1}$. Thus, the expected value of the geometric mean is $\left(\frac{n}{n+1}\right)^n$.

Note that the *arithmetic* mean is constant at 1/2 and does not depend on the sample size.

3.9.13

x	y	$f_{X,Y}$	xy	$xyf_{X,Y}$
1	1	1/36	1	1/36
1	2	1/36	2	2/36
1	3	1/36	3	3/36
1	4	1/36	4	4/36
1	5	1/36	5	5/36
1	6	1/36	6	6/36
2	2	2/36	4	8/36
2	3	1/36	6	6/36
2	4	1/36	8	8/36
2	5	1/36	10	10/36
2	6	1/36	12	12/36
3	3	3/36	9	27/36
3	4	1/36	12	12/36
3	5	1/36	15	15/36
3	6	1/36	18	18/36
4	4	4/36	16	64/36
4	5	1/36	20	20/36
4	6	1/36	24	24/36
5	5	5/36	25	125/36
5	6	1/36	30	30/36
6	6	6/36	36	216/36

$E(XY)$ is the sum of the last column = $\frac{616}{36}$. Clearly $E(X) = 7/2$.

$E(Y) = 1\frac{1}{36} + 2\frac{2}{36} + 3\frac{5}{36} + 4\frac{7}{36} + 5\frac{9}{36} + 6\frac{11}{36} = \frac{161}{36}$

$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{616}{36} - \frac{7}{2}\cdot\frac{161}{36} = \frac{105}{72}$

3.9.14 $\text{Cov}(aX + b, cY + d) = E[(aX + b)(cY + d)] - E(aX + b)E(cY + d)$
$= E(acXY + adX + bcY + bd) - [aE(X) + b][cE(Y) + d]$
$= acE(XY) + adE(X) + bcE(Y) + bd - acE(X)E(Y) - adE(X) - bcE(Y) - bd$
$= ac[E(XY) - E(X)E(Y)] = ac\text{Cov}(X, Y)$

3.9.15 $\int_0^{2\pi} \cos x dx = \int_0^{2\pi} \sin x dx = \int_0^{2\pi} (\cos x)(\sin x)\, dx = 0$, so $E(X) = E(Y) = E(XY) = 0$.

Then Cov(X, Y) = 0. But X and Y are functionally dependent, $Y = \sqrt{1 - X^2}$, so they are probabilistically dependent.

3.9.16 $E(XY) = \int_0^1 y \int_{-y}^{y} x\, dxdy = \int_0^1 y \left[\frac{x^2}{2}\right]_{-y}^{y} dy = \int_0^1 y \cdot 0\, dy = 0$

$E(X) = \int_0^1 \int_{-y}^{y} x\, dxdy = 0$, so Cov($X$, Y) = 0. However, X and Y are dependent since

$P(-1/2 < x < 1/2, 0 < Y < 1/2) = P(0 < Y < 1/2) \neq P(-1/2 < x < 1/2)P(0 < Y < 1/2)$

3.9.17 The random variables are independent and have the same exponential pdf, so Var($X + Y$) = Var(X) + Var(Y). By Question 3.6.11, Var(X) = Var(Y) = $\frac{1}{\lambda^2}$, so Var($X + Y$) = $\frac{2}{\lambda^2}$.

3.9.18 From Question 3.9.3, we have $E(X + Y) = E(X) + E(Y) = 5/9 + 11/18 = 21/18 = 7/6$.

$E[(X + Y)^2] = \int_0^1 \int_0^1 (x+y)^2 \frac{2}{3}(x + 2y)dxdy = \frac{2}{3}\int_0^1 \int_0^1 (x^2 + 2xy + y^2)(x + 2y)dxdy$

$\frac{2}{3}\int_0^1 \int_0^1 (x^3 + 2x^2y + xy^2 + 2x^2y + 4xy^2 + 2y^3)dxdy$

$= \frac{2}{3}\int_0^1 \int_0^1 (x^3 + 4x^2y + 5xy^2 + 2y^3)dxdy = \frac{2}{3}\int_0^1 \left(\frac{1}{4}x^4 + \frac{4}{3}x^3y + \frac{5}{2}x^2y^2 + 2xy^3\right)\Big|_0^1 dy$

$= \frac{2}{3}\int_0^1 \left(\frac{1}{4} + \frac{4}{3}y + \frac{5}{2}y^2 + 2y^3\right)dy = \frac{2}{3}\left(\frac{1}{4} + \frac{4}{6} + \frac{5}{6} + \frac{2}{4}\right) = \frac{3}{2}$

Then $\text{Var}(X + Y) = E[(X + Y)^2] - E(X + Y)^2 = \frac{3}{2} - \left(\frac{7}{6}\right)^2 = \frac{5}{36}$.

3.9.19 First note that $E\left[\left(\sqrt{Y_1Y_2}\right)^2\right] = E[Y_1Y_2] = E(Y_1)E(Y_2) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$, since the Y_i are independent, for $i = 1, 2$.

$E\left(\sqrt{Y_1Y_2}\right) = E\left(\sqrt{Y_1}\right)E\left(\sqrt{Y_2}\right) = \frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9}$, since $E(Y_i) = \int_0^1 \sqrt{y} \cdot 1\, dy = \frac{2}{3}$, $i = 1, 2$.

Then Var $\left(\sqrt{Y_1Y_2}\right) = \frac{1}{4} - \left(\frac{4}{9}\right)^2 = \frac{17}{324}$.

3.9.20 $E(W) = E(4X + 6Y) = 4E(X) + 6E(Y) = 4np_X + 6mp_Y$
$\text{Var}(W) = \text{Var}(4X + 6Y) = 16\text{Var}(X) + 36\text{Var}(Y) = 16np_X(1 - p_X) + 36mp_Y(1 - p_Y)$

3.9.21 Let U_i be the number of calls during the i-th hour in the normal nine hour work day. Then $U = U_1 + U_2 + \ldots + U_9$ is the number of calls during this nine hour period. $E(U) = 9(7) = 63$. For a Poisson random variable, the variance is equal to the mean, so Var(U) = 9(7) = 63. Similarly, if V is the number of calls during the off hours, $E(V) = \text{Var}(V) = 15(4) = 60$.
Let the total cost be the random variable $W = 50U + 60V$.
Then $E(W) = E(50U + 60V) = 50E(U) + 60E(V) = 50(63) + 60(60) = 6750$;
$\text{Var}(W) = \text{Var}(50U + 60V) = 50^2\text{Var}(U) + 60^2\text{Var}(V) = 50^2(63) + 60^2(60) = 373{,}500$.

3.9.22 $L = \sum_{i=1}^{50} B_i + \sum_{i=1}^{49} M_i$, where B_i is the length of the i-th brick, and M_i is the thickness of the i-th mortar separation. Assume all of the B_i and M_i are independent. By Theorem 3.9.5,

$\text{Var}(L) = 50\text{Var}(B_1) + 49\text{Var}(M_1) = 50\left(\frac{1}{32}\right)^2 + 49\left(\frac{1}{16}\right)^2 = 0.240$. Thus, the standard deviation of L is 0.490.

3.9.23 Let R_i be the resistance of the i-th resistor, $1 \le i \le 6$. Assume the R_i are independent and each has standard deviation σ. Then the variance of the circuit resistance is $\text{Var}\left(\sum_{i=1}^{6} R_i\right) = 6\sigma^2$. The circuit must have $6\sigma^2 \le (0.4)^2$ or $\sigma \le 0.163$.

3.9.24 Let p be the probability the gambler wins a hand. Let T_k be his winnings on the k-th hand. Then $E(T_k) = kp$. Also, $E\left(T_k^2\right) = k^2p$, so $\text{Var}(T_k) = k^2p - (kp)^2 = k^2(p - p^2)$.

The total winnings $T = \sum_{k=1}^{n} T_k$, so $E(T) = \sum_{k=1}^{n} kp = \frac{n(n+1)}{2}p$.

$\text{Var}(T) = \sum_{k=1}^{n} k^2(p - p^2) = \frac{n(n+1)(2n+1)}{6}(p - p^2)$

Section 3.10: Order Statistics

3.10.1 $P(Y_3' < 5) = \int_0^5 f_{Y_3'}(y)dy = \int_0^5 \frac{4!}{(3-1)!(4-3)!}\frac{y}{10}^{3-1}\left(1 - \frac{y}{10}\right)^{4-3}\frac{1}{10}dy$

$= \frac{12}{10^4}\int_0^5 y^2(10 - y)dy = \frac{12}{10^4}\left[\frac{10}{3}y^3 - \frac{1}{4}y^4\right]_0^5 = \frac{12}{10^4}\left[\frac{10}{3}5^3 - \frac{1}{4}5^4\right] = 5/16$

3.10.2 First find F_Y: $F_Y(y) = \int_0^y 3t^2dt = y^3$. Then $P(Y_5' > 0.75) = 1 - P(Y_5' < 0.75)$.

But $P(Y_5' < 0.75) = \int_0^{0.75} \frac{6!}{(5-1)!(6-5)!}(y^3)^{5-1}(1 - y^3)^{6-5}3y^2dy$

$= \int_0^{0.75} \frac{6!}{4!}(y^3)^4(1 - y^3)3y^2dy = \int_0^{0.75} \frac{6!}{4!}(y^3)^4(1 - y^3)3y^2dy$

$= \int_0^{0.75} 90(y^{14})(1 - y^3)dy = 90\left[\frac{y^{15}}{15} - \frac{y^{18}}{18}\right]_0^{0.75} = 0.052,$

so $P(Y_5' > 0.75) = 1 - P(Y_5' < 0.75) = 1 - 0.052 = 0.948$.

3.10.3 $P(Y_2' > y_{60}) = 1 - P(Y_2' < y_{60}) = 1 - P(Y_1 < y_{60}, Y_2 < y_{60})$
$= 1 - P(Y_1 < y_{60})P(Y_2 < y_{60}) = 1 - (0.60)(0.60) = 0.64$

3.10.4 The complement of the event is $P(Y_1' > 0.6) \cup P(Y_5' < 0.6)$.

These are disjoint events, so their probability is $P(Y_1' > 0.6) + P(Y_5' < 0.6)$.

But $P(Y_1' > 0.6) = P(Y_1, Y_2, Y_3, Y_4, Y_5 > 0.6) = [P(Y > 0.6)]^5 = \left(\int_{0.6}^{1} 2y\,dy\right)^5 = (0.64)^5 = 0.107$

Also, $P(Y_5' < 0.6) = P(Y_1, Y_2, Y_3, Y_4, Y_5 < 0.6) = [P(Y < 0.6)]^5 = \left(\int_{.0}^{0.6} 2y\,dy\right)^5 = (0.36)^5 = 0.006$

The desired probability is $1 - 0.107 - 0.006 = 0.887$.

3.10.5 $P(Y_1' > m) = P(Y_1, \ldots, Y_n > m) = \left(\dfrac{1}{2}\right)^n$

$P(Y_n' > m) = 1 - P(Y_n' < m) = 1 - P(Y_1, \ldots, Y_n < m)$

$= 1 - P(Y_1 < m) \cdot \ldots \cdot P(Y_n < m) = 1 - \left(\dfrac{1}{2}\right)^n$

If $n \geq 2$, the latter probability is greater.

3.10.6 $P(Y_{\min} < 0.2) = \int_0^{0.2} n f_Y(y)[1 - F_Y(y)]^{n-1} dy$ by Theorem 3.10.1.

Since $F_Y(y) = 1 - e^{-y}$, $\int_0^{0.2} n f_Y(y)[1 - F_Y(y)]^{n-1} dy = \int_0^{0.2} ne^{-y}[1-(1-e^{-y})]^{n-1}$

$= \int_0^{0.2} ne^{-ny} dy = -e^{-ny}\Big|_0^{0.2} = 1 - e^{-0.2n}$

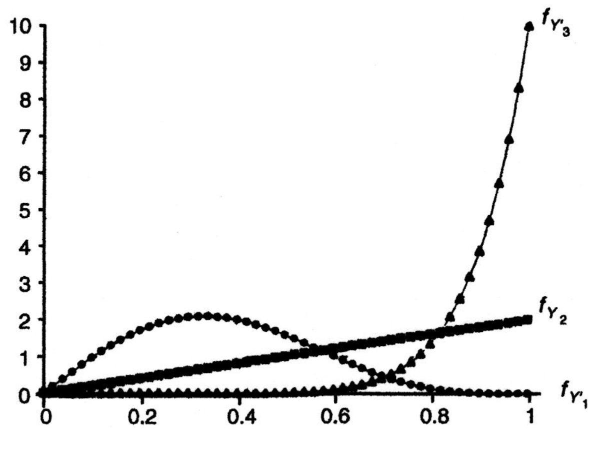
But $(1 - e^{-0.2n}) > 0.9$ if $e^{-0.2n} < 0.1$, which is equivalent to $n > -\dfrac{1}{0.2}\ln 0.1 = 11.513$. The smallest n satisfying this inequality is 12.

3.10.7 $P(0.6 < Y_4' < 0.7) = F_{Y_4'}(0.7) - F_{Y_4'}(0.6) = \int_{0.6}^{0.7} \dfrac{6!}{(4-1)!(6-4)!} y^{4-1}(1-y)^{6-4}(1)dy$ (by Theorem 3.10.2) $= \int_{0.6}^{0.7} 60y^3(1-y)^2 dy = \int_{0.6}^{0.7} 60(y^3 - 2y^4 + y^5)dy = (15y^4 - 24y^5 + 10y^6)\Big|_{0.6}^{0.7}$

$= 0.74431 - 0.54432 = 0.19999$

3.10.8 First note that $F_Y(y) = \int_0^y 2t\,dt = y^2$. Then by Theorem 3.10.2,

$f_{Y_1'}(y) = 5(2y)(1-y^2)^{5-1} = 10y(1-y^2)^4$. By this same result, $f_{Y_5'}(y) = 5(2y)(y^2)^{5-1} = 10y^9$.



3.10.9 $P(Y_{\min} > 20) = P(Y_1 > 20, Y_2 > 20, \ldots, Y_n > 20) = P(Y_1 > 20)P(Y_2 > 20) \ldots P(Y_n > 20)$
$= [P(Y > 20)]^n$. But 20 is the median of Y, so $P(Y > 20) = 1/2$. Thus, $P(Y_{\min} > 20) = (1/2)^n$.

3.10.10 $P(Y_{\min} = Y_n) = P(Y_n < Y_1, Y_n < Y_2, \ldots, Y_n < Y_{n-1}) = P(Y_n < Y_1)P(Y_n < Y_2) \ldots P(Y_n < Y_{n-1}) = \left(\frac{1}{2}\right)^n$

3.10.11 The graphed pdf is the function $f_Y(y) = 2y$, so $F_Y(y) = y^2$
Then $f_{Y_4'}(y) = 20y^6(1 - y^2)2y = 40y^7(1 - y^2)$ and $F_{Y_4'}(y) = 5y^8 - 4y^{10}$.
$P(Y_4' > 0.75) = 1 - F_{Y_4'}(0.75) = 1 - 0.275 = 0.725$
The probability that none of the schools will have fewer than 10% of their students bused is
$P(Y_{\min} > 0.1) = 1 - F_{Y_{\min}}(0.1) = 1 - \int_0^{0.1} 10y(1-y^2)^4\,dy = 1 - \left[-(1-y^2)^5\right]_0^{0.1} = 0.951$ (see Question 3.10.8).

3.10.12 Using the solution to Question 3.10.6, we can, in a similar manner, establish that $f_{Y_1'}(y) = n\lambda e^{-n\lambda y}$. The mean of such an exponential random variable is the inverse of its parameter, or $1/n\lambda$.

3.10.13 If $Y_1, Y_2, \ldots Y_n$ is a random sample from the uniform distribution over [0, 1], then by Theorem 3.10.2, the quantity $\frac{n!}{(i-1)!(n-i)!}\left[F_Y(y)\right]^{i-1}\left[1-F_Y(y)\right]^{n-i} f_y(y)$
$= \frac{n!}{(i-1)!(n-i)!} y^{i-1}(1-y)^{n-i}$ is the pdf of the i-th order statistic.
Thus, $1 = \int_0^1 \frac{n!}{(i-1)!(n-i)!} y^{i-1}(1-y)^{n-i}\,dy = \frac{n!}{(i-1)!(n-i)!}\int_0^1 y^{i-1}(1-y)^{n-i}\,dy$
or, equivalently, $\int_0^1 y^{i-1}(1-y)^{n-i}\,dy = \frac{(i-1)!(n-i)!}{n!}$.

3.10.14 $E(Y_i') = \int_0^1 y \cdot \frac{n!}{(i-1)!(n-i)!} y^{i-1}(1-y)^{n-i}\,dy = \frac{n!}{(i-1)!(n-i)!}\int_0^1 y^i(1-y)^{n-i}\,dy$
$= \frac{n!}{(i-1)!(n-i)!}\int_0^1 y^{(i+1)-1}(1-y)^{(n+1)-(i+1)}\,dy = \frac{n!}{(i-1)!(n-i)!}\frac{[(i+1)-1]![(n+1)-(i+1)]!}{(n+1)!}$
where this last equality comes from the result in Question 3.10.13.
Thus, $E(Y_i') = \frac{n!}{(i-1)!(n-i)!}\frac{i!(n-1)!}{(n+i)!} = \frac{i}{(n+1)}$

3.10.15 This question translates to asking for the probability that a random sample of three independent uniform random variables on [0, 1] has range $R \le 1/2$. Example 3.10.6 establishes that $F_R(r) = 3r^2 - 2r^3$. The desired probability is $F_R(1/2) = 3(1/2)^2 - 2(1/2)^3 = 0.5$.

3.10.16 This question requires finding the probability that a random sample of three independent exponential random variables on [0, 10] has range $R \le 2$. From Equation 3.10.5, we find the joint pdf of $Y_{\min}$ and $Y_{\max}$ to be $3[F_Y(v) - F_Y(u)]f_Y(u)f_Y(v) = 3[(1 - e^{-v}) - (1 - e^{-u})]e^{-u}e^{-v}$
$= 3(e^{-2u-v} - e^{-u-2v})$, $u \le v$.

$P(R \le 2)$ is obtained by integrating the joint pdf over a strip such as pictured in Figure 3.10.4, but the strip is infinite in extent. Thus,

$P(R \le 2) = \int_0^\infty \int_u^{u+2} 3(e^{-2u-v} - e^{-u-2v})\,dv\,du$. The inner integral is

$$\int_u^{u+2} (e^{-2u-v} - e^{-u-2v})\,dv = e^{-2u}(-e^{-v})\Big|_u^{u+2} - e^{-u}\left(-\frac{1}{2}e^{-2v}\right)\Big|_u^{u+2}$$

$$= e^{-2u}(e^{-u} - e^{-u-2}) - \frac{1}{2}e^{-u}(e^{-2u} - e^{-2u-4}) = e^{-3u}\left(\frac{1}{2} - e^{-2} + \frac{1}{2}e^{-4}\right)$$

Then $P(R \le 2) = 3\left(\frac{1}{2} - e^{-2} + \frac{1}{2}e^{-4}\right)\int_0^\infty e^{-3u}\,du = \frac{1}{2} - e^{-2} + \frac{1}{2}e^{-4} = 0.374.$

Section 3.11: Conditional Densities

3.11.1 $p_X(x) = \dfrac{x+1+x\cdot 1}{21} + \dfrac{x+2+x\cdot 2}{21} = \dfrac{3+5x}{21}$, $x = 1, 2$

$$p_{Y|x}(y) = \frac{p_{X,Y}(x,y)}{p_X(x)} = \frac{x+y+xy}{3+5x}, \; y = 1, 2$$

3.11.2 The probability that $X = x$ and $Y = y$ is the probability of y 4's on the first two rolls and $x - y$ rolls on the last four rolls. These events are independent, so

$$p_{X,Y}(x, y) = \binom{2}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{2-y}\binom{4}{x-y}\left(\frac{1}{6}\right)^{x-y}\left(\frac{5}{6}\right)^{4-x+y} \text{ for } y \le x$$

Then $$p_{Y|x}(y) = \frac{p_{X,Y}(x,y)}{p_X(x)} = \frac{\binom{2}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{2-y}\binom{4}{x-y}\left(\frac{1}{6}\right)^{x-y}\left(\frac{5}{6}\right)^{4-x+y}}{\binom{6}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{6-x}} = \frac{\binom{2}{y}\binom{4}{x-y}}{\binom{6}{x}},$$

$0 \le y \le \min(2, x)$, which we recognize as a hypergeometric distribution.

3.11.3 $p_{Y|x}(y) = \dfrac{p_{X,Y}(x,y)}{p_X(x)} = \dfrac{\binom{8}{x}\binom{6}{y}\binom{4}{3-x-y}}{\binom{18}{3}} \div \dfrac{\binom{8}{x}\binom{10}{3-x}}{\binom{18}{3}} = \dfrac{\binom{6}{y}\binom{4}{3-x-y}}{\binom{10}{3-x}}$, with $0 \le y \le 3 - x$

3.11.4 $P(X = 2|Y = 2) = \dfrac{P(X=2, Y=2)}{P(Y=2)}$

$$= \frac{\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{52}{5}} \div \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} = \frac{\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{4}{2}\binom{48}{3}} = 0.015$$

3.11.5 **(a)** $1/k = \sum_{x=1}^{3}\sum_{y=1}^{3}(x+y) = 36$, so $k = 1/36$

(b) $p_X(x) = \frac{1}{36}\sum_{y=1}^{3}(x+y) = \frac{1}{36}(3x+6)$

$$p_{Y|x}(1) = \frac{p_{X,Y}(x,1)}{p_X(x)} = \frac{\frac{1}{36}(x+1)}{\frac{1}{36}(3x+6)} = \frac{x+1}{3x+6},\ x = 1, 2, 3$$

3.11.6 **(a)** $p_{X,Y}(x, y) = p_{Y|x}(y)p_X(x) = \binom{x}{y}\left(\frac{1}{2}\right)^x\left(\frac{1}{3}\right),\ y \le x$

(b) $p_Y(0) = \left(\frac{1}{3}\right)\sum_{x=1}^{3}\binom{x}{0}\left(\frac{1}{2}\right)^x = \frac{7}{24}$

$p_Y(1) = \left(\frac{1}{3}\right)\sum_{x=1}^{3}\binom{x}{1}\left(\frac{1}{2}\right)^x = \frac{11}{24}$

$p_Y(2) = \left(\frac{1}{3}\right)\sum_{x=2}^{3}\binom{x}{2}\left(\frac{1}{2}\right)^x = \frac{5}{24}$

$p_Y(3) = \left(\frac{1}{3}\right)\binom{3}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{24}$

3.11.7 $p_Z(z) = \frac{1\cdot 1+1\cdot z+1\cdot z}{54} + \frac{1\cdot 2+1\cdot z+2\cdot z}{54} + \frac{2\cdot 1+2\cdot z+1\cdot z}{54} + \frac{2\cdot 2+2\cdot z+2\cdot z}{54}$

$= \frac{9+12z}{54},\ z = 1, 2$

Then $p_{X,Y|z}(x,y) = \frac{xy+xz+yz}{9+12z},\ x = 1, 2 \quad y = 1, 2 \quad z = 1, 2$

3.11.8 $p_{W,X}(1, 1) = P(\{(1, 1, 1)\}) = 3/54$
$p_{W,X}(2, 2) = 33/54$; $P(W = 2, X = 2) = P(X = 2)$. But $P(X = 2) = P(Z = 2)$ by symmetry, and from Question 3.11.7, this probability is 33/54. Then $p_{W,X}(2, 1) = 1 - 3/54 - 33/54 = 18/54$
Finally, $p_{W|1}(1) = (3/54)/(21/54) = 1/7$; $p_{W|1}(2) = (18/54)/(21/54) = 6/7$; and
$p_{W|2}(2) = (33/54)/(33/54) = 1$

3.11.9 $p_{X|x+y=n}(x) = \frac{P(X=k, X+Y=n)}{P(X+Y=n)} = \frac{P(X=k, Y=n-k)}{P(X+Y=n)}$

$$= \frac{e^{-\lambda}\frac{\lambda^k}{k!}e^{-\mu}\frac{\mu^{n-k}}{(n-k)!}}{e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^n}{n!}} = \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{n-k}$$

but the right hand term is a binomial probability with parameters n and $\lambda/(\lambda+\mu)$.

3.11.10 Let U be the number of errors made by Compositor A in the 100 pages. Then the pdf of U is Poisson with parameter $\lambda = 200$. Similarly, if V is the number of errors made by Compositor B, then the pdf for V is Poisson with $\mu = 300$. From the previous question, $P(U \le 259 | U + V = 520)$ is binomial with parameter $n = 520$ and $p = 200/(200 + 300) = 2/5$.

The desired probability is $\sum_{k=0}^{259} \binom{520}{k} \left(\frac{2}{5}\right)^k \left(\frac{3}{5}\right)^{520-k}$

3.11.11 $P(X > s + t | X > t) = \dfrac{P(X > s+t \text{ and } X > t)}{P(X > t)} = \dfrac{P(X > s+t)}{P(X > t)}$

$$= \frac{(1/\lambda)\int_{s+t}^{\infty} e^{-x/\lambda}dx}{(1/\lambda)\int_{t}^{\infty} e^{-x/\lambda}dx} = \frac{-(1/\lambda)e^{-x/\lambda}\Big|_{s+t}^{\infty}}{-(1/\lambda)e^{-x/\lambda}\Big|_{t}^{\infty}} = \frac{(1/\lambda)e^{-(s+t)/\lambda}}{(1/\lambda)e^{-t/\lambda}} = e^{-s/\lambda} = \int_s^{\infty} (1/\lambda)e^{-x/\lambda}dx = P(X > s)$$

3.11.12 **(a)** $f_X(x) = \int_x^{\infty} 2e^{-x}e^{-y}dy = 2e^{-2x}$, $x > 0$, so $P(X < 1) = \int_0^1 2e^{-2x}dx = 1 - e^{-2} = 1 - 0.135 = 0.865$

Also, $P(X < 1, Y < 1) = \int_0^1\int_0^x 2e^{-(x+y)}dydx = \int_0^1 2e^{-x}\left[-e^{-y}\right]_0^x dx = \int_0^1 (2e^{-x} - 2e^{-2x})dx$

$= -2e^{-x} + e^{-2x}\Big|_0^1 = 0.400$. Then the conditional probability is $\dfrac{0.400}{0.865} = 0.462$

(b) $P(Y < 1 | X = 1) = 0$, since the joint pdf is defined with y always larger than x.

(c) $f_{Y|x}(y) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{2e^{-(x+y)}}{2e^{-2x}} = e^x e^{-y}$, $x < y$

(d) $E(Y | x) = \int_x^{\infty} ye^x e^{-y}dy = e^x\int_x^{\infty} ye^{-y}dy = e^x[-e^{-y}(y+1)]_x^{\infty} = e^x[e^{-x}(x+1)] = x+1$

3.11.13 $f_X(x) = \int_0^1 (x + y)dy = xy + \dfrac{y^2}{2}\Big|_0^1 = x + \dfrac{1}{2}$, $0 \le x \le 1$

$f_{Y|x}(y) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{x+y}{x+\frac{1}{2}}$, $0 \le y \le 1$

3.11.14 $f_X(x) = \int_0^{1-x} 2\,dy = 2y\Big|_0^{1-x} = 2(1 - x)$, $0 \le x \le 1$

$f_{Y|x}(y) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{2}{2(1-x)} = \dfrac{1}{1-x}$, $0 \le y \le 1 - x$

For each x, the conditional pdf does not depend on y, so it is a uniform pdf.

3.11.15 $f_{X,Y}(x, y) = f_{Y|x}(y)f_X(x) = \left(\dfrac{2y+4x}{1+4x}\right)\dfrac{1}{3}(1+4x) = \dfrac{1}{3}(2y+4x)$

$f_Y(y) = \int_0^1 \dfrac{1}{3}(2y+4x)dx = \dfrac{1}{3}(2xy+2x^2)\Big|_0^1 = \dfrac{1}{3}(2y+2)$, with $0 \le y \le 1$

3.11.16 (a) $f_X(x) = \int_0^1 \frac{2}{5}(2x+3y)dy = \frac{2}{5}\left(2xy + \frac{3}{2}y^2\right)\Big|_0^1 = \frac{4}{5}x + \frac{3}{5}$, with $0 \le x \le 1$

(b) $f_{Y|x}(y) = \dfrac{\frac{2}{5}(2x+3y)}{\frac{4}{5}x+\frac{3}{5}} = \dfrac{4x+6y}{4x+3}$, $0 \le y \le 1$

(c) $f_{Y|\frac{1}{2}}(y) = \frac{1}{5}(2+6y)$

$P(1/4 \le Y \le 3/4) = \int_{1/4}^{3/4} \frac{1}{5}(2+6y)dy = 10/20 = 1/2$

3.11.17 $f_Y(y) = \int_0^y 2\,dx = 2y$

$f_{X|y}(x) = \dfrac{2}{2y} = \dfrac{1}{y}$, $0 < x < y$

$f_{X|\frac{3}{4}}(x) = \dfrac{1}{\frac{3}{4}} = \dfrac{4}{3}$, $0 < x < 3/4$

$P(0 < X < ½|\ Y = 3/4) = \int_0^{1/2} \frac{4}{3}dx = \frac{2}{3}$

3.11.18 $f_Y(y) = \int_0^y \frac{xy}{2}dx = \frac{x^2 y}{4}\Big|_0^y = \frac{y^3}{4}$, $0 < y < 2$

$f_{X|y}(x) = \dfrac{xy}{2}\Big/\dfrac{y^3}{4} = \dfrac{2x}{y^2}$, $0 \le x < y \le 2$

$f_{X|\frac{3}{2}}(x) = \dfrac{2x}{(3/2)^2} = \dfrac{8}{9}x$, $0 \le x < 3/2$

$P(X < 1|\ Y = 3/2) = \int_0^1 \frac{8}{9}x\,dx = \frac{4}{9}x^2\Big|_0^1 = \frac{4}{9}$

3.11.19 $f_{X_4,X_5}(x_4,x_5) = \int_0^1\int_0^1\int_0^1 32x_1x_2x_3x_4x_5\ dx_1dx_2dx_3 = 4x_4x_5$, $0 < x_4, x_5 < 1$

$f_{X_1,X_2,X_3|x_4,x_5}(x_1,x_2,x_3) = \dfrac{32x_1x_2x_3x_4x_5}{4x_4x_5} = 8x_1x_2x_3$, $0 < x_1, x_2, x_3 < 1$

Note: the five random variables are independent, so the conditional pdfs are just the marginal pdfs.

3.11.20 (a) $f_X(x) = \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)dy = \frac{6}{7}\left(x^2 y + \frac{xy^2}{4}\right)\Big|_0^2 = \frac{6}{7}(2x^2 + x)$

(b) $P(X > 2Y) = \int_0^1 \int_0^{\frac{1}{2}x} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dydx$

$= \int_0^1 \left[\frac{6}{7}\left(x^2 y + \frac{xy^2}{4}\right)\right]_0^{\frac{1}{2}x} dx = \int_0^1 \frac{6}{7}\left(\frac{9}{16}x^3\right)dx = \frac{27}{224}$

(c) $P(Y > 1 \mid X > 1/2) = \dfrac{P(X > 1/2, Y > 1)}{P(X > 1/2)}$

First calculate the numerator: $P(X > 1/2,\ Y > 1) = \int_{1/2}^1 \int_1^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dydx = \frac{55}{112}$

We know f_X from part **(a)** so the denominator is $P(X > 1/2) = \int_{1/2}^1 \frac{6}{7}(2x^2 + x)dx = \frac{23}{28}$

The conditional probability requested is $\frac{55}{112} \Big/ \frac{23}{28} = \frac{55}{92}$.

Section 3.12: Moment-Generating Functions

3.12.1 Let X be a random variable with $p_X(k) = 1/n$, for $k = 0, 1, 2, \ldots, n - 1$, and 0 otherwise.

$M_X(t) = E(e^{tX}) = \sum_{k=0}^{n-1} e^{tk} p_X(k) = \sum_{k=0}^{n-1} e^{tk}\frac{1}{n} = \frac{1}{n}\sum_{k=0}^{n-1}(e^t)^k = \frac{1 - e^{nt}}{n(1 - e^t)}$.

(Recall that $1 + r + \ldots + r^{n-1} = \frac{1 - r^n}{1 - r}$).

3.12.2 $f_X(-3) = 6/10$; $f_X(5) = 4/10$.
$M_X(t) = E(e^{tX}) = e^{-3t}(6/10) + e^{5t}(4/10)$

3.12.3 For the given binomial random variable,

$E(e^{tX}) = M_X(t) = \left(1 - \frac{1}{3} + \frac{1}{3}e^t\right)^{10}$. Set $t = 3$ to obtain $E(e^{3X}) = \frac{1}{3^{10}}(2 + e^3)^{10}$

3.12.4 $M_X(t) = \sum_{k=0}^{\infty} e^{tk}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^k = \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{3e^t}{4}\right)^k$

$= \frac{1}{4}\frac{1}{1 - \frac{3e^t}{4}} = \frac{1}{4 - 3e^t}$, $0 < e^t < 4/3$

3.12.5 **(a)** Normal with $\mu = 0$ and $\sigma^2 = 12$ **(b)** Exponential with $\lambda = 2$
(c) Binomial with $n = 4$ and $p = ½$ **(d)** Geometric with $p = 0.3$

3.12.6 $M_Y(t) = E(e^{tY}) = \int_0^1 e^{ty}\, y\, dy + \int_1^2 e^{ty}(2-y)\, dy$

$$= \left(\frac{1}{t}y - \frac{1}{t^2}\right)e^{ty}\Big|_0^1 + \frac{2}{t}e^{ty}\Big|_1^2 - \left(\frac{1}{t}y - \frac{1}{t^2}\right)e^{ty}\Big|_1^2$$

$$= \left(\frac{1}{t} - \frac{1}{t^2}\right)e^t - \left(-\frac{1}{t^2}\right) + \frac{2}{t}e^{2t} - \frac{2}{t}e^t - \left(\frac{2}{t} - \frac{1}{t^2}\right)e^{2t} + \left(\frac{1}{t} - \frac{1}{t^2}\right)e^t$$

$$= \frac{1}{t^2} + \frac{1}{t^2}e^{2t} - \frac{2}{t^2}e^t = \frac{1}{t^2}(e^t - 1)^2$$

3.12.7 $M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!} = e^{\lambda(e^t - 1)}$

3.12.8 $M_Y(t) = E(e^{tY}) = \int_0^{\infty} e^{ty}\, y e^{-y} dy = \int_0^{\infty} y e^{-y(1-t)} dy = \frac{1}{1-t}\int_0^{\infty} y(1-t)e^{-y(1-t)} dy$

$$= \left(\frac{1}{1-t}\right)\left(\frac{1}{1-t}\right) = \frac{1}{(1-t)^2},$$

since the integral is the mean of an exponential pdf with parameter $(1 - t)$, which is $\frac{1}{1-t}$.

3.12.9 $M_Y^{(1)}(t) = \frac{d}{dt}e^{t^2/2} = te^{t^2/2}$

$M_Y^{(2)}(t) = \frac{d}{dt}te^{t^2/2} = t(te^{t^2/2}) + e^{t^2/2} = (t^2+1)e^{t^2/2}$

$M_Y^{(3)}(t) = \frac{d}{dt}(t^2+1)e^{t^2/2} = (t^2+1)te^{t^2/2} + 2te^{t^2/2}$, and $E(Y^3) = M_Y^{(3)}(0) = 0$

3.12.10 From Example 3.12.3, $M_Y(t) = \frac{\lambda}{(\lambda - t)}$ and $M_Y^{(1)}(t) = \frac{\lambda}{(\lambda - t)^2}$.

Successive differentiation gives $M_Y^{(4)}(t) = \frac{(4!)\lambda}{(\lambda - t)^5}$. Then $E(Y^4) = M_Y^{(4)}(0) = \frac{(4!)\lambda}{\lambda^5} = \frac{24}{\lambda^4}$.

3.12.11 $M_Y^{(1)}(t) = \frac{d}{dt}e^{at + b^2t^2/2} = (a + b^2 t)e^{at+b^2t^2/2}$, so $M_Y^{(1)}(0) = a$

$M_Y^{(2)}(t) = (a + b^2 t)^2 e^{at+b^2t^2/2} + b^2 e^{at+b^2t^2/2}$, so $M_Y^{(2)}(0) = a^2 + b^2$.

Then $\text{Var}(Y) = (a^2 + b^2) - a^2 = b^2$

3.12.12 Successive differentiation of $M_Y(t)$ gives

$M_Y^{(4)}(t) = \alpha^4 k(k+1)(k+2)(k+3)(1-\alpha t)^{-k-4}$. Thus, $E(Y^4) = M_Y^{(4)}(0) = \alpha^4 k(k+1)(k+2)(k+3)$

3.12.13 The moment generating function of Y is that of a normal variable with mean $\mu = -1$ and variance $\sigma^2 = 8$. Then $E(Y^2) = \text{Var(Y)} + \mu^2 = 8 + 1 = 9$.

3.12.14 $M_Y^{(1)}(t) = \dfrac{d}{dt}(1-t/\lambda)^{-r} = (-r)(1-t/\lambda)^{-r-1}(-\lambda) = \lambda r(1-t/\lambda)^{-r-1}$

$M_Y^{(2)}(t) = \dfrac{d}{dt}\lambda r(1-t/\lambda)^{-r-1} = (-r-1)\lambda r(1-t/\lambda)^{-r-2}(-\lambda) = \lambda^2 r(r+1)(1-t/\lambda)^{-r-2}$

Continuing in this manner yields $M_Y^{(k)}(t) = \lambda^r \dfrac{(r+k-1)!}{(r-1)!}(1-t/\lambda)^{-r-k}$.

Then $E(Y^k) = M_Y^{(k)}(0) = \lambda^r \dfrac{(r+k-1)!}{(r-1)!}$.

3.12.15 $M_Y(t) = \displaystyle\int_a^b e^{ty}\frac{1}{b-a}dy = \frac{1}{(b-a)t}e^{ty}\Big|_a^b = \frac{1}{(b-a)t}(e^{tb}-e^{at})$ for $t \neq 0$

$M_Y^{(1)}(t) = \dfrac{1}{(b-a)}\left[\dfrac{be^{tb}-ae^{at}}{t} - \dfrac{e^{tb}-e^{at}}{t^2}\right]$

$E(Y) = \lim\limits_{t\to 0} M_Y^{(1)}(t) = \dfrac{1}{(b-a)}\lim\limits_{t\to 0}\left[\dfrac{be^{tb}-ae^{at}}{t} - \dfrac{e^{tb}-e^{at}}{t^2}\right]$. Applying L'Hospital's rule gives

$E(Y) = \dfrac{1}{(b-a)}\left[(b^2-a^2) - \dfrac{b^2-a^2}{2}\right] = \dfrac{(a+b)}{2}$

3.12.16 $M_Y^{(1)}(t) = \dfrac{(1-t^2)2e^{2t}-(-2t)e^{2t}}{(1-t^2)^2} = 2\dfrac{(1+t-t^2)e^{2t}}{(1-t^2)^2}$, so $E(Y) = M_Y^{(1)}(0) = 2$.

$M_Y^{(2)}(t) = \dfrac{2(1-t^2)^2[(1-2t)e^{2t}+2(1+t-t^2)e^{2t}] - 2(1-t^2)(-2t)2(1+t-t^2)e^{2t}}{(1-t^2)^4}$

so $M_Y^{(2)}(0) = 6$. Thus $\text{Var}(Y) = E(Y^2) - \mu^2 = 6 - 4 = 2$.

3.12.17 Let $Y = \dfrac{1}{\lambda}V$, where $f_V(y) = ye^{-y}$, $y \geq 0$. Question 3.12.8 establishes that $M_V(t) = \dfrac{1}{(1-t)^2}$. By Theorem 3.12.3(a), $M_Y(t) = M_V(t/\lambda) = 1(1-t/\lambda)^2$.

3.12.18 $M_{Y_1+Y_2+Y_3}(t) = M_{Y_1}(t)M_{Y_2}(t)M_{Y_3}(t) = \left(\dfrac{1}{(1-t/\lambda)^2}\right)^3 = \dfrac{1}{(1-t/\lambda)^6}$

3.12.19 (a) Let X and Y be two Poisson variables with parameters λ and μ, respectively.
Then $M_X(t) = e^{-\lambda+\lambda e^t}$ and $M_Y(t) = e^{-\mu+\mu e^t}$.
$M_{X+Y}(t) = M_X(t)M_Y(t) = e^{-\lambda+\lambda e^t}e^{-\mu+\mu e^t} = e^{-(\lambda+\mu)+(\lambda+\mu)e^t}$.
This last expression is that of a Poisson variable with parameter $\lambda + \mu$, which is then the distribution of $X + Y$.

(b) Let X and Y be two exponential variables, with parameters λ and μ, respectively.

Then $M_X(t) = \dfrac{\lambda}{(\lambda - t)}$ and $M_Y(t) = \dfrac{\mu}{(\mu - t)}$.

$M_{X+Y}(t) = M_X(t)M_Y(t) = \dfrac{\lambda}{(\lambda - t)}\dfrac{\mu}{(\mu - t)}$.

This last expression is not that of an exponential variable, and the distribution of $X + Y$ is not exponential.

(c) Let X and Y be two normal variables, with parameters μ_1, σ_1^2 and μ_2, σ_2^2 respectively.

Then $M_X(t) = e^{\mu_1 t + \sigma_1^2 t^2/2}$ and $M_Y(t) = e^{\mu_2 t + \sigma_2^2 t^2/2}$.

$M_{X+Y}(t) = M_X(t)M_Y(t) = e^{\mu_1 t + \sigma_1^2 t^2/2} e^{\mu_2 t + \sigma_2^2 t^2/2} = e^{(\mu_1 + \mu_2)t + (\sigma_1^2 + \sigma_2^2)t^2/2}$.

This last expression is that of a normal variable with parameters $\mu_1 + \mu_2$ and σ_1^2 and σ_2^2, which is then the distribution of $X + Y$.

3.12.20 From the moment-generating function of X, we know that it is binomial with $n = 5$ and $p = 3/4$. Then $P(X \le 2) = (1/4)^5 + 5(3/4)(1/4)^4 + 10(3/4)^2(1/4)^3 = 0.104$

3.12.21 Let $S = \sum_{i=1}^{n} Y_i$. Then $M_s(t) = \prod_{i=1}^{n} M_{Y_i}(t) = \left(e^{\mu t + \sigma^2 t^2/2}\right)^n = e^{n\mu t + n\sigma^2 t^2/2}$.

$M_{\bar{Y}}(t) = M_{S/n}(t) = M_S(t/n) = e^{\mu t + (\sigma^2/n)t^2/2}$. Thus $\bar{Y}$ is normal with mean μ and variance σ^2/n.

3.12.22 From the moment-generating function of W, we know that $W = X + Y$, where X is Poisson with parameter 3, and Y is binomial with parameters $n = 4$ and $p = 1/3$. Also, X and Y are independent. Then $P(W \le 1) = p_X(0)p_Y(0) + p_X(0)p_Y(1) + p_X(1)p_Y(0)$
$= (e^{-3})(2/3)^4 + (e^{-3})4(1/3)(2/3)^3 + (3e^{-3})(2/3)^4 = 0.059$

3.12.23 (a) $M_W(t) = M_{3X}(t) = M_X(3t) = e^{-\lambda + \lambda e^{3t}}$. This last term is not the moment-generating function of a Poisson random variable, so W is not Poisson.

(b) $M_W(t) = M_{3X+1}(t) = e^t M_X(3t) = e^t e^{-\lambda + \lambda e^{3t}}$. This last term is not the moment-generating function of a Poisson random variable, so W is not Poisson.

3.12.24 (a) $M_W(t) = M_{3Y}(t) = M_Y(3t) = e^{\mu(3t) + \sigma^2(3t)^2/2} = e^{(3\mu)t + 9\sigma^2 t^2/2}$.

This last term is the moment-generating function of a normal random variable with mean 3μ and variance $9\sigma^2$, which is then the distribution of W.

(b) $M_W(t) = M_{3Y+1}(t) = e^t M_Y(3t) = e^t e^{\mu(3t) + \sigma^2(3t)^2/2} = e^{(3\mu+1)t + 9\sigma^2 t^2/2}$. This last term is the moment-generating function of a normal random variable with mean $3\mu + 1$ and variance $9\sigma^2$, which is then the distribution of W.

Chapter 4: Special Distributions

Section 4.2: The Poisson Distribution

4.2.1 $p = P(\text{word is misspelled}) = \dfrac{1}{3250}$; $n = 6000$. Let x = number of words misspelled. Using the exact binomial analysis, $P(X = 0) = \binom{6000}{0}\left(\dfrac{1}{3250}\right)^0\left(\dfrac{3249}{3250}\right)^{6000} = 0.158$. For the Poisson approximation, $\lambda = 6000\left(\dfrac{1}{3250}\right) = 1.846$, so $P(X = 0) \doteq \dfrac{e^{-1.846}(1.846)^0}{0!} = 0.158$. The agreement is not surprising because n is so large and p is so small (recall Example 4.2.1).

4.2.2 Let X = number of prescription errors. Then $\lambda = np = 10\cdot\dfrac{905}{289{,}411} = 0.0313$, and $P(X \geq 1) =$ $1 - P(X = 0) \doteq 1 - \dfrac{e^{-0.0313}(0.0313)^0}{0!} = 0.031$.

4.2.3 Let X = number born on Poisson's birthday. Since $n = 500$, $p = \dfrac{1}{365}$, and $\lambda = 500 \cdot \dfrac{1}{365} =$ 1.370, $P(X \leq 1) = P(X = 0) + P(X = 1) \doteq \dfrac{e^{-1.370}(1.370)^0}{0!} + \dfrac{e^{-1.370}(1.370)^1}{1!} = 0.602$.

4.2.4 **(a)** Let X = number of chromosome mutations. Given that $n = 20{,}000$ and $p = \dfrac{1}{10{,}000}$ (so $\lambda = 2$), $P(X = 3) \doteq e^{-2}2^3/3! = 0.18$.

(b) Listed in the table are values of $P(X \geq k)$ under the assumption that $p = \dfrac{1}{10{,}000}$. If X is on the order of 5 or 6, the credibility of that assumption becomes highly questionable.

k	$P(X \geq k)$
3	0.3233
4	0.1429
5	0.0527
6	0.0166

4.2.5 Let X = number of items requiring a price check. If $p = P(\text{item requires price check}) = 0.01$ and $n = 10$, a binomial analysis gives $P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{10}{0}(0.01)^0(0.99)^{10} = 0.10$. Using the Poisson approximation, $\lambda = 10(0.01) = 0.1$ and $P(X \geq 1) = 1 - P(X = 0)$ $\doteq 1 - \dfrac{e^{-0.1}(0.1)^0}{0!} = 0.10$. The exact model that applies here is the hypergeometric, rather than the binomial, because p is a function of the previous items purchased. However, the variation in p will be essentially zero for the 10 items purchased, so the binomial and hypergeometric models in this case will be effectively the same.

4.2.6 Let X = number of policy-holders who will die next year. Since $n = 120$, $p = \frac{1}{150}$, and $\lambda = \frac{120}{150} = 0.8$, P(company will pay at least \$150,000 in benefits) = $P(X \geq 3)$ = $1 - P(X \leq 2) \doteq 1 - \sum_{k=0}^{2} \frac{e^{-0.8}(0.8)^k}{k!} = 0.047$.

4.2.7 Let X = number of pieces of luggage lost. Given that $n = 120$, $p = \frac{1}{200}$, (so $\lambda = 120 \cdot \frac{1}{200} = 0.6$), $= P(X \geq 2) = 1 - P(X \leq 1) = 1 - \sum_{k=0}^{1} \frac{e^{-0.6}(0.6)^k}{k!} = 0.122$.

4.2.8 Let X = number of cancer cases. If $n = 9500$ and $p = \frac{1}{1{,}000{,}000}$, then $\lambda = \frac{9{,}500}{1{,}000{,}000} = 0.0095$ and $P(X \geq 2) = 1 - P(X \leq 1) = 1 - \sum_{k=0}^{1} \frac{e^{-0.0095}(0.0095)^k}{k!} = 0.00005$. The fact that the latter is so small suggests that a lineman's probability of contracting cancer is considerably higher than the one in a million value for p characteristic of the general population.

4.2.9 Let X = number of solar systems with intelligent life and let $p = P$(solar system is inhabited). For n = 100,000,000,000, $P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{100{,}000{,}000{,}000}{0} p^0 \cdot (1-p)^{100{,}000{,}000{,}000}$.
Solving $1 - (1-p)^{100{,}000{,}000{,}000} = 0.50$ gives $p = 6.9 \times 10^{-12}$. Alternatively, it must be true that $1 - \frac{e^{-\lambda}\lambda^0}{0!} = 0.50$, which implies that $\lambda = -\ln(0.50) = 0.69$. But $0.69 = np = 1 \times 10^{11} \cdot p$, so $p = 6.9 \times 10^{-12}$.

4.2.10 The average number of fatalities per corps-year = $\frac{109(0) + 65(1) + 22(2) + 3(3) + 1(4)}{200} = 0.61$, so the presumed Poisson model is $p_X(k) = \frac{e^{-0.61}(0.61)^k}{k!}$, $k = 0, 1, \ldots$ Evaluating $p_X(k)$ for $k = 0, 1, 2, 3$, and 4+ shows excellent agreement between the observed proportions and the corresponding Poisson probabilities.

No. of deaths, k	Frequency	Proportion	$p_X(k)$
0	109	0.545	0.5434
1	65	0.325	0.3314
2	22	0.110	0.1011
3	3	0.015	0.0206
4+	1	0.005	0.0035
	200	1.000	1.0000

4.2.11 The observed number of major changes = 0.44 (= $\bar{x} = \frac{1}{356}$[237(0) + 90(1) + 22(2) + 7(3)]), so the presumed Poisson model is $p_X(k) = \frac{e^{-0.44}(0.44)^k}{k!}$, $k = 0, 1, \ldots$ Judging from the agreement evident in the accompanying table between the set of observed proportions and the values for $p_X(k)$, the hypothesis that X is a Poisson random variable is entirely credible.

No. of changes, k	Frequency	Proportion	$p_X(k)$
0	237	0.666	0.6440
1	90	0.253	0.2834
2	22	0.062	0.0623
3+	7	0.020	0.0102
	356	1.000	1.0000

4.2.12 Since $\bar{x} = \frac{1}{40}[9(0) + 13(1) + 10(2) + 5(3) + 2(4) + 1(5)] = 1.53$, $p_X(k) = \frac{e^{-1.53}(1.53)^k}{k!}$, $k = 0, 1, \ldots$ Yes, the Poisson appears to be an adequate model, as indicated by the close agreement between the observed proportions and the values of $p_X(k)$.

No. of bags lost, k	Frequency	Proportion	$p_X(k)$
0	9	0.225	0.2165
1	13	0.325	0.3313
2	10	0.250	0.2534
3	5	0.125	0.1293
4	2	0.050	0.0494
5+	1	0.025	0.0201
	40	1.000	1.0000

4.2.13 The average of the data is $\frac{1}{113}[82(0) + 25(1) + 4(2) + 0(3) + 2(4) \doteq 0.363$. Then use the model $e^{-0.363}\frac{0.363^k}{k!}$. Usual statistical practice suggests collapsing the low frequency categories, in this case, $k = 2, 3, 4$. The result is the following table.

No. of countires, k	Frequency	$p_X(k)$	Expected frequency
0	82	0.696	78.6
1	25	0.252	28.5
2+	6	0.052	5.9

The level of agreement between the observed and expected frequencies suggests that the Poisson is a good model for these data.

4.2.14 **(a)** The model $p_X(k) = e^{-2.157}(2.157)^k/k!$, $k = 0, 1, \ldots$ fits the data fairly well (where $\bar{x} = 2.157$), but there does appear to be a slight tendency for deaths to "cluster"— that is, the values 0, 5, 6, 7, 8, and 9 are all over-represented.

No. of deaths, k	Frequency	$p_X(k)$	Expected frequency
0	162	0.1157	126.8
1	267	0.2495	273.5
2	271	0.2691	294.9
3	185	0.1935	212.1
4	111	0.1043	114.3
5	61	0.0450	49.3
6	27	0.0162	17.8
7	8	0.0050	5.5
8	3	0.0013	1.4
9	1	0.0003	0.3
10+	0	0.0001	0.1

(b) Deaths may not be independent events in all cases, and the fatality rate may not be constant.

4.2.15 If the mites exhibit any sort of "contagion" effect, the independence assumption implicit in the Poisson model will be violated. Here, $\bar{x} = \frac{1}{100}[55(0) + 20(1) + \ldots + 1(7)] = 0.81$, but $p_X(k) = e^{-0.81}(0.81)^k/k!$, $k = 0, 1, \ldots$ does not adequately approximate the infestation distribution.

No. of infestations, k	Frequency	Proportion	$p_X(k)$
0	55	0.55	0.4449
1	20	0.20	0.3603
2	21	0.21	0.1459
3	1	0.01	0.0394
4	1	0.01	0.0080
5	1	0.01	0.0013
6	0	0	0.0002
7+	1	0.01	0.0000
		1.00	1.0000

4.2.16 Let X = number of repairs needed during an eight-hour workday. Since $E(X) = \lambda = 8 \cdot \frac{1}{5} = 1.6$, $P(\text{expenses} \leq \$100) = P(X \leq 2) = \sum_{k=0}^{2} \frac{e^{-1.6}(1.6)^k}{k!} = 0.783$.

4.2.17 Let X = number of transmission errors made in next half-minute. Since $E(X) = \lambda = 4.5$, $P(X > 2) = 1 - P(X \leq 2) = 1 - \sum_{k=0}^{2} \frac{e^{-4.5}(4.5)^k}{k!} = 0.826$.

4.2.18 If $P(X = 0) = e^{-\lambda}\lambda^0/0! = e^{-\lambda} = \frac{1}{3}$, then $\lambda = 1.10$. Therefore, $P(X \geq 2) = 1 - P(X \leq 1) = 1 - e^{-1.10}(1.10)^0/0! - e^{-1.10}(1.10)^1/1! = 0.301$.

4.2.19 Let X = number of flaws in 40 sq. ft. Then $E(X) = 4$ and $P(X \geq 3) = 1 - P(X \leq 2) =$
$1 - \sum_{k=0}^{2} \frac{e^{-4}4^k}{k!} = 0.762$.

4.2.20 Let X = number of particles counted in next two minutes. Since the rate at which the particles are counted <u>per minute</u> is $4.017\left(=\frac{482}{120}\right)$, $E(X) = 8.034$ and $P(X = 3) = \frac{e^{-8.034}(8.034)^3}{3!} = 0.028$.
Now, suppose X = number of particles counted <u>in one minute</u>. Then P(3 particles are counted in next two minutes) = $P(X = 3) \cdot P(X = 0) + P(X = 2) \cdot P(X = 1) + P(X = 1) \cdot P(X = 2) + P(X = 0) \cdot P(X = 3) = 0.028$, where $\lambda = 4.017$.

4.2.21 **(a)** Let X = number of accidents in next five days. Then $E(X) = 0.5$ and $P(X = 2) = e^{-0.5}(0.5)^2/2! = 0.076$.

(b) No. P(4 accidents occur during next two weeks) =
$P(X = 4) \cdot P(X = 0) + P(X = 3) \cdot P(X = 1) + P(X = 2) \cdot P(X = 2) + P(X = 1) \cdot P(X = 3) + P(X = 0) \cdot P(X = 4)$.

4.2.22 If $P(X = 1) = P(X = 2)$, then $e^{-\lambda}\lambda^1/1! = e^{-\lambda}\lambda^2/2!$, which implies that $2\lambda = \lambda^2$, or, equivalently, $\lambda = 2$. Therefore, $P(X = 4) = e^{-2}2^4/4! = 0.09$.

4.2.23 $P(X \text{ is even}) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^{2k}}{(2k)!} = e^{-\lambda}\left\{1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \frac{\lambda^6}{6!} + \cdots\right\} = e^{-\lambda} \cdot \cosh \lambda =$
$e^{-\lambda}\left(\frac{e^{\lambda} + e^{-\lambda}}{2}\right) = \frac{1}{2}(1 + e^{-2\lambda})$.

4.2.24 $f_{X+Y}(w) = \sum_{k=0}^{\infty} p_k(k)p_Y(w-k) = \sum_{k=0}^{w} e^{-\lambda}\frac{\lambda^k}{k!}e^{-\mu}\frac{\mu^{w-k}}{(w-k)!}$
$= e^{-(\lambda+\mu)}\sum_{k=0}^{w}\frac{1}{k!(w-k)!}\lambda^k\mu^{w-k} = e^{-(\lambda+\mu)}\frac{1}{w!}\sum_{k=0}^{w}\frac{w!}{k!(w-k)!}\lambda^k\mu^{w-k} = e^{-(\lambda+\mu)}\frac{1}{w!}(\lambda+\mu)^w$
The last expression is the w^{th} term of the Poisson pdf with parameter $\lambda + \mu$.

4.2.25 From Definition 3.11.1 and Theorem 3.7.1, $P(X_2 = k) = \sum_{x_1=k}^{\infty}\binom{x_1}{k}p^k(1-p)^{x_1-k} \cdot \frac{e^{-\lambda}\lambda^{x_1}}{x_1!}$.
Let $y = x_1 - k$. Then $P(X_2 = k) = \sum_{y=0}^{\infty}\binom{y+k}{k}p^k(1-p)^y \cdot \frac{e^{-\lambda}\lambda^{y+k}}{(y+k)!} = \frac{e^{-\lambda}(\lambda p)^k}{k!} \cdot \sum_{y=0}^{\infty}\frac{[\lambda(1-p)]^y}{y!} =$
$\frac{e^{-\lambda}(\lambda p)^k}{k!} \cdot e^{\lambda(1-p)} = \frac{e^{-\lambda p}(\lambda p)^k}{k!}$.

4.2.26 **(a)** Yes, because the Poisson assumptions are probably satisfied—crashes are independent events and the crash rate is likely to remain constant.

(b) Since $\lambda = 2.5$ crashes per year, $P(X \geq 4) = 1 - P(X \leq 3) = 1 - \sum_{k=0}^{3}\frac{e^{-2.5}(2.5)^k}{k!} = 0.24$.

(c) Let Y = interval (in yrs.) between next two crashes. By Theorem 4.2.3,
$P(Y < 0.25) = \int_0^{0.25} 2.5e^{-2.5y}\,dy = 1 - 0.535 = 0.465.$

4.2.27 Let X = number of deaths in a week. Based on the daily death rate, $E(X) = \lambda = 0.7$. Let Y = interval (in weeks) between consecutive deaths. Then $P(Y > 1) = \int_1^{\infty} 0.7e^{-0.7y}\,dy = -e^{-u}\Big|_{0.7}^{\infty}$
$= 0.50.$

4.2.28 Given that $f_Y(y) = 0.027e^{-0.027}$, $P(Y_1 + Y_2 < 40) = \int_0^{40} (0.027)^2\, ye^{-0.027y}\,dy = \int_0^{1.08} ue^{-u}\,du =$
$e^{-u}(-u-1)\Big|_0^{1.08} = 1 - 0.706 = 0.29$ (where $u = 0.027y$).

4.2.29 Let X = number of bulbs burning out in 1 hour. Then $E(X) = \lambda = 0.11$. Let Y = number of hours a bulb remains lit. Then $P(Y < 75) = \int_0^{75} 0.011e^{-0.011y}\,dy = -e^{-u}\Big|_0^{0.825} = 0.56$. (where $u = 0.011y$).
Since $n = 50$ bulbs are initially online, the expected number that will fail to last at least 75 hours is $50 \cdot P(Y < 75)$, or 28.

4.2.30 Assume that 29 long separations and 7 short separations are to be randomly arranged. In order for "bad things to come in fours" three of the short separations would have to occur at least once in the 30 spaces between and around the 29 long separations. For that to happen, either (1) 3 short separations have to occur in one space and the remaining 4 shorts in 4 other spaces (2) 3 short separations occur in one space, 2 short separations occur in another space, and 1 short separation occurs in each of two spaces or (3) 3 short separations occur in each of two spaces and the remaining short occurs in a third space. The combined probability of these three possibilities is

$$\frac{\binom{30}{5}\binom{5}{1} + \binom{30}{4}\dfrac{4!}{2!1!1!} + \binom{30}{3}\binom{3}{1}}{\binom{36}{29}} = 0.126$$

Section 4.3: The Normal Distribution

4.3.1 **(a)** 0.5782 **(b)** 0.8264 **(c)** 0.9306 **(d)** 0.0000

4.3.2 **(a)** $0.9808 - 0.5000 = 0.4808$
(b) $0.4562 - 0.2611 = 0.1951$
(c) $1 - 0.1446 = 0.8554 = P(Y < 1.06)$
(d) 0.0099
(e) $P(Z \geq 4.61) < P(Z \geq 3.9) = 1 - 1.0000 = 0.0000$

4.3.3 **(a)** Both are the same because of the symmetry of $f_Z(z)$.

(b) Since $f_Z(z)$ is decreasing for all $z > 0$, $\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$ is larger than $\int_a^{a+1} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$.

4.3.4 **(a)** $\int_0^{1.24} e^{-z^2/2} dz = \sqrt{2\pi} \int_0^{1.24} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \sqrt{2\pi}(0.8925 - 0.5000) = 1.234$

(b) $\int_{-\infty}^{\infty} 6e^{-z^2/2} dz = 6\sqrt{2\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = 6\sqrt{2\pi}$

4.3.5 **(a)** –0.44 **(b)** 0.76 **(c)** 0.41 **(d)** 1.28 **(e)** 0.95

4.3.6 From Appendix Table A.1, $z_{.25} = 0.67$ and $z_{.75} = -0.67$, so $Q = 0.67 - (-0.67) = 1.34$.

4.3.7 Let X = number of decals purchased in November. Then X is binomial with n = 74,806 and $p = 1/12$.
$P(50X < 306{,}000) = P(X < 6120) = P(X \le 6119)$. Using the DeMoivre-Laplace approximation with continuity correction gives

$$P(X \le 6119) \doteq P\left(Z \le \frac{6119.5 - 74{,}806(1/12)}{\sqrt{74{,}806(1/12)(11/12)}}\right) = P(Z \le -1.51) = 0.0655$$

4.3.8 Let X = number of usable cabinets in next shipment. Since $np = 1600(0.80) = 1280$ and $\sqrt{np(1-p)} = \sqrt{1600(0.80)(0.20)} = 16$, P(shipment causes no problems) = $P(1260 \le X \le 1310)$ =

$$P\left(\frac{1259.5-1280}{16} \le \frac{X-1280}{16} \le \frac{1310.5-1280}{16}\right) \doteq P(-1.28 \le Z \le 1.91) = 0.8716.$$

4.3.9 Let X = number of voters challenger receives. Given that $n = 400$ and $p = P$(voter favors challenger) = 0.45, $np = 180$ and $np(1 - p) = 99$.

(a) P(tie) = $P(X = 200) = P(199.5 \le X \le 200.5)$ =

$$P\left(\frac{199.5-180}{\sqrt{99}} \le \frac{X-180}{\sqrt{99}} \le \frac{200.5-180}{\sqrt{99}}\right) \doteq P(1.96 \le Z \le 2.06) = 0.0053.$$

(b) P(challenger wins) = $P(X > 200) = P(X \ge 200.5)$ =

$$P\left(\frac{X-180}{\sqrt{99}} \ge \frac{200.5-180}{\sqrt{99}}\right) \doteq P(Z \ge 2.06) = 0.0197.$$

4.3.10 **(a)** Let X = number of shots made in next 100 attempts.
Since $p = P$(attempt is successful) = 0.70, $P(75 \le X \le 80)$ =

$$\sum_{k=75}^{80} \binom{100}{k} (0.70)^k (0.30)^{100-k}.$$

(b) With $np = 100(0.70) = 70$ and $np(1-p) = 100(0.70)(0.30) = 21$, $P(75 \le X \le 80) = P(74.5 \le X \le 80.5)$ = $P\left(\frac{74.5-70}{\sqrt{21}} \le \frac{X-70}{\sqrt{21}} \le \frac{80.5-70}{\sqrt{21}}\right) = P(0.98 \le Z \le 2.29) = 0.1525.$

4.3.11 Let $p = P$(person dies by chance in the three months following birthmonth) = $\frac{1}{4}$. Given that n = 747, np = 186.75, and $np(1-p)$ = 140.06, $P(X \geq 344) = P(X \geq 343.5) =$
$P\left(\frac{X-186.75}{\sqrt{140.06}} \geq \frac{343.5-186.75}{\sqrt{140.06}}\right) = P(Z \geq 13.25) = 0.0000$. The fact that the latter probability is so small strongly discredits the hypothesis that people die randomly with respect to their birthdays.

4.3.12 Let X = number of correct guesses (out of n = 1500 attempts). Since five choices were available for each guess (recall Figure 4.3.4), $p = P$(correct answer) = 1/5, if ESP is not a factor. Then $P(X \geq 326) = P(X \geq 325.5) = P\left(\frac{X-1500(1/5)}{\sqrt{1500(1/5)(4/5)}} \geq \frac{325.5-300}{\sqrt{240}}\right) \doteq P(Z \geq 1.65) =$ 0.0495. Based on these results, there is certainly <u>some</u> evidence that ESP may be increasing the probability of a correct guess, but the magnitude of $P(X \geq 326)$ is not so small that it precludes the possibility that chance is the only operative factor.

4.3.13 No, the normal approximation is inappropriate because the values of n (= 10) and p (= 0.7) fail to satisfy the condition $n > 9\frac{p}{1-p} = 9\frac{0.7}{0.3} = 21$.

4.3.14 Let X = number of fans buying hot dogs. To be determined is the smallest value of c for which $P(X > c) \leq 0.20$. Assume that no one eats more than one hot dog. Then X is a binomial random variable with n = 42,200 and $p = P$(fan buys hot dog) = 0.38. Since np = 16,036 and $\sqrt{np(1-p)}$ = 99.7, $P(X > c) = 0.20 = P(X \geq c+1) \doteq P\left(Z \geq \frac{c+1-\frac{1}{2}-16{,}036}{99.7}\right)$.

But $P(Z \geq 0.8416) = 0.20$, so $0.8416 = \left(\frac{c+1-\frac{1}{2}-16{,}036}{99.7}\right)$, from which it follows that c = 16,119.

4.3.15 $P(|X-E(X)| \leq 5) = P(-5 \leq X-100 \leq 5) = P\left(\frac{-5.5}{\sqrt{50}} \leq \frac{X-100}{\sqrt{50}} \leq \frac{5.5}{\sqrt{50}}\right) \doteq$
$P(-0.78 \leq Z \leq 0.78) = 0.5646$.

For binomial data, the central limit theorem and DeMoivre-Laplace approximations differ only if the continuity correction is used in the DeMoivre-Laplace approximation.

4.3.16 Let X_i = face showing on ith die, $i = 1, 2, \ldots, 100$, and let $X = X_1 + X_2 + \ldots + X_{100}$. Following the approach taken in Example 3.9.5 gives $E(X) = 350$. Also, $\text{Var}(X_i) = E\left(X_i^2\right) - [E(X_i)]^2 =$ $\frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2)-\left(3\frac{1}{2}\right)^2 = \frac{35}{12}$, so $\text{Var}(X) = \frac{3500}{12}$. By the central limit theorem, then, $P(X > 370) = P(X \geq 371) = P(X \geq 370.5) = P\left(\frac{X-350}{\sqrt{3500/12}} \geq \frac{370.5-350}{\sqrt{3500/12}}\right) \doteq P(Z \geq 1.20)$ $= 0.1151$.

4.3.17 For the given X, $E(X) = 5(18/38) + (-5)(20/38) = -10/38 = -0.263$.
$\text{Var}(X) = 25(18/38) + (25)(20/38) - (-10/38)^2 = 24.931$, $\sigma = 4.993$.

Then $P(X_1 + X_2 + \ldots + X_{100} > -50)$
$= P\left(\frac{X_1+X_2+\ldots+X_{100}-100(-0.263)}{\sqrt{100}(4.993)} > \frac{-50-100(-0.263)}{10(4.993)}\right) \doteq P(Z > -0.47)$
$= 1 - 0.3192 = 0.6808$

4.3.18 If X_i is a Poisson random variable with parameter λ_i, then $E(X_i) = \text{Var}(X_i) = \lambda_i$. Let $X = X_1 + X_2 + \ldots + X_n$ be a sum of independent Poissons. Then $E(X) = \sum_{i=1}^{n}\lambda_i = \text{Var}(X)$. If $\lambda = \sum_{i=1}^{n}\lambda_i$, the ratio in Theorem 4.3.2 reduces to $\frac{X-\lambda}{\sqrt{\lambda}}$.

4.3.19 Let X = number of chips ordered next week. Given that $\lambda = E(X) = 50$, P(company is unable to fill orders) $= P(X \geq 61) = P(X \geq 60.5) = P\left(\frac{X-50}{\sqrt{50}} \geq \frac{60.5-50}{\sqrt{50}}\right)$ $P(Z \geq 1.48) = 0.0694$.

4.3.20 Let X = number of leukemia cases diagnosed among 3000 observers. If $\lambda = E(X) = 3$, $P(X \geq 8) = 1 - P(X \leq 7) = 1 - \sum_{k=0}^{7}\frac{e^{-3}3^k}{k!} = 1 - 0.9881 = 0.0119$. Using the central limit theorem, $1 - P(X \leq 7) = 1 - P(X \leq 7.5) = 1 - P\left(\frac{X-3}{\sqrt{3}} \leq \frac{7.5-3}{\sqrt{3}}\right) \doteq 1 - P(Z \leq 2.60) = 0.0047$. The approximation is not particularly good because λ is small. In general, if λ is less than 5, the normal approximation should not be used. Both analyses, though, suggest that the observer's risk of contracting leukemia was increased because of their exposure to the test.

4.3.21 No, only 84% of drivers are likely to get at least 25,000 miles on the tires. If X denotes the mileage obtained on a set of Econo-Tires, $P(X \geq 25{,}000) =$
$P\left(\frac{X-30{,}000}{5000} \geq \frac{25{,}000-30{,}000}{5000}\right) = P(Z \geq -1.00) = 0.8413$.

4.3.22 Let Y denote a child's IQ. Then P(child needs special services) $= P(Y < 80) + P(Y > 135) =$
$P\left(\frac{Y-100}{16} < \frac{80-100}{16}\right) + P\left(\frac{Y-100}{16} > \frac{135-100}{16}\right) = P(Z < -1.25) + P(Z > 2.19) =$
$0.1056 + 0.0143 = 0.1199$. It follows that $1400 \times 0.1199 \times \$1750 = \$293{,}755$ should be added to Westbank's special ed budget.

4.3.23 Let Y = donations collected tomorrow. Given that $\mu = \$20{,}000$ and $\sigma = \$5{,}000$,
$P(Y > \$30{,}000) = P\left(\frac{Y-\$20{,}000}{\$5{,}000} > \frac{\$30{,}000-\$20{,}000}{\$5{,}000}\right) = P(Z > 2.00) = 0.0228.$

4.3.24 Let Y = pregnancy duration (in days). Ten months and five days is equivalent to 310 days. The credibility of San Diego Reader's claim hinges on the magnitude of $P(Y \geq 310)$—the smaller that probability is, the less believable her explanation becomes. Given that $\mu = 266$ and $\sigma = 16$, $P(Y \geq 310) = P\left(\frac{Y-266}{16} \geq \frac{310-266}{16}\right) = P(Z \geq 2.75) = 0.0030$. While the latter does not rule out the possibility that San Diego Reader is telling the truth, pregnancies lasting 310 or more days are extremely unlikely.

4.3.25 **(a)** Let Y_1 and Y_2 denote the scores made by a random nondelinquent and delinquent, respectively. Then $E(Y_1) = 60$ and $\text{Var}(Y_1) = 10^2$; also, $E(Y_2) = 80$ and $\text{Var}(Y_2) = 5^2$. Since 75 is the cutoff between teenagers classified as delinquents or nondelinquents,
P(nondelinquent is misclassified as delinquent) $= P(Y_1 > 75) = P\left(Z > \frac{75-60}{10}\right) = 0.0668.$
Similarly, P(delinquent is misclassified as nondelinquent) $= P(Y_2 < 75) = P\left(Z < \frac{75-80}{5}\right) =$
0.1587.

4.3.26 Let Y denote the cross-sectional area of a tube. Then $p = P$(tube does not fit properly) =
$P(Y < 12.0) + P(Y > 13.0) = 1 - P(12.0 \leq Y \leq 13.0) = 1 - P\left(\frac{12.0-12.5}{0.2} \leq \frac{Y-12.5}{0.2} \leq \frac{13.0-12.5}{0.2}\right) =$
$1 - P(-2.50 \leq Z \leq 2.50) = 1 - 0.9876 = 0.0124$. Let X denote the number of tubes (out of 1000) that will not fit. Since X is a binomial random variable, $E(X) = np = 1000(0.0124) = 12.4$.

4.3.27 Let Y = freshman's verbal SAT score. Given that $\mu = 565$ and $\sigma = 75$, $P(Y > 660) =$
$P\left(\frac{Y-565}{75} > \frac{660-565}{75}\right) = P(Z > 1.27) = 0.1020$. It follows that the expected <u>number</u> doing better is 4250(0.1020), or 434.

4.3.28 Let A^* and B^* denote the lowest A and the lowest B, respectively. Since the top 20% of the grades will be A's, $P(Y < A^*) = 0.80$, where Y denotes a random student's score. Equivalently,
$P\left(Z < \frac{A^*-70}{12}\right) = 0.80$. From Appendix Table A.1, though, $P(Z < 0.84) = 0.7995 \doteq 0.80$.
Therefore, $0.84 = \frac{A^*-70}{12}$, which implies that $A^* = 80$.

Similarly, $P(Y < B^*) = 0.54 = P\left(Z < \dfrac{B^* - 70}{12}\right)$. But $P(Z < 0.10) = 0.5398 \doteq 0.54$, so $0.10 = \dfrac{B^* - 70}{12}$, implying that $B^* = 71$.

4.3.29 If $P(20 \le Y \le 60) = 0.50$, then $P\left(\dfrac{20-40}{\sigma} \le \dfrac{Y-40}{\sigma} \le \dfrac{60-40}{\sigma}\right) = 0.50 = P\left(\dfrac{-20}{\sigma} \le Z \le \dfrac{20}{\sigma}\right)$. But $P(-0.67 \le Z \le 0.67) = 0.4972 \doteq 0.50$, which implies that $0.67 = \dfrac{20}{\sigma}$. The desired value for σ, then, is $\dfrac{20}{0.67}$, or 29.85.

4.3.30 Let Y = a random 18-year-old woman's weight. Since $\mu = \dfrac{103.5 + 144.5}{2} = 124$, $P(103.5 \le Y \le 144.5) = 0.80 = P\left(\dfrac{103.5-124}{\sigma} \le \dfrac{Y-124}{\sigma} \le \dfrac{144.5-124}{\sigma}\right) = P\left(\dfrac{-20.5}{\sigma} \le Z \le \dfrac{20.5}{\sigma}\right)$. According to Appendix Table A.1, $P(-1.28 \le Z \le 1.28) \doteq 0.80$, so $\dfrac{20.5}{\sigma} = 1.28$, implying that $\sigma = 16.0$ lbs.

4.3.31 Let Y = analyzer reading for driver whose true blood alcohol concentration is 0.9. Then $P(\text{analyzer mistakenly shows driver to be sober}) = P(Y < 0.08) = P\left(\dfrac{Y-0.9}{0.004} < \dfrac{0.08-0.09}{0.004}\right) = P(Z < -2.50) = 0.0062$. The "0.075%" driver should ask to take the test twice. The "0.09%" driver has a greater chance of not being charged by taking the test only once. As, n the number of times the test taken, increases, the precision of the average reading increases. It is to the sober driver's advantage to have a reading as precise as possible; the opposite is true for the drunk driver.

4.3.32 The normed score for Michael is $\dfrac{75-62.0}{7.6} = 1.71$; the normed score for Laura is $\dfrac{92-76.3}{10.8} = 1.45$. So, even though Laura made 17 points higher on the test, the company would be committed to hiring Michael.

4.3.33 By the first corollary to Theorem 4.3.3, $P(\bar{Y} > 103) = P\left(\dfrac{\bar{Y}-100}{16/\sqrt{9}} > \dfrac{103-100}{16/\sqrt{9}}\right) = \text{P}(Z > 0.56) = 0.2877$. For any arbitrary Y_i, $P(Y_i > 103) = P\left(\dfrac{Y_i - 100}{16} > \dfrac{103-100}{16}\right) = P(Z > 0.19) = 0.4247$. Let X = number of Y_i's that exceed 103. Since X is a binomial random variable with $n = 9$ and $p = P(Y_i > 103) = 0.4247$, $P(X = 3) = \dbinom{9}{3}(0.4247)^3(0.5753)^6 = 0.23$.

4.3.34 If $P(1.9 \le \bar{Y} \le 2.1) \ge 0.99$, then $P\left(\frac{1.9-2}{2/\sqrt{n}} \le Z \le \frac{2.1-2}{2/\sqrt{n}}\right) \ge 0.99$. But $P(-2.58 \le Z \le 2.58)$ $\doteq 0.99$, so $2.58 = \frac{2.1-2}{2/\sqrt{n}}$, which implies that $n = 2663$.

4.3.35 Let Y_i = resistance of ith resistor, $i = 1, 2, 3$, and let $Y = Y_1 + Y_2 + Y_3$ = circuit resistance. By the first corollary to Theorem 4.3.3, $E(Y) = 6 + 6 + 6 = 18$ and $\text{Var}(Y) = (0.3)^2 + (0.3)^2 + (0.3)^2 = 0.27$. Therefore, $P(Y > 19) = P\left(\frac{Y-18}{\sqrt{0.27}} > \frac{19-18}{\sqrt{0.27}}\right) = P(Z > 1.92) = 0.0274$. Suppose $P(Y > 19)$ $= 0.005 = P\left(Z > \frac{19-18}{\sqrt{3\sigma^2}}\right)$. From Appendix Table A.1, $P(Z > 2.58) \doteq 0.005$, so $2.58 = \frac{19-18}{\sqrt{3\sigma^2}}$, which implies that the minimum "precision" of the manufacturing process would have to be σ= 0.22 ohms.

4.3.36 Let Y_P and Y_C denote a random piston diameter and cylinder diameter, respectively. Then P(pair needs to be reworked) = $P(Y_P > Y_C) = P(Y_P - Y_C > 0)$

$$= P\left(\frac{Y_P - Y_C - (40.5-41.5)}{\sqrt{(0.3)^2+(0.4)^2}} > \frac{0-(40.5-41.5)}{\sqrt{(0.3)^2+(0.4)^2}}\right) = P(Z > 2.00) = 0.0228, \text{ or } 2.28\%.$$

4.3.37 $M_{\bar{Y}}(t) = M_{Y_1+\ldots Y_n}\left(\frac{t}{n}\right) = \prod_{i=1}^{n} M_{Y_i}\left(\frac{t}{n}\right) = \prod_{i=1}^{n} e^{\mu t/n + \sigma^2 t^2/2n^2} = e^{\mu t + \sigma^2 t^2/2n}$, but the latter is the moment-generating function for a normal random variable whose mean is μ and whose variance is σ^2/n. Similarly, if $Y = a_1Y_1 + \ldots + a_nY_n$, $M_Y(t) = \prod_{i=1}^{n} M_{a_iY_i}(t) = \prod_{i=1}^{n} M_{Y_i}(a_it) =$ $\prod_{i=1}^{n} e^{\mu_i a_i t + \sigma_i^2 a_i^2 t^2/2} = e^{\sum_{i=1}^{n} a_i\mu_i t + \sum_{i=1}^{n} a_i^2\sigma_i^2 t^2/2}$. By inspection, Y has the moment-generating function of a normal random variable for which $E(Y) = \sum_{i=1}^{n} a_i\mu_i$ and $\text{Var}(Y) = \sum_{i=1}^{n} a_i^2\sigma_i^2$.

4.3.38 $P(\bar{Y} \ge \bar{Y}^*) = P(\bar{Y} - \bar{Y}^* \ge 0)$, where $E(\bar{Y} - \bar{Y}^*) = E(\bar{Y}) - E(\bar{Y}^*) = 2 - 1 = 1$.

Also, $\text{Var}(\bar{Y} - \bar{Y}^*) = \text{Var}(\bar{Y}) - \text{Var}(\bar{Y}^*) = \frac{2^2}{9} + \frac{1^2}{4} = \frac{25}{36}$, because $\bar{Y}$ and $\bar{Y}^*$ are independent.

Therefore, $P(\bar{Y} \ge \bar{Y}^*) = P\left(\frac{\bar{Y} - \bar{Y}^* - 1}{\sqrt{25/36}} \ge \frac{0-1}{\sqrt{25/36}}\right) = P(Z \ge -1.20) = 0.8849$

Section 4.4: The Geometric Distribution

4.4.1 Let $p = P$(return is audited in a given year) = 0.30 and let X = year of first audit. Then P(Jody escapes detection for at least 3 years) = $P(X \geq 4) = 1 - P(X \leq 3)$ =

$$1 - \sum_{k=1}^{3} (0.70)^{k-1}(0.30) = 0.343.$$

4.4.2 If X = attempt at which license is awarded and $p = P$(driver passes test on any given attempt) = 0.10, then $p_X(k) = (0.90)^{k-1}(0.10)$, $k = 1, 2, \ldots$; $E(X) = \dfrac{1}{p} = \dfrac{1}{0.10} = 10$.

4.4.3 No, the expected frequencies (= $5 \cdot p_X(k)$) differ considerably from the observed frequencies, especially for small values of k. The observed number of 1's, for example, is 4, while the expected number is 12.5.

k	Obs. Freq.	$p_X(k) = \left(\frac{3}{4}\right)^{k-1}\left(\frac{1}{4}\right)$	$50 \cdot p_X(k)$ = Exp. freq.
1	4	0.2500	12.5
2	13	0.1875	9.4
3	10	0.1406	7.0
4	7	0.1055	5.3
5	5	0.0791	4.0
6	4	0.0593	3.0
7	3	0.0445	2.2
8	3	0.0334	1.7
9+	1	0.1001	5.0
	50	1.0000	50.0

4.4.4 If $p = P$(child is a girl) and X = birth order of first girl, then $E(X) = \dfrac{1}{p} = \dfrac{1}{\frac{1}{2}} = 2$. Barring any medical restrictions, it would not be unreasonable to model the appearance of a couple's first girl (or boy) by the geometric probability function. The most appropriate value for p, though, would not be exactly $\frac{1}{2}$ (although census figures indicate that it would be close to $\frac{1}{2}$).

4.4.5 $F_X(t) = P(X \leq t) = p\sum_{s=0}^{[t]}(1-p)^s$. But $\sum_{s=0}^{[t]}(1-p)^s = \dfrac{1-(1-p)^{[t]}}{1-(1-p)} = \dfrac{1-(1-p)^{[t]}}{p}$, and the result follows.

4.4.6 Let X = roll on which sum of 4 appears for first time. Since $p = P(\text{sum} = 4) = \frac{3}{216}$. $p_X(k) = \left(\frac{213}{216}\right)^{k-1} \cdot \frac{3}{216}$, $k = 1, 2, \ldots$ Using the expression for $F_X(k)$ given in Question 4.4.5, we can write $P(65 \le X \le 75) = F_X(75) - F_X(64) = 1 - \left(1-\frac{3}{216}\right)^{[75]} - \left(1-\left(1-\frac{3}{216}\right)^{[64]}\right) = \left(\frac{213}{216}\right)^{64} - \left(\frac{213}{216}\right)^{75} = 0.058$.

4.4.7 $P(n \le Y \le n+1) = \int_n^{n+1} \lambda e^{-\lambda y}\,dy = (1-e^{-\lambda y})\Big|_n^{n+1} = e^{-\lambda n} - e^{-\lambda(n+1)} = e^{-\lambda n}(1-e^{-\lambda})$
Setting $p = 1 - e^{-\lambda}$ gives $P(n \le Y \le n+1) = (1-p)^n p$.

4.4.8 Let the random variable X^* denote the number of trials preceding the first success. By inspection, $p_{X^*}(t) = (1-p)^k p$, $k = 0, 1, 2, \ldots$ Also, $M_{X^*}(t) = \sum_{k=0}^{\infty} e^{tk} \cdot (1-p)^k p = p\sum_{k=0}^{\infty}[(1-p)e^t]^k = p \cdot \left(\frac{1}{1-(1-p)e^t}\right) = \frac{p}{1-(1-p)e^t}$. Let X denote the geometric random variable defined in Theorem 4.4.1. Then $X^* = X - 1$, and $M_{X^*}(t) = e^{-t}M_X(t) = e^{-t} \cdot \frac{pe^t}{1-(1-p)e^t} = \frac{p}{1-(1-p)e^t}$.

4.4.9 $M_X(t) = pe^t[1-(1-p)e^t]^{-1}$, so $M_X^{(1)}(t) = pe^t(-1)[1-(1-p)e^t]^{-2} \cdot (-(1-p)e^t) + [1-(1-p)e^t]^{-1}pe^t$. Setting $t = 0$ gives $M_X^{(1)}(0) = E(X) = \frac{1}{p}$. Similarly, $M_X^{(2)}(t) = p(1-p)e^{2t}(-2)[1-(1-p)e^t]^{-3} \cdot (-(1-p)e^t) + [1-(1-p)e^t]^{-2}p(1-p)e^{2t} \cdot 2 + [1-(1-p)e^t]^{-1}pe^t + pe^t(-1)[1-(1-p)e^t]^{-2} \cdot (-(1-p)e^t)$ and $M_X^{(2)}(0) = E(X^2) = \frac{2-p}{p^2}$.
Therefore, $\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}$

4.4.10 No, because $M_X(t) = M_{X_1}(t) \cdot M_{X_2}(t)$ does not have the form of a geometric moment-generating function.

4.4.11 Let $M_X^*(t) = E(t^X) = \sum_{k=1}^{\infty} t^k \cdot (1-p)^{k-1} p = \frac{p}{1-p}\sum_{k=1}^{\infty}[t(1-p)]^k = \frac{p}{1-p}\sum_{k=0}^{\infty}[t(1-p)]^k - \frac{p}{1-p} = \frac{p}{1-p}\left[\frac{1}{1-t(1-p)}\right] - \frac{p}{1-p} = \frac{pt}{1-t(1-p)}$ = factorial moment-generating function for X.
Then $M_X^{*(1)}(t) = pt(-1)[1-t(1-p)]^{-2}(-(1-p)) + [1-t(1-p)]^{-1}p = \frac{p}{[1-t(1-p)]^2}$.

When $t = 1$, $M_X^{*(1)}(1) = E(X) = \dfrac{1}{p}$. Also, $M_X^{*(2)}(t) = \dfrac{2p(1-p)}{[1-t(1-p)]^3}$ and $M_X^{*(2)}(1) = \dfrac{2-2p}{p^2} =$

$E[X(X-1)] = E(X^2) - E(X)$. Therefore, $\text{Var}(X) = E(X^2) - [E(X)]^2 = \dfrac{2-2p}{p^2} + \dfrac{1}{p} - \left(\dfrac{1}{p}\right)^2$

$= \dfrac{1-p}{p^2}$.

Section 4.5: The Negative Binomial Distribution

4.5.1 Let X = number of houses needed to achieve fifth invitation. If $p = P$(saleswoman receives invitation at a given house) = 0.30, $p_X(k) = \binom{k-1}{4}(0.30)^4(0.70)^{k-1-4}(0.30)$, $k = 5, 6, \ldots$ and

$P(X < 8) = P(5 \le X \le 7) = \sum_{k=5}^{7}\binom{k-1}{4}(0.30)^5(0.70)^{k-5} = 0.029$.

4.5.2 Let $p = P$(missile scores direct hit) = 0.30. Then P(target will be destroyed by seventh missile fired) = P(exactly three direct hits occur among first six missiles <u>and</u> seventh missile scores direct hit) = $\binom{6}{3}(0.30)^3(0.70)^3(0.30) = 0.056$.

4.5.3 Darryl might have actually done his homework, but there is reason to suspect that he did not. Let the random variable X denote the toss where a head appears for the second time. Then $p_X(k) = \binom{k-1}{1}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{k-2}$, $k = 2, 3, \ldots$, but that particular model fits the data almost perfectly, as the table shows. Agreement this good is often an indication that the data have been fabricated.

k	$p_X(k)$	Obs. freq.	Exp. freq.
2	1/4	24	25
3	2/8	26	25
4	3/16	19	19
5	4/32	13	12
6	5/64	8	8
7	6/128	5	5
8	7/256	3	3
9	8/512	1	2
10	9/1024	1	1

4.5.4 Let $p = P$(defective is produced by improperly adjusted machine) = 0.15. Let X = item at which machine is readjusted. Then $p_X(k) = \binom{k-1}{2}(0.15)^2(0.85)^{k-1-2}(0.15) = \binom{k-1}{2}(0.15)^3(0.85)^{k-3}$, $k = 3, 4, \ldots$ It follows that $P(X \ge 5) = 1 - P(X \le 4)$

$= 1 - [P(X = 3) + P(X = 4)] = 0.988$ and $E(X) = \dfrac{3}{0.15} = 20$.

4.5.5 $E(X) = \sum_{k=r}^{\infty} k\binom{k-1}{r-1}p^r(1-p)^{k-r} = \frac{r}{p}\sum_{k=r}^{\infty}\binom{k}{r}p^{r+1}(1-p)^{k-r} = \frac{r}{p}$.

4.5.6 Let Y denote the number of trials to get the rth success, and let X denote the number of trials in excess of r to get the rth success. Then $X = Y - r$. Substituting into Theorem 4.5.1 gives $p_X(k) = \binom{k+r-1}{r-1}p^r(1-p)^k = \binom{k+r-1}{k}p^r(1-p)^k$, $k = 0, 1, 2, \ldots$

4.5.7 Here $X = Y - r$, where Y has the negative binomial distribution as described in Theorem 4.5.1. Using the properties (1), (2), and (3) given by the theorem, we can write $E(X) = E(Y - r)$ $= E(Y) - E(r) = \frac{r}{p} - r = \frac{r(1-p)}{p}$ and $\text{Var}(X) = \text{Var}(Y - r) = \text{Var}(Y) + \text{Var}(r)$ $= \frac{r(1-p)}{p^2} + 0 = \frac{r(1-p)}{p^2}$. Also, $M_X(t) = M_{Y-r}(t) = e^{-rt}M_Y(t) = e^{-rt}\left[\frac{pe^t}{1-(1-p)e^t}\right]^r = \left[\frac{p}{1-(1-p)e^t}\right]^r$.

4.5.8 For each X_i, $M_{X_i}(t) = \left[\frac{(4/5)e^t}{1-(1-4/5)e^t}\right]^3$, $i = 1, 2, 3$. If $X = X_1 + X_2 + X_3$, it follows that $M_X(t) = \prod_{i=1}^{3} M_{X_i}(t) = \left[\frac{(4/5)e^t}{1-(1-4/5)e^t}\right]^9$, which implies that $p_X(k) = \binom{k-1}{8}\left(\frac{4}{5}\right)^9\left(\frac{1}{5}\right)^{k-9}$, $k = 9, 10, \ldots$ Then $P(10 \le X \le 12) = \sum_{k=10}^{12} p_X(k) = 0.66$.

4.5.9 $M_X^{(1)}(t) = r\left[\frac{pe^t}{1-(1-p)e^t}\right]^{r-1}$ $[pe^t[1-(1-p)e^t]^{-2}(1-p)e^t + [1-(1-p)e^t]^{-1}pe^t]$. When $t = 0$,

$M_X^{(1)}(0) = E(X) = r\left[\frac{p(1-p)}{p^2} + \frac{p}{p}\right] = \frac{r}{p}$.

4.5.10 $M_X(t) = \prod_{i=1}^{k} M_{X_i}(t) = \prod_{i=1}^{k}\left[\frac{pe^t}{1-(1-p)e^t}\right]^{r_i} = \left[\frac{pe^t}{1-(1-p)e^t}\right]^{r^*}$, where $r^* = \sum_{i=1}^{k} r_i$. Also,

$E(X) = E(X_1 + X_2 + \ldots + X_k) = E(X_1) + E(X_2) + \ldots + E(X_k) = \sum_{i=1}^{k}\frac{r_i}{p} = \frac{r^*}{p}$ and

$\text{Var}(X) = \sum_{i=1}^{k}\text{Var}(X_i) = \sum_{i=1}^{k}\frac{r_i(1-p)}{p^2} = \frac{r^*(1-p)}{p^2}$.

Section 4.6: The Gamma Distribution

4.6.1 Let Y_i = lifetime of ith gauge, i = 1, 2, 3. By assumption, $f_{Y_i}(y) = 0.001e^{-0.001y}$, $y > 0$. Define the random variable $Y = Y_1 + Y_2 + Y_3$ to be the lifetime of the system. By Theorem 4.6.1, $f_Y(y) = \dfrac{(0.001)^3}{2}y^2e^{-0.001y}$, $y > 0$.

4.6.2 The time until the 24th breakdown is a gamma random variable with parameters $r = 24$ and $\lambda = 3$. The mean of this random variable is $r/\lambda = 24/3 = 8$ months.

4.6.3 If $E(Y) = \dfrac{r}{\lambda} = 1.5$ and $\text{Var}(Y) = \dfrac{r}{\lambda^2} = 0.75$, then $r = 3$ and $\lambda = 2$, which makes $f_Y(y) = 4y^2e^{-2y}$, $y > 0$. Then $P(1.0 \le Y_\text{i} \le 2.5) = \int_{1.0}^{2.5} 4y^2e^{-2y}dy = 0.55$. Let X = number of Y_i's in the interval (1.0, 2.5). Since X is a binomial random variable with $n = 100$ and $p = 0.55$, $E(X) = np = 55$.

4.6.4 $f_{\lambda Y}(y) = \dfrac{1}{\lambda}f_Y(y/\lambda) = \dfrac{1}{\lambda}\dfrac{\lambda^r}{\Gamma(r)}\left(\dfrac{y}{\lambda}\right)^{r-1}e^{-\lambda(y/\lambda)} = \dfrac{1}{\Gamma(r)}y^{r-1}e^{-y}$

4.6.5 To find the maximum of the function $f_Y(y) = \dfrac{\lambda^r}{\Gamma(r)}y^{r-1}e^{-\lambda y}$, differentiate it with respect to y and set it equal to 0; that is

$$\frac{df_Y(y)}{dy} = \frac{d}{dy}\frac{\lambda^r}{\Gamma(r)}y^{r-1}e^{-\lambda y} = \frac{\lambda^r}{\Gamma(r)}[(r-1)y^{r-2}e^{-\lambda y} - \lambda y^{r-1}e^{-\lambda y}] = 0$$

This implies $\dfrac{\lambda^r}{\Gamma(r)}y^{r-2}e^{-\lambda y}[(r-1)-\lambda y] = 0$, whose solution is $y_\text{mode} = \dfrac{r-1}{\lambda}$. Since the derivative is positive for $y < y_\text{mode}$, and negative for $y > y_\text{mode}$, then there is a maximum.

4.6.6 Let Z be a standard normal random variable. Then $E(Z^2) = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2e^{-z^2/2}dz$ $= \sqrt{\dfrac{2}{\pi}}\int_0^{\infty} z^2e^{-z^2/2}dz = 1$. Let $y = z^2$. Then $E(Z^2) = \dfrac{2}{\sqrt{\pi}}\Gamma\left(1+\dfrac{1}{2}\right) = \dfrac{2}{\sqrt{\pi}}\left(\dfrac{1}{2}\right)\Gamma\left(\dfrac{1}{2}\right)$, which implies that $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$.

4.6.7 $\Gamma\left(\dfrac{7}{2}\right) = \dfrac{5}{2}\Gamma\left(\dfrac{5}{2}\right) = \dfrac{5}{2}\dfrac{3}{2}\Gamma\left(\dfrac{3}{2}\right) = \dfrac{5}{2}\dfrac{3}{2}\dfrac{1}{2}\Gamma\left(\dfrac{1}{2}\right) = \dfrac{15}{8}\Gamma\left(\dfrac{1}{2}\right)$ by Theorem 4.6.2, part 2.

Further, $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$ by Question 4.6.6.

4.6.8 $E(Y^m) = \int_0^\infty y^m \cdot \frac{\lambda^r}{(r-1)!} y^{r-1} e^{-\lambda y} dy = \int_0^\infty \frac{\lambda^r}{(r-1)!} y^{m+r-1} e^{-\lambda y} dy$

$= \frac{(m+r-1)!}{\lambda^m (r-1)!} \int_0^\infty \frac{\lambda^{m+r}}{(m+r-1)!} y^{m+r-1} e^{-\lambda y} dy = \frac{(m+r-1)!}{\lambda^m (r-1)!}$.

4.6.9 Write the gamma moment-generating function in the form $M_Y(t) = (1 - t/\lambda)^{-r}$. Then $M_Y^{(1)}(t) = -r(1 - t/\lambda)^{-r-1}(-1/\lambda) = (r/\lambda)(1 - t/\lambda)^{-r-1}$ and $M_Y^{(2)}(t) = (r/\lambda)(-r-1)(1 - t/\lambda)^{r-2} \cdot (-1/\lambda)$ $= (r/\lambda^2)(r+1)(1 - t/\lambda)^{-r-2}$. Therefore, $E(Y) = M_Y^{(1)}(0) = \frac{r}{\lambda}$ and $\text{Var}(Y) = M_Y^{(2)}(0) - \left[M_Y^{(1)}(0)\right]^2$ $= \frac{r(r+1)}{\lambda^2} - \frac{r^2}{\lambda^2} = \frac{r}{\lambda^2}$.

4.6.10 $M_Y(t) = (1 - t/\lambda)^{-r}$ so $M_Y^{(1)}(t) = \frac{d}{dt}(1 - t/\lambda)^{-r} = r(1 - t/\lambda)^{-r-1}(-1/\lambda) = \frac{r}{\lambda}(1 - t/\lambda)^{-r-1}$ and

$M_Y^{(2)}(t) = \frac{d}{dt}\frac{r}{\lambda}(1 - t/\lambda)^{-r-1} = \frac{r}{\lambda}(-r-1)(1 - t/\lambda)^{-r-2}(-1/\lambda) = \frac{r(r+1)}{\lambda^2}(1 - t/\lambda)^{-r-2}$

For an arbitrary integer $m \geq 2$, we can generalize the above to see that

$M_Y^{(m)}(t) = \frac{r(r+1)...(r+m-1)}{\lambda^m}(1 - t/\lambda)^{-r-m}$. Then $E(Y^m) = M_Y^{(m)}(0) = \frac{r(r+1)...(r+m-1)}{\lambda^m}$.

But note that $\frac{r(r+1)...(r+m-1)}{\lambda^m} = \frac{\Gamma(r+m)}{\Gamma(r)\lambda^m}$. The right hand side of the equation is equal to the expression in Question 4.6.8 when r is an integer.

Chapter 5: Estimation

Section 5.2: Estimating Parameters: The Method of Maximum Likelihood and Method of Moments

5.2.1 $L(\theta) = \prod_{i=1}^{8} \theta^{k_i}(1-\theta)^{1-k_i} = \theta^{\sum_{i=1}^{8} k_i}(1-\theta)^{8-\sum_{i=1}^{8} k_i} = \theta^5(1-\theta)^3$

$\frac{dL(\theta)}{d\theta} = \theta^5 3(1-\theta)^2(-1) + 5\theta^4(1-\theta)^3 = \theta^4(1-\theta)^2(-8\theta+5) \cdot \frac{dL(\theta)}{d\theta} = 0$ implies $\theta_e = 5/8$

5.2.2 $L(p) = p(1-p)(1-p)p(1-p) = p^2(1-p)^3$

$L(1/3) = \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = \frac{8}{243}$ is greater than $L(1/2) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$, so $p_e = 1/3$.

5.2.3 $L(\theta) = \prod_{i=1}^{4} \lambda e^{-\lambda y_i} = \lambda^4 e^{-\lambda\sum_{i=1}^{4} y_i} = \lambda^4 e^{-32.8\lambda}$.

$\frac{dL(\lambda)}{d\lambda} = \lambda^4(-32.8)e^{-32.8\lambda} + 4\lambda^3 e^{-32.8\lambda} = \lambda^3 e^{-32.8\lambda}(4 - 32.8\lambda)$

$\frac{dL(\lambda)}{d\lambda} = 0$ implies $\lambda_e = 4/32.8 = 0.122$

5.2.4 $L(\theta) = \prod_{i=1}^{n} \frac{\theta^{2k_i} e^{-\theta^2}}{k_i!} = \frac{\theta^{2\sum_{i=1}^{n} k_i} e^{-n\theta^2}}{\prod_{i=1}^{n} k_i!}$.

$\ln L(\theta) = \left(2\sum_{i=1}^{n} k_i\right)(\ln \theta) - n\theta^2 + \ln \prod_{i=1}^{n} k_i!$

$\frac{d \ln L(\theta)}{d\theta} = 0$ implies $\frac{2\sum_{i=1}^{n} k_i}{\theta} - 2n\theta = \frac{2\sum_{i=1}^{n} k_i - 2n\theta^2}{\theta} = 0$ or $\theta_e = \sqrt{\frac{\sum_{i=1}^{n} k_i}{n}}$

5.2.5 $L(\theta) = \prod_{i=1}^{3} \frac{y_i^3 e^{-y_i/\theta}}{6\theta^4} = \frac{\left(\prod_{i=1}^{3} y_i^3\right) e^{-\sum_{i=1}^{3} y_i/\theta}}{216\theta^{12}}$.

$\ln L(\theta) = \ln \prod_{i=1}^{3} y_i^3 - \frac{1}{\theta}\sum_{i=1}^{3} y_i - \ln 216 - 12 \ln \theta$

$\frac{d \ln L(\theta)}{d\theta} = \frac{1}{\theta^2}\sum_{i=1}^{3} y_i - \frac{12}{\theta} = \frac{\sum_{i=1}^{3} y_i - 12\theta}{\theta^2}$

$$\frac{d\ln L(\theta)}{d\theta} = 0 \text{ implies } \frac{\sum_{i=1}^{3} y_i - 12\theta}{\theta^2} = \frac{8.8-12\theta}{\theta^2} = 0 \text{ or } \theta_e = 0.733$$

5.2.6 $L(\theta) = \prod_{i=1}^{4} \frac{\theta}{2\sqrt{y_i}} e^{-\theta\sqrt{y_i}} = \frac{\theta^4}{16\prod_{i=1}^{4}\sqrt{y_i}} e^{-\theta\sum_{i=1}^{4}\sqrt{y_i}}$

$$\ln L(\theta) = 4\ln \theta - \ln\left(16\prod_{i=1}^{4}\sqrt{y_i}\right) - \theta\sum_{i=1}^{4}\sqrt{y_i}$$

$$\frac{d\ln L(\theta)}{d\theta} = \frac{4}{\theta} - \sum_{i=1}^{4}\sqrt{y_i}.$$

$$\frac{d\ln L(\theta)}{d\theta} = 0 \text{ implies } \theta_e = \frac{4}{\sum_{i=1}^{4}\sqrt{y_i}} = \frac{4}{8.766} = 0.456$$

5.2.7 $L(\theta) = \prod_{i=1}^{5} \theta y_i^{\theta-1} = \theta^5 \left(\prod_{i=1}^{5} y_i\right)^{\theta-1}.$

$$\ln L(\theta) = 5\ln\theta + (\theta-1)\sum_{i=1}^{5}\ln y_i$$

$$\frac{d\ln L(\theta)}{d\theta} = \frac{5}{\theta} + \sum_{i=1}^{5}\ln y_i = \frac{5+\theta\sum_{i=1}^{5}\ln y_i}{\theta}$$

$$\frac{d\ln L(\theta)}{d\theta} = 0 \text{ implies } \frac{5-0.625\theta}{\theta} = 0 \text{ or } \theta_e = 8.00$$

5.2.8 $L(p) = \prod_{i=1}^{n}(1-p)^{k_i-1}p = (1-p)^{\sum_{i=1}^{n}k_i - n}p^n$

$$\ln L(p) = \left(\sum_{i=1}^{n}k_i - n\right)\ln(1-p) + n\ln p$$

$$\frac{\ln L(p)}{dp} = -\frac{\sum_{i=1}^{n}k_i - n}{1-p} + \frac{n}{p} \text{ and } \frac{\ln L(p)}{dp} = 0 \text{ implies } p_e = \frac{n}{\sum_{i=1}^{n}k_i}$$

For the data, $n = 1011$, and $\sum_{i=1}^{n}k_i = 1(678) + 2(227) + 3(56) + 4(28) + 5(8) + 6(14) = 1536$, so $p_e = \frac{1011}{1536} = 0.658$. The table gives the comparison of observed and expected frequencies.

No. of Occupants	Observed Frequency	Expected Frequency
1	678	665.2
2	227	227.5
3	56	77.8
4	28	26.6
5	8	9.1
6+	14	4.8

5.2.9 **(a)** From the Comment following Example 5.2.1,

$$\lambda_e = \frac{1}{n}\sum_{i=1}^{n} k_i = \frac{1}{59}[1(19) + 2(12) + 3(13) + 4(9)] = 2.00 .$$

(b) For example, the expected frequency for the k = 3 class is $59 \cdot e^{-2}\frac{2^3}{3!} = 10.6$

The full set of expected values is given in column of the following table.

No. of No-hitters	Observed Frequency	Expected Frequency
0	6	8.0
1	19	16.0
2	12	16.0
3	13	10.6
4+	9	8.4

The last expected frequency has been chosen to make that column sum to n = 59.

The techniques to be introduced in Chapter 10 would support the Poisson model. The difference between observed and expected frequencies is in part because the game of baseball changed significantly over the years from 1950 to 2008. Thus, the parameter λ would not be constant over this period.

5.2.10 **(a)** $L(\theta) = \left(\frac{1}{\theta}\right)^n$, if $0 \le y_1, y_2, \ldots, y_n \le \theta$, and 0 otherwise. Thus $\theta_e = y_{\max}$, which for these data is 14.2.

(b) $L(\theta) = \left(\frac{1}{\theta_2 - \theta_1}\right)^n$, if $\theta_1 \le y_1, y_2, \ldots, y_n \le \theta_2$, and 0 otherwise. Thus $\theta_{1e} = y_{\min}$ and $\theta_{2e} = y_{\max}$. For these data, $\theta_{1e} = 1.8$, $\theta_{2e} = 14.2$.

5.2.11 $L(\theta) = \prod_{i=1}^{6}\frac{2y_i}{1-\theta^2} = \frac{64\prod_{i=1}^{n} y_i}{(1-\theta^2)^6}$, if $\theta \le y_1, y_2, \ldots, y_n \le 1$ and 0 otherwise. If $\theta > y_{\min}$, then $L(\theta) = 0$. So $\theta_e \le y_{\min}$. Also, to maximize $L(\theta)$, minimize the denominator, which in turn means maximize θ. Thus $\theta_e \ge y_{\min}$. We conclude that $\theta_e = y_{\min}$, which for these data is 0.92.

5.2.12 $L(\theta) = \prod_{i=1}^{n} \frac{2y_i}{\theta^2} = 2^n \left(\prod_{i=1}^{n} y_i\right)\theta^{-2n}$, if $0 \le y_1, y_2, \ldots, y_n \le \theta$, and 0 otherwise. To maximize $L(\theta)$ maximize θ. Since each $y_i \le \theta$ for $1 \le i \le n$, the maximum value for θ under these constraints is the maximum of the y_i, or $\theta_e = y_{\max}$.

5.2.13 $L(\theta) = \prod_{i=1}^{25} \theta k^{\theta}\left(\frac{1}{y_i}\right)^{\theta+1} = \theta^{25}k^{25\theta}\left(1/\prod_{i=1}^{25} y_i\right)^{\theta+1}$, $y_i \ge k$, $1 \le i \le 25$

$$\ln L(\theta) = 25 \ln \theta + 25\theta \ln k - (\theta + 1) \sum_{i=1}^{n} \ln y_1$$

$$\frac{d \ln L(\theta)}{d\theta} = \frac{25}{\theta} + 25 \ln k - \sum_{i=1}^{n} \ln y_i \text{, so } \theta_e = \frac{25}{-25 \ln k + \sum_{i=1}^{n} \ln y_i}$$

5.2.14 **(a)** $L(\alpha,\beta) = \prod_{i=1}^{n} \alpha\beta y_i^{\beta-1} e^{-\alpha y_i^{\beta}} = \alpha^n \beta^n \left(\prod_{i=1}^{n} y_i\right)^{\beta-1} e^{-\alpha \sum_{i=1}^{n} y_i^{\beta}}$

$$\text{In } L(\alpha,\beta) = n \ln \alpha + n \ln \beta + (\beta - 1)\ln\left(\prod_{i=1}^{n} y_i\right) - \alpha \sum_{i=1}^{n} y_i^{\beta}$$

$$\frac{\partial \ln L(\alpha,\beta)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} y_i^{\beta}$$

Setting $\frac{\partial \ln L(\alpha,\beta)}{\partial \alpha} = 0$ gives $\alpha_e = \frac{n}{\sum_{i=1}^{n} y_i^{\beta}}$

(b) The other equation is $\frac{\partial \ln L(\alpha,\beta)}{\partial \beta} = \frac{n}{\beta} + \ln \prod_{i=1}^{n} y_i - \alpha\beta \sum_{i=1}^{n} y_i^{\beta-1} = 0$

Setting $\frac{\partial \ln L(\alpha,\beta)}{\partial \alpha} = 0$ provides the other equation.

Solving the two simultaneously would be done by numerical methods.

5.2.15 Let $\theta = \sigma^2$, so $L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2}\frac{(y_i-\mu)^2}{\theta}} = 2\pi^{-n/2}\theta^{-n/2} e^{-\frac{1}{2}\frac{1}{\theta}\sum_{i=1}^{n}(y_i-\mu)^2}$

$$\ln L(\theta) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln \theta - \frac{1}{2}\frac{1}{\theta}\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\frac{d \ln L(\theta)}{d\theta} = -\frac{n}{2\theta} + \frac{1}{2}\frac{1}{\theta^2}\sum_{i=1}^{n}(y_i - \mu)^2 = \frac{1}{2}\frac{-n\theta + \sum_{i=1}^{n}(y_i - \mu)^2}{\theta^2}$$

Setting $\frac{d \ln L(\theta)}{d\theta} = 0$ gives $\theta_e = \sigma_e^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mu)^2$

5.2.16 $E(Y)=\int_0^\theta y\dfrac{2y}{\theta^2}\,dy=\dfrac{2y^3}{3\theta^2}\Bigg|_0^\theta=\dfrac{2}{3}\theta$. Setting $\dfrac{2}{3}\theta=\overline{y}$ gives $\theta_e=\dfrac{3}{2}\overline{y}$ = 75. The maximum likelihood estimate is $y_{\max}$ = 92.

5.2.17 $E(Y) = \int_0^1 y(\theta^2+\theta)y^{\theta-1}(1-y)dy=(\theta^2+\theta)\int_0^1 y^\theta(1-y)dy = \dfrac{\theta}{\theta+2}$. Set $\dfrac{\theta}{\theta+2}=\overline{y}$, which yields $\theta_e=\dfrac{2\overline{y}}{1-\overline{y}}$

5.2.18 For Y Poisson, $E(Y) = \lambda$. Then $\lambda_e=\overline{y}$ = 13/6. The maximum likelihood estimate is the same.

5.2.19 For Y exponential, $E(Y) = 1/\lambda$. Then $1/\lambda = \overline{y}$ implies $\lambda_e=1/\overline{y}$.

5.2.20 $E(Y) = \theta_1$ so $\theta_{1e}=\overline{y}$.

$E(Y^2) = \int_{\theta_1-\theta_2}^{\theta_1+\theta_2} y^2\dfrac{1}{2\theta_2}dy=\dfrac{1}{2\theta_2}\left[\dfrac{y^3}{3}\right]_{\theta_1-\theta_2}^{\theta_1+\theta_2}=\theta_1^2+\dfrac{1}{3}\theta_2^2$

Substitute $\theta_{1e}=\overline{y}$ into the equation $\theta_1^2+\dfrac{1}{3}\theta_2^2=\dfrac{1}{n}\sum_{i=1}^n y_i^2$ to obtain $\theta_{2e}=\sqrt{3\left(\dfrac{1}{n}\sum_{i=1}^n y_i^2-\overline{y}^2\right)}$.

5.2.21 $E(Y) = \int_k^\infty y\theta k^\theta\left(\dfrac{1}{y_i}\right)^{\theta+1}dy=\theta k^\theta\int_k^\infty y^{-\theta}dy=\dfrac{\theta k}{\theta-1}$

Setting $\dfrac{\theta k}{\theta-1}=\overline{y}$ gives $\theta_e=\overline{y}/(\overline{y}-k)$

5.2.22 $E(X) = 0\cdot\theta^0(1-\theta)^{1-0}+1\cdot\theta^1(1-\theta)^{1-1}=\theta$.
Then $\theta_e=\overline{y}$, which for the given data is 2/5.

5.2.23 $E(Y) = \mu$, so $\mu_e=\overline{y}$. $E(Y^2) = \sigma^2+\mu^2$. Then substitute $\mu_e=\overline{y}$ into the equation for $E(Y^2)$ to obtain $\sigma_e^2+\overline{y}^2=\dfrac{1}{n}\sum_{i=1}^n y_i^2$ or $\sigma_e^2=\dfrac{1}{n}\sum_{i=1}^n y_i^2-\overline{y}^2$

5.2.24 From Theorem 4.5.1, $E(X) = r/p$. $\text{Var}(X) = \dfrac{r(1-p)}{p^2}$.

Then $E(X^2) = \text{Var}(X) + E(X)^2 = \dfrac{r(1-p)+r^2}{p^2}$

Set $\dfrac{r}{p}=\overline{x}$ to obtain $r=p\overline{x}$, and substitute into the equation $\dfrac{r(1-p)+r^2}{p^2}=\dfrac{1}{n}\sum_{i=1}^n x_i^2$

to obtain an equation in p: $\dfrac{p\overline{x}(1-p)+(p\overline{x})^2}{p^2}=\dfrac{1}{n}\sum_{i=1}^n x_i^2$.

Equivalently, $p\bar{x} - p^2\bar{x} + p^2\bar{x}^2 - p^2\frac{1}{n}\sum_{i=1}^{n} x_i^2 = 0$. Solving for p gives $p_e = \dfrac{\bar{x}}{\bar{x} + \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2}$.

Then $r_e = \dfrac{\bar{x}^2}{\bar{x} + \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2}$.

5.2.25 From Chapter 4, $E(X) = 1/p$. Setting $1/p = \bar{x}$, gives $p_e = \dfrac{1}{\bar{x}}$. For the given data, $p_e = 0.479$. The expected frequencies are:

No. of clusters/song	Observed frequency	Expected frequency
1	132	119.8
2	52	62.4
3	34	32.5
4	9	16.9
5	7	8.8
6	5	4.6
7	5	2.4
8	6	2.6

The last expected frequency has been chosen to make that column sum to n = 59.

5.2.26 $\text{Var}(Y) = \hat{\sigma}^2$ implies $E(Y^2) - E(Y)^2 = \frac{1}{n}\sum_{i=1}^{n} y_i^2 - \bar{y}^2$. However by the first given equation, $E(Y)^2 = \bar{y}^2$. Removing these equal terms from the equation above gives the second equation of Definition 5.2.3, or $E(Y^2) = \frac{1}{n}\sum_{i=1}^{n} y_i^2$.

Section 5.3: Interval Estimation

5.3.1 The confidence interval is $\left(\bar{y} - z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}, \bar{y} + z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}\right)$

$= \left(107.9 - 1.96\dfrac{15}{\sqrt{50}},\ 107.9 + 1.96\dfrac{15}{\sqrt{50}}\right) = (103.7, 112.1)$.

5.3.2 The confidence interval is $\left(\bar{y} - z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}, \bar{y} + z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}\right) = \left(0.766 - 1.96\dfrac{0.09}{\sqrt{19}}, 0.766 + 1.96\dfrac{0.09}{\sqrt{19}}\right)$ = (0.726, 0.806). The value of 0.80 is believable.

5.3.3 The confidence interval is $\left(\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = \left(70.833 - 1.96\frac{8.0}{\sqrt{6}}, 70.833 + 1.96\frac{8.0}{\sqrt{6}}\right)$ = (64.432, 77.234). Since 80 does not fall within the confidence interval, that men and women metabolize methylmercury at the same rate is not believable.

5.3.4 The confidence interval is $\left(\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$

$= \left(188.4 - 1.96\frac{40.7}{\sqrt{38}}, 188.4 + 1.96\frac{40.7}{\sqrt{38}}\right)$ = (175.46, 201.34). Since 192 does fall in the confidence interval, there is doubt the diet has an effect.

5.3.5 The length of the confidence interval is $2z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = \frac{2(1.96)(14.3)}{\sqrt{n}} = \frac{56.056}{\sqrt{n}}$. For $\frac{56.056}{\sqrt{n}} \le 3.06$,

$n \ge \left(\frac{56.056}{3.06}\right)^2 = 335.58$, so take $n = 336$.

5.3.6 **(a)** $P(-1.64 < Z < 2.33) = 0.94$, a 94% confidence level.
(b) $P(-\infty < Z < 2.58) = 0.995$, a 99.5% confidence level.
(c) $P(-1.64 < Z < 0) = 0.45$, a 45% confidence level.

5.3.7 The probability that the given interval will contain μ is $P(-0.96 < Z < 1.06) = 0.6869$. The probability of four or five such intervals is binomial with $n = 5$ and $p = 0.6869$, so the probability is $5(0.6869)^4(0.3131) + (0.6869)^5 = 0.501$.

5.3.8 The given interval is symmetric about $\bar{y}$.

5.3.9 The interval given is correctly *calculated*. However, the data do not appear to be normal, so claiming that it is a 95% confidence interval would not be correct.

5.3.10 $\left(\frac{192}{540} - 1.96\sqrt{\frac{(192/540)(1-192/540)}{540}},\ \frac{192}{540} + 1.96\sqrt{\frac{(192/540)(1-192/540)}{540}}\right) = (0.316, 0.396)$

5.3.11 Let p be the probability that a viewer would watch less than a quarter of the advertisements during Super Bowl XXIX. The confidence interval for p is

$$\left(\frac{281}{1015} - 1.64\sqrt{\frac{(281/1015)(1-281/1015)}{1015}},\ \frac{281}{1015} + 1.64\sqrt{\frac{(281/1015)(1-281/1015)}{1015}}\right)$$

= (0.254, 0.300)

5.3.12 Budweiser would use the sample proportion 0.54 alone as the estimate. Schlitz would construct the 95% confidence interval (0.36, 0.56) to claim that values < 0.50 are believable.

5.3.13 In closest integer to 0.63(2253) is 1419. This gives the confidence interval

$$\left(\frac{1419}{2253}-1.96\sqrt{\frac{(1419/2253)(1-1419/2253)}{2253}},\quad \frac{1419}{2253}+1.96\sqrt{\frac{(1419/2253)(1-1419/2253)}{2253}}\right)$$

= (0.610, 0.650). Since 0.54 is not in the interval, the increase can be considered significant.

5.3.14 $\frac{k}{n}-0.67\sqrt{\frac{(k/n)(1-k/n)}{n}}=0.57$

$\frac{k}{n}+0.67\sqrt{\frac{(k/n)(1-k/n)}{n}}=0.63$

Adding the two equations gives $2\frac{k}{n}=1.20$ or $\frac{k}{n}=0.60$

Substituting the value for $\frac{k}{n}$ into the first equation above gives $0.60-0.67\sqrt{\frac{(0.60)(1-0.60)}{n}}$ = 0.57. Solving this equation for n gives $n = 120$.

5.3.15 $2.58\sqrt{\frac{p(1-p)}{n}}\le 2.58\sqrt{\frac{1}{4n}}\le 0.01$, so take $n\ge\frac{(2.58)^2}{4(0.01)^2}=16{,}641$

5.3.16 For Foley to win the election, he needed to win at least 8088 of the absentee votes, since 8088 > 8086 = 2174 + (14,000 − 8088). If X is the number of absentee votes for Foley, then it is binomial with n = 14,000 and p to be determined.

$$P(X\ge 8088)=P\left(\frac{X-14{,}000p}{\sqrt{14{,}000p(1-p)}}\ge\frac{8088-14{,}000p}{\sqrt{14{,}000p(1-p)}}\right)=P\left(Z\ge\frac{8088-14{,}000p}{\sqrt{14{,}000p(1-p)}}\right).$$

For this probability to be 0.20, $\frac{8088-14{,}000p}{\sqrt{14{,}000p(1-p)}}=z_{.20}=0.84$. This last equation can be solved by the quadratic formula or trial and error to obtain the approximate solution p = 0.5742.

5.3.17 Both intervals have confidence level approximately 50%.

5.3.18 $g(p)=p-p^2$. $g'(p)=1-2p$. Setting $g'(p)=0$ gives $p=1/2$. Also, $g''(p)=-2$. Since the second derivative is negative at p = 1/2, a maximum occurs there. The maximum value of $g(p)$ is $g(1/2)=1/4$.

5.3.19 The margin of error is $\frac{1.96}{2\sqrt{998}}=0.031$. The number of in-favor responses was the closest integer to 0.59(998) or 589. The 95% confidence interval is

$$\left(\frac{589}{998}-1.96\sqrt{\frac{(589/998)(1-589/998)}{998}},\quad \frac{589}{998}+1.96\sqrt{\frac{(589/998)(1-589/998)}{998}}\right)$$

= (0.559, 0.621)

5.3.20 From Definition 5.3.1, $d=\frac{1.96}{2\sqrt{202}}=0.069$. The sample proportion is 86/202 = 0.426. The largest believable value is 0.426 + 0.069 = 0.495, so we should not accept the notion that the true proportion is as high as 50%.

5.3.21 If X is hypergeometric, then Var(X/n) = $\dfrac{p(1-p)}{n}\dfrac{N-n}{N-1}$.

As before $p(1-p) \le 1/4$. Thus, in Definition 5.3.1, substitute $d = \dfrac{1.96}{2\sqrt{n}}\sqrt{\dfrac{N-n}{N-1}}$.

5.3.22 **(a)** The 90% confidence interval is

$$\left(\frac{126}{350} - 1.64\sqrt{\frac{(126/350)(1-126/350)}{350}},\quad \frac{126}{350} + 1.64\sqrt{\frac{(126/350)(1-126/350)}{350}}\right)$$

= (0.318, 0.402)

(b) Use for the margin of error $1.64\sqrt{\dfrac{(126/350)(1-126/350)}{350}}\sqrt{\dfrac{3000-350}{3000-1}}$

= 0.039, which gives a confidence interval (0.321, 0.399)

5.3.23 If n is such that $0.06 = \dfrac{1.96}{2\sqrt{n}}$, then n is the smallest integer $\ge \dfrac{1.96^2}{4(0.06)^2} = 266.8$. Take $n = 267$.

If n is such that $0.03 = \dfrac{1.96}{2\sqrt{n}}$, then n is the smallest integer $\ge \dfrac{1.96^2}{4(0.03)^2} = 1067.1$. Take $n = 1068$.

5.3.24 For candidate A, the believable values for the probability of winning fall in the range (0.52 – 0.05, 0.52 + 0.05) = (0.47, 0.57). For candidate B, the believable values for the probability of winning fall in the range (0.48 – 0.05, 0.48 + 0.05) = (0.43, 0.53). Since 0.50 falls in both intervals, there is a sense in which the candidates can be considered tied.

5.3.25 Case 1: n is the smallest integer greater than

$\dfrac{z_{.02}^2}{4(0.05)^2} = \dfrac{2.05^2}{4(0.05)^2} = 420.25$, so take $n = 421$.

Case 2: n is the smallest integer greater than

$\dfrac{z_{.04}^2}{4(0.04)^2} = \dfrac{1.75^2}{4(0.04)^2} = 478.5$, so take $n = 479$.

5.3.26 Take n to be the smallest integer $\ge \dfrac{z_{.005}^2\, p(1-p)}{(0.05)^2} = \dfrac{2.58^2(0.40)(0.60)}{(0.05)^2} = 639.01$,

so $n = 640$.

5.3.27 Take n to be the smallest integer $\ge \dfrac{z_{.10}^2}{4(0.02)^2} = \dfrac{1.28^2}{4(0.02)^2} = 1024$.

5.3.28 **(a)** Take n to be the smallest integer $\ge \dfrac{z_{.075}^2}{4(0.03)^2} = \dfrac{1.44^2}{4(0.03)^2} = 576$.

(b) Take n to be the smallest integer $\ge \dfrac{z_{.075}^2\, p(1-p)}{(0.03)^2} = \dfrac{1.44^2(0.10)(0.90)}{(0.03)^2} = 207.36$, so let $n = 208$.

Section 5.4: Properties of Estimators

5.4.1 $P(|\hat{\theta} - 3| > 1.0) = P(\hat{\theta} < 2) + P(\hat{\theta} > 4)$
$= P(\hat{\theta} = 1.5) + P(\hat{\theta} = 4.5) = P((1,2)) + P((4,5)) = 2/10$

5.4.2 For the uniform variable Y, $F_Y(y) = \dfrac{y}{\theta}$. By Theorem 3.10.1, $F_{\hat{\theta}}(y) = F_{Y_{\max}}(y) = \dfrac{y^n}{\theta^n}$, $0 \le y \le \theta$.

(a) For $n = 6$ and $\theta = 3$, $P(|\hat{\theta} - 3| < 0.2) = F_{Y_{\max}}(3) - F_{Y_{\max}}(2.8) = 1 - \dfrac{2.8^6}{3^6} = 1 - 0.661 = 0.339$

(b) For $n = 3$and $\theta = 3$, $P(|\hat{\theta} - 3| < 0.2) = F_{Y_{\max}}(3) - F_{Y_{\max}}(2.8) = 1 - \dfrac{2.8^3}{3^3} = 1 - 0.813 = 0.187$

5.4.3 $P(X < 250) = P\left(\dfrac{X - 500(0.52)}{\sqrt{500(0.52)(0.48)}} < \dfrac{250 - 500(0.52)}{\sqrt{500(0.52)(0.48)}}\right) = P(Z < -0.90) = 0.1841$

5.4.4 $P(19.0 < \overline{Y} < 21.0) = P\left(\dfrac{19.0 - 20}{10/\sqrt{16}} < Z < \dfrac{21.0 - 20}{10/\sqrt{16}}\right) = P(-0.40 < Z < 0.40) = 0.6554 - 0.3446$
$= 0.3108$

5.4.5 $E(\overline{X}) = E\left(\dfrac{1}{n}\sum_{i=1}^{n} X_i\right) = \dfrac{1}{n}\sum_{i=1}^{n} E(X_i) = \dfrac{1}{n}\sum_{i=1}^{n} \lambda = \lambda$

In general, the sample mean is an unbiased estimator of the mean μ.

5.4.6 $f_{Y_{\min}}(y) = n\dfrac{1}{\theta}\left(1 - \dfrac{y}{\theta}\right)^{n-1}$, so $E(Y_{\min}) = n\dfrac{1}{\theta}\int_0^{\theta} y\left(1 - \dfrac{y}{\theta}\right)^{n-1} dy$

Integration by parts yields $E(Y_{\min}) = \dfrac{\theta}{n+1}$. An unbiased estimator would be $(n + 1)Y_{\min}$.

5.4.7 First note that $F_Y(y) = 1 - e^{-(y-\theta)}$, $\theta \le y$. By Theorem 3.10.1, $f_{Y_{\min}}(y) = ne^{-n(y-\theta)}$, $\theta \le y$.

Then $E(Y_{\min}) = \int_{\theta}^{\infty} y \cdot ne^{-(y-\theta)}\, dy = \int_0^{\infty} (u + \theta) \cdot ne^{-u}\, du$, the last equality arising from the substitution $u = y - \theta$.

Thus, $E(Y_{\min}) = \int_0^{\infty} (u + \theta) \cdot ne^{-u}\, du = \int_0^{\infty} u \cdot ne^{-u}\, du + \theta\int_0^{\infty} ne^{-u}\, du = \dfrac{1}{n} + \theta$.

Finally, $E(Y_{\min} - \dfrac{1}{n}) = \dfrac{1}{n} + \theta - \dfrac{1}{n} = \theta$.

5.4.8 **(a)** $f_{Y_3'}(y) = 12\left(\dfrac{y}{\theta}\right)^2\left(1 - \dfrac{y}{\theta}\right)\dfrac{1}{\theta} = \dfrac{12}{\theta^4}[y^2(\theta - y)]$

$E(Y_3') = \dfrac{3}{5}\theta$, so the unbiased estimator is $\dfrac{5}{3}Y_3'$.

(b) $\frac{5}{3}Y_3' = \frac{5}{3}18 = 30$

(c) Suppose the sample were 10, 14, 18, 31. The estimate for θ is 30, but the largest observation 31 falls outside of the [0, 30] interval.

5.4.9 $E(Y) = 2\int_0^{1/\theta} y^2\theta^2 dy = \frac{2}{3}\left(\frac{1}{\theta}\right)$.

$E[c(Y_1 + 2Y_2)] = c[E(Y_1) + 2E(Y_2)] = c\left[\frac{2}{3}\left(\frac{1}{\theta}\right) + \frac{4}{3}\left(\frac{1}{\theta}\right)\right] = 2c\left(\frac{1}{\theta}\right)$. For the estimator to be unbiased, $2c = 1$ or $c = 1/2$.

5.4.10 $E(Y^2) = \int_0^{\theta} y^2 \frac{1}{\theta} dy = \frac{\theta^2}{3}$, so $3Y^2$ is unbiased.

5.4.11 $E(W^2) = \text{Var}(W) + E(W)^2 = \text{Var}(W) + \theta^2$. Thus, W^2 is unbiased only if $\text{Var}(W) = 0$, which in essence means that W is constant.

5.4.12 For each i, $E[(Y_i - \mu)^2] = \sigma^2$ by definition of σ^2.

$$E\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu)^2\right] = \frac{1}{n}\sum_{i=1}^{n}E\left[(Y_i - \mu)^2\right] = \frac{1}{n}n\sigma^2 = \sigma^2$$

5.4.13 $f_{\frac{n+1}{n}Y_{\max}}(y) = \frac{n}{n+1}f_{Y_{\max}}\left(\frac{n}{n+1}y\right) = \frac{n}{n+1}\frac{n}{\theta}\frac{n^{n-1}}{(n+1)^{n-1}}\frac{y^{n-1}}{\theta^{n-1}} = \frac{n^{n+1}}{(n+1)^n}\frac{y^{n-1}}{\theta^n}$

The median of this distribution is the number m such that

$$1/2 = \int_0^m \frac{n^{n+1}}{(n+1)^n}\frac{y^{n-1}}{\theta^n}dy = \frac{n^n}{(n+1)^n}\frac{y^n}{\theta^n}\bigg|_0^m = \frac{n^n}{(n+1)^n}\frac{m^n}{\theta^n}$$

Solving for m gives $m = \frac{1}{\sqrt[n]{2}}\frac{(n+1)}{n}\theta$. The estimator is unbiased only when $n = 1$.

5.4.14 $f_{Y_{\min}}(y) = nf_Y(y)(1 - F_Y(y))^{n-1} = n\frac{1}{\theta}e^{-y/\theta}[1-(1-e^{-y/\theta})]^{n-1} = n\frac{1}{\theta}e^{-ny/\theta}$. Then

$f_{nY_{\min}}(y) = \frac{1}{n}f_{Y_{\min}}\left(\frac{y}{n}\right) = \frac{1}{n}n\frac{1}{\theta}e^{-n\frac{y}{n}/\theta} = \frac{1}{\theta}e^{-y/\theta}$. $E(nY_{\min}) = \theta$, so $nY_{\min}$ is unbiased. Also, the sample mean $\frac{1}{n}\sum_{i=1}^{n}Y_i$ is unbiased (see Question 5.4.5).

5.4.15 $E(\overline{W}^2) = \text{Var}(\overline{W}) + E(\overline{W})^2 = \frac{\sigma^2}{n} + \mu^2$, so $\lim_{n\to\infty} E(\overline{W}^2) = \lim_{n\to\infty}\left(\frac{\sigma^2}{n} + \mu^2\right) = \mu^2$

5.4.16 $\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$, so $E(\hat{\theta}_n) = \frac{n-1}{n}\sigma^2$. This estimator is asymptotically unbiased since $\lim_{n\to\infty} E(\hat{\theta}_n) = \lim_{n\to\infty}\frac{n-1}{n}\sigma^2 = \sigma^2$.

5.4.17 **(a)** $E(\hat{p}_i) = E(X_1) = p$, since X_1 is binomial with $n = 1$. $E(\hat{p}_2) = E\left(\frac{X}{n}\right) = \frac{1}{n}np = p$, since X is binomial.

(b) $\text{Var}(\hat{p}_1) = p(1-p)$; $\text{Var}(X) = np(1-p)$. Then $\text{Var}(\hat{p}_2) = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$, which is smaller than $\text{Var}(\hat{p}_1)$ by a factor of n.

5.4.18 From the solution to Question 5.4.2, $f_{Y_{\max}}(y) = 5\frac{y^4}{\theta^5}$.

$$E(Y_{\max}) = \int_0^\theta y\cdot 5\frac{y^4}{\theta^5}\,dy = \frac{5}{6}\theta$$

$$E(Y_{\max}^2) = \int_0^\theta y^2\cdot 5\frac{y^4}{\theta^5}\,dy = \frac{5}{7}\theta^2$$

$$\text{Var}(Y_{\max}) = \frac{5}{7}\theta^2 - \left(\frac{5}{6}\theta\right)^2 = \frac{5}{36(7)}\theta^2$$

$$\text{Var}\left(\hat{\theta}_1\right) = \text{Var}\left(\frac{6}{5}\cdot Y_{\max}\right) = \frac{36}{25}\frac{5}{36(7)}\theta^2 = \frac{1}{35}\theta^2$$

By symmetry, $\text{Var}(Y_{\min}) = \text{Var}(Y_{\max})$.

$\text{Var}(\hat{\theta}_2) = \text{Var}(6\cdot Y_{\min}) = 36\text{Var}(Y_{\min}) = \frac{36}{35}\theta^2$. Thus, $\frac{6}{5}\cdot Y_{\max}$ has smaller variance. This result makes sense intuitively, since efficiency here depends on the size of the constant needed to make the estimator unbiased.

5.4.19 **(a)** $f_{Y_{\min}}(y) = nf_Y(y)(1 - F_Y(y))^{n-1} = n\frac{1}{\theta}e^{-y/\theta}[1-(1-e^{-y/\theta})]^{n-1} = n\frac{1}{\theta}e^{-ny/\theta}$.

Then $f_{nY_{\min}}(y) = \frac{1}{n}f_{Y_{\min}}\left(\frac{y}{n}\right) = \frac{1}{n}n\frac{1}{\theta}e^{-n\frac{y}{n}/\theta} = \frac{1}{\theta}e^{-y/\theta}$. Thus, Y_1 and $nY_{\min}$ have the same distribution; they are both gamma variables with parameters 1 and $1/\theta$. Then $E(Y_1) = E(Y_{\min}) = \theta$ .Also, the sample mean $\frac{1}{n}\sum_{i=1}^{n}Y_i$ is unbiased (see Question 5.4.5).

(b) Since Y_1 and $nY_{\min}$ are both gamma variables with parameters 1 and $1/\theta$, then $\text{Var}(Y_1) = \text{Var}(nY_{\min}) = \theta^2$. Further, $\text{Var}(\overline{Y}) = \text{Var}(Y) = \theta^2 / n$

(c) Relative efficiency of $\hat{\theta}_1$ to $\hat{\theta}_3$ is $\text{Var}(\hat{\theta}_3)/\text{Var}(\hat{\theta}_1) = (\theta^2/n^2)/\theta^2) = 1/n^2$

Relative efficiency of $\hat{\theta}_2$ to $\hat{\theta}_3$ is $\text{Var}(\hat{\theta}_3)/\text{Var}(\hat{\theta}_2) = (\theta^2/n^2)/\theta^2) = 1/n^2$

5.4.20 $\text{Var}(\hat{\lambda}_1) = \text{Var}(X_1) = \lambda$. $\text{Var}(\hat{\lambda}_2) = \text{Var}(\overline{X}) = \lambda/n$.
$\text{Var}(\hat{\lambda}_2)/\text{Var}(\hat{\lambda}_1) = (\lambda/n)/\lambda = 1/n$

5.4.21 $E(Y_{\max}) = \int_0^\theta y \cdot n\frac{y^{n-1}}{\theta^n}\,dy = \frac{n}{n+1}\theta$

$$E(Y_{\max}^2) = \int_0^\theta y^2 \cdot n\frac{y^{n-1}}{\theta^n}\,dy = \frac{n}{n+2}\theta^2$$

$$\text{Var}(Y_{\max}) = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 = \frac{n}{(n+1)^2(n+2)}\theta^2$$

$$\text{Var}\left(\hat{\theta}_1\right) = \text{Var}\left(\frac{n+1}{n}\cdot Y_{\max}\right) = \frac{(n+1)^2}{n^2}\frac{n}{(n+1)^2(n+2)}\theta^2 = \frac{1}{n(n+2)}\theta^2$$

By symmetry, $\text{Var}(Y_{\min}) = \text{Var}(Y_{\max})$.

$$\text{Var}(\hat{\theta}_2) = \text{Var}\left((n+1)Y_{\min}\right) = (n+1)^2\frac{n}{(n+1)^2(n+2)}\theta^2 = \frac{n}{(n+2)}\theta^2$$

$$\text{Var}(\hat{\theta}_2)/\text{Var}(\hat{\theta}_1) = \frac{n\theta^2}{(n+2)} / \frac{\theta^2}{n(n+2)} = n^2$$

5.4.22 We seek the value of c that minimizes $\text{Var}[cW_1 + (1-c)W_2]$.
But $\text{Var}[cW_1 + (1-c)W_2] = c^2\text{Var}(W_1) + (1-c)^2\,\text{Var}(W_2) = c^2\sigma_1^2 + (1-c)^2\sigma_2^2$

Differentiate this expression with respect to c to obtain $\frac{\partial[c^2\sigma_1^2 + (1-c)^2\sigma_2^2]}{\partial c} = 2c\sigma_1^2 - 2(1-c)\sigma_2^2$

Setting this derivative equal to zero gives $c = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$

Section 5.5: Minimum-Variance Estimators: The Cramér-Rao Lower Bound

5.5.1 $\ln f_Y(Y;\theta) = -\ln\theta - Y/\theta$

$$\frac{\partial \ln f_Y(Y;\theta)}{\partial\theta} = -\frac{1}{\theta} + Y/\theta^2$$

$$\frac{\partial^2 \ln f_Y(Y;\theta)}{\partial\theta^2} = \frac{1}{\theta^2} - 2Y/\theta^3$$

$E\left[\frac{\partial^2 \ln f_Y(Y;\theta)}{\partial\theta^2}\right] = \frac{1}{\theta^2} - 2\theta/\theta^3 = \frac{-1}{\theta^2}$, so the Cramer-Rao bound is θ^2/n.

Also, $\text{Var}(\hat{\theta}) = Var(\overline{Y}) = \text{Var}(Y)/n = \theta^2/n$, so $\hat{\theta}$ is a best estimator.

5.5.2 $\ln f_X(X;\lambda) = -\lambda + X\ln\lambda - \ln X!$

$$\frac{\partial \ln f_X(X;\lambda)}{\partial\lambda} = -1 + X/\lambda$$

$$\frac{\partial^2 \ln f_X(X;\lambda)}{\partial\lambda^2} = -X/\lambda^2$$

$E\left[\dfrac{\partial^2 \ln f_X(X;\lambda)}{\partial\lambda^2}\right] = -\lambda/\lambda^2 = -1/\lambda$, so the Cramer-Rao bound is λ/n.

Also, $\text{Var}(\hat{\lambda}) = \text{Var}(\bar{X}) = \text{Var}(X)/n = \lambda/n$, so $\hat{\theta}$ is an efficient estimator.

5.5.3 $\ln f_Y(Y;\mu) = -\ln\sqrt{2\pi}\sigma - \dfrac{1}{2}\dfrac{(Y-\mu)^2}{\sigma^2}$

$$\frac{\partial \ln f_Y(Y;\mu)}{\partial\mu} = \frac{(Y-\mu)}{\sigma^2}$$

$$\frac{\partial^2 \ln f_Y(Y;\mu)}{\partial\mu^2} = \frac{-1}{\sigma^2}$$

$E\left[\dfrac{\partial^2 \ln f_Y(Y;\mu)}{\partial\mu^2}\right] = \dfrac{-1}{\sigma^2}$, so the Cramer-Rao bound is σ^2/n.

Also, $\text{Var}(\hat{\mu}) = \text{Var}(\bar{Y}) = \text{Var}(Y)/n = \sigma^2/n$, so $\hat{\mu}$ is an efficient estimator.

5.5.4 $\ln f_Y(Y;\theta) = -\ln\theta$

$$\frac{\partial \ln f_Y(Y;\theta)}{\partial\theta} = \frac{-1}{\theta}$$

$E\left[\left(\dfrac{\partial \ln f_Y(Y;\theta)}{\partial\theta}\right)^2\right] = \dfrac{1}{\theta^2}$, so the Cramer-Rao bound is $\dfrac{\theta^2}{n}$. From Question 5.4.21,

$\text{Var}(\hat{\theta}) = \dfrac{\theta^2}{n(n+2)}$, which is smaller than the Cramer-Rao bound. This occurs because Theorem 5.5.1 is not necessarily valid if the interval where the pdf is nonzero depends on the parameter.

5.5.5 $\ln f_X(X;\theta) = (X-1)\ln(\theta-1) - X\ln\theta$

$$\frac{\partial \ln f_X(X;\theta)}{\partial\theta} = \frac{X-1}{\theta-1} - \frac{X}{\theta}$$

$$\frac{\partial^2 \ln f_X(X;\theta)}{\partial\theta^2} = -\frac{X-1}{(\theta-1)^2} + \frac{X}{\theta^2}$$

$$E\left(\frac{\partial^2 \ln f_X(X;\theta)}{\partial\theta^2}\right) = -\frac{E(X)-1}{(\theta-1)^2} + \frac{E(X)}{\theta^2} = -\frac{1}{\theta-1} + \frac{1}{\theta} = \frac{-1}{\theta(\theta-1)}$$

so the Cramer-Rao bound is $\dfrac{\theta(\theta-1)}{n}$.

Also, $\text{Var}(\bar{X}) = \dfrac{\text{Var}(X)}{n} = \dfrac{\theta(\theta-1)}{n}$ so $\bar{X}$ is an efficient estimator.

5.5.6 **(a)** Y is a gamma random variable with parameters r and $1/\theta$ so $E(Y) = r\theta$.

Let $\hat{\theta} = \dfrac{1}{r}\bar{Y} = \dfrac{1}{rn}\displaystyle\sum_{i=1}^{n} Y_i$. Then $E(\hat{\theta}) = \dfrac{1}{rn}\displaystyle\sum_{i=1}^{n} E(Y_i) = \dfrac{1}{rn} nr\theta = \theta$

(b) $\ln f_Y(Y;\theta) = -\ln(r-1)! - r\ln\theta + (r-1)\ln Y - Y/\theta$

$$\frac{\partial \ln f_Y(Y;\theta)}{\partial\theta} = -r/\theta + Y/\theta^2$$

$$\frac{\partial^2 \ln f_Y(Y;\theta)}{\partial\theta^2} = r/\theta^2 - 2Y/\theta^3$$

$$E\left[\frac{\partial^2 \ln f_Y(Y;\theta)}{\partial\theta^2}\right] = r/\theta^2 - 2(r\theta)/\theta^3 = -r/\theta^2$$

The Cramer-Rao bound is θ^2/rn.

$\text{Var}(\hat{\theta}) = \left(\frac{1}{rn}\right)^2 \sum_{i=1}^{n} \text{Var}(Y_i) = \left(\frac{1}{rn}\right)^2 nr\theta^2 = \theta^2/rn$, so $\hat{\theta}$ is a minimum-variance estimator.

5.5.7 $E\left(\frac{\partial^2 \ln f_W(W;\theta)}{\partial\theta^2}\right) = \int_{-\infty}^{\infty} \frac{\partial}{\partial\theta}\left(\frac{\partial \ln f_W(w;\theta)}{\partial\theta}\right) f_W(w;\theta)\, dw$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial\theta}\left(\frac{1}{f_W(w;\theta)} \frac{\partial f_W(w;\theta)}{\partial\theta}\right) f_W(w;\theta)\, dw$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{f_W(w;\theta)} \frac{\partial^2 f_W(w;\theta)}{\partial\theta^2} - \frac{1}{\left(f_W(w;\theta)\right)^2}\left(\frac{\partial f_W(w;\theta)}{\partial\theta}\right)^2\right] f_W(w;\theta)\, dw$$

$$= \int_{-\infty}^{\infty} \frac{\partial^2 f_W(w;\theta)}{\partial\theta^2} dw - \int_{-\infty}^{\infty} \frac{1}{(f_W(w;\theta))^2}\left(\frac{\partial f_W(w;\theta)}{\partial\theta}\right)^2 f_W(w;\theta)\, dw$$

$$= 0 - \int_{-\infty}^{\infty} \left(\frac{\partial \ln f_W(w;\theta)}{\partial\theta}\right)^2 f_W(w;\theta)\, dw$$

The 0 occurs because 1 = dw, so 0 = $\dfrac{\partial^2 \int_{-\infty}^{\infty} f_W(w;\theta)dw}{\partial\theta^2} = \int_{-\infty}^{\infty} \dfrac{\partial^2 f_W(w;\theta)}{\partial\theta^2} dw$

The above argument shows that $E\left(\dfrac{\partial^2 \ln f_W(W;\theta)}{\partial\theta^2}\right) = -E\left(\dfrac{\partial \ln f_W(W;\theta)}{\partial\theta}\right)^2$.

Multiplying both sides of the equality by n and inverting gives the desired equality.

Section 5.6: Sufficient Estimators

5.6.1 $\prod_{i=1}^{n} p_X(k_i;p) = \prod_{i=1}^{n} (1-p)^{k_i - 1} p = (1-p)^{\left(\sum_{i=1}^{n} k_i\right) - n} p^n$

Let $g\left(\sum_{i=1}^{n} k_i; p\right) = (1-p)^{\left(\sum_{i=1}^{n} k_i\right) - n} p^n$ and $u(k_1, \ldots, k_n) = 1$.

By Theorem 5.6.1, the statistic $\sum_{i=1}^{n} X_i$ is sufficient

5.6.2 $P((1,1,0)|\, X_1 + 2X_2 + 3X_3 = 3)$

$$= \frac{P((1,1,0) \text{ and } X_1 + 2X_2 + 3X_3 = 3)}{P(X_1 + 2X_2 + 3X_3 = 3)} = \frac{P((1,1,0))}{P((1,1,0),(0,0,1))} = \frac{p^2(1-p)}{p^2(1-p) + p(1-p)^2} = p.$$

Since the conditional probability does depend on the parameter p, the statistic cannot be sufficient, by Definition 5.6.1.

5.6.3 We must show $P(W_1 = w_1, W_2 = w_2, \ldots, W_n = w_n \,|\, g(\hat{\theta}) = \theta_e)$ does not depend on θ. But $P(W_1 = w_1, W_2 = w_2, \ldots, W_n = w_n \,|\, g(\hat{\theta}) = \theta_e) = P(W_1 = w_1, W_2 = w_2, \ldots, W_n = w_n \,|\, \hat{\theta} = g^{-1}(\theta_e))$, which does not depend on θ, because $\hat{\theta}$ is sufficient.

5.6.4 $\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{y_i^2}{\sigma^2}} = \left[(\sigma^2)^{-n/2} e^{-\frac{1}{2}\frac{1}{\sigma^2}\left(\sum_{i=1}^{n} y_i^2\right)}\right][2\pi^{-n/2}]$, so $\sum_{i=1}^{n} Y_i^2$ is sufficient by Theorem 5.6.1.

5.6.5 $$\prod_{i=1}^{n} \frac{1}{(r-1)!\theta^r} y_i^{r-1} e^{-y_i/\theta} = \frac{1}{[(r-1!]^n} \frac{1}{\theta^{rn}} \left(\prod_{i=1}^{n} y_i\right)^{r-1} = \left[\frac{1}{\theta^{rn}} e^{-\frac{1}{\theta}\sum_{i=1}^{n} y_i}\right] \frac{1}{[(r-1)!]^n} \left(\prod_{i=1}^{n} y_i\right)^{r-1}$$

so $\sum_{i=1}^{n} Y_i$ is a sufficient statistic for θ. So also is $\frac{1}{r}\overline{Y}$. (See Question 5.6.3)

5.6.6 $L = \prod_{i=1}^{n} f_Y(y_i;\theta) = \prod_{i=1}^{n} \theta y_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} y_i\right)^{\theta-1}$, and $\ln L = n \cdot \ln \theta + (\theta - 1)\sum_{i=1}^{n} \ln y_i$

$$\frac{d \ln L}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \ln y_i$$

Setting $\frac{d \ln L}{d\theta} = 0$ gives $\theta_e = \frac{-n}{\sum_{i=1}^{n} \ln y_i} = \frac{-n}{\ln\left(\prod_{i=1}^{n} y_i\right)}$, which is a function of $\prod_{i=1}^{n} y_i$.

5.6.7 **(a)** Write the pdf in the form $f_Y(y) = e^{-(y-\theta)} \cdot I_{[\theta,\infty]}(y)$, where $I_{[\theta,\infty]}(y)$ is the indicator function introduced in Example 5.6.2. Then the likelihood function is

$$L(\theta) = \prod_{i=1}^{n} e^{-(y_i-\theta)} \cdot I_{[\theta,\infty]}(y_i) = e^{-\sum_{i=1}^{n} y_i} e^{n\theta} \prod_{i=1}^{n} I_{[\theta,\infty]}(y_i)$$

But the likelihood function is $\prod_{i=1}^{n} e^{-(y_i-\theta)} = \begin{cases} e^{n\theta} e^{-\sum_{i=1}^{n} y_i} \text{ if } \theta \le y_1, y_2, \ldots y_n \\ 0 \text{ otherwise} \end{cases}$

$$L(\theta) = \left(e^{-\sum_{i=1}^{n} y_i}\right)\left[e^{n\theta} \cdot I_{[\theta,\infty]}(y_{\min})\right]$$

Thus the likelihood function decomposes in such a way that the factor involving θ only contains the y_i's through $y_{\min}$. By Theorem 5.6.1, $y_{\min}$ is sufficient.

(b) We need to show that the likelihood function given $y_{\max}$ is independent of θ.

But the likelihood function is $\prod_{i=1}^{n} e^{-(y_i-\theta)} = \begin{cases} e^{\theta} e^{-\sum_{i=1}^{n} y_i} & \text{if } \theta \le y_1, y_2, \ldots y_n \\ 0 & \text{otherwise} \end{cases}$

Regardless of the value of $y_{\max}$, the expression for the likelihood does depend on θ. If any one of the y_i, is less than θ, the expression is 0. Otherwise, it is non-zero.

5.6.8 Write the pdf in the form $f_Y(y) = \dfrac{1}{\theta} \cdot I_{[0,\theta]}(y)$. Then the likelihood function is

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} \cdot I_{[0,\theta]}(y_i) = \frac{1}{\theta^n} \prod_{i=1}^{n} I_{[0,\theta]}(y_i)$$

But $\prod_{i=1}^{n} I_{[0,\theta]}(y_i) = I_{[0,\theta]}(y_{\max})$, so the likelihood function factors into

$$L(\theta) = (1)\left[\frac{1}{\theta^n} \cdot I_{[\theta,\infty]}(y_{\max})\right]$$

Thus the likelihood function decomposes in such a way that the factor involving θ only contains the y_i's through $y_{\max}$. By Theorem 5.6.1, $y_{\max}$ is sufficient.

5.6.9 $\prod_{i=1}^{n} g_W(w_i;\theta) = \prod_{i=1}^{n} e^{K(w_i)p(\theta)+S(w_i)+q(\theta)} = e^{\left(\sum_{i=1}^{n} K(w_i)\right)p(\theta)+\sum_{i=1}^{n} S(w_i)+nq(\theta)} = \left(e^{\left(\sum_{i=1}^{n} K(w_i)\right)p(\theta)+nq(\theta)}\right)\left(e^{\sum_{i=1}^{n} S(w_i)}\right)$,

so $\sum_{i=1}^{n} K(W_i)$ is a sufficient statistic for θ by Theorem 5.6.1.

5.6.10 $\lambda e^{-\lambda y} = e^{\ln \lambda - \lambda y} = e^{y(-\lambda) + \ln\lambda}$. Take $K(y) = y$, $p(\lambda) = -\lambda$, $S(y) = 0$, and $q(\lambda) = \ln \lambda$.

Then $\sum_{i=1}^{n} Y_i$ is sufficient.

5.6.11 $\theta/(1 + y)^{\theta+1} = e^{\ln\theta-(\theta+1)\ln(1+y)} = e^{[\ln(1+y)](-\theta-1)+\ln\theta}$. Take $K(y) = \ln (1 + y)$, $p(\theta) = -\theta - 1$, and $q(\theta) = \ln \theta$. Then $\sum_{i=1}^{n} K(Y_i) = \sum_{i=1}^{n} \ln(1+Y_i)$ is sufficient for θ.

Section 5.7: Consistency

5.7.1 $P(16 < \overline{Y} < 20) = 0.90$ is equivalent to $P\left(\dfrac{16-18}{5.0/\sqrt{n}} < Z < \dfrac{20-18}{5.0/\sqrt{n}}\right) = 0.90$,

or $P(-0.40\sqrt{n} < Z < 0.40\sqrt{n}) = 0.90$. Then $0.40\sqrt{n} = 1.64$ or $n = \left(\dfrac{1.64}{0.40}\right)^2 = 16.81$,

so take $n = 17$.

5.7.2 Since $\mu = 0$, for each i, $E\left(Y_i^2\right) = \sigma^2$. By the weak law of large numbers demonstrated in Example 5.7.2, $\frac{1}{n}\sum_{i=1}^{n} Y_i^2$ is a consistent estimator of the mean of the Y_i^2, in this case σ^2. However, the proof given in the example requires that $\text{Var}\left(Y_i^2\right) < \infty$. This follows from an application of the moment generating function for the normal distribution since the fourth derivative of the moment generating function evaluated at 0 is finite.

5.7.3 **(a)** $P(Y_1 > 2\lambda) = \int_{2\lambda}^{\infty} \lambda e^{-\lambda y} dy = e^{-2\lambda^2}$. Then $P(|Y_1 - \lambda| < \lambda/2) < 1 - e^{-2\lambda^2} < 1$.

Thus, $\lim_{n\to\infty} P(|Y_1 - \lambda| < \lambda/2) < 1$.

(b) $P\left(\sum_{i=1}^{n} Y_i > 2\lambda\right) \geq P(Y_1 > 2\lambda) = e^{-2\lambda^2}$. The proof now proceeds along the lines of Part **(a)**.

5.7.4 **(a)** Let $\mu_n = E(\hat{\theta}_n)$. $E\left[(\hat{\theta}_n - \theta)^2\right] = E\left[(\hat{\theta}_n - \mu_n + \mu_n - \theta)^2\right]$

$= E\left[(\hat{\theta}_n - \mu_n)^2 + (\mu_n - \theta)^2 + 2(\hat{\theta}_n - \mu_n)(\mu_n - \theta)\right]$

$= E[(\hat{\theta}_n - \mu_n)^2] + E[(\mu_n - \theta)^2] + 2(\mu_n - \theta)E[(\hat{\theta}_n - \mu_n)]$

$= E\left[(\hat{\theta}_n - \mu_n)^2\right] + (\mu_n - \theta)^2 + 0$

or $E\left[(\hat{\theta}_n - \theta)^2\right] = E\left[(\hat{\theta}_n - \mu_n)^2\right] + (\mu_n - \theta)^2$

The left hand side of the equation tends to 0 by the squared-error consistency hypothesis. Since the two summands on the right hand side are non-negative, each of them must tend to zero also. Thus,
$\lim_{n\to\infty} (\mu_n - \theta)^2 = 0$, which implies $\lim_{n\to\infty} (\mu_n - \theta) = 0$, or $\lim_{n\to\infty} \mu_n = \theta$.

(b) By Part (a) $\lim_{n\to\infty} \mu_n - \theta = 0$.

For any $\varepsilon > 0$, $\lim_{n\to\infty} P\left(\left|\hat{\theta}_n - \theta\right| \geq \varepsilon\right) = \lim_{n\to\infty} P\left(\left|(\hat{\theta}_n - \mu_n) - (\mu_n - \theta)\right| \geq \varepsilon\right)$

$= \lim_{n\to\infty} P\left(\left|\hat{\theta}_n - \mu_n\right| \geq \varepsilon\right) \leq \frac{\text{Var}(\hat{\theta}_n)}{\varepsilon^2}$ by Chebyshev's Inequality.

But $\frac{\text{Var}(\hat{\theta}_n)}{\varepsilon^2} = \frac{E[(\hat{\theta}_n - \mu_n)^2]}{\varepsilon^2}$, and by Part (a), $\lim_{n\to\infty} E\left[(\hat{\theta}_n - \mu_n)^2\right] = 0$,

so $\lim_{n\to\infty} P\left(\left|\hat{\theta}_n - \theta\right| \geq \varepsilon\right) = 0$. Thus, $\hat{\theta}_n$ is consistent.

5.7.5 $E\left[(Y_{\max} - \theta)^2\right] = \int_0^{\theta} (y - \theta)^2 \frac{n}{\theta}\left(\frac{y}{\theta}\right)^{n-1} dy$

$= \frac{n}{\theta^n}\int_0^{\theta} (y^{n+1} - 2\theta y^n + \theta^2 y^{n-1}) dy = \frac{n}{\theta^n}\left(\frac{\theta^{n+2}}{n+2} - \frac{2\theta^{n+2}}{n+1} + \frac{\theta^{n+2}}{n}\right) = \left(\frac{n}{n+2} - \frac{2n}{n+1} + 1\right)\theta^2$

Then $\lim_{n\to\infty} E\left[(Y_{\max}-\theta)^2\right] = \lim_{n\to\infty}\left(\frac{n}{n+2}-\frac{2n}{n+1}+1\right)\theta^2 = 0$ and the estimator is squared error consistent.

5.7.6 Because the symmetry of the pdf, the mean of the sample median is μ. Chebyshev's inequality applies and $P\left(\left|Y'_{n+1}-\mu\right|<\varepsilon\right) > 1-\frac{\mathrm{Var}(Y'_{n+1})}{\varepsilon^2}$.

Now, $\lim_{n\to\infty}\mathrm{Var}(Y'_{n+1}) = \lim_{n\to\infty}\frac{1}{8[f_Y(\mu;\mu)]^2 n}$ =0, so $\lim_{n\to\infty} P\left(\left|Y'_{n+1}-\mu\right|<\varepsilon\right) = 1$, and Y_{n+1} is consistent for μ.

Section 5.8: Bayesian Estimation

5.8.1 The numerator of $g_\Theta(\theta \mid X = k)$ is

$$p_X(k|\theta)f_\Theta(\theta) = [(1-\theta)^{k-1}\theta]\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\theta^{r-1}(1-\theta)^{s-1} = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\theta^{r}(1-\theta)^{s+k-2}$$

The term $\theta^r(1-\theta)^{s+k-2}$ is the variable part of the beta distribution with parameters $r + 1$ and $s + k$ -1, so that is the pdf of $g_\Theta(\theta| X = k)$.

5.8.2 The Bayes estimate is the mean of the posterior distribution, a beta pdf with a parameters $k + 4$ and $n - k + 102$. The mean of this pdf is $\frac{k+4}{k+4+n-k+102} = \frac{k+4}{n+106}$.

Note we can write $\frac{k+4}{n+106} = \frac{n}{n+106}\left(\frac{k}{n}\right)+\frac{106}{n+106}\left(\frac{4}{106}\right)$

5.8.3 **(a)** Following the pattern of Example 5.8.2, we can see that the posterior distribution is a beta pdf with parameters $k + 135$ and $n - k + 135$.

(b) The mean of the Bayes pdf given in part (a) is $\frac{k+135}{k+135+n-k+135} = \frac{k+135}{n+270}$,

Note we can write $\frac{k+135}{n+270} = \frac{n}{n+270}\left(\frac{k}{n}\right)+\frac{270}{n+270}\left(\frac{135}{270}\right) = \frac{n}{n+270}\left(\frac{k}{n}\right)+\frac{270}{n+270}\left(\frac{1}{2}\right)$

5.8.4 $\frac{k+1}{n+2} = \frac{n}{n+2}\left(\frac{k}{n}\right)+\frac{2}{n+2}\left(\frac{1}{2}\right)$

5.8.5 In each case the estimator is biased, since the mean of the estimator is a weighted average of the unbiased maximum likelihood estimator and a non-zero constant. However, in each case, the weighting on the maximum likelihood estimator tends to 1 as n tends to ∞, so these estimators are asymptotically unbiased.

5.8.6 The numerator of $g_\Theta(\theta | X = k)$ is $f_Y(y | \theta) f_\Theta(\theta)$

$$= \frac{\theta^r}{\Gamma(r)} y^{r-1} e^{-\theta y} \frac{\mu^s}{\Gamma(s)} \theta^{s-1} e^{-\mu\theta} = \frac{\mu^s y^{r-1}}{\Gamma(r)\Gamma(s)} \theta^{r+s-1} e^{-(y+\mu)\theta}$$

We recognize the part involving θ as the variable part of the gamma distribution with parameters $r + s$ and $y + \mu$, so that is $g_\Theta(\theta | X = k)$.

5.8.7 Since the sum of gamma random variables is gamma, then W is gamma with parameters nr and λ. Then $g_\Theta(\theta | W = k)$ is a gamma pdf with parameters $nr + s$ and $\sum_{i=1}^{n} y_i + \mu$.

5.8.8 The Bayes estimate is the mean of the posterior pdf, which in this case is $\dfrac{nr + s}{\sum_{i=1}^{n} y_i + \mu}$

5.8.9 $p_X(k | \theta) f_\Theta(\theta) = \binom{n}{k} \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \theta^{k+r-1}(1-\theta)^{n-k+s-1}$,

so $p_X(k | \theta) = \binom{n}{k} \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 \theta^{k+r-1}(1-\theta)^{n-k+s-1} d\theta$,

$$= \binom{n}{k} \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \frac{\Gamma(k+r)\Gamma(n-k+s)}{\Gamma(n+r+s)} = \frac{n!}{k!(n-k)!} \frac{(r+s-1)!}{(r-1)!(s-1)!} \frac{(k+r-1)!(n-k+s-1)!}{(n+r+s-1)!}$$

$$= \frac{(k+r-1)!}{k!(r-1)!} \frac{(n-k+s-1)!}{(n-k)!(s-1)!} \frac{n!(r+s-1)!}{(n+r+s-1)!}$$

$$= \binom{k+r-1}{k} \binom{n-k+s-1}{n-k} \Big/ \binom{n+r+s-1}{n}$$

Chapter 6: Hypothesis Testing

Section 6.2: The Decision Rule

6.2.1 **(a)** Reject H_0 if $\dfrac{\bar{y}-120}{18/\sqrt{25}} \le -1.41$; $z = -1.61$; reject H_0.

(b) Reject H_0 if $\dfrac{\bar{y}-42.9}{3.2/\sqrt{16}}$ is either ≤ -2.58 or ≥ 2.58; $z = 2.75$; reject H_0.

(c) Reject H_0 if $\dfrac{\bar{y}-14.2}{4.1/\sqrt{9}} \ge 1.13$; $z = 1.17$; reject H_0.

6.2.2 Let μ = true average IQ of students after drinking Brain-Blaster. To test H_0: $\mu = 95$ versus H_1: $\mu \ne 95$ at the $\alpha = 0.06$ level of significance, the null hypothesis should be rejected if $z = \dfrac{\bar{y}-95}{15/\sqrt{22}}$ is either ≤ -1.88 or ≥ 1.88. Equivalently, H_0 will be rejected if $\bar{y}$ is either

1) $\le 95 - (1.88)\dfrac{15}{\sqrt{22}} = 89.0$ or **2)** $\ge 95 + (1.88)\dfrac{15}{\sqrt{22}} = 101.0$.

6.2.3 **(a)** No, because the observed z could fall <u>between</u> the 0.05 and 0.01 cutoffs.
(b) Yes. If the observed z exceeded the 0.01 cutoff, it would necessarily exceed the 0.05 cutoff.

6.2.4 Assuming there is no reason to suspect that the polymer would shorten a tire's lifetime, the alternative hypothesis should be H_1: $\mu > 32{,}500$. At the $\alpha = 0.05$ level, H_0 should be rejected if the test statistic exceeds $z_{.05} = 1.64$. But $z = \dfrac{33{,}800-32{,}500}{4000/\sqrt{15}} = 1.26$, implying that the observed mileage increase is not statistically significant.

6.2.5 No, because two-sided cutoffs (for a given α) are further away from 0 than one-sided cutoffs.

6.2.6 By definition, $\alpha = P(29.9 \le \bar{Y} \le 30.1 \mid H_0 \text{ is true}) = P\left(\dfrac{29.9-30}{6.0/\sqrt{16}} \le \dfrac{\bar{Y}-30}{6.0/\sqrt{16}} \le \dfrac{30.1-30}{6.0/\sqrt{16}}\right) =$ $P(-0.07 \le Z \le 0.07) = 0.056$. The interval (29.9, 30.1) is a poor choice for C because it rejects H_0 for the $\bar{y}$-values that are most compatible with H_0 (that is, closest to $\mu_0 = 30$). Since the alternative is two-sided, H_0 should be rejected if $\bar{y}$ is either

1) $\le 30 - 1.91\cdot\dfrac{6.0}{\sqrt{16}} = 27.1$ or **2)** $\ge 30 + 1.91\cdot\dfrac{6.0}{\sqrt{16}} = 32.9$.

6.2.7 **(a)** H_0 should be rejected if $\dfrac{\bar{y}-12.6}{0.4/\sqrt{30}}$ is either ≤ -1.96 or ≥ 1.96. But $\bar{y} = 12.76$ and $z = 2.19$, suggesting that the machine should be readjusted.
(b) The test assumes that the y_i's constitute a random sample from a normal distribution. Graphed, a histogram of the 30 y_i's shows a mostly bell-shaped pattern. There is no reason to suspect that the normality assumption is not being met.

6.2.8 **(a)** Obs. $z = -1.61$, so P-value $= P(Z \le -1.61) = 0.0537$.
(b) Obs. $z = 2.75$, so P-value $= P(Z \le -2.75) + P(Z \ge 2.75) = 0.0030 + 0.0030 = 0.0060$.
(c) Obs. $z = 1.17$, so P-value $= P(Z \ge 1.17) = 0.1210$.
Yes, all the P-values agree with the decisions reached in Question 6.2.1.

6.2.9 P-value $= P(Z \le -0.92) + P(Z \ge 0.92) = 0.3576$; H_0 would be rejected if α had been set at any value greater than or equal to 0.3576.

6.2.10 Let μ = true average blood pressure when taking statistics exams. Test H_0: $\mu = 120$ versus H_1: $\mu > 120$. Given that $\sigma = 12$, $n = 50$ and $\bar{y} = 125.2$, $z = \dfrac{125.2 - 120}{12/\sqrt{50}} = 3.06$. The corresponding P-value is approximately 0.001 (= $P(Z \ge 3.06)$), so H_0 would be rejected for any usual choice of α.

6.2.11 H_0 should be rejected if $\dfrac{\bar{y} - 145.75}{9.50/\sqrt{25}}$ is either ≤ -1.96 or ≥ 1.96. Here, $\bar{y} = 149.75$ and $z = 2.10$, so the difference between \$145.75 and \$149.75 <u>is</u> statistically significant.

Section 6.3: Testing Binomial Data—H_0: $p = p_o$

6.3.1 **(a)** Given that the technique worked $k = 24$ times during the $n = 52$ occasions it was tried,
$z = \dfrac{24 - 52(0.40)}{\sqrt{52(0.40)(0.60)}} = 0.91$. The latter is not larger than $z_{.05} = 1.64$, so H_0: $p = 0.40$ would not be rejected at the $\alpha = 0.05$ level. These data do not provide convincing evidence that transmitting predator sounds helps to reduce the number of whales in fishing waters.

(b) P-value $= P(Z \ge 0.91) = 0.1814$; H_0 would be rejected for any $\alpha \ge 0.1814$.

6.3.2 Let $p = P(A/\text{HeJ}$ mouse is right-pawed). Test H_0: $p = 0.67$ versus H_1: $p \ne 0.67$. For $\alpha = 0.05$, H_0 should be rejected if z is either ≤ -1.96 or ≥ 1.96. Here, $n = 35$ and k = number of right-pawed HeJ mice = 18, so $z = \dfrac{18 - 35(0.67)}{\sqrt{35(0.67)(0.33)}} = -1.96$, implying that H_0 should be rejected.

6.3.3 Let $p = P$(current supporter is male). Test H_0: $p = 0.65$ versus H_1: $p < 0.65$. Since $n = 120$ and k = number of male supporters = 72, $z = \dfrac{72 - 120(0.65)}{\sqrt{120(0.65)(0.35)}} = -1.15$, which is not less than or equal to $-z_{.05}$ (= -1.64), so H_0: $p = 0.65$ would not be rejected.

6.3.4 The null hypothesis would be rejected if $z = \dfrac{k - 200(0.45)}{\sqrt{200(0.45)(0.55)}} \ge 1.08$ (= $z_{.14}$). For that to happen, $k \ge 200(0.45) + 1.08 \cdot \sqrt{200(0.45)(0.55)} \doteq 98$.

6.3.5 Let $p = P(Y_i \le 0.69315)$. Test H_0: $p = \frac{1}{2}$ versus H_1: $p \ne \frac{1}{2}$. Given that $k = 26$ and $n = 60$, P-value $= P(X \le 26) + P(X \ge 34) = 0.3030$.

6.3.6 Let $p = P$(person dies if month preceding birthmonth). Test H_0: $p = \frac{1}{12}$ versus H_1: $p < \frac{1}{12}$. Given that $\alpha = 0.05$, H_0 should be rejected if $z \le -1.64$. In this case, $z = \frac{16 - 348(1/12)}{\sqrt{348(1/12)(11/12)}}$ $= -2.52$, which suggests that people do not necessarily die randomly with respect to the month in which they were born. More specifically, there appears to be a tendency to "postpone" dying until the next birthday has passed.

6.3.7 Reject H_0 if $k \ge 4$ gives $\alpha = 0.50$; reject H_0 if $k \ge 5$ gives $\alpha = 0.23$; reject H_0 if $k \ge 6$ gives $\alpha = 0.06$; reject H_0 if $k \ge 7$ gives $\alpha = 0.01$.

6.3.8 Let A_1 be the event that "$k \ge 8$" is the rejection region being used, and let A_2 be the event that "$k = 9$" is the rejection region being used. Define B to be the event that H_0: $p = 0.6$ is rejected. From Theorem 2.4.1, $P(\text{b}) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2)$. But $P(B|A_1) = 0.060466 + 0.010078$ $= 0.070544$ and $P(B|A_2) = 0.010078$. If p denotes the probability that A_1 occurs, $0.05 =$ desired α $= 0.070544 \cdot p + 0.010078 \cdot (1 - p)$, which implies that $p = 0.66$. It follows that the probability of rejecting H_0 will be 0.05 if the "$k \ge 8$" decision rule is used 66% of the time and the "$k = 9$" decision rule is used the remaining 34% of the time.

6.3.9 **(a)** $\alpha = P(\text{reject } H_0|H_0 \text{ is true}) = P(X \le 3|p = 0.75) = \sum_{k=0}^{3}\binom{7}{k}(0.75)^k(0.25)^{7-k} = 0.07$

(b)

p	$P(X \le 3\|p)$
0.75	0.07
0.65	0.20
0.55	0.39
0.45	0.61
0.35	0.80
0.25	0.93
0.15	0.99

Section 6.4: Type I and Type II Errors

6.4.1 As described in Example 6.2.1, H_0: $\mu = 494$ is to be tested against H_1: $\mu \ne 494$ using ± 1.96 as the $\alpha = 0.05$ cutoffs. That is, H_0 is rejected if $\frac{\bar{y} - 494}{124/\sqrt{86}} \le -1.96$ or if $\frac{\bar{y} - 494}{124/\sqrt{86}} \ge 1.96$. Equivalently, the null hypothesis is rejected if $\bar{y} \le 467.8$ or if $\bar{y} \ge 520.2$. Therefore,

$1 - \beta = P(\text{reject } H_0|\ \mu = 500) = P(\bar{Y} \le 467.8|\mu = 500) + P(\bar{Y} \ge 520.2|\mu = 500)$

$= P\left(Z \le \frac{467.8 - 500}{124/\sqrt{86}}\right) + P\left(Z \ge \frac{520.2 - 500}{124/\sqrt{86}}\right) = P(Z \le -2.41) + P(Z \ge 1.51)$

$= 0.0080 + 0.0655 = 0.0735$.

6.4.2 When $\alpha = 0.10$, H_0 is rejected if $\dfrac{\bar{y} - 25.0}{2.4/\sqrt{30}} \ge 1.28$ ($= z_{.10}$). Solving for $\bar{y}$ shows that the decision rule can be re-expressed as "Reject H_0 if $\bar{y} \ge 25.0 + 1.28 \cdot \dfrac{2.4}{\sqrt{30}} = 25.561$."

6.4.3 The null hypothesis in Question 6.2.2 is rejected if $\bar{y}$ is either ≤ 89.0 or ≥ 101.0. Suppose $\mu = 90$. Since $\sigma = 15$ and $n = 22$, $1 - \beta = P(\bar{Y} \le 89.0) + P(\bar{Y} \ge 101.0)$

$= P\left(Z \le \dfrac{89.0 - 90}{15/\sqrt{22}}\right) + P\left(Z \ge \dfrac{101.0 - 90}{15/\sqrt{22}}\right) = P(Z \le -0.31) + P(Z \ge 3.44) = 0.3783 + 0.0003$

$= 0.3786$.

6.4.4 For $n = 16$, $\sigma = 4$, and $\alpha = 0.05$, H_0: $\mu = 60$ should be rejected in favor of a two-sided H_1 if either $\bar{y} \le 60 - 1.96 \cdot \dfrac{4}{\sqrt{16}} = 58.04$ or $\bar{y} \ge 60 + 1.96 \cdot \dfrac{4}{\sqrt{16}} = 61.96$. Then, for arbitrary μ, $1 - \beta = P(\bar{Y} \le 58.04 \mid \mu) + P(\bar{Y} \ge 61.96 \mid \mu)$. Selected values of ($\mu$, $1 - \beta$) that would lie on the power curve are listed in the accompanying table.

μ	$1 - \beta$
56	0.9793
57	0.8508
58	0.5160
59	0.1700
60	0.05 ($=\alpha$)
61	0.1700
62	0.5160
63	0.8508
64	0.9793

6.4.5 H_0 should be rejected if $z = \dfrac{\bar{y} - 240}{50/\sqrt{25}} \le -2.33$ or, equivalently, if $\bar{y} \le 240 - 2.33 \cdot \dfrac{50}{\sqrt{25}} = 216.7$. Suppose $\mu = 220$. Then $\beta = P(\text{accept } H_0 | H_1 \text{ is true}) = P(\bar{Y} > 216.7 \mid \mu = 220)$

$= P\left(Z > \dfrac{216.7 - 220}{50/\sqrt{25}}\right) = P(Z > -0.33) = 0.6293$.

6.4.6 **(a)** In order for α to be 0.07, $P(60 - \bar{y}^* \le \bar{Y} \le 60 + \bar{y}^* | \mu = 60) = 0.07$. Equivalently,

$P\left(\dfrac{60 - \bar{y}^* - 60}{8.0/\sqrt{36}} \le \dfrac{\bar{Y} - 60}{8.0/\sqrt{36}} \le \dfrac{60 + \bar{y}^* - 60}{8.0/\sqrt{36}}\right) = P(-0.75\bar{y}^* \le Z \le 0.75\bar{y}^*) = 0.07$.

But $P(-0.09 \le Z \le 0.09) = 0.07$, so $0.75\bar{y}^* = 0.09$, which implies that $\bar{y}^* = 0.12$.

(b) $1 - \beta = P(\text{reject } H_0 | H_1 \text{ is true}) = P(59.88 \le \bar{Y} \le 60.12 | \mu = 62)$

$= P\left(\dfrac{59.88 - 62}{8.0/\sqrt{36}} \le Z \le \dfrac{60.12 - 62}{8.0/\sqrt{36}}\right) = P(-1.59 \le Z \le -1.41) = 0.0793 - 0.0559$

$= 0.0234$.

(c) For $\alpha = 0.07$, $\pm z_{\alpha/2} = \pm 1.81$ and H_0 should be rejected if $\bar{y}$ is either

1) $\leq 60 - 1.81 \cdot \dfrac{8.0}{\sqrt{36}} = 57.59$ or 2) $\geq 60 + 1.81 \cdot \dfrac{8.0}{\sqrt{36}} = 62.41$. Suppose $\mu = 62$. Then

$1 - \beta = P(\bar{Y} \leq 57.59 | \mu = 62) + P(\bar{Y} \geq 62.41 | \mu = 62) = P(Z \leq -3.31) + P(Z \geq 0.31)$
$= 0.0005 + 0.3783 = 0.3788$.

6.4.7 For $\alpha = 0.10$, H_0: $\mu = 200$ should be rejected if $\bar{y} \leq 200 - 1.28 \cdot \dfrac{15.0}{\sqrt{n}}$. Also,

$$1 - \beta = P\left(\bar{Y} \leq 200 - 1.28 \cdot \frac{15.0}{\sqrt{n}} \middle| \mu = 197\right) = 0.75, \text{ so } P\left(\frac{200 - 1.28 \cdot 15.0/\sqrt{n} - 197}{15.0/\sqrt{n}}\right) = 0.75.$$

But $P(Z \leq 0.67) = 0.75$, implying that $\dfrac{200 - 1.28 \cdot 15.0/\sqrt{n} - 197}{15.0/\sqrt{n}} = 0.67$. It follows that the smallest n satisfying the conditions placed on α and $1 - \beta$ is 95.

6.4.8 If $n = 45$, H_0 will be rejected when $\bar{y}$ is either (1) $\leq 10 - 1.96 \cdot \dfrac{4}{\sqrt{45}} = 8.83$ or

(2) $\geq 10 + 1.96 \cdot \dfrac{4}{\sqrt{45}} = 11.17$. When $\mu = 12$, $\beta = P(\text{accept } H_o | H_1 \text{ is true})$

$$= P(8.83 \leq \bar{Y} \leq 11.17 | \mu = 12) = P\left(\frac{8.83 - 12}{4/\sqrt{45}} \leq Z \leq \frac{11.17 - 12}{4/\sqrt{45}}\right) = P(-5.32 \leq Z \leq -1.39) = 0.0823.$$

It follows that a sample of size $n = 45$ is sufficient to keep β smaller than 0.20 when $\mu = 12$.

6.4.9 Since H_1 is one-sided, H_0 is rejected when $\bar{y} \geq 30 + z_\alpha \cdot \dfrac{9}{\sqrt{16}}$. Also, $1 - \beta$ = power

$$= P\left(\bar{Y} \geq 30 + z_\alpha \cdot \frac{9}{\sqrt{16}} \middle| \mu = 34\right) = 0.85. \text{ Therefore, } 1 - \beta = P\left(Z \geq \frac{30 + z_\alpha \cdot 9/\sqrt{16} - 34}{9/\sqrt{16}}\right)$$

$= 0.85$. But $P(Z \geq -1.04) = 0.85$, so $\dfrac{30 + z_\alpha \cdot 9/\sqrt{16} - 34}{9/\sqrt{16}} = -1.04$, implying that $z_\alpha = 0.74$.

Therefore, $\alpha = 0.23$.

6.4.10 **(a)** $P(\text{Type I error}) = P(\text{reject } H_0 | H_0 \text{ is true}) = P(Y \geq 3.20 | \lambda = 1) = \int_{3.20}^{\infty} e^{-y} dy = 0.04$.

(b) $P(\text{Type II error}) = P(\text{accept } H_0 | H_1 \text{ is true}) = P\left(Y < 3.20 \middle| \lambda = \dfrac{4}{3}\right) = \int_0^{3.20} \dfrac{3}{4} e^{-3y/4} dy = \int_0^{2.4} e^{-u} du$

$= 0.91$.

6.4.11 In this context, α is the proportion of incorrect decisions made on innocent suspects—that is, $\dfrac{9}{140}$, or 0.064. Similarly, β is the proportion of incorrect decisions made on guilty suspects—here, $\dfrac{15}{140}$, or 0.107. A Type I error (convicting an innocent defendant) would be considered more serious than a Type II error (acquitting a guilty defendant).

6.4.12 Let X = number of white chips in sample. Then $\alpha = P(X \geq 2|\text{urn has 5 white and 5 red})$ $= \binom{5}{2}\binom{5}{1}\Big/\binom{10}{3} + \binom{5}{3}\binom{5}{0}\Big/\binom{10}{3} = \frac{1}{2}$. When the urn is 60% white,

$\beta = P(X \leq 1|\text{urn has 6 white and 4 red}) = \binom{6}{0}\binom{4}{3}\Big/\binom{10}{3} + \binom{6}{1}\binom{4}{2}\Big/\binom{10}{3} = \frac{1}{3}$. When the urn is 70% white, $\beta = P(X \leq 1|\text{urn has 7 white and 3 red}) = \binom{7}{0}\binom{3}{3}\Big/\binom{10}{3} + \binom{7}{1}\binom{3}{2}\Big/\binom{10}{3} = \frac{11}{60}$.

6.4.13 For a uniform pdf, $f_{Y_{\max}}(y) = \frac{5}{\theta}\left(\frac{y}{\theta}\right)^4$, $0 \leq y \leq \theta$ when $n = 5$.

Therefore, $\alpha = P(\text{reject } H_0|H_0 \text{ is true}) = P(Y_{\max} \geq k|\theta = 2) = \int_k^2 \frac{5}{2}\left(\frac{y}{2}\right)^4 dy = 1 - \frac{k^5}{32}$. For α to be 0.05, $k = 1.98$.

6.4.14 Level of significance = $\alpha = P(\text{reject } H_0|H_0 \text{ is true}) = P(Y \geq 0.90|f_Y(y) = 2y, 0 \leq y \leq 1)$ $= \int_{0.90}^1 2y\,dy = 0.19$.

6.4.15 $\beta = P(\text{accept } H_0|H_1 \text{ is true}) = P(X \leq n-1|p) = 1 - P(X = n|p) = 1 - \binom{n}{n}p^n(1-p)^0 = 1 - p^n$.

When $\beta = 0.05$, $p = \sqrt[n]{0.95}$.

6.4.16 If H_0 is true, $X = X_1 + X_2$ has a binomial distribution with $n = 6$ and $p = \frac{1}{2}$.

Therefore, $\alpha = P(\text{reject } H_0|H_0 \text{ is true}) = P\left(X \geq 5\middle| p = \frac{1}{2}\right) = \sum_{k=5}^{6}\binom{6}{k}\left(\frac{1}{2}\right)^k\left(1 - \frac{1}{2}\right)^{6-k}$

$= 7/2^6 = 0.11$.

6.4.17 $1 - \beta = P(\text{reject } H_0|H_1 \text{ is true}) = P\left(Y \leq \frac{1}{2}\middle|\theta\right) = \int_0^{1/2}(1+\theta)y^\theta dy = y^{\theta+1}\Big|_0^{1/2} = \left(\frac{1}{2}\right)^{\theta+1}$

6.4.18 **(a)** $\alpha = P(\text{reject } H_0|H_0 \text{ is true}) = P(X \leq 2|\lambda = 6) = \sum_{k=0}^{2}\frac{e^{-6}6^k}{k!} = 0.062$.

(b) $\beta = P(\text{accept } H_0|H_1 \text{ is true}) = P(X \geq 3|\lambda = 4) = 1 - P(X \leq 2|\lambda = 4) = 1 - \sum_{k=0}^{2}\frac{e^{-4}4^k}{k!}$

$= 1 - 0.238 = 0.762$

6.4.19 $P(\text{Type II error}) = \beta = P(\text{accept } H_0|H_1 \text{ is true}) = P\left(X \leq 3\middle| p = \frac{1}{2}\right) = \sum_{k=1}^{3}\left(1 - \frac{1}{2}\right)^{k-1}\cdot\frac{1}{2} = \frac{7}{8}$

6.4.20 $\beta = P(\text{accept } H_0|H_1 \text{ is true}) = P(Y < \ln 10|\lambda) = \int_0^{\ln 10} \lambda e^{-\lambda y} dy = 1 - e^{-\lambda \ln 10} = 1 - 10^{-\lambda}$

6.4.21 $\alpha = P(\text{reject } H_0|H_0 \text{ is true}) = P(Y_1 + Y_2 \le k|\theta = 2)$. When H_0 is true, Y_1 and Y_2 are uniformly distributed over the square defined by $0 \le Y_1 \le 2$ and $0 \le Y_2 \le 2$, so the joint pdf of Y_1 and Y_2 is a plane parallel to the Y_1Y_2-axis at height $\frac{1}{4}\left(= f_{Y_1}(y_1)\cdot f_{Y_2}(y_2) = \frac{1}{2}\cdot\frac{1}{2}\right)$. By geometry, α is the volume of the triangular wedge in the lower left-hand corner of the square over which Y_1 and Y_2 are defined. The hypotenuse of the triangle in the Y_1Y_2-plane has the equation $y_1 + y_2 = k$. Therefore, α = area of triangle × height of wedge = $\frac{1}{2}\cdot k\cdot k\cdot\frac{1}{4} = k^2/8$.
For α to be 0.05, $k = \sqrt{0.4} = 0.63$.

6.4.22 $\alpha = P(\text{reject } H_0|H_0 \text{ is true}) = P(Y_1Y_2 \le k^*|\theta = 2)$. If $\theta = 2$, the joint pdf of Y_1 and Y_2 is the horizontal plane $f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{4}$, $0 \le y_1 \le 2$, $0 \le y_2 \le 2$. Therefore, $\alpha = P(Y_1Y_2 \le k^*|\theta = 2)$
$= 2\cdot\frac{k^*}{2}\cdot\frac{1}{4} + \int_{k^*/2}^{2}\int_0^{k^*/y_1}\frac{1}{4}dy_2dy_1 = \frac{k^*}{4} + \int_{k^*/2}^{2}\frac{k^*}{4y_1}dy_1 = \frac{k^*}{4} + \left(\frac{k^*}{4}\ln y_1\Big|_{k^*/2}^{2}\right) =$
$\frac{k^*}{4} + \frac{k^*}{4}\ln 2 - \frac{k^*}{4}\ln\frac{k^*}{2}$. By trial and error, $k^* = 0.087$ makes $\alpha = 0.05$.

Section 6.5: A Notion of Optimality: The Generalized Likelihood Ratio

6.5.1 $L(\hat{\omega}) = \prod_{i=1}^{n}(1-p_0)^{k_i-1}p_0 = p_0^n(1-p_0)^{\sum_{i=1}^{n}k_i-n} = p_0^n(1-p_0)^{k-n}$, where $k = \sum_{i=1}^{n}k_i$. From Case Study 5.2.1, the maximum likelihood estimate for p is $p_e = \frac{n}{k}$. Therefore, $L(\hat{\Omega}) = \left(\frac{n}{k}\right)^n\left(1-\frac{n}{k}\right)^{k-n}$, and the generalized likelihood ratio for testing H_0: $p = p_0$ versus
H_1: $p \ne p_0$ is the quotient $L(\hat{\omega})/L(\hat{\Omega})$.

6.5.2 Let $y = \sum_{i=1}^{10}y_i$. Then $L(\hat{\omega}) = \prod_{i=1}^{10}\lambda_o e^{-\lambda_o y_i} = \lambda_0^{10}e^{-\lambda_0\sum_{i=1}^{10}y_i} = \lambda_0^{10}e^{-\lambda_0 y}$.
Also, $L(\lambda) = \prod_{i=1}^{10}\lambda e^{-\lambda y_i} = \lambda^{10}e^{-\lambda y}$, so $\ln L(\lambda) = 10\ln\lambda - \lambda y$ and $\frac{d\ln L(\lambda)}{d\lambda} = \frac{10}{\lambda} - y$. Setting the latter equal to 0 implies that the maximum likelihood estimate for λ is $\lambda_e = \frac{10}{y}$. Therefore,
$L(\hat{\Omega}) = \left(\frac{10}{y}\right)^{10}e^{-\left(\frac{10}{y}\right)y} = (10/y)^{10}e^{-10}$. The generalized likelihood ratio, then, is the quotient
$\lambda_0^{10}e^{-\lambda_0 y}/(10/y)^{10}e^{-10} = (\lambda_0 e/10)^{10}y^{10}e^{-\lambda_0 y}$. It follows that H_0 should be rejected if $\lambda =$
$y^{10}e^{-\lambda_0 y} \le \lambda^*$, where λ^* is chosen so that $\int_0^{\lambda^*} f_\lambda(\lambda|H_0 \text{ is true})d\lambda = 0.05$.

6.5.3 $L(\hat{\omega}) = \prod_{i=1}^{n}(1/\sqrt{2\pi})e^{-\frac{1}{2}(y_i-\mu_0)^2} = (2\pi)^{-n/2}e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu_0)^2}$. Since $\bar{y}$ is the maximum likelihood estimate for μ (recall the first derivative taken in Example 5.2.4), $L(\hat{\Omega}) = (2\pi)^{-n/2}e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\bar{y})^2}$. Here the generalized likelihood ratio reduces to $\lambda = L(\hat{\omega})/L(\hat{\Omega}) = e^{-\frac{1}{2}((\bar{y}-\mu_0)/(1/\sqrt{n}))^2}$. The null hypothesis should be rejected if $e^{-\frac{1}{2}((\bar{y}-\mu_0)/(1/\sqrt{n}))^2} \le \lambda^*$ or, equivalently, if $|(\bar{y}-\mu_0)|/(1/\sqrt{n}) > \lambda^{**}$, where values for λ^{**} come from the standard normal pdf, $f_Z(z)$.

6.5.4 To test H_0: $\mu = \mu_0$ versus H_1: $\mu = \mu_1$, the "best" critical region would consist of all those samples for which $\prod_{i=1}^{n}(1/\sqrt{2\pi})^n e^{-\frac{1}{2}(y_i-\mu_0)^2} / \prod_{i=1}^{n}(1/\sqrt{2\pi})^n e^{-\frac{1}{2}(y_i-\mu_1)^2} \le k$. Equivalently, H_0 should be rejected if $\sum_{i=1}^{n}(y_i-\mu_0)^2 - \sum_{i=1}^{n}(y_i-\mu_1)^2 > 2\ln k$. Simplified, the latter becomes $2(\mu_1-\mu_0)\sum_{i=1}^{n}y_i > 2\ln k + n\left(\mu_1^2-\mu_0^2\right)$. Consider the case where $\mu_1 < \mu_0$. Then $\mu_1 - \mu_0 < 0$, and the decision rule reduces to rejecting H_0 when $\bar{y} < \dfrac{2\ln k + n\left(\mu_1^2-\mu_0^2\right)}{2n(\mu_1-\mu_0)}$.

6.5.5 **(a)** $\lambda = \left(\dfrac{1}{2}\right)^n/[(x/n)^x(1-x/n)^{n-x}] = 2^{-n}x^{-x}(n-x)^{x-n}n^n$. Rejecting H_0 when $0 < \lambda \le \lambda^*$ is equivalent to rejecting H_0 when $x\ln x + (n-x)\ln(n-x) \ge \lambda^{**}$.

(b) By inspection, $x\ln x + (n-x)\ln(n-x)$ is symmetric in x. Therefore, the left-tail and right-tail critical regions will be equidistant from $p = \dfrac{1}{2}$, which implies that H_0 should be rejected if $\left|x - \dfrac{1}{2}\right| \ge k$, where k is a function of α.

6.5.6 If $\hat{\theta}$ is a sufficient statistic for θ, it follows from Theorem 5.6.1 that $L(\theta) = g(\hat{\theta};\theta)\cdot b(w_1, w_2, \ldots, w_n)$, where $b(w_1, w_2, \ldots, w_n)$ is does not involve θ. Therefore, the choice of θ that maximizes $L(\theta)$ is necessarily the same θ that maximizes $g(\hat{\theta};\theta)$, which, in turn, implies that the critical regions of likelihood ratio tests are functions of sufficient statistics.

Chapter 7: Inferences Based on the Normal Distribution

Section 7.3: Deriving the Distribution of $\dfrac{\overline{Y}-\mu}{S/\sqrt{n}}$

7.3.1 Clearly, $f_U(u) > 0$ for all $u > 0$. To verify that $f_U(u)$ is a pdf requires proving that $\int_0^\infty f_U(u)du = 1$. But $\int_0^\infty f_U(u)du = \dfrac{1}{\Gamma(n/2)}\int_0^\infty \dfrac{1}{2^{n/2}}u^{n/2-1}e^{-u/2}du =$

$= \dfrac{1}{\Gamma(n/2)}\int_0^\infty \left(\dfrac{u}{2}\right)^{n/2-1} e^{-u/2}(du/2) = \dfrac{1}{\Gamma\left(\dfrac{n}{2}\right)}\int_0^\infty v^{n/2-1}e^{-v}dv$, where $v = \dfrac{u}{2}$ and $dv = \dfrac{du}{2}$.

By definition, $\Gamma\left(\dfrac{n}{2}\right) = \int_0^\infty v^{n/2-1}e^{-v}dv$. Thus, $\int_0^\infty f_U(u)dy = \dfrac{1}{\Gamma(n/2)}\cdot\Gamma\left(\dfrac{n}{2}\right) = 1$.

7.3.2 Substituting $\dfrac{n}{2}$ and $\dfrac{1}{2}$ for r and λ, respectively, in the moment-generating function for a gamma pdf gives $M_{\chi_n^2}(t) = (1-2t)^{-n/2}$. Also, $M_{\chi_n^2}^{(1)}(t) = (-n/2)\ (1-2t)^{-n/2-1}(-2) = n(1-2t)^{-n/2-1}$ and $M_{\chi_n^2}^{(2)}(t) = \left(-\dfrac{n}{2}-1\right)(n)(1-2t)^{-n/2-2}(-2) = (n^2+2n)\cdot(1-2t)^{-n/2-2}$, so $M_{\chi_n^2}^{(1)}(0) = n$ and $M_{\chi_n^2}^{(2)}(0) = n^2+2n$. Therefore, $E(\chi_n^2) = n$ and $\text{Var}\left(\chi_n^2\right) = n^2+2n-n^2 = 2n$.

7.3.3 If $\mu = 50$ and $\sigma = 10$, $\sum_{i=1}^{3}\left(\dfrac{Y_i-50}{10}\right)^2$ should have a χ_3^2 distribution, implying that the numerical value of the sum is likely to be between, say, $\chi^2_{.025,3}$ (=0.216) and $\chi^2_{.975,3}$ (= 9.348). Here, $\sum_{i=1}^{3}\left(\dfrac{Y_i-50}{10}\right)^2 = \left(\dfrac{65-50}{10}\right)^2 + \left(\dfrac{30-50}{10}\right)^2 + \left(\dfrac{55-50}{10}\right)^2 = 6.50$, so the data are not inconsistent with the hypothesis that the Y_i's are normally distributed with $\mu = 50$ and $\sigma = 10$.

7.3.4 Let $Y = \dfrac{(n-1)S^2}{\sigma^2}$. Then Var($Y$) = $\text{Var}\left(\chi^2_{n-1}\right) = 2(n-1) = \dfrac{(n-1)^2\,\text{Var}(S^2)}{\sigma^4}$. If follows that $\text{Var}(S^2) = \dfrac{2\sigma^4}{n-1}$.

7.3.5 Since $E(S^2) = \sigma^2$, it follows from Chebyshev's inequality that $P(|S^2-\sigma^2| < \varepsilon) > 1 - \dfrac{\text{Var}(S^2)}{\varepsilon^2}$. But $\text{Var}(S^2) = \dfrac{2\sigma^4}{n-1} \to 0$ as $n \to \infty$. Therefore, S^2 is consistent for σ^2.

7.3.6 Let $Y = \chi^2_{200}$. Then $\dfrac{Y-200}{\sqrt{400}} \doteq Z$, in which case $P\left(\dfrac{Y-200}{\sqrt{400}} \le -0.25\right) \doteq 0.40$. Equivalently, $Y \le 200 - 0.25\sqrt{400} = 195$, implying that $\chi^2_{.40,200} \doteq 195$.

7.3.7 **(a)** 0.983 **(b)** 0.132 **(c)** 9.00

7.3.8 $P\left(2.51 < \dfrac{V/7}{U/9} < 3.29\right) = P(2.51 < F_{7,9} < 3.29) = P(F_{7,9} < 3.29) - P(F_{7,9} \le 2.51) = 0.95 - 0.90$ $= 0.05$. But $P(3.29 < F_{7,9} < 4.20) = 0.975 - 0.95 = 0.025$.

7.3.9 **(a)** 6.23 **(b)** 0.65 **(c)** 9 **(d)** 15 **(e)** 2.28

7.3.10 Since the samples are independent and $S^2/\sigma^2 = \chi^2_{n-1}/(n-1)$, it follows that $\dfrac{S_1^2}{\sigma^2}/\dfrac{S_2^2}{\sigma^2} = S_1^2/S_2^2$ has an $F_{n-1,n-1}$ distribution. As n increases, the distributions of the unbiased estimators S_1^2 and S_2^2 become increasingly concentrated around σ^2 (recall Question 7.3.4), implying that F ratios converge to 1 as the two sample sizes get large.

7.3.11 $F = \dfrac{V/m}{U/n}$, where U and V are independent χ^2 random variables with m and n degrees of freedom, respectively. Then $\dfrac{1}{F} = \dfrac{U/n}{V/m}$, which implies that $\dfrac{1}{F}$ has an F distribution with n and m degrees of freedom.

7.3.12 If $P(a \le F_{m,n} \le b) = q$, then $P\left(a \le \dfrac{1}{F_{n,m}} \le b\right) = q = P\left(\dfrac{1}{b} \le F_{n,m} \le \dfrac{1}{a}\right)$. From Appendix Table A.4, $P(0.052 \le F_{2,8} \le 4.46) = 0.90$. Also, $P(0.224 \le F_{8,2} \le 19.4) = 0.90$. But $\dfrac{1}{4.46} = 0.224$ and $\dfrac{1}{0.052} = 19.23 \doteq 19.4$.

7.3.13 To show that $f_{T_n}(t)$ converges to $f_Z(t)$ requires proving that $\left(1+\dfrac{t^2}{n}\right)^{-(n+1)/2}$ converges to $e^{-t^2/2}$ and $\dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}$ converges to $1/\sqrt{2\pi}$. To verify the first limit, write $\left(1+\dfrac{t^2}{n}\right)^{-(n+1)/2}$

$= \left\{\left(1+\dfrac{1}{n/t^2}\right)^{n/t^2}\right\}^{-t^2/2} \cdot \left(1+\dfrac{t^2}{n}\right)^{-1/2}$. As n gets large, the last factor approaches 1 and the product approaches $(e^1)^{-t^2/2} = e^{-t^2/2}$. Also, for large n, $n! \doteq \sqrt{2\pi n}\,n^n e^{-n}$. Equivalently, $\Gamma(r) \doteq \sqrt{\dfrac{2\pi}{r}}\left(\dfrac{r}{e}\right)^r$ if r is large. An application of the latter equation shows that $\Gamma\left(\dfrac{n+1}{2}\right)\Big/\Gamma\left(\dfrac{n}{2}\right)$ converges to $\sqrt{\dfrac{n}{2}}$, which means that the constant in $f_{T_n}(t)$ converges to $1/\sqrt{2\pi}$, the constant in the standard normal pdf.

7.3.14 $\int_0^\infty \frac{1}{1+x^2}dx$ is the integral of the variable portion of a T_1 pdf over the upper half of its range.

Since $\int_0^\infty \frac{\Gamma(1)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)(1+t^2)}dt = \frac{1}{2}$, it follows that $\int_0^\infty \frac{1}{1+x^2}dx = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(1)} = \frac{\pi}{2}$.

7.3.15 Let T be a Student t random variable with n degrees of freedom. Then
$E(T^{2k}) = C\int_{-\infty}^{\infty} t^{2k}\frac{1}{\left(1+\frac{t^2}{n}\right)^{(n+1)/2}}dt$, where C is the product of the constants appearing in the definition of the Student t pdf. The change of variable $y = t/\sqrt{n}$ results in the integral

$E(T^{2k}) = C^*\int_{-\infty}^{\infty} y^{2k}\frac{1}{\left(1+y^2\right)^{(n+1)/2}}dy$ for some constant C^*.

Because of the symmetry of the integrand, $E(T^{2k})$ is finite if the integral $\int_0^\infty \frac{y^{2k}}{\left(1+y^2\right)^{(n+1)/2}}dy$ is finite.

But $\int_0^\infty \frac{y^{2k}}{\left(1+y^2\right)^{(n+1)/2}}dy < \int_0^\infty \frac{(1+y^2)^k}{\left(1+y^2\right)^{(n+1)/2}}dy = \int_0^\infty \frac{1}{\left(1+y^2\right)^{(n+1)/2-k}}dy = \int_0^\infty \frac{1}{\left(1+y^2\right)^{\frac{n-2k}{2}+\frac{1}{2}}}dy$

To apply the hint, take $\alpha = 2$ and $\beta = \frac{n-2k}{2}+\frac{1}{2}$. Then $2k < n$, $\beta > 0$, and $\alpha\beta > 1$, so the the integral is finite.

Section 7.4: Drawing Inferences About μ

7.4.1 **(a)** 0.15 **(b)** 0.80 **(c)** 0.85 **(d)** 0.99 – 0.15 = 0.84

7.4.2 **(a)** 2.508 **(b)** 1.079 **(c)** 1.7056 **(d)** 4.3027

7.4.3 Both differences represent intervals associated with 5% of the area under $f_{T_n}(t)$. Because the pdf is closer to the horizontal axis the further t is away from 0, the difference $t_{.05,n} - t_{.10,n}$ is the larger of the two.

7.4.4 Since $\frac{\overline{Y}-27.6}{S/\sqrt{9}}$ is a Student t random variable with 8 df, $P\left(-1.397 \le \frac{\overline{Y}-27.6}{S/\sqrt{9}} \le 1.397\right) = 0.80$ and $P\left(-1.8595 \le \frac{\overline{Y}-27.6}{S/\sqrt{9}} \le 1.8595\right) = 0.90$ (see Appendix Table A.2).

7.4.5 $P\left(\left|\frac{\overline{Y}-15.0}{S/\sqrt{11}}\right| \geq k\right) = 0.05$ implies that $P\left(-k \leq \frac{\overline{Y}-15.0}{S/\sqrt{11}} \leq k\right) = 0.95$. But $\frac{\overline{Y}-15.0}{S/\sqrt{11}}$ is a Student t random variable with 10 df. From Appendix Table A.2, $P(-2.2281 \leq T_{10} \leq 2.2281)$ $= 0.95$, so $k = 2.2281$.

7.4.6 $P(90.6 - k(S) \leq \overline{Y} \leq 90.6 + k(S)) = 0.99 =$

$$P\left(\frac{90.6-k(S)-90.6}{S/\sqrt{20}} \leq \frac{\overline{Y}-90.6}{S/\sqrt{20}} \leq \frac{90.6+k(S)-90.6}{S/\sqrt{20}}\right) = P\left(\frac{k(S)}{S/\sqrt{20}} \leq T_{19} \leq \frac{k(S)}{S/\sqrt{20}}\right) =$$

$P(-2.8609 \leq T_{19} \leq 2.8609)$, so $\frac{k(S)}{S/\sqrt{20}} = 2.8609$, implying that $k(S) = \frac{2.8609\cdot S}{\sqrt{20}}$.

7.4.7 Since $n = 20$ and the confidence interval has level 90%, $t_{\alpha/2,n-1} = t_{.05,19} = 1.7291$. For these data $\sum_{i=1}^{20} y_i = 20.22$ and $\sum_{i=1}^{20} y_i^2 = 23.014$. Then $\overline{y} = 20.22/20 = 1.011$ and $s =$ $\sqrt{\frac{20(23.014)-(20.22)^2}{20(19)}} = 0.368$. The confidence interval is

$$\left(\overline{y}-t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}, \overline{y}+t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}\right) = \left(1.011-1.7291\frac{0.368}{\sqrt{20}}, 1.011+1.7291\frac{0.368}{\sqrt{20}}\right) = (0.869, 1.153).$$

7.4.8 Given that $n = 16$ and the confidence level is 95%, $t_{\alpha/2,n-1} = t_{.025,15} = 2.1314$.

Here $\sum_{i=1}^{16} y_i = 24{,}256$, so $\overline{y} = \frac{1}{16}(24{,}256) = 1516.0$, and s is given to be 369.02.

The confidence interval is $\left(1516.0-2.1314\frac{369.02}{\sqrt{16}}, 1516.0+2.1314\frac{369.02}{\sqrt{16}}\right)$

$= (\$1319.4, \$1712.6)$.

7.4.9 **(a)** Let μ = true average age at which scientists make their greatest discoveries.

Since $\sum_{i=1}^{12} y_i = 425$ and $\sum_{y=1}^{12} y_i^2 = 15{,}627$, $\overline{y} = \frac{1}{12}(425) = 35.4$ and $s = \sqrt{\frac{12(15{,}627)-(425)^2}{12(11)}}$ $= 7.2$. Also, $t_{\alpha/2,n-1} = t_{.025,11} = 2.2010$, so the 95% confidence interval for μ is the range $\left(35.4 - 2.2010\cdot\frac{7.2}{\sqrt{12}}, 35.4 + 2.2010\cdot\frac{7.2}{\sqrt{12}}\right)$, or (30.8 yrs, 40.0 yrs).

(b) The graph of date versus age shows no obvious patterns or trends. The assumption that μ has remained constant over time is believable.

7.4.10 Given that $n = 61$ and the confidence level is 99%, $t_{\alpha/2,n-1} = t_{.005,60} = 2.6603$.

Then $\overline{y} = \frac{1}{60}(6450) = 105.7$, $s = \sqrt{\frac{61(684{,}900)-(6450)^2}{61(60)}} = 6.94$

The 99% confidence interval for μ is $\left(105.7 - 2.6603 \cdot \frac{6.94}{\sqrt{61}}, 105.7 + 2.6603 \cdot \frac{6.94}{\sqrt{61}}\right)$

= (103.3, 108.1).

7.4.11 For n = 24, $t_{\alpha/2,n-1} = t_{.05,23}$ = 1.7139. For these data $\bar{y}$ = 4645/24 = 193.54 and

$s = \sqrt{\frac{24(959,265) - (4645)^2}{24(23)}}$ = 51.19.

The confidence interval is $\left(193.54 - 1.7139\frac{51.19}{\sqrt{24}}, 193.54 + 1.7139\frac{51.19}{\sqrt{24}}\right)$ = (175.6, 211.4).

The medical and statistical definition of "normal" differ somewhat. There are people with medically norm platelet counts who appear in the population less than 10% of the time.

7.4.12 Given that n = 16, $t_{\alpha/2,n-1} = t_{.025,15}$ = 2.1315, so $\left(\bar{y} - 2.1315 \cdot \frac{s}{\sqrt{16}}, \bar{y} + 2.1315 \cdot \frac{s}{\sqrt{16}}\right)$

= (44.7, 49.9). Therefore, 49.9 – 44.7 = 5.2 = 2(2.1315) · $\frac{s}{\sqrt{16}}$, implying that s = 4.88. Also,

because the confidence interval is centered around the sample mean, $\bar{y} = \frac{44.7 + 49.9}{2}$ = 47.3.

7.4.13 No, because the length of a confidence interval for μ is a function of s, as well as the confidence coefficient. If the sample standard deviation for the second sample were sufficiently small (relative to the sample standard deviation for the first sample), the 95% confidence interval would be shorter than the 90% confidence interval.

7.4.14 The range spanned by, say, a 99% confidence interval for μ would be a reasonable set of values for the company's true average revenue. With n = 9, $\bar{y}$ = \$59,540, and s = \$6,680, the 99%

confidence interval for μ is $\left(59{,}540 - 3.554 \cdot \frac{6{,}860}{\sqrt{9}}, 59{,}540 + 3.554 \cdot \frac{6{,}860}{\sqrt{9}}\right)$

= (\$51,867, \$67,213).

7.4.15 **(a)** 0.95 **(b)** 0.80 **(c)** 0.945 **(d)** 0.95

7.4.16 **(a)** Given that n = 336 and the confidence level is 95%, we use the normal tables and take

$z_{\alpha/2} = z_{.025}$ = 1.96. Then $\bar{y} = \frac{1}{336}(1392.60)$ = 4.14, $s = \sqrt{\frac{336(10{,}518.84) - (1392.6)^2}{336(335)}}$

= 3.764. Theorem 7.4.1 implies that the 95% confidence interval for μ is

$\left(4.14 - 1.96 \cdot \frac{3.764}{\sqrt{336}}, 4.14 + 1.96 \cdot \frac{3.764}{\sqrt{336}}\right)$ = (3.74, 4.54).

(b) The normality assumption is egregiously violated for these data; note that the data's histogram is sharply skewed. Theorem 7.4.1 is *not* appropriate.

7.4.17 Let μ = true average FEV$_1$/VC ratio for exposed workers. Since $\sum_{i=1}^{19} y_i = 14.56$ and $\sum_{i=1}^{19} y_i^2 = 11.2904$, $\bar{y} = \frac{14.56}{19} = 0.766$ and $s = \sqrt{\frac{19(11.2904)-(14.56)^2}{19(18)}} = 0.0859$. To test H_0: $\mu = 0.80$ versus H_1: $\mu < 0.80$ at the $\alpha = 0.05$ level of significance, reject the null hypothesis if $t \le -t_{.05,18} = -1.7341$. But $t = \frac{0.766-0.80}{0.0859/\sqrt{19}} = -1.71$, so we fail to reject H_0.

7.4.18 At the $\alpha = 0.05$ level, H_0: $\mu = 132.4$ should be rejected in favor of H_1:$\mu \ne 132.4$ if $\left|\frac{\bar{y}-132.4}{s/\sqrt{84}}\right| \ge t_{.025,83} = 1.9890$. But $t = \frac{143.8-132.4}{6.9/\sqrt{84}} = 17.4$, making it clear that the skull differences between Etruscans and native Italians are too great to be ascribed to chance. A histogram of the 84 y_i's shows a distinctly bell-shaped pattern, so the normality assumption implicit in the t test appears to be satisfied.

7.4.19 Let μ = true average GMAT increase earned by students taking the review course. The hypotheses to be tested are H_0: $\mu = 40$ versus H_1: $\mu < 40$. Here, $\sum_{i=1}^{15} y_i = 556$ and $\sum_{i=1}^{15} y_i^2 = 20{,}966$, so $\bar{y} = \frac{556}{15} = 37.1$, $s = \sqrt{\frac{15(20{,}966)-(556)^2}{15(14)}} = 5.0$, and $t = \frac{37.1-40}{5.0/\sqrt{15}} = -2.25$. Since $-t_{.05,14} = -1.7613$, H_0 should be rejected at the $\alpha = 0.05$ level of significance, suggesting that the MBAs ′R Us advertisement may be fraudulent.

7.4.20 H_0: $\mu = 0.618$ should be rejected in favor of a two-sided H_1 at the 0.01 level of significance if $|t| \ge t_{.005,33} = 2.7333$. Given that $\bar{y} = 0.6373$ and $s = 0.14139$, the t statistic is $\frac{0.6373-0.618}{0.14139/\sqrt{34}} = 0.80$, so H_0 is not rejected. These data do not rule out the possibility that national flags embrace the Golden Rectangle as an aesthetic standard.

7.4.21 Let u = true average pit depth associated with plastic coating. To test H_0: $\mu = 0.0042$ versus H_1: $\mu < 0.0042$ at the $\alpha = 0.05$ level, we should reject the null hypothesis if $t \le -t_{.05,9} = -1.8331$. For the 10 y_i's, $\bar{y} = \frac{0.0390}{10} = 0.0039$. Also, $s = 0.00383$, so $t = \frac{0.0039-0.0042}{0.00383/\sqrt{10}} = -2.48$. Since H_0 is rejected , these data support the claim that the plastic coating is an effective corrosion retardant.

7.4.22 The set of μ_0's for which H_0: $\mu = \mu_0$ would <u>not</u> be rejected at the $\alpha = 0.05$ level of significance is the same as the 95% confidence interval for μ.

7.4.23 Because of the skewed shape of $f_Y(y)$, and if the sample size was small, it would not be unusual for all the y_i's to lie close together near 0. When that happens, $\bar{y}$ will be less than μ, s will be considerably smaller than $E(S)$, and the t ratio will be further to the left of 0 than $f_{T_{n-1}}(t)$ would predict.

7.4.24 Both sets of ratios would have distributions similar to that of a Student t random variable with 2 df. Because of the shapes of the two $f_Y(y)$'s, though, both distributions would be skewed to the right, particularly so for $f_Y(y) = 4y^3$ (recall the answer to Question 7.4.23).

7.4.25 As n increases, Student t pdfs converge to the standard normal, $f_Z(z)$ (see Question 7.3.13).

7.4.26 Only (c) would raise any serious concerns. There the sample size is small and the y_i's are showing a markedly skewed pattern. Those are the two conditions that particularly "stress" the robustness property of the t-ratio (recall Figure 7.4.6).

Section 7.5: Drawing Inferences About σ^2

7.5.1 **(a)** 23.685 **(b)** 4.605 **(c)** 2.700

7.5.2 **(a)** 0.95 **(b)** 0.90 **(c)** 0.975 – 0.025 = 0.95 **(d)** 0.99

7.5.3 **(a)** 2.088 **(b)** 7.261 **(c)** 14.041 **(d)** 17.539

7.5.4 **(a)** 13 **(b)** 19 **(c)** 31 **(d)** 17

7.5.5 $$\chi^2_{.95,200} \doteq 200\left(1-\frac{2}{9(200)}+1.64\sqrt{\frac{2}{9(200)}}\right)^3 = 233.9$$

7.5.6 $P\left(\frac{S^2}{\sigma^2}<2\right) = P\left(\frac{(n-1)S^2}{\sigma^2}<2(n-1)\right) = P\left(\chi^2_{n-1}<2(n-1)\right)$. Values from the 0.95 column in a χ^2 table show that for each $n < 8$, $P\left(\chi^2_{n-1}<2(n-1)\right)<0.95$. But for $n = 9$, $\chi^2_{.95,8}$ = 15.507, which means that $P\left(\chi^2_8<16\right)$ > 0.95.

7.5.7 $$P\left(\chi^2_{\alpha/2,n-1} \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{1-\alpha/2,n-1}\right) = 1-\alpha = P\left(\frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}\right),$$ so

$\left(\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}, \frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}\right)$ is a 100(1 – α)% confidence interval for σ^2.

Taking the square root of both sides gives a 100(1 – α)% confidence interval for σ.

7.5.8 If n = 19 and σ^2 = 12.0, $\frac{18S^2}{12.0}$ has a χ^2 distribution with 18 df, so for example

$$P\left(8.231 \le \frac{18S^2}{12.0} \le 31.526\right)$$

= 0.95 = $P(5.49 \le S^2 \le 21.02)$.

7.5.9 **(a)** $\sum_{i=1}^{16} y_i = 1514$, so $\bar{y} = \frac{1514}{16} = 94.6$. $\sum_{i=1}^{16} y_i^2 = 154{,}398$, so $s^2 = \frac{16(154{,}398)-(1514)^2}{16(15)}$ $= 742.38$. Since $\chi^2_{.025,15} = 6.262$ and $\chi^2_{.975,15} = 27.488$, a 95% confidence interval for σ is $\left(\sqrt{\frac{15(742.38)}{27.488}}, \sqrt{\frac{15(742.38)}{6.262}}\right)$, or (20.1, 42.2).

(b) Given that $\chi^2_{.05,15} = 7.261$ and $\chi^2_{.95,15} = 24.966$, the two one-sided confidence intervals for σ are $\left(0, \sqrt{\frac{15(742.38)}{7.261}}\right) = (0, 39.2)$ and $\left(\sqrt{\frac{15(742.38)}{24.966}}, \infty\right) = (21.1, \infty)$.

7.5.10 $\sum_{i=1}^{16} y_i = 29.98$, so $\bar{y} = \frac{29.98}{10} = 2.998$. $\sum_{i=1}^{16} y_i^2 = 91.609$, so $s^2 = \frac{10(91.609)-(29.98)^2}{10(9)}$ $= 0.1921$. Since $\chi^2_{.025,9} = 2.700$ and $\chi^2_{.975,9} = 19.023$, a 95% confidence interval for σ is $\left(\sqrt{\frac{9(0.1921)}{19.023}}, \sqrt{\frac{9(0.1921)}{2.700}}\right) = (0.302, 0.800)$.

Since the standard deviation for 1-year CD rates of 0.262 falls below the interval, we have evidence that the variability of 5-year CD interest rates is higher.

7.5.11 Experimenters often prefer confidence intervals for σ (as opposed to σ^2) because they are expressed in the same units as the data, which makes them easier to interpret.

7.5.12 **(a)** If $\frac{\chi^2_{n-1}-(n-1)}{\sqrt{2(n-1)}} \doteq Z$, then $P\left(-z_{\alpha/2} \le \frac{\chi^2_{n-1}-(n-1)}{\sqrt{2(n-1)}} \le z_{\alpha/2}\right) \doteq 1-\alpha$

$$= P\left(n-1-z_{\alpha/2}\sqrt{2(n-1)} \le \frac{(n-1)S^2}{\sigma^2} \le n-1+z_{\alpha/2}\sqrt{2(n-1)}\right)$$

$$= P\left(\frac{(n-1)S^2}{n-1+z_{\alpha/2}\sqrt{2(n-1)}} \le \sigma^2 \le \frac{(n-1)S^2}{n-1-z_{\alpha/2}\sqrt{2(n-1)}}\right), \text{ so}$$

$\left(\frac{(n-1)s^2}{n-1+z_{\alpha/2}\sqrt{2(n-1)}}, \frac{(n-1)s^2}{n-1-z_{\alpha/2}\sqrt{2(n-1)}}\right)$ is an approximate $100(1-\alpha)$% confidence interval for σ^2. Likewise, $\left(\frac{\sqrt{n-1}s}{\sqrt{n-1+z_{\alpha/2}\sqrt{2(n-1)}}}, \frac{\sqrt{n-1}s}{\sqrt{n-1-z_{\alpha/2}\sqrt{2(n-1)}}}\right)$ is an approximate $100(1-\alpha)$% confidence interval for σ.

(b) For the data in Table 7.5.1, $n = 19$ and $s = \sqrt{733.4} = 27.08$, so the formula in Part a gives $\left(\frac{\sqrt{18}(27.08)}{\sqrt{18+1.96\sqrt{36}}}, \frac{\sqrt{18}(27.08)}{\sqrt{18-1.96\sqrt{36}}}\right)$ = (21.1 million years, 46.0 million years) as the approximate 95% confidence interval for σ.

7.5.13 If $\left(\frac{(n-1)s^2}{\chi^2_{.95,n-1}}, \frac{(n-1)s^2}{\chi^2_{.05,n-1}}\right)$ = (51.47, 261.92), then $\chi^2_{.95,n-1}/\chi^2_{.05,n-1} = \frac{261.90}{51.47}$ = 5.088. A trial and error inspection of the χ^2 table shows that $\chi^2_{.95,9}/\chi^2_{.05,9} = \frac{16.919}{3.325}$ = 5.088, so n = 10, which means that $\frac{9s^2}{3.325}$ = 261.92. Therefore, s = 9.8.

7.5.14 **(a)** $M_Y(t) = \frac{1}{1-\theta t}$. Let $X = \frac{2n\bar{Y}}{\theta} = \frac{2\sum_{i=1}^{n} Y_i}{\theta}$. Then $M_X(t) = \prod_{i=1}^{n} M_{Y_i}\left(\frac{2t}{\theta}\right) = \left(\frac{1}{1-2t}\right)^{2n/2}$, implying that X is a χ^2_{2n} random variable. See Theorem 4.6.5.

(b) $P\left(\chi^2_{\alpha/2,2n} \le \frac{2n\bar{Y}}{\theta} \le \chi^2_{1-\alpha/2,2n}\right) = 1-\alpha$, so $\left(\frac{2n\bar{y}}{\chi^2_{1-\alpha/2,2n}}, \frac{2n\bar{y}}{\chi^2_{\alpha/2,2n}}\right)$ is a $100(1-\alpha)\%$ confidence interval for θ.

7.5.15 Test H_0: $\sigma^2 = 30.4^2$ versus H_1: $\sigma^2 < 30.4^2$. The test statistic in this case is
$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{18(733.4)}{30.4^2}$ = 14.285. The critical value is $\chi^2_{\alpha,n-1} = \chi^2_{.05,18}$ = 9.390.
Since the test statistic is not less than the critical value, we accept the null hypothesis, and we cannot assume that the potassium-argon method is more precise.

7.5.16 Test $H_0: \sigma^2 = 1$ versus $H_1: \sigma^2 > 1$.
The sample variance is $\frac{30(19{,}195.7938) - (758.62)^2}{30(29)}$ = 0.425
The test statistic is $\chi^2 = \frac{29(0.425)}{1}$ = 12.325. The critical value is $\chi^2_{.95,29}$ = 42.557.
Since 12.325 < 42.557, we accept the null hypothesis and assume the machine is working properly.

7.5.17 **(a)** Test $H_0: \mu = 10.1$ versus $H_1: \mu > 10.1$
The test statistic is $\frac{\bar{y}-\mu_0}{s/\sqrt{n}} = \frac{11.5-10.1}{10.17/\sqrt{24}}$ = 0.674
The critical value is $t_{\alpha,n-1} = t_{0.05,23}$ = 1.7139.

Since 0.674 < 1.7139, we accept the null hypothesis. We cannot ascribe the increase of the portfolio yield over the benchmark to the analyst's system for choosing stocks.

(b) Test $H_0: \sigma^2 = 15.67$ versus $H_0: \sigma^2 < 15.67$
The test statistic is $\chi^2 = \frac{23(10.17^2)}{15.67^2}$ = 9.688. The critical value is $\chi^2_{.05,23}$ = 13.091.
Since the test statistic of 9.688 is less than the critical value of 13.091, we reject the null hypothesis. The analyst's method of choosing stocks does seem to result in less volatility

Chapter 8: Types of Data: A Brief Overview

Section 8.2: Classifying Data

8.2.1 Regression data

8.2.2 Two-sample data

8.2.3 One-sample data

8.2.4 Randomized block data

8.2.5 Regression data

8.2.6 Paired data

8.2.7 *k*-sample data

8.2.8 Regression data

8.2.9 One-sample data

8.2.10 Regression data

8.2.11 Regression data

8.2.12 One-sample data

8.2.13 Two-sample data

8.2.14 Regression data

8.2.15 k-sample data

8.2.16 Paired data

8.2.17 Categorical data

8.2.18 Paired data

8.2.19 Two-sample data

8.2.20 Categorical data

8.2.21 Paired data

8.2.22 *k*-sample data

8.2.23 Categorical data

8.2.24 Randomized block data

8.2.25 Categorical data

8.2.26 Two-sample data

8.2.27 Categorical data

8.2.28 Two-sample data

8.2.29 Paired data

8.2.30 *k*-sample data

8.2.31 Randomized block data

8.2.32 Two-sample data

Chapter 9: Two-Sample Inferences

Section 9.2: Testing $H_0 : \mu_X = \mu_Y$

9.2.1 $s_X^2 = 13.79$ and $s_Y^2 = 15.61$

$$s_p = \sqrt{\frac{(n-1)s_X^2 + (m-1)s_Y^2}{n+m-2}} = \sqrt{\frac{11(13.79) + 8(15.61)}{12+9-2}} = 3.82$$

$$t = \frac{\bar{x} - \bar{y}}{s_p\sqrt{1/n + 1/m}} = \frac{29.8 - 26.9}{3.82\sqrt{1/12 + 1/9}} = 1.72$$

Since $t = 1.72 < t_{.01,19} = 2.539$, accept H_0.

9.2.2 For large samples, use the approximate z statistic $z = \dfrac{\bar{x} - \bar{y}}{\sqrt{\dfrac{s_X^2}{n} + \dfrac{s_Y^2}{m}}} = \dfrac{-4.7 - (-1.6)}{\sqrt{\dfrac{7.05^2}{77} + \dfrac{5.36^2}{79}}} = -3.09$

Since $z = -3.09 < -1.64 = -z_{.05}$, reject H_0.

9.2.3 For large samples, use the approximate z statistic $z = \dfrac{\bar{x} - \bar{y}}{\sqrt{\dfrac{s_X^2}{n} + \dfrac{s_Y^2}{m}}} = \dfrac{189.0 - 177.2}{\sqrt{\dfrac{34.2^2}{476} + \dfrac{33.3^2}{592}}} = 5.67$

Since $5.67 > 1.64 = z_{.05}$, reject H_0.

9.2.4 $$z = \frac{\bar{x} - \bar{y}}{\sqrt{\dfrac{s_X^2}{n} + \dfrac{s_Y^2}{m}}} = \frac{491 - 498}{\sqrt{\dfrac{119^2}{1126} + \dfrac{129^2}{5042}}} = -1.76$$

Since $-1.76 < -1.64 = z_{.05}$, reject H_0.

9.2.5 $$z = \frac{\bar{x} - \bar{y}}{\sqrt{\dfrac{s_X^2}{n} + \dfrac{s_Y^2}{M}}} = \frac{4.17 - 4.61}{\sqrt{\dfrac{3.70^2}{93} + \dfrac{4.28^2}{28}}} = -0.491$$

Since $-z_{.005} = -2.58 < -0.491 < 2.58 = z_{.005}$, accept H_0.

9.2.6 $$t = \frac{\bar{x} - \bar{y}}{s_p\sqrt{1/n + 1/m}} = \frac{65.2 - 75.5}{13.9\sqrt{1/9 + 1/12}} = -1.68$$

Since $-t_{.05,19} = -1.7291 < t = -1.68$, accept H_0.

9.2.7 $$s_p = s_p = \sqrt{\frac{(n-1)s_X^2 + (m-1)s_Y^2}{n+m-2}} = \sqrt{\frac{3(266.9^2) + 3(224.3^2)}{4+4-2}} = 246.52$$

$$t = \frac{\bar{x} - \bar{y}}{s_p\sqrt{1/n + 1/m}} = \frac{1133.0 - 1013.5}{246.52\sqrt{1/4 + 1/4}} = 0.69$$

Since $-t_{.025,6} = -2.4469 < t = 0.69 < t_{.025,6} = 2.4469$, accept H_0.

9.2.8 $s_p = \sqrt{\dfrac{5(15.1^2)+8(8.1^2)}{6+9-2}} = 11.317$

$t = \dfrac{70.83-79.33}{11.317\sqrt{1/6+1/9}} = -1.43$

Since $-t_{.005,13} = -3.0123 < t = -1.43 < t_{.005,13} = 3.0123$, accept H_0.

9.2.9 $s_p = \sqrt{\dfrac{30(1.469^2)+56(1.350^2)}{31+57-2}} = 1.393$

$t = \dfrac{3.10-2.43}{1.393\sqrt{1/31+1/57}} = 2.16$

Since $t = 2.16 > t_{.025,86}$=1.9880, reject H_0.

9.2.10 Let H_0: $\mu_X - 1 = \mu_Y$ and H_1: $\mu_X - 1 < \mu_Y$.

$s_p = \sqrt{\dfrac{10(12)^2+10(16^2)}{10+10-2}} = 14.9$

$t = \dfrac{(2.1-1)-1.6}{14.9\sqrt{1/10+1/10}} = -0.08$

Since $-t_{.05,18} = -1.7341 < -0.08 = t$, accept H_0.

9.2.11 **(a)** Reject H_0 if $|t| > t_{.005,15} = 2.9467$, so we seek the smallest value of $|\bar{x}-\bar{y}|$ such that

$t = \dfrac{|\bar{x}-\bar{y}|}{s_p\sqrt{1/n+1/m}} = \dfrac{|\bar{x}-\bar{y}|}{15.3\sqrt{1/6+1/11}} > 2.9467$, or $|\bar{x}-\bar{y}| > (15.3)(0.5075)(2.9467)$
$= 22.880$

(b) Reject H_0 if $t > t_{.05,19} = 1.7291$, so we seek the smallest value of $\bar{x}-\bar{y}$ such that

$t = \dfrac{\bar{x}-\bar{y}}{s_p\sqrt{1/n+1/m}} = \dfrac{\bar{x}-\bar{y}}{214.9\sqrt{1/13+1/8}} > 1.7291$, or $\bar{x}-\bar{y} > (214.9)(0.4494)(1.7291)$
$=166.990$

9.2.12 $z = \dfrac{\bar{x}-\bar{y}}{\sqrt{\sigma_X^2/n+\sigma_Y^2/m}} = \dfrac{81.6-79.9}{\sqrt{17.6/10+22.9/20}} = 1.00$

The P-value is $2P(Z \geq 1.00) = 2(1 - 0.8413) = 2(0.1587) = 0.3174$

9.2.13 **(a)** Let X be the interstate route; Y, the town route.

$P(X > Y) = P(X - Y > 0)$. $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) = 6^2 + 5^2 = 61$.

$P(X-Y>0) = P\left(\dfrac{X-Y-(33-35)}{\sqrt{61}} > \dfrac{2}{\sqrt{61}}\right) = P(Z \geq 0.26) = 1 - 0.6026 = 0.3974$

(b) $\text{Var}(\bar{X}-\bar{Y}) = \text{Var}(\bar{X}) + \text{Var}(\bar{Y}) = 6^2/10 + 5^2/10 = 61/10$

$P(\bar{X}-\bar{Y}) > 0 = P\left(\dfrac{\bar{X}-\bar{Y}-(33-35)}{\sqrt{61/10}} > \dfrac{2}{\sqrt{61/10}}\right) = P(Z > 0.81) = 1 - 0.7910 = 0.2090$

9.2.14 $E(\bar{X} - \bar{Y}) = E(\bar{X}) - E(\bar{Y}) = \mu_X - \mu_Y$

$\text{Var}(\bar{X} - \bar{Y}) = \text{Var}(\bar{X}) + \text{Var}(\bar{Y}) = \sigma_X^2 / n + \sigma_Y^2 / m$

Also, we know that $\bar{X} - \bar{Y}$ is normal. The Z variable in Equation 9.2.1 is a Z transformation of a normal variable, and thus is standard normal.

9.2.15 It follows from Example 5.4.4 that $E(S_X^2) = E(S_Y^2) = \sigma^2$.

$$E(S_p^2) = \frac{(n-1)E(S_X^2) + (m-1)E(S_Y^2)}{n+m-2} = \frac{(n-1)\sigma^2 + (m-1)\sigma^2}{n+m-2} = \sigma^2$$

9.2.16 Take $\omega = \{(\mu_X, \mu_Y): -\infty < \mu_X = \mu_Y < \infty\}$. Since the X's and Y's are normal and independent,

$$L(\omega) = \prod_{i=1}^{n} f_X(x_i) \prod_{i=1}^{m} f_Y(y_i) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{n+m} \exp\left[-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i - \mu)^2 + \sum_{i=1}^{m}(y_i - \mu)^2\right)\right],$$

where $\mu = \mu_X = \mu_Y$. Differentiating the expression with respect to μ and setting it equal to 0 yields the maximum likelihood estimate $\hat{\mu} = \dfrac{\sum_{i=1}^{n} x_i + \sum_{i=1}^{m} y_i}{n+m} = \dfrac{n\bar{x} + m\bar{y}}{n+m}$. Substituting $\hat{\mu}$ for μ in $L(\omega)$ gives the numerator of the generalized likelihood ratio. After algebraic simplification, we obtain $L(\hat{\omega}) = \left(\dfrac{1}{\sqrt{2\pi}\sigma}\right)^{n+m} \exp\left[-\dfrac{1}{2\sigma^2}\left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{m} y_i^2 - \dfrac{(n\bar{x} + m\bar{y})^2}{n+m}\right)\right]$.

The likelihood function unrestricted by the null hypothesis is

$$L(\Omega) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{n+m} \exp\left[-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i - \mu_X)^2 + \sum_{i=1}^{m}(y_i - \mu_Y)^2\right)\right]$$

Solving $\dfrac{\partial \ln L(\Omega)}{\partial \mu_X} = 0$ and $\dfrac{\partial \ln L(\Omega)}{\partial \mu_Y} = 0$ gives $\hat{\mu}_X = \bar{x}$ and $\hat{\mu}_Y = \bar{y}$.

Substituting those values into $L(\Omega)$ gives $L(\hat{\Omega})$, which simplifies to

$$L(\hat{\Omega}) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{n+m} \exp\left[-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 + \sum_{i=1}^{m} y_i^2 - m\bar{y}^2\right)\right]$$

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \exp\left[-\frac{1}{2\sigma^2}\left(n\bar{x}^2 + m\bar{y}^2 - \frac{(n\bar{x} + m\bar{y})^2}{n+m}\right)\right] = \exp\left[-\frac{1}{2\sigma^2}\left(\frac{nm(\bar{x} - \bar{y})^2}{n+m}\right)\right].$$

Rejecting H_0 when $0 < \lambda < \lambda^*$ is equivalent to $\ln \lambda < \ln \lambda^*$ or $-2\ln \lambda > -2\ln\lambda^* = \lambda^{**}$.

But $-2\ln \lambda = \dfrac{(\bar{x} - \bar{y})^2}{\sigma^2\left(\dfrac{1}{n} + \dfrac{1}{m}\right)}$. Thus, we reject H_0 when $\dfrac{\bar{x} - \bar{y}}{\sigma\sqrt{\dfrac{1}{n} + \dfrac{1}{m}}} < -\sqrt{\lambda^{**}}$

or $\dfrac{\bar{x} - \bar{y}}{\sigma\sqrt{\dfrac{1}{n} + \dfrac{1}{m}}} > \sqrt{\lambda^{**}}$, and we recognize this as a Z test when σ^2 is known.

9.2.17 For the data given, $\bar{x} = 545.45$, $s_X = 428$, and $\bar{y} = 241.82$, $s_Y = 183$. Then

$$t = \frac{\bar{x}-\bar{y}}{\sqrt{s_X^2/n+s_Y^2/m}} = \frac{545.45-241.82}{\sqrt{428^2/11+183^2/11}} = 2.16$$

Let $\hat{\theta} = \dfrac{s_X^2}{s_Y^2} = \dfrac{428^2}{183^2} = 5.47$. The degrees of freedom associated with this statistic is greatest

integer in $\nu = \dfrac{\left(\hat{\theta}+\dfrac{n}{m}\right)^2}{\dfrac{1}{(n-1)}\hat{\theta}^2+\dfrac{1}{(m-1)}\left(\dfrac{n}{m}\right)^2} = \dfrac{\left(5.47+\dfrac{11}{11}\right)^2}{\dfrac{1}{(11-1)}(5.47)^2+\dfrac{1}{(11-1)}\left(\dfrac{11}{11}\right)^2} = 13.5$

Thus, the greatest integer is 13. Since $t = 2.16 > t_{.05,13} = 1.7709$, reject H_0.

9.2.18 Decreasing the degrees of freedom also decreases the power of the test.

9.2.19 **(a)** The sample standard deviation for the first data set is approximately 3.15; for the second, 3.29. These values seem close enough to permit the use of Theorem 9.2.2.

(b) Intuitively, the states with the comprehensive law should have fewer deaths. However, the average for these data is 8.1, which is larger than the average of 7.0 for the states with a more limited law.

9.2.20 For the data given, $\bar{x} = 29.43$, $s_X^2 = 93.073$, and $\bar{y} = 35.73$, $s_Y^2 = 234.946$. Then

$$t = \frac{\bar{x}-\bar{y}}{\sqrt{s_X^2/n+s_Y^2/m}} = \frac{29.43-35.73}{\sqrt{93.073/9+234.946/7}} = -0.95$$

Let $\hat{\theta} = \dfrac{s_X^2}{s_Y^2} = \dfrac{93.073}{234.946} = 0.396$. The degrees of freedom associated with this statistic is greatest

integer in $\nu = \dfrac{\left(0.396+\dfrac{9}{7}\right)^2}{\dfrac{1}{(9-1)}(0.396)^2+\dfrac{1}{(6-1)}\left(\dfrac{9}{7}\right)^2} = 8.08$, that is, 8

Since $t = -0.95 > -t_{.01,8} = -2.896$, do not reject H_0.

Section 9.3: Testing $H_0: \sigma_X^2 = \sigma_Y^2$—The *F* Test

9.3.1 From the case study, $s_X^2 = 115.9929$ and $s_Y^2 = 35.7604$. The observed $F = \dfrac{35.7604}{115.9929} = 0.308$. Since $F_{.025,11,11} = 0.288 < 0.308 < 3.47 = F_{.975,11,11}$, we can assume that the variances are equal.

9.3.2 The observed $F = \dfrac{0.2578}{0.0982} = 2.625$. Since $F_{.025,4,7} = 0.110 < 2.625 < 5.52 = F_{.975,4,7}$, we can accept H_0 that the variances are equal.

9.3.3 **(a)** The critical values are $F_{.025,19,19}$ and $F_{.975,19,19}$. These values are not tabulated, but in this case, we can approximate them by $F_{.025,20,20} = 0.406$ and $F_{.975,20,20} = 2.46$. The observed $F = 2.41/3.52 = 0.685$. Since $0.406 < 0.685 < 2.46$, we can assume that the variances are equal.

(b) Since $t = 2.662 > t_{.025,38} = 2.0244$, reject H_0.

9.3.4 The observed $F = \dfrac{3.18^2}{5.67^2} = 0.315$. Since $F_{.025,9,9} = 0.248 < 0.315 < 4.03 = F_{.975,9,9}$, we can accept H_0 that the variances are equal.

9.3.5 $F = 0.20^2/0.37^2 = 0.292$. Since $F_{.025,9,9} = 0.248 < 0.292 < 4.03 = F_{.975,.9,9}$, accept H_0.

9.3.6 The observed $F = 398.75/274.52 = 1.453$. Let $\alpha = 0.05$. The critical values are $F_{.025,13,11}$ and $F_{.975,13,11}$. These values are not in Table A.4, so approximate them by $F_{.025,12,11} = 0.301$ and $F_{.975,12,11} = 3.47$. Since $0.301 < 1.453 < 3.47$, accept H_0 that the variances are equal. Theorem 9.2.2 is appropriate.

9.3.7 Let $\alpha = 0.05$. $F = 65.25/227.77 = 0.286$. Since $F_{.025,8,5} = 0.208 < 0.286 < 6.76 = F_{.975,8,5}$, accept H_0. Thus, Theorem 9.2.2 is appropriate.

9.3.8 For these data, $s_X^2 = 56.86$ and $s_Y^2 = 66.5$. The observed $F = 66.5/56.86 = 1.170$. Since $F_{.025,8,8} = 0.226 < 1.170 < 4.43 = F_{.975,\,8,\,8}$, we can accept H_0 that the variances are equal. Thus, Theorem 9.2.2 can be used, as it has the hypothesis that the variances are equal.

9.3.9 If $\sigma_X^2 = \sigma_Y^2 = \sigma^2$, the maximum likelihood estimator for σ^2 is

$$\hat{\sigma}^2 = \frac{1}{n+m}\left(\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{i=1}^{m}(y_i - \bar{y})^2\right).$$

Then $L(\hat{\omega}) = \left(\dfrac{1}{2\pi\hat{\sigma}^2}\right)^{(n+m)/2} e^{-\frac{1}{2\hat{\sigma}^2}\left(\sum_{i=1}^{n}(x_i-\bar{x})^2+\sum_{i=1}^{m}(y_i-\bar{y})^2\right)} = \left(\dfrac{1}{2\pi\hat{\sigma}^2}\right)^{(n+m)/2} e^{-(n+m)/2}$

If $\sigma_X^2 \neq \sigma_Y^2$ the maximum likelihood estimators for σ_X^2 and σ_Y^2 are

$$\hat{\sigma}_X^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \text{ and } \hat{\sigma}_Y^2 = \frac{1}{m}\sum_{i=1}^{m}(y_i - \bar{y})^2 .$$

Then $L(\hat{\Omega}) = \left(\dfrac{1}{2\pi\hat{\sigma}_X^2}\right)^{n/2} e^{-\frac{1}{2\hat{\sigma}_X^2}\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)} \left(\dfrac{1}{2\pi\hat{\sigma}_Y^2}\right)^{m/2} e^{-\frac{1}{2\hat{\sigma}_Y^2}\left(\sum_{i=1}^{m}(y_i-\bar{y})^2\right)}$

$$= \left(\frac{1}{2\pi\hat{\sigma}_X^2}\right)^{n/2} e^{-m/2} \left(\frac{1}{2\pi\hat{\sigma}_Y^2}\right)^{m/2} e^{-n/2}$$

The ratio $\lambda = \dfrac{L(\hat{\omega})}{L(\hat{\Omega})} = \dfrac{(\hat{\sigma}_X^2)^{n/2}(\hat{\sigma}_Y^2)^{m/2}}{(\hat{\sigma}^2)^{(n+m)/2}}$ equates to the expression given in the statement of the question.

9.3.10 Since μ_X and μ_Y are known, the maximum likelihood estimator uses μ_X instead of $\bar{x}$ and μ_Y instead of $\bar{y}$. For the GLRT, λ is as in Question 9.3.9 with those substitutions.

Section 9.4: Binomial Data: Testing $H_0: p_X = p_Y$

9.4.1 $p_e = \dfrac{x+y}{n+m} = \dfrac{55+40}{200+200} = 0.2375$

$$z = \frac{\dfrac{x}{n} - \dfrac{y}{m}}{\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n} + \dfrac{\hat{p}(1-\hat{p})}{m}}} = \frac{\dfrac{55}{200} - \dfrac{40}{200}}{\sqrt{\dfrac{0.2375(0.7625)}{200} + \dfrac{0.2375(0.7625)}{200}}} = 1.76$$

Since $-1.96 < z = 1.76 < 1.96 = z_{.025,}$ accept H_0.

9.4.2 $p_e = \dfrac{x+y}{n+m} = \dfrac{66+93}{423+423} = 0.188$

$$z = \frac{\dfrac{x}{n} - \dfrac{y}{m}}{\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n} + \dfrac{\hat{p}(1-\hat{p})}{m}}} = \frac{\dfrac{66}{423} - \dfrac{93}{423}}{\sqrt{\dfrac{0.188(0.812)}{423} + \dfrac{0.188(0.812)}{423}}} = -2.38$$

For this experiment, H_0: $p_X = p_Y$ and H_1: $p_X < p_Y$. Since $z = -2.38 < -1.64 = -z_{.05}$, reject H_0.

9.4.3 Let $\alpha = 0.05$. $p_e = \dfrac{24+27}{29+32} = 0.836$

$$z = \frac{\dfrac{24}{29} - \dfrac{27}{32}}{\sqrt{\dfrac{0.836(0.164)}{29} + \dfrac{0.836(0.164)}{32}}} = -0.17$$

For this experiment, H_0: $p_X = p_Y$ and H_1: $p_X \neq p_Y$. Since $-1.96 < z = -0.17 < 1.96 = z_{.025}$, accept H_0 at the 0.05 level of significance.

9.4.4 $p_e = \dfrac{53+705}{91+1117} = 0.627$

$$z = \frac{\dfrac{53}{91} - \dfrac{705}{1117}}{\sqrt{\dfrac{0.627(0.373)}{91} + \dfrac{0.627(0.373)}{1117}}} = -0.92$$

Since $-2.58 < z = -0.92 < 2.58 = z_{.005}$, accept H_0 at the 0.01 level of significance.

9.4.5 $p_e = \dfrac{1033+344}{1675+660} = 0.590$

$$z = \frac{0.617 - 0.521}{\sqrt{\dfrac{0.590(0.410)}{1675} + \dfrac{0.590(0.410)}{660}}} = 4.25$$

Since $z = 4.25 > 2.33 = z_{.01}$, reject H_0 at the 0.01 level of significance.

9.4.6 $p_e = \dfrac{60+48}{100+100} = 0.54$

$$z = \frac{\frac{60}{100} - \frac{48}{100}}{\sqrt{\frac{0.54(0.46)}{100} + \frac{0.54(0.46)}{100}}} = 1.70$$

The P value is $P(Z \le -1.70) + P(Z \ge 1.70) = 2(1 - 0.9554) = 0.0892$.

9.4.7 $p_e = \dfrac{2915+3086}{4134+4471} = 0.697$

$$z = \frac{\frac{2915}{4134} - \frac{3086}{4471}}{\sqrt{\frac{0.697(0.303)}{4134} + \frac{0.697(0.303)}{4471}}} = 1.50$$

Since $-1.96 < z = 1.50 < 1.96 = z_{.025}$, accept H_0 at the 0.05 level of significance.

9.4.8 $p_e = \dfrac{175+100}{609+160} = 0.358$

$$z = \frac{\frac{175}{609} - \frac{100}{160}}{\sqrt{\frac{0.358(0.642)}{609} + \frac{0.358(0.642)}{160}}} = -7.93.$$ Since $z = -7.93 < -1.96 = -z_{.025}$, reject H_0.

9.4.9 $p_e = \dfrac{78+50}{300+200} = 0.256$

$$z = \frac{\frac{78}{300} - \frac{50}{200}}{\sqrt{\frac{0.256(0.744)}{300} + \frac{0.256(0.744)}{200}}} = 0.25.$$ In this situation, H_1 is $p_X > p_Y$.

Since $z = 0.25 < 1.64 = z_{.05}$, accept H_0. The player is right.

9.4.10 From Equation 9.4.1,

$$\lambda = \frac{[(55+60)/(160+192)]^{(55+60)}[1-(55+60)/(160+192)]^{(160+192-55-60)}}{(55/160)^{55}[1-(55/160)]^{105}(60/192)^{60}[1-(60/192)]^{132}}$$

$$= \frac{115^{115}(237^{237})(160^{160})(192^{192})}{352^{352}(55^{55})(105^{105})(60^{60})(132^{132})}.$$ We calculate $\ln \lambda$, which is -0.1935.

Then $-2\ln \lambda = 0.387$. Since $-2\ln \lambda = 0.387 < 6.635 = \chi^2_{.99,1}$, accept H_0.

Section 9.5: Confidence Intervals for the Two-Sample Problem

9.5.1 The center of the confidence interval is $\bar{x}-\bar{y}$ = 6.7 – 5.6 = 1.1. $s_p = \sqrt{\dfrac{8(0.54^2)+6(0.36^2)}{14}}$ = 0.47. The radius is $t_{\alpha/2,n+m-2}s_p\sqrt{\dfrac{1}{n}+\dfrac{1}{m}} = 1.7613(0.47)\sqrt{\dfrac{1}{9}+\dfrac{1}{7}}$ = 0.42. The confidence interval is (1.1 – 0.42, 1.1 + 0.42) = (0.68, 1.52). Since 0 is not in the interval, we can reject the null hypothesis that $\mu_X = \mu_Y$.

9.5.2 The center of the confidence interval is $\bar{x}-\bar{y}$ = 83.96 – 84.84 = –0.88. The radius is $t_{\alpha/2,n+m-2}s_p\sqrt{\dfrac{1}{n}+\dfrac{1}{m}} = 2.2281(11.2)\sqrt{\dfrac{1}{5}+\dfrac{1}{7}}$ = 14.61. The confidence interval is (–0.88 – 14.61, –0.88 + 14.61) = (–15.49, 13.73). Since the confidence interval contains 0, the data do not suggest that the dome makes a difference.

9.5.3 In either case, the center of the confidence interval is $\bar{x}-\bar{y}$ = 18.6 – 21.9 = –3.3.

For the assumption of equal variances, calculate $s_p = \sqrt{\dfrac{11(115.9929)+11(35.7604)}{22}}$ = 8.71

The radius of the interval is $t_{.005,22}s_p\sqrt{\dfrac{1}{12}+\dfrac{1}{12}} = 2.8188(8.71)\sqrt{\dfrac{1}{12}+\dfrac{1}{12}}$ = 10.02

The confidence interval is (–3.3 – 10.02, –3.3 + 10.02) = (–13.32, 6.72).

For the case of unequal variances, the radius of the interval is

$t_{.005,17}\sqrt{\dfrac{s_X^2}{12}+\dfrac{s_Y^2}{12}} = 2.8982\sqrt{\dfrac{115.9929}{12}+\dfrac{35.7604}{12}}$ = 10.31

The confidence interval is (–3.3 – 10.31, –3.3 + 10.31) = (–13.61, 7.01).

9.5.4 Equation (9.5.1) is $P\left(-t_{\alpha/2,n+m-2} \le \dfrac{\bar{X}-\bar{Y}-(\mu_X-\mu_Y)}{S_p\sqrt{\dfrac{1}{n}+\dfrac{1}{m}}} \le t_{\alpha/2,n+m-2}\right) = 1-\alpha$

so $P\left(-t_{\alpha/2,n+m-2}S_p\sqrt{\dfrac{1}{n}+\dfrac{1}{m}} \le \bar{X}-\bar{Y}-(\mu_X-\mu_Y) \le t_{\alpha/2,n+m-2}S_p\sqrt{\dfrac{1}{n}+\dfrac{1}{m}}\right) = 1-\alpha$, or

$$P\left(-(\bar{X}-\bar{Y})-t_{\alpha/2,n+m-2}S_p\sqrt{\frac{1}{n}+\frac{1}{m}} \le -(\mu_X-\mu_Y) \le -(\bar{X}-\bar{Y})+t_{\alpha/2,n+m-2}S_p\sqrt{\frac{1}{n}+\frac{1}{m}}\right) = 1-\alpha.$$

Multiplying the inequality above by –1 gives the inequality of the confidence interval of Theorem 9.5.1.

9.5.5 Begin with the statistic $\bar{X}-\bar{Y}$, which has $E(\bar{X}-\bar{Y}) = \mu_X - \mu_Y$ and $\text{Var}(\bar{X}-\bar{Y}) = \sigma_X^2/n+\sigma_Y^2/m$.

Then $P\left(-z_{\alpha/2} \le \dfrac{\bar{X}-\bar{Y}-(\mu_X-\mu_Y)}{\sqrt{\sigma_X^2/n+\sigma_Y^2/m}} \le z_{\alpha/2}\right) = 1-\alpha$, which implies

$$P\left(-z_{\alpha/2}\sqrt{\sigma_X^2/n+\sigma_Y^2/m} \le \bar{X}-\bar{Y}-(\mu_X-\mu_Y) \le z_{\alpha/2}\sqrt{\sigma_X^2/n+\sigma_Y^2/m}\right) = 1-\alpha.$$

Solving the inequality for $\mu_X - \mu_Y$ gives

$$P\left(\overline{X} - \overline{Y} - z_{\alpha/2}\sqrt{\sigma_X^2/n + \sigma_Y^2/m} \le \mu_X - \mu_Y \le \overline{X} - \overline{Y} + z_{\alpha/2}\sqrt{\sigma_X^2/n + \sigma_Y^2/m}\right) = 1 - \alpha.$$

Thus the confidence interval is $\left(\overline{x} - \overline{y} - z_{\alpha/2}\sqrt{\sigma_X^2/n + \sigma_Y^2/m}, \overline{x} - \overline{y} + z_{\alpha/2}\sqrt{\sigma_X^2/n + \sigma_Y^2/m}\right)$.

9.5.6 The observed ratio is $F = \dfrac{s_X^2}{s_Y^2} = \dfrac{0.0002103}{0.0000955} = 2.20$. The confidence interval is $\left(\dfrac{s_X^2}{s_Y^2}F_{0.025,9,7}, \dfrac{s_X^2}{s_Y^2}F_{0.975,9,7}\right) = \left(0.238(2.20), 4.82(2.20)\right) = (0.52, 10.60)$. Because the confidence interval contains 1, it supports the assumption of Case Study 9.2.1 that the variances are equal.

9.5.7 The confidence interval is $\left(\dfrac{s_X^2}{s_Y^2}F_{.025,5,7}, \dfrac{s_X^2}{s_Y^2}F_{.975,5,7}\right) = \left(\dfrac{137.4}{340.3}(0.146), \dfrac{137.4}{340.3}(5.29)\right)$ $= (0.06, 2.14)$. Since the confidence interval contains 1, we can accept H_0 that the variances are equal, and Theorem 9.2.1 applies.

9.5.8 Since $\dfrac{S_Y^2/\sigma_Y^2}{S_X^2/\sigma_X^2}$ has an F distribution with $m - 1$ and $n - 1$ degrees of freedom,

$$P\left(F_{\alpha/2,m-1,n-1} \le \frac{S_Y^2/\sigma_Y^2}{S_X^2/\sigma_X^2} \le F_{1-\alpha/2,m-1,n-1}\right) = P\left(\frac{S_X^2}{S_Y^2}F_{\alpha/2,m-1,n-1} \le \frac{\sigma_X^2}{\sigma_Y^2} \le \frac{S_X^2}{S_Y^2}F_{1-\alpha/2,m-1,n-1}\right) = 1 - \alpha.$$

The inequality provides the confidence interval of Theorem 9.5.2.

9.5.9 The center of the confidence interval is $\dfrac{x}{n} - \dfrac{y}{m} = \dfrac{126}{782} - \dfrac{111}{758} = 0.015$. The radius is

$$z_{.025}\sqrt{\frac{\left(\frac{x}{n}\right)\left(1-\frac{x}{n}\right)}{n} + \frac{\left(\frac{y}{m}\right)\left(1-\frac{y}{m}\right)}{m}} = 1.96\sqrt{\frac{\left(\frac{126}{782}\right)\left(1-\frac{126}{782}\right)}{782} + \frac{\left(\frac{111}{758}\right)\left(1-\frac{111}{758}\right)}{758}} = 0.036.$$

The 95% confidence interval is $(0.015 - 0.036, 0.015 + 0.036) = (-0.021, 0.051)$

Since 0 is in the confidence interval, one cannot conclude a significantly different frequency of headaches.

9.5.10 The center of the confidence interval is $\dfrac{x}{n} - \dfrac{y}{m} = \dfrac{55}{160} - \dfrac{60}{192} = 0.031$. The radius is

$$z_{.10}\sqrt{\frac{\left(\frac{x}{n}\right)\left(1-\frac{x}{n}\right)}{n} + \frac{\left(\frac{y}{m}\right)\left(1-\frac{y}{m}\right)}{m}} = 1.28\sqrt{\frac{\left(\frac{55}{160}\right)\left(1-\frac{55}{160}\right)}{160} + \frac{\left(\frac{60}{192}\right)\left(1-\frac{60}{192}\right)}{192}} = 0.064.$$

The 80% confidence interval is $(0.031 - 0.064, 0.031 + 0.064) = (-0.033, 0.095)$

9.5.11 The approximate normal distribution implies that

$$P\left(-z_\alpha \le \frac{\frac{X}{n}-\frac{Y}{m}-(p_X-p_Y)}{\sqrt{\frac{(X/n)(1-X/n)}{n}+\frac{(Y/m)(1-Y/m)}{m}}} \le z_\alpha\right)=1-\alpha$$

or $P\left(-z_\alpha\sqrt{\frac{(X/n)(1-X/n)}{n}+\frac{(Y/m)(1-Y/m)}{m}} \le \frac{X}{n}-\frac{Y}{m}-(p_X-p_Y)\right.$

$\left.\le z_\alpha\sqrt{\frac{(X/n)(1-X/n)}{n}+\frac{(Y/m)(1-Y/m)}{m}}\right)=1-\alpha$ which implies that

$P\left(-\left(\frac{X}{n}-\frac{Y}{m}\right)-z_\alpha\sqrt{\frac{(X/n)(1-X/n)}{n}+\frac{(Y/m)(1-Y/m)}{m}} \le -(p_X-p_Y)\right.$

$\left.\le -\left(\frac{X}{n}-\frac{Y}{m}\right)+z_\alpha\sqrt{\frac{(X/n)(1-X/n)}{n}+\frac{(Y/m)(1-Y/m)}{m}}\right)=1-\alpha$

Multiplying the inequality by -1 yields the confidence interval.

9.5.12 The center of the confidence interval is $\frac{x}{n}-\frac{y}{m}=\frac{106}{3522}-\frac{13}{115}=-0.083$. The radius is

$$z_{.025}\sqrt{\frac{\left(\frac{x}{n}\right)\left(1-\frac{x}{n}\right)}{n}+\frac{\left(\frac{y}{m}\right)\left(1-\frac{y}{m}\right)}{m}}=1.96\sqrt{\frac{\left(\frac{106}{3522}\right)\left(1-\frac{106}{3522}\right)}{3522}+\frac{\left(\frac{13}{115}\right)\left(1-\frac{13}{115}\right)}{115}}=0.058$$

The 95% confidence interval is (–0.083 – 0.058, –0.083 + 0.058) = (–0.141, –0.025)
Since the confidence interval lies to the left of 0, there is statistical evidence that the suicide rate among women members of the American Chemical Society is higher.

Chapter 10: Goodness-of-Fit Tests

Section 10.2: The Multinomial Distribution

10.2.1 Let X_i = number of students with a score of i, i = 1, 2, 3, 4, 5. Then $P(X_1 = 0, X_2 = 0, X_3 = 1, X_4 = 2, X_5 = 3) = \dfrac{6!}{0!0!1!2!3!}(0.116)^0(0.325)^0(0.236)^1(0.211)^2(0.112)^3 = 0.000886$.

10.2.2 Let X_1 = number of round and yellow phenotypes, X_2 = number of round and green phenotypes, and so on. Then $P(X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1) = \dfrac{4!}{1!1!1!1!}\left(\dfrac{9}{16}\right)^1\left(\dfrac{3}{16}\right)^1\left(\dfrac{3}{16}\right)^1\left(\dfrac{1}{16}\right)^1 = 0.0297$.

10.2.3 Let Y denote a person's blood pressure and let X_1, X_2, and X_3 denote the number of individuals with blood pressures less than 140, between 140 and 160, and over 160, respectively. If $\mu = 124$ and $\sigma = 13.7$, $p_1 = P(Y < 140) = P\left(Z < \dfrac{140-124}{13.7}\right) = 0.8790$,

$p_2 = P(140 \le Y \le 160) = P\left(\dfrac{140-124}{13.7} \le Z \le \dfrac{160-124}{13.7}\right) = 0.1167$, and $p_3 = 1 - p_1 - p_2 = 0.0043$.

Then $P(X_1 = 6, X_2 = 3, X_3 = 1) = \dfrac{10!}{6!3!1!}(0.8790)^6(0.1167)^3(0.0043)^1 = 0.00265$.

10.2.4 Let Y denote a recruit's IQ and let X_i denote the number of recruits in class i, i = 1, 2, 3. Then $p_1 = P(\text{class I}) = P(Y < 90) = P\left(Z < \dfrac{90-100}{16}\right) = 0.2643$, $p_2 = P(\text{class II}) = P(90 \le Y \le 110)$

$= P\left(\dfrac{90-100}{16} \le Z \le \dfrac{110-100}{16}\right) = 0.4714$, and $p_3 = P(\text{class III}) = P(Y > 110) = 1 - p_1 - p_2$

$= 0.2643$. From Theorem 10.2.1, $P(X_1 = 2, X_2 = 4, X_3 = 1) = \dfrac{7!}{2!4!1!}(0.2643)^2(0.4714)^4(0.2643)^1$

$= 0.0957$.

10.2.5 Let Y denote the distance between the pipeline and the point of impact. Let X_1 denote the number of missiles landing within 20 yards to the left of the pipeline, let X_2 denote the number of missiles landing within 20 yards to the right of the pipeline, and let X_3 denote the number of missiles for which $|y| > 20$. By the symmetry of $f_Y(y)$, $p_1 = P(-20 \le Y \le 0) = \dfrac{5}{18} = P(0 \le Y \le 20) = p_2$,

so $p_3 = P(|Y| > 20) = 1 - \dfrac{5}{18} - \dfrac{5}{18} = \dfrac{8}{18}$. Therefore, $P(X_1 = 2, X_2 = 4, X_3 = 0)$

$= \dfrac{6!}{2!4!0!}\left(\dfrac{5}{18}\right)^2\left(\dfrac{5}{18}\right)^4\left(\dfrac{8}{18}\right)^0 = 0.00689$.

10.2.6 Let X_i, $i = 1, 2, 3, 4, 5$, denote the number of outs, singles, doubles, triples, and home runs, respectively, that the player makes in 5 at-bats. Then P(two-outs, two singles, one double) $= P(X_1 = 2, X_2 = 2, X_3 = 1, X_4 = 0, X_5 = 0) = \dfrac{5!}{2!2!1!0!0!} \cdot (0.713)^2(0.270)^2(0.010)^1(0.002)^0(0.005)^0$ $= 0.0111$.

10.2.7 **(a)** $p_1 = P\left(0 \le Y < \dfrac{1}{4}\right) = \displaystyle\int_0^{1/4} 3y^2 dy = \dfrac{1}{64}$, $p_2 = P\left(\dfrac{1}{4} \le Y < \dfrac{1}{2}\right) = \displaystyle\int_{1/4}^{1/2} 3y^2 dy = \dfrac{7}{64}$,

$p_3 = P\left(\dfrac{1}{2} \le Y < \dfrac{3}{4}\right) = \displaystyle\int_{1/2}^{3/4} 3y^2 dy = \dfrac{19}{64}$, and $p_4 = P\left(\dfrac{3}{4} \le Y \le 1\right) = \displaystyle\int_{3/4}^{1} 3y^2 dy = \dfrac{37}{64}$.

Then $f_{X_1,X_2,X_3,X_4}(3,7,15,25) = P(X_1 = 3, X_2 = 7, X_3 = 15, X_4 = 25)$

$= \dfrac{50!}{3!7!15!25!}\left(\dfrac{1}{64}\right)^3\left(\dfrac{7}{64}\right)^7\left(\dfrac{19}{64}\right)^{15}\left(\dfrac{37}{64}\right)^{25}$.

(b) By Theorem 10.2.2, X_3 is a binomial random variable with parameters n (= 50) and $p_3 \left(= \dfrac{19}{64}\right)$. Therefore, $\text{Var}(X_3) = np_3(1 - p_3) = 50\left(\dfrac{19}{64}\right)\left(\dfrac{45}{64}\right) = 10.44$.

10.2.8 $M_{X_1,X_2,X_3}(t_1,t_2,t_3) = \sum\sum\sum e^{t_1k_1+t_2k_2+t_3k_3} \cdot \dfrac{n!}{k_1!k_2!k_3!} \cdot p_1^{k_1} p_2^{k_2} p_3^{k_3}$

$= \sum\sum\sum \dfrac{n!}{k_1!k_2!k_3!}\left(p_1e^{t_1}\right)^{k_1}\left(p_2e^{t_2}\right)^{k_2}\left(p_3e^{t_3}\right)^{k_3}$, where the summation extends over all the values of (k_1, k_2, k_3) such that $k_i \ge 0$, $i = 1, 2, 3$ and $k_1 + k_2 + k_3 = n$. Recall Newton's binomial expansion. Applied here, it follows that the triple sum defining the moment-generating function for (X_1, X_2, X_3) can also be written $\left(p_1e^{t_1} + p_2e^{t_2} + p_3e^{t_3}\right)^n$.

10.2.9 Assume that $M_{X_1,X_2X_3}(t_1,t_2,t_3) = \left(p_1e^{t_1} + p_2e^{t_2} + p_3e^{t_3}\right)^n$. Then $M_{X_1,X_2,X_3}(t_1,0,0)$ $= E(e^{t_1X_1}) = \left(p_1e^{t_1} + p_2 + p_3\right)^n = (1 - p_1 + p_1e^{t_1})^n$ is the mgf for X_1. But the latter has the form of the mgf for a binomial random variable with parameters n and p_1.

10.2.10 The log of the likelihood vector $(k_1, k_2, \ldots, k_t)$ is $\log L = \log\ p_1^{k_1} p_2^{k_2} \ldots p_t^{k_t} = k_1 \log p_1 +$ $k_2 \log p_2 + \ldots + k_t \log p_t$, where the p_i's are constrained by the condition that $\displaystyle\sum_{i=1}^{t} p_i = 1$. Finding the MLE for the p_i's can be accomplished using Lagrange multipliers. Differentiating $\log L - \lambda\displaystyle\sum_{i=1}^{t} p_i$ with respect to each p_i gives $\dfrac{\partial}{\partial p_i}\left[\log L - \lambda\displaystyle\sum_{i=1}^{t} p_i\right] = \dfrac{k_i}{p_i} - \lambda$, $i = 1, 2, \ldots, t$. But these derivatives equal 0 only if $\dfrac{k_i}{p_i} = \lambda$ for all i. The latter equations, together with the fact that $\displaystyle\sum_{i=1}^{t} p_i = 1$, imply that $\hat{p}_i = \dfrac{k_i}{n}$, $i = 1, 2, \ldots, t$.

Section 10.3: Goodness-of-Fit Tests: All Parameters Known

10.3.1 $\sum_{i=1}^{t}\frac{(X_i-np_i)^2}{np_i}=\sum_{i=1}^{t}\frac{(X_i^2-2np_iX_i+n^2p_i^2)}{np_i}=\sum_{i=1}^{t}\frac{X_i^2}{np_i}-2\sum_{i=1}^{t}X_i+n\sum_{i=1}^{t}p_i=\sum_{i=1}^{t}\frac{X_i^2}{np_i}-n$.

10.3.2 If the hypergeometric model applies, $\pi_1 = P$(0 whites are drawn) = $\binom{4}{0}\binom{6}{2}\Big/\binom{10}{2}=\frac{15}{45}$,

$\pi_2 = P$(1 white is drawn) = $\binom{4}{1}\binom{6}{1}\Big/\binom{10}{2}=\frac{24}{45}$, and $\pi_3 = P$(2 whites are drawn)

= $\binom{4}{2}\binom{6}{0}\Big/\binom{10}{2}=\frac{6}{45}$. Let p_1, p_2, and p_3 denote the actual probabilities of drawing 0, 1, and 2 white chips, respectively. To test H_0: $p_1=\frac{15}{45}$, $p_2=\frac{24}{45}$, $p_3=\frac{6}{45}$ versus H_1: at least one $p_i \neq \pi_i$, reject H_0 if $d \geq \chi^2_{1-\alpha,k-1}=\chi^2_{.90,2}=4.605$.

Here, $d=\frac{(35-100(15/45))^2}{100(15/45)}+\frac{(55-100(24/45))^2}{100(24/45)}+\frac{(10-100(6/45))^2}{100(6/45)}=0.96$, so H_0 (and the hypergeometric model) would not be rejected.

10.3.3 If the sampling is presumed to be <u>with</u> replacement, the number of white chips selected would follow a binomial distribution. Specifically, $\pi_1 = P$(0 whites are drawn) = $\binom{2}{0}\left(\frac{4}{10}\right)^0\left(\frac{6}{10}\right)^2$

= 0.36, $\pi_2 = P$(1 white is drawn) = $\binom{2}{1}\left(\frac{4}{10}\right)^1\left(\frac{6}{10}\right)^1$ = 0.48, and $\pi_3 = P$(2 whites are drawn)

= $\binom{2}{2}\left(\frac{4}{10}\right)^2\left(\frac{6}{10}\right)^0$ = 0.16. The form of the $\alpha = 0.10$ decision rule is reject H_0 if $d \geq \chi^2_{1-\alpha,k-1}=\chi^2_{.90,2}=4.605$. In this case, though,

$d=\frac{(35-100(0.36))^2}{100(0.36)}+\frac{(55-100(0.48))^2}{100(0.48)}+\frac{(10-100(0.16))^2}{100(0.16)}=3.30$. The null hypothesis that the sampling occurred with replacement is not rejected.

10.3.4 If births occur randomly in time, then $\pi_1 = P$(baby is born between midnight and 4 A.M.) = $\frac{1}{6}$ and $\pi_2 = P$(baby is born at a "convenient" time) = $1-\pi_1=\frac{5}{6}$. Let p_1 and p_2 denote the actual probabilities of birth during those two time periods. The null hypothesis to be tested is H_0: $p_1=\frac{1}{6}$, $p_2=\frac{5}{6}$. At the $\alpha = 0.05$ level of significance, H_0 should be rejected if $d \geq \chi^2_{.95,1}=3.841$. Given that $n = 2650$ and that X_1 = number of births between midnight and 4 A.M. = 494, it follows that $d=\frac{(494-2650(1/6))^2}{2650(1/6)}+\frac{(2156-2650(5/6))^2}{2650(5/6)}=7.44$. Since the latter exceeds 3.841, we reject the hypothesis that births occur uniformly in all time periods.

10.3.5 Let $p = P$(baby is born between midnight and 4 A.M.). Test H_0: $p = \frac{1}{6}$ versus H_1: $p \neq \frac{1}{6}$. Let $n = 2650$ be the number of births and $k = 494$ is the number of babies born between midnight and 4 A.M. From Theorem 6.3.1, H_0 should be rejected if z is either ≤ -1.96 or $\geq 1.96 = z_{.025}$. Here $z = \dfrac{494 - 2650(1/6)}{\sqrt{2650(1/6)(5/6)}} = 2.73$, so H_0 is rejected. These two test procedures are equivalent: If one rejects H_0, so will the other. Notice that $z_{.025}^2 = (1.96)^2 = 3.84$ $= \chi^2_{.95,1}$ and (except for a small rounding error) $z^2 = (2.73)^2 = 7.45 = \chi^2 = 7.44$.

10.3.6 In the terminology of Theorem 10.3.1, $X_1 = 1383$ = number of schizophrenics born in first quarter and X_2 = number of schizophrenics born after the first quarter. By assumption, $n\pi_1 = 1292.1$ and $n\pi_2 = 3846.9$ (where $n = 5139$). The null hypothesis that birth month is unrelated to schizophrenia is rejected if $d \geq \chi^2_{.95,1} = 3.841$. But $d = \dfrac{(1383-1292.1)^2}{1292.1} + \dfrac{(3756-3846.9)^2}{3846.9}$ $= 8.54$, so H_0 is rejected, suggesting that month of birth may, indeed, be a factor in the incidence of schizophrenia.

10.3.7 Listed in the accompanying table are the observed and expected numbers of M&Ms of each color. Let p_1 = true proportion of browns, p_2 = true proportion of yellows, and so on.

Color	Observed Frequency	p_i	Expected Frequency
Brown	455	0.3	458.1
Yellow	343	0.2	305.4
Red	318	0.2	305.4
Orange	152	0.1	152.7
Blue	130	0.1	152.7
Green	129	0.1	152.7
	1527		1527

To test H_0: $p_1 = 0.30, p_2 = 0.20, \ldots, p_6 = 0.10$ versus H_1: at least one $p_i \neq \pi_i$, reject H_0 if $d \geq \chi^2_{.95,5} = 11.070$. But $d = \dfrac{(455-458.1)^2}{458.1} + \ldots + \dfrac{(129-152.7)^2}{152.7} = 12.23$, so H_0 is rejected (these particular observed frequencies are not consistent with the company's intended probabilities).

10.3.8 Let the random variable X denote the length of a World Series. Then $P(X = 4) = \pi_1$ $= P$(AL wins in 4) + P(NL wins in 4) = $2 \cdot P$(AL wins in 4) = $2\left(\frac{1}{2}\right)^4 = \frac{1}{8}$. Similarly, $P(X = 5) = \pi_2 = 2 \cdot P$(AL wins in 5) = $2 \cdot P$(AL wins exactly 3 of first 4 games) $\cdot$ P(AL wins 5th game)
$= 2 \cdot \binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 \cdot \frac{1}{2} = \frac{1}{4}$. Also, $P(X = 6) = \pi_3 = 2 \cdot P$(AL wins exactly 3 of first 5 games) $\cdot$ P(AL wins 6th game) = $2 \cdot \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right) = \frac{5}{16}$, and $P(X = 7) = \pi_4$

$= 1 - P(X = 4) - P(X = 5) - P(X = 6) = \frac{5}{16}$. Listed in the table is the information necessary for calculating the goodness-of-fit statistic d. The "Bernoulli model" is rejected if $d \geq \chi^2_{.90,3} = 6.251$. For these data, $d = \frac{(9-6.25)^2}{6.25} + \frac{(11-12.50)^2}{12.50} + \frac{(8-15.625)^2}{15.625} + \frac{(22-15.625)^2}{15.625} = 7.71$, so H_0 is rejected.

Number of Games	Observed Frequency	Expected Frequency
4	9	6.25
5	11	12.5
6	8	15.625
7	22	15.625
	50	50

10.3.9 Let $p_i = P$(horse starting in post position i wins), $i = 1, 2, \ldots, 8$. One relevant null hypothesis to test would be that p_i is not a function of i—that is, H_0: $p_1 = p_2 = \ldots = p_8 = \frac{1}{8}$ versus H_1: at least one $p_i \neq \frac{1}{8}$. If $\alpha = 0.05$, H_0 should be rejected if $d \geq \chi^2_{.95,7} = 14.067$. Each $E(X_i)$ in this case is $144 \cdot \frac{1}{8} = 18.0$, so $d = \frac{(32-18.0)^2}{18.0} + \frac{(21-18.0)^2}{18.0} + \ldots + \frac{(11-18.0)^2}{18.0} = 18.22$. Since $18.22 \geq \chi^2_{.95,7}$, we reject H_0 (which is not surprising because faster horses are often awarded starting positions close to the rail).

10.3.10 Listed is the frequency distribution for the 70 y_i's using classes of width 10 starting at 220. If normality holds, each π_i is an integral of the normal pdf having $\mu = 266$ and $\sigma = 16$.

Duration	Observed Frequency	π_i	Expected Frequency
$220 \leq y < 230$	1	0.0122	0.854
$230 \leq y < 240$	5	0.0394	2.758
$240 \leq y < 250$	10	0.1071	7.497
$250 \leq y < 260$	16	0.1933	13.531
$260 \leq y < 270$	23	0.2467	17.269
$270 \leq y < 280$	7	0.2119	14.833
$280 \leq y < 290$	6	0.1226	8.582
$290 \leq y < 300$	2	0.0668	4.676
	70		70

For example, $\pi_2 = P(230 \leq Y < 240) = P\left(\frac{230-266}{16} \leq \frac{Y-266}{16} < \frac{240-266}{16}\right)$
$= P(-2.25 \leq Z < -1.63) = 0.0394$. To account for all the area under $f_Y(y)$, the intervals defining the first and last classes need to be extended to $-\infty$ and $+\infty$, respectively. That is, $\pi_1 = P(-\infty < Y < 230)$ and $\pi_8 = P(290 \leq Y < \infty)$. Some of the expected frequencies ($= 70 \cdot \pi_i$) are too small (i.e., less than 5) for the χ^2 approximation to be fully adequate.

The first three classes need to be combined, giving 0.854 + 2.758 + 7.497 = 11.109. Also, the last two class should be combined to yield 8.582 + 2.758 = 13.258. With $t = 5$ final classes, then, the normality assumption is rejected if $d \geq \chi^2_{.90,4} = 7.779$. Here, $d =$

$$\frac{(16-11.109)^2}{11.109}+\ldots+\frac{(8-13.258)^2}{13.258}$$

= 10.73, so we would reject the null hypothesis that pregnancy durations are normally distributed.

10.3.11 Let the random variable Y denote the prison time served by someone convicted of grand theft auto. In the accompanying table is the frequency distribution for a sample of 50 y_i's, together with expected frequencies based on the null hypothesis that $f_Y(y) = \frac{1}{9}y^2$, $0 \leq y \leq 3$. For example, $E(X_1) = 50 \cdot \pi_1 = 50\int_0^1 \frac{1}{9}y^2 dy = 1.85$. Combining the first two intervals (because $E(X_1) < 5$) yields $k = 2$ final classes, so H_0: $f_Y(y) = \frac{1}{9}y^2$, $0 \leq y \leq 3$ should be rejected if $d \geq \chi^2_{.95,1} = 3.841$. But $d = \frac{(24-14.81)^2}{14.81}+\frac{(26-35.19)^2}{35.19} = 8.10$, implying that the proposed quadratic pdf does not provide a good model for describing prison time.

Prison Time	Observed Frequency	p_i	Expected Frequency
$0 \leq y < 1$	8	1/27	1.85
$1 \leq y < 2$	16	7/27	12.96
$2 \leq y < 3$	26	19/27	35.19
	50		50.00

Section 10.4: Goodness-of-Fit Tests: Parameters Unknown

10.4.1 Let $p = P$(voter says "yes"). Then $\hat{p} = \frac{\text{number of yeses}}{\text{number of voters}} = \frac{30(0)+56(1)+73(2)+41(3)}{600} =$ 0.54, so the H_0 model to be tested is $P(i \text{ yeses}) = \binom{3}{i}(0.54)^i(0.46)^{3-i}$, $i = 0, 1, 2, 3$. Detailed in the accompanying table are the relevant observed and expected frequencies. At the $\alpha = 0.05$ level, the binomial model should be rejected if $d_1 \geq \chi^2_{.95,4-1-1} = 5.991$.

No. Saying "Yes"	Observed Frequency	$\hat{p}_i$	Expected Frequency
0	30	0.097	19.4
1	56	0.343	68.6
2	73	0.402	80.4
3	41	0.157	31.4
	200		200.0

But $d_1 = \frac{(30-19.4)^2}{19.4}+\frac{(56-68.6)^2}{68.6}+\frac{(73-80.4)^2}{80.4}+\frac{(41-31.4)^2}{31.4} = 11.72$, implying that the binomial model is inadequate in this particular context (probably because the trials are not likely to be independent, which is one of the model's assumptions).

10.4.2 For the Poisson pdf, $\hat{\lambda} = \dfrac{59(0)+27(1)+9(2)+1(3)}{96} = 0.50$ so the hypotheses being tested are H_0: $P(i \text{ vacancies}) = e^{-0.50}(0.50)^i/i!$, $i = 0, 1, 2, \ldots$ vs. H_1: $P(i \text{ vacancies}) \neq e^{-0.50}(0.50)^i/i!$, $i = 0, 1, 2, \ldots$ As the table indicates, the original frequency distribution needs to have several classes combined because the expected frequencies are too small.

No. of Vacancies	Observed Frequency	$\hat{p}_i$	Expected Frequency
0	59	0.607	58.27
1	27	0.303	29.09
2	9	0.076	7.30
3	1	0.013	1.25
4+	0	0.001	0.10
	96		96.00

The Poisson fit in this case is exceptionally good. Given that $\alpha = 0.01$, H_0 should be rejected if $d_1 \geq \chi^2_{.99,3-1-1} = 6.635$, but $d_1 = \dfrac{(59-58.27)^2}{58.27} + \dfrac{(27-29.09)^2}{29.09} + \dfrac{(10-8.65)^2}{8.65} = 0.37$.

10.4.3 Here the H_0 model is $P(y \text{ infected plants}) = e^{-\hat{\lambda}}(\hat{\lambda})^i / i!$, $i = 0, 1, 2, \ldots$, where $\hat{\lambda} = \dfrac{38(0)+57(1)+\ldots+1(12)}{270} = 2.53$. As the table clearly shows, the Poisson model is inappropriate for these data. The disagreements between the observed and expected frequencies are considerable—$d_1 = \dfrac{(38-21.52)^2}{21.52} + \ldots + \dfrac{(28-11.88)^2}{11.88} = 46.75$, which greatly exceeds the $\alpha = 0.05$ critical value, $\chi^2_{.95,7-1-1} = 11.070$. The independence assumption would not hold if the infestation was contagious (which is likely to be the case).

No. of Infected Plants	Observed Frequency	$\hat{p}_i$	Expected Frequency
0	38	0.0797	21.52
1	57	0.2015	54.41
2	68	0.2549	68.82
3	47	0.215	58.05
4	23	0.136	36.72
5	9	0.0688	18.58
6	10	0.0290	7.83
7	7	0.0105	2.84
8	3	0.0033	0.89
9	4	0.0009	0.24
10	2	0.0002	0.05
11	1	0.0001	0.03
12	1		0
		0	
13+	0	0	0
	270		270.00

10.4.4 Let $\hat{\lambda} = \dfrac{109(0)+65(1)+22(2)+3(3)+4(1)}{200}$ = 0.61. Then the model to be fit under H_0 is the Poisson pdf, $p_X(i) = e^{-0.61}(0.61)^i/i!$, i = 0, 1, 2, … Using t = 4 final classes (the combined "4.8" is close enough to 5 for the χ^2 approximation to be adequate), we should reject H_0 if $d_1 \geq \chi^2_{.99,4-1-1}$ = 9.210. In the table, the observed and expected frequencies are in excellent agreement, so d_1 will be very small (and the Poisson model will not be rejected). Specifically,

$$d_1 = \frac{(109-108.7)^2}{108.7} + \frac{(65-66.3)^2}{66.3} + \frac{(22-20.2)^2}{20.2} + \frac{(4-4.8)^2}{4.8} = 0.32.$$

No. of Deaths	Observed Frequency	$\hat{p}_i$	Expected Frequency
0	109	0.5434	108.7
1	65	0.3314	66.3
2	22	0.1011	20.2
3	3	0.0206	4.1
4+	1	0.0035	0.7
	200		200.0

10.4.5 Under H_0, the intervals between shutdowns should be described by an exponential pdf, $f_Y(y) = \hat{\lambda}e^{-\hat{\lambda}y}, y > 0$, where $\hat{\lambda} = 1/\bar{y}$ (recall Theorem 4.2.3). Here, the sample mean can be approximated by assigning each observation in a range a value equal to the midpoint of that range. Therefore, $\bar{y} = \dfrac{130(0.5)+41(1.5)+\ldots+1(7.5)}{211}$ = 1.22, which makes $\hat{\lambda}$ = 0.82. Moreover, each $\hat{p}_i$ is an area under $f_Y(y)$. For example, $\hat{p}_1 = \int_0^1 0.82e^{-0.82y}\,dy = 0.56$.

The complete set of $\hat{p}_i$'s and estimated expected frequencies are listed in the accompanying table. Using t = 5 final classes, we should reject the exponential model if $d_1 \geq \chi^2_{.95,5-1-1}$ = 7.815. But

$$d_1 = \frac{(130-118.16)^2}{118.16} + \ldots + \frac{(7-8.01)^2}{8.01} = 4.2, \text{ so } H_0 \text{ is not rejected.}$$

Interval	Observed Frequency	$\hat{p}_i$	Expected Frequency
0 £ y < 1	130	0.560	118.16
1 £ y < 2	41	0.246	51.91
2 £ y < 3	25	0.109	23.00
3 £ y < 4	8	0.047	9.92
4 £ y < 5	2	0.021	4.43
5 £ y < 6	3	0.010	2.11
6 £ y < 7	1	0.004	0.84
$y \geq 7$	1	0.003	0.63
	211		211.00

10.4.6 Below is the set of observed and expected frequencies, the latter based on the null hypothesis that the states' SAT scores are normally distributed with $\bar{y}$ = 949.4 and s = 68.4. With t = 4 classes and two estimated parameters, H_0 should be rejected if $d_1 \geq \chi^2_{.95,4-1-2}$ = 3.841. For these data,

$d_1 = \frac{(18-12.2)^2}{12.2}+\frac{(10-13.7)^2}{13.7}+\frac{(6-13.5)^2}{13.5}+\frac{(17-11.6)^2}{11.6}$ = 10.44, suggesting that the normality assumption is unwarranted.

Range	Observed Frequency	$\hat{p}_i$	Expected Frequency
≥ 900	18	0.2389	12.2
901-950	10	0.2691	13.7
950-1000	6	0.2654	13.5
≥ 1001	17	0.2266	11.6
	51		51.0

10.4.7 If $p = P$(child is a boy), $\hat{p} = \frac{\text{number of boys}}{\text{number of children}} = \frac{24(0)+64(1)+32(2)}{240}$ = 0.533, so the hypotheses to be tested are H_0: $P(i \text{ boys}) = \binom{2}{i}(0.533)^i(0.467)^{2-i}$, i = 0, 1, 2, versus

H_1: $P(i \text{ boys}) \neq \binom{2}{i}(0.533)^i(0.467)^{2-i}$, i = 0, 1, 2. Summarized in the table are the observed and expected numbers of families with 0, 1, and 2 boys. Given that t = number of classes = 3 and that one parameter has been estimated, H_0 should be rejected if $d_1 \geq \chi^2_{.95,3-1-1}$ = 3.841. But

$d_1 = \frac{(24-26.2)^2}{26.2}+\frac{(64-59.7)^2}{59.7}+\frac{(32-34.1)^2}{34.1}$ = 0.62, implying that the binomial model should not be rejected.

No. of Boys	Observed Frequency	$\hat{p}_i$	Expected Frequency
0	24	0.2181	26.2
1	64	0.4978	59.7
2	32	0.2841	34.1
	120		120.0

10.4.8 The table gives the observed frequencies for 100 supposedly random choices from the [0, 1] interval, as well as the expected values of 10 for each category. With 10 classes and no parameters estimated, H_0 should be rejected if $d_1 \geq \chi^2_{.95,10-1}$ = 16.919. For these data,

$d_1 = \frac{(12-10)^2}{10}+\frac{(9-10)^2}{10}+\cdots+\frac{(8-10)^2}{10}$ = 1.8

We can accept the null hypothesis that the data come from a uniform pdf over [0, 1].

Interval	Observed Frequency	Expected frequency
.000-.099	12	10
.100-.199	9	10
.200-.299	11	10
.300-.399	8	10
.400-.499	11	10
.500-.599	10	10
.600-.699	11	10
.700-.799	9	10
.800-.899	11	10
.900-.999	8	10
	100	100

10.4.9 Given that $\hat{\lambda}$ = 3.87, the model to fit under H_0 is $p_X(i) = e^{-3.87}(3.87)^i/i!$, $i = 0, 1, 2, \ldots$ Multiplying the latter probabilities by 2608 gives the complete set of estimated expected frequencies, as shown in the table. No classes need to be combined, so $t = 12$ and one parameter has been estimated. Let $\alpha = 0.05$. Then H_0 should be rejected if $d_1 \geq \chi^2_{.95,12-1-1} = 18.307$. But $d_1 = \dfrac{(57-54.51)^2}{54.51} + \dfrac{(203-210.47)^2}{210.47} + \ldots + \dfrac{(6-6.00)^2}{6.00}$ = 12.92, implying that the Poisson model should not be rejected.

No. Detected	Observed Frequency	$\hat{p}_i$	Expected Frequency
0	57	0.0209	54.51
1	203	0.0807	210.47
2	383	0.1562	407.37
3	525	0.2015	525.51
4	532	0.1949	508.30
5	408	0.1509	393.55
6	273	0.0973	253.76
7	139	0.0538	140.31
8	45	0.0260	67.81
9	27	0.0112	29.21
10	10	0.0043	11.21
11+	6	0.0023	6.00
	2608		2608.00

10.4.10 Take $\hat{\lambda}$ to be the mean of the data or 0.363. The model to be fit, then, is the Poisson pdf with parameter 0.363. The table gives the observed frequencies, the estimated probabilities and the estimated frequencies. Note that the last three classes should be collapsed, giving a total of three classes. With one parameter estimated, we should reject H_0 if $d_1 \geq \chi^2_{.95,3-1-1} = 3.841$. The data gives $d_1 = \dfrac{(82-78.6)^2}{78.6} + \dfrac{(25-28.5)^2}{28.5} + \dfrac{(6-5.9)^2}{5.9}$ = 0.58 and we can accept the Poisson model for these data.

No. of Years	Observed Frequency	$\hat{p}_i$	Expected Frequency
0	82	0.6956	78.6
1	25	0.2525	28.5
2	4	0.0458	5.2
3	0	0.0055	0.6
4	2	0.0006	0.1
	113		113.0

10.4.11 The MLE for p is the reciprocal of the sample mean. Here, $\hat{p} = \dfrac{50}{4(1)+13(2)+\ldots+1(9)} = 0.26$, so the H_0 model becomes $p_X(i) = (1-0.26)^{i-1}(0.26)$, $i = 1, 2, \ldots$ Combining the last five classes (see the accompanying table) makes $t = 5$. Let $\alpha = 0.05$. Then H_0 should be rejected if $d_1 \geq \chi^2_{.95,5-1-1} = 7.815$. In this case, $d_1 = \dfrac{(4-13.00)^2}{13.00} + \dfrac{(13-9.62)^2}{9.62} + \ldots + \dfrac{(16-15.00)^2}{15.00} = 9.23$, which suggests that the 50 observations did not come from a geometric pdf.

Outcome	Observed Frequency	$\hat{p}_i$	Expected Frequency
1	4	0.2600	13.00
2	13	0.1924	9.62
3	10	0.1424	7.12
4	7	0.1054	5.27
5	5	0.0780	3.90
6	4	0.0577	2.89
7	3	0.0427	2.14
8	3	0.0316	1.58
9+	1	0.0898	4.49
	50		50.00

10.4.12 If the lottery is fair, all of the $\binom{n}{50}$ possible samples of size 50 are equally likely. Since $1/\binom{n}{50}$ is largest when n is as small as possible, the MLE for n is $y_{\max}$—in this case, 115. It follows that the H_0 probability associated with a given range of numbers—that is, $\hat{p}_i$—should equal the number of numbers in the interval divided by 115. The table shows a frequency distribution for the 50 numbers drawn, together with the corresponding estimated expected frequencies. Let $\alpha = 0.05$. Since $t = 6$, the equally-likely model should be rejected if $d_1 \geq \chi^2_{.95,6-1-1} = 9.488$. But $d_1 = \dfrac{(8-8.25)^2}{8.25} + \dfrac{(6-8.70)^2}{8.70} + \ldots + \dfrac{(11-6.95)^2}{6.95} = 3.79$. At the $\alpha = 0.05$ level, then, we should accept the presumption that the lottery was fair.

Number Drawn	Observed Frequency	$\hat{p}_i$	Expected Frequency
$1 \leq y \leq 19$	8	0.165	8.25
$20 \leq y \leq 39$	6	0.174	8.70
$40 \leq y \leq 59$	7	0.174	8.70
$60 \leq y \leq 79$	10	0.174	8.70
$80 \leq y \leq 99$	8	0.174	8.70
$100 \leq y \leq 115$	11	0.139	6.95
	50		50.00

Section 10.5: Contingency Tables

10.5.1 To test H_0: Telephone listing and home ownership are independent at the $\alpha = 0.05$ level, reject H_0 if $d_2 \geq \chi^2_{.95,(2-1)(2-1)} = 3.841$. If α is increased to 0.10, the critical value reduces to $\chi^2_{.90,1} = 2.706$. Based on the expected frequencies predicted by H_0 (see the accompanying table), $d_2 = 2.77$ so H_0 is rejected at the $\alpha = 0.10$ level, but not at the $\alpha = 0.05$ level.

	Listed	Unlisted	
Own	628 (619.20)	146 (154.80)	774
Rent	172 (180.80)	54 (45.20)	226
	800	200	1000

10.5.2 At the $\alpha = .05$ level, H_0: Type of company and importance of work force are independent is rejected if $d_2 \geq \chi^2_{.95,(2-1)(2-1)} = 3.841$. But $d_2 = \frac{(168-163.79)^2}{163.79} + \ldots + \frac{(26-21.79)^2}{21.79} = 1.54$, so H_0 is not rejected.

	Manufacturing	Other	
Important	168 (163.79)	73 (77.21)	241
Not Important	42 (46.21)	26 (21.79)	68
	210	99	309

10.5.3 To test H_0: Delinquency and birth order are independent versus H_1: Delinquency and birth order are dependent at the $\alpha = 0.01$ level, reject the null hypothesis if $d_2 \geq \chi^2_{.99,(4-1)(2-1)} = 11.345$. Here, $d_2 = \frac{(24-45.59)^2}{45.59} + \ldots + \frac{(70-84.05)^2}{84.05} = 42.25$, suggesting that delinquency and birth order <u>are</u> related.

	Delinquent	Not Delinquent	
Oldest	24 (45.59)	450 (428.41)	474
In Between	29 (32.80)	312 (308.20)	341
Youngest	35 (23.66)	211 (222.34)	246
Only	23 (8.95)	70 (84.05)	93
	111	1043	1154

10.5.4 To test the null hypothesis that the outcome of the pregnancy and the time of the infection are independent, reject H_0 if $d_2 \geq \chi^2_{.99,(2-1)(2-1)} = 6.635$. Looking at the contingency table, it appears that the risk of an abnormal birth increases dramatically when a rubella infection is contracted early in a pregnancy (compare the "observed" of 59 with the "expected" of 30.06). The value of the test statistic confirms that suspicion—

$d_2 = \dfrac{(143-171.94)^2}{171.94} + \ldots + \dfrac{(27-55.94)^2}{55.94} = 50.34$, implying that H_0 should be rejected.

	Early Infection	Late Infection	
Normal	143 (171.94)	349 (320.06)	492
Abnormal	59 (30.06)	27 (55.94)	86
	202	376	578

10.5.5 The null hypothesis that regular use of aspirin and breast cancer are independent should be rejected at the $\alpha = 0.05$ level if $d_2 \geq \chi^2_{.95,(2-1)(2-1)} = 3.841$. These data suggest that the two are <u>not</u> independent (and H_0 should be rejected)—

$d_2 = \dfrac{(301-325.48)^2}{325.48} + \ldots + \dfrac{(1075-1099.48)^2}{1099.48} = 4.79$

	Aspirin Use	Not Aspirin	
Breast Cancer	301 (325.48)	1141 (1116.52)	1442
Control Group	345 (320.52)	1075 (1099.48)	1420
	646	2216	2862

10.5.6 Let $\alpha = 0.05$. To test H_0: Children's blood pressures are independent of their parent's blood pressures versus H_1: Children's blood pressures are not independent of their parent's blood pressures, reject the null hypothesis if $d_2 \geq \chi^2_{.95,(3-1)(3-1)} = 9.488$.

Here, $d_2 = \dfrac{(14-11.12)^2}{11.12} + \ldots + \dfrac{(12-8.83)^2}{8.83} = 3.81$, so H_0 would not be rejected. Based on these data, attempts to use one group to screen for high-risk individuals in the other group are not likely to be successful

		Child's Blood Pressure			
		Lower	Middle	Upper	
Father's Blood Pressure	Lower	14 (11.12)	11 (11.48)	8 (10.40)	33
	Middle	11 (10.45)	11 (10.78)	9 (9.77)	31
	Upper	6 (9.43)	10 (9.74)	12 (8.83)	28
		31	32	29	92

10.5.7 Given that $\alpha = 0.05$, the null hypothesis that early upbringing and aggressiveness later in life are independent is rejected if $d_2 \geq \chi^2_{.95,(2-1)(2-1)} = 3.841$. But $d_2 = \dfrac{(27-40.25)^2}{40.25} + \ldots + \dfrac{(93-106.25)^2}{106.25} = 12.61$, so H_0 <u>is</u> rejected—mice raised by foster mothers appear to be more aggressive than mice raised by their natural mothers.

	Natural Mother	Foster Mother	
No. Fighting	27	47	74
	(40.25)	(33.75)	
No. Not Fighting	140	93	233
	(126.75)	(106.25)	
	167	140	307

10.5.8 The null hypothesis that enrollment rates are independent of racial groups is rejected at the $\alpha = 0.05$ level if $d_2 \geq \chi^2_{.95,(4-1)(2-1)} = 7.815$.

For these data, $d_2 = \dfrac{(2592-2622.49)^2}{2622.49} + \ldots + \dfrac{(399-379.63)^2}{379.63} = 10.29$, implying that the differences in enrollment rates from race to race are statistically significant.

	Admitted	Enrolled	
White	2592	1481	4073
	(2622.49)	(1450.51)	
Afr.-Amer.	159	78	237
	(152.60)	(84.40)	
Hispanic	800	375	1175
	(756.55)	(418.45)	
Asian	667	399	1066
	(686.37)	(379.63)	
	4218	2333	6551

10.5.9 Let $\alpha = 0.05$. To test the null hypothesis that annual return and portfolio turnover are independent, we should reject H_0 if $d_2 \geq \chi^2_{.95,(2-1)(2-1)} = 3.841$. Based on the table below, the value of d_2 in this case is $2.20\left(= \dfrac{(11-13.86)^2}{13.86} + \ldots + \dfrac{(24-26.86)^2}{26.86}\right)$, so the appropriate conclusion would be that annual return and portfolio turnover <u>are</u> independent.

		Annual Return		
		≤ 10%	> 10%	
Portfolio	≥ 100%	11	10	21
Return		(13.86)	(7.14)	
	<100%	55	24	79
		(52.14)	(26.86)	
		66	34	100

10.5.10 (a) From Theorem 9.4.1, $p_e = \dfrac{x+y}{n+m} = \dfrac{14+5}{141+44} = 0.1027$, and

$$z = \frac{(5/44)-(14/141)}{\sqrt{\dfrac{(0.1027)(0.8973)}{141}+\dfrac{(0.1027)(0.8973)}{44}}} = 0.28$$

(b) The test statistic for a 2×2 contingency table has a χ^2 distribution with 1 degree of freedom. From Definition 7.3.1, the square of a standard normal random variable also has a χ^2 distribution with 1 degree of freedom. It follows that he square of the observed z statistic calculated from a set of two sample binomial data will equal the observed χ^2 statistic for the same data. For the same reason, the square of the critical value for a two sided z test at level α will equal the critical value for a χ^2 test done on a 2×2 contingency table with the same α.

Chapter 11: Regression

Section 11.2: The Method of Least Squares

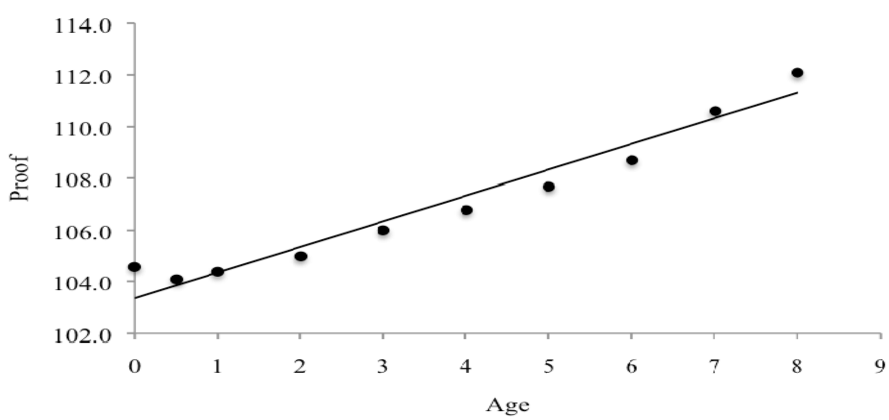
11.2.1 $$b = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{15(20{,}127.47) - (249.8)(1{,}200.6)}{15(4200.56) - (249.8)^2} = 3.291$$

$$a = \frac{\sum_{i=1}^{n} y_i - b\sum_{i=1}^{n} x_i}{n} = \frac{1{,}200.6 - 3.291(249.8)}{15} = 25.234$$

Then $y = 25.234 + 3.291x$; $y(18) = 84.5$°F

11.2.2 $$b = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{10(3973.35) - (36.5)(1070)}{10(204.25) - (36.5)^2} = 0.9953$$

$$a = \frac{\sum_{i=1}^{n} y_i - b\sum_{i=1}^{n} x_i}{n} = \frac{1070 - 0.9953(36.5)}{10} = 103.367$$

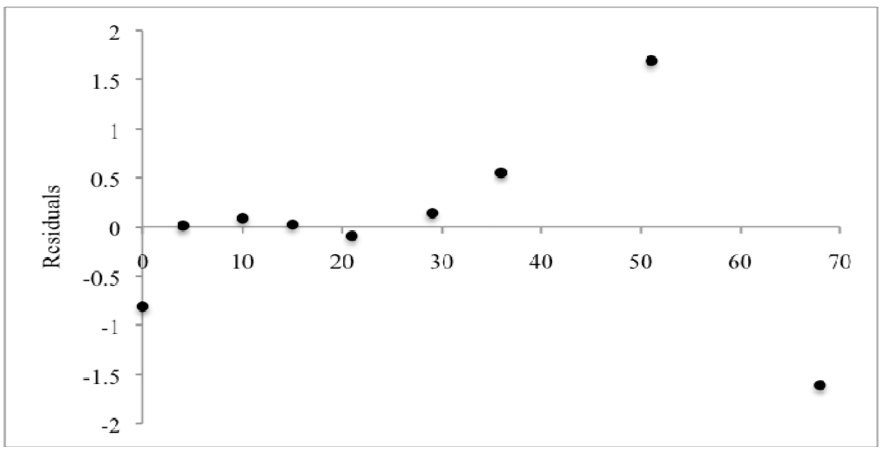


11.2.3 $$b = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{9(24{,}628.6) - (234)(811.3)}{9(10{,}144) - (234)^2} = 0.8706$$

$$a = \frac{\sum_{i=1}^{n} y_i - b\sum_{i=1}^{n} x_i}{n} = \frac{811.3 - 0.8706(234)}{9} = 67.5088$$

As an example of calculating a residual, consider $x_2 = 4$. Then the corresponding residual is $y_2 - \hat{y}_2 = 71.0 - [67.5088 + 0.8706(4)] = 0.0098$. The complete set of residuals, rounded to two decimal places is

x_i	$y_i - \hat{y}_i$
0	–0.81
4	0.01
10	0.09
15	0.03
21	–0.09
29	0.14
36	0.55
51	1.69
68	–1.61



A straight line appears to fit these data.

11.2.4 In the first graph, all of the residuals are positive. The residuals in the second graph alternate from positive to negative. Neither graph would normally occur from linear models.

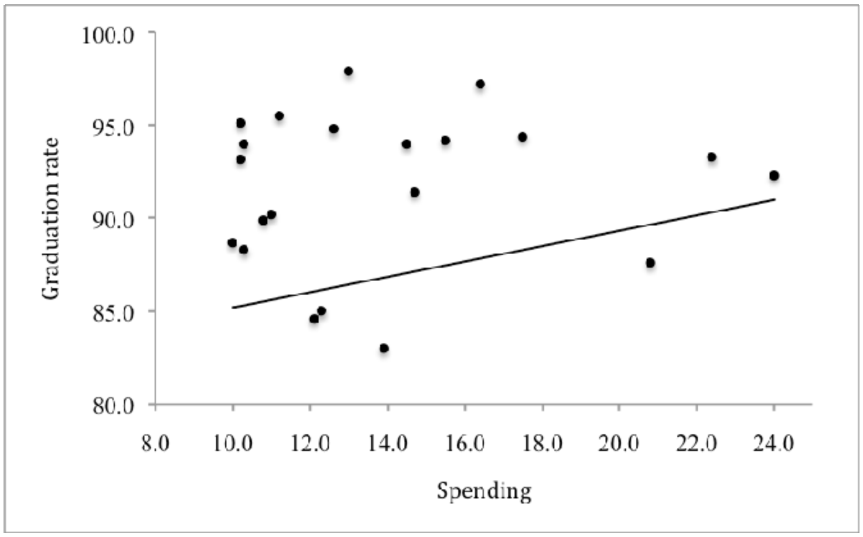
11.2.5 The value 12 is too "far" from the data observed

11.2.6 The problem here is the gap in x values, leaving some doubt as to the x–y relationship.

11.2.7 $b = \dfrac{26(31,402) - (360)(2256.6)}{26(5365.08) - (360)^2} = 0.412$

$a = \dfrac{2256.6 - 0.412(360)}{26} = 81.088$

The least squares line is $81.088 + 0.412x$. The plot of the data and least square line is:

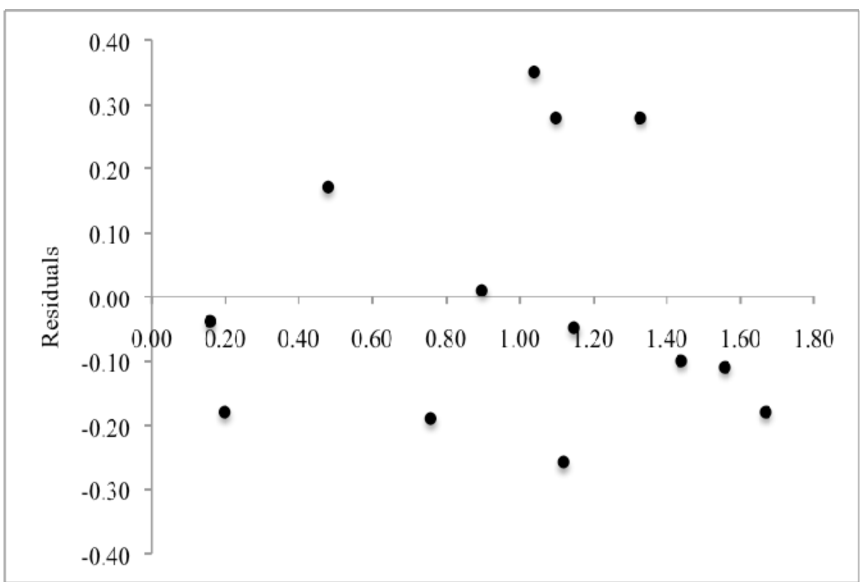


11.2.8 **(a)** The sums needed are $\sum_{i=1}^{13} x_i = 12.91$, $\sum_{i=1}^{13} x_i^2 = 15.6171$, $\sum_{i=1}^{13} y_i = 25.29$, $\sum_{i=1}^{13} x_i y_i = 29.8762$

Then $b = \dfrac{13(29.8762) - (12.91)(25.29)}{13(15.6171) - (12.91)^2} = 1.703$; $a = \dfrac{1}{13}(25.29) - \dfrac{1.703}{9}(12.91) = 0.255$

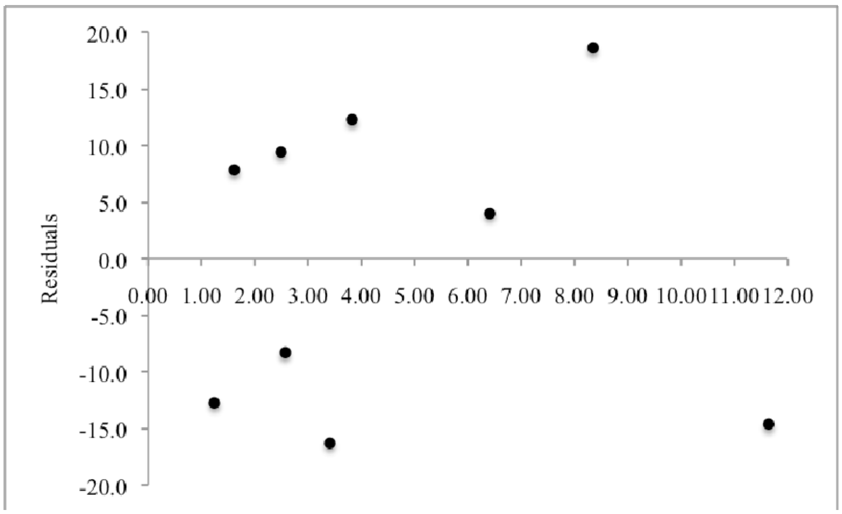
The least squares line is $y = 0.255 + 1.703x$.

(b) The residuals do not show a strong pattern, suggesting that a straight line fit is appropriate.



11.2.9 $b = \dfrac{9(7,439.37) - (41.56)(1,416.1)}{9(289.4222) - (41.56)^2} = 9.23$

$a = \dfrac{(1,416.1) - 9.23(41.56)}{9} = 114.72$



A linear relationship seems reasonable.

11.2.10 The x values spread evenly across their range, and the scatter diagram has a linear trend. Fitting this data with a straight line seems appropriate.

11.2.11 $b = \dfrac{11(1141) - (111)(100)}{11(1277) - (111)^2} = 0.84$

$a = \dfrac{1072 - 0.84(111)}{11} = 0.61$

The least squares line is $y = 0.61 + 0.84x$. The residuals given in the table below are large relative to the x values, which suggests that the linear fit is inadequate.

x_i	$y_i - \hat{y}_i$
7	−3.5
13	−1.5
14	−1.4
6	−0.7
14	2.6
15	1.8
4	3.0
8	2.7
7	−2.5
9	0.8
14	−1.4

11.2.12 Using Cramer's rule we obtain

$$b = \frac{\begin{vmatrix} n & \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i y_i \end{vmatrix}}{\begin{vmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{vmatrix}} = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i\right)}$$

which is essentially the form of b in Theorem 11.2.1. The first row of the matrix equation is $na + \left(\sum_{i=1}^{n} x_i\right)b = \sum_{i=1}^{n} y_i$. Solving this equation for a in terms of b gives the expression in Theorem 11.2.1 for a.

11.2.13 When $\bar{x}$ is substituted for x in the least-squares line equation, we obtain
$y = a + b\bar{x} = \bar{y} - b\bar{x} + b\bar{x} = \bar{y}$

11.2.14 The desired b is that value minimizing the equation $L = \sum_{i=1}^{n} (y_i - bx_i)^2$.

$\frac{dL}{db} = \sum_{i=1}^{n} 2(y_i - bx_i)(-x_i)$, and setting $\frac{dL}{db} = 0$ gives $\sum_{i=1}^{n} (x_i y_i - bx_i^2) = 0$. The solution of this equation is $b = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$.

11.2.15 For these data $\sum_{i=1}^{n} d_i v_i = 95{,}161.2$, and $\sum_{i=1}^{n} d_i^2 = 2{,}685{,}141$.

Then $H = \frac{\sum_{i=1}^{n} d_i v_i}{\sum_{i=1}^{n} d_i^2} = \frac{95{,}161.2}{2{,}685{,}141} = 0.03544$.

11.2.16 (a) We seek the a value that minimizes the equation $L = \sum_{i=1}^{n}(y_i - a - b^* x_i)^2$.

$$\frac{dL}{da} = \sum_{i=1}^{n} 2(y_i - a - b^* x_i)(-1)$$

Setting $\frac{dL}{da} = 0$ gives $\sum_{i=1}^{n}(y_i - a - b^* x_i) = 0$.

The solution of this equation is $a = \frac{\sum_{i=1}^{n} y_i}{n} - b^* \frac{\sum_{i=1}^{n} x_i}{n} = \overline{y} - b^* \overline{x}$

(b) We seek the b that minimizes the equation

$$L = \sum_{i=1}^{n}(y_i - a^* - bx_i)^2$$

$$\frac{dL}{db} = \sum_{i=1}^{n} 2(y_i - a^* - bx_i)(-x_i)$$

Setting $\frac{dL}{db} = 0$ gives $\sum_{i=1}^{n}(x_i y_i - a^* x_i - bx_i^2) = 0$.

The solution of this equation is $b = \frac{\sum_{i=1}^{n} x_i y_i - a^* \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2}$.

11.2.17 $b = \frac{\sum_{i=1}^{n} x_i y_i - a^* \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2} = \frac{1513 - 100(45)}{575.5} = -5.19$, so $y = 100 - 5.19x$.

11.2.18 (a) The sums needed are $\sum_{i=1}^{13} x_i^2 = 54{,}437$, $\sum_{i=1}^{13} x_i y_i = 3{,}329.4$.

Then $b = \frac{3329.4}{54{,}437} = 0.0612$.

(b) $y(120) = 0.612(\$120) = \$7.34$ million

11.2.19 $L = \sum_{i=1}^{n}(y_i - a - bx_i - c\sin x_i)^2$. To find the a, b, and c, solve the following set of equations.

(1) $\frac{dL}{da} = \sum_{i=1}^{n} 2(y_i - a - bx_i - c\sin x_i)(-1) = 0$ or

$$na + \left(\sum_{i=1}^{n} x_i\right)b + \left(\sum_{i=1}^{n} \sin x_i\right)c = \sum_{i=1}^{n} y_i$$

(2) $\frac{dL}{db} = \sum_{i=1}^{n} 2(y_i - a - bx_i - c\sin x_i)(-x_i) = 0$ or

$$\left(\sum_{i=1}^{n} x_i\right)a + \left(\sum_{i=1}^{n} x_i^2\right)b + \left(\sum_{i=1}^{n} x_i \sin x_i\right)c = \sum_{i=1}^{n} x_i y_i$$

(3) $\frac{dL}{dc} = \sum_{i=1}^{n} 2(y_i - a - bx_i - c\sin x_i)(-\cos x_i) = 0$ or

$$\left(\sum_{i=1}^{n} \cos x_i\right)a + \left(\sum_{i=1}^{n} x_i \cos x_i\right)b + \left(\sum_{i=1}^{n} (\cos x_i)(\sin x_i)\right)c = \sum_{i=1}^{n} y_i \cos x_i$$

11.2.20 (a) One choice for the model is $y = ae^{bx}$. Then ln y is linear with x. Using Theorem 11.2.1 on the pairs (x_i, ln y_i) gives

$$b = \frac{n\sum_{i=1}^{n} x_i \ln y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} \ln y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{10\sum_{i=1}^{n} 137.97415 - (35)(41.35720)}{10(169) - 35^2} = -0.14572$$

$$\ln a = \frac{\sum_{i=1}^{n} \ln y_i - b\sum_{i=1}^{n} x_i}{n} = \frac{41.35720 - (-0.14572)(35)}{10} = 4.64574.$$

Then a rounded to three decimal places is $e^{4.64574} = 104.140$. The desired exponential fit is $y = 104.140e^{-0.146x}$. This model fits the data well. However, note that the initial percentage by this model is 104.141, when we know it must be 100. This discrepancy suggests using Question 11.2.16 where $a^* = 100$. In this case,

$$b = \frac{\sum_{i=1}^{n} x_i \ln y_i - \ln a^*\left(\sum_{i=1}^{n} x_i\right)}{\sum_{i=1}^{n} x_i^2} = \frac{137.97415 - 4.60517(35)}{169} = -0.13732.$$

This model is $y = 100e^{-0.137x}$.

(b) For the first model, the half life is the solution to $50 = 104.140e^{-0.146x}$, or $\ln(50/104.140) = -0.146x$, so $x = 5.025$.
For the second model, the half life is the solution to $0.5 = e^{-0.137x}$ or $\ln 0.5 = -0.137x$, so $x = 5.059$.

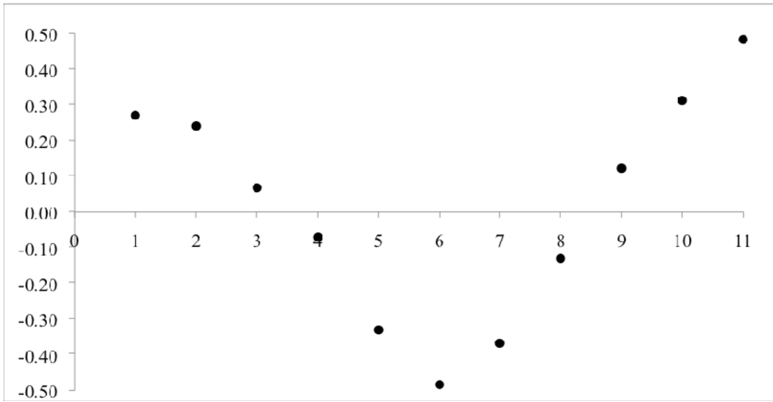
11.2.21 (a) To fit the model $y = ae^{bx}$, we note that ln y is linear with x. Then

$$b = \frac{11(126.33786) - (66)(20.16825)}{11(506) - 66^2} = 0.0484$$

$$\ln a = \frac{20.16825 - (0.0484)(66)}{11} = 1.5431$$

Then $a = e^{1.5431} = 4.6791$, and the model is $y = 4.6791e^{0.0484x}$.

(b) The model predicts $y = 4.6791e^{0.0484(12)}$ = \$8.363 trillion, short of the projected figure.

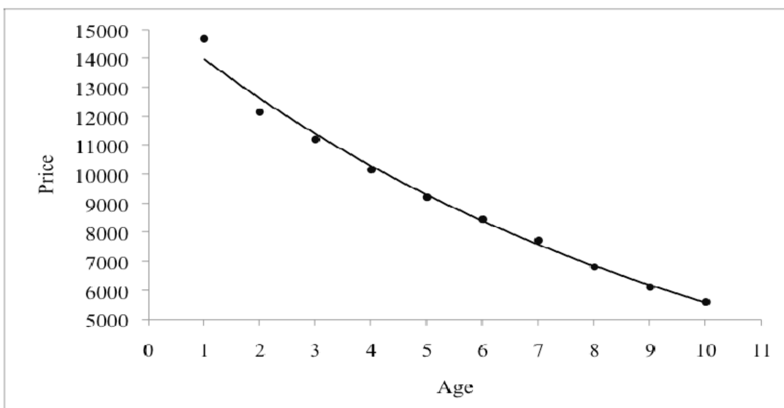


(c) The pattern of the residuals suggests a poor fit.

11.2.22 (a) $b = \dfrac{10(491.332) - (55)(90.862)}{10(385) - 55^2} = -0.1019$

$\ln a = \dfrac{90.862 - (-0.1019)(55)}{10} = 9.9467$

Then $a = e^{9.9467}$ = 15470. 6506, and the model is $y = 15470.\ 6506e^{-0.1019x}$



(b) $y = 15470.\ 6506e^{-0.1019(11)}$ = \$5043

(c) The exponential curve, which fits the data very well, predicts that a car 0 years old will have a value of \$15471, but the selling price is \$16,200. The difference \$16,200 - \$15471 = \$729 could be considered initial depreciation.

11.2.23 $b = \dfrac{10(133.68654) - (55)(22.78325)}{10(385) - 55^2} = 0.1285$

$\ln a = \dfrac{158.58560 - (0.12847)(190)}{20} = 6.70885$

Then $a = e^{6.70885}$ = 819.4, and the model is $y = 819.4e^{0.1285x}$.

11.2.24 **(a)** If $\frac{dy}{dx} = by$, then $\frac{1}{y}\frac{dy}{dx} = b$. Integrate both sides of the latter equality with respect to x:

$\int \frac{1}{y}\frac{dy}{dx}dx = \int bdx$, which implies that $\ln y = bx + C$.

Now apply the function e^x to both sides to get $y = e^{\beta_1 x}e^c = ae^{\beta_1 x}$, where $a = e^c$.

(b) x on the abscissa, $\ln y$ on the ordinate

11.2.25 $b = \dfrac{7(0.923141) - (-0.067772)(7.195129)}{7(0.0948679) - (-0.067772)^2} = 10.538;$

$\log a = \frac{1}{7}(7.195129) - \frac{10.538}{7}(-0.067772) = 1.1299$

Then $a = 10^{1.1299} = 13.487$. The model is $13.487x^{10.538}$.

The table below gives a comparison of the model values and the observed y_i's.

x_i	y_i	Model
0.98	25.000	10.901
0.74	0.950	0.565
1.12	200.000	44.522
1.34	150.000	294.677
0.87	0.940	3.109
0.65	0.090	0.144
1.39	260.000	433.516

11.2.26 **(a)** $b = \dfrac{n\sum_{i=1}^{n}\log x_i \cdot \log y - \left(\sum_{i=1}^{n}\log x_i\right)\left(\sum_{i=1}^{n}\log y_i\right)}{n\left(\sum_{i=1}^{n}\log^2 x_i\right) - \left(\sum_{i=1}^{n}\log x_i\right)^2}$

$= \dfrac{15(156.03811) - (41.77441)(52.79857)}{15(126.6045) - 41.77441^2} = 0.87644$

$\log a = \dfrac{\sum_{i=1}^{n}\log y_i - b\left(\sum_{i=1}^{n}\log x_i\right)}{n} = \dfrac{52.79857 - 0.87644(41.77441)}{15} = 1.07905$

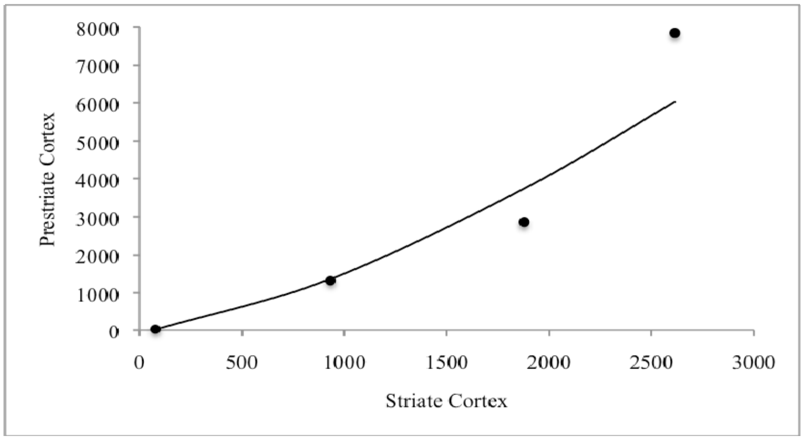
$a = 11.99637$, and the model is $y = 11.99637x^{0.87644}$

(b) $y(2500) = 11.99637(2500)^{0.87644} = 11{,}406$

11.2.27 $b = \dfrac{4(36.95941) - (11.55733)(12.08699)}{4(34.80999) - 11.55733^2} = 1.43687$

$\log a = \dfrac{12.08699 - 1.43687(11.55733)}{4} = -1.12985$

$a = 10^{-1.12985} = 0.07416$. The model is $y = 0.07416x^{1.43687}$



11.2.28 $b = \dfrac{n\sum_{i=1}^{n}(1/x_i)y_i - \left(\sum_{i=1}^{n}1/x_i\right)\left(\sum_{i=1}^{n}y_i\right)}{n\left(\sum_{i=1}^{n}(1/x_i)^2\right) - \left(\sum_{i=1}^{n}1/x_i\right)^2} = \dfrac{7(435.625)-(8.01667)(169.1)}{7(21.35028)-8.01667^2} = 19.82681$

$a = \dfrac{\sum_{i=1}^{n}y_i - b\left(\sum_{i=1}^{n}1/x_i\right)}{n} = \dfrac{169.7-19.82681(8.01667)}{7} = 1.53643$

One quarter mile = 0.25(5,280) = 1,320 feet.
$y(1.32) = 1.53643 + (19.82681)(1/1.32) = 16.557$, or \$16,557

11.2.29 **(d)** If $y = \dfrac{1}{a+bx}$, then $\dfrac{1}{y} = a + bx$ and $1/y$ is linear with x.

(e) If $y = \dfrac{x}{a+bx}$, then $\dfrac{1}{y} = \dfrac{a+bx}{x} = b + a\dfrac{1}{x}$, and $1/y$ is linear with $1/x$.

(f) If $y = 1 - e^{-x^b/a}$, then $1 - y = e^{-x^b/a}$, and $\dfrac{1}{1-y} = e^{x^b/a}$. Taking ln of both sides gives

$\ln\dfrac{1}{1-y} = x^b/a$. Taking ln again yields

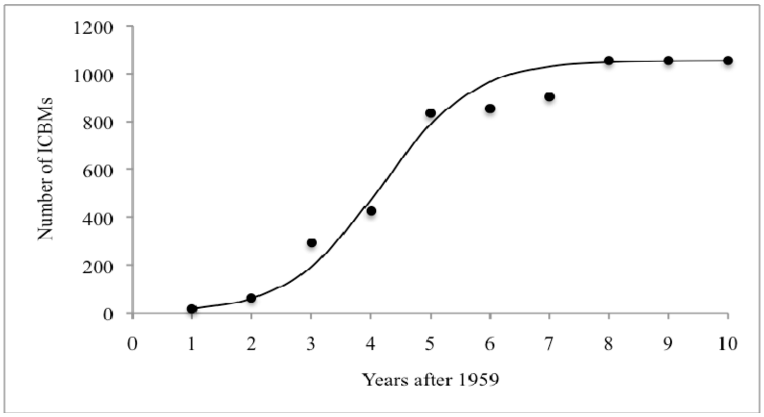
$\ln\ \ln\dfrac{1}{1-y} = -\ln a + b\ln x$, and $\ln\ \ln\dfrac{1}{1-y}$ is linear with $\ln x$.

11.2.30 Let $y' = \ln\left(\dfrac{1055-y}{y}\right)$. We find the linear relationship between x and y'. The needed sums are

$\sum_{i=1}^{10}x_i = 55$, $\sum_{i=1}^{10}x_i^2 = 385$, $\sum_{i=1}^{10}y_i' = -17.28636$, $\sum_{i=1}^{10}x_iy_i' = -201.76600$.

$b = \dfrac{10(-201.76600)-55(-17.28636)}{10(385)-55^2} = -1.29322$

$a = \dfrac{-17.28636-(-1.29322)(55)}{10} = 5.38407$

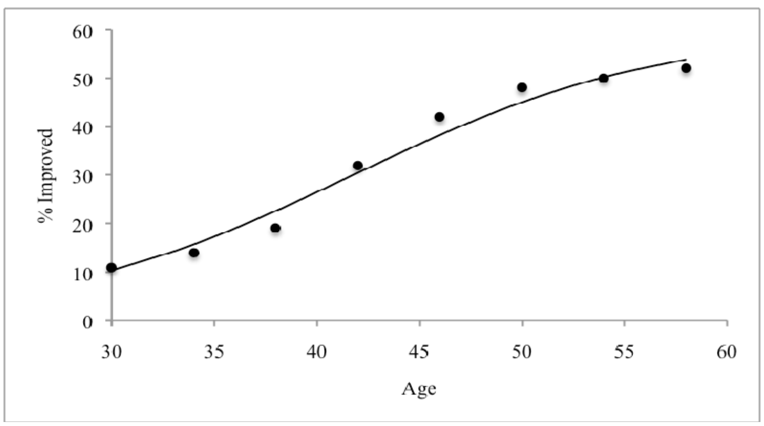


11.2.31 Let $y' = \ln\left(\dfrac{60-y}{y}\right)$. We find the linear relationship between x and y'. The needed sums are

$$\sum_{i=1}^{8} x_i = 352,\ \sum_{i=1}^{8} x_i^2 = 16160,\ \sum_{i=1}^{8} y_i' = -2.39572,\ \sum_{i=1}^{n} x_i y_i' = -194.88216.$$

$$b = \frac{8(-194.88216) - 352(-2.39572)}{8(16160) - 352^2} = -0.13314$$

$$a = \frac{-2.39572 - (-0.13314)(352)}{8} = 5.55870$$



Section 11.3: The Linear Model

11.3.1 $\beta_1 = \dfrac{4(93) - 10(40.2)}{4(30) - 10^2} = -1.5$

$$\beta_0 = \frac{(40.2) - (-1.5)(10)}{4} = 13.8$$

Thus, $y = 13.8 - 1.5x$. $t = \dfrac{\hat{\beta}_1 - \beta_1^0}{s/\sqrt{\sum_{i=1}^{4}(x_i - \bar{x})^2}} = \dfrac{-1.5 - 0}{2.114/\sqrt{5}} = -1.59.$

Since $-t_{.025,2} = -4.3027 < t = -1.59 < 4.3027 = t_{.025,2}$, accept H_0.

11.3.2 **(a)** The radius of the confidence interval = $t_{.025,24}\dfrac{s}{\sqrt{\sum_{i=1}^{26}(x_i-\bar{x})^2}} = 2.0639\dfrac{11.788481}{\sqrt{380.464615}} = 1.247$

The center is $\beta_1 = 0.412$, and the confidence interval is (-0.835, 1.659)

(b) Since 0 is in the confidence interval, we cannot reject H_0 at the 0.05 level of significance.

(c) See the solution to Question 11.2.7. The linear fit for $11 \le x \le 14$ is not very good, suggesting a search for other contributing variables in that x range.

11.3.3 $t = \dfrac{\beta_1-\beta_1^0}{s/\sqrt{\sum_{i=1}^{15}(x_i-\bar{x})^2}} = \dfrac{3.291-0}{3.829/\sqrt{40.55733}} = 5.47.$

Since $t = 5.47 > t_{0.005,13} = 3.0123$, reject H_0.

11.3.4 To minimize the width of the interval, we must maximize $\sum_{i=1}^{n}(x_i-\bar{x})^2$. To accomplish this, take half of the x_i to be 0 and half to be +5.

11.3.5 $\text{Var}(\hat{\beta}_1) = \sigma^2/\sum_{i=1}^{9}(x_i-\bar{x})^2 = 45/60 = 0.75$. The standard deviation of $\hat{\beta}_1 = \sqrt{0.75} = 0.866$.

$P\left(\left|\hat{\beta}_1-\beta_1\right|<1.5\right) = P\left(\dfrac{\left|\hat{\beta}_1-\beta_1\right|}{0.866}<\dfrac{1.5}{0.866}\right) = P(|Z| < 1.73)$ for the standard normal random variable Z. $P(Z > 1.73) = 1 - 0.9582 = 0.0418$, so $P(|Z| < 1.73) = 1 - 2(0.0418) = 0.9164$

11.3.6 $\sum_{i=1}^{n}(Y_i-\hat{\beta}_0-\hat{\beta}_1x_i)^2 = \sum_{i=1}^{n}[Y_i-(\bar{Y}-\hat{\beta}_1\bar{x})-\hat{\beta}_1x_i]^2 = \sum_{i=1}^{n}[(Y_i-\bar{Y})-\hat{\beta}_1(x_i-\bar{x})]^2$

$$= \sum_{i=1}^{n}(Y_i-\bar{Y})^2+\hat{\beta}_1^2\sum_{i=1}^{n}(x_i-\bar{x})^2-2\hat{\beta}_1\sum_{i=1}^{n}(x_i-\bar{x})(Y_i-\bar{Y})$$

$$= \sum_{i=1}^{n}(Y_i-\bar{Y})^2+\hat{\beta}_1\frac{\sum_{i=1}^{n}(x_i-\bar{x})(Y_i-\bar{Y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\sum_{i=1}^{n}(x_i-\bar{x})^2-2\hat{\beta}_1\sum_{i=1}^{n}(x_i-\bar{x})(Y_i-\bar{Y})$$

$$= \sum_{i=1}^{n}(Y_i-\bar{Y})^2-\hat{\beta}_1\sum_{i=1}^{n}(x_i-\bar{x})(Y_i-\bar{Y}) = \sum_{i=1}^{n}Y_i^2-n\bar{Y}^2-\hat{\beta}_1\left[\sum_{i=1}^{n}x_iY_i-\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}Y_i\right)\right]$$

$$= \sum_{i=1}^{n}Y_i^2-\hat{\beta}_1\sum_{i=1}^{n}x_iY_i-n\bar{Y}^2+\frac{1}{n}\hat{\beta}_1\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}Y_i\right) = \sum_{i=1}^{n}Y_i^2-\hat{\beta}_1\sum_{i=1}^{n}x_iY_i-\left(\bar{Y}-\hat{\beta}_1\bar{x}\right)\left(\sum_{i=1}^{n}Y_i\right)$$

$$= \sum_{i=1}^{n}Y_i^2-\hat{\beta}_1\sum_{i=1}^{n}x_iY_i-\hat{\beta}_0\sum_{i=1}^{n}Y_i$$

11.3.7 The radius of the confidence interval is $t_{.05,7}\dfrac{s\sqrt{\sum_{i=1}^{n}x_i^2}}{\sqrt{9}\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}} = 1.8946\dfrac{(0.959)\sqrt{10144}}{\sqrt{9}\sqrt{4060}} = 0.957.$

The center of the interval is $\hat{\beta}_0 = 67.508$. The interval = (66.551, 68.465).

11.3.8 Since there is no reason to believe that radioactivity decreases cancer rates, the test should be H_0: $\beta_1 = 0$ versus H_1: $\beta > 0$.

$t = \dfrac{\hat{\beta}_1-\beta_1^0}{s/\sqrt{\sum_{i=1}^{9}(x_i-\bar{x})^2}} = \dfrac{9.23-0}{14.010/\sqrt{97.508}} = 6.51$. Since $t = 6.51 > t_{.05,7} = 1.8946$, reject H_0.

11.3.9 $t = \dfrac{\hat{\beta}_1-\beta_1^0}{s/\sqrt{\sum_{i=1}^{11}(x_i-\bar{x})^2}} = \dfrac{0.84-0}{2.404/\sqrt{156.909}} = 4.38$. Since $t = 4.38 > t_{.025,9} = 2.2622$, reject H_0.

11.3.10 $E(\bar{Y}) = \dfrac{1}{n}\sum_{i=1}^{n}E(Y_i|x_i) = \dfrac{1}{n}\sum_{i=1}^{n}(\beta_0+\beta_1x_i) = \dfrac{1}{n}n\beta_0 + \beta_1\dfrac{1}{n}\sum_{i=1}^{n}x_i = \beta_0 + \beta_1\bar{x}$

11.3.11 By Theorem 11.3.2, $E(\hat{\beta}_0) = \beta_0$, and $\operatorname{Var}(\hat{\beta}_0) = \dfrac{\sigma^2\sum_{i=1}^{n}x_i}{n\sum_{i=1}^{n}(x_i-\bar{x})^2}$.

Now, $(\hat{\beta}_0-\beta_0)/\sqrt{\operatorname{Var}(\hat{\beta}_0)}$ is normal, so $P\left(-z_{\alpha/2} < (\hat{\beta}_0-\beta_0)/\sqrt{\operatorname{Var}(\hat{\beta}_0)} < z_{\alpha/2}\right) = 1-\alpha$.

Then the confidence interval is $(\hat{\beta}_0 - z_{\alpha/2}\sqrt{\operatorname{Var}(\hat{\beta}_0)}, \hat{\beta}_0 + z_{\alpha/2}\sqrt{\operatorname{Var}(\hat{\beta}_0)}$, or

$$\left(\hat{\beta}_0 - z_{\alpha/2}\frac{\sigma\sqrt{\sum_{i=1}^{n}x_i}}{\sqrt{n}\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}}, \hat{\beta}_0 + z_{\alpha/2}\frac{\sigma\sqrt{\sum_{i=1}^{n}x_i}}{\sqrt{n}\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}}\right)$$

11.3.12 Refer to the four assumptions in the subsection "A Special Case".

(1) Normality of the data cannot be assessed from the scatter plot
(2) The standard deviation does not appear to be the same for the three data sets.
(3) The means could be collinear
(4) Independence of the underlying random variables cannot be assessed from the scatter plot.

11.3.13 Reject the null hypothesis if the statistic is < $\chi^2_{\alpha/2,n-2} = \chi^2_{.025,22} = 10.982$ or >$\chi^2_{1-\alpha/2,n-2} = \chi^2_{.975,22}$ = 36.781. The observed chi square is $\dfrac{(n-2)s^2}{\sigma_0^2} = \dfrac{(24-2)(18.2)}{12.6}$ = 31.778, so do not reject H_0.

11.3.14 Case Study 11.3.1 provides the value of s^2 = 2181.66. Then the confidence interval for σ^2 is

$$\left(\frac{(n-2)s^2}{\chi^2_{1-\alpha/2,n-2}}, \frac{(n-2)s^2}{\chi^2_{\alpha/2,n-2}}\right) = \left(\frac{(19)(2181.66)}{\chi^2_{.95,19}}, \frac{(19)(2181.66)}{\chi^2_{.05,19}}\right) = \left(\frac{41,451.54}{30.144}, \frac{41,451.54}{10.177}\right)$$

= (1375.12, 4097.22)

11.3.15 The value of is given to be 2.31, so s^2 = 5.3361. Then the confidence interval for σ^2 is

$$\left(\frac{(n-2)s^2}{\chi^2_{1-\alpha/2,n-2}}, \frac{(n-2)s^2}{\chi^2_{\alpha/2,n-2}}\right) = \left(\frac{(6)(5.3361)}{\chi^2_{.95,7}}, \frac{(6)(5.3361)}{\chi^2_{.05,7}}\right) = \left(\frac{32.0166}{14.067}, \frac{32.0166}{2.167}\right)$$

= (2.276, 14.775)

11.3.16 (a) The radius of the confidence interval is $t_{.25,16}s\sqrt{\dfrac{1}{n} + \dfrac{(x-\overline{x})^2}{\sum_{i=1}^{18}(x_i-\overline{x})^2}}$

$$= 2.1199(0.202)\sqrt{\frac{1}{18} + \frac{(14.0-15.0)^2}{96.38944}} = 0.110$$

The center is $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x =$ –0.104 + 0.988(14.0) = 13.728.
The confidence interval is (13.62, 13.84).

(b) The radius of the prediction interval is

$$t_{.025,16}s\sqrt{1 + \frac{1}{n} + \frac{(x-\overline{x})^2}{\sum_{i=1}^{18}(x_i-\overline{x})^2}} = 2.1199(0.202)\sqrt{1 + \frac{1}{18} + \frac{(14.0-15.0)^2}{96.38944}} = 0.442$$

The center is $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$ = –0.104 + 0.988(14.0) = 13.728. The confidence interval is (13.29, 14.17).

11.3.17 The radius of the 95% confidence interval is $2.0687(0.0113)\sqrt{\dfrac{1}{25} + \dfrac{(2.750-2.643)^2}{0.0367}}$ = 0.0139.

The center is $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$ = 0.308 + 0.642(2.750) = 2.0735. The confidence interval is (2.0596, 2.0874)

11.3.18 The radius of the 99% confidence interval is $2.8609(46.708)\sqrt{\dfrac{1}{21} + \dfrac{(2500-2148.095)^2}{13056523.81}}$

= 31.932. The center is $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$ = 15.771 + 0.060(2500) = 165.771. The confidence interval is (133.839, 197.703). If the official were interested in a specific country, the prediction interval would be of more use.

11.3.19 The radius of the 95% confidence interval for $E(Y\,|\,102)$ is $2.1448(19.601)\sqrt{\frac{1}{16}+\frac{(102-89.75)^2}{901}}$ = 20.12. The center is $\hat{y}=\hat{\beta}_0+\hat{\beta}_1 x$ = -107.91 + 2.96(102) = 194.01. The 95% confidence interval is (194.01 - 20.12, 194.01 + 20.12) = (173.89, 214.13)

The radius of the prediction interval is $2.1448(19.601)\sqrt{1+\frac{1}{16}+\frac{(102-89.75)^2}{901}}$ = 46.61

The prediction interval is (194.01 – 46.61, 194.01 + 46.61) = (147.40, 240.62)
Thus, the prediction interval contains the Harvard median salary of 215,000, while the confidence interval does not.

11.3.20 The radius of the 95% confidence interval for $E(Y\,|\,9.00)$ is $2.3646(14.010)\sqrt{\frac{1}{9}+\frac{(9-4.618)^2}{97.507}}$ = 18.387. The center is $\hat{y}=\hat{\beta}_0+\hat{\beta}_1 x$ = 114.72 + 9.23(9) = 197.79. The confidence interval is (197.79 – 18.387, 197.79 + 18.387) = (179.40, 216.18).

The radius of the 95% prediction interval is $2.3646(14.010)\sqrt{1+\frac{1}{9}+\frac{(9-4.618)^2}{97.507}}$ = 37.888.

The prediction interval is (197.79 – 37.888, 197.79 + 37.888) = (159.90, 235.68)

11.3.21 The test statistic is $t=\dfrac{\hat{\beta}_1-\hat{\beta}_1^*}{s\sqrt{\dfrac{1}{\sum_{i=1}^{6}(x_i-\bar{x})^2}+\dfrac{1}{\sum_{i=1}^{8}(x_i^*-\bar{x}^*)^2}}}$, where $s=\sqrt{\dfrac{5.983+13.804}{6+8-4}}$ = 1.407.

Then $t=\dfrac{0.606-1.07}{1.407\sqrt{\frac{1}{31.33}+\frac{1}{46}}}=-1.42$. Since the observed ratio is not less than $-t_{.05,10}=-1.8125$ the difference in slopes can be ascribed to chance. These data do not support further investigation.

11.3.22 $s=\sqrt{\frac{1}{3+4}(3s^2+4s^{*2})}=\sqrt{\frac{1}{7}[3(0.9058^2)+4(1.2368^2)]}$ = 1.1071.

Then $t=\dfrac{-3.4615+2.7373}{1.1071\sqrt{\frac{1}{26}+\frac{1}{39.3333}}}=-2.59$. Since $t=-2.59<-t_{.025,7}=-2.3646$, reject H_0.

11.3.23 The form given in the text is $\text{Var}(\hat{Y}) = \sigma^2\left[\frac{1}{n} + \frac{(x-\overline{x})^2}{\sum_{i=1}^{n}(x_i-\overline{x})^2}\right]$. Putting the sum in the brackets over a least common denominator gives $\frac{1}{n} + \frac{(x-\overline{x})^2}{\sum_{i=1}^{n}(x_i-\overline{x})^2} = \frac{\sum_{i=1}^{n}(x_i-\overline{x})^2 + n(x-\overline{x})^2}{n\sum_{i=1}^{n}(x_i-\overline{x})^2}$

$$= \frac{\sum_{i=1}^{n}x_i^2 - n\overline{x}^2 + n(x^2+\overline{x}^2 - 2x\overline{x})}{n\sum_{i=1}^{n}(x_i-\overline{x})^2} = \frac{\sum_{i=1}^{n}x_i^2 + nx^2 - 2nx\overline{x}}{n\sum_{i=1}^{n}(x_i-\overline{x})^2} = \frac{\sum_{i=1}^{n}x_i^2 + nx^2 - 2x\sum_{i=1}^{n}x_i}{n\sum_{i=1}^{n}(x_i-\overline{x})^2}$$

$= \frac{\sum_{i=1}^{n}(x_i-x)^2}{n\sum_{i=1}^{n}(x_i-\overline{x})^2}$. Thus $\text{Var}(\hat{Y}) = \frac{\sigma^2\sum_{i=1}^{n}(x_i-x)^2}{n\sum_{i=1}^{n}(x_i-\overline{x})^2}$.

11.3.24 $\sum_{i=1}^{n}(\hat{Y}_i - \overline{Y})^2 = \sum_{i=1}^{n}(\hat{\beta}_0 + \hat{\beta}_1 x_i - \overline{Y})^2 =$

$\sum_{i=1}^{n}(\overline{Y} - \hat{\beta}_1\overline{x} + \hat{\beta}_1 x_i - \overline{Y})^2 = \sum_{i=1}^{n}(\hat{\beta}_1 x_i - \hat{\beta}_1\overline{x})^2 = \hat{\beta}_1^2\sum_{i=1}^{n}(x_i-\overline{x})^2$.

An application of Equation 11.3.3 completes the proof.

Section 11.4: Covariance and Correlation

11.4.1 $E(XY) = 1\frac{1+2(1)}{22} + 2\frac{2+2(1)}{22} + 3\frac{1+2(3)}{22} + 6\frac{2+2(3)}{22}$ = 80/22 = 40/11

$E(X) = 1\frac{10}{22} + 2\frac{12}{22}$ = 34/22=17/11

$E(X^2) = 1\frac{10}{22} + 4\frac{12}{22}$ = 58/22=29/11

$E(Y) = 1\frac{7}{22} + 3\frac{15}{22}$ = 52/22=26/11

$E(Y^2) = 1\frac{7}{22} + 9\frac{15}{22}$ = 142/22 = 71/11

Cov(XY) = 40/11 – (17/11)(26/11) = –2/121

Var(X) = 29/11 – $(17/11)^2$ = 30/121

Var(Y) = 71/11 – $(26/11)^2$ = 105/121

$\rho(X,Y) = \frac{-2/121}{\sqrt{30/121}\sqrt{105/121}} = \frac{-2}{\sqrt{3150}} = \frac{-2}{15\sqrt{14}}$ = -0.036

11.4.2 $E(XY) = \int_0^1\int_0^1 xy(x+y)dydx = \int_0^1\left(\frac{x^2}{2}+\frac{x}{3}\right)dx = \left.\frac{x^3}{6}+\frac{x^2}{6}\right|_0^1 = 1/3.$

$f_X(x) = x + \frac{1}{2}$, so $E(X) = \int_0^1 x\left(x+\frac{1}{2}\right)dx = 7/12$

$E(X^2) = \int_0^1 x^2\left(x+\frac{1}{2}\right)dx = 5/12$. $\text{Var}(X) = 5/12 - (7/12)^2 = 11/144$.

By symmetry Y has the same moments, so $\text{Cov}(X, Y) = 1/3 - (7/12)(7/12) = -1/144$. Then

$\rho = \frac{-1/144}{\left(\sqrt{11}/12\right)^2} = -1/11.$

11.4.3 $f_X(x) = \int_0^x 8xy\, dy = 4x^3$. $E(X) = \int_0^1 x(4x^3)dx = 4/5$. $E(X^2) = \int_0^1 x^2(4x^3)dx = 2/3$.

$\text{Var}(X) = 2/3 - (4/5)^2 = 2/75$

$f_Y(y) = \int_y^1 8xy\, dx = 4(y - y^3)$. $E(Y) = 4\int_0^1 (y - y^4)dy = 8/15$. $E(Y^2) = 4\int_0^1 (y^3 - y^5)dy = 1/3$.

$\text{Var}(Y) = 1/3 - (8/15)^2 = 11/225$.

$E(XY) = \int_0^1\int_0^x 8x^2y^2\, dy\, dx = 4/9$. $\text{Cov}(X,Y) = \frac{4}{9} - \frac{4}{5}\cdot\frac{8}{15} = \frac{8}{450}$

$\rho = \frac{8/450}{\sqrt{2/75}\sqrt{11/225}} = 0.492$

11.4.4 $E(XY) = 2\left(\frac{1}{2}+\frac{1}{8}\right)+3\frac{1}{4}+8\frac{1}{8} = 3$

$E(X) = 1\frac{3}{4}+2\frac{1}{4} = 5/4$

$E(X^2) = 1\frac{3}{4}+4\frac{1}{4} = 7/4$

$E(Y) = 1\frac{1}{8}+2\frac{1}{2}+3\frac{1}{4}+4\frac{1}{8} = 19/8$

$E(Y^2) = 1\frac{1}{8}+4\frac{1}{2}+9\frac{1}{4}+16\frac{1}{8} = 51/8$

$\text{Cov}(XY) = 3 - (5/4)(19/8) = 1/32$

$\text{Var}(X) = 7/4 - (5/4)^2 = 3/16$

$\text{Var}(Y) = 51/8 - (19/8)^2 = 47/64$

$\rho(X, Y) = \frac{1/32}{\sqrt{3/16}\sqrt{47/64}} = \frac{1}{\sqrt{3}\sqrt{47}} = 0.0842$

11.4.5 $\rho(a + bX, c + dY) = \frac{\text{Cov}(a+bX, c+dY)}{\sqrt{\text{Var}(a+bX)\text{Var}(c+dY)}} = \frac{bd\text{Cov}(X,Y)}{\sqrt{b^2\text{Var}(X)d^2\text{Var}(Y)}}$, the equality in the numerator stemming from Question 3.9.14. Since $b > 0$, $d > 0$, this last expression is $\frac{bd\text{Cov}(X,Y)}{bd\sigma_X\sigma_Y} = \frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y} = \rho(X,Y)$.

11.4.6 To find $\rho(X, Y)$, we need the first four moments of X.

$$E(X) = \sum_{k=1}^{n} k\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} k = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}$$

$$E(X^2) = \sum_{k=1}^{n} k^2\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} k^2 = \frac{1}{n}\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

$$E(X^3) = \frac{1}{n}\sum_{k=1}^{n} k^3 = \frac{1}{n}\frac{n^2(n+1)^2}{4} = \frac{n(n+1)^2}{4}$$

$$E(X^4) = \frac{1}{n}\sum_{k=1}^{n} k^4 = \frac{1}{n}\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = \frac{(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Note: We have already encountered the sums of the integers to the first and second powers. The sums for the third and fourth powers can be found in such books of mathematical tables as the <u>CRC Standard Mathematical Tables and Formulae.</u>

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = E(X^3) - E(X)E(X^2)$$

$$= \frac{n(n+1)^2}{4} - \frac{(n+1)}{2}\frac{(n+1)(2n+1)}{6} = \frac{(n+1)^2(n-1)}{12}$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{(n+1)(n-1)}{12}$$

$$\text{Var}(Y) = E(Y^2) - E(Y)^2 = E(X^4) - E(X^2)^2 = \frac{(n+1)(2n+1)(3n^2+3n-1)}{30} - \frac{(n+1)^2(2n+1)^2}{36}$$

$$= \frac{(n+1)(2n+1)(8n^2+3n-11)}{180} = \frac{(n+1)(2n+1)(n-1)(8n+11)}{180}$$

$$\rho(X, Y) = \frac{\dfrac{(n+1)^2(n-1)}{12}}{\sqrt{\dfrac{(n+1)(n-1)}{12}\dfrac{(n+1)(2n+1)(n-1)(8n+11)}{180}}} = \frac{\sqrt{15}(n+1)}{\sqrt{(2n+1)(8n+11)}}$$

$$\lim_{n\to\infty}\rho(X,Y) = \lim_{n\to\infty}\frac{\sqrt{15}(n+1)}{\sqrt{(2n+1)(8n+11)}} = \lim_{n\to\infty}\frac{\sqrt{15}\left(1+\dfrac{1}{n}\right)}{\sqrt{\left(2+\dfrac{1}{n}\right)\left(8+\dfrac{11}{n}\right)}} = \frac{\sqrt{15}}{4}$$

11.4.7 **(a)** $\text{Cov}(X + Y, X - Y) = E[(X + Y)(X - Y)] - E(X + Y)E(X - Y) = E[X^2 - Y^2] - (\mu_X + \mu_Y)(\mu_X - \mu_Y)$

$$= E(X^2) - \mu_X^2 - E(Y^2) - \mu_Y^2$$

(b) $\rho(X + Y) = \dfrac{\text{Cov}(X+Y, X-Y)}{\sqrt{\text{Var}(X+Y)\text{Var}(X-Y)}}$.

By part (a) $\text{Cov}(X + Y, X - Y) = \text{Var}(X) - \text{Var}(Y)$.
$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) = \text{Var}(X) + \text{Var}(Y) + 0$.
Similarly, $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y)$. Then

$$\rho(X+Y) = \frac{\text{Var}(X) - \text{Var}(Y)}{\sqrt{(\text{Var}(X)+\text{Var}(Y))(\text{Var}(X)+\text{Var}(Y))}} = \frac{\text{Var}(X) - \text{Var}(Y)}{\text{Var}(X) + \text{Var}(Y)}$$

11.4.8 Multiply the numerator and denominator of Equation 11.4.1 by n^2 to obtain

$$R = \frac{n\sum_{i=1}^{n} X_iY_i - \left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} Y_i\right)}{\sqrt{n\sum_{i=1}^{n}(X_i - \bar{X})^2}\sqrt{n\sum_{i=1}^{n}(Y_i - \bar{Y})^2}} = \frac{n\sum_{i=1}^{n} X_iY_i - \left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} Y_i\right)}{\sqrt{n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2}\sqrt{n\sum_{i=1}^{n} Y_i^2 - \left(\sum_{i=1}^{n} Y_i\right)^2}}$$

11.4.9 By Equation 11.4.2 $r = \dfrac{n\sum_{i=1}^{n} x_iy_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sqrt{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}\sqrt{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2}}$

$$= \frac{n\sum_{i=1}^{n} x_iy_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \cdot \frac{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}{\sqrt{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}\sqrt{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2}}$$

$$= \hat{\beta}_1 \frac{\sqrt{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}}{\sqrt{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2}}$$

11.4.10 $r = \dfrac{n\sum_{i=1}^{n} x_iy_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sqrt{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}\sqrt{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2}}$

$$= \frac{21(7{,}319{,}602) - (45{,}110)(3042.2)}{\sqrt{21(109{,}957{,}100) - (45{,}110)^2}\sqrt{21(529{,}321.58) - (3042.2)^2}} = 0.730$$

Since $r^2 = (0.730)^2 = 0.5329$, we can say that 53.3% of the variability is explained by cigarette consumption.

11.4.11 $r = \dfrac{n\sum_{i=1}^{n} x_iy_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sqrt{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}\sqrt{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2}}$

$$= \frac{12(480{,}565) - (4936)(1175)}{\sqrt{12(3{,}071{,}116) - (4936)^2}\sqrt{12(123{,}349) - (1175)^2}} = -0.030.$$

The data do not suggest that altitude affects home run hitting.

11.4.12 $r = \dfrac{10(325.08)-(123.1)(25.80)}{\sqrt{10(1529.63)-(123.1)^2}\sqrt{10(74.00)-(25.80)^2}} = 0.726$

11.4.13 $r = \dfrac{17(4,759,470)-(7,973)(8,517)}{\sqrt{17(4,611,291)-(7,973)^2}\sqrt{17(5,421,917)-(8,517)^2}} = 0.762$. The amount of variation attributed to the linear regression is $r^2 = (0.762)^2 = 0.581$, or 58.1%.

11.4.14 $r = \dfrac{18(221.37)-(-0.9)(160.2)}{\sqrt{18(92.63)-(-0.9)^2}\sqrt{18(6437.68)-(160.2)^2}} = 0.337$. The amount of variation attributed to the linear regression is $r^2 = (0.337)^2 = 0.114$, or 11.4%.

Section 11.5: The Bivariate Normal Distribution

11.5.1 Y is a normal random variable with $E(Y) = 6$ and $\text{Var}(Y) = 10$. Then $P(5 < Y < 6.5) =$ $P\left(\dfrac{5-6}{\sqrt{10}} < Z < \dfrac{6.5-6}{\sqrt{10}}\right) = P(-0.32 < Z < 0.16) = 0.5636 - 0.3745 = 0.1891$. By Theorem 11.5.1,

$Y|2$ is normal with $E(Y|2) = \mu_Y + \dfrac{\rho\sigma_Y}{\sigma_X}(2-\mu_X) = 6 + \dfrac{\frac{1}{2}\sqrt{10}}{2}(2-3) = 5.209$

$\text{Var}(Y|2) = (1-\rho^2)\sigma_Y^2 = (1-0.25)10 = 7.5$, so the standard deviation of Y is $\sqrt{7.5} = 2.739$.

$P(5 < Y|2 < 6.5) = P\left(\dfrac{5-5.209}{2.739} < Z < \dfrac{6.5-5.209}{2.739}\right) = P(-0.08 < Z < 0.47) = 0.6808 - 0.4681$

$= 0.2127$

11.5.2 **(a)** The lemma on page 424 can be used to show that X and $Y - \rho X$ are bivariate normal. Thus it suffices to show that $\text{Cov}(X, Y - \rho X) = 0$.

$\text{Cov}(X, Y-\rho X) = E[X(Y-\rho X)] - E(X)E(Y-\rho X) = E(XY) - \rho E(X^2) - E(X)E(Y) + \rho E(X)^2$

$= \text{Cov}(X, Y) - \rho\text{Var}(X) = \text{Cov}(X, Y) - \dfrac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}}\text{Var}(X)$

$= \text{Cov}(X, Y) - \text{Cov}(X, Y) = 0$, since $\text{Var}(X) = \text{Var}(Y)$.

(b) The lemma on page 424 can be used to show that $X+Y$ and $X-Y$ are bivariate normal. By Question 11.4.7, $\text{Cov}(X + Y, X - Y) = \text{Var}(X) - \text{Var}(Y)$. Since the variances are equal, $\text{Cov}(X + Y, X - Y) = 0$, and the two variables are independent.

11.5.3 **(a)** $f_{X+Y}(t) = \dfrac{1}{2\pi\sqrt{1-\rho^2}}\displaystyle\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{1}{1-\rho^2}\right)\left[(t-y)^2 - 2\rho(t-y)y + y^2\right]\right\}dy$

The expression in the brackets can be expanded and rewritten as
$t^2 + 2(1+\rho)y^2 - 2t(1+\rho)y = t^2 + 2(1+\rho)[y^2 - ty]$

$= t^2 + 2(1+\rho)\left[y^2 - ty + \frac{t^2}{4}\right] - \frac{1}{2}(1+\rho)t^2 = \frac{1-\rho}{2}t^2 + 2(1+\rho)(y-t/2)^2$. Placing this expression into the exponent gives

$$f_{X+Y}(t) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{1}{1-\rho^2}\right)\frac{1-\rho}{2}t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{1}{1-\rho^2}\right)2(1+\rho)(y-t/2)^2} dy$$

$$= f_{X+Y}(t) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{t^2}{2(1+\rho)}\right)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{(y-t/2)^2}{(1+\rho)/2}\right)} dy.$$

The integral is that of a normal pdf with mean $t/2$ and $\sigma^2 = (1+\rho)/2$.
Thus, the integral equals $\sqrt{2\pi(1+\rho)/2} = \sqrt{\pi(1+\rho)}$.
Putting this into the expression for f_{X+Y} gives

$f_{X+Y}(t) = \frac{1}{\sqrt{2\pi}\sqrt{2(1+\rho)}} e^{-\frac{1}{2}\left(\frac{t^2}{2(1+\rho)}\right)}$, which is the pdf of a normal variable with $\mu = 0$ and $\sigma^2 = 2(1+\rho)$.

(b) $E(X+Y) = c\mu_X + d\mu_Y$; $\text{Var}(X+Y) = c^2\sigma_X^2 + d^2\sigma_Y^2 + 2cd\sigma_X\sigma_Y\rho(X,Y)$

11.5.4 $E(Y|55) = \mu_Y + \frac{\rho\sigma_Y}{\sigma_X}(55-\mu_X) = 11 + \frac{0.6\sqrt{2.6}}{\sqrt{1.2}}(55-56) = 10.117$

$\text{Var}(Y|55) = (1-\rho^2)\sigma_Y^2 = (1-0.6^2)2.6 = 1.664$, so the standard deviation of Y is $\sqrt{1.664} = 1.290$.

$P(10 \le Y \le 10.5|x = 55) = P\left(\frac{10-10.117}{1.290} \le Z \le \frac{10.5-10.117}{1.290}\right) = P(-0.09 \le Z \le 0.30)$

$= 0.6179 - 0.4641 = 0.1538$

The mean of $\bar{Y}$ also $= 10.117$. However, the standard deviation is $1.290/\sqrt{4} = 0.645$. Then

$P(10.5 \le \bar{Y} \le 11|x = 55) = P\left(\frac{10.5-10.117}{0.645} \le Z \le \frac{11-10.117}{0.645}\right) = P(0.59 \le Z \le 1.37)$

$= 0.9147 - 0.7224 = 0.1923$

11.5.5 $E(X) = E(Y) = 0$; $\text{Var}(X) = 4$; $\text{Var}(Y) = 1$; $\rho(X, Y) = 1/2$; $k = 1/(2\pi\sqrt{3})$

11.5.6 $-(ax^2 - 2uxy + by^2) = -\frac{1}{2}\left(\frac{1}{1-\rho^2}\right)\left(\frac{x^2}{\sigma_X^2} - 2\rho\frac{x}{\sigma_X}\frac{y}{\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)$

so we get the following equations:

$a = \frac{1}{2}\left(\frac{1}{1-\rho^2}\right)\frac{1}{\sigma_X^2}$; $b = \frac{1}{2}\left(\frac{1}{1-\rho^2}\right)\frac{1}{\sigma_Y^2}$; and $u = \frac{1}{2}\left(\frac{1}{1-\rho^2}\right)\frac{\rho}{\sigma_X\sigma_Y}$. From the first two equations we obtain $\frac{1}{\sigma_X} = \sqrt{2a(1-\rho^2)}$ and $\frac{1}{\sigma_Y} = \sqrt{2b(1-\rho^2)}$. Substituting these values in the expression for u gives $u = \frac{1}{2}\left(\frac{1}{1-\rho^2}\right)\rho\sqrt{2a(1-\rho^2)}\sqrt{2b(1-\rho^2)} = \rho\sqrt{a}\sqrt{b}$, or $\rho = \frac{u}{\sqrt{a}\sqrt{b}}$.

From this equation, we get the only conditions on the parameters a, b, and u:

$$\left|\frac{u}{\sqrt{a}\sqrt{b}}\right| = |\rho| \le 1 \text{ or } |u| \le \sqrt{a}\sqrt{b}$$

Then we can solve for σ_X^2 and σ_Y^2 in terms of a, b, and ρ: $\sigma_X^2 = \frac{1}{2}\left(\frac{1}{1-\rho^2}\right)\frac{1}{a}$;

$$\sigma_Y^2 = \frac{1}{2}\left(\frac{1}{1-\rho^2}\right)\frac{1}{b}$$

11.5.7 $r = -0.453$. $T_{18} = \frac{\sqrt{n-2}\,r}{\sqrt{1-r^2}} = \frac{\sqrt{18}(-0.453)}{\sqrt{1-(-0.453)^2}} = -2.16$

Since $-t_{.005,18} = -2.8784 < T_{18} = -2.16 < 2.8784 = t_{.005,18}$, accept H_0.

11.5.8 $r = \frac{14(710{,}499)-(2458)(4097)}{\sqrt{14(444{,}118)-(2458)^2}\sqrt{14(1{,}262{,}559)-(4097)^2}} = -0.312$

$$T_{12} = \frac{\sqrt{n-2}\,r}{\sqrt{1-r^2}} = \frac{\sqrt{12}(-0.312)}{\sqrt{1-(-0.312)^2}} = -1.14$$

Since $-t_{.025,12} = -2.1788 < T_{12} = -1.14 < 2.1788 = t_{.025,12}$, accept H_0. The data sets appear to be independent.

11.5.9 From Question 11.4.11, $r = -0.030$. $T_{10} = \frac{\sqrt{10}(-0.030)}{\sqrt{1-(-0.030)^2}} = -0.09$.

Since $-t_{.025,10} = -2.2281 < T_{10} = -0.09 < 2.2281 = t_{.025,10}$, accept H_0.

11.5.10 From Question 11.4.13, $r = 0.762$. $T_{15} = \frac{\sqrt{15}(0.762)}{\sqrt{1-(0.762)^2}} = 4.56$.

Since $T_{15} = 4.56 > 2.9467 = t_{.005,15}$ reject H_0.

11.5.11 $r = \frac{10(1349.66)-(18.33)(738)}{\sqrt{10(34.1267)-(18.33)^2}\sqrt{10(54756)-(738)^2}} = -0.249$

Since the correlation coefficient is negative, there is no need to test $H_1: \rho > 0$.
To see if there is any effect at all, one could test against $H_1: \rho \ne 0$. In that case the test statistic is $\frac{\sqrt{8}(-0.249)}{\sqrt{1-(-0.249)^2}} = 0.727$. Since the test statistic lies between $-t_{.05,8} = -1.8595$ and $t_{.05,8} = 1.8595$, do not reject H_0

Chapter 12: The Analysis of Variance

Section 12.2: The *F* test

12.2.1 Here $n = 10$ and $k = 4$. To test H_0: $\mu_A = \mu_B = \mu_C = \mu_D$ at the $\alpha = 0.05$ level, reject the null hypothesis if $F \geq F_{.95,4-1,10-4} = 4.76$. At the $\alpha = 0.10$ level, H_0 is rejected if $F \geq F_{.90,3,6} = 3.29$. As the ANOVA table shows, the observed F falls between the two cutoffs, meaning that H_0 is rejected at the $\alpha = 0.10$ level, but not at the $\alpha = 0.05$ level.

Source	df	SS	MS	F
Model	3	61.33	20.44	3.94
Error	6	31.17	5.19	
Total	9	92.50		

12.2.2 Let μ_1, μ_2, and μ_3 denote the true average magnetic field declinations characteristic of the time periods 1669, 1780, and 1865 respectively. To test H_0: $\mu_1 = \mu_2 = \mu_3$ at the $\alpha = 0.05$ level, reject the null hypothesis if $F \geq F_{.95,3-1,9-3} = 5.14$. For these data, $F = 15.28$, implying that the magnetic field has shifted over the time period spanned by the eruptions. (Of course, the fact that $P = 0.004$ is less than $\alpha = 0.05$ also implies that H_0 should be rejected).

Source	df	SS	MS	F	P
Time	2	90.03	45.01	15.28	0.004
Error	6	17.67	2.95		
Total	8	107.70			

12.2.3 For these $n = 30$ observations and $k = 3$ treatment groups, $C = T_{..}^2/n = (422.9)^2/30$

$= 5961.48$, SSTOT $= \sum_{j=1}^{3}\sum_{i=1}^{10} y_{ij}^2 - C = 914.1$, and SSTR $= (121.4)^2/10 + (176.1)^2/10$

$+ (125.4)^2/10 - 5961.48 = 186.0$. To test the null hypothesis that the three types of stocks have equal price-earnings ratios, reject H_0 if $F \geq F_{.99,3-1,30-3} = F_{.99,2,27}$. The latter is not a cutoff that appears in Table A.4 of the Appendix. However, its value can be bounded by cutoffs with similar degrees of freedom that are listed: $F_{.99,2,30} = 5.39 < F_{.99,2,27} < F_{.99,2,24} = 5.61$. According to the ANOVA table, the observed F ratio equals 3.45, which implies that H_0 should not be rejected.

Source	df	SS	MS	F
Sector	2	186.0	93.0	3.45
Error	27	728.2	27.0	
Total	29	914.1		

12.2.4 Let μ_i = true average yield for Variety i, $i = 1, 2, \ldots, 5$. Then H_0: $\mu_1 = \mu_2 = \ldots = \mu_5$ should be rejected at the $\alpha = 0.05$ level if $F \geq F_{.95,5-1,15-5} = 3.48$. The ANOVA table shows that $F = 6.39$, which implies that the differences among the sample means are statistically significant.

Source	df	SS	MS	F	P
Varieties	4	270.3	67.6	6.39	0.008
Error	10	105.7	10.6		
Total	14	375.9			

12.2.5 To test at the $\alpha = 0.01$ level of significance the null hypothesis that the four tribes were contemporaries of one another, H_0 should be rejected if $F \geq F_{.99,4-1,12-4} = 7.59$ (or if $P < 0.01$). According to the ANOVA table, F is less than 7.59 (and P is greater than 0.01), so H_0 should not be rejected.

Source	df	SS	MS	F	P
Tribe	3	504167	168056	3.70	0.062
Error	8	363333	45417		
Total	11	867500			

12.2.6

Source	df	SS	MS	F
Group	2	275.4	137.7	7.37
Error	12	224.2	18.7	
Total	14	499.6		

Since the observed $F = 7.37 > 3.89 = F_{.95,2,12}$, then reject the null hypothesis that the means are equal.

12.2.7

Source	df	SS	MS	F
Treatment	4	271.36	67.84	6.40
Error	10	106.00	10.60	
Total	14	377.36		

12.2.8 The sample variances for Treatments A, C, and D are much smaller than the sample variance for Treatment B, suggesting that the assumption that σ^2 is the same for all treatment levels may not be true.

12.2.9 SSTOT = $\sum_{j=1}^{k}\sum_{i=1}^{n_j}(Y_{ij}-\overline{Y}_{..})^2 = \sum_{j=1}^{k}\sum_{i=1}^{n_j}(Y_{ij}^2 - 2Y_{ij}\overline{Y}_{..} + \overline{Y}_{..}^2) = \sum_{j=1}^{k}\sum_{i=1}^{n_j}Y_{ij}^2 - 2\overline{Y}_{..}\sum_{j=1}^{k}\sum_{i=1}^{n_j}Y_{ij} + n\overline{Y}_{..}^2$

$= \sum_{j=1}^{k}\sum_{i=1}^{n_j}Y_{ij}^2 - 2n\overline{Y}_{..}^2 + n\overline{Y}_{..}^2 = \sum_{j=1}^{k}\sum_{i=1}^{n_j}Y_{ij}^2 - n\overline{Y}_{..}^2 = \sum_{j=1}^{k}\sum_{i=1}^{n_j}Y_{ij}^2 - C$, where $C = T_{..}^2/n$.

Also, SSTR = $\sum_{j=1}^{k}\sum_{i=1}^{n_j}(\overline{Y}_{.j}^2 - \overline{Y}_{..})^2 = \sum_{j=1}^{k}n_j(\overline{Y}_{.j}^2 - 2\overline{Y}_{.j}\overline{Y}_{..} + \overline{Y}_{..}^2) = \sum_{j=1}^{k}T_{.j}^2/n_j - 2\overline{Y}_{..}\sum_{j=1}^{k}n_j\overline{Y}_{.j} + n\overline{Y}_{..}^2$

$= \sum_{j=1}^{n}T_{.j}^2/n_j - 2n\overline{Y}_{..}^2 + n\overline{Y}_{..}^2 = \sum_{j=1}^{k}T_{.j}^2/n_j - C.$

12.2.10 SSTR/σ^2 = $(1/\sigma^2)\sum_{j=1}^{k}n_j(\overline{Y}_{.j}-\overline{Y}_{..})^2 = (1/\sigma^2)\sum_{j=1}^{k}n_j[(\overline{Y}_{.j}-\mu)-(\overline{Y}_{..}-\mu)]^2$

$$= (1/\sigma^2)\left[\sum_{j=1}^{k}n_j[(\overline{Y}_{.j}-\mu)^2 - n(\overline{Y}_{..}-\mu)^2\right] = \sum_{j=1}^{k}\left(\frac{\overline{Y}_{.j}-\mu}{\sigma/\sqrt{n_j}}\right)^2 - \left(\frac{\overline{Y}_{..}-\mu}{\sigma/\sqrt{n}}\right)^2.$$

Since the $\left(\dfrac{\overline{Y}_{.j}-\mu}{\sigma/\sqrt{n_j}}\right)$'s are independent normal random variables, and since $\left(\dfrac{\overline{Y}_{..}-\mu}{\sigma/\sqrt{n}}\right)$ can be written as a linear combination of the $\left(\dfrac{\overline{Y}_{.j}-\mu}{\sigma/\sqrt{n_j}}\right)$, it follows from Fisher's lemma that SSTR/σ^2 has a χ^2 distribution with $k-1$ df.

12.2.11 Analyzed with a two-sample t test, the data in Question 9.2.8 require that H_0: $\mu_X = \mu_Y$ be rejected (in favor of a two-sided H_1) at the $\alpha = 0.05$ level if $|t| \geq t_{.025,6+9-2} = 2.1604$. Evaluating the test statistic gives $t = (70.83-79.33)/11.31\sqrt{1/6+1/9} = -1.43$, which implies that H_0 should not be rejected. The ANOVA table for the same data shows that $F = 2.04$. But $(-1.43)^2 = 2.04$. Moreover, H_0 would be rejected with the analysis of variance if $F \geq F_{.95,1,13} = 4.667$. But $(2.1604)^2 = 4.667$.

Source	df	SS	MS	F
Sex	1	260	260	2.04
Error	13	1661	128	
Total	14	1921		

12.2.12

Source	df	SS	MS	F	P
Author	1	0.002185	0.002185	15.04	0.001
Error	16	0.002325	0.000145		
Total	17	0.004510			

From Case Study 9.2.1, $t = 3.88$; except for a small rounding error, the square of the observed t ratio is the same as the observed F ratio: $(3.88)^2 = 15.05 \doteq 15.04$.

12.2.13

Source	df	SS	MS	F	P
Law	1	16.333	16.333	1.58	0.2150
Error	46	475.283	10.332		
Total	47	491.617			

The F critical value is 4.05. For the pooled two-sample t test, the observed t ratio is -1.257, and the critical value is 2.1029. Note that $(-1.257)^2 = 1.58$ (rounded to two decimal places) which is the observed F ratio. Also, $2.0129^2 = 4.05$ (rounded to two decimal places), which is the F critical value.

Section 12.3: Multiple Comparisons: Tukey's Method

12.3.1 For the data in Case Study 12.2.1, $k = 4$, $r = 6$, MSE = 79.74, and $D = Q_{.05,4,20}/\sqrt{6}$ $= 3.96/\sqrt{6} = 1.617$. Let μ_1, μ_2, μ_3, and μ_4 denote the true average heart rates for Non-smokers, Light smokers, Moderate smokers, and Heavy smokers, respectively. Substituting into Theorem 12.3.1 gives the six different Tukey intervals summarized in the table below.

Pairwise Difference	Tukey interval	Conclusion
$\mu_1 - \mu_2$	(−15.27, 13.60)	NS
$\mu_1 - \mu_3$	(−23.77, 5.10)	NS
$\mu_1 - \mu_4$	(−33.77, −4.90)	Reject
$\mu_2 - \mu_3$	(−22.94, 5.94)	NS
$\mu_2 - \mu_4$	(−32.94, −4.06)	Reject
$\mu_3 - \mu_4$	(−24.44, 4.44)	NS

12.3.2 Let μ_1, μ_2, and μ_3 denote the true average price-earnings ratios for stocks in the financial, industrial, and utility sectors, respectively. The data's ANOVA table shows that MSE = 27.0. Moreover, in the terminology of Theorem 12.3.1, $k = 3$, $r = 10$, and

Source	df	SS	MS	F
Sector	2	186.0	93.0	3.45
Error	27	728.2	27.0	
Total	29	914.1		

$D = Q_{.05,3,27}/\sqrt{10} = 3.51/\sqrt{10} = 1.11$. The corresponding 95% Tukey confidence intervals are listed in the table.

Pairwise Difference	Tukey interval	Conclusion
$\mu_1 - \mu_2$	(−11.234, 0.294)	NS
$\mu_1 - \mu_3$	(−6.164, 5.364)	NS
$\mu_2 - \mu_3$	(−0.694, 10.834)	NS

12.3.3 Let μ_C, μ_A, and μ_M denote the true average numbers of contaminant particles in IV fluids produced by Cutter, Abbott, and McGaw, respectively. According to the analysis of variance, H_0: $\mu_C = \mu_A = \mu_M$ is rejected at the $\alpha = 0.05$ level (since the P value is less than 0.05).

Source	df	SS	MS	F	P
Company	2	113646	56823	5.81	0.014
Error	15	146754	9784		
Total	17	260400			

The three 95% Tukey confidence intervals (based on $k = 3$, $r = 6$, and $D = Q_{.05,3,15}/\sqrt{6} = 3.67/\sqrt{6}$ $= 1.498$) show that Abbott and McGaw have the only pairwise difference (204.50 − 396.67 = −192.17) that is statistically significant.

Pairwise Difference	Tukey interval	Conclusion
$\mu_C - \mu_A$	(−78.9, 217.5)	NS
$\mu_C - \mu_M$	(−271.0, 25.4)	NS
$\mu_A - \mu_M$	(−340.0, −44.0)	Reject

12.3.4 Since $k = 5$ and $r = 3$, $D = Q_{.05,5,10}/\sqrt{3} = 4.65/\sqrt{3} = 2.685$. Also, from the ANOVA table,

Source	df	SS	MS	F
Varieties	4	270.3	67.6	6.39
Error	10	105.7	10.6	
Total	14	375.9		

MSE = 10.6. Substituting into Theorem 12.3.1, then, gives the following set of 95% Tukey confidence intervals:

Pairwise Difference	Tukey interval	Conclusion
$\mu_1 - \mu_2$	(−14.861, 2.594)	NS
$\mu_1 - \mu_3$	(−19.594, −2.139)	Reject
$\mu_1 - \mu_4$	(−8.094, 9.361)	NS
$\mu_1 - \mu_5$	(−11.727, 5.727)	NS
$\mu_2 - \mu_3$	(−13.461, 3.994)	NS
$\mu_2 - \mu_4$	(−1.961, 15.494)	NS
$\mu_2 - \mu_5$	(−5.594, 11.861)	NS
$\mu_3 - \mu_4$	(2.773, 20.227)	Reject
$\mu_3 - \mu_5$	(−0.861, 16.594)	NS
$\mu_4 - \mu_5$	(−12.361, 5.094)	NS

12.3.5 Since $k = 3$ and $r = 3$, $D = Q_{.05,3,6}/\sqrt{3} = 4.34/\sqrt{3} = 2.506$. MSE = 41.666, so the radius of the intervals is $D\sqrt{\text{MSE}} = 2.506\sqrt{41.667} = 16.17$.

Pairwise Difference	Tukey interval	Conclusion
$\mu_1 - \mu_2$	(−29.5, 2.8)	NS
$\mu_1 - \mu_3$	(−56.2, −23.8)	Reject
$\mu_2 - \mu_3$	(−42.8, −10.5)	Reject

12.3.6 No. Look at the Tukey intervals constructed in Question 12.3.4, for example. The hypotheses H_0: $\mu_1 = \mu_3$ and H_0: $\mu_3 = \mu_4$ are rejected, but H_0: $\mu_1 = \mu_4$ is not.

12.3.7 Longer. As k gets larger, the number of possible pairwise comparisons increases. To maintain the same overall probability of committing at least one Type I error, the individual intervals would need to be widened.

Section 12.4: Testing Subhypotheses with Contrasts

12.4.1

Source	df	SS	MS	F
Tube	2	510.7	255.4	11.56
Error	42	927.7	22.1	
Total	44	1438.4		

Subhypothesis	Contrast	SS	F
H_0: $\mu_A = \mu_C$	$C_1 = \mu_A - \mu_C$	264.0	11.95
H_0: $\mu_B = \dfrac{\mu_A + \mu_C}{2}$	$C_2 = \dfrac{1}{2}\mu_A - \mu_B + \dfrac{1}{2}\mu_C$	246.7	11.16

H_0: $\mu_A = \mu_B = \mu_C$ is strongly rejected ($F_{.99,2,42} \doteq F_{.99,2,40} = 5.18$). Theorem 12.4.1 holds true for orthogonal contrasts C_1 and C_2—$SS_{C_1} + SS_{C_2} = 264.0 + 246.7 = 510.7 = SSTR$. Also, both subhypotheses would be rejected—their F ratios exceed $F_{.99,1,42}$.

12.4.2 To test H_0: $(\mu_1 + \mu_2 + \mu_3)/3 = (\mu_4 + \mu_5)/2$, let $C = \frac{1}{3}\mu_1 + \frac{1}{3}\mu_2 + \frac{1}{3}\mu_3 - \frac{1}{2}\mu_4 - \frac{1}{2}\mu_5$. Then

$$\hat{C} = \sum_{j=1}^{5} c_j \overline{Y}_{.j} = \frac{1}{3}(48.933) + \frac{1}{3}(55.067) + \frac{1}{3}(59.800) - \frac{1}{2}(48.300) - \frac{1}{2}(51.933) = 4.483.$$ Also,

$$SS_C = (4.483)^2 / \left[\frac{(1/3)^2}{3} + \frac{(1/3)^2}{3} + \ldots + \frac{(-1/2)^2}{3}\right] = 72.34.$$ From the ANOVA table,

$SSE = 105.7$ and $n - k = 10$, so H_0 should be rejected if $F \geq F_{.95,1,10} = 4.96$.

Source	df	SS	MS	F
Varieties	4	270.3	67.6	6.39
Error	10	105.7	10.6	
Total	14	375.9		

But, $F = \dfrac{72.34/1}{105.7/10} = 6.84$, implying that the subhypothesis is rejected at the $\alpha = 0.05$ level.

12.4.3 Let μ_1, μ_2, μ_3, and μ_4 denote the true average heart rates for Non-smokers, Light smokers, Moderate smokers, and Heavy smokers, respectively. To test H_0: $(\mu_2 + \mu_3)/2 = \mu_4$, let $C = \frac{1}{2}\mu_2 + \frac{1}{2}\mu_3 - \mu_4$, so $\hat{C} = \frac{1}{2}(63.2) + \frac{1}{2}(71.7) - 1(81.7) = -14.25$. Also,

$$SS_C = (-14.25)^2 / \left[\frac{(1/2)^2}{6} + \frac{(1/2)^2}{6} + \frac{(-1)^2}{6}\right] = 812.25.$$ From the ANOVA table on p. 604, SSE = 1594.833 and $n - k = 20$. Therefore, H_0 should be rejected if $F \geq F_{.95,1,20} = 4.35$. Here $F = \dfrac{812.25/1}{1594.833/20} = 10.19$, so H_0: $(\mu_2 + \mu_3)/2 = \mu_4$ is rejected at the $\alpha = 0.05$ level.

12.4.4 Let μ_1, μ_2, and μ_3 denote the profitability for small medium, and large companies, respectively. To test $H_0: \dfrac{\mu_1+\mu_2}{2} = \mu_3$ against the alternative that they are not equal, let

$C = \dfrac{1}{2}\mu_1 + \dfrac{1}{2}\mu_2 - \mu_3$, so $\hat{C} = \dfrac{1}{2}(5.5) + \dfrac{1}{2}(7.1) - 1(8.8) = -2.5$. Also,

$SS_C = (-2.5)^2 / \left[\dfrac{(1/2)^2}{7} + \dfrac{(1/2)^2}{7} + \dfrac{(-1)^2}{7}\right] = 29.167$. The given SSE = 147.17429 and $n - k = 18$.

Therefore, H_0 should be rejected if $F \geq F_{.90,1,18}$, which is approximately 3.00.

Here $F = \dfrac{29.167/1}{147.17429\ /18} = 3.57$, so reject the null hypothesis.

12.4.5

	μ_A	μ_B	μ_C	μ_D	$\sum_{j=1}^{4} c_j$
C_1	1	–1	0	0	0
C_2	0	0	1	–1	0
C_3	$\frac{11}{12}$	$\frac{11}{12}$	–1	$\frac{-5}{6}$	0

C_1 and C_3 are orthogonal because $\dfrac{1(11/12)}{6} + \dfrac{(-1)(11/12)}{6} = 0$; also, C_2 and C_3 are orthogonal because $\dfrac{1(-1)}{6} + \dfrac{(-1)(-5/6)}{5} = 0$. $\hat{C}_3 = -2.293$ and $SS_{C_3} = 8.97$. But $SS_{C_1} + SS_{C_2} + SS_{C_3} = 4.68 + 1.12 + 8.97 = 14.77 =$ SSTR.

12.4.6 To test H_0: $\mu_A = \mu_B$ and H_0: $(\mu_A + \mu_B)/2 = (\mu_C + \mu_D)/2$, the two associated contrasts are $C_1 = \mu_A - \mu_B$ and $C_2 = \dfrac{1}{2}\mu_A + \dfrac{1}{2}\mu_B - \dfrac{1}{2}\mu_C - \dfrac{1}{2}\mu_D$. Clearly, the two are orthogonal. By inspection, a third orthogonal contrast is $C_3 = \mu_C - \mu_D$, which tests the subhypothesis H_0: $\mu_C = \mu_D$. Since $\bar{y}_A = 26.00$ $\bar{y}_B = 40.33$, $\bar{y}_C = 22.00$, and $\bar{y}_D = 28.67$, $\hat{C}_1 = -14.33$, $\hat{C}_2 = 7.83$, and $\hat{C}_3 = -6.67$.

Therefore, $SS_{C_1} = (-14.33)^2/\left[\dfrac{(1)^2}{3} + \dfrac{(-1)^2}{3}\right] = 308.01$,

$$SS_{C_2} = (7.83)^2 / \left[\dfrac{\left(\frac{1}{2}\right)^2}{3} + \dfrac{\left(\frac{1}{2}\right)^2}{3} + \dfrac{\left(-\frac{1}{2}\right)^2}{3} + \dfrac{\left(-\frac{1}{2}\right)^2}{3}\right] = 183.95, \text{ and}$$

$SS_{C_3} = (-6.67)^2 / \left[\dfrac{(1)^2}{3} + \dfrac{(-1)^2}{3}\right] = 66.73$. According to the ANOVA table, SSTR = 559(= 308.01 + 183.95 + 66.73). Each subhypothesis should be rejected if $F \geq F_{.95,1,8} = 5.32$.

Source	df	SS	MS	F
Brand	3	559	186	0.66
Error	8	2241	280	
Total	11	2800		

For C_1, $F = \frac{308.01/1}{2241/8} = 1.10$; for C_2, $F = \frac{183.95/1}{2241/8} = 0.66$; and for C_3, $F = \frac{66.73/1}{2241/8} = 0.24$. It follows that none of the three subhypotheses should be rejected.

Section 12.5: Data Transformations

12.5.1 Replace each observation by its square root. At the $\alpha = 0.05$ level, H_0: $\mu_A = \mu_B$ is rejected. (For $\alpha = 0.01$, though, we would fail to reject H_0).

Source	df	SS	MS	F	P
Developer	1	1.836	1.836	6.23	0.032
Error	10	2.947	0.295		
Total	11	4.783			

12.5.2 For each treatment group, $\bar{y}_{.j} = s_j^2$. The latter is the relationship that would be expected if the underlying random variable has a Poisson distribution (recall Theorem 4.2.2). Therefore, each observation should be replaced by its square root before doing the analysis of variance (as justified in Example 12.5.1).

12.5.3 Since Y_{ij} is a binomial random variable based on $n = 20$ trials, each data point should be replaced by the arcsin of $(y_{ij}/20)^{\frac{1}{2}}$. Based on those transformed observations, H_0: $\mu_A = \mu_B = \mu_C$ is strongly rejected ($P < 0.001$).

Source	df	SS	MS	F	P
Launcher	2	0.30592	0.15296	22.34	0.000
Error	9	0.06163	0.00685		
Total	11	0.36755			

Appendix 12.A.3: The Distribution of $\frac{SSTR/(k-1)}{SSE/(n-k)}$ When H_1 Is True

12.A.3.1 The F test will have greater power against H_1** because the latter yields a larger noncentrality parameter than does H_1*.

12.A.3.2 By definition, $\mu = \frac{1}{n}\sum_{j=1}^{k} n_j\mu_j$. If a requirement is imposed to the effect that μ must remain constant, then H_1 is inadmissible because the H_0 set of μ_j's imply that $\mu = 0$, whereas under H_1, $\sum_{j=1}^{k}\mu_j = +1$. However, if no such condition is imposed on μ, then H_1 is admissible.

12.A.3.3 $M_V(t) = (1-2t)^{-r/2}e^{\gamma t(1-2t)^{-1}}$, so $M_V^{(1)}(t) = (1-2t)^{-r/2} \cdot e^{\gamma t(1-2t)^{-1}}$ $[\gamma t(-1)(1-2t)^{-2}(-2) + (1-2t)^{-1}\gamma] + e^{\gamma t(1-2t)^{-1}}\left(-\frac{r}{2}\right)(1-2t)^{-(r/2)-1}(-2)$. Therefore, $E(V) = M_V^{(1)}(0) = \gamma + r$.

12.A.3.4 By Question 12.A.3.3, the expected value of V, a noncentral χ^2 random variable with r degrees of freedom and noncentrality parameter γ is $E(V) = \gamma + r$. Therefore, $\gamma = E(V) - r$.

12.A.3.5 $M_V(t) = \prod_{i=1}^{n}(1-2t)^{-r_i/2}e^{\gamma_i t/(1-2t)} = (1-2t)^{-\sum_{i=1}^{n} r_i/2} \cdot e^{\left(\sum_{i=1}^{n}\gamma_i\right)t/(1-2t)}$, which implies that V has a noncentral χ^2 distribution with $\sum_{i=1}^{n} r_i$ df and with noncentrality parameter $\sum_{i=1}^{n}\gamma_i$.

Chapter 13: Randomized Block Designs

Section 13.2: The *F* Test for a Randomized Block Design

13.2.1

Source	df	SS	MS	F	P
States	1	61.63	61.63	7.20	0.0178
Students	14	400.80	28.63	3.34	0.0155
Error	14	119.87	8.56		
Total	29	582.30			

The critical value $F_{.95,1,14}$ is approximately 4.6. Since the F statistic = 7.20 > 4.6, reject H_0.

13.2.2

Source	df	SS	MS	F	P
Networks	2	17.11	8.56	8.72	0.0168
Cities	3	4.69	1.56	1.59	0.2868
Error	6	5.89	0.98		
Total	11	27.69			

Since the F statistic = 8.72 > $F_{.90,2,6}$ = 3.46, reject H_0.

13.2.3

Source	df	SS	MS	F	P
Additive	1	0.03	0.03	4.19	0.0865
Batch	6	0.02	0.00	0.41	0.8483
Error	6	0.05	0.01		
Total	13	0.10			

Since the F statistic = 4.19 < $F_{.95,1,6}$ = 5.99, accept H_0.

13.2.4

Source	df	SS	MS	F	P
Region	2	4.04	2.02	8.28	0.0188
Year	3	3.82	1.27	5.22	0.0414
Error	6	1.46	0.24		
Total	11	9.33			

Since the F statistic = 8.28 > $F_{.95,2,6}$ = 5.14, reject H_0.

13.2.5 From the Table 13.2.9, we obtain MSE = 6.00. The radius of the Tukey interval is $D\sqrt{\text{MSE}} = (Q_{.05,3,22}/\sqrt{b})\sqrt{6.00} = (3.56/\sqrt{12})\sqrt{6.00} = 2.517$. The Tukey intervals are

Pairwise Difference	$\bar{y}_{.s} - \bar{y}_{.t}$	Tukey Interval	Conclusion
$\mu_1 - \mu_2$	−2.41	(−4.93, 0.11)	NS
$\mu_1 - \mu_3$	−0.54	(−3.06, 1.98)	NS
$\mu_2 - \mu_3$	1.87	(−0.65, 4.39)	NS

From this analysis and that of Case Study 13.2.3, we find that the significant difference occurs not for overall means testing or pairwise comparisons, but for the comparison of "during the full moon" with "not during the full moon".

13.2.6

Source	df	SS	MS	F	P
Quarter	3	14.05	4.68	0.25	0.860
Year	4	43.56	10.89	0.58	0.683
Error	12	255.30	18.78		
Total	19	282.91			

Since the F statistic for treatments = 0.25 < $F_{.95,3,12}$ = 3.49, accept H_0 that yields are not affected by the quarter. Since the F statistic for blocks = 0.58 < $F_{.95,4,12}$ = 3.26, accept H_0 that yields are not affected by the year.

13.2.7 From Question 13.2.2 we obtain the value MSE = 0.98. The radius of the interval is $D\sqrt{\text{MSE}} = (Q_{.05,3,6}/\sqrt{b})\sqrt{0.98} = (4.34/\sqrt{4})\sqrt{0.98}$ = 2.148. The Tukey intervals are

Pairwise Difference	$\bar{y}_{.s} - \bar{y}_{.t}$	Tukey Interval	Conclusion
$\mu_1 - \mu_2$	2.925	(0.78, 5.07)	Reject
$\mu_1 - \mu_3$	1.475	(−0.67, 3.62)	NS
$\mu_2 - \mu_3$	−1.450	(−3.60, 0.70)	NS

13.2.8 **(a)**

Source	df	SS	MS	F	P
System	3	1709.60	569.87	100.59	0.0000
Subject	9	193.53	21.50	3.80	0.0034
Error	27	152.96	5.67		
Total	39	2056.10			

Since the F statistic = 100.59 > $F_{.95,3,27}$ = 2.96, reject H_0.

(b) The radius of the interval is $D\sqrt{\text{MSE}} = (Q_{.05,4,27}/\sqrt{b})\sqrt{5.67}$. Since $Q_{.05,4,27}$ is not in the table, we use the slightly larger value $Q_{.05,4,24}$ = 3.90. Then the radius of the interval is $(3.90/\sqrt{10})\sqrt{5.67}$ = 2.937. The Tukey intervals are

Pairwise Difference	$\bar{y}_{.s} - \bar{y}_{.t}$	Tukey Interval	Conclusion
$\mu_1 - \mu_2$	18.30	(15.36, 21.24)	Reject
$\mu_1 - \mu_3$	11.42	(8.48, 14.36)	Reject
$\mu_1 - \mu_4$	9.58	(6.64, 12.52)	Reject
$\mu_2 - \mu_3$	−6.88	(−9.82, −3.94)	Reject
$\mu_2 - \mu_4$	−8.72	(−11.66, −5.78)	Reject
$\mu_3 - \mu_4$	−1.84	(−4.78, 1.10)	NS

13.2.9 **(a)**

Source	df	SS	MS	F	P
Sleep stages	2	16.99	8.49	4.13	0.0493
Shrew	5	195.44	39.09	19.00	0.0001
Error	10	20.57	2.06		
Total	17	233.00			

Since the F statistic = 4.13 > $F_{.95,\,2,\,10}$ = 4.10, reject H_0.

(b) The contrast associated with the subhypothesis is $C_1 = -\frac{1}{2}\mu_1 - \frac{1}{2}\mu_2 + \mu_3$, and

$\hat{C}_1 = -\frac{1}{2}(21.1) - \frac{1}{2}(19.1) + 18.983 = -1.117$. $SS_{C_1} = \dfrac{(-1.117)^2}{\left(-\frac{1}{2}\right)^2/6 + \left(-\frac{1}{2}\right)^2/6 + (1)^2/6}$ = 4.99.

$F = \dfrac{SS_{C_1}}{MSE} = \dfrac{4.99}{2.06}$ = 2.42. Since the observed F ratio = 2.42 < $F_{.95,\,1,\,10}$ = 4.96, accept the subhypothesis. Let a second orthogonal contrast be $C_2 = \mu_1 - \mu_2$. $\hat{C}_2$ = 21.1 − 19.1 = 2.0.

$SS_{C_2} = \dfrac{2.0^2}{(1)^2/6 + (-1)^2/6}$ = 12.0

Then SSTR = 16.99 = 4.99 + 12.00 = $SS_{C_1} + SS_{C_2}$

13.2.10 $C_1 = \mu_1 - \mu_3$ and $C_2 = \mu_2 - \mu_4$. Then take $C_3 = \frac{1}{2}\mu_1 - \frac{1}{2}\mu_2 + \frac{1}{2}\mu_3 - \frac{1}{2}\mu_4$. This contrast tests the subhypothesis $\dfrac{\mu_1 + \mu_3}{2} = \dfrac{\mu_2 + \mu_4}{2}$. $\hat{C}_1$ = 17.6 − 19.86 = −2.7. $SS_{C_1} = \dfrac{(-2.7)^2}{(1)^2/5 + (-1)^2/5}$ = 18.225.

$F = \dfrac{SS_{C_1}}{MSE} = \dfrac{18.225}{2.48}$ = 7.35. Since the observed F ratio = 7.35 > $F_{.90,\,1,\,12}$ = 3.18, reject the subhypothesis.

$\hat{C}_2$ = 20.3 − 16.4 = 3.9. $SS_{C_2} = \dfrac{(3.9)^2}{(1)^2/5 + (-1)^2/5}$ = 38.025

$F = \dfrac{SS_{C_2}}{MSE} = \dfrac{38.025}{2.48}$ = 15.33. Since the observed F ratio = 15.33 > $F_{.90,\,1,\,12}$ = 3.18, reject the subhypothesis.

$\hat{C}_3$ = (17.16 − 20.3 + 19.86 − 16.4)/2 = 0.16.

$SS_{C_3} = \dfrac{(0.16)^2}{\left(\frac{1}{2}\right)^2/5 + \left(-\frac{1}{2}\right)^2/5 + \left(\frac{1}{2}\right)^2/5 + \left(-\frac{1}{2}\right)^2/5}$ = 0.128

$F = \dfrac{SS_{C_3}}{MSE} = \dfrac{0.128}{2.48}$ = 0.05. Since the observed F ratio = 0.05 < $F_{.90,\,1,\,12}$ = 3.18, accept the subhypothesis.

Note that $SS_{C_1} + SS_{C_2} + SS_{C_3}$ = 18.225 + 38.025 + 0.128 = 56.378 = SSTR.

13.2.11 Equation 13.2.2: SSTR = $\sum_{i=1}^{b}\sum_{j=1}^{k}(\overline{Y}_{.j}-\overline{Y}_{..})^2 = b\sum_{j=1}^{k}(\overline{Y}_{.j}-\overline{Y}..)^2 = b\sum_{j=1}^{k}(\overline{Y}_{.j}^2-2\overline{Y}_{.j}\overline{Y}_{..}+\overline{Y}_{..}^2)$

$$= b\sum_{j=1}^{k}\overline{Y}_{.j}^2 - 2b\overline{Y}_{..}\sum_{j=1}^{k}\overline{Y}_{.j} + bk\overline{Y}_{..}^2 = b\sum_{j=1}^{k}\frac{T_{.j}^2}{b^2} - \frac{2T_{..}^2}{bk} + \frac{T_{..}^2}{bk} = \sum_{j=1}^{k}\frac{T_{.j}^2}{b} - \frac{T_{..}^2}{bk} = \sum_{j=1}^{k}\frac{T_{.j}^2}{b} - c$$

Equation 13.2.3: SSB = $\sum_{i=1}^{b}\sum_{j=1}^{k}(\overline{Y}_{i.}-\overline{Y}_{..})^2 = k\sum_{i=1}^{b}(\overline{Y}_{i.}-\overline{Y}_{..})^2 = k\sum_{i=1}^{b}(\overline{Y}_{i.}^2-2\overline{Y}_{i.}\overline{Y}_{..}+\overline{Y}_{..}^2)$

$$= k\sum_{i=1}^{b}\overline{Y}_{i.}^2 - 2k\overline{Y}_{..}\sum_{i=1}^{b}\overline{Y}_{i.} + bk\overline{Y}_{..}^2 = k\sum_{i=1}^{b}\frac{T_{i.}^2}{k^2} - \frac{2T_{..}^2}{bk} + \frac{T_{..}^2}{bk} = \sum_{i=1}^{b}\frac{T_{i.}^2}{k} - \frac{T_{..}^2}{bk} = \sum_{i=1}^{b}\frac{T_{i.}^2}{k} - c$$

Equation 13.2.4: SSTOT = $\sum_{i=1}^{b}\sum_{j=1}^{k}(Y_{ij}-\overline{Y}_{..})^2 = \sum_{i=1}^{b}\sum_{j=1}^{k}(Y_{ij}^2-2Y_{ij}\overline{Y}_{..}+\overline{Y}_{..}^2)$

$$= \sum_{i=1}^{b}\sum_{j=1}^{k}Y_{ij}^2 - 2\overline{Y}_{..}\sum_{i=1}^{b}\sum_{j=1}^{k}Y_{ij} + bk\overline{Y}_{..}^2 = \sum_{i=1}^{b}\sum_{j=1}^{k}Y_{ij}^2 - \frac{2T_{..}^2}{bk} + \frac{T_{..}^2}{bk} = \sum_{i=1}^{b}\sum_{j=1}^{k}Y_{ij}^2 - c$$

13.2.12 Fix $i = s$. For $i \neq s$, $\frac{\partial L}{\partial \beta_i} = 0$, and $\frac{\partial L}{\partial \beta_s} = \sum_{j=1}^{k}2(y_{sj}-\beta_s-\mu_j)(-1)$. Setting $\frac{\partial L}{\partial \beta_s} = 0$ gives

$$\hat{\beta}_s + \frac{1}{k}\sum_{j=1}^{k}\hat{\mu}_j = \frac{1}{k}\sum_{j=1}^{k}y_{sj} = \overline{y}_{s.} \qquad (13.2.12.1)$$

Fix $j = t$. For $j \neq t$, $\frac{\partial L}{\partial \mu_j} = 0$, and $\frac{\partial L}{\partial \mu_t} = \sum_{i=1}^{b}2(y_{it}-\beta_i-\mu_t)(-1)$. Setting $\frac{\partial L}{\partial \mu_t} = 0$ gives

$$\frac{1}{b}\sum_{i=1}^{b}\hat{\beta}_i + \hat{\mu}_t = \frac{1}{b}\sum_{i=1}^{b}y_{it} = \overline{y}_{.t} \qquad (13.2.12.2)$$

Rather than use a formal method to solve this system of bk equations in bk unknowns, let us make the educated guess that $\hat{\mu}_j = \overline{y}_{.j}$. Substituting this into Equation 13.2.12.1 gives

$\hat{\beta}_s + \frac{1}{k}\sum_{j=1}^{k}\overline{y}_{.j} = \overline{y}_{s.}$, or $\hat{\beta}_s = \overline{y}_{s.} - \overline{y}_{..}$. Placing $\hat{\beta}_s = \overline{y}_{s.} - \overline{y}_{..}$ and $\hat{\mu}_t = \overline{y}_{.t}$ into Equation 13.2.12.2 gives $\frac{1}{b}\sum_{i=1}^{b}(\overline{y}_{s.}-\overline{y}_{..}) + \overline{y}_{.t} = \overline{y}_{.t}$. Since the first term = 0, we do have equality.

13.2.13 **(a)** False. $\sum_{i=1}^{b}\overline{Y}_{i.} = \frac{1}{k}\sum_{i=1}^{b}\sum_{j=1}^{k}Y_{ij}$. $\sum_{j=1}^{k}\overline{Y}_{.j} = \frac{1}{b}\sum_{i=1}^{b}\sum_{j=1}^{k}Y_{ij}$. The two expressions are equal only when $b = k$.

(b) False. If neither treatment levels nor blocks are significant, it is possible to have F variables $\frac{\text{SSTR}/(k-1)}{\text{SSE}/(b-1)(k-1)}$ and $\frac{\text{SSB}/(b-1)}{\text{SSE}/(b-1)(k-1)}$ both < 1. In that case both SSTR and SSB are less than SSE.

13.2.14 (a) From Theorem 13.2.2, when H_0: $\beta_1 = \beta_2 = \ldots = \beta_b$ is true, SSB/σ^2 has a chi-square distribution with $b - 1$ degrees of freedom. Thus, $E(\text{SSB}/\sigma^2) = b - 1$ or $E(\text{SSE}) = (b - 1)\sigma^2$.

(b) From Theorem 13.2.2, SSE/σ^2 has a chi-square distribution with $(b - 1)(k - 1)$ degrees of freedom. Thus, $E(\text{SSE}/\sigma^2) = (b - 1)(k - 1)$ or $E(\text{SSE}) = (b - 1)(k - 1)\sigma^2$.

Section 13.3: The Paired *t* Test

13.3.1 Test $H_0: \mu_D = 0$ vs $H_1: \mu_D > 0$.

$$\bar{d} = \frac{28.17}{19} = 1.483;\ s_D^2 = \frac{b\sum_{i=1}^{b} d_i^2 - \left(\sum_{i=1}^{b} d_i\right)^2}{b(b-1)} = \frac{19(370.8197) - (28.17)^2}{19(18)} = 18.281$$

$$t = \frac{\bar{d}}{s_D/\sqrt{b}} = \frac{1.483}{\sqrt{18.281}/\sqrt{19}} = 1.51.$$ Since $1.51 < 1.7341 = t_{.05,18}$, do not reject H_0.

13.3.2 Test H_0: $\mu_D = 0$ vs. H_1: $\mu_D < 0$.

$$s_D^2 = \frac{b\sum_{i=1}^{b} d_i^2 - \left(\sum_{i=1}^{b} d_i\right)^2}{b(b-1)} = \frac{13(216) - (-42)^2}{13(12)} = 6.69$$

$$t = \frac{\bar{d}}{s_D/\sqrt{b}} = \frac{-3.23}{\sqrt{6.69}/\sqrt{13}} = -4.50.$$

Since the observed $t = -4.50 < -1.7823 = -t_{.05,12}$, we can conclude that the mothered lambs learn more quickly.

13.3.3 Test H_0: $\mu_D = 0$ vs. H_1: $\mu_D \neq 0$.

$$s_D^2 = \frac{b\sum_{i=1}^{b} d_i^2 - \left(\sum_{i=1}^{b} d_i\right)^2}{b(b-1)} = \frac{12(2.97) - (1.3)^2}{12(11)} = 0.257$$

$$t = \frac{\bar{d}}{s_D/\sqrt{b}} = \frac{1.108}{\sqrt{0.257}/\sqrt{12}} = 0.74.$$

$\alpha = 0.05$: Since $-t_{.025,11} = -2.2010 < 0.74 < 2.2010 = t_{.025,11}$, accept H_0.
$\alpha = 0.01$: Since $-t_{.005,\,11} = -3.1058 < 0.74 < 3.1058 = t_{.005,11}$ accept H_0.

13.3.4 Test H_0: $\mu_D = 0$ vs. H_1: $\mu_D < 0$.

$$s_D^2 = \frac{15(363) - (-43)^2}{15(14)} = 17.124$$

$$t = \frac{-2.867}{\sqrt{17.124}/\sqrt{15}} = -2.68$$

Since $-2.68 < -1.7613 = -t_{.05,14}$, reject H_0.

13.3.5 Test H_0: $\mu_D = 0$ vs. H_1: $\mu_D \neq 0$.

$$s_D^2 = \frac{7(0.1653) - (-0.69)^2}{7(6)} = 0.01621$$

$$t = \frac{-0.09857}{\sqrt{0.01621}/\sqrt{7}} = -2.048.$$

Since $-t_{.025,6} = -2.4469 < -2.048 < 2.4469 = t_{.025,6}$ accept H_0.
The square of the observed Student t statistic = $(-2.048)^2 = 4.194$ = the observed F statistic.
Also, $(t_{.025,6})^2 = (2.4469)^2 = 5.987 = F_{.95,1,6}$
Conclusion: the two-sided test for paired data is equivalent to the randomized block design test for 2 treatments.

13.3.6 The quotient $\dfrac{\bar{D}-\mu_D}{\sigma_D/\sqrt{b}}$ is standard normal. Also, $(b-1)\,S_D^2/\sigma_D^2$ is chi-square with $b-1$ degrees of freedom. Then $\dfrac{\bar{D}-\mu_D}{\sigma_D/\sqrt{b}} / \sqrt{\dfrac{(b-1)S_D^2/\sigma_D^2}{b-1}} = \dfrac{\bar{D}-\mu_D}{S_D/\sqrt{b}}$ is a Student t variable with $b-1$ degrees of freedom. So $P\left(-t_{\alpha/2,b-1} < \dfrac{\bar{D}-\mu_D}{S_D/\sqrt{b}} < t_{\alpha/2,b-1}\right) = 1-\alpha$ or

$P\left(\bar{D} - t_{\alpha/2,b-1}\dfrac{S_D}{\sqrt{b}} < \mu_D < \bar{D} + t_{\alpha/2,b-1}\dfrac{S_D}{\sqrt{b}}\right) = 1-\alpha$. The desired confidence interval is

$$\left(\bar{d} - t_{\alpha/2,b-1}\frac{s_D}{\sqrt{b}}, \bar{d} + t_{\alpha/2,b-1}\frac{s_D}{\sqrt{b}}\right)$$

For the data of Case Study 13.3.1, the 95% confidence interval is

$$\left(0.47 - 2.2622\frac{\sqrt{0.662}}{\sqrt{10}},\ 0.47 + 2.2622\frac{\sqrt{0.662}}{\sqrt{10}}\right) = (-0.11, 1.05).$$

13.3.7 The 95% confidence interval is $\left(\bar{d} - t_{.025,11}\dfrac{s_D}{\sqrt{b}}, \bar{d} + t_{.025,11}\dfrac{s_D}{\sqrt{b}}\right)$

$$= \left(0.108 - 2.2010\frac{\sqrt{0.257}}{\sqrt{12}}, 0.108 + 2.2010\frac{\sqrt{0.257}}{\sqrt{12}}\right) = (-0.21, 0.43)$$

13.3.8 From the proof of Theorem 7.4.2 (or refer to Question 13.3.6), if T_n is a Student t variable, then T_n^2 is an F variable with 1 and n degrees of freedom. Thus, for a Student t variable with n degrees of freedom and any α, $P(-u < T_n < u) = P(T_n^2 < u^2) = P(F < u^2)$, where F has 1 and n degrees of freedom. Thus, $t_{\alpha/2,n}^2 = F_{1-\alpha,1,n}$. In the case of randomized block design or the paired two samples, $n = b - 1$.
Next we must show that the square of the paired two sample t statistic equals the randomized block design F for $k = 2$. To do so, SSTR and SSE need to be expressed in terms of X's and Y's.
To that end note that SSTR = $\sum_{i=1}^{b}\sum_{j=1}^{2}(\bar{Y}_{.j} - \bar{Y}_{..})^2 = b\left[(\bar{Y}_{.1} - \bar{Y}_{..})^2 + (\bar{Y}_{.2} - \bar{Y}_{..})^2\right]$.

$\bar{Y}_{..} = \dfrac{\sum_{i=1}^{b} X_i + \sum_{i=1}^{b} Y_i}{2b} = \dfrac{1}{2}\bar{X} + \dfrac{1}{2}\bar{Y}$, and $\bar{Y}_{.1} = \bar{X}, \bar{Y}_{.2} = \bar{Y}$.

Thus, SSTR = $b\left[\left(\bar{X} - \left(\dfrac{1}{2}\bar{X} + \dfrac{1}{2}\bar{Y}\right)\right)^2 + \left(\bar{Y} - \left(\dfrac{1}{2}\bar{X} + \dfrac{1}{2}\bar{Y}\right)\right)^2\right]$

$= b\left[\left(\dfrac{1}{2}\bar{X} - \dfrac{1}{2}\bar{Y}\right)^2 + \left(\dfrac{1}{2}\bar{Y} - \dfrac{1}{2}\bar{X}\right)^2\right] = \dfrac{b}{2}(\bar{X} - \bar{Y})^2 = \dfrac{b}{2}\bar{D}^2$

From Equation 13.2.1, SSE = $\sum_{i=1}^{b}\sum_{j=1}^{2}(Y_{ij} - \bar{Y}_{.j})^2 - \sum_{i=1}^{b}\sum_{j=1}^{2}(\bar{Y}_{i.} - \bar{Y}_{..})^2$

$= \sum_{i=1}^{b}(Y_{il} - \bar{Y}_{.1})^2 + \sum_{i=1}^{b}(Y_{i2} - \bar{Y}_{.2})^2 - 2\sum_{i=1}^{b}(\bar{Y}_{i.} - \bar{Y}_{..})^2 = \sum_{i=1}^{b}(X_i - \bar{X})^2 + \sum_{i=1}^{b}(Y_i - \bar{Y})^2 - 2\sum_{i=1}^{b}(\bar{Y}_{i.} - \bar{Y}_{..})^2$.

Now $(\bar{Y}_{i.} - \bar{Y}_{..})^2 = \left(\dfrac{X_i + Y_i}{2} - \dfrac{\bar{X} + \bar{Y}}{2}\right)^2 = \dfrac{1}{4}\left((X_i - \bar{X}) + (Y_i - \bar{Y})\right)^2$. So

SSE = $\sum_{i=1}^{b}(X_i - \bar{X})^2 + \sum_{i=1}^{b}(Y_i - \bar{Y})^2 - \dfrac{1}{2}\sum_{i=1}^{b}\left((X_i - \bar{X}) + (Y_i - \bar{Y})\right)^2$

$= \dfrac{1}{2}\sum_{i=1}^{b}(X_i - \bar{X})^2 + \dfrac{1}{2}\sum_{i=1}^{b}(Y_i - \bar{Y})^2 - \sum_{i=1}^{b}(X_i - \bar{X})(Y_i - \bar{Y})$

$= \dfrac{1}{2}\sum_{i=1}^{b}\left((X_i - \bar{X}) - (Y_i - \bar{Y})\right)^2 = \dfrac{1}{2}\sum_{i=1}^{b}(D_i - \bar{D})^2$

The *F* statistic is $\dfrac{\text{SSTR}}{\text{SSE}/(b-1)} = \dfrac{\dfrac{b}{2}\bar{D}^2}{\dfrac{1}{2}\sum_{i=1}^{b}(D_i - \bar{D})^2/(b-1)} = \dfrac{\bar{D}^2}{\dfrac{1}{b-1}\sum_{i=1}^{b}(D_i - \bar{D})^2/b} = \left(\dfrac{\bar{D}}{s_D/\sqrt{b}}\right)^2$,

which is the square of the *t* statistic.

Chapter 14: Nonparametric Statistics

Section 14.2: The Sign Test

14.2.1 Here $x = 8$ of the $n = 10$ groups were larger than the hypothesized median of 9. The P-value is $P(X \geq 8) + P(X \leq 2) = 0.000977 + 0.009766 + 0.043945 + 0.043945 + 0.009766 + 0.000977$ $= 2(0.054688) = 0.109376$

14.2.2 The number of data values greater than 0.12 is $x = 2$. The test statistic is

$z = 2 - \dfrac{2-15/2}{\sqrt{15/4}} = -2.84$. The P-value is $P(Z \leq -2.84) = 0.0023$.

For the exact binomial test, the P-value is $P(X \leq 2)$, where X is a binomial random variable with parameters $n = 15$ and $p = \dfrac{1}{2}$. In this case $P(X \leq 2) = 0.0037$.

14.2.3 The median of $f_Y(y)$ is 0.693. There are $x = 22$ values that exceed the hypothesized median of 0.693. The test statistic is $z = \dfrac{22-50/2}{\sqrt{50/4}} = -0.85$.

Since $-z_{0.025} = -1.96 < -0.85 < z_{0.025} = 1.96$, do not reject H_0.

14.2.4 The critical value for the test is the solution to the equation $z_\alpha = \dfrac{y_+^{**} - \frac{1}{2}n}{\sqrt{n/4}}$ or $1.64 = \dfrac{y_+^{**} - \frac{1}{2}(22)}{\sqrt{22/4}}$.

Then $y_+^{**} = 14.85$.

$P(Y > 10 \mid \mu = 11) = P\left(\dfrac{Y-11}{6} > \dfrac{10-11}{6} \middle| \mu = 11\right) = P(Z > -0.17) = 1 - 0.4325 = 0.5675$.

Thus, when $\mu = 11$, Y_+ is binomial with $n = 22$ and $p = 0.5675$. The power of the test is

$P(Y_+ \geq 14.85 \mid \mu = 11) = P\left(\dfrac{Y_+ - 22(0.5675)}{\sqrt{22(0.5675)(0.4325)}} > \dfrac{14.85 - 22(0.5675)}{\sqrt{22(0.5675)(0.4325)}}\right) = P(Z > 1.02)$

$= 1 - 0.8461 = 0.1539$.

14.2.5 $P(Y_+ = y_+) = \dbinom{7}{y_+}\dfrac{1}{2^7}$. These values are given in the table.

y_+	$P(Y_+ = y_+)$
0	1/128
1	7/128
2	21/128
3	35/128
4	35/128
5	21/128
6	7/128
7	1/128

Possible levels for a one-sided test: 1/128, 8/128, 29/128, etc.

14.2.6

$w/1$	$w/1 - 0.618$	sign	$w/1$	$w/1 - 0.618$	sign
0.693	0.075	+	0.654	0.036	+
0.662	0.044	+	0.615	–0.003	–
0.690	0.072	+	0.668	0.050	+
0.606	–0.012	–	0.601	–0.017	–
0.570	–0.048	–	0.576	–0.042	–
0.749	0.131	+	0.670	0.052	+
0.672	0.054	+	0.606	–0.012	–
0.628	0.010	+	0.611	–0.007	–
0.609	–0.009	–	0.553	–0.065	–
0.844	0.226	+	0.933	0.315	+

Note that $\sum_{k=0}^{5}\binom{20}{k}\left(\frac{1}{2}\right)^{20} = \sum_{k=15}^{20}\binom{20}{k}\left(\frac{1}{2}\right)^{20} = 0.021$. The test is to reject H_0 if $y_+ \le 5$ or $y_+ \ge 15$. This gives an $\alpha =$ of 0.042, which is as close as to the desired level of significance of 0.05 as can be achieved with the sign test. From the table, we see that $y_+ = 11$. Since $5 < y_+ = 11 < 15$, accept H_0.

14.2.7

y_i	$y_i - 0.80$	sign	y_i	$y_i - 0.80$	sign
0.61	–0.19	–	0.78	–0.02	–
0.70	–0.10	–	0.84	0.04	+
0.63	–0.17	–	0.83	0.03	+
0.76	–0.04	–	0.82	0.02	+
0.67	–0.13	–	0.74	–0.06	–
0.72	–0.08	–	0.85	0.05	+
0.64	–0.16	–	0.73	–0.07	–
0.82	0.02	+	0.85	0.05	+
0.88	0.08	+	0.87	0.07	+
0.82	0.02	+			

$\sum_{k=0}^{6}\binom{19}{k}\left(\frac{1}{2}\right)^{19} = 0.0835$, while $\sum_{k=0}^{7}\binom{19}{k}\left(\frac{1}{2}\right)^{19} = 0.1796$. Thus, the closest test to one with $\alpha = 0.10$ is to reject H_0 if $y_+ \le 6$. This test has $\alpha = 0.0835$. Since $y_+ = 9$, accept H_0.
Since the observed t statistic $= -1.71 < -1.330 = -t_{.10,18}$, the t test rejects H_0.

14.2.8 For this question, we have H_0: $p = \frac{1}{2}$ vs. H_1: $p < \frac{1}{2}$. The table below gives a value of $u = 3$, the number of times that $x_i > y_i$. The test statistic is then $z = \frac{3 - 15/2}{\sqrt{15/4}} = -2.32$.

Since $-2.32 < -1.64 = -z_{.05}$, we reject H_0. We can entertain the possibility that hypnosis enhances ESP.

Waking State, x_i	Hypnotic State, y_i	$x_i > y_i$
18	25	no
19	20	no
16	26	no
21	26	no
16	20	no
20	23	no
20	14	yes
14	18	no
11	18	no
22	20	yes
19	22	no
29	27	yes
16	19	no
27	27	no
15	21	yes

14.2.9 $z = \dfrac{y_+ - \frac{1}{2}n}{\sqrt{n/4}} = \dfrac{19 - \frac{1}{2}(28)}{\sqrt{28/4}} = 1.89$. Accept H_0, since $-z_{.025} = -1.96 < 1.89 < 1.96 = z_{.025}$.

14.2.10 Assign + if $Y_i \le 6$, and – otherwise. If Y_+ is the number of + signs, then it is binomial with $n = 36$ and $p = 0.25$. The desired test is to reject H_0 if $Y_+ \le Y_+^*$ or $Y_+ \ge Y_+^{**}$ where

$P(Y_+ \le Y_+^*) = P(Y_+ \ge Y_+^{**}) = \alpha/2$.

Then $-1.96 = \dfrac{Y_+^* - 36(0.25)}{\sqrt{36(0.25)(0.75)}}$, or $Y_+^* = 3.91$. Similarly, $1.96 = \dfrac{Y_+^{**} - 36(0.25)}{\sqrt{36(0.25)(0.75)}}$,

or $Y_+^{**} = 14.09$. If 7 is the true 25th percentile, then $\theta = 28$.

$P(Y_i \le 6) = 6/28 = 0.214$. In this case Y_+ is binomial with $n = 36$ and $p = 0.214$. Assume the 25th percentile is 7. The probability of the Type II error is $P(3.91 < Y_+ < 14.09|$ 25th percentile is 7)

$= P\left(\dfrac{3.91 - 36(0.214)}{\sqrt{36(0.214)(0.786)}} < Z < \dfrac{14.09 - 36(0.214)}{\sqrt{36(0.214)(0.786)}}\right) = P(-1.54 < Z < 2.60) = 0.9953 - 0.0618$

$= 0.9335$.

14.2.11 From Table 13.3.1, the number of pairs where $x_i > y_i$ is 7. The P-value for this test is $P(U \ge 7) + P(U \le 3) = 2(0.17186) = 0.343752$. Since the P value exceeds $\alpha = 0.05$, do not reject the null hypothesis, which is the conclusion of Case Study 13.3.1.

Section 14.3: Wilcoxon Tests

14.3.1

x_i	y_i	$y_i - x_i$	$\lvert y_i - x_i \rvert$	r_i	z_i	$r_i z_i$
1458	1424	−34	34	1	0	0
1353	1501	148	148	5	1	5
2209	1495	−714	714	8	0	0
1804	1739	−65	65	2	0	0
1912	2031	119	119	4	1	4
1366	934	−432	432	7	0	0
1598	1401	−197	197	6	0	0
1406	1339	−67	67	3	0	0

The sum of the $r_i z_i$ column is 9. From Table A.6, the critical values of 7 and 29 give $\alpha = 0.148$. Since $7 < w = 9 < 29$, accept H_0.

14.3.2 Section 14.3 gives the expression for $\prod_{i=1}^{4}(1+e^{it})$. Then $\prod_{i=1}^{5}(1+e^{it})$ is $(1 + e^{5t})$ times that expression, or $1 + e^t + e^{2t} + 2e^{3t} + 2e^{4t} + 2e^{5t} + 2e^{6t} + 2e^{7t} + e^{8t} + e^{9t} + e^{10t} + e^{5t} + e^{6t} + e^{7t} + 2e^{8t} + 2e^{9t} + 2e^{10t} + 2e^{11t} + 2e^{12t} + e^{13t} + e^{14t} + e^{15t} = 1 + e^t + e^{2t} + 2e^{3t} + 2e^{4t} + 3e^{5t} + 3e^{6t} + 3e^{7t} + 3e^{8t} + 3e^{9t} + 3e^{10t} + 2e^{11t} + 2e^{12t} + e^{13t} + e^{14t} + e^{15t}$.
The pdf for each integer i is the coefficient of e^{it} divided by $2^5 = 32$. The possible α levels are $P(W = 15) = 1/32$, $P(W \geq 14) = 1/32 + 1/32 = 1/16$, $P(W \geq 13) = 1/32 + 1/32 + 1/32 = 3/32$, etc.

14.3.3

x_i	y_i	$y_i - x_i$	$\lvert y_i - x_i \rvert$	r_i	z_i	$r_i z_i$
16.5	16.9	0.4	0.4	12.5	1	12.5
17.6	17.2	−0.4	0.4	12.5	0	0
16.9	17.0	0.1	0.1	2	1	2
15.8	16.1	0.3	0.3	8.5	1	8.5
18.4	18.2	−0.2	0.2	4.5	0	0
17.5	17.7	0.2	0.2	4.5	1	4.5
17.6	17.9	0.3	0.3	8.5	1	8.5
16.1	16.0	−0.1	0.1	2	0	0
16.8	17.3	0.5	0.5	14	1	14
15.8	16.1	0.3	0.3	8.5	1	8.5
16.8	16.5	−0.3	0.3	8.5	0	0
17.3	17.6	0.3	0.3	8.5	1	8.5
18.1	18.4	0.3	0.3	8.5	1	8.5
17.9	17.2	−0.7	0.7	15	0	0
16.4	16.5	0.1	0.1	2	1	2

w = sum of the $r_1\, z_i$ column is 77.5. The mean of W is $n(n + 1)/4 = 60$. The variance of $W = n(n + 1)(2n + 1)/24 = 310$. The observed Z statistic $w' = \dfrac{77.5-60}{\sqrt{310}} = 0.99$.

Since $-1.96 < w' = 0.99 < 1.96 = z_{.025}$, accept H_0.

14.3.4

y_i	$\lvert y_i \rvert$	r_i	z_i	$r_i z_i$
−6	6	4.5	0	0
10	10	10.5	1	10.5
9	9	8.5	1	8.5
−8	8	7	0	0
−6	6	4.5	0	0
−2	2	1.5	0	0
20	20	12	1	12
−7	7	6	0	0
5	5	3	1	3
−9	9	8.5	0	0
−10	10	10.5	0	0
−2	2	1.5	0	0

w = sum of the $r_i\, z_i$ column = 34. From Table A.6, the critical values of 14 and 64 give a test with level of significance 0.052. Since $14 < w = 34 < 64$, accept H_0.

14.3.5

y_i	$y_i - 0.80$	$\lvert y_i - 0.80 \rvert$	r_i	z_i	$r_i z_i$
0.61	−0.19	0.19	19	0	0
0.70	−0.10	0.10	15	0	0
0.63	−0.17	0.17	18	0	0
0.76	−0.04	0.04	6.5	0	0
0.67	−0.13	0.13	16	0	0
0.72	−0.08	0.08	13.5	0	0
0.64	−0.16	0.16	17	0	0
0.82	0.02	0.02	2.5	1	2.5
0.88	0.08	0.08	13.5	1	13.5
0.82	0.02	0.02	2.5	1	2.5
0.78	−0.02	0.02	2.5	0	0
0.84	0.04	0.04	6.5	1	6.5
0.83	0.03	0.03	5	1	5
0.82	0.02	0.02	2.5	1	2.5
0.74	−0.06	0.06	10	0	0
0.85	0.05	0.05	8.5	1	8.5
0.73	−0.07	0.07	11.5	0	0
0.85	0.05	0.05	8.5	1	8.5
0.87	0.07	0.07	11.5	1	11.5

w = sum of the $r_i\, z_i$ column = 61. The mean of W is $n(n + 1)/4 = 95$. The variance of $W = n(n + 1)(2n + 1)/24 = 617.5$. The observed Z statistic $w' = \dfrac{61-95}{\sqrt{617.5}} = -1.37$.

Since $w' = -1.37 < -1.28 = -z_{.10}$, reject H_0. The sign test accepted H_0.

14.3.6

x_i	y_i	$y_i - x_i$	$\lvert y_i - x_i \rvert$	r_i	z_i	$r_i z_i$
14.6	13.8	−0.8	0.8	7	0	0
17.3	15.4	−1.9	1.9	10	0	0
10.9	11.3	0.4	0.4	3.5	1	3.5
12.8	11.6	−1.2	1.2	8.5	0	0
16.6	16.4	−0.2	0.2	1	0	0
12.2	12.6	0.4	0.4	3.5	1	3.5
11.2	11.8	0.6	0.6	6	1	6
15.4	15.0	−0.4	0.4	3.5	0	0
14.8	14.4	−0.4	0.4	3.5	0	0
16.2	15.0	−1.2	1.2	8.5	0	0

w = sum of the $r_i\, z_i$ column = 13. From Table A.6, the critical values of 8 and 47 give a test with level of significance 0.048. Since $8 < w = 13 < 47$, accept H_0. The sign test also accepted H_0.

14.3.7 The signed rank test should have more power since it uses a greater amount of the information in the data.

14.3.8

No of trials	r_i	z_i	$r_i z_i$
2	5	1	5
3	9	1	9
5	16.5	1	16.5
3	9	1	9
2	5	1	5
1	2	1	2
1	2	1	2
5	16.5	1	16.5
3	9	1	9
1	2	1	2
7	21	1	21
3	9	1	9
5	16.5	1	16.5
3	9	0	0
11	25	0	0
10	24	0	0
5	16.5	0	0
5	16.5	0	0
4	12.5	0	0
2	5	0	0
7	21	0	0
5	16.5	0	0
4	12.5	0	0
8	23	0	0
12	26	0	0
7	21	0	0
			$w' = 122.5$

The hypothesis is that lambs with mothers learn faster, that is, test H_0: $\mu_X = \mu_Y$ vs. H_1: $\mu_X < \mu_Y$. The array for the rank sum test yields $w' = 122.5$. The test statistic is

$z = \dfrac{122.5 - 175.5}{\sqrt{380.25}} = -2.72$. Since $-2.72 < -1.64$, accept H_0.

14.3.9 A reasonable assumption is that alcohol abuse shortens life span. In that case, reject H_0 if the test statistic is less than $-z_{.05} = -1.64$. From the table below, we see that $w' = 72.5$.

The test statistic is $z = \dfrac{72.5 - 99}{\sqrt{198}} = -1.88$, so reject H_0.

Age at death	r_i	z_i	$r_i z_i$
48	2	1	2
66	7	1	7
71	12	1	12
65	5.5	1	5.5
56	3	1	3
67	9	1	9
67	9	1	9
70	11	1	11
77	14	1	14
65	5.5	0	0
87	18	0	0
32	1	0	0
77	14	0	0
89	20	0	0
86	17	0	0
77	14	0	0
84	16	0	0
64	4	0	0
88	19	0	0
90	21	0	0
67	9	0	0
			$w' = 72.5$

14.3.10 Case Study 9.3.1 asks whether sensory deprivation has any effect on the alpha-wave pattern. Thus, we should use the data in Table 9.3.1 to test H_0: $\mu_X = \mu_Y$ vs. H_1: $\mu_X \neq \mu_Y$. In that case, reject H_0 if the test statistic is less than $-z_{.025} = -1.96$ or greater than $z_{.025} = 1.96$. From the table below, we see that $w' = 140.5$. The test statistic is $z = \dfrac{140.5 - 105}{\sqrt{175}} = 2.68$, so reject H_0.

Frequencies	r_i	z_i	$r_i z_i$
10.7	15.5	1	15.5
10.7	15.5	1	15.5
10.4	12	1	12
10.9	17.5	1	17.5
10.5	14	1	14
10.3	9.5	1	9.5
9.6	5.5	1	5.5
11.1	19	1	19
11.2	20	1	20
10.4	12	1	12
9.6	5.5	0	0
10.4	12	0	0
9.7	7	0	0
10.3	9.5	0	0
9.2	2	0	0
9.3	3	0	0
9.9	8	0	0
9.5	4	0	0
9.0	1	0	0
10.9	17.5	0	0
			0
			$w' = 140.5$

Section 14.4: The Kruskal-Wallis Test

14.4.1

Group I	Rank	Group II	Rank	Group III	Rank
3	2.5	10	9	20	15
2	1	4	4	9	7
6	6	11	11	18	13
10	9	14	12	19	14
10	9	3	2.5		
5	5				

The sum of the second column in the table is $r_{.1} = 32.5$; the fourth column, $r_{.2} = 38.5$; and the sixth column $r_{.3} = 49$. The test statistic is

$$b = \frac{12}{n(n+1)}\left(\frac{r_{.1}^2}{n_1}+\frac{r_{.2}^2}{n_2}+\frac{r_{.3}^2}{n_3}\right)-3(n+1) = \frac{12}{15(16)}\left(\frac{32.5^2}{6}+\frac{38.5^2}{5}+\frac{49^2}{4}\right)-3(16) = 5.64.$$

Since $5.64 < 5.991 = \chi_{.95,2}$, accept H_0.

14.4.2

With Dome	Rank	Without Dome	Rank
100.0	12	76.4	2
58.6	1	84.2	7
93.5	10	96.5	11
83.6	4.5	88.8	9
84.1	6	85.3	8
		79.1	3
		83.6	4.5

Summing the second column in the table gives $r_{.1} = 33.5$, and the sum of the fourth column is $r_{.2} = 44.5$.

The test statistic is $b = \dfrac{12}{n(n+1)}\left(\dfrac{r_{.1}^2}{n_1} + \dfrac{r_{.2}^2}{n_2}\right) - 3(n+1) = \dfrac{12}{12(13)}\left(\dfrac{33.5^2}{5} + \dfrac{44.5^2}{7}\right) - 3(13) = 0.3.$

Since $b = 0.03 < 3.841 = \chi^2_{.95,1}$ accept H_0.

14.4.3

Women	Rank	Men	Rank
52	1	72	5
69	3	88	14
73	6	87	12
88	13	74	7
87	11	78	9
56	2	70	4
		78	10
		93	15
		74	8

Summing the second column in the table gives $r_{.1} = 36$, and the sum of the fourth column is $r_{.2} = 84$. The test statistic is $b = \dfrac{12}{n(n+1)}\left(\dfrac{r_{.1}^2}{n_1} + \dfrac{r_{.2}^2}{n_2}\right) - 3(n+1) = \dfrac{12}{15(16)}\left(\dfrac{36^2}{6} + \dfrac{84^2}{9}\right) - 3(16)$ $= 2.00$. Since $b = 7.38 > 3.841 = \chi^2_{.95,1}$, reject H_0.

14.4.4

Twain	Rank	QCS	Rank
0.225	13	0.209	6
0.262	18	0.205	4
0.217	8.5	0.196	1
0.240	17	0.210	7
0.230	15	0.202	3
0.229	14	0.207	5
0.235	16	0.224	12
0.217	8.5	0.223	11
		0.220	10
		0.201	2

Summing the second column in the table gives $r_{.1} = 110$, and the sum of the fourth column is

$r_{.2} = 61$. The test statistic is $b = \frac{12}{n(n+1)}\left(\frac{r_{.1}^2}{n_1}+\frac{r_{.2}^2}{n_2}\right) - 3(n+1) = \frac{12}{18(19)}\left(\frac{110^2}{8}+\frac{61^2}{10}\right) - 3(19)$ $= 9.13$. Since $b = 9.13 > 3.841 = \chi^2_{.95,1}$, reject H_0.

14.4.5

Non-Smokers	Rank	Light Smokers	Rank	Moderate Smokers	Rank	Heavy Smokers	Rank
69	13	55	2	66	10.5	91	23
52	1	60	7	81	20.5	72	16
71	15	78	18	70	14	81	20.5
58	4.5	58	4.5	77	17	67	12
59	6	62	8	57	3	95	24
65	9	66	10.5	79	19	84	22

$$b = \frac{12}{n(n+1)}\left(\frac{r_{.1}^2}{n_1}+\frac{r_{.2}^2}{n_2}+\frac{r_{.3}^2}{n_3}+\frac{r_{.4}^2}{n_4}\right)-3(n+1) = \frac{12}{24(25)}\left(\frac{48.5^2}{6}+\frac{50^2}{6}+\frac{84^2}{6}+\frac{117.5^2}{6}\right)-3(25)$$

$= 10.715$. Since $b = 10.715 > 7.815 = \chi^2_{.95,3}$, reject H_0.

14.4.6 **(a)**

Plant 1	Rank	Plant 2	Rank	Plant 3	Rank
905	13.5	1109	26	571	5
1018	20	1155	28	1346	29
905	13.5	835	9	292	3
886	11	1152	27	825	8
958	16	1036	22	676	6
1056	24	926	15	541	4
904	12	1029	21	818	7
856	10	1040	23	90	1
1070	25	959	17	2246	30
1006	19	996	18	104	2

$$b = \frac{12}{n(n+1)}\left(\frac{r_{.1}^2}{n_1}+\frac{r_{.2}^2}{n_2}+\frac{r_{.3}^2}{n_3}\right) - 3(n+1) = \frac{12}{30(31)}\left(\frac{164^2}{10}+\frac{206^2}{10}+\frac{95^2}{10}\right)-3(31) = 8.11$$

Since $b = 8.11 > 5.991 = \chi^2_{.95,2}$, reject H_0.

(b) The ANOVA table is

Source	df	SS	MS	F
Plant	2	403931.27	201965.63	1.40
Error	27	3891027.4	144112.13	
Total	29	4294958.7		

Since $F = 1.40 < 3.35 = F_{.95,2,27}$, accept H_0.

(c) The change has no effect on the ranks, so b is the same.

(d)

Source	df	SS	MS	F
Plant	2	678906.87	339453.43	4.24
Error	27	2161202.6	80044.541	
Total	29	2840109.5		

Since $F = 4.24 > 3.35 = F_{.95,2,27}$, reject H_0.

14.4.7

Powdered	Rank	Moderate	Rank	Coarse	Rank
146	8.5	150	14.5	141	4
152	16	144	6	138	2
149	12.5	148	10.5	142	5
161	21	155	19	146	8.5
158	20	154	17.5	139	3
149	12.5	150	14.5	145	7
154	17.5	148	10.5	137	1

$$b = \frac{12}{n(n+1)}\left(\frac{r_{.1}^2}{n_1}+\frac{r_{.2}^2}{n_2}+\frac{r_{.3}^2}{n_3}\right)-3(n+1) = \frac{12}{21(22)}\left(\frac{108^2}{7}+\frac{92.5^2}{7}+\frac{30.5^2}{7}\right)-3(22)=12.48$$

Since $b = 12.48 > 5.991 = \chi^2_{.95,2}$, reject H_0.

14.4.8 $$\sum_{j=1}^{k}\left(\frac{n-n_j}{n}\right)Z_j^2 = \sum_{j=1}^{k}\left(\frac{n-n_j}{n}\right)\left(\frac{\frac{R_{.j}}{n_j}-\frac{n+1}{2}}{\sqrt{\frac{(n+1)(n-n_j)}{12n_j}}}\right)^2$$

$$= \sum_{j=1}^{k}\left(\frac{n-n_j}{n}\right)\frac{\left(\frac{R_{.j}}{n_j}-\frac{n+1}{2}\right)^2}{\frac{(n+1)(n-n_j)}{12n_j}} = \sum_{j=1}^{k}\frac{12n_j}{n(n+1)}\left(\frac{R_{.j}}{n_j}-\frac{n+1}{2}\right)^2$$

$$= \sum_{j=1}^{k}\frac{12n_j}{n(n+1)}\left(\frac{R_{.j}^2}{n_j^2}-\frac{n+1}{n_j}R_{.j}+\frac{(n+1)^2}{4}\right)$$

$$= \frac{12}{n(n+1)}\sum_{j=1}^{k}\frac{R_{.j}^2}{n_j}-\frac{12}{n}\sum_{j=1}^{k}R_{.j}+\frac{3}{n}(n+1)\sum_{j=1}^{k}n_j = \frac{12}{n(n+1)}\sum_{j=1}^{k}\frac{R_{.j}^2}{n_j}-\frac{12}{n}\frac{n(n+1)}{2}+\frac{3}{n}(n+1)n$$

$$= \frac{12}{n(n+1)}\sum_{j=1}^{k}\frac{R_{.j}^2}{n_j}-6(n+1)+3(n+1) = B$$

Section 14.5: The Friedman Test

14.5.1

36 lb.	Rank	54 lb.	Rank	72 lb.	Rank	108 lb.	Rank	144 lb.	Rank
7.62	3	8.14	5	7.76	4	7.17	1	7.46	2
8.00	4	8.15	5	7.73	3	7.57	1	7.68	2
7.93	5	7.87	4	7.74	2	7.80	3	7.21	1

$$g = \frac{12}{bk(k+1)}\sum_{j=1}^{5} r_{.j}^2 - 3b(k+1) = \frac{12}{3(5)(6)}(12^2 + 14^2 + 9^2 + 5^2 + 5^2) - 3(3)(6)$$

Since $g = 8.8 < 9.488 = \chi^2_{.95,4}$, accept H_0.

14.5.2

Before	Rank	During	Rank	After	Rank
6.4	3	5.0	1	5.8	2
7.1	1	13.0	3	9.2	2
6.5	1	14.0	3	7.9	2
8.6	2	12.0	3	7.7	1
8.1	2	6.0	1	11.0	3
10.4	2	9.0	1	12.9	3
11.5	1	13.0	2	13.5	3
13.8	2	16.0	3	13.1	1
15.4	1	25.0	3	15.8	2
15.7	3	13.0	1	13.3	2
11.7	1	14.0	3	12.8	2
15.8	2	20.0	3	14.5	1

$$g = \frac{12}{bk(k+1)}\sum_{j=1}^{3} r_{.j}^2 - 3b(k+1) = \frac{12}{12(3)(4)}(21^2 + 27^2 + 24^2) - 3(12)(4) = 1.5$$

Since $g = 1.5 < 5.991 = \chi^2_{.95,2}$ accept H_0.

14.5.3

PcrCh1	Rank	Davies	Rank	AOAC	Rank
0.598	1	0.628	2	0.632	3
0.614	1	0.628	2	0.630	3
0.600	1.5	0.600	1.5	0.622	3
0.580	1	0.612	3	0.584	2
0.596	1	0.600	2	0.650	3
0.592	1	0.628	3	0.606	2
0.616	1	0.628	2	0.644	3
0.614	1	0.644	2.5	0.644	2.5
0.604	1	0.644	3	0.624	2
0.608	1	0.612	2	0.619	3
0.602	1	0.628	2	0.632	3
0.614	1	0.644	3	0.616	2

$$g = \frac{12}{bk(k+1)}\sum_{j=1}^{3} r_{.j}^2 - 3b(k+1) = \frac{12}{12(3)(4)}(12.5^2 + 28^2 + 31.5^2) - 3(12)(4) = 17.0$$

Since $g = 17.0 > 5.991 = \chi^2_{.95,2}$ reject H_0.

14.5.4

High	Rank	Intermediate	Rank	Low	Rank
0.30	3	0.11	1	0.12	2
0.20	1	0.24	2	0.28	3
0.17	2	0.13	1	0.20	3
0.25	2	0.36	3	0.15	1
0.27	2	0.20	1	0.31	3
0.19	3	0.12	1	0.16	2
0.27	3	0.19	1	0.20	2
0.23	3	0.08	1	0.17	2
0.37	3	0.18	1.5	0.18	1.5
0.29	3	0.20	1.5	0.20	1.5

$g = \frac{12}{bk(k+1)}\left(r_{.1}^2 + r_{.2}^2 + r_{.3}^2\right) - 3b(k+1) = \frac{12}{10(3)(4)}\left(25^2 + 14^2 + 21^2\right) - 3(10)(4) = 6.2$ Since $g = 6.2 > 5.991 = \chi_{.95,2}$, reject H_0. Analysis of variance of these data would yield an observed F = 3.558. The critical value is $F_{0.95,2,18} = 3.555$. Thus, analysis of variance would also reject the null hypothesis.

14.5.5

Contact Desensitization	Rank	Demonstration Participation	Rank	Live Modeling	Rank
8	3	2	2	–2	1
11	3	1	2	0	1
9	2	12	3	6	1
16	3	11	2	2	1
24	3	19	2	11	1

$$g = \frac{12}{5(3)(4)}\left(14^2 + 11^2 + 5^2\right) - 3(5)(4) = 8.4$$

Since $g = 8.4 < 9.210 = \chi_{.99,2}$, accept H_0. On the other hand, using analysis of variance, the null hypothesis would be rejected at this level.

14.5.6 The following two equations will be needed in what follows.

(1) $r_{..} = \sum_{j=1}^{k} r_{.j} = b\frac{k(k+1)}{2} = \frac{bk(k+1)}{2}$

(2) $\bar{r}_{..} = \frac{1}{bk}\frac{bk(k+1)}{2} = \frac{k+1}{2}$

$$\sum_{j=1}^{k}(\bar{r}_{.j} - \bar{r}_{..})^2 = \sum_{j=1}^{k}\frac{r_{.j}^2}{b^2} - 2\sum_{j=1}^{k}\frac{r_{.j}}{b}\bar{r}_{..} + \sum_{j=1}^{k}\bar{r}_{..}^2 = \frac{1}{b^2}\sum_{j=1}^{k}r_{.j}^2 - 2\frac{k+1}{2}\frac{1}{b}\sum_{j=1}^{k}r_{.j} + k\frac{(k+1)^2}{4}$$

$$= \frac{1}{b^2}\sum_{j=1}^{k}r_{.j}^2 - (k+1)\frac{1}{b}\frac{bk(k+1)}{2} + k\frac{(k+1)^2}{4} = \frac{1}{b^2}\sum_{j=1}^{k}r_{.j}^2 - \frac{k(k+1)^2}{4}$$

So, $\frac{12b}{k(k+1)}\sum_{j=1}^{k}(\bar{r}_{.j} - \bar{r}_{..})^2 = \frac{12b}{k(k+1)}\frac{1}{b^2}\sum_{j=1}^{k}r_{.j}^2 - \frac{12b}{k(k+1)}\frac{k(k+1)^2}{4}$

$$= \frac{12}{bk(k+1)}\sum_{j=1}^{k}r_{.j}^2 - 3b(k+1) = g.$$

Section 14.6: Testing for Randomness

14.6.1 **(a)**

% Change January	$\text{sgn}(y_i - y_{i-1})$	% Change January	$\text{sgn}(y_i - y_{i-1})$
2.0	+	–2.9	+
2.3	–	0.7	–
0.6	–	0.0	+
–0.9	+	1.4	+
0.5	–	1.5	–
–1.8	–	–1.5	+
–2.1	+	2.2	+
–0.9	+	4.9	–
2.5	–	–2.3	–
0.3	–	–4.6	+
–0.7	+	2.8	–
1.2	–	0.9	–
–3.4	+	–2.0	–
2.6	–	–2.4	+
1.3	–	3.2	–
0.7	+	2.4	–
0.8	+	–1.9	+
3.1	–	–1.6	
0.2	–		

For these data, the number of runs $w = 23$. The test statistic is $Z = \dfrac{W - E(W)}{\sqrt{\text{Var}(W)}}$, where $E(W) = (2n - 1)/3$ and $\text{Var}(W) = (16n - 29)/90$. For these data, $E(W) = 24.33$ and $\text{Var}(W) = 6.26$. Then $z = \dfrac{23 - 24.33}{\sqrt{6.26}} = -0.53$. Since $-z_{.025} = -1.96 < -0.53 < 1.96 = z_{.025}$, accept H_0 and assume the sequence is random.

(b) The number of runs is $w = 21$ and $z = \dfrac{21 - 24.33}{\sqrt{6.26}} = -1.33$.
Since $-z_{.025} = -1.96 < -1.33 < 1.96 = z_{.025}$, accept H_0 and assume the sequence is random.

14.6.2 The number of runs is $w = 17$ and $z = \dfrac{17 - 15.67}{\sqrt{3.94}} = 0.67$.
Since $-z_{.025} = -1.96 < 0.67 < 1.96 = z_{.025}$, accept H_0 and assume the sequence is random.

14.6.3 The number of runs is $w = 19$, so $z = 1.68$.
Since $-z_{.025} = -1.96 < 1.68 < 1.96 = z_{.025}$, accept H_0 and assume the sequence is random.

14.6.4 The number of runs is $w = 19$, so $z = -0.30$.
Since $-z_{.025} = -1.96 < -0.30 < 1.96 = z_{.025}$, accept H_0 and assume the sequence is random.

14.6.5 For these data, $w = 25$, and $z = -0.51$. Since $-z_{0.025} = -1.96 < -0.51 < 1.96 = z_{.025}$, accept H_0 at the 0.05 level of significance and assume the sequence is random.

14.6.6

Range	$\text{sgn}(y_i - y_{i-1})$
1.7	+
2.9	–
1.4	+
2.1	+
4.4	–
4.1	–
2.7	–
2.3	+
4.7	+
9.2	–
3.0	+
3.6	+
4.4	+
9.7	–
4.2	–
2.7	–
0.1	+
4.3	–
1.0	+
1.3	

From the table, we count the number of runs to be $w = 11$.

Then, $z = \dfrac{11-13}{\sqrt{3.23}} = -1.11$. Since $-z_{.025} = -1.96 < -1.11 < 1.96 = z_{.025}$, accept H_0 at the 0.05 level of significance and assume the sequence of ranges is random.