STAT 403 Spring 2018
HW03

Nan Tang
1662478

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Q1

Q1-a

```
data("chickwts")

feed_linseed_dt <- subset(chickwts, feed == 'linseed')

> t.test(x=feed_linseed_dt, mu=240, alternative='two.sided')

One Sample t-test

data:  sample_wt
t = -1.4092, df = 11, p-value = 0.1864
alternative hypothesis: true mean is not equal to 240
95 percent confidence interval:
 185.561 251.939
sample estimates:
mean of x
   218.75
```

We can perceive from summary that the p-value calculated from our sample is 0.1864, which is greater than 0.1. Therefore, under the test size $\alpha = 0.1$, we cannot reject the null hypothesis.

Q1-b

```
> mean(feed_linseed_dt$weight)
[1] 218.75
> sd(feed_linseed_dt$weight)
[1] 52.2357
```

Mean weight of chicken fed by linseed is 218.75
Standard deviation is 52.2358

Q1-c

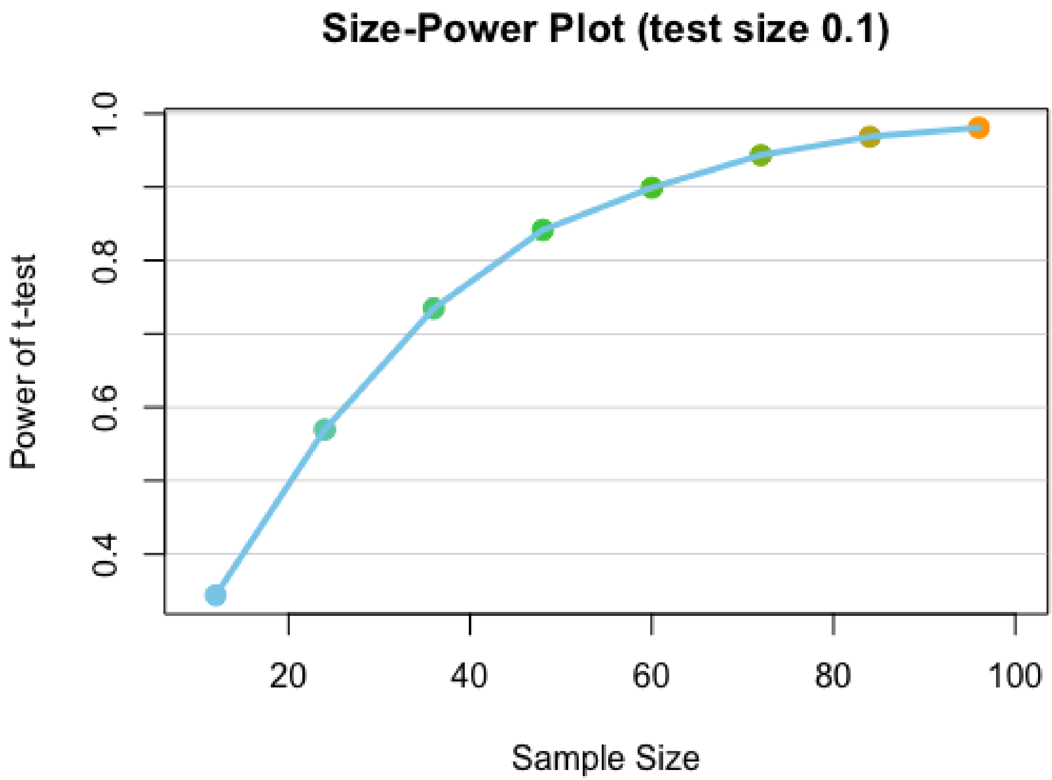
```
sim_size <- 10000
alpha <- 0.1
sim_result <- rep(NA, sim_size)
for (i in 1:sim_size) {
  sample <- rnorm(n=12, mean=220, sd=52)
  t_result <- t.test(x=sample, mu=240, alternative='two.sided')
  sim_result[i] <- t_result$p.value < alpha
}
t_power <- sum(sim_result) / sim_size

> t_power
[1] 0.3377
```

The power of the t-test under the condition that true population mean is 220, is the probability of rejecting H_0: $\mu_0 = 240$ with test size $\alpha = 0.1$.

Base on my simulation, 3377 of 10000 simulations rejected the null. Since the proportion of rejection among simulations is an unbiased estimator of test power, we can say the power of t-test under this condition is $\frac{3377}{10000} = 0.3377$.

Q1-d

```
sim_size <- 10000
sample_sz_seq <- c(12,24,36,48,60,72,84,96)
power_seq <- rep(NA, length(sample_sz_seq))

for (i in 1:length(sample_sz_seq)) {
  sim_result <- rep(NA, sim_size)
  for (j in 1:sim_size) {
    sample <- rnorm(n=sample_sz_seq[i], mean=220, sd=52)
    t_result <- t.test(x=sample, mu=240, alternative='two.sided')
    sim_result[j] <- t_result$p.value < alpha
  }
  t_power <- sum(sim_result) / sim_size
  power_seq[i] <- t_power
}

size_power_comp <- data.frame(cbind(sample_sz_seq, power_seq))
colnames(size_power_comp)[1] <- 'size'
colnames(size_power_comp)[2] <- 'power'

col_ramp = colorRampPalette(c("skyblue","limegreen"))

plot(x=size_power_comp$size, y=size_power_comp$power, col=col_ramp(nrow(size_power_comp)
     pch=20, cex=2, xlim=c(10, 100), main='Size-Power Plot (test size 0.1)',
     ylab='Power of t-test', xlab='Sample Size')
lines(x=size_power_comp$size, y=size_power_comp$power, lwd=3, col="skyblue")
grid(nx=NA,ny=NULL,lty=1,lwd=0.5,col="gray")
```



Under the condition of same true parameter and same test size, the power of t-test has a logarithmic growth as sample size increases.

Q1-e

When estimating the power of test, the variance of Monte Carlo simulation is

$$Var(\bar{D}_N) = \frac{\beta(1-\beta)}{N}$$

N denotes number of simulations, β is power of t-test. Under the same sample size and test size, the power is constant.

$$Var(\bar{D}_{N=10000}) = \frac{\beta(1-\beta)}{10000}$$
$$Var(\bar{D}_{N=1000000}) = \frac{\beta(1-\beta)}{1000000}$$
$$\frac{Var(\bar{D}_{10000})}{Var(\bar{D}_{1000000})} = 100$$

Thereby we know that the Monte Carlo Variance of 10000 simulations is 100 times larger than error of 1000000 simulations. The Monte Carlo error is $sd(\bar{D}_N) = \sqrt{Var(\bar{D}_N)}$, therefore, error of $N = 10000$ is $\sqrt{100} = 10$ times bigger than error of $N = 1000000$

We assumed the Monte Carlo error at N = 10000 is 5×10^{-3}, therefore, the error of Monte Carlo simulation with N = 1000000 will be 5×10^{-4}.

Q2

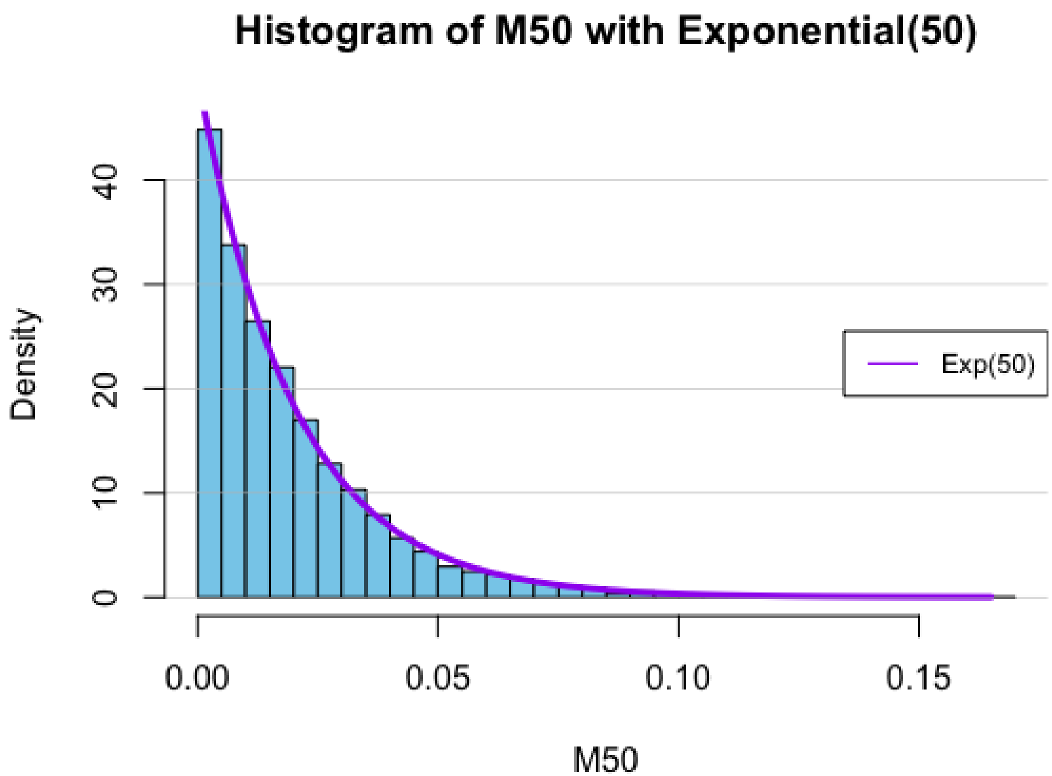
Q2-a

```
sim_size <- 10000
sample_size <- 50
minunif_eva <- rep(NA, sim_size)

for (i in 1:sim_size) {
  sample <- runif(sample_size)
  minunif_eva[i] <- min(sample)
}
> mean(minunif_eva)
[1] 0.01959125
> sd(minunif_eva)
[1] 0.01923361
```

Mean of M_{50} is 0.01959125
Standard deviation of M_{50} is 0.01923361

Q2-b

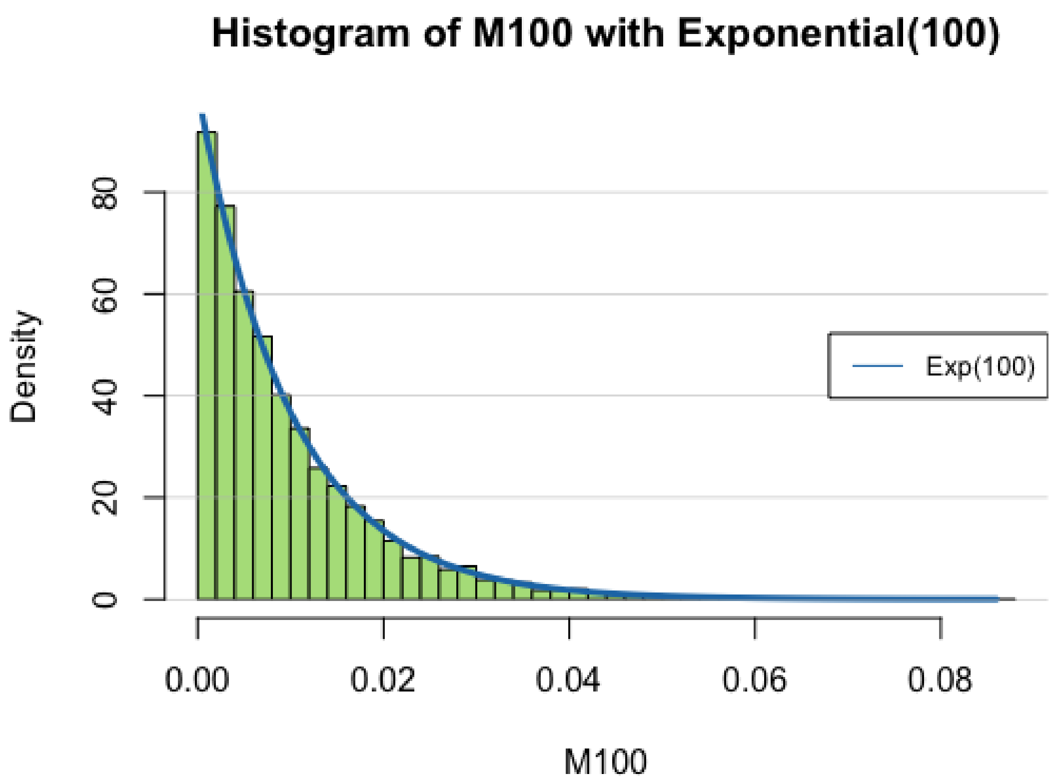
```
x_value <- seq(from=0, to=max(minunif_eva), by=0.001)
exp50 <- dexp(x_value, 50)

hist(minunif_eva, breaks=50, col='skyblue', probability=T,
     main='Histogram of M50 with Exponential(50)', xlab='M50')
lines(x=x_value, y=exp50, lwd=3, col='purple')
grid(nx=NA,ny=NULL,lty=1,lwd=0.5,col="gray")
legend('right', legend='Exp(50)',
       col='purple', lty=1:2, cex=0.8)
```



The distribution of $M_{50} = Min(U_1, U_2, ...U_{50})$ seems to fit the density curve of $exp(50)$.

Q2-c

```
sample_size <- 100
minunif_eva <- rep(NA, sim_size)

for (i in 1:sim_size) {
  sample <- runif(sample_size)
  minunif_eva[i] <- min(sample)
}

x_value <- seq(from=0, to=max(minunif_eva), by=0.001)
exp50 <- dexp(x_value, 100)

hist(minunif_eva, breaks=50, col='#b2df8a', probability=T,
     main='Histogram of M100 with Exponential(100)', xlab='M100')
lines(x=x_value, y=exp50, lwd=3, col='#1f78b4')
grid(nx=NA,ny=NULL,lty=1,lwd=0.5,col="gray")
legend('right', legend='Exp(100)',
       col='#1f78b4', lty=1:2, cex=0.8)
```



The distribution of $M_{100} = Min(U_1, U_2, ...U_100)$ seems to fit the density curve of $exp(100)$.

Q2-d

Find the cdf of nM_n

$$\begin{aligned}
P(nM_n \geq x) &= P(M_n \geq \frac{x}{n}) \\
&= P(U_1 \geq \frac{x}{n}, U_2 \geq \frac{x}{n}, ...U_n \geq \frac{x}{n}) \\
&= P(U_1 \geq \frac{x}{n})P(U_2 \geq \frac{x}{n})...P(U_n \geq \frac{x}{n}) \\
&= (1 - P(U_1 < \frac{x}{n}))(1 - P(U_2 < \frac{x}{n}))...(1 - P(U_n < \frac{x}{n})) \\
&= (1 - F_U(\frac{x}{n}))^n, (F_U(\frac{x}{n}) = \frac{x}{n}) \\
&= (1 - \frac{x}{n})^n
\end{aligned}$$

$$\begin{aligned}
F_{nM_n}(x) &= 1 - P(nM_n \geq x) \\
&= 1 - (1 - \frac{x}{n})^n
\end{aligned}$$

The exponential function tells us that

$$\lim_{n \to \infty} (1 + (-\frac{x}{n}))^n = e^{-x}$$

As n goes to infinity, the cdf and pdf of nM_n turn out to be

$$\begin{aligned}
\lim_{n \to \infty} F_{nM_n}(x) &= 1 - e^{-x} \\
\lim_{n \to \infty} f_{nM_n}(x) &= \frac{\partial F_{nM_n}}{\partial x} \\
&= e^{-x}
\end{aligned}$$

The pdf of $Exp(1)$ is also defined as $\epsilon(x) = e^{-x}$, therefore,

$$\lim_{n \to \infty} f_{nM_n} = e^{-x} = \epsilon(x)$$

This proves that as n goes to infinity $n \cdot M_n$ converses in distribution to $exp(1)$.