STAT 403 Spring 2018
HW02

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April 13, 2019

Q1

Q1-1

$$p(x) = \begin{cases} 6x(1-x), & \text{if } x \in [0,1] \\ 1, & \text{otherwise} \end{cases}$$

For $x \in [1, 0]$, the cdf of X is

$$\begin{aligned} F_X(x) &= \int_0^x p(t)dt \\ &= \int_0^x 6t(1-t)dt \\ &= \int_0^x 6t - 6t^2 dt \\ &= 3x^2 - 2x^3 \end{aligned}$$

Otherwise, if $x > 1, F(x) = 1$; if $x < 0, F(x) = 0$, since $p(x) = 0$ if $x \notin [0, 1]$.

$$F_X(x) = \begin{cases} 3x^2 - 2x^3, & \text{if } x \in [0,1] \\ 1, & \text{if } x > 1 \\ 0, & \text{if } x < 0 \end{cases}$$

Q1-2

For $x \in [0, 1]$, the edf of X is

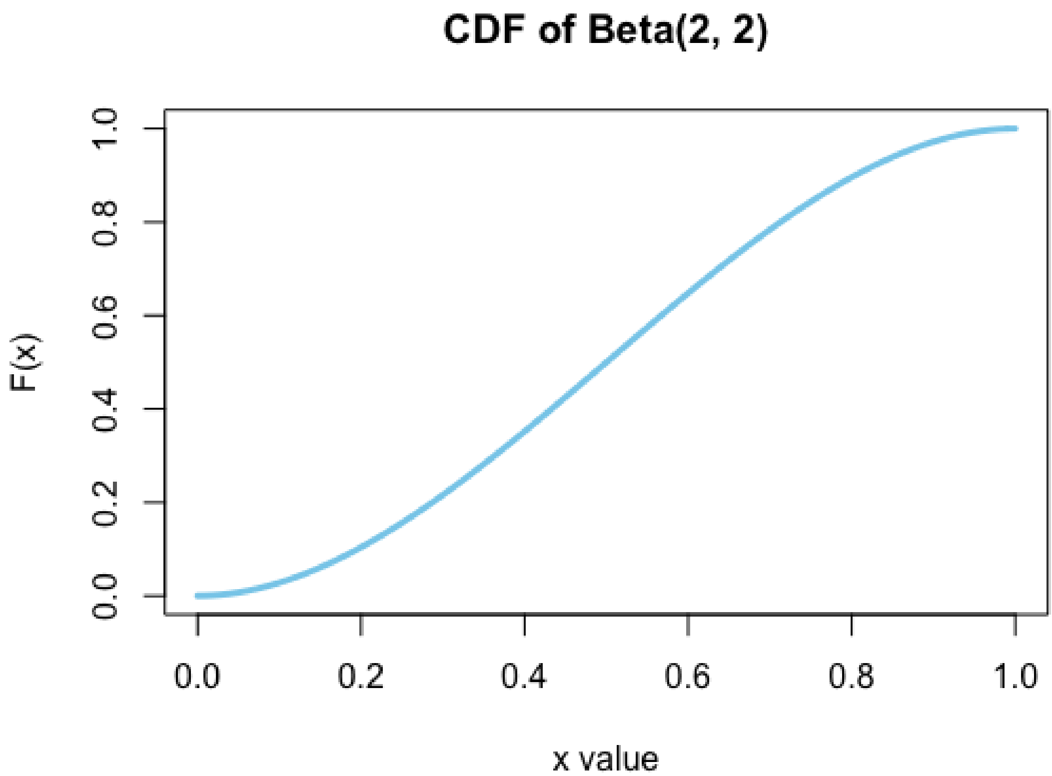
$$\widehat{F_n}(x) = \frac{1}{n} \sum_{i=1}^{n} I(x_i \leq x)$$

$$
\begin{aligned}
\mathbb{E}(\widehat{F_n}) &= \mathbb{E}(\frac{1}{n}\sum_{i=1}^{n} I(x_i \leq x)) \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(x_i \leq x) \\
&= \mathbb{E}(x_i \leq x) \\
&= P(x_i < x) \\
&= F_X(x), \text{ since } (x_i < x) \text{ follows Bernoulli distribution} \\
&= 3x^2 - 2x^3
\end{aligned}
$$

$$
\begin{aligned}
Var(\widehat{F_n}) &= Var(\frac{1}{n}\sum_{i=1}^{n} I(x_i \leq x)) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var(I(x_i \leq x)) \\
&= \frac{1}{n} Var(P(x_i < x)) \\
&= \frac{1}{n} F(x)(1 - F(x)), \text{ since } Var(Bern(p)) = p(1-p) \\
&= \frac{(3x^2 - 2x^3)(1 - 3x^2 + 2x^3)}{n}
\end{aligned}
$$

Q1-3

```
x_base <- seq(0, 1, 0.01)
beta_cdf <- function(x) {return(3 * x^2 - 2 * x^3)}
plot(x_base, beta_cdf(x_base), type='l', col='skyblue', lwd=3,
     xlim=c(0, 1), main='CDF of Beta(2, 2)', xlab='x value', ylab='F(x)')
```



Q2

Q2-1

Given $U \sim Uniform(0, 1)$ and $W = -2log(U)$.
The cdf of U $F_U(x) = x, x \in [0, 1]$.

First find the cdf of W.

$$
\begin{aligned}
F_W(x) &= P(W < x) \\
P(W < x) &= P(-2log(U) < x) \\
&= P(e^{-2log(U)} < e^x) \\
&= P(U^{-2} < e^x) \\
&= P(U \geq e^{-\frac{x}{2}}) \\
&= 1 - P(U < e^{-\frac{x}{2}}) \\
&= 1 - e^{-\frac{x}{2}}
\end{aligned}
$$

Note that the cdf of $Exp(0.5)$, $F = 1 - e^{-\frac{x}{2}}$, which is equal to the cdf of W, therefore, we can conclude that W and $Exp(0.5)$ have the same distribution.