STAT 403 Spring 2018
HW01

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Q1

Q1-a

$$\begin{aligned} Bias(\bar{X}) &= \mathbb{E}(\bar{X}) - \beta \\ &= \mathbb{E}(\frac{1}{n}\sum_{i=1}^{n} X_i) - \beta \\ &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(X_i) - \beta \\ &= \mathbb{E}(X_i) - \beta \end{aligned}$$

Note that X follows an exponential distribution, the expected value for any sample from X, X_i is equal to β.

$$Bias(\bar{X}) = \beta - \beta = 0$$

The sample average $\bar{X}$ is an unbiased estimator.

$$\begin{aligned} Var(\bar{X}) &= Var(\frac{1}{n}\sum_{i=1}^{n} X_i)) \\ &= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) \end{aligned}$$

The variance of X_i from exponential distribution is equal to β^2.

$$
\begin{aligned}
Var(\bar{X}) &= \frac{1}{n^2}\sum_{i=1}^{n}\beta^2 \\
&= \frac{\beta^2}{n}
\end{aligned}
$$

The variance of $\bar{X}$ is $\frac{\beta^2}{n}$.

Q1-b

$$
MSE(\bar{X}) = Var(X) + Bias(X)^2 = \frac{\beta^2}{n} + 0 = \frac{\beta^2}{n}
$$

The mean square error of $\bar{X}$ is $\frac{\beta^2}{n}$.

Q1-c

Note that $\lim_{n\to\infty} Var(\bar{X}) = \lim_{n\to\infty}\frac{\beta^2}{n} = 0$, since β is fixed, and we have proved $\bar{X}$ is unbiased, $\bar{X}$ can be considered as a consistent estimator that converges to β as sample size n increases.

Q1-d

$$
\begin{aligned}
Bias(a\bar{X}) &= \mathbb{E}(a\bar{X}) - \beta \\
&= \mathbb{E}(\frac{a}{n}\sum_{i=1}^{n}X_i) - \beta \\
&= \frac{a}{n}\sum_{i=1}^{n}\mathbb{E}(X_i) - \beta \\
&= a\mathbb{E}(X_i) - \beta \\
&= (a-1)\beta
\end{aligned}
$$

$$
\begin{aligned}
Var(a\bar{X}) &= Var(\frac{a}{n}\sum_{i=1}^{n}X_i)) \\
&= \frac{a^2}{n^2}\sum_{i=1}^{n}Var(X_i) \\
&= \frac{a^2\beta^2}{n}
\end{aligned}
$$

$$
\begin{aligned}
MSE(a\bar{X}) &= Var(a\bar{X}) + Bias(a\bar{X})^2 \\
&= \frac{a^2\beta^2}{n} + (a-1)^2\beta^2 \\
&= \beta^2(\frac{a^2}{n} + (a-1)^2)
\end{aligned}
$$

The mean square error for estimator $a\bar{X}$ is $\beta^2(\frac{a^2}{n} + (a-1)^2)$.

Q1-e

$$
\frac{\partial MSE(a\bar{X}))}{\partial a} = 2(a-1)\beta^2 + \frac{2a\beta^2}{n}
$$

$$
\frac{\partial^2 MSE(a\bar{X}))}{\partial^2 a} = 2\beta^2 + \frac{2\beta^2}{n} > 0
$$

Since second derivative of $MSE(a\bar{X})$ is positive, we can get a minimum value of $MSE(a\bar{X})$ when $\frac{\partial MSE(a\bar{X}))}{\partial a} = 0$, then we obtain

$$
\begin{aligned}
2(a-1)\beta^2 + \frac{2a\beta^2}{n} &= 0 \\
a &= \frac{n}{n+1}
\end{aligned}
$$

Substitute the value of a into $MSE(a\bar{X})$

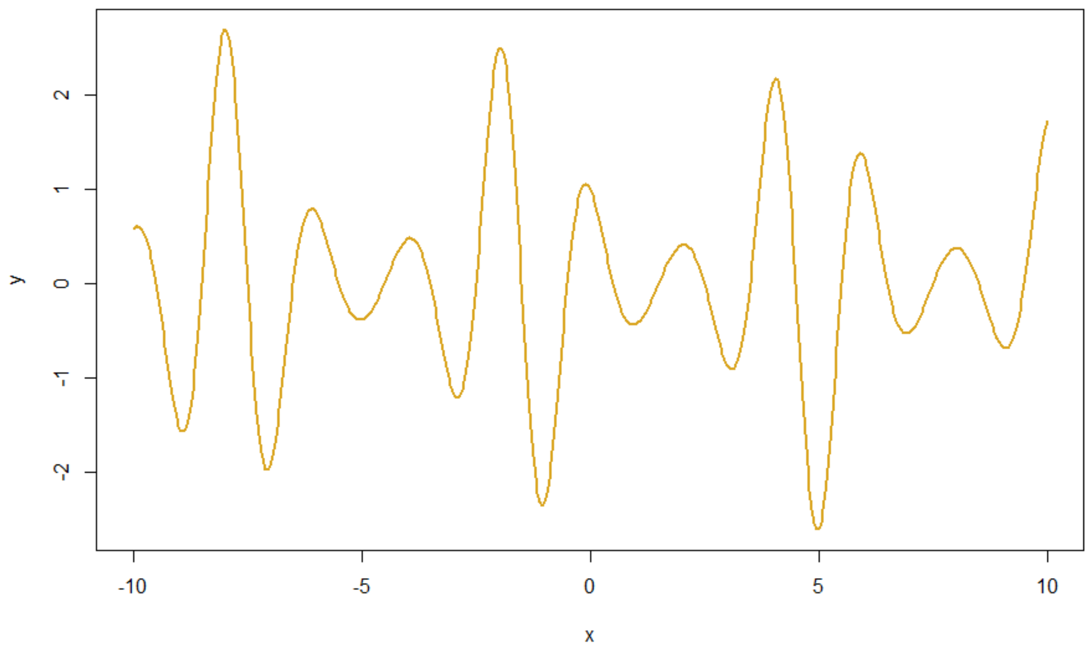
$$
\begin{aligned}
MSE(a\bar{X}) &= \beta^2(\frac{(\frac{n}{n+1})^2}{n} + \frac{1}{(n+1)^2}) \\
&= \frac{\beta}{n+1} < \frac{\beta^2}{n} = MSE(\bar{X})
\end{aligned}
$$

The mean square error of $\bar{X}$ has its minimum value when $a = \frac{n}{n+1}$. Taking this value of a minimizes $MSE(a\bar{X})$ so that the MSE of estimator $a\bar{X}$ is less than MSE of $\bar{X}$.

Q2

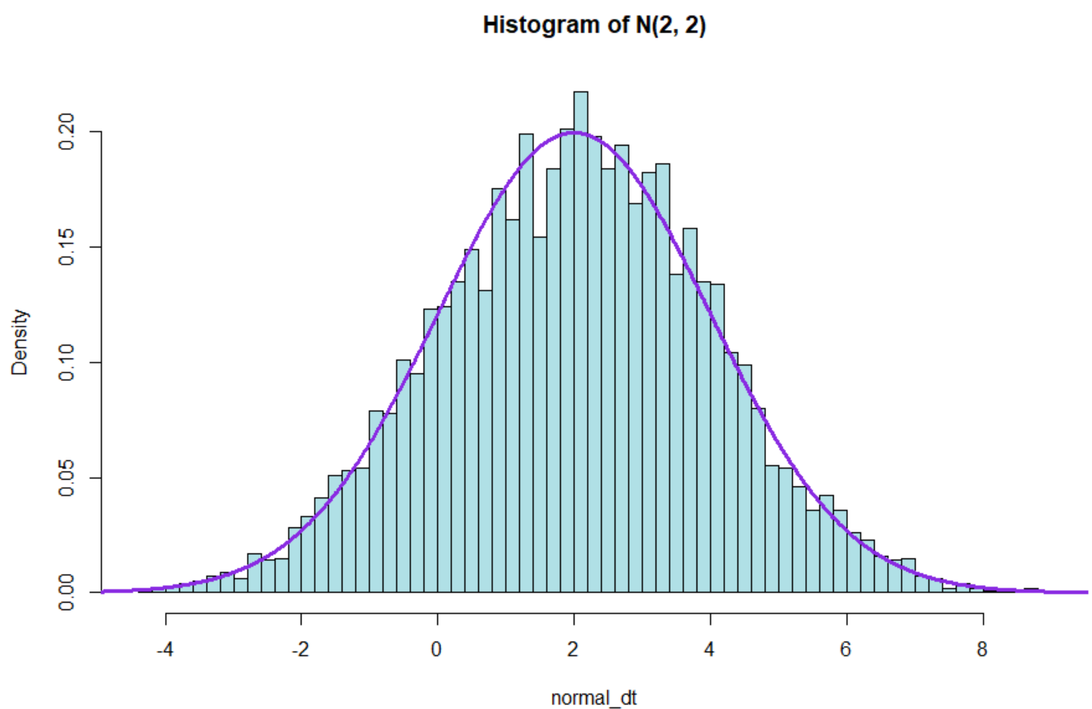
```
x_value <- seq(-10, 10, 0.05)
y_value <- exp(-sin(x_value)) * cos(pi * x_value)

plot(x_value, y_value, type = 'l', lwd = 2, col = 'goldenrod',
     xlab = 'x', ylab = 'y')
```



Q3

```
normal_dt <- rnorm(5000, 2, 2)
hist(normal_dt, breaks=50, main='Histogram of N(2, 2)', probability=TRUE,
     col='powderblue')
x_nor_value <- seq(-5, 10, 0.01)
y_nor_value <- dnorm(x_nor_value, 2, 2)
lines(x_nor_value, y_nor_value, lwd=3, col='blueviolet')
```



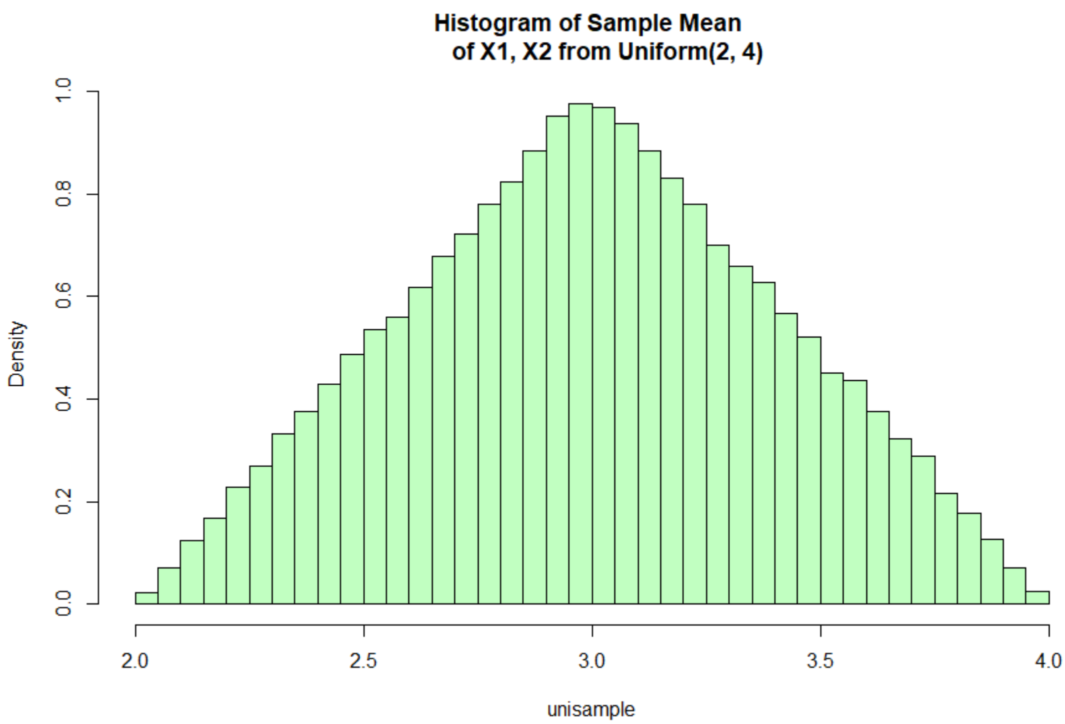
Q4

Q4-a

```
iterations <- 100000
unisample <- rep(NA, iterations)

for(i in 1:iterations) {
  uni_avg <- mean(runif(2, 2, 4))
  unisample[i] <- uni_avg
}

hist(unisample, breaks=50, main='Histogram of Sample Mean
     of X1, X2 from Uniform(2, 4)', probability=TRUE, col='palegreen')
```



Q4-b

```
hist(unisample, breaks=50, main='Histogram of Sample Mean
     of X1, X2 from Uniform(2, 4)', probability=TRUE, col='darkseagreen1')
lines(c(2:3), c(0:1), lwd=3, col='deepskyblue')
lines(c(3:4), c(1:0), lwd=3, col='deepskyblue')
```