STAT 403 Spring 2018
HW04

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May 1, 2019

Q1

Q1-a

$$
\begin{aligned}
Bias(\bar{X}_n) &= \mathbb{E}(\bar{X}_n) - \lambda \\
&= \mathbb{E}(\frac{\sum_{i=1}^{100} X_i}{100}) - \lambda \\
&= \frac{\sum_{i=1}^{100}(\mathbb{E}(X_i))}{100} - \lambda \\
&= \frac{\sum_{i=1}^{100} \lambda}{100} - \lambda \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
Var(\bar{X}_n) &= Var(\frac{\sum_{i=1}^{100} X_i}{100}) \\
&= \frac{\sum_{i=1}^{100} Var(X_i)}{100^2} \\
&= \frac{\sum_{i=1}^{100} \lambda}{100^2} \\
&= \frac{\lambda}{100}
\end{aligned}
$$

$X_i \sim Po(\lambda = 4)$, $Var(\bar{X}_n) = \frac{4}{100}$, $Bias(\bar{X}_n) = 0$

Q1-b

$$
\begin{aligned}
SE(\bar{X}_n) =& \sqrt{Var(\bar{X}_n)} \\
=& \sqrt{\frac{\lambda}{100}}
\end{aligned}
$$

$\widehat{\lambda} = \bar{X}_n$ is consistent in estimating λ, and sample size is large enough, thereby we can consider SE as

$$
SE(\bar{X}_n) = \sqrt{\frac{\bar{X}_n}{100}}
$$

By CLT, $\bar{X}_100$ follows approximately normal distribution with $\mu = \lambda$ and $\sigma = SE(\bar{X}_n)$

$$
\begin{aligned}
P(-c \leq \bar{X}_n \leq c) &= 0.9 \\
P(-\frac{c-\lambda}{SE} \leq \frac{\bar{X}_n - \lambda}{SE} \leq \frac{c-\lambda}{SE}) &= 0.9 \\
P(-\frac{c-\lambda}{SE} \leq Z \leq \frac{c-\lambda}{SE}) &= 0.9 \\
\frac{c-\lambda}{SE} &= Z_{1-\frac{\alpha}{2}} \approx 1.64 \\
P(-1.64 \leq \frac{\bar{X}_n - \lambda}{SE} \leq 1.64) &= 0.9 \\
P(\bar{X}_n - 1.64 \cdot SE \leq \lambda \leq \bar{X}_n + 1.64 \cdot SE) &= 0.9 \\
P(\bar{X}_n - 1.64 \cdot \sqrt{\frac{\bar{X}_n}{100}} \leq \lambda \leq \bar{X}_n + 1.64 \cdot \sqrt{\frac{\bar{X}_n}{100}}) &= 0.9
\end{aligned}
$$

Here we get the 90% confidence interval for λ,

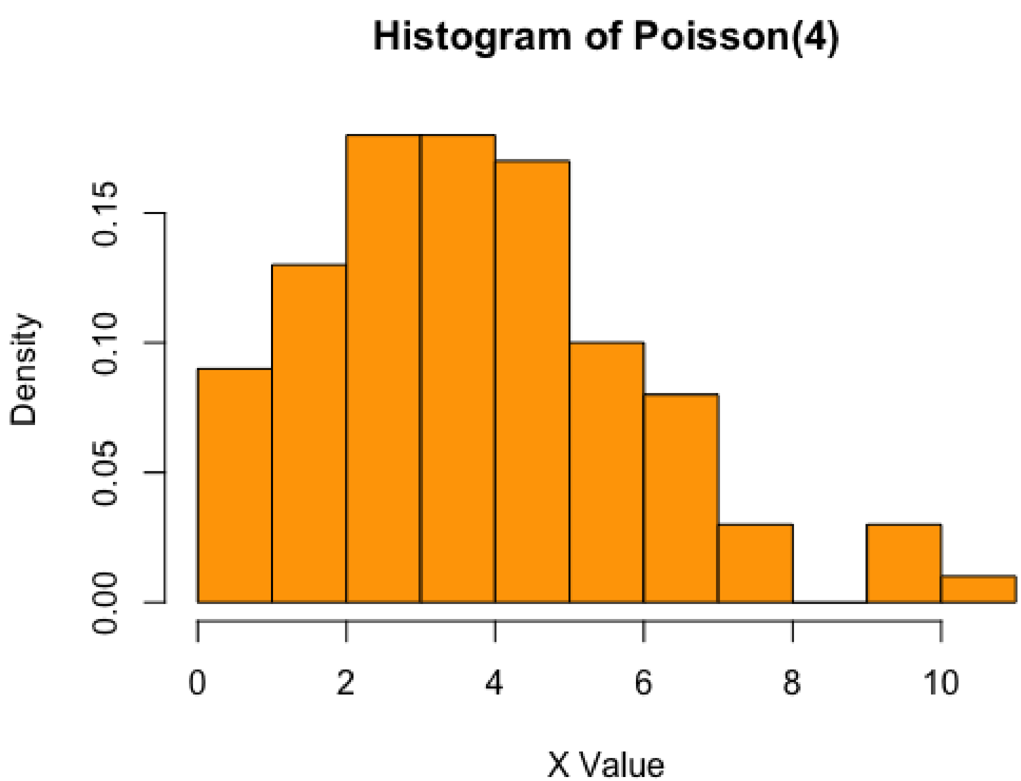
$$
CI = [\bar{X}_n - 1.64 \cdot \frac{\sqrt{\bar{X}_n}}{10}, \bar{X}_n + 1.64 \cdot \frac{\sqrt{\bar{X}_n}}{10}]
$$

Q1-c

```
po4_dt <- rpois(100, 4)

hist(po4_dt, main='Histogram of Poisson(4)', probability = T, xlab='X Value',
     ylab='Density', col='orange')

lambda_est <- mean(po4_dt)
> lambda_est
[1] 4.24
```



In this scenarios, MLE is equal to 4.24

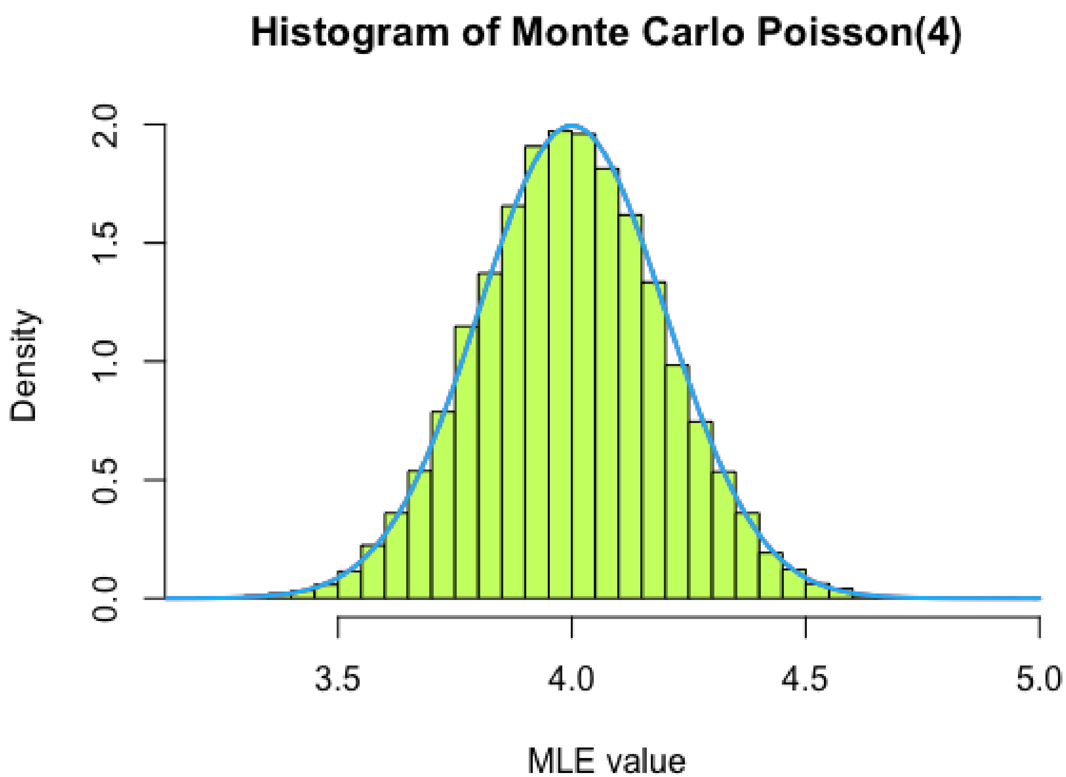
0.1 Q1-d

```
sim_size <- 10000
sim_result <- rep(NA, sim_size)

for (i in 1:sim_size) {
  po_dt <- rpois(100, 4)
  sim_result[i] <- mean(po_dt)
}

norm_base <- seq(3.0, 5.0, 0.01)
norm_dt <- dnorm(norm_base, 4, 0.2)

hist(sim_result, probability = T, breaks=30, col='darkolivegreen1',
     main='Histogram of Monte Carlo Poisson(4)', xlab='MLE value', ylab='Density')
lines(norm_base, norm_dt, lwd=2, col='deepskyblue')
```



The distribution of simulated MLE fits the normal curve $N(4, 0.2^2)$.

Q1-e

```
> sum(sim_result >= 3.5 & sim_result <= 4.5) / sim_size
[1] 0.9876
```

Among 10000 simulations, 9876 of them fell in the interval $[3.5, 4.5]$, the fraction is 0.9876.

From previous part, we perceived that the distribution of MLE follows a normal distribution. The mean of this distribution is equal to $\lambda = 4$, since λ_{MLE} is an unbiased estimator for $lambda$. The variance of MLE is, as we calculated in problem a, $\frac{4}{100}$. The distribution of λ_{MLE} turns out to be $Normal(4, 0.04)$.

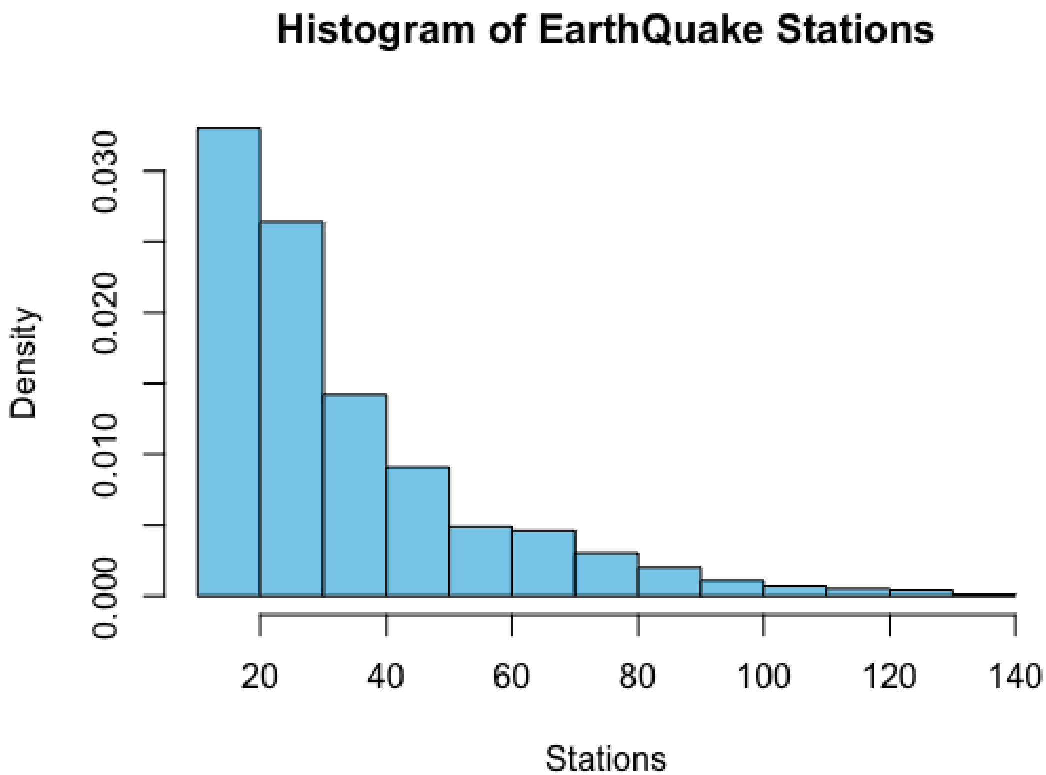
The proportion of simulated MLE that falls into the interval $[3.5, 4.5]$ can be represented by

$$
\begin{aligned}
P(3.5 \leq \lambda_{MLE} \leq 4.5) &= P(\frac{3.5 - \mathbb{E}(\lambda_{MLE})}{SE(\lambda_{MLE})} \leq Z \leq \frac{4.5 - \mathbb{E}(\lambda_{MLE})}{SE(\lambda_{MLE})}) \\
&= P(\frac{3.5 - 4}{0.2} \leq Z \leq \frac{4.5 - 4}{0.2}) \\
&= \Phi(2.5) - \Phi(-2.5) \\
&\approx 0.9876
\end{aligned}
$$

Q2-a

```
quakes <- read.table('fijiquakes.dat', sep='', header=T)

hist(quakes$stations, probability=T, main='Histogram of EarthQuake Stations',
     xlab='Stations', col='skyblue')
```



Q2-b

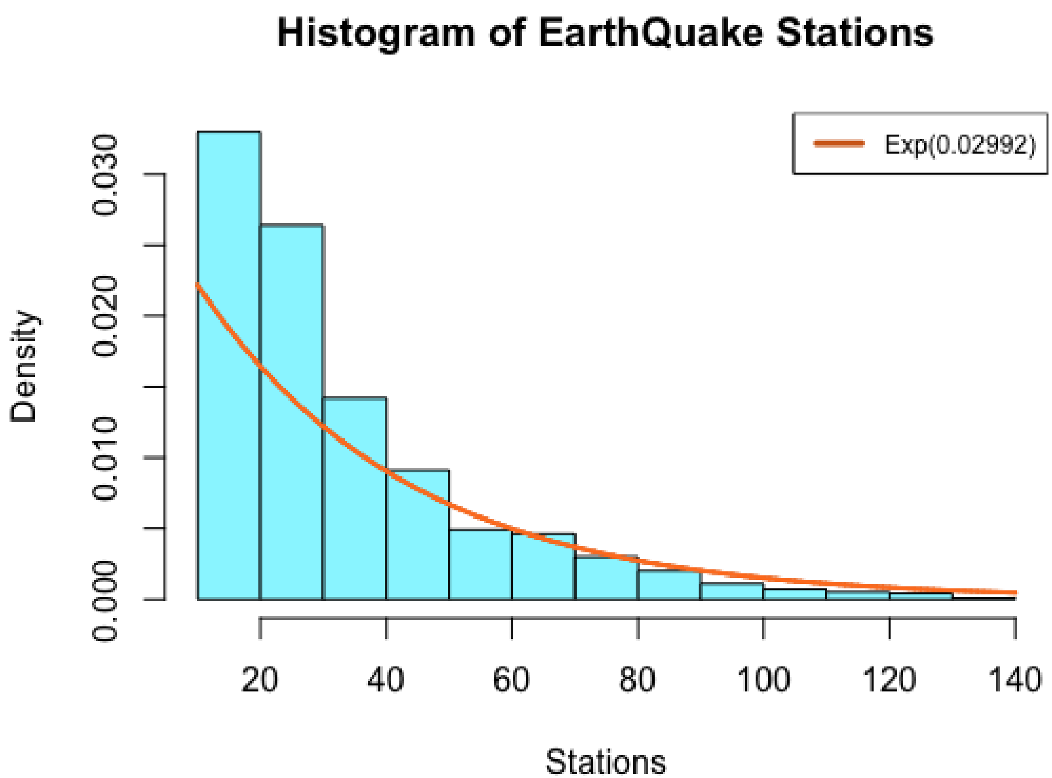
The MLE for λ for exponential distribution is $\frac{n}{\sum_{i=1}^{n} X_i} = \frac{1}{X_n}$, so we choose $\frac{1}{X_n}$ as fitted value for ratio parameter.

```
lambda_est <- 1/mean(quakes$stations)
> lambda_est
[1] 0.02992399
```

The estimated ration parameter is 0.02992

```
exp_base <- seq(10, 140, 0.01)
exp_dt <- dexp(exp_base, lambda_est)

hist(quakes$stations, probability=T, main='Histogram of EarthQuake Stations',
     xlab='Stations', col='cadetblue1')
lines(exp_base, exp_dt, lwd=2, col='chocolate1')
legend('topright', legend=('Exp(0.02992)'), col='chocolate', lwd=3, cex=0.75)
```



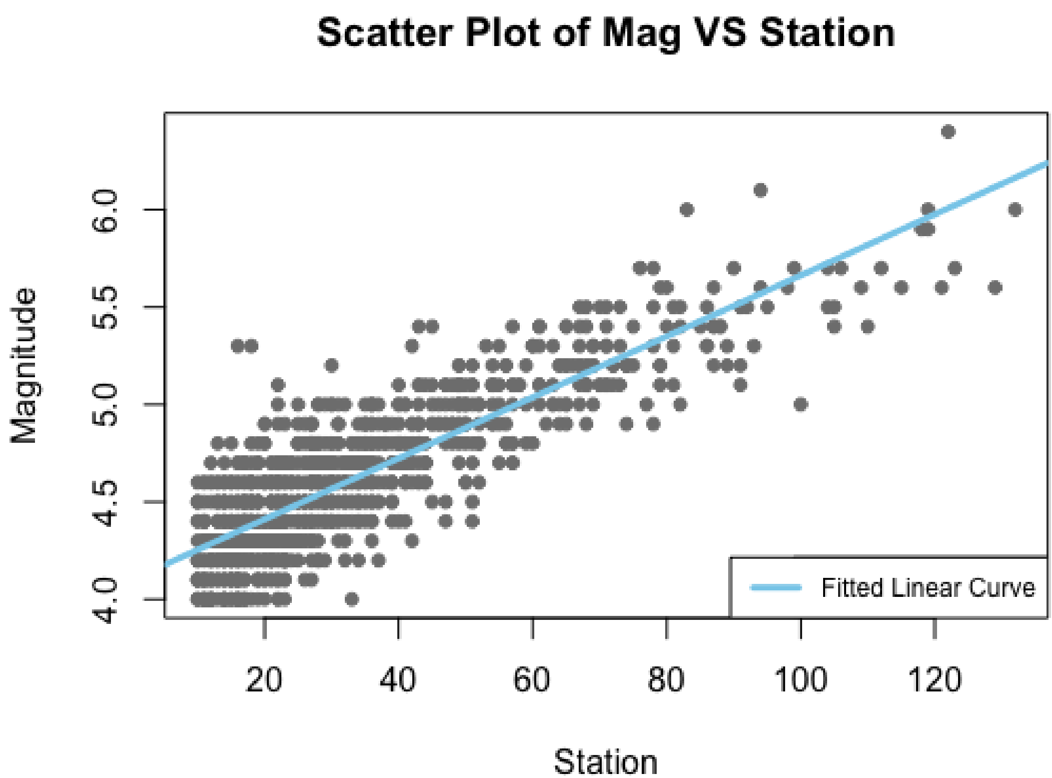
Both station distribution and the exponential distribution skew to right. However, station's distribution seems to be denser on the right tail.

Q2-c

```
linear_reg = lm(quakes$mag~quakes$stations)
> summary(linear_reg)$coeff[2,1]
[1] 0.01565421
```

The linear model of mag versus stations has a slope of 0.0157.

```
plot(x=quakes$stations, y=quakes$mag, pch=19, cex=0.8, col='gray50', xlab='Station',
     ylab='Magnitude', main='Scatter Plot of Mag VS Station')
abline(linear_reg, lwd=3, col='skyblue')
legend('bottomright', legend=('Fitted Linear Curve'), col='skyblue', lwd=3, cex=0.75)
```



Q2-d

The 95% confidence interval of fitted slope β_1 is

$$[\widehat{\beta}_1 - Z_{0.975} \cdot sd(\widehat{\beta}_1), \widehat{\beta}_1 + Z_{0.975} \cdot sd(\widehat{\beta}_1)]$$

```
beta1 <- summary(linear_reg)$coeff[2,1]
sd_beta1 <- summary(linear_reg)$coeff[2,2]
lowerbd <- beta1 - qnorm(0.975) * sd_beta1
upperbd <- beta1 + qnorm(0.975) * sd_beta1
> c(lowerbd, upperbd)
[1] 0.01505533 0.01625310
```

The 95% CI of fitted slope is $[0.01505533, 0.01625310]$.