Lect 6-1

```
type_one <- c(65, 81, 57, 66, 82, 82, 67, 59, 75, 70)
type_two <- c(64, 71, 83, 59, 65, 56, 69, 74, 82, 79)
n1 <- length(type_one)
n2 <- length(type_two)
s1 <- sd(type_one)
s2 <- sd(type_two)
```

a

$H_0 : \mu_1 = \mu_2$
$H_1 : \mu_1 \neq \mu_2$, a two-tails t-test

With the assumption of equal variance, we can calculate the pooled sample variance.

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{(n_1 - 1) + (n_2 - 1)}$$
$$t = \frac{(\bar{Y_1} - \bar{Y_2}) - (\mu_1 - \mu_2)}{s_p\sqrt{1/n_1 + 1/n_2}}$$
$$t \sim t_{(n_1+n_2-2)}$$

Sample standard error $se = s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$

```
sp <- sqrt(((n1-1)*s1^2 + (n2-1)*s2^2)/(n1 + n2 - 2))
se <- sp * sqrt(1/n1 + 1/n2)
t_obs <- (mean(type_one) - mean(type_two)) / se

> mean(type_one) - mean(type_two)
[1] 0.2
> sp
[1] 9.315459
> se
[1] 4.166
> t_obs
[1] 0.04800768
```

Plug in the data, we get $(\bar{Y_1} - \bar{Y_2}) = 0.2$, $s_p = 9.315459$, standard error $= 4.166$, observed t-statistic $= 0.04800768$

```
alpha <- 0.05
low_cv <- qt(alpha/2, df = (n1 + n2 - 2))
up_cv <- qt(1 - alpha/2, df = (n1 + n2 - 2))

> low_cv
[1] -2.100922
> up_cv
[1] 2.100922
```

The lower critical value for t-statistic is -2.100922 and the upper critical value is 2.100922. Under the significant level of 0.05, rejection regions are $t_{obs} < -2.100922$ or $t_{obs} > 2.100922$. Because $t_{obs} = 0.048$ does not fall in any of the above regions, therefore, this sample data cannot be significant evidence against H_0.

```
CI_low <- (mean(type_one) - mean(type_two)) - qt(1 - alpha/2, df = (n1 + n2 - 2)) * se
CI_up <- (mean(type_one) - mean(type_two)) + qt(1 - alpha/2, df = (n1 + n2 - 2)) * se
> CI_low
[1] -8.552441
> CI_up
[1] 8.952441
```

We are 95% confident that true difference in population means fall in the region $[-8.552441, 8.952441]$. Since $H_0 : \mu_1 - \mu_2 = 0$ is within this confidence interval, the sample data does not provide significant evidence to reject H_0.

```
p_val <- (1 - pt(t_obs, df=n1+n2-2)) * 2
> p_val
[1] 0.9622388
```

Observed t-statistic has a p-value of 0.9622388, which is larger than significant level $\alpha = 0.05$. Therefore, sample data fail to provide significant evidence to reject H_0

b

$H_0 : \sigma_1 = \sigma_2$
$H_1 : \sigma_1 \neq \sigma_2$

A two variance two-tail F-test can be applied here. $F = s_1^2/s_2^2$ with $df1 = 9, df2 = 9$

```
F_obs <- s1^2 / s2^2
p_val <- pf(F_obs, df1=n1-1, df2=n2-1) +
  (1 - pf(1 + abs(F_obs - 1), df1=n1-1, df2=n2-1))

> F_obs
[1] 0.9782168
> p_val
[1] 0.9746426
```

The sample F-statistic has a value of 0.9782168, having a p-value of 0.9746. Under significance level of 0.05, the data fail to provide strong evidence against H_0, since p-value $> \alpha$.

The α level confidence interval of F-statistic is

$$
\begin{aligned}
P(F_{\frac{\alpha}{2}} < F_{obs} < F_{1-\frac{\alpha}{2}}) &= 1-\alpha \\
P(F_{\frac{\alpha}{2}} < \frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} < F_{1-\frac{\alpha}{2}}) &= 1-\alpha \\
P(\frac{s_2^2}{s_1^2}F_{\frac{\alpha}{2}} < \frac{\sigma_2^2}{\sigma_1^2} < \frac{s_2^2}{s_1^2}F_{1-\frac{\alpha}{2}}) &= 1-\alpha \\
P(\frac{s_1^2}{s_2^2}\frac{1}{F_{1-\frac{\alpha}{2}}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2}\frac{1}{F_{\frac{\alpha}{2}}}) &= 1-\alpha
\end{aligned}
$$

The lower confidence limit is $\frac{s_1^2}{s_2^2}\frac{1}{F_{1-\frac{\alpha}{2}}}$, the upper confidence limit is $\frac{s_1^2}{s_2^2}\frac{1}{F_{\frac{\alpha}{2}}}$

```
CI_low <- s1^2 / s2^2 * 1 / qf(1 - alpha/2, df1=n1-1, df2=n2-1)
CI_up <- s1^2 / s2^2 * 1 / qf(alpha/2, df1=n1-1, df2=n2-1)

> CI_low
[1] 0.2429752
> CI_up
[1] 3.938295
```

We are 95% confident that $\sigma_1^2/sigma_2^2$ is in $[0.2429752, 3.938295]$.
Since $H_0 : \frac{\sigma_1^2}{\sigma_2^2} = 1$ falls in the 95% confidence interval, the observed statistic fail to provide strong evidence against the null.

```
low_cv <- qf(alpha/2, df1=n1-1, df2=n2-1)
up_cv <- qf(1 - alpha/2, df1=n1-1, df2=n2-1)
> low_cv
[1] 0.2483859
> up_cv
[1] 4.025994
```

The rejection region under significance level 95% is $s_1^2/s_2^2 < 0.2483859$ or $s1^2/s_2^2 > 4.025994$. Since observed F-statistic 0.978 does not fall in any of the rejection regions, we may conclude that the sample data fail to show significant evidence against H_0.

Lect 6-2

$$
\begin{aligned}
SS_{Treat} &= n\sum_i^a(\bar{Y_{i.}} - \bar{Y_{..}})^2 \\
&= n\sum_i^a(\bar{Y_{i.}}^2 + \bar{Y_{..}}^2 - 2\bar{Y_{i.}}\bar{Y_{..}}) \\
&= n\sum_i^a \bar{Y_{i.}}^2 + n\sum_i^a \bar{Y_{..}}^2 - 2n\sum_i^a \bar{Y_{i.}}\bar{Y_{..}}
\end{aligned}
$$

Note that $\bar{Y_{i.}} = \frac{1}{n}\sum_j^n Y_{ij}$, and $Y_{i.} = \sum_j^n Y_{ij}$

$$\begin{aligned}
n\sum_i^a \bar{Y_{i.}}^2 &= n\sum_i^a (\frac{1}{n}\sum_j^n Y_{ij})^2 \\
&= n(\frac{1}{n})^2 \sum_i^a (\sum_j^n Y_{ij})^2 \\
&= \frac{1}{n}\sum_i^a Y_{i.}^2
\end{aligned}$$

Note that $\bar{Y_{..}} = \frac{1}{na}\sum_i^a \sum_j^n Y_{ij}$, and $Y_{..} = \sum_i^a \sum_j^n Y_{ij}$

$$\begin{aligned}
n\sum_i^a \bar{Y_{..}}^2 &= n\bar{Y_{..}}^2 \sum_i^a 1 \\
&= na\bar{Y_{..}}^2 \\
&= na(\frac{1}{na}\sum_i^a \sum_j^n Y_{ij})^2 \\
&= \frac{1}{na} Y_{..}^2
\end{aligned}$$

$$\begin{aligned}
2n\sum_i^a \bar{Y_{i.}}\bar{Y_{..}} &= 2n\sum_i^a (\frac{1}{n}\sum_j^n X_{ij})(\frac{1}{na}\sum_i^a \sum_j^n Y_{ij}) \\
&= 2n \cdot \frac{1}{n} \cdot \frac{1}{na}(\sum_i^a \sum_j^n X_{ij})(\sum_i^a \sum_j^n Y_{ij}) \\
&= \frac{2}{na} Y_{..}^2
\end{aligned}$$

$$\begin{aligned}
SS_{Treat} &= n\sum_i^a \bar{Y_{i.}}^2 + n\sum_i^a \bar{Y_{..}}^2 - 2n\sum_i^a \bar{Y_{i.}}\bar{Y_{..}} \\
&= \frac{1}{n}\sum_i^a Y_{i.}^2 + \frac{1}{na}Y_{..}^2 - \frac{2}{na}\bar{Y_{..}}^2 \\
&= \frac{1}{n}\sum_i^a Y_{i.}^2 - \frac{1}{N}Y_{..}^2
\end{aligned}$$

Lect 6-3

$$
\begin{aligned}
F &= \frac{an}{a-1}\frac{\sum_i^a(\bar{Y_{i.}}-\bar{Y_{..}})^2}{\sum_i^a s_i^2}\\
\text{when (a = 2)}\ F &= 2n\frac{(\bar{Y_{1.}}-\bar{Y_{..}})^2+(\bar{Y_{2.}}-\bar{Y_{..}})^2}{s_1^2+s_2^2}\\
&= 2n\frac{\bar{Y_{1.}}^2+\bar{Y_{..}}^2-2\bar{Y_{1.}}\bar{Y_{..}}+\bar{Y_{2.}}^2+\bar{Y_{..}}^2-2\bar{Y_{2.}}\bar{Y_{..}}}{s_1^2+s_2^2}\\
&= 2n\frac{\bar{Y_{1.}}^2+\bar{Y_{2.}}^2+2\bar{Y_{..}}^2-2\bar{Y_{1.}}\bar{Y_{..}}-2\bar{Y_{2.}}\bar{Y_{..}}}{s_1^2+s_2^2}
\end{aligned}
$$

Note that $\bar{Y_{..}} = \frac{1}{2}(\bar{Y_{1.}}+\bar{Y_{2.}})$, substitute this value into the equation.

$$
\begin{aligned}
2\bar{Y_{..}}^2 &= 2(\frac{1}{2}(\bar{Y_{1.}}+\bar{Y_{2.}}))^2\\
&= \frac{1}{2}(\bar{Y_{1.}}+\bar{Y_{2.}})^2\\
&= \frac{1}{2}(\bar{Y_{1.}}^2+\bar{Y_{2.}}^2+2\bar{Y_{1.}}\bar{Y_{2.}})\\
&= \frac{1}{2}\bar{Y_{1.}}^2+\frac{1}{2}\bar{Y_{2.}}^2+\bar{Y_{1.}}\bar{Y_{2.}}
\end{aligned}
$$

$$
\begin{aligned}
-2\bar{Y_{1.}}\bar{Y_{..}} &= -2\bar{Y_{1.}}(\frac{1}{2}(\bar{Y_{1.}}+\bar{Y_{2.}}))\\
&= -\bar{Y_{1.}}^2-\bar{Y_{1.}}\bar{Y_{2.}}
\end{aligned}
$$

$$
\begin{aligned}
-2\bar{Y_{2.}}\bar{Y_{..}} &= -2\bar{Y_{2.}}(\frac{1}{2}(\bar{Y_{1.}}+\bar{Y_{2.}}))\\
&= -\bar{Y_{2.}}^2-\bar{Y_{1.}}\bar{Y_{2.}}
\end{aligned}
$$

$$
\begin{aligned}
F &= 2n\frac{\bar{Y_{1.}}^2+\bar{Y_{2.}}^2+\frac{1}{2}\bar{Y_{1.}}^2+\frac{1}{2}\bar{Y_{2.}}^2+\bar{Y_{1.}}\bar{Y_{2.}}-\bar{Y_{1.}}^2-\bar{Y_{1.}}\bar{Y_{2.}}-\bar{Y_{2.}}^2-\bar{Y_{1.}}\bar{Y_{2.}}}{s_1^2+s_2^2}\\
&= 2n\frac{\frac{1}{2}\bar{Y_{1.}}^2+\frac{1}{2}\bar{Y_{2.}}^2-\bar{Y_{1.}}\bar{Y_{2.}}}{s_1^2+s_2^2}\\
&= n\frac{\bar{Y_{1.}}^2+\bar{Y_{2.}}^2-2\bar{Y_{1.}}\bar{Y_{2.}}}{s_1^2+s_2^2}\\
&= \frac{(\bar{Y_{1.}}-\bar{Y_{2.}})^2}{(s_1^2+s_2^2)/n}
\end{aligned}
$$

Note that $s_p^2 = \frac{1}{2}(s_1^2+s_2^2)$, substitute $2\cdot s_p^2$ for $(s_1^2+s_2^2)$

$$
F = \frac{(\bar{Y_{1.}}-\bar{Y_{2.}})^2}{s_p^2\cdot 2/n} = t^2\text{, when a = 2}
$$

Lect 6-4

$$
\begin{aligned}
E(\bar{Y^2}) - E(\bar{Y}^2) &= E(\frac{1}{n}\sum_i^n Y_i^2) - E((\frac{1}{n}\sum_i^n Y_i)^2) \\
&= \frac{1}{n}E(\sum_i^n Y_i^2) - \frac{1}{n^2}E((\sum_i^n Y_i)^2) \\
&= \frac{1}{n}\sum_i^n E(Y_i^2) - \frac{1}{n^2}E((\sum_i^n Y_i)(\sum_i^n Y_i))
\end{aligned}
$$

Note that $Y_i's$ are identical, therefore, $E(Y_i^2) = E(Y^2)$

$$
\begin{aligned}
\frac{1}{n}\sum_i^n E(Y_i^2) &= \frac{1}{n}E(Y^2)\sum_i^n 1 \\
&= E(Y^2) \\
\frac{1}{n^2}E((\sum_i^n Y_i)(\sum_i^n Y_i)) &= \frac{1}{n^2}E(\sum_i^n Y_i^2 + 2\cdot\sum_{j>i}^n Y_iY_j) \\
&= \frac{1}{n^2}E(\sum_i^n Y_i^2) + \frac{2}{n^2}E(\sum_{j>i}^n Y_iY_j) \\
&= \frac{1}{n^2}\sum_i^n E(Y_i^2) + \frac{2}{n^2}\sum_{j>i}^n E(Y_iY_j)
\end{aligned}
$$

Note that $Y_i's$ are independent, $E(Y_iY_j) = E(Y_i)E(Y_j)$. $Y_i's$ are identical, therefore, $E(Y_i) = E(Y_j) = E(Y)$.

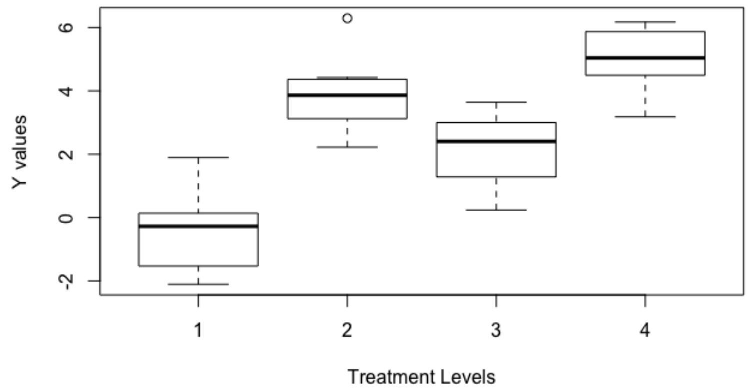
$$
\begin{aligned}
\frac{1}{n^2}E((\sum_i^n Y_i)(\sum_i^n Y_i)) &= \frac{2}{n^2}E(Y_i)E(Y_j)\sum_i^n 1 + \frac{1}{n^2}E(Y)^2\sum_{j>i}^n 1 \\
&= \frac{1}{n}E(Y^2) + \frac{2}{n^2}\frac{n(n-1)}{2}E(Y)^2 \\
&= \frac{1}{n}E(Y^2) + \frac{n-1}{n}E(Y)^2
\end{aligned}
$$

$$
\begin{aligned}
E(\bar{Y^2}) - E(\bar{Y}^2) &= \frac{1}{n}\sum_i^n E(Y_i^2) - \frac{1}{n^2}E((\sum_i^n Y_i)(\sum_i^n Y_i)) \\
&= E(Y^2 - \frac{1}{n}E(Y^2) - \frac{n-1}{n}E(Y)^2 \\
&= \frac{n-1}{n}(E(Y^2) - E(Y)^2) \\
&= \frac{n-1}{n}Var(Y)
\end{aligned}
$$

Lect 7-1

a

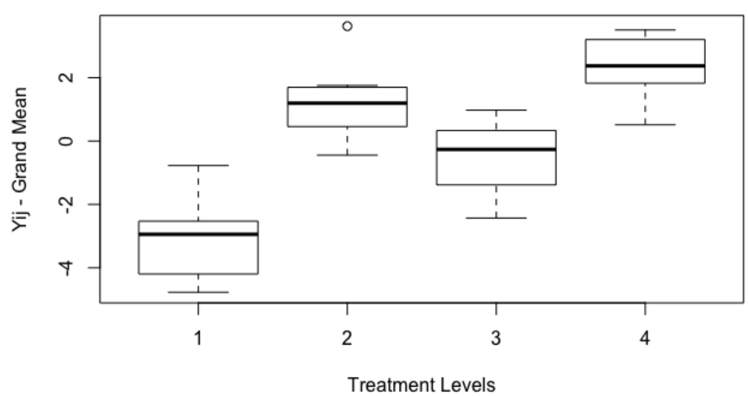
```
tr_level <- as.factor(rep(c(1,2,3,4), each=10))
pool_dt <- data.frame(tr_level, as.vector(t(y)))
plot(pool_dt, xlab='Treatment Levels', ylab='Y values')
```



If the hypothesis looks like $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$, H_1 : there are at least two means are different, then base on this boxplot, X do have effect on Y, since the range of 1 doesn't have any overlap with box of 4, indicating that the true mean μ_1 are unlikely equal to μ_4

b

```
grand_mean <- mean(pool_dt[,2])
new_dt <- data.frame(tr_level, pool_dt[,2] - grand_mean)
plot(new_dt, xlab='Treatment Levels', ylab='Yij - Grand Mean')
```



The distribution of data on boxplot resembles boxplot in part a, the range of y_1 does not overlap range of y_4, indicating that the true mean μ_1 are unlikely equal to μ_4. Therefore, the sample data provide significant evidence against the equal means hypothesis, showing that X has effect on Y.

c

```
y_bar <- apply(y, 1, mean)
y_effect <- y_bar - grand_mean

> y_bar
[1] -0.3537095  3.9038404  2.1095378  5.0102435
> y_effect
[1] -3.0211875  1.2363624 -0.5579403  2.3427655
```

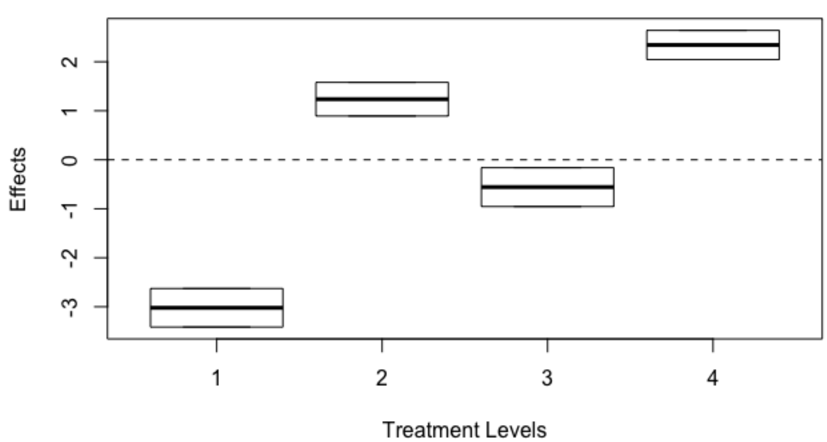
d

```
y_sd <- apply(y, 1, sd)
y_se <- y_sd / sqrt(10)

> y_se
[1] 0.3933592 0.3433781 0.3964595 0.2976147
```

e

```
one_sd_bd <- c(y_effect + y_se, y_effect - y_se)
tr_level <- as.factor(rep(c(1,2,3,4), 2))
effect_est <- data.frame(tr_level, one_sd_bd)
plot(effect_est, xlab='Treatment Levels', ylab='Effects')
abline(h=0, lty=2)
```



From the plot we can perceive that none of the intervals $\bar{\tau_i} \pm se$ covers 0. In this case, it looks like none of the effect can be zero, i.e. zero is in none of the confidence intervals, providing evidence that X has effect on Y.

f

$$SS_{treat} = n \sum_{i}^{a} (\bar{Y_{i.}} - \bar{Y_{..}})$$

$$SS_e = (n-1) \sum_{i}^{a} s_i^2$$

```
SStr <- 10 * sum((y_bar - grand_mean)^2)
SSe <- (10 - 1) * sum(y_sd^2)
> SStr
[1] 164.5601
> SSe
[1] 46.65552
```

$$F_{obs} = \frac{MS_{tr}}{MS_e} = \frac{SS_{tr}/(a-1)}{SS_e/(N-a)}$$

```
F_obs <- (SStr / (4-1)) / (SSe / (40 - 4))
> F_obs
[1] 42.32557
```

The F-statistic is 42.33.

```
> qf(0.95, df1=3, df2=36, lower.tail = T)
[1] 2.866266
```

The critical value is 2.866266, which implies if $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$, then the probability of getting more extreme F-statistic than 2.866266 is only 0.05. Since observed F-statistic is 42.33, falling in rejection region, it is a strong evidence that at least one of the effects is non-zero.

Lect 7-2

a

$$\begin{aligned}
E(\widehat{\tau_i}) &= E(\bar{Y_{i.}} - \bar{Y_{..}}) \\
&= E(\frac{1}{n} \sum_{j}^{n} Y_{ij} - \frac{1}{na} \sum_{j}^{n} \sum_{k}^{a} Y_{kj}) \\
&= \frac{1}{n} \sum_{j}^{n} E(Y_{ij}) - \frac{1}{na} \sum_{j}^{n} \sum_{k}^{a} E(Y_{kj}) \\
&= \frac{1}{n} \sum_{j}^{n} E(\mu + \tau_i + \epsilon_{ij}) - \frac{1}{na} \sum_{j}^{n} \sum_{k}^{a} E(Y_{kj})
\end{aligned}$$

Note that in this model $E(\epsilon_{ij}) = 0$, μ and τ_i are parameters, $E(Y_{kj}) = \mu$ for $k = 1, 2, ...a$, Y_{kj} is pooled data.

$$
\begin{aligned}
E(\widehat{\tau_i}) &= \frac{1}{n}(\sum_{j}^{n} 1)(\mu + \tau_i + 0) - \frac{1}{na}(\sum_{j}^{n}\sum_{k}^{a} 1)\mu \\
&= \tau_i
\end{aligned}
$$

$$
\begin{aligned}
Var(\widehat{\tau_i}) &= V(\bar{Y_{i.}} - \bar{Y_{..}}) \\
&= V(\frac{1}{n}\sum_{j}^{n} Y_{ij} - \frac{1}{na}\sum_{k}^{a}\sum_{j}^{n} Y_{kj}) \\
&= V(\frac{1}{n}\sum_{j}^{n}(\mu + \tau_i + \epsilon_{ij}) - \frac{1}{na}\sum_{k}^{a}\sum_{j}^{n}(\mu + \tau_k + \epsilon_{kj})) \\
&= V(\mu + \tau_i + \sum_{j}^{n} \epsilon_{ij} - \mu - \bar{\tau}_k - \sum_{k}^{a}\sum_{j}^{n} \epsilon_{kj})
\end{aligned}
$$

Note that μ, τ_i, $\bar{\tau}_i$ are parameters, and $Var(x + c) = Var(x)$ for any constant c.

$$
Var(\widehat{\tau_i}) = Var(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}}) = \frac{a-1}{an}\sigma_\epsilon^2
$$

b

With assumption that $\widehat{\tau_i}$ is normal, σ_ϵ is known.
Then $Z = \frac{(\widehat{\tau_i}) - (\tau_i)}{\sqrt{\frac{a-1}{an} \cdot \sigma_\epsilon^2}} \sim N(0, 1)$, $\bar{Y_{i.}} - \bar{Y_{..}}$ is an unbiased estimator for τ_i.

$$
\begin{aligned}
&P(Z_{\frac{\alpha}{2}} < Z_{obs} < Z_{\frac{\alpha}{2}}) = 1 - \alpha \\
&P(Z_{\frac{\alpha}{2}} < \frac{(\bar{Y_{i.}} - \bar{Y_{..}}) - (\tau_i)}{\sqrt{\frac{a-1}{an} \cdot \sigma_\epsilon^2}} < Z_{\frac{\alpha}{2}}) = 1 - \alpha \\
&P(Z_{\frac{\alpha}{2}}\sqrt{\frac{a-1}{an} \cdot \sigma_\epsilon^2} < \bar{Y_{i.}} - \bar{Y_{..}} - \tau_i < Z_{1-\frac{\alpha}{2}}\sqrt{\frac{a-1}{an} \cdot \sigma_\epsilon^2}) = 1 - \alpha \\
&P(Z_{\frac{\alpha}{2}}\sqrt{\frac{a-1}{an} \cdot \sigma_\epsilon^2} - (\bar{Y_{i.}} - \bar{Y_{..}}) < -\tau_i < Z_{1-\frac{\alpha}{2}}\sqrt{\frac{a-1}{an} \cdot \sigma_\epsilon^2} - (\bar{Y_{i.}} - \bar{Y_{..}})) = 1 - \alpha \\
&P((\bar{Y_{i.}} - \bar{Y_{..}}) - Z_{\frac{\alpha}{2}}\sqrt{\frac{a-1}{an} \cdot \sigma_\epsilon^2} > \tau_i > (\bar{Y_{i.}} - \bar{Y_{..}}) - Z_{1-\frac{\alpha}{2}}\sqrt{\frac{a-1}{an} \cdot \sigma_\epsilon^2}) = 1 - \alpha \\
&P((\bar{Y_{i.}} - \bar{Y_{..}}) + Z_{\frac{\alpha}{2}}\sqrt{\frac{a-1}{an} \cdot \sigma_\epsilon^2} < \tau_i < (\bar{Y_{i.}} - \bar{Y_{..}}) - Z_{\frac{\alpha}{2}}\sqrt{\frac{a-1}{an} \cdot \sigma_\epsilon^2}) = 1 - \alpha
\end{aligned}
$$

We are $(1 - \alpha)\%$ confident that the effect τ_i at $X = i$ is in the interval
$[(\bar{Y_{i.}} - \bar{Y_{..}}) + Z_{\frac{\alpha}{2}}\sqrt{\frac{a-1}{an} \cdot \sigma_\epsilon^2}, (\bar{Y_{i.}} - \bar{Y_{..}}) - Z_{\frac{\alpha}{2}}\sqrt{\frac{a-1}{an} \cdot \sigma_\epsilon^2}]$

c

$$\begin{aligned}
Var(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}}) &= V(\frac{1}{n}\sum_{j}^{n}\epsilon_{ij} - \frac{1}{na}\sum_{k}^{a}\sum_{j}^{n}\epsilon_{kj}) \\
&= V(\frac{1}{n}\sum_{j}^{n}\epsilon_{ij} - \frac{1}{na}(\sum_{k\neq i}^{a}\sum_{j}^{n}\epsilon_{kj} + \sum_{j}^{n}\epsilon_{ij})) \\
&= V(\sum_{j}^{n}\epsilon_{ij}(\frac{1}{n} - \frac{1}{na}) - \frac{1}{na}\sum_{k\neq i}\sum_{j}^{n}\epsilon_{kj}) \\
&= V(\sum_{j}^{n}\epsilon_{ij}(\frac{1}{n} - \frac{1}{na})) + V(\frac{1}{na}\sum_{k\neq i}^{a}\sum_{j}^{n}\epsilon_{kj}) - 2Cov(\sum_{j}^{n}\epsilon_{ij}(\frac{1}{n} + \frac{1}{na}), \frac{1}{na}\sum_{k\neq i}\sum_{j}^{n}\epsilon_{kj})
\end{aligned}$$

Since ϵ is independent, and $\sum_{j}^{n}\epsilon_{ij}$ are disjoint with $\sum_{k\neq i}^{a}\sum_{j}^{n}\epsilon_{kj}$, $Cov(\sum_{j}^{n}\epsilon_{ij}(\frac{1}{n}+\frac{1}{na}), \frac{1}{na}\sum_{k\neq i}\sum_{j}^{n}\epsilon_{kj})$ is equal to zero.

$$\begin{aligned}
Var(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}}) &= V(\sum_{j}^{n}\epsilon_{ij}(\frac{1}{n} - \frac{1}{na})) + V(\frac{1}{na}\sum_{k\neq i}^{a}\sum_{j}^{n}\epsilon_{kj}) \\
&= (\frac{a-1}{na})^2\sum_{j}^{n}V(\epsilon_{ij}) + \frac{1}{(na)^2}\sum_{k\neq i}^{a}\sum_{j}^{n}V(\epsilon_{kj})
\end{aligned}$$

Note that ϵ is iid and follows normal distribution with variance of σ_ϵ. $Var(\epsilon_{ij}) = Var(\epsilon_{kj}) = Var(\epsilon) = \sigma_\epsilon^2$

$$\begin{aligned}
Var(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}}) &= \frac{(a-1)^2}{n^2a^2}\sigma^2\sum_{j}^{n}1 + \frac{1}{n^2a^2}\sigma\sum_{k\neq i}^{2}\sum_{j}^{n}1 \\
&= (\frac{(a-1)^2}{na^2} + \frac{(a-1)}{na^2})\sigma^2 \\
&= \frac{a-1}{na}\sigma^2
\end{aligned}$$

Lect 7-3

a

$$
\begin{aligned}
E(MS_{Tr}) &= E(\frac{n}{a-1}\sum_i^a(\bar{Y_{i.}} - \bar{Y_{..}})^2) \\
&= \frac{n}{a-1}E(\sum_i^a(\frac{1}{n}\sum_j^n Y_{ij} - \frac{1}{na}\sum_i^a\sum_j^n Y_{ij})^2) \\
&= \frac{n}{a-1}E(\sum_i^a(\frac{1}{n}\sum_j^n(\mu+\tau_i+\epsilon_{ij}) - \frac{1}{na}\sum_i^a\sum_j^n \mu+\tau_i+\epsilon_{ij})^2) \\
&= \frac{n}{a-1}E[\sum_i^a(\mu+\tau_i+(\frac{1}{n}\sum_j^n\epsilon_{ij}) - \mu - \frac{1}{n}\sum_j^n(\frac{1}{a}\sum_i^a\tau_i + \frac{1}{a}\sum_i^a\epsilon_{ij}))^2] \\
&= \frac{n}{a-1}E[\sum_i^a(\mu+\tau_i+\bar{\epsilon_{i.}} - \mu - \bar{\tau}_i - \frac{1}{an}\sum_j^n\sum_i^a\epsilon_{ij})^2] \\
&= \frac{n}{a-1}E[\sum_i^a(\mu+\tau_i+\bar{\epsilon_{i.}} - \mu - \bar{\tau_.} - \bar{\epsilon_{..}})^2] \\
&= \frac{n}{a-1}E[\sum_i^a(\tau_i+\bar{\epsilon_{i.}} - \bar{\tau_.} - \bar{\epsilon_{..}})^2]
\end{aligned}
$$

b

$$
\begin{aligned}
E(\sum_i^a(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}})^2) &= \sum_i^a E((\bar{\epsilon_{i.}} - \bar{\epsilon_{..}})^2) \\
&= \sum_i^a(Var(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}}) + E(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}})^2)
\end{aligned}
$$

We have proved $Var(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}}) = \frac{a-1}{na}\sigma_\epsilon^2$.

$E(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}}) = E(\bar{\epsilon_{i.}}) - E(\bar{\epsilon_{..}}) = \frac{1}{n}\sum_j^n E(\epsilon_{ij})$, since ϵ is iid, $E(\epsilon_{ij}) = E(\epsilon) = 0$, $E(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}}) = 0$

$$
E(\sum_i^a(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}})^2) = \sum_i^a \frac{a-1}{na}\sigma_\epsilon^2 = \frac{a-1}{n}\sigma_\epsilon^2
$$

c

$$
\begin{aligned}
E(\sum_i^a(\tau_i - \bar{\tau_.})^2) &= \sum_i^a E(\tau_i - \bar{\tau_.})^2) \\
&= \sum_i^a(Var(\tau_i - \bar{\tau_.}) + E(\tau_i - \bar{\tau_.})^2)
\end{aligned}
$$

Note that τ_i's are parameters, $t\bar{a}u_.$ is constant, therefore, $Var(\tau_i - \bar{\tau_.}) = 0$. $E(\tau_i) = \tau_i$, $E(\bar{\tau_.}) = \bar{\tau_.}$

$$E(\sum_i^a (\tau_i - \bar{\tau_.})^2) = \sum_i^a (0 + (\tau_i - \bar{\tau_.})^2) = \sum_i^a (\tau_i - \bar{\tau_.})^2$$

d

$$\begin{aligned} 2E[\sum_i^a (\tau_i - \bar{\tau_.})(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}})] &= 2\sum_i^a E[(\tau_i - \bar{\tau_.})(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}})] \\ &= 2\sum_i^a (\tau_i - \bar{\tau_.})E(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}}) \end{aligned}$$

τ_i and $\bar{\tau_.}$ are constant numbers, therefore, $E(\tau_i - \bar{\tau_.}) = \tau_i - \bar{\tau_.}$. We have proved that $E(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}}) = 0$

$$2E[\sum_i^a (\tau_i - \bar{\tau_.})(\bar{\epsilon_{i.}} - \bar{\epsilon_{..}})] = 2\sum_i^a (\tau_i - \bar{\tau_.}) \cdot 0 = 0$$

Lect 7-4

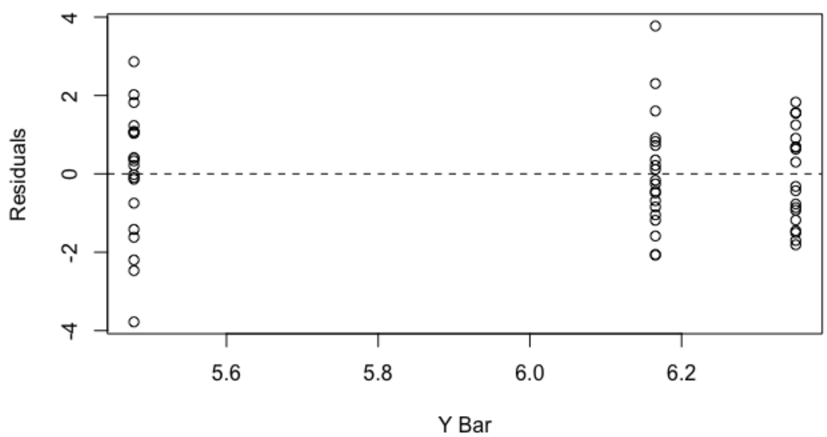
a

```
y_m <- matrix(nrow=3, ncol=20)
y_m[1,] <- c(5.32, 6, 5.12, 7.08, 5.48, 6.52, 4.09, 6.28, 7.77,
           5.68, 8.47, 4.58, 4.11, 5.72, 5.91, 6.89, 6.99, 4.98, 9.94, 6.38)
y_m[2,] <- c(4.73, 5.81, 5.69, 3.86, 4.06, 6.56, 8.34, 3.01, 6.71,
           6.51, 1.70, 5.89, 6.55, 5.34, 5.88, 7.50, 3.28, 5.38, 7.30, 5.46)
y_m[3,] <- c(7.03, 4.65, 6.65, 5.49, 6.98, 4.85, 7.26, 5.92, 5.58,
            7.91, 4.90, 4.54, 8.18, 5.42, 6.03, 7.04, 5.17, 7.60, 7.90, 7.91)
A <- as.factor(rep(c(1, 2, 4), each=20))
y <- c(y_m[1,] , y_m[2,], y_m[3,]  )
lm_1 <- lm(y~A)
summary.aov(lm_1)
```

Under the null hypothesis, observed F-statistic has a p-value of 0.143, which is greater than 0.05. This ANOVA F-test fail to provide significant evidence against H_0, we may conclude that X does not have significant effect on Y.

b

```
y_bars <- apply(y_m, 1, mean)
plot(x=rep(y_bars, each=20), y=lm_1$residuals, xlab='Y Bar', ylab='Residuals', pch=1)
abline(h=0, lty=2)
> y_bars
[1] 6.1655 5.4780 6.3505
```



In all these three levels, the residuals distribute randomly across the x-axis, therefore, the equivalent variance assumption is not violated.

c

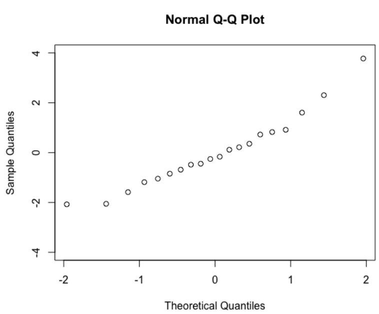


Figure 1: 1 hour

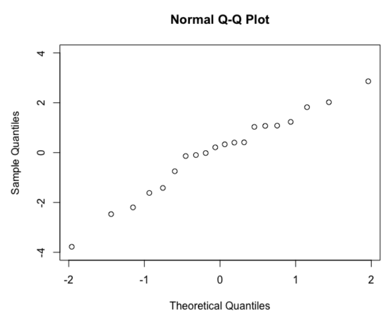


Figure 2: 2 hours

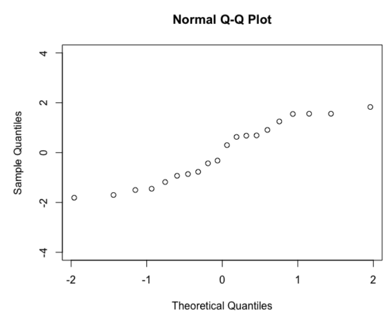


Figure 3: 4 hours

```
dt_1 <- data.frame(A, lm_1$residuals)
qqnorm(dt_1[dt_1$A==1,]$lm_1.residuals, ylim=c(-4, 4))
qqnorm(dt_1[dt_1$A==2,]$lm_1.residuals, ylim=c(-4, 4))
qqnorm(dt_1[dt_1$A==4,]$lm_1.residuals, ylim=c(-4, 4))
```

We may perceive that the slope of third qqplot is slightly different from other two, but in general, their have similar slopes, therefore, equivalent variance assumption is not violated.

d

Given the assumption that $Y_{ij} \sim N(\mu_i, \sigma_\epsilon)$, by CLT, $\bar{Y_{i.}} \sim N(\mu_i, \frac{\sigma}{\sqrt{n}})$.
Since σ_ϵ^2 is unknown, we use $MSE = \frac{\sum_i^a s_i^2}{a}$ as an estimator of σ_ϵ^2.

Therefore, $\frac{\bar{Y_{i.}}-\mu_i}{\sigma/\sqrt{n}} \sim t_{N-a}$
CI for μ_i is $\bar{Y_{i.}} \pm t_{\alpha/2,N-a} \cdot \sqrt{MSE/n}$

```
alpha <- 0.05
n <- 20
y1_mean <- mean(y_m[1,])
y_sds <- apply(y_m, 1, sd)
MSE <- sum(y_sds^2) / 3
cv_low <- y1_mean + sqrt(MSE / n) * qt(alpha/2, df=(60-3))
cv_high <- y1_mean - sqrt(MSE / n) * qt(alpha/2, df=(60-3))

> cv_high
[1] 6.814768
> cv_low
[1] 5.516232
```

We are 95% confident μ_1 is in the interval $[5.516232, 6.814768]$.

e

Before applying t-test on $\mu_2 - \mu_3$, we need to check if their population variance is the same. A two variance F-test is applied here.

```
y2_sd <- sd(y_m[2,])
y3_sd <- sd(y_m[3,])
F_obs <- y2_sd^2 / y3_sd^2
p_val <- pf(1 - (F_obs-1), df1=n-1, df2=n-1) +
  (1 - pf(F_obs, df1=n-1, df2=n-1))

> p_val
[1] 0.1091745
```

The p-value of F-statistic is 0.109, which is not a strong evidence against equivalent variance assumption. Therefore, t-test can be done on these two population means.

CI for $\mu_i - \mu_j$ is $(\bar{Y_{i.}} - \bar{Y_{j.}}) \pm t_{\alpha/2,N-a} \cdot \sqrt{2MSE/n}$

```
y2_mean <- mean(y_m[2,])
y3_mean <- mean(y_m[3,])
cv_low <- y2_mean - y3_mean + sqrt(2 * MSE / n) * qt(alpha/2, df=(60-3))
cv_high <- y2_mean - y3_mean - sqrt(2 * MSE / n) * qt(alpha/2, df=(60-3))

> cv_low
[1] -1.790704
> cv_high
[1] 0.04570407
```

We are 95% confident that $\mu_2 - \mu_3$ is in the interval $[-1.790704, 0.04570407]$.

f

$\Gamma = \mu_1 - \frac{1}{2}\mu_2 - \frac{1}{2}\mu_3$, $\vec{c} = (1, -\frac{1}{2}, -\frac{1}{2})$

$H_0 : \Gamma = 0$, $H_1 : \Gamma \neq 0$, assume $\sigma_1 = \sigma_2 = \sigma_3$

We use $\widehat{\Gamma} = \sum_i^a c_i \bar{Y}_i$ as an unbiased estimator for Γ, $MSE = \frac{1}{a}\sum_i^a s_i^2$ as an estimator for σ_ϵ^2.

$t = \frac{\widehat{\Gamma}-\Gamma_0}{\sqrt{Var(\widehat{\Gamma})}} = \frac{\widehat{\Gamma}-\Gamma_0}{\sqrt{MSE \cdot \sum_i^a c_i^2/n}} \sim t_{N-a}$

We will reject H_0 in favor of $\Gamma \neq 0$ if t-statistic $t > t_{1-\frac{\alpha}{2},N-a}$ or $t < t_{\frac{\alpha}{2},N-a}$

```
c <- c(1, -1/2, -1/2)
gamma_est <- sum(y_bars * c)
MSE <- sum(y_sds^2) / 3
t <- gamma_est / (sqrt(MSE * sum(c^2) / n))
low_cv <- qt(alpha/2, df=n*3 - 3)
up_cv <- qt(1-alpha/2, df=n*3 - 3)

> t
[1] 0.632705
> low_cv
[1] -2.002465
> up_cv
[1] 2.002465
```

The rejection regions are $t < -2.002465$ or $t > 2.002465$, the t-statistic from observed values (0.632705) does not fall in any of these regions, therefore, under significant level of 0.05, the contrast test fail to provide strong evidence against H_0.

g

$$P(t_{\frac{1}{2}\alpha,N-a} < t < t_{1-\frac{1}{2}\alpha,N-a}) = 1-\alpha$$
$$P(t_{\frac{1}{2}\alpha,N-a} < \frac{\widehat{\Gamma}-\Gamma_0}{\sqrt{MSE \cdot \sum_i^a c_i^2/n}} < t_{1-\frac{1}{2}\alpha,N-a}) = 1-\alpha$$
$$P(\widehat{\Gamma} + t_{\frac{1}{2}\alpha,N-a}\sqrt{MSE \cdot \sum_i^a c_i^2/n} < \Gamma < \widehat{\Gamma} - t_{\frac{1}{2}\alpha,N-a}\sqrt{MSE \cdot \sum_i^a c_i^2/n}) = 1-\alpha$$

```
ci_low <- gamma_est + (sqrt(MSE * sum(c^2) / n)) * qt(alpha/2, df=n*3 - 3)
ci_up <- gamma_est - (sqrt(MSE * sum(c^2) / n)) * qt(alpha/2, df=n*3 - 3)

> ci_low
[1] -0.5439381
> ci_up
[1] 1.046438
```

We are 95% confident that Γ is in the interval $[-0.5439381, 1.046438]$ as computed from observed data. Since this CI include the null hypothesis value $\Gamma = 0$, this constant test fail to show strong evidence against H_0.

Lect 8-1

a

$H_0 : \mu_1 = \mu_2 = \mu_3$, H_1 : there are at least two means are different.

```
a <- 3
n <- 5
ct_m <- matrix(nrow=a, ncol=n)
ct_m[1,] <- c(9 ,12, 10, 8, 15)
ct_m[2,] <- c(20, 21, 23, 17, 30)
ct_m[3,] <- c(6, 5, 8, 16, 7)
y_means <- apply(ct_m, 1, mean)
grand_mean <- mean(ct_m)
y_sds <- apply(ct_m, 1, sd)
SStr <- n * sum((y_means - grand_mean)^2)
SSe <- (n-1) * sum(y_sds^2)
F_obs <- (SStr / (a-1)) / (SSe / (n*a - a))
p_val <- pf(F_obs, df1=a-1, df2=a*n - a, lower.tail = F)

> SStr
[1] 543.6
> SSe
[1] 202.8
> F_obs
[1] 16.08284
> p_val
[1] 0.0004023258
```

Observed f-statistic is 16.08284, which has a p-value of 0.0004023258, much less than 0.001. This one way ANOVA F-test provides significant evidence against H_0 in favor of the alternative that there exist at least two means are different.

b

```
A <- as.factor(rep(c(1,2,3), each=n))
ct <- as.vector(t(ct_m))
lm_2 <- lm(ct~A)
aov(ct~A)
summary.aov(lm_2)
```

By $lm()$, SStreat = 543.6, SSe = 202.8, F-statistic = 16.08, and p-value = 0.000402. Conclusion is same as above.

c

Apply a F-test on $H_0 : \sigma_1 = \sigma_3$, $H_1 : \sigma_1 \neq \sigma_3$

```
y1_sd <- sd(ct_m[1,])
y3_sd <- sd(ct_m[3,])
F_obs <- y1_sd^2 / y3_sd^2
p_val <- (1 - pf(1 + (1 - F_obs), df1=n-1, df2=n-1)) +
  pf(F_obs, df1=n-1, df2=n-1)

> p_val
[1] 0.5273811
```

The p-value is 0.5273811, which is much greater than 0.05, indicating that sample data fail to provide strong evidence against equivalent variances assumption. A two-sample t-test can be applied here.

CI for $\mu_1 - \mu_3$ is $(\bar{Y_{1.}} - \bar{Y_{3.}}) \pm t_{\alpha/2,N-a} \cdot \sqrt{2MSE/n}$

```
MSE <- sum(y_sds^2) / a
y1_bar <- mean(ct_m[1,])
y3_bar <- mean(ct_m[3,])
ci_low <- (y1_bar - y3_bar) + qt(alpha/2, df=a*(n-1)) * sqrt(2 * MSE / n)
ci_up <- (y1_bar - y3_bar) - qt(alpha/2, df=a*(n-1)) * sqrt(2 * MSE / n)

> ci_low
[1] -3.264913
> ci_up
[1] 8.064913
```

We are 95% confident that $\mu_1 - \mu_3$ is in the interval [-3.264913, 8.064913]. Since this interval covers 0, the observed data fail to provide evidence against H_0.

d

$H_0 : \mu_1 - \mu_3 = 0$, $H_1 : \mu_1 - \mu_3 \neq 0$

under the null hypothesis, $t_obs = \frac{\bar{Y_{1.}} - \bar{Y_{3.}}}{\sqrt{2 \cdot MSE/n}} \sim t_{N-a}$

```
t_obs <- (y1_bar - y3_bar) / sqrt(2 * MSE / n)
p_val <- 2 * (1 - pt(t_obs, df=n*a - a))

> p_val
[1] 0.374155
```

Under the null hypothesis, p-value of observed t is 0.374155, greater than 0.05. Therefore, the sample fail to provide strong evidence against equivalent means.

e

In part b, $H_0: \mu_1 - \mu_3$, $\vec{c} = (1, 0, -1)$, another orthogonal vector is $\vec{d} = (1, -2, 1)$

```
u <- c(-1, 0, 1, 0)
v <- c(-1, 2, -1, 0)
w <- c(-1, -1, -1, 3)
u_ssc <- (sum(u * y_bars))^2 / (sum(u^2) / n)
v_ssc <- (sum(v * y_bars))^2 / (sum(v^2) / n)
w_ssc <- (sum(w * y_bars))^2 / (sum(w^2) / n)
sum(c(u_ssc, v_ssc, w_ssc))

> c(c_ssc, d_ssc)
[1] 14.4 529.2
> sum(c(c_ssc + d_ssc))
[1] 543.6
```

Sum of contrast sum of squares is equal to SS_{tr}.

f

```
> c(c_ssc, d_ssc)
[1] 14.4 529.2
```

In ANOVA F-test, larger SS_{tr} will lead to more statistical significance (smaller p-value). Since the second contrast vector contributes more to the SS_{tr}, we can say the second contrast vector is contributing more to significance.

g

$H_0: \Gamma = 0$, $H_1: \Gamma \neq 0$

$$t_obs = \frac{\widehat{\Gamma} - \Gamma_0}{\sqrt{MSE \cdot |\vec{c}|/n}} \sim t_{N-a}$$

$$\widehat{\Gamma} = \sum_{i}^{a} c_i \cdot \bar{Y_{i.}}, \Gamma_0 = 0$$

For vector $\vec{c} = (1, 0, -1)$

```
c <- c(1, 0, -1)
MSE <- sum(y_sds^2) / a
c_gamma <- sum(c * y_means)
t_obs <- c_gamma / sqrt(MSE * sum(c^2) / n)
p_val <- 2 * (1 - pt(t_obs, df=n * (a-1)))

> p_val
[1] 0.3777015
```

$\widehat{\Gamma} = 2.4$, with p-value of 0.378, which is greater than 0.05. This contrast test fail to provide significant evidence against $H_0 : \Gamma = 0$.

For vector $\vec{d} = (1, -2, 1)$

```
d <- c(1, -2, 1)
d_gamma <- sum(d * y_means)
t_obs <- d_gamma / sqrt(MSE * sum(d^2) / n)
p_val <- 2 * pt(t_obs, df=n * (a-1))

> p_val
[1] 0.0002290025
```

By this contrast, $\widehat{\Gamma} = -25.2$, with p-value of 0.00023, which is much less than 0.05. This contrast test provides significant evidence against H_0, in favor of $\Gamma \neq 0$.

Lect 8-2

c_i's, μ_i's are constant, assume ϵ iid, $\epsilon \sim N(0,1)$, $E(\epsilon_{ij}) = E(\epsilon) = 0$, $Var(\epsilon_{ij}) = Var(\epsilon) = \sigma_\epsilon^2$

$$
\begin{aligned}
E(SS_c) &= E(\frac{(\sum_i^a c_i \bar{Y_{i.}})^2}{|\vec{c}|^2/n}) \\
&= \frac{1}{|\vec{c}|^2/n} E((\sum_i^a c_i \bar{Y_{i.}})^2) \\
&= \frac{1}{|\vec{c}|^2/n} (Var(\sum_i^a c_i \bar{Y_{i.}}) + E(\sum_i^a c_i \bar{Y_{i.}})^2)
\end{aligned}
$$

$$
\begin{aligned}
Var(\sum_i^a c_i \bar{Y_{i.}}) &= \sum_i^a c_i^2 \cdot Var(\bar{Y_{i.}}) \\
&= \sum_i^a c_i^2 \cdot Var(\frac{1}{n} \sum_j^n \mu_i + \epsilon_{ij}) \\
&= \sum_i^a c_i^2 \cdot Var(\frac{1}{n} \sum_j^n \epsilon_{ij}) \\
&= \frac{1}{n^2} \sum_i^a c_i^2 \cdot \sum_j^n Var(\epsilon_{ij}) \\
&= \frac{1}{n^2} \sum_i^a c_i^2 \cdot \sigma_\epsilon^2 \sum_j^n 1 \\
&= \frac{1}{n} |\vec{c}|^2 \cdot \sigma_\epsilon^2
\end{aligned}
$$

$$
\begin{aligned}
E(\sum_i^a c_i \bar{Y_{i.}}) &= \sum_i^a c_i E(\bar{Y_{i.}}) \\
&= \sum_i^a c_i E(\frac{1}{n}\sum_j^n \mu_i + \epsilon_{ij}) \\
&= \sum_i^a c_i(\mu_i + \frac{1}{n}\sum_j^n E(\epsilon_i j)) \\
&= \sum_i^a c_i(\mu_i) = \Gamma
\end{aligned}
$$

$$
\begin{aligned}
E(SS_c) &= \frac{1}{|\vec{c}|^2/n}(Var(\sum_i^a c_i \bar{Y_{i.}}) + E(\sum_i^a c_i \bar{Y_{i.}})^2) \\
&= \frac{1}{|\vec{c}|^2/n} \cdot \frac{1}{n}|\vec{c}|^2 \cdot \sigma_\epsilon^2 + \frac{1}{|\vec{c}|^2/n} \cdot \Gamma^2 \\
&= \sigma_\epsilon^2 + \frac{\Gamma^2}{|\vec{c}|^2/n}
\end{aligned}
$$

Lect 8-3

a

```
a = 4
n = 4
y_m <- matrix(nrow=a, ncol=n)
y_m[1,] = c(143, 141, 150, 146)
y_m[2,] = c(152, 149, 137, 143)
y_m[3,] = c(134, 136, 132, 127)
y_m[4,] = c(129, 127, 132, 129)
c <- c(1, 1, -1, -1)
d <- c(1, -1, 0, 0)
e <- c(0, 0, 1, -1)
y_bars <- apply(y_m, 1, mean)
c_ssc <- (sum(c * y_bars))^2 / (sum(c^2) / n)
d_ssc <- (sum(d * y_bars))^2 / (sum(d^2) / n)
e_ssc <- (sum(e * y_bars))^2 / (sum(e^2) / n)

> c(c_ssc, d_ssc, e_ssc)
[1] 826.5625   0.1250  18.0000
> sum(c(c_ssc, d_ssc, e_ssc))
[1] 844.6875
```

Sum of three contrast sum of squares is equal to SS_{tr}

b

choose $\vec{f} = (1, -1, 1, -1)$, $\vec{g} = (0, 1, 0, -1)$, $\vec{h} = (1, 0, -1, 0)$

```
f <- c(1, -1, 1, -1)
g <- c(0, 1, 0, -1)
h <- c(1, 0, -1, 0)
f_ssc <- (sum(f * y_bars))^2 / (sum(f^2) / n)
g_ssc <- (sum(g * y_bars))^2 / (sum(g^2) / n)
h_ssc <- (sum(h * y_bars))^2 / (sum(h^2) / n)

> c(f_ssc, g_ssc, h_ssc)
[1] 7.5625 512.0000 325.1250
> sum(c(c_ssc, g_ssc, h_ssc))
[1] 844.6875
```

Sum of three contrast sum of squares is equal to SS_{tr}

c

choose $\vec{u} = (-1, 0, 1, 0)$, $\vec{v} = c(-1, 2, -1, 0)$, $\vec{w} = c(-1, -1, -1, 3)$

```
u <- c(-1, 0, 1, 0)
v <- c(-1, 2, -1, 0)
w <- c(-1, -1, -1, 3)
u_ssc <- (sum(u * y_bars))^2 / (sum(u^2) / n)
v_ssc <- (sum(v * y_bars))^2 / (sum(v^2) / n)
w_ssc <- (sum(w * y_bars))^2 / (sum(w^2) / n)

> c(u_ssc, v_ssc, w_ssc)
[1] 325.1250 117.0417 402.5208
> sum(c(u_ssc, v_ssc, w_ssc))
[1] 844.6875
```

Sum of three contrast sum of squares is equal to SS_{tr}