Lect 9-1

Given the model $Y_{ij} = \mu_i \epsilon_{ij}$ where $\epsilon \sim N(0, \sigma_\epsilon)$. Then $Y_{ij} \sim N(\mu_i, \sigma_\epsilon)$

The likelihood function of data Y_{ij}'s is written as

$$\begin{aligned}
L(\mu_i, \sigma_\epsilon | Y) &= \prod_i^a \prod_j^n \text{pdf of Y} \\
&= \prod_i^a \prod_j^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{Y_{ij}-\mu_i}{\sigma_\epsilon})^2} \\
&= e^{-\frac{1}{2}\sum_i^a \sum_j^n (\frac{Y_{ij}-\mu_i}{\sigma})^2 - \frac{1}{2} log(2\pi\sigma^2)}
\end{aligned}$$

To find the maximum point of likelihood function, find minimum value of exponent $-\frac{1}{2}\sum_i^a \sum_j^n (\frac{Y_{ij}-\mu_i}{\sigma})^2 - \frac{1}{2} log(2\pi\sigma^2)$

$$\begin{aligned}
\frac{\partial exponent}{\partial \mu_i} &= \partial - \frac{1}{2}\sum_i^a \sum_j^n (\frac{Y_{ij}-\mu_i}{\sigma})^2 - \frac{1}{2} log(2\pi\sigma^2)/\partial\mu_i \\
&= -\frac{1}{2}\sum_i^a \sum_j^n \partial(\frac{Y_{ij}-\mu_i}{\sigma})^2/\partial\mu_i
\end{aligned}$$

Note that μ_k only counts for kth level, therefore,

$$\begin{aligned}
\frac{1}{2}\sum_i^a \sum_j^n \partial(\frac{Y_{ij}-\mu_i}{\sigma})^2/\partial\mu_k &= -\frac{1}{2}\sum_j^n \partial(\frac{Y_{kj}-\mu_k}{\sigma})^2/\partial\mu_k \\
&= \frac{1}{\sigma^2}\sum_j^n (Y_{kj} - \mu_k)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial exponent}{\partial \sigma_\epsilon} &= \partial - \frac{1}{2}\sum_i^a \sum_j^n (\frac{Y_{ij}-\mu_i}{\sigma})^2 - \frac{1}{2} log(2\pi\sigma^2)/\partial\sigma_\epsilon \\
&= \frac{1}{\sigma^3}\sum_i^a \sum_j^n (Y_{ij} - \mu_i)^2 - \frac{N}{\sigma}
\end{aligned}$$

Let $\frac{\partial L}{\partial \mu_i} = 0$ and $\frac{\partial L}{\partial \sigma_\epsilon} = 0$

$$\begin{aligned}
\frac{1}{\sigma^2}\sum_j^n (Y_{kj} - \mu_k) &= 0 \\
\sum_j^n (Y_{kj} - \mu_k) &= 0 \\
Y_{i.} - n\hat{\mu}_i &= 0 \\
\hat{\mu}_i &= \bar{Y_{i.}}
\end{aligned}$$

$$\frac{1}{\sigma^3}\sum_i^a\sum_j^n(Y_{ij}-\mu_i)^2-\frac{N}{\sigma}=0$$

$$\sigma_\epsilon^2=\frac{1}{N}\sum_i^a\sum_j^n(Y_{ij}-\hat{\mu_i})^2$$

Plug in the estimator of μ_i as $\bar{Y_{i.}}$

$$\begin{aligned}\hat{\sigma_\epsilon}&=\frac{1}{N}\sum_i^a\sum_j^n(Y_{ij}-\bar{Y_{i.}})^2\\&=\frac{SSE}{N}\approx\frac{SSE}{N-a}=MSE\end{aligned}$$

The most likelihood estimator $\hat{\mu_i}=\bar{Y_{i.}}$, $\hat{\sigma_\epsilon}=\frac{SSE}{N}$

Lect 9-2

a

From the partial derivative on μ and τ_i, we get equations

$$\begin{aligned}Y_{..}-N\hat{\mu}-n\hat{\tau_.}&=0\\Y_{i.}-n\hat{\mu}-n\hat{\tau_i}&=0\end{aligned}$$

Plug in the constraint $\hat{\tau_.}=0$ to equation one

$$\begin{aligned}Y_{..}-N\hat{\mu}&=0\\\hat{\mu}&=\bar{Y_{..}}\end{aligned}$$

Plug in the estimator for μ to equations $Y_{i.}-n\hat{\mu}-n\hat{\tau_i}=0$

$$\begin{aligned}Y_{i.}-n\bar{Y_{..}}-n\hat{\tau_i}&=0\\\hat{\tau_i}&=\bar{Y_{i.}}-\bar{Y_{..}}\end{aligned}$$

$$\hat{Y_{ij}}=\hat{\mu}+\hat{\tau_i}=\bar{Y_{..}}+\bar{Y_{i.}}-\bar{Y_{..}}=\bar{Y_{i.}}$$

b

Use any constraint $\hat{\tau_k}=c$ where c is a constant, $k\in 1,2,...a$. Plug in this constraint to equation $Y_{k.}-n\hat{\mu}-n\hat{\tau_k}=0$

$$\begin{aligned}Y_{k.}-n\hat{\mu}&=c\\\hat{\mu}&=\bar{Y_{k.}}-\frac{c}{n}\end{aligned}$$

Plug in the constraint $\hat{\mu} = \bar{Y_{k.}} - \frac{c}{n}$ in to equation $Y_{i.} - n\hat{\mu} - n\hat{\tau_i} = 0$ for $i \neq k$

$$Y_{i.} - n\bar{Y_{k.}} + c - n\hat{\tau_i} = 0$$
$$\tau_i = \bar{Y_{i.}} - \bar{Y_{k.}} + \frac{c}{n}$$

$$Y_{ij} = \hat{\mu} + \hat{\tau_i} = \bar{Y_{k.}} - \frac{c}{n} + \bar{Y_{i.}} - \bar{Y_{k.}} + \frac{c}{n} = \bar{Y_{i.}}$$

Therefore, in general, constraint doesn't change prediction Y_{ij}.

Lect 9-3

a

From the partial derivative on μ and τ_i, we get equations

$$Y_{..} - N\hat{\mu} - n\hat{\tau_.} = 0$$
$$Y_{i.} - n\hat{\mu} - n\hat{\tau_i} = 0$$

Plug in the constraint $\hat{\tau_.} = 0$ to equation one

$$Y_{..} - N\hat{\mu} = 0$$
$$\hat{\mu} = \bar{Y_{..}}$$

Plug in the estimator for μ to equations $Y_{i.} - n\hat{\mu} - n\hat{\tau_i} = 0$

$$Y_{i.} - n\bar{Y_{..}} - n\hat{\tau_i} = 0$$
$$\hat{\tau_i} = \bar{Y_{i.}} - \bar{Y_{..}}$$

b

From part a, we have known that estimator for τ_i is $\bar{Y_{i.}} - \bar{Y_{..}}$, then $\hat{\tau_1} = \bar{Y_{1.}} - \bar{Y_{..}}$, $\hat{\tau_2} = \bar{Y_{2.}} - \bar{Y_{..}}$, $\hat{\tau_3} = \bar{Y_{3.}} - \bar{Y_{..}}$

$$\hat{\tau_1} - \hat{\tau_2} = \bar{Y_{1.}} - \bar{Y_{..}} - \bar{Y_{2.}} + \bar{Y_{..}} = \bar{Y_{1.}} - \bar{Y_{2.}}$$
$$\hat{\tau_1} - \hat{\tau_3} = \bar{Y_{1.}} - \bar{Y_{..}} - \bar{Y_{3.}} + \bar{Y_{..}} = \bar{Y_{1.}} - \bar{Y_{3.}}$$
$$\hat{\tau_2} - \hat{\tau_3} = \bar{Y_{2.}} - \bar{Y_{..}} - \bar{Y_{3.}} + \bar{Y_{..}} = \bar{Y_{2.}} - \bar{Y_{3.}}$$

c

From the partial derivative on μ and τ_i, we get equations

$$Y_{..} - N\hat{\mu} - n\hat{\tau_.} = 0$$
$$Y_{i.} - n\hat{\mu} - n\hat{\tau_i} = 0, \text{ i = 1,2,3 ... a}$$

Plug in the constraint $\hat{\tau_3} = 0$ to equation $Y_{3.} - n\hat{\mu} - n\hat{\tau_3} = 0$

$$Y_{3.} - n\hat{\mu} = 0$$
$$\hat{\mu} = \frac{Y_{3.}}{n} = \bar{Y_{3.}}$$

Plug in the constraint $\hat{\mu} = \bar{Y_{3.}}$ to equations $Y_{i.} - n\hat{\mu} - n\hat{\tau_i} = 0$ for $i = 1, 2, 4...a$

$$Y_{i.} - n\bar{Y_{3.}} - n\hat{\tau_i} = 0$$
$$\tau_i = \bar{Y_{i.}} - \bar{Y_{3.}}$$

d

From part c, we have known that estimator for τ_i is $\bar{Y_{i.}} - \bar{Y_{3.}}$, then $\hat{\tau_1} = \bar{Y_{1.}} - \bar{Y_{3.}}$, $\hat{\tau_2} = \bar{Y_{2.}} - \bar{Y_{3.}}$, $\hat{\tau_3} = 0$

$$\begin{aligned}
&\hat{\tau_1} - \hat{\tau_2} = \bar{Y_{1.}} - \bar{Y_{3.}} - \bar{Y_{2.}} + \bar{Y_{3.}} = \bar{Y_{1.}} - \bar{Y_{2.}} \\
&\hat{\tau_1} - \hat{\tau_3} = \bar{Y_{1.}} - \bar{Y_{3.}} \\
&\hat{\tau_2} - \hat{\tau_3} = \bar{Y_{2.}} - \bar{Y_{3.}}
\end{aligned}$$

e

estimator for $\mu + \tau_1$ with constraint in part a is

$$\mu + \tau_1 = \bar{Y_{..}} + \bar{Y_{1.}} - \bar{Y_{..}} = \bar{Y_{1.}}$$

estimator for $\mu + \tau_1$ with constraint in part c is

$$\mu + \tau_1 = \bar{Y_{3.}} + \bar{Y_{1.}} - \bar{Y_{3.}} = \bar{Y_{1.}}$$

estimator for $2\tau_1 - \tau_2 - \tau_3$ with constraint in part a is

$$2\tau_1 - \tau_2 - \tau_3 = 2\bar{Y_{1.}} - 2\bar{Y_{..}} - \bar{Y_{2.}} + \bar{Y_{..}} - \bar{Y_{3.}} + \bar{Y_{..}} = 2\bar{Y_{1}}. - \bar{Y_{2.}} - \bar{Y_{3.}}$$

estimator for $2\tau_1 - \tau_2 - \tau_3$ with constraint in part c is

$$2\tau_1 - \tau_2 - \tau_3 = 2\bar{Y_{1.}} - 2\bar{Y_{3.}} - \bar{Y_{2.}} + \bar{Y_{3.}} = 2\bar{Y_{1}}. - \bar{Y_{2.}} - \bar{Y_{3.}}$$

estimator for $\mu + \tau_1 + \tau_2$ with constraint in part a is

$$\mu + \tau_1 + \tau_2 = \bar{Y_{..}} + \bar{Y_{1.}} - \bar{Y_{..}} + \bar{Y_{2.}} - \bar{Y_{..}} = \bar{Y_{1.}} + \bar{Y_{2.}} - \bar{Y_{..}}$$

estimator for $\mu + \tau_1 + \tau_2$ with constraint in part c is

$$\mu + \tau_1 + \tau_2 = \bar{Y_{3.}} + \bar{Y_{1.}} - \bar{Y_{3.}} + \bar{Y_{2.}} - \bar{Y_{3.}} = \bar{Y_{1.}} + \bar{Y_{2.}} - \bar{Y_{3.}}$$

First and second contrasts do not depend on the constraint, implying first and second contrasts are estimable.

Lect 10-1

a

$Y_{ij} \sim N(\mu, \sigma_\epsilon)$

$$
\begin{aligned}
L(\mu, \sigma_\epsilon | Y) &= \prod_i^a \prod_j^n \text{pdf of Y} \\
&= \prod_i^a \prod_j^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{Y_{ij}-\mu}{\sigma_\epsilon})^2} \\
&= e^{-\frac{1}{2}\sum_i^a \sum_j^n (\frac{Y_{ij}-\mu}{\sigma})^2 - \frac{1}{2} log(2\pi\sigma^2)}
\end{aligned}
$$

To find the maximum point of likelihood function, find minimum value of exponent $-\frac{1}{2}\sum_i^a \sum_j^n (\frac{Y_{ij}-\mu}{\sigma})^2 - \frac{1}{2} log(2\pi\sigma^2)$

$$
\begin{aligned}
\frac{\partial exponent}{\partial \mu} &= \partial - \frac{1}{2}\sum_i^a \sum_j^n (\frac{Y_{ij}-\mu}{\sigma})^2 - \frac{1}{2} log(2\pi\sigma^2)/\partial \mu_i \\
&= -\frac{1}{2}\sum_i^a \sum_j^n \partial(\frac{Y_{ij}-\mu}{\sigma})^2/\partial \mu_i \\
&= \sum_i^a \sum_j^n \frac{Y_{ij}-\mu}{\sigma^2} \\
&= (\sum_i^a \sum_j^n Yij - N \cdot \mu)/\sigma^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial exponent}{\partial \sigma_\epsilon} &= \partial - \frac{1}{2}\sum_i^a \sum_j^n (\frac{Y_{ij}-\mu}{\sigma})^2 - \frac{1}{2} log(2\pi\sigma^2)/\partial \sigma_\epsilon \\
&= \frac{1}{\sigma^3}\sum_i^a \sum_j^n (Y_{ij}-\mu)^2 - \frac{N}{\sigma}
\end{aligned}
$$

Let $\frac{\partial L}{\partial \mu} = 0$ and $\frac{\partial L}{\partial \sigma_\epsilon} = 0$

$$
\begin{aligned}
(\sum_i^a \sum_j^n Yij - N \cdot \hat{\mu})/\sigma^2 = 0 \\
\sum_i^a \sum_j^n Yij - N \cdot \hat{\mu} = 0 \\
\hat{\mu} = Y_{..}/N = \bar{Y_{..}}
\end{aligned}
$$

Plug in the estimator of μ as $\bar{Y_{..}}$

$$\begin{aligned}
\hat{\sigma_\epsilon} &= \frac{1}{N}\sum_i^a\sum_j^n (Y_{ij} - \bar{Y_{..}})^2 \\
&= \frac{SSE}{N}
\end{aligned}$$

The most likelihood estimator $\hat{\mu_i} = \bar{Y_{..}}$, $\hat{\sigma_\epsilon} = \frac{SSE}{N}$

Using the estimator of μ and σ_ϵ, $SSE(\mu) = \sum_i^a \sum_j^n (Y_{ij} - \hat{\mu})^2$ or $N \cdot \hat{\sigma}^2$

b

Know that $\hat{\mu} = \bar{Y_{..}}$, $\hat{\tau_i} = \bar{Y_{i.}} - \bar{Y_{..}}$

$$\begin{aligned}
SSE_{\mu,\tau_i} &= \sum_i^a\sum_j^n (Y_{ij} - \hat{\mu} - \hat{\tau_i})^2 \\
&= \sum_i^a\sum_j^n (Y_{ij} - \bar{Y_{..}} - \bar{Y_{i.}} + \bar{Y_{..}})^2 \\
&= \sum_i^a\sum_j^n (Y_{ij} - \bar{Y_{i.}})^2 \\
&= \sum_i^a\sum_j^n (Y_{ij} - \hat{\mu})^2 \\
&= N \cdot \hat{\sigma}^2
\end{aligned}$$

c

$$
\begin{aligned}
SSE(\mu) - SSE(\mu,\tau) &= [\sum_i^a \sum_j^n (Y_{ij} - \bar{Y_{..}})^2] - [\sum_i^a \sum_j^n (Y_{ij} - Y_{i.})^2] \\
&= \sum_i^a \sum_j^n [(Y_{ij} - \bar{Y_{..}})^2 - (Y_{ij} - \bar{Y_{i.}})^2] \\
&= \sum_i^a \sum_j^n [(Y_{ij} - \bar{Y_{..}} + Y_{ij} - \bar{Y_{i.}})(Y_{ij} - \bar{Y_{..}} - Y_{ij} + \bar{Y_{i.}})] \\
&= \sum_i^a \sum_j^n [(2Y_{ij} - \bar{Y_{..}} - \bar{Y_{i.}})(\bar{Y_{i.}} - \bar{Y_{..}})] \\
&= \sum_i^a [\sum_j^n (2Y_{ij} - \bar{Y_{..}} - \bar{Y_{i.}})](\bar{Y_{i.}} - \bar{Y_{..}}) \\
&= \sum_i^a [(2Y_{i.} - n\bar{Y_{..}} - Y_{i.})](\bar{Y_{i.}} - \bar{Y_{..}}) \\
&= \sum_i^a n(\bar{Y_{i.}} - \bar{Y_{..}})(\bar{Y_{i.}} - \bar{Y_{..}}) \\
&= n\sum_i^a (\bar{Y_{i.}} - \bar{Y_{..}})^2 = SStr
\end{aligned}
$$

Then $F = \frac{MStr}{MSE} = \frac{SS_{treat}/(a-1)}{SSE(\mu,\tau_i)/(N-a)} = \frac{SSE(\mu)-SSE(\mu,\tau_i)/(a-1)}{SSE(\mu,\tau_i)/(N-a)}$

Lect 10-2

$$
\begin{aligned}
E(MS_{treat}) &= E(\frac{b\sum_i^a(\bar{Y_{i.}}-\bar{Y_{..}})^2}{a-1}) \\
&= \frac{b}{a-1}\sum_i^a E((\bar{Y_{i.}}-\bar{Y_{..}})^2) \\
&= \frac{b}{a-1}\sum_i^a E((\frac{\sum_j^b Y_{ij}}{b}-\frac{\sum_i^a\sum_j^b Y_{ij}}{ab})^2) \\
&= \frac{b}{a-1}\sum_i^a E((\frac{\sum_j^b \mu+\alpha_i+\beta_j+\epsilon_{ij}}{b}-\frac{\sum_i^a\sum_j^b \mu+\alpha_i+\beta_j+\epsilon_{ij}}{ab})^2) \\
&= \frac{b}{a-1}\sum_i^a E((\frac{\sum_j^b \mu+\alpha_i+\beta_j+\epsilon_{ij}}{b}-\frac{\sum_i^a\sum_j^b \mu+\alpha_i+\beta_j+\epsilon_{ij}}{ab})^2) \\
&= \frac{b}{a-1}\sum_i^a E((\frac{b\mu+b\alpha_i+\sum_j^b\beta_j+\epsilon_{ij}}{b}-\frac{ab\mu+\sum_i^a b\alpha_i+\sum_j^b a\beta_j+\sum_i^a\sum_j^b\epsilon_{ij}}{ab})^2) \\
&= \frac{b}{a-1}\sum_i^a E((\mu+\alpha_i+\bar{\beta}_.+\bar{\epsilon_{i.}}-\mu-\bar{\alpha}_.-\bar{\beta}_.-\bar{\epsilon_{..}})^2) \\
&= \frac{b}{a-1}\sum_i^a E((\alpha_i-\bar{\alpha}_.+\bar{\epsilon_{i.}}-\bar{\epsilon_{..}})^2) \\
&= \frac{b}{a-1}\sum_i^a E((\alpha_i-\bar{\alpha}_.)^2+2(\alpha_i-\bar{\alpha}_.)(\bar{\epsilon_{i.}}-\bar{\epsilon_{..}})+(\bar{\epsilon_{i.}}-\bar{\epsilon_{..}})^2) \\
&= \frac{b}{a-1}\sum_i^a [E((\alpha_i-\bar{\alpha}_.)^2)+E(2(\alpha_i-\bar{\alpha}_.)(\bar{\epsilon_{i.}}-\bar{\epsilon_{..}}))+E(\bar{\epsilon_{i.}}-\bar{\epsilon_{..}})^2)]
\end{aligned}
$$

Note that α_i's are parameter $E(2(\alpha_i-\bar{\alpha}_.)(\bar{\epsilon_{i.}}-\bar{\epsilon_{..}})) = 2(\alpha_i-\bar{\alpha}_.)E(\bar{\epsilon_{i.}}-\bar{\epsilon_{..}})$. Since $\epsilon_i j$ are iid's, $E(\bar{\epsilon_{i.}}-\bar{\epsilon_{..}})=0$, then $E(2(\alpha_i-\bar{\alpha}_.)(\bar{\epsilon_{i.}}-\bar{\epsilon_{..}}))=0$.

$$
\begin{aligned}
E(MS_{treat}) &= \frac{b}{a-1}\sum_i^a [E((\alpha_i-\bar{\alpha}_.)^2)-\bar{\epsilon_{..}}))+E(\bar{\epsilon_{i.}}-\bar{\epsilon_{..}})^2)] \\
&= \frac{b}{a-1}\sum_i^a [(\alpha_i-\bar{\alpha}_.)^2+\frac{1}{b}(1-\frac{1}{a}\sigma^2)] \\
&= \frac{b}{a-1}\sum_i^a (\alpha_i-\bar{\alpha}_.)^2+\sigma_\epsilon^2
\end{aligned}
$$

Lect 10-3

a

Row means and variances $\bar{Y_{1.}}=\frac{4+6}{2}=5$, $\bar{Y_{2.}}=\frac{10+2}{2}=6$, $s_{1.}^2=\frac{1}{2}(4-6)^2=2$, $s_{2.}^2=\frac{1}{2}(10-2)^2=32$

Col means and variances $\bar{Y_{.1}} = \frac{4+10}{2} = 7$, $\bar{Y_{.2}} = \frac{6+2}{2} = 4$, $s_{.1}^2 = \frac{1}{2}(4-10)^2 = 18$, $s_{.2}^2 = \frac{1}{2}(6-2)^2 = 8$

$\bar{Y}_{..} = \frac{4+6+10+2}{4} = 5.5$, $SST = \sum_i^2 \sum_j^2 (Y_{ij} - \bar{Y}_{..})^2 = 35$

b

For row quantities, variance of row means is $\frac{1}{2}((5-5.5)^2 + (6-5.5)^2) = 0.25$. Sum of row variance is $2 + 32 = 34$.

$SStr = 2\cdot$ variance of row means $= 0.5$, $SSE =$ sum of row variance $= 34$, $SST = n \cdot SStr + (n-1) \cdot SSE = 4\cdot$ variance of row means $+2\cdot$ sum of row variance $= 1 + 34 = 35 = SST$

c

For row quantities, variance of col means is $\frac{1}{2}((7-5.5)^2 + (4-5.5)^2) = 2.25$. Sum of col variance is $18 + 8 = 26$.

$SStr = 2\cdot$ variance of col means $= 4.5$, $SSE =$ sum of col variance $= 26$, $SST = n \cdot SStr + (n-1) \cdot SSE = 4\cdot$ variance of col means $+2\cdot$ sum of col variance $= 9 + 26 = 35 = SST$

Lect 10-4

```
b = 5
a = 4
data = matrix(0, ncol = b, nrow = a)
data[1,] = c(73, 68, 74, 71, 67)
data[2,] = c(73, 67, 75, 72, 70)
data[3,] = c(75, 68, 78, 73, 68)
data[4,] = c(73, 71, 75, 75, 69)
```

a

```
y <- as.vector(t(data))
A <- as.factor(rep(c(1:a), each=b))
B <- as.factor(rep(c(1:b), times=a))

SSE <- summary.aov(lm(y~A+B))[[1]]$`Sum Sq`[3]
F_block <- summary.aov(lm(y~A+B))[[1]]$`F value`[2]
F_treat <- summary.aov(lm(y~A+B))[[1]]$`F value`[1]


> SSE
[1] 21.8
> F_block
[1] 21.6055
> F_treat
[1] 2.376147
```

SSE is 21.8, F-value for block factor is 21.6, F-value for treatment factor is 2.376

b

```
lm1 <- summary.aov(lm(y~B))[[1]]
SSE1 <- lm1$`Sum Sq`[2]

> SSE1
[1] 34.75
```

Exclude treatment factor, the SSE of reduced model is 34.75

c

```
lm2 <- summary.aov(lm(y~A))[[1]]
SSE2 <- lm2$`Sum Sq`[2]

> SSE2
[1] 178.8
```

Exclude block factor, the SSE of reduced model is 178.8

d

$$F_{treat} = \frac{(SSE(\text{without treatment factor}) - SSE(\text{with treatment factor}))/(a-1)}{SSE(\text{with treatment factor})/(a-1)(b-1)}$$

```
> ((SSE1 - SSE)/(a-1)) / (SSE/((a-1)*(b-1)))
[1] 2.376147
```

This F ratio is equal to F ration of treatment factor in part a.

$$F_{block} = \frac{(SSE(\text{without block factor}) - SSE(\text{with block factor}))/(b-1)}{SSE(\text{with block factor})/(a-1)(b-1)}$$

```
> ((SSE2 - SSE)/(b-1)) / (SSE/((a-1)*(b-1)))
[1] 21.6055
```

This F ratio is equal to F ration of block factor in part a.

Lect 11-1

a

```
y <- as.vector(t(data))
grand_mean <- mean(y)
SSTr <- sum((apply(data, 1, mean) - grand_mean)^2) * b
SSE <- sum(apply(data, 1, var)) * (b-1)
```

```
F_ratio <- (SSTr / (a-1)) / (SSE / (a*b - a))
p_val <- pf(F_ratio, df1 = a-1, df2=a*b - a, lower.tail = F)

> p_val
[1] 0.764377
```

By CRD model, under H_0 that treatment factor has no effect on strength, p-value of ANOVA one-way F-test is 0.76, which is insignificant to reject null hypothesis.

b

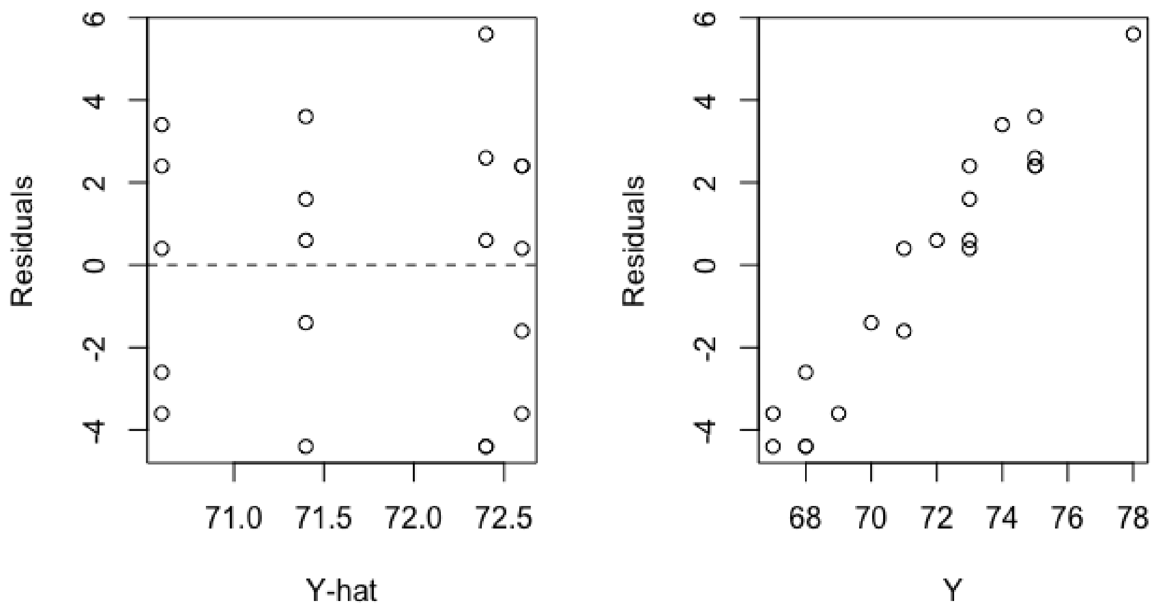
```
A <- as.factor(rep(c(1:a), each=b))
B <- as.factor(rep(c(1:b), times=a))

yi. <- apply(data, 1, mean)
y.j <- apply(data, 2, mean)
SStr <- b * sum((yi. - grand_mean)^2)
SSbl <- a * sum((y.j - grand_mean)^2)
SST <- sum((y - grand_mean)^2)
SSE <- SST - SStr - SSbl
F_ratio <- (SStr / (a-1)) / (SSE / ((a-1)*(b-1)))
p_val <- pf(F_ratio, df1=a-1, df2=(a-1) * (b-1), lower.tail = F)

> p_val
[1] 0.1211445
```

By RCBD model, under H_0 that treatment factor has no effect on strength, p-value of ANOVA one-way F-test is 0.121, which is insignificant to reject null hypothesis.

c

In CRD, $\hat{Y_{ij}} = \hat{\mu} + \hat{\tau_i} = \bar{Y_{i.}}$

```
y_hat <- rep(yi., each=b)
resid <- y - y_hat
par(mfrow=c(1,2))
plot(x=y_hat, y=resid, xlab='Y-hat', ylab='Residuals')
abline(h=0, lty=2)
plot(x=y, y=resid, xlab='Y', ylab='Residuals')
```

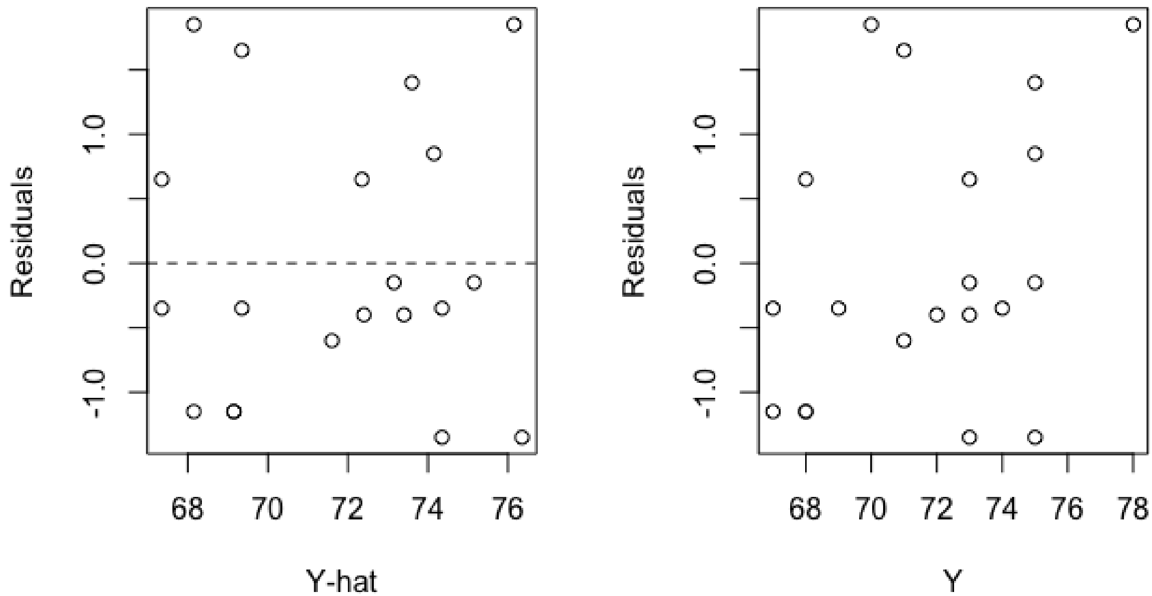


The plot of residuals vs $\hat{Y}$ distributed randomly, implying the equal variance assumption is not violate.

d

In RCBD, $\hat{Y_{ij}} = \hat{\mu} + \hat{\alpha_i} + \hat{\beta_j} = \bar{Y_{i.}} + \bar{Y_{.j}} - \bar{Y_{..}}$

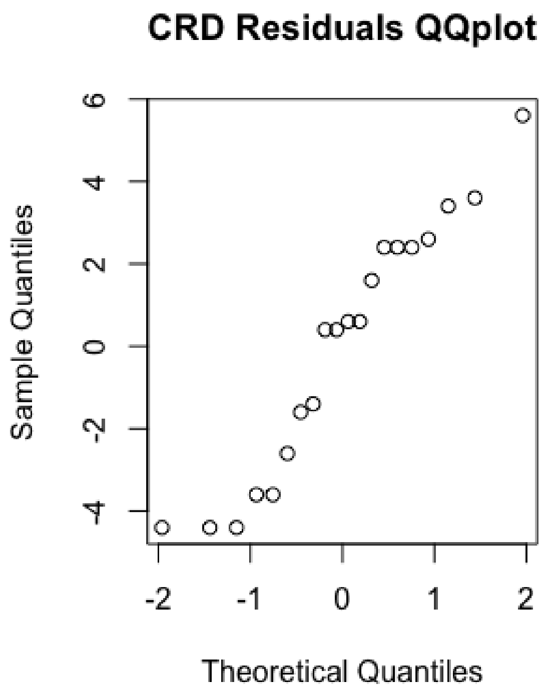
```
y_hat <- rep(yi., each=b) + rep(y.j, times=a) - grand_mean
resid <- y - y_hat
par(mfrow=c(1,2))
plot(x=y_hat, y=resid, xlab='Y-hat', ylab='Residuals')
abline(h=0, lty=2)
plot(x=y, y=resid, xlab='Y', ylab='Residuals')
```

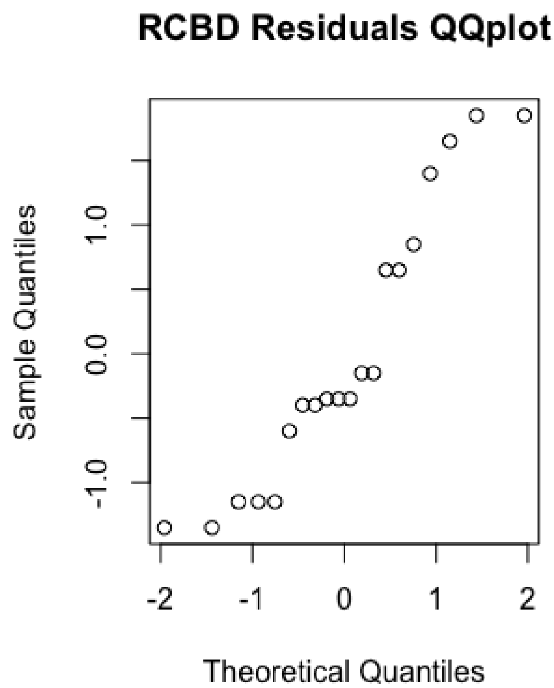


The plot of residuals vs $\hat{Y}$ distributed randomly, implying the equal variance assumption is not violate.

e

```
y_hat1 <- rep(yi., each=b)
resid1 <- y - y_hat1
y_hat2 <- rep(yi., each=b) + rep(y.j, times=a) - grand_mean
resid2 <- y - y_hat2
par(mfrow=c(1,2))
qqnorm(resid1, main='CRD Residuals QQplot')
qqnorm(resid2, main='RCBD Residuals QQplot')
```





The qqplots for both models have the shape of a line, implying the normality assumption is not violated.

Lect 11-2

A	B	C
B	C	A
C	A	B

A	B	C
C	A	B
B	C	A

A	C	B
B	A	C
C	B	A

A	C	B
C	B	A
B	A	C

B	A	C
A	C	B
C	B	A

B	A	C
C	B	A
A	C	B

B	C	A
A	B	C
C	A	B

B	C	A
C	A	B
A	B	C

C	A	B
A	B	C
B	C	A

C	B	A
A	C	B
B	A	C

C	B	A
B	A	C
A	C	B

C	A	B
B	C	A
A	B	C

Lect 11-3

A	B	C	D
B	C	D	A
C	D	A	B
D	A	B	C

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

A	B	C	D
B	D	A	C
C	A	D	B
D	C	B	A

A	B	C	D
B	A	D	C
C	D	B	A
D	C	A	B