hw-lect9-1   3.49 (8th ed.), 3.41 (7th ed.)

You can enter the numbers here, but I'll do it generally.

The Least-squares normal eqs are (Lect 10)

$y_{..} - an\hat{\mu} - n\hat{\tau}_. = 0 \quad y_{i.} - n\hat{\mu} - n\hat{\tau}_i = 0$, but I'll suppress

The hat (^) for simplicity.

$y_{..} - an\mu - n\tau_. = 0 \quad y_{i.} - n\mu - n\tau_i = 0$,

a) Impose $\boxed{\tau_. = 0}$. Then $y_{..} - an\mu = 0 \qquad y_{i.} - n\mu - n\tau_i = 0$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{cases} \mu = \frac{1}{an} y_{..} = \bar{y}_{..} \implies y_{i.} - n\bar{y}_{..} - n\tau_i = 0 \\ \qquad\qquad\qquad\qquad \Downarrow \\ \tau_i = \bar{y}_{i.} - \bar{y}_{..} \Leftarrow \tau_i = \frac{1}{n}(y_{i.} - n\bar{y}_{..}) \end{cases}$$

$$\left.\begin{aligned} \tau_1 - \tau_2 &= (\bar{y}_{1.} - \cancel{\bar{y}_{..}}) - (\bar{y}_{2.} - \cancel{\bar{y}_{..}}) = \bar{y}_{1.} - \bar{y}_{2.} \\ \text{Also, } \tau_1 - \tau_3 &= (\bar{y}_{1.} - \bar{y}_{..}) - (\bar{y}_{3.} - \bar{y}_{..}) = \bar{y}_{1.} - \bar{y}_{3.} \\ \tau_2 - \tau_3 &= \qquad\qquad\qquad\qquad = \bar{y}_{2.} - \bar{y}_{3.} \end{aligned}\right\}$$

b) Impose $\boxed{\tau_3 = 0}$. Then

$$\begin{cases} y_{..} - an\mu - n(\tau_1 + \tau_2) = 0 & \text{I} \\ y_{1.} - n\mu - n\tau_1 = 0 & \text{II} \\ y_{2.} - n\mu - n\tau_2 = 0 & \text{III} \\ y_{3.} - n\mu = 0 \implies \mu = \bar{y}_{3.} \end{cases}$$

III $\implies \tau_2 = \frac{1}{n}(y_{2.} - n\mu) = \frac{1}{n}(y_{2.} - n\bar{y}_{3.}) \implies \tau_2 = \bar{y}_{2.} - \bar{y}_{3.}$

II $\implies \tau_1 = \frac{1}{n}(y_{1.} - n\mu) = \frac{1}{n}(y_{1.} - n\bar{y}_{3.}) \implies \tau_1 = \bar{y}_{1.} - \bar{y}_{3.}$

The estimates are completely different.

But $\tau_1 - \tau_2 = (\bar{y}_{1.} - \bar{y}_{3.}) - (\bar{y}_{2.} - \bar{y}_{3.}) = \bar{y}_{1.} - \bar{y}_{2.}$ is same as in part a

In fact consider the other contrasts in τ_i :

$\tau_1 - \tau_2 = \bar{y}_{1.} - \bar{y}_{2.}$

$\tau_1 - \tau_3 = \bar{y}_{1.} - \bar{y}_{3.} - 0 = \bar{y}_{1.} - \bar{y}_{3.}$

$\tau_2 - \tau_3 = \bar{y}_{2.} - \bar{y}_{3.} - 0 = \bar{y}_{2.} - \bar{y}_{3.}$

These are all equal to the contrasts in part a.

Because these contrasts $\tau_i - \tau_j$ do not depend on the choice of the constraint, one says "they are <u>estimable</u>."

c)

	$\tau_. = 0$	$\tau_3 = 0$
$\mu + \tau_1$	$\bar{y}_{..} + \bar{y}_{1.} - \bar{y}_{..} = \bar{y}_{1.}$	$\bar{y}_{3.} + \bar{y}_{1.} - \bar{y}_{3.} = \bar{y}_{1.}$
$2\tau_1 - \tau_2 - \tau_3$	$2(\bar{y}_{1.} - \cancel{\bar{y}_{..}}) - (\bar{y}_{2.} - \cancel{\bar{y}_{..}}) - (\bar{y}_{3.} - \cancel{\bar{y}_{..}}) = 2\bar{y}_{1.} - \bar{y}_{2.} - \bar{y}_{3.}$	$2(\bar{y}_{1.} - \bar{y}_{3.}) - (\bar{y}_{2.} - \bar{y}_{3.}) = 2\bar{y}_{1.} - \bar{y}_{2.} - \bar{y}_{3.}$
$\mu + \tau_1 + \tau_3$	$\cancel{\bar{y}_{..}} + \bar{y}_{1.} - \cancel{\bar{y}_{..}} + \bar{y}_{2.} - \bar{y}_{..} = \underbrace{\bar{y}_{1.} + \bar{y}_{2.} - \bar{y}_{..}}_{(1-\frac{1}{a})(\bar{y}_{1.} + \bar{y}_{2.}) - \frac{1}{a}\bar{y}_{3.}}$ $\neq$	$\cancel{\bar{y}_{3.}} + \bar{y}_{1.} - \cancel{\bar{y}_{3.}} = \bar{y}_{1.}$

$\therefore$
- $\mu + \tau_1$ is estimable
- $2\tau_1 - \tau_2 - \tau_3$ is estimable
- $\mu + \tau_1 + \tau_3$ is <u>not</u> estimable

hw-lect 9-2

Consider the "means model": $Y_{ij} = \mu_i + \epsilon_{ij}$, $\epsilon \sim N(0, \sigma_\epsilon^2)$

Find the maximum likelihood (or least square) estimates of μ_i, σ_ϵ.

Show all the steps, like done above.

Note: you cannot write $\mu_i = \mu + \tau_i$ here. In this model μ_i is the param.

then Likelihood of data $= \prod_{i=1}^{a} \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma_\epsilon^2}} e^{-\frac{1}{2}\left(\frac{Y_{ij}-\mu_i}{\sigma_\epsilon}\right)^2}$

$$= e^{-\frac{1}{2}\sum_i^a \sum_j^n \left(\frac{Y_{ij}-\mu_i}{\sigma_\epsilon}\right)^2 - \frac{1}{2} an \log(2\pi\sigma_\epsilon^2)}$$

$$\frac{\partial}{\partial \mu_i}[\text{exponent}] \sim \sum_j^n (Y_{ij} - \mu_i)\Big|_{\hat{\mu}_i, \hat{\sigma}_\epsilon} = 0 \Rightarrow \boxed{\hat{\mu}_i = \bar{Y}_{i.}}$$

$$\frac{\partial}{\partial \sigma_\epsilon}[\ldots] \sim \sum_i \sum_j (Y_{ij} - \mu_i)^2 \frac{1}{\sigma_\epsilon^3} - an\frac{1}{\sigma_\epsilon}\Big|_{\ldots} = 0 \Rightarrow \boxed{\hat{\sigma}_\epsilon = \text{same as before}}$$

Note: In contrast to the "effects model" $Y_{ij} = \mu + \tau_i + \epsilon_{ij}$, no constraint is necessary for the means model. The effects model is said to be <u>over-parameterized</u>; the introduction of the constraint is necessary to handle the "extra" parameters.

hw-lect9-3

a) For the reduced model $y_{ij} = \mu + \epsilon_{ij}$, find the Max. Like. estimate of μ and σ_ϵ^2 (and ϵ_{ij}), in terms of sums or averages of y_{ij}, and find the value of SSE at those estimated values. That quantity is what I denoted $SSE(\mu)$ (although a better notation is $SSE(\hat{\mu})$.)

where $SSE(\hat{\mu}) = \sum_{ij} \hat{\epsilon}_{ij}^2 = \sum_{ij} (y_{ij} - \hat{\mu})^2$

b) For the full model $y_{ij} = \mu + \tau_i + \epsilon_{ij}$, we have already estimated $\hat{\mu}, \hat{\tau}_i, \sigma_\epsilon^2$ (and ϵ_{ij}). Use those expressions to compute $SSE(\mu, \tau)$, again better written as $SSE(\hat{\mu}, \hat{\tau}) = \sum_{ij} \hat{\epsilon}_{ij}^2 = \cdots$

c) Show that the F-ratio given in terms of these SSE values is the same as the ANOVA F-ratio $= \frac{MSTr}{MSE}$, used before. The advantage of writing the F-ratio in terms of the SSE's of full and reduced models is that it allows us to test other effects (not just the τ_i); later!

a) $y_{ij} = \mu + \epsilon_{ij}$ Likelihood $= e^{-\frac{1}{2}\sum_i^a \sum_j^n \left(\frac{y_{ij}-\mu}{\sigma_\epsilon}\right)^2 - \frac{1}{2} an \log(2\pi\sigma_\epsilon^2)}$

$$\frac{\partial}{\partial\mu}[\text{exponent}]\Big|_{\hat{\mu},\hat{\sigma}_\epsilon} = 0 \Rightarrow \frac{\partial}{\partial\mu}\sum_{ij}(y_{ij}-\mu)^2\Big|_{\hat{\mu}} = 0 \Rightarrow \sum_{ij}(y_{ij}-\hat{\mu}) = 0 \Rightarrow \hat{\mu} = \bar{y}_{..}$$

$$\frac{\partial}{\partial\sigma_\epsilon}[\quad] = 0 \Rightarrow \text{same as usual} \Rightarrow \hat{\sigma}_\epsilon^2 = \frac{1}{an}\sum_{ij}(y_{ij}-\bar{y}_{i.})^2$$

$$SSE[\hat{\mu}] = \sum_{ij}\hat{\epsilon}_{ij}^2 = \sum_{ij}(y_{ij}-\hat{\mu})^2 = \underline{\underline{\sum_{ij}(y_{ij}-\bar{y}_{..})^2}} \quad [= SST]$$

b) $\hat{\mu} = \bar{Y}_{..}$, $\hat{\tau}_i = \bar{Y}_{i.} - \bar{Y}_{..}$, $\hat{\sigma}_\epsilon = \frac{1}{an} \sum_{ij} (Y_{ij} - \bar{Y}_{i.})^2$

$$SSE(\hat{\mu}, \hat{\tau}) = \sum_{ij} \hat{\epsilon}_{ij} = \sum_{ij} (Y_{ij} - \hat{\mu} - \hat{\tau}_i)^2 = \sum_{ij} (Y_{ij} - \cancel{\bar{Y}_{..}} - \bar{Y}_{i.} + \cancel{\bar{Y}_{..}})^2$$

$$= \underline{\underline{\sum_{ij} (Y_{ij} - \bar{Y}_{i.})^2}}$$

c) $$F = \frac{[SSE(\hat{\mu}) - SSE(\hat{\mu}, \hat{\tau})]/(a-1)}{SSE(\hat{\mu}, \hat{\tau})/(an-a)} = \frac{\left[\sum_{ij} (Y_{ij} - \bar{Y}_{..})^2 - \sum_{ij} (Y_{ij} - \bar{Y}_{i.})^2\right]/(a-1)}{\sum_{ij} (Y_{ij} - \bar{Y}_{i.})^2 / (an-a)}$$

$$\sum_{ij} \left[(Y_{ij} - \bar{Y}_{..})^2 - (Y_{ij} - \bar{Y}_{i.})^2 \right]$$

$$= \sum_{ij} \left\{ (\cancel{Y_{ij}} - \bar{Y}_{..}) - (\cancel{Y_{ij}} - \bar{Y}_{i.}) \right\} \left\{ (Y_{ij} - \bar{Y}_{..}) + (Y_{ij} - \bar{Y}_{i.}) \right\}$$

$$= \sum_{ij} (\bar{Y}_{i.} - \bar{Y}_{..})(2Y_{ij} - \bar{Y}_{i.} - \bar{Y}_{..})$$

$$= \sum_i (\bar{Y}_{i.} - \bar{Y}_{..}) \underbrace{\sum_j^n (2Y_{ij} - \bar{Y}_{i.} - \bar{Y}_{..})}_{2Y_{i.} - n\bar{Y}_{i.} - n\bar{Y}_{..} = n(\bar{Y}_{i.} - \bar{Y}_{..})} = n \sum_i (\bar{Y}_{i.} - \bar{Y}_{..})^2 = SS_{Tr}$$

$$2Y_{i.} = 2n\bar{Y}_{i.}$$

$$F = \frac{SS_{Tr}/(a-1)}{SSE/(an-a)} = \text{same } F \text{ ratio as before.}$$

hw-lect 9-4

The following should be done ALL by hand, not even R for arithmetic. But to simplify the computation of variances, use the fact that for y_1, y_2, one has $s^2 = \frac{1}{2}(y_1 - y_2)^2$. Now, consider the data in the following table.

a) Find the row-means & row-vars, and put them here, respectively.

$$\begin{array}{ccc} & \multicolumn{2}{c}{\text{Coupon}} \\ & 1 & 2 \\ \text{Tip} \quad 1 & 4 & 6 \\ 2 & 10 & 2 \end{array}$$

Find the col-means & col-vars, and write them here, respectively.

Find $\bar{Y}_{..}$ and SST, and put them here, respectively.

$$\begin{array}{cccccc} & \multicolumn{2}{c}{\text{Coupon}} & \bar{Y}_{i.} & s_i^2 \\ & 1 & 2 & \swarrow & \swarrow \\ \text{Tip} \quad 1 & 4 & 6 & 5 & 2 \\ 2 & 10 & 2 & 6 & 32 \\ \bar{Y}_{.j} \rightarrow & 7 & 4 & (\frac{11}{2}) & \\ s_j^2 \rightarrow & 18 & 8 & & 35 \end{array}$$

$\frac{1}{2}(10-2)^2$

$(4-\frac{11}{2})^2 + (6-\frac{11}{2})^2 + (10-\frac{11}{2})^2 + (2-\frac{11}{2})^2$

$= \frac{1}{4}(9 + 1 + 81 + 49)$

b) For the row-quantities, find the variance of the means, and the sum of the vars. According to the ANOVA decomposition, these 2 numbers are supposed to be related to SST. Confirm that relationship.

var. of the means $= \frac{1}{2}(5-6)^2 = 1/2$

sum of the vars $= 2 + 32 = 34$

$SS_{tr} + SSE = SST$

$2(\frac{1}{2}) + 34 = 35$ ✓.

c) Same as b) but between SST and the col-means and col-variances?

var. of the means $= \frac{1}{2}(7-4)^2 = 9/2$

sum of the vars $= 18 + 8 = 26$

$SS_{Tr} + SSE = SST$

$2(\frac{9}{2}) + 26 = 35$ ✓

The point of this exercise is to get you comfortable with all the quantities appearing here. The difference between CRD and RCBD is in the expected value of all these quantities.