

Introduction: Applied Regression ¹

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¹(based on lectures of Elena Erosheva)

Basics

- ▶ Syllabus: Text, schedule, grades.
- ▶ Lectures and lab sessions, laptops, homework.
- ▶ Project.

Course Projects

- ▶ STAT/CSSS 504 is a project-based course.
- ▶ Students identify a research question and a corresponding data set, and carry out a regression analysis to answer the research question.
- ▶ Everyone proposes a project idea. Instructor selects projects. Best projects start with a question or idea, then find data.
- ▶ If not enough viable project ideas are proposed, there will be an in-class final exam and the class schedule will be revised accordingly.
- ▶ Project groups deliver short oral presentations in class.
- ▶ Project groups present final results in poster format during the finals week.

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- ▶ To gain practical skills necessary to formulate a research question, carry out analyses, interpret results, and present findings addressing the research question from a regression study.
- ▶ To become a critical consumer of research that employs regression techniques.

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- ▶ Robust regression.
- ▶ Logistic regression.

Basic ideas of regression

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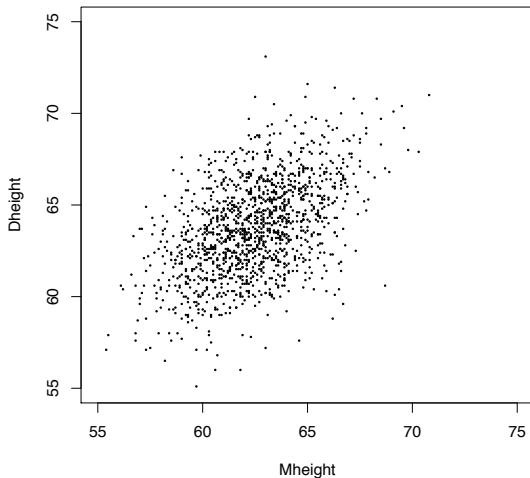
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- ▶ When $p = 1$, the analysis is called **simple** regression; when $p > 1$, it is called **multiple** regression.

Inheritance of Heights: Mothers and Daughters

From R package `alr4` dataset `Heights`.



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Linear regression

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Note: the function f is not necessarily linear in the predictors!

Linear regression

Which equation is not a linear regression function?

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon$$

$$y = \beta_0 + \beta_1 \log(x_1) + \beta_2 x_2 + \beta_3 \log(x_1)x_2 + \beta_4 x_2^2 + \epsilon$$

$$y = \beta_0 + \beta_1 x_1 + x_2^{\beta_2} + \epsilon$$

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- ▶ inherent randomness (unpredictable aspects of Y).

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- ▶ What is the relationship between mothers' and daughters' heights?
- ▶ What is the relationship between education and voting Democrat?

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On the other hand, **description and inference** objectives are characteristic of problems where understanding the mechanism is the key issue and predictions are by-products.

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- ▶ univariate graphical summaries (boxplots, histograms, density plots),
- ▶ scatterplots.

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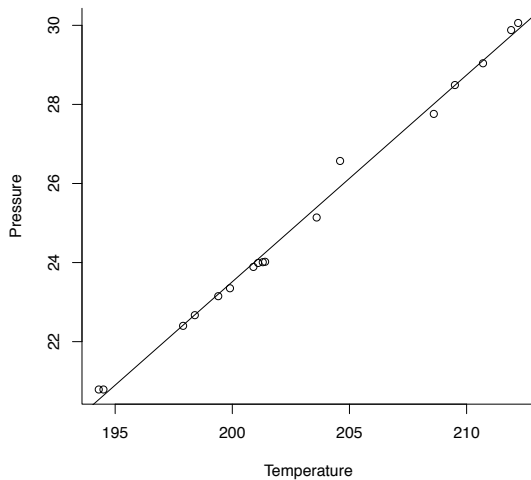
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Assuming we have already examined numerical summaries and univariate plots, let us look at the scatterplots.

Data: Forbes in R package `alr4`.

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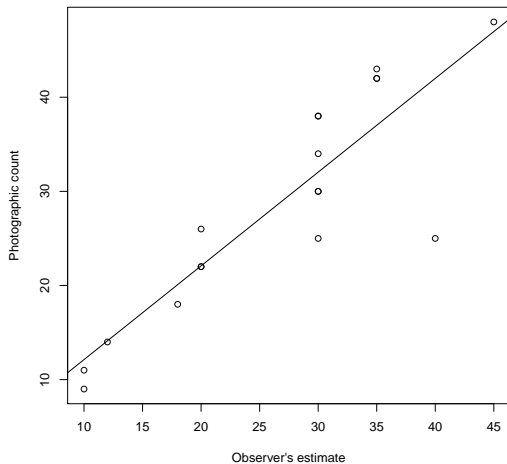


Example: Forbes data

Forbes Data: Observations from the scatterplot.

- ▶ Points appear to lie close to a line, however some curvature can be seen (by theory, $\log(\text{pressure})$ is linearly related to temperature).
- ▶ One point does not “fit”.

Example: Snow Geese



Data: snowgeese in R package alr3.

Example: Snow Geese

Snow geese: Observations from the scatterplot.

- ▶ Small sample size; some x values have multiple y values recorded; some data points may be duplicated (we are not able to see this on the plot).
- ▶ Although a non-constant variance (heteroscedasticity) is not easily spotted on the plot due to a relatively small sample size, we expect it to be present because estimation errors by wildlife service members are likely to increase with the size of a flock.

Example: Inheritance of height

- ▶ The sample size is $n = 1375$ (pairs of mothers and daughters).

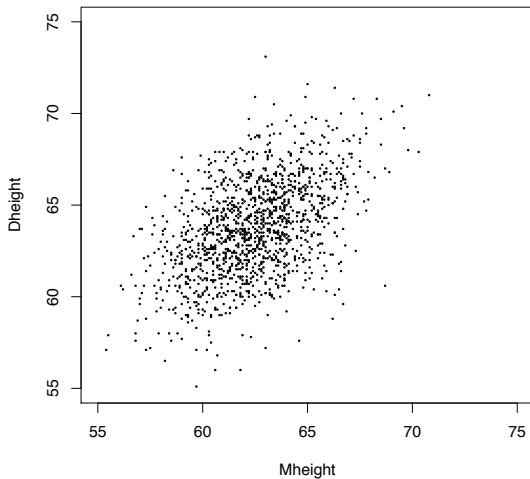
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- ▶ For the graph, data were jittered (uniform, $U(-0.5, 0.5)$, random noise added to mothers' and daughters' heights).

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Data: Heights from R package alr4.

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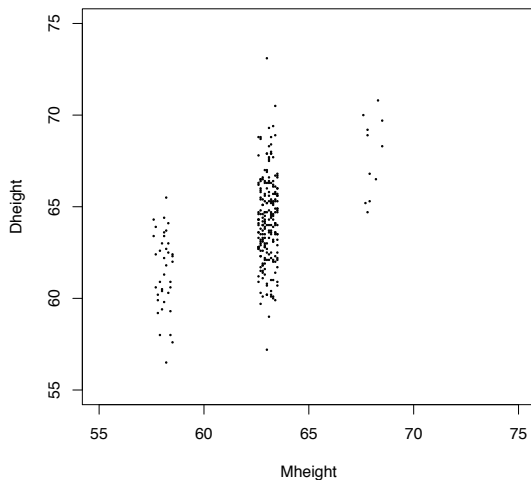
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- ▶ The scatter appears elliptically shaped (rather typical if (X, Y) is a bivariate normal random vector).
- ▶ What about variance in the daughter's height for short, about average, and tall mothers?

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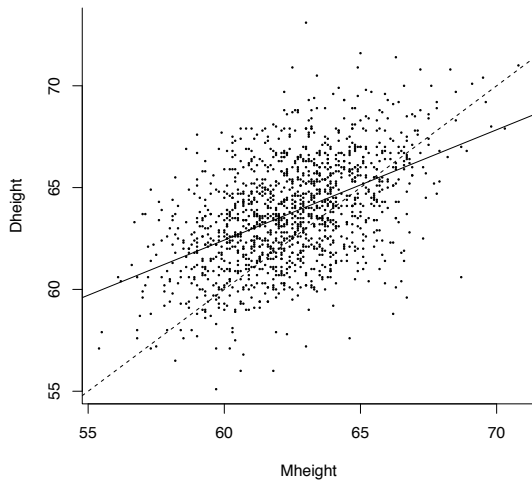
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Inheritance of height: Examining daughters' heights for mothers who are about 58, 64 and 68 inches tall, we find that the mean is increasing.

- ▶ The variance might be about the same (notice many more data points in the middle).
- ▶ The next figure illustrates two possible regression lines.

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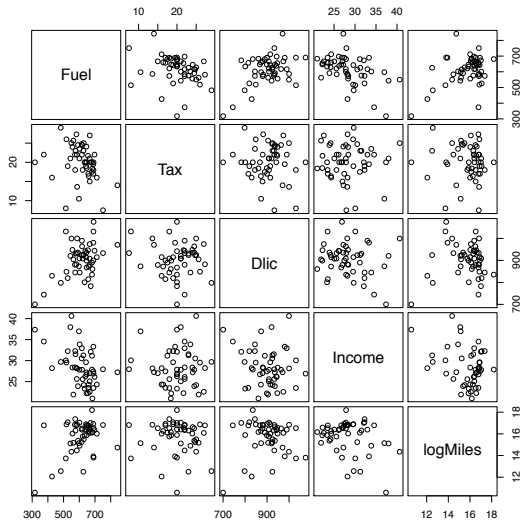
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For multiple regression, scatterplot matrices can be useful.

Data: `fuel2001` from R package `alr4`.

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- ▶ Because marginal relationships among the pairs of the predictors is weak, marginal plots for fuel versus the predictors are informative for the multiple regression problem.

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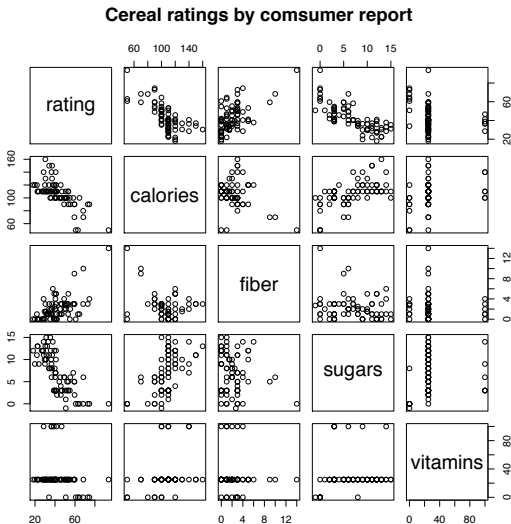
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- ▶ rating - a rating of the cereals
- ▶ calories - calories per serving
- ▶ fiber - grams of dietary fiber
- ▶ sugars - grams of sugars
- ▶ vitamins - vitamins and minerals: 0, 25, or 100, indicating the typical percentage of FDA recommended daily intake.

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- ▶ The rating seems to be related to calories and sugars, however calories and sugar content also seem to be related to each other.
- ▶ Note: If three or more predictors were linearly related, such as

$$X_1 + X_2 - X_3 \approx 0,$$

we would not be able to see this sort of relationship on a matrix of scatterplots.