Reconstruction of 3-D Graphics Using Kansa's Method

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Second-Order Boundary Value Problem

Reconstruct the 3-D graphic by solving the following second-order boundary value problem,

$$\Delta u(x, y, z) - \lambda u(x, y, z) = f(x, y, z) \quad (x, y, z) \in \Omega$$

$$u(x, y, z) = g(x, y, z) \quad (x, y, z) \in \partial\Omega,$$

where λ is a nonnegative scalar. Reconstruct the surface using Kansa's method with the given boundary points and chosen Radial Basis Function. For the following results, we will be using the Normalized Multiquadric Radial Basis Function (NMQ RBF),

$$\phi(r, c) = \sqrt{1 + r^2 c^2},$$

Kansa's Method

The solution can be approximated by a linear combination of NMQ RBF

$$u(x, y, z) \simeq \hat{u}(x, y, z) = \sum_{i=1}^{n} a_i \sqrt{1 + r^2 c^2} = g(x, y, z),$$

where c is the chosen shape parameter and r is the Euclidean distance,

$$r = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}.$$

The centers $\{(x_i, y_i, z_i)\}_1^n$, are comprised of n_i interiors points and n_b boundary points. Note that $n = n_i + n_b$.

The Laplacian of the approximate solution is then,

$$\Delta \hat{u}(x, y, z) = \sum_{i=1}^{n} a_i \frac{3c^2 + 2c^4 r^2}{(1 + r^2 c^2)^{3/2}}.$$

Second-Order Boundary Value Problem Approximation

The approximation of the second-order boundary value problem is given by

$$\Delta \hat{u}(x,y,z) - \lambda \hat{u}(x,y,z) =$$

$$\sum_{i=1}^{n} a_i \frac{3c^2 + 2c^4 r^2}{(1 + r^2 c^2)^{3/2}} - \lambda \sum_{i=1}^{n} a_i \sqrt{1 + r^2 c^2} = f(x,y,z).$$

for $(x,y,z) \in \Omega$ and

$$\hat{u}(x,y,z) = \sum_{i=1}^{n} a_i \sqrt{1 + r^2 c^2} = g(x,y,z),$$

for $(x,y,z) \in \partial\Omega$.

Collocation Method

Using the Collocation method, n_i equations are produced using the interior points $\{(x_j, y_j, z_j)\}$, for $j = 1, \ldots, n_i$,

$$\sum_{i=1}^{n} a_i \frac{3c^2 + 2c^4((x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2)}{(1 + c^2(x_j - x_i)^2 + c^2(y_j - y_i)^2 + c^2(z_j - z_i)^2)^{3/2}}$$

$$-\lambda \sum_{i=1}^{n} a_i \sqrt{1 + c^2(x_j - x_i)^2 + c^2(y_j - y_i)^2 + c^2(z_j - z_i)^2} = f(x_j, y_j, z_j),$$

and n_b equations are produced using the boundary points $\{(x_j, y_j, z_j)\}$, for $j = n_i + 1, \ldots, n$,

$$\sum_{i=1}^{n} a_i \sqrt{1 + c^2(x_j - x_i)^2 + c^2(y_j - y_i)^2 + c^2(z_j - z_i)^2} = g(x_j, y_j, z_j).$$

Collocation Method

By allowing Ψ and Φ to represent block matrices with entries,

$$\Psi_{i,j} = \frac{3c^2 + 2c^4((x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2)}{(1 + c^2(x_j - x_i)^2 + c^2(y_j - y_i)^2 + c^2(z_j - z_i)^2)^{3/2}}$$

$$-\lambda\sqrt{1 + c^2(x_j - x_i)^2 + c^2(y_j - y_i)^2 + c^2(z_j - z_i)^2},$$

$$\text{for } j = 1, \ldots, n_i, \;\; i = 1, \ldots, n,$$

$$\Phi_{i,j} = \sqrt{1 + c^2(x_j - x_i)^2 + c^2(y_j - y_i)^2 + c^2(z_j - z_i)^2},$$

$$\text{for } j = n_i + 1, \ldots, n, \;\; i = 1, \ldots, n,$$

the previous equations can be represented in matrix form,

$$\begin{bmatrix} \Psi \\ \Phi \end{bmatrix} \boldsymbol{a} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{bmatrix}.$$

Reconstructing the Surface

By solving the following $n \times n$ system of equations,

$$\begin{bmatrix} \Psi \\ \Phi \end{bmatrix} \boldsymbol{a} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{bmatrix},$$

the coefficients $\boldsymbol{a}_i$ for $i = 1, \ldots, n$ are produced. To reconstruct the surface of our 3-D graphic, we need the coefficients and K constructed uniform 3-D evaluation points in the domain. The approximate solution is given by,

$$\hat{u}(x_k, y_k, z_k) = \sum_{i=1}^{n} a_i \sqrt{1 + r^2 c^2} \quad \text{for } k = 1, \ldots, K,$$

where K is chosen based on number of evaluation points desired.

Creating Interior Points

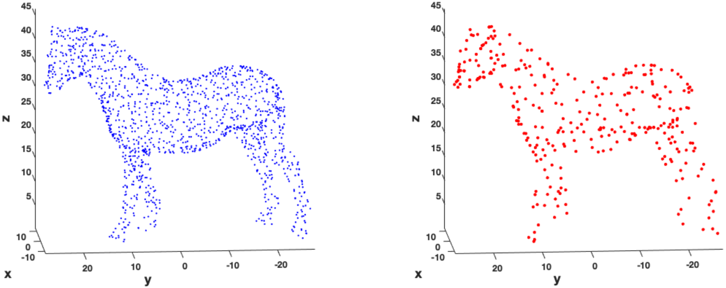
If the normal vectors are not given, we must find them in order to create interior points. For the results below, the normal vectors were given.

We are given $290,898$ boundary points and the corresponding $96,966$ normal vectors. Only 1 of the 3 boundary points per normal vector is needed. Let the reduced boundary points be represented by $\boldsymbol{b}_i = \{(bx_i, by_i, bz_i)\}_1^{n_b}$ and the normal vectors represented by $\boldsymbol{v}_i = \{(vx_i, vy_i, vz_i)\}_1^{n_b}$. Then the interior points can be produced by,

$$(x_i, y_i, z_i) = \boldsymbol{b}_i - h * \boldsymbol{v}_i, \quad \text{for } i = 1, \ldots, n_i,$$

where h is the spacing from the boundary points, n_i is the number of interior points, and n_b is the number of boundary points.

Boundary and Interior Points



(a) Given n_b boundary points, $n_b = 1,376$. (b) Produced n_i interior points, $n_i = 319$.

Figure: The figure in 1a contains the chosen boundary points from the given data set. The figure in 1b contains the interior points created from every 5^{th} boundary point and its corresponding normal vector with spacing $h = 0.01$.

Boundary and Interior Points

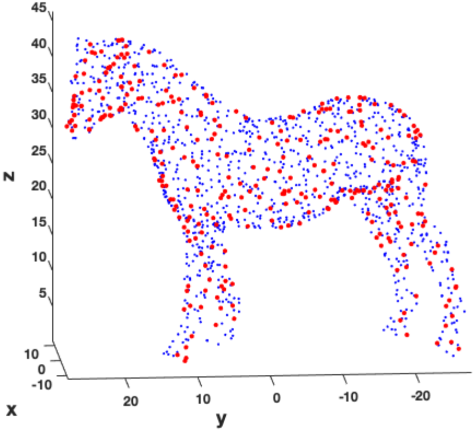


Figure: The boundary points and interior points are used to create the centers, $\{(x_i, y_i, z_i)\}_{i=1}^{n}$, where $n = n_i + n_b$.

Applying Kansa's Method

The block matrix Ψ is produced using interior points, $\{(x_j, y_j, z_j)\}_1^{n_i}$, where $\Delta\hat{u}(x_j, y_j, z_j) - \lambda\hat{u}(x_j, y_j, z_j) =$

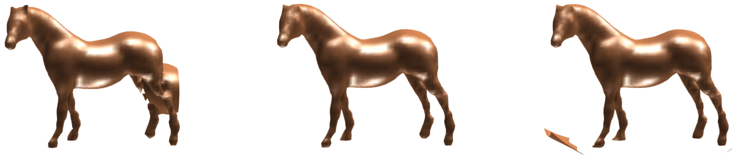
$$\sum_{i=1}^{n} a_i \frac{3c^2 + 2c^4 r^2}{(1 + r^2c^2)^{3/2}} - \lambda \sum_{i=1}^{n} a_i \sqrt{1 + r^2c^2} = f(x_j, y_j, z_j).$$

The block matrix Φ is produced using boundary points, $\{(x_j, y_j, z_j)\}_{n_i+1}^{n}$, where $\hat{u}(x_j, y_j, z_j) =$

$$\sum_{i=1}^{n} a_i \sqrt{1 + r^2c^2} = g(x_j, y_j, z_j).$$

For this boundary value problem, $\boldsymbol{f}$ and $\boldsymbol{g}$ are vectors of length n_i and n_b, respectively, containing entries of *ones*. The value of λ is nonnegative and c is the shape parameter, both which will be chosen based upon results.

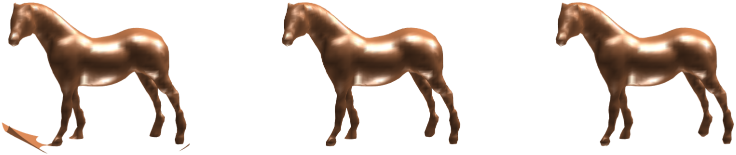
Second-Order Results - Change in c



(a) $c = 7,\ \lambda = 10$ (b) $c = 25,\ \lambda = 10$ (c) $c = 32,\ \lambda = 10$

Figure: For $\lambda = 10$, the values for the shape parameter must be $10 \leq c \leq 28$. All results use 319 interior points and $1,376$ boundary points.

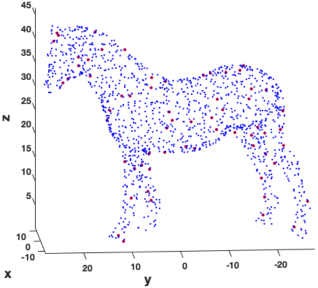
Second-Order Results - Change in λ



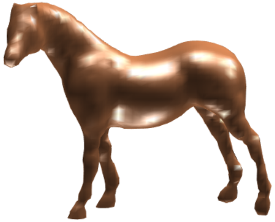
(a) $\lambda = 6,\ c = 28$ (b) $\lambda = 10,\ c = 28$ (c) $\lambda = 1000,\ c = 28$

Figure: After testing multiple values of λ and c, it can be concluded that if c is on the smaller end of the interval, then λ can be as small as 3. However, if c is on the larger end of the interval, then λ must be no smaller than 10. All results use 319 interior points and $1,376$ boundary points.

Second-Order Results - Improved Efficiency



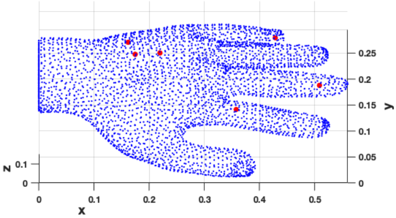
(a) $n_b = 1,376,\ n_i = 63$



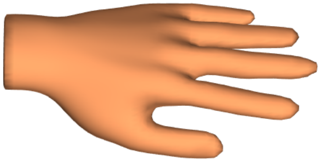
(b) $\lambda = 50,\ c = 5$

Figure: The figure in 5a contains the given n_b boundary points and the produced n_i interior points. The figure in 5b shows the reconstruction of the surface with only 63 interior points, reducing the number of interior points needed by more than four times. Notice the value of c is smaller than the previously stated domain.

Second-Order Result - Hand



(a) Boundary points and Interior points



(b) Surface Reconstruction

Figure: The figure in 6a contains the chosen $3,698$ boundary points and the chosen 6 interior points from the given data set. The figure in 6b contains the surface reconstruction with $\lambda = 4000$ and $c = 1000$.

Original Data

(a) Original Point Cloud Data

(b) 16,091 3-D Points

Figure: The figure in 7a contains the original 321,813 boundary points. The figure in 7b contains every 20th boundary point. The point cloud data was created using the Scandy Pro 3D Scanner app on an Iphone X.

Clean Point Cloud Data

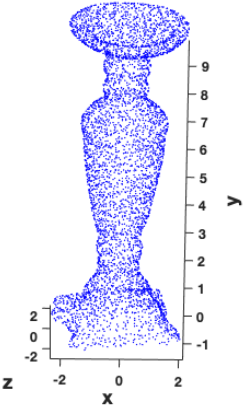
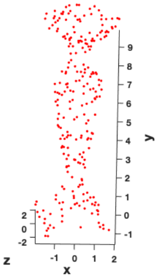
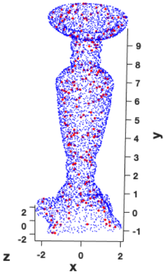


Figure: The chosen 4, 920 boundary points. Normal vectors were produced using these boundary points in order to create interior points.

Boundary and Interior Points



(a) Produced Interior Points



(b) Boundary and Interior Points

Figure: The figure in 9a contains the 246 interior points created from every 20^{th} boundary point and its corresponding normal vector with spacing $h = 0.25$. The figure in 9b contains the points used as our centers.

Surface Reconstruction - NMQ RBF

Figure: Surface reconstruction using the Normalized MultiQuadric Radial Basis Function, $\sqrt{1 + r^2 c^2}$. The shape parameters used were $c = 1000$ and $\lambda = 100$.

Inverse MultiQuadric (IMQ) RBF

The Inverse MultiQuadric Radial Basis Function is given by

$$\phi(r, c) = \frac{1}{\sqrt{r^2 + c^2}},$$

where c is the shape parameter and r is the Euclidean distance. The solution can be approximated by a linear combination of IMQ RBF

$$u(x, y, z) \simeq \hat{u}(x, y, z) = \sum_{i=1}^{n} a_i \frac{1}{\sqrt{r^2 + c^2}} = g(x, y, z).$$

The Laplacian of the approximate solution is then given by

$$\Delta \hat{u}(x, y, z) = \frac{r^2 - 2c^2}{(r^2 + c^2)^{3/2}}.$$

Inverse MultiQuadric (IMQ) RBF Approximation

The approximation of the second-order boundary value problem is given by

$$\Delta\hat{u}(x,y,z) - \lambda\hat{u}(x,y,z) =$$

$$\sum_{i=1}^{n} a_i \frac{r^2 - 2c^2}{(r^2+c^2)^{3/2}} - \lambda \sum_{i=1}^{n} a_i \frac{1}{\sqrt{r^2+c^2}} = f(x,y,z).$$

for $(x,y,z) \in \Omega$ and

$$\hat{u}(x,y,z) = \sum_{i=1}^{n} a_i \frac{1}{\sqrt{r^2+c^2}} = g(x,y,z),$$

for $(x,y,z) \in \partial\Omega$.

For this boundary value problem, $\boldsymbol{f}$ and $\boldsymbol{g}$ are vectors of length n_i and n_b, respectively, containing entries of *ones*. The value of λ is nonnegative and c is the shape parameter, both which will be chosen based upon results.

Gaussian RBF

The Gaussian Radial Basis Function is given by

$$\phi(r, c) = \mathbf{e}^{-r^2 c^2},$$

where c is the shape parameter and r is the Euclidean distance. The solution can be approximated by a linear combination of Gaussian RBF

$$u(x, y, z) \simeq \hat{u}(x, y, z) = \sum_{i=1}^{n} a_i \mathbf{e}^{-r^2 c^2} = g(x, y, z).$$

The Laplacian of the approximate solution is then given by

$$\Delta \hat{u}(x, y, z) = 4c^2 \mathbf{e}^{-c^2 r^2}(c^2 r^2 - 1).$$

Gaussian RBF Approximation

The approximation of the second-order boundary value problem is given by

$$\Delta \hat{u}(x,y,z) - \lambda \hat{u}(x,y,z) =$$

$$\sum_{i=1}^{n} a_i 4c^2 \mathbf{e}^{-c^2 r^2}(c^2 r^2 - 1) - \lambda \sum_{i=1}^{n} a_i \mathbf{e}^{-c^2 r^2} = f(x,y,z).$$

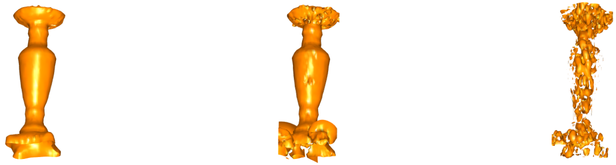
for $(x,y,z) \in \Omega$ and

$$\hat{u}(x,y,z) = \sum_{i=1}^{n} a_i \mathbf{e}^{-c^2 r^2} = g(x,y,z),$$

for $(x,y,z) \in \partial\Omega$.

For this boundary value problem, $\boldsymbol{f}$ and $\boldsymbol{g}$ are vectors of length n_i and n_b, respectively, containing entries of *ones*. The value of λ is nonnegative and c is the shape parameter, both which will be chosen based upon results.

Comparing Results - Candle Holder



(a) $\lambda = 100,\ c = 1000$ (b) $\lambda = 1000,\ c = 0.3$ (c) $\lambda = 10000,\ c = 5$

Figure: The figure in 11a uses NMQ RBF, the figure in 11b uses IMQ RBF, and the figure in 11c uses Gaussian RBF. All results use 246 interior points and $4,920$ boundary points.

Dr. Biswas' Cloud Point Data

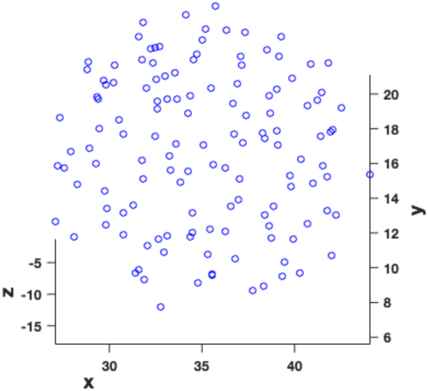
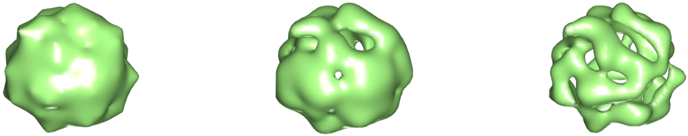


Figure: Given Point Cloud Data from Dr. Biswas containing 132 boundary points.

Comparing Results



(a) $\lambda = 100,\ c = 1000$ (b) $\lambda = 100,\ c = 3$ (c) $\lambda = 100,\ c = 0.4$

Figure: The figure in 13a uses NMQ RBF, the figure in 13b uses IMQ RBF, and the figure in 13c uses Gaussian RBF. All results use 132 boundary points.

Conclusion

- A drawback from this method is determining how to choose c and λ.
- The shape parameter values can be restricted into a domain once multiple values are tested and results are concluded.
- If the number of interior points change, or the values of the interior points change, new optimal intervals for c and λ must be produced.
- Another drawback from this method is the need for interior points. The more difficult the shape of the 3-D graphic, the more interior points that are needed.
- This method requires solving $n \times n$ system of equations, where $n = n_i + n_b$. Using less interior points means solving a smaller system of equations, i.e. more efficient.

For More Information...

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