Chapter 4: Exercise 2, Exercise 8, and Exercise 22

2. For each of the following two statements, decide whether it is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

(a) Suppose we are given an instance of the Minimum Spanning Tree Problem on a graph G, with edge costs that are all positive and distinct. Let T be a minimum spanning tree for this instance. Now suppose we replace each edge cost c_e by its square, c_e^2, thereby creating a new instance of the problem with the same graph but different costs.

True or false? T must still be a minimum spanning tree for this new instance.

(b) Suppose we are given an instance of the Shortest s-t Path Problem on a directed graph G. We assume that all edge costs are positive and distinct. Let P be a minimum-cost s-t path for this instance. Now suppose we replace each edge cost c_e by its square, c_e^2, thereby creating a new instance of the problem with the same graph but different costs.

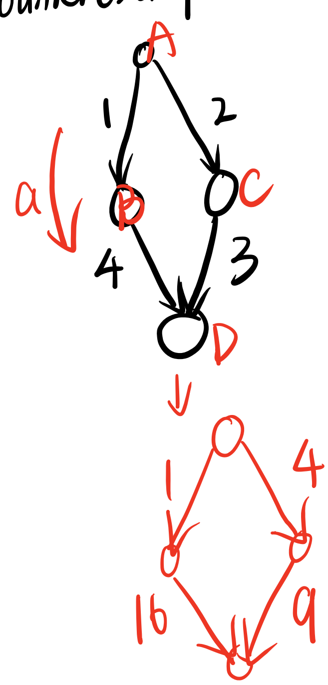
True or false? P must still be a minimum-cost s-t path for this new instance.

2. (a) True

Consider a kruscal's Algorithm, we retrieve the edge in ascending order until all nodes are included. So if the weight of the edge doubled, since the weight is positive and distinct, it's order remain unchanged. so follow the algorithm, we'll get exactly the same Spanning Tree.

(b) False

Counterexample:



path a is a shortest path from node A to node D.

but after squaring the weight, it's'n't a shortest path anymore.

8. Suppose you are given a connected graph G, with edge costs that are all distinct. Prove that G has a unique minimum spanning tree.

Proof by contradiction:

- suppose there are two minimum spanning tree T_1、T_2.
- suppose the edge set of T_1 and T_2 are

  $T_1 \{e_1, e_2, e_3 \cdots\}$
  $T_2 \{f_1, f_2, f_3 \cdots\cdots\}$ > assume they are in accending order

  easy to conclude that there are at least one pair of edge $\langle e_i, f_i \rangle$ that differs.

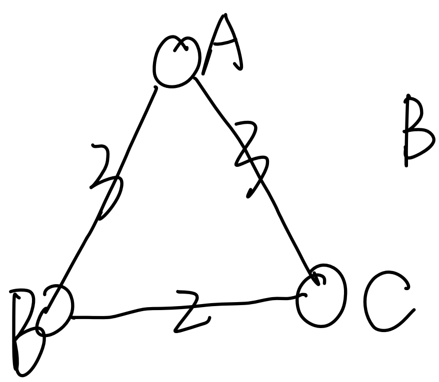
- assume k is the smallest value for i. Since all edges have distinct values, so $e_i \neq f_i$, assume $e_i < f_i$.
- Then for T_2, we can simply add e_i and delete one edge to delete the cycle, thus we can get a smaller spanning Tree.

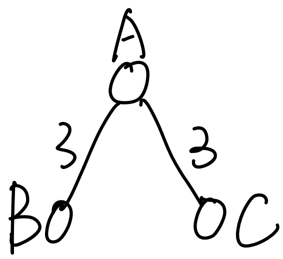
So $T_1 = T_2$. the spanning tree is unique.

22. Consider the Minimum Spanning Tree Problem on an undirected graph $G = (V, E)$, with a cost $c_e \geq 0$ on each edge, where the costs may not all be different. If the costs are not all distinct, there can in general be many distinct minimum-cost solutions. Suppose we are given a spanning tree $T \subseteq E$ with the guarantee that for every $e \in T$, e belongs to *some* minimum-cost spanning tree in G. Can we conclude that T itself must be a minimum-cost spanning tree in G? Give a proof or a counterexample with explanation.

False

Counterexample





is a spanning tree

but it isn't a MST.