

# CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Define:  $f: X \to Y$  by  $f((i,j,k)) = \{i,j,k\}$  **Claim**: f is a **bijection** (why) so |X| = |Y| f is a bijection because f is one-to-one if  $(i,j,k) \neq (i',j',k') \Rightarrow f((i,j,k)) \neq f((i',j',k'))$ f is onto

if  $\gamma$  is a 3-element subset then it can be written as  $\gamma = \{i, j, k\}$  where i < j < k so  $f((i, j, k)) = \gamma$ .

## Inclusion-Exclusion Principle Recall

- This can be used to determine the number of onto functions
  - A, B are two sets with |A| = m and |B| = n.
  - (a) How many onto functions are there from A to B?
  - (b) How many functions are there from A to B that map nothing to at least one element of B?

$$\#(a) + \#(b) = n^m$$

Set  $E_i$  – set of functions that map nothing to element i of B

$$\begin{aligned}
\#(b) &= |\cup_{i=1}^{n} E_{i}| \\
&= \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_{1} < i_{2} < \dots < i_{k} \leq n} |E_{i_{1}} \cap E_{i_{2}} \cap \dots \cap E_{i_{k}}| \\
&= \sum_{k=1}^{n} (-1)^{k+1} {n \choose k} (n-k)^{m}
\end{aligned}$$



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Note that the case of k = n is special;

An *n*-element permutation of a set N of size |N| = n is what we earlier simply called a permutation.



• How many three-element permutations of  $\{1, 2, \ldots, n\}$  are there?



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n choices for first number



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Ex: When n = 4, there are 4 \times 3 \times 2 = 24
3 -element permutations of \{1, 2, 3, 4\}
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L = \{123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243, 312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432\}.
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$$L = \{123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243, 312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432\}.$$

Note: This type of "dictionary" ordering of tuples (assuming that we treat numbers the same as letters) is called a *lexicographic ordering* and is used quite often.



■ **Theorem** If N is a positive integer and k is an integer with  $1 \le k \le n$ , then there are

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1)$$

k-element permutations with n distinct elements.



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$$(\# 3\text{-element perms}) = 6 \times (\# 3\text{-element subsets})$$

$$P(n,3) = 3! \cdot C(n,3)$$



#### Binomial Coefficient

■ **Theorem** For integers n and k with  $0 \le k \le n$ , the number of k-element subsets of an n-element set is

$$\binom{n}{k} = C(n, k) = \frac{P(n, k)}{k!} = \frac{n!}{k!(n-k)!}.$$

This is the number of k-combinations of a set with n elements.



$$\binom{n}{0} = 1$$
 only one set of size 0.

$$\binom{n}{n} = 1$$
 only one set of size  $n$ .

 $\binom{n}{k} = \binom{n}{n-k}$  Obvious from equation. Can you think of a simple bijection that explains this?



 $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ 



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#### Use Sum Rule

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S_i = \text{set of all } i \text{ subsets of } \{1,2,\ldots,n\}
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Use Sum Rule

Let 
$$P = \text{set of all subsets of } \{1,2,\ldots,n\}$$
  
 $S_i = \text{set of all } i \text{ subsets of } \{1,2,\ldots,n\}$ 

$$\Rightarrow |P| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i}$$



Let  $L = L_1 L_2 \dots L_n$  be a list of size n from  $\{0, 1\}$ If  $\mathcal{L} = \text{set of all such lists} \Rightarrow |\mathcal{L}| = 2^n$ There is a *bijection* between  $\mathcal{L}$  and P so  $|P| = 2^n$  and we are done.

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Define the following function  $f: \mathcal{L} \to P$ 

If  $L \in \mathcal{L}$  then f(L) is the set  $S \subseteq \{1, 2, ..., n\}$  defined by  $i \in S \Leftrightarrow L_i = 1$ 

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f is a *bijection* between  $\mathcal L$  and P (why?) so  $|\mathcal L|=|P|$ 

Ex: 
$$n = 5$$

$$f(10101) = \{1, 3, 5\}, \ f(11101) = \{1, 2, 3, 5\}, \ f(00000) = \emptyset$$

# Binomial Coefficients

$n^{k}$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1



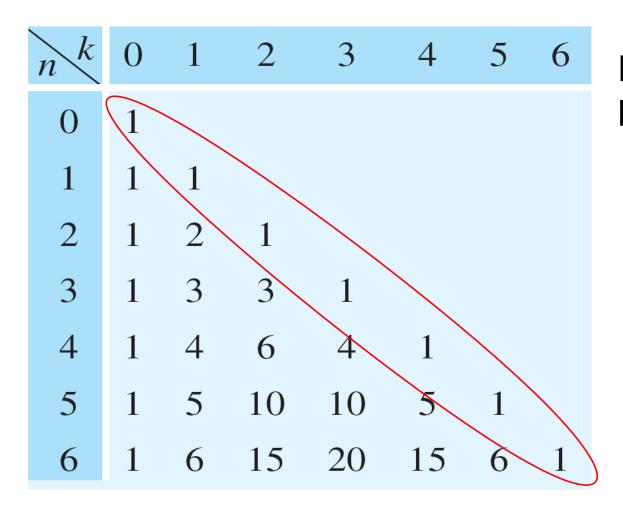
## Binomial Coefficients

$n^{k}$			2			5	6
0	$\sqrt{1}$		1 3 6 10 15				
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Each row begins with a 1 because  $\binom{n}{0} = 1$ 



## Binomial Coefficients



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### **Binomial Coefficients**

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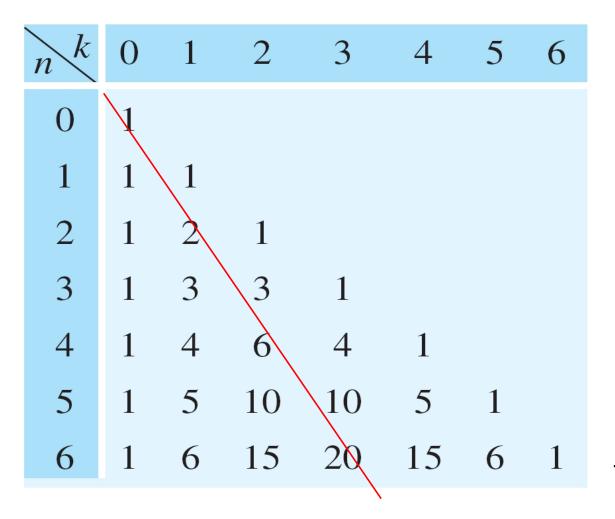
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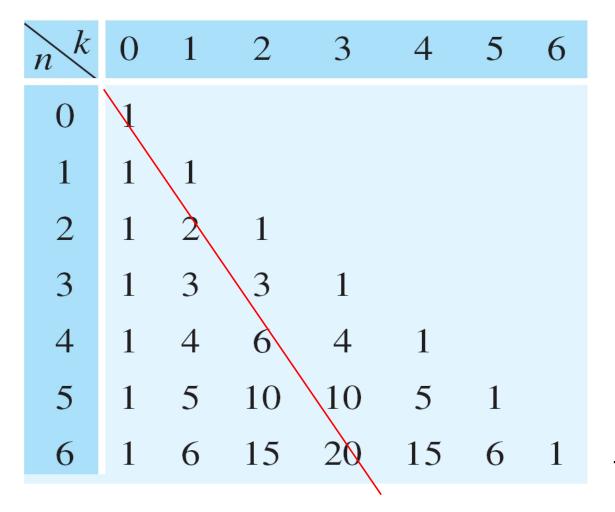
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### Binomial Coefficients



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Each row ends with a 1 because  $\binom{n}{n} = 1$ .

Each row increases at first then decreases.

Second half of each row is the reverse of the first half. Sum of items on n-th row is  $2^n$ 



#### Take the table

$n^{k}$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1



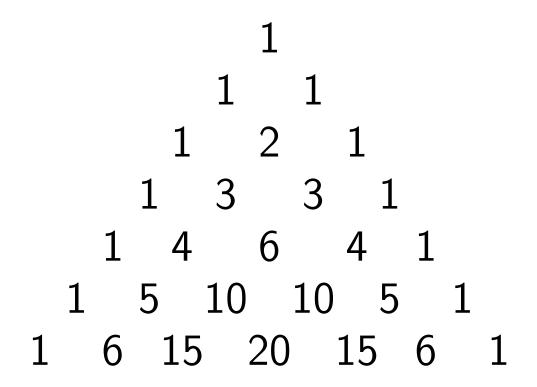
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5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

and shift each row slightly so that middle element is in middle





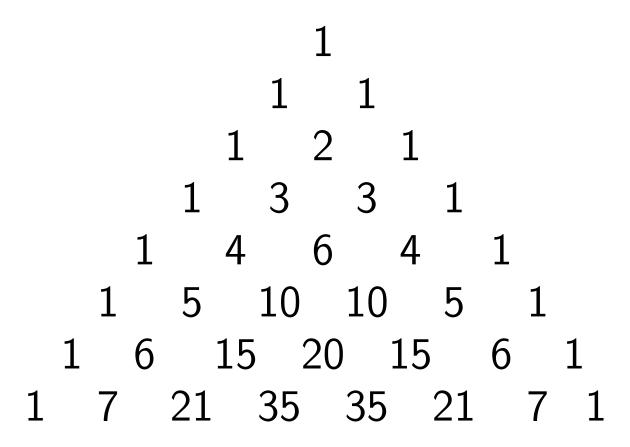


What is the next row in the table?



```
10 10
      15 20 15
1 7 21 35 35 21
```





#### **Pascal identity**

Each (non-1) entry in Pascal's

Triangle is the sum of
the two entries directly above it

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A purely *algebraic* proof (manipulating formulas) is possible.

We will use a combinatorial proof.



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$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Therefore, each term (left and right) represents the number of subsets of a particular size chosen from an appropriately sized set.



$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



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Number of k-subsets of an n-element set.



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Number of k-subsets of an n-element set.

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Try to use sum principle to explain relationship among these three terms.

Example: 
$$n = 5$$
,  $k = 2$ 

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$



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Consider  $S = \{A, B, C, D, E\}$ .



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Set  $S_1$  of 2-subsets of S

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}\}.$$



$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

Consider  $S = \{A, B, C, D, E\}$ .

Set  $S_1$  of 2-subsets of S can be partitioned into 2 disjoint parts.

 $S_2$  the 2-subsets that contain E and

 $S_3$ , the set of 2-subsets that do not contain E.

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}\}.$$



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If n and k are integers satisfying 0 < k < n, then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$



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**Proof:** Apply sum rule.

Let  $S_1$  be set of all k-element subsets.

To apply sum rule, partition  $S_1$  into  $S_2$  and  $S_3$ .

Let  $S_2$  be set of k-element subsets that contain  $x_n$ .

Let  $S_3$  be set of k-element subsets that don't contain  $x_n$ 



#### Blaise Pascal

Born 1623; Died 1662

French Mathematician

A Founder of Probability Theory

Inventor of one of the first mechanical calculating machines

Pascal Programming Language named for him





$$(x+y) = \binom{1}{0}x + \binom{1}{1}y$$



$$(x+y) = \binom{1}{0}x + \binom{1}{1}y$$

$$(x+y)^2 = x^2 + 2xy + y^2 = {2 \choose 0}x^2 + {2 \choose 1}x^1y^1 + {2 \choose 2}y^2$$



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$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$= {3 \choose 0}x^3 + {3 \choose 1}x^2y + {3 \choose 2}xy^2 + {3 \choose 3}y^3$$



Number of k-element subsets of an n-element set is called a binomial coefficient because of its role in the algebraic expansion of a binomial  $(x + y)^n$ .



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**The Binomial Theorem** For any integer  $n \geq 0$ ,

$$(x+y)^{n} = \binom{n}{0}x^{n} + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^{2} + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^{n}$$



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$$(x+y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i.$$



#### The Binomial Theorem

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$$(x+y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i.$$

#### **Proof?**



## Application of the Binomial Theorem

We may use the Binomial Theorem to prove

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$



Suppose we have k labels of one kind, e.g., red and n-k labels of another, e.g., blue. In how many different ways can we apply these labels to n objects?



Suppose we have k labels of one kind, e.g., red and n-k labels of another, e.g., blue. In how many different ways can we apply these labels to n objects?

Show that if we have  $k_1$  labels of one kind, e.g., red,  $k_2$  labels of a second kind, e.g., blue, and  $k_3 = n - k_1 - k_2$  labels of a third kind, then there are  $\frac{n!}{k_1!k_2!k_3!}$  ways to apply these labels to n objects



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Show that if we have  $k_1$  labels of one kind, e.g., red,  $k_2$  labels of a second kind, e.g., blue, and  $k_3 = n - k_1 - k_2$  labels of a third kind, then there are  $\frac{n!}{k_1!k_2!k_3!}$  ways to apply these labels to n objects

What is the coefficient of  $x^{k_1}y^{k_2}z^{k_3}$  in  $(x+y+z)^n$ ?



There are  $\binom{n}{k_1}$  ways to choose the red items There are then  $\binom{n-k_1}{k_2}$  ways to choose the blue items from the remaining  $n-k_1$ . The remaining  $k_3$  items get labelled a third color.



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Using the *product rule* the total number of labellings is

$$\binom{n}{k_1} \binom{n-k_1}{k_2} = \frac{n!}{k_1!(n-k_1)!} \frac{(n-k_1)!}{(k_2)!(n-k_1-k_2)!}$$

$$= \frac{n!}{k_1!k_2!(n-k_1-k_2)!} = \frac{n!}{k_1!k_2!k_3!}$$



• When  $k_1 + k_2 + k_3 = n$ , we call

$$\frac{n!}{k_1!k_2!k_3!}$$

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What is the coefficient of  $x^{k_1}y^{k_2}z^{k_3}$  in  $(x+y+z)^n$ ?



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This will be very similar to the analysis of hashing *n* keys into a table of size 365.



 $\blacksquare$   $A_n$  – "there are n students in a room and at least two of them share a birthday."

Sample space:  $|S| = 365^n$ 



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$$\#A_n + \#B_n = 365^n$$



n	$A_{n}$	$B_n$	n	$A_n$	$B_n$
1	0.00000000	1.00000000	16	0.28360400	0.71639599
2	0.00273972	0.99726027	17	0.31500766	0.68499233
3	0.00820416	0.99179583	18	0.34691141	0.65308858
4	0.01635591	0.98364408	19	0.37911852	0.62088147
5	0.02713557	0.97286442	20	0.41143838	0.58856161
6	0.04046248	0.95953751	21	0.44368833	0.55631166
7	0.05623570	0.94376429	22	0.47569530	0.52430469
8	0.07433529	0.92566470	23	0.50729723	0.49270276
9	0.09462383	0.90537616	24	0.53834425	0.46165574
10	0.11694817	0.88305182	25	0.56869970	0.43130029
11	0.14114137	0.85885862	26	0.59824082	0.40175917
12	0.16702478	0.83297521	27	0.62685928	0.37314071
13	0.19441027	0.80558972	28	0.65446147	0.34553852
14	0.22310251	0.77689748	29	0.68096853	0.31903146
15	0.25290131	0.74709868	30	0.70631624	0.29368375
			I		or OF SO

Event A: at least two people in the room have the same birthday
Event B: no two people in the room have the same birthday

$$Pr[A] = 1 - Pr[B]$$

$$\Pr[B] = \left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdot \dots \cdot \left(1 - \frac{n-1}{365}\right)$$
$$= \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right).$$

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$$p(n; H) := 1 - \prod_{i=1}^{n-1} (1 - \frac{i}{H})$$



Since  $e^x = 1 + x + \frac{x^2}{2!} + \cdots$ , for  $|x| \ll 1$ ,  $e^x \approx 1 + x$ 



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Recall that 
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This probability can be approximated as

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Let n(p; H) be the smallest number of values we have to choose, such that the probability for finding a collision is at least p. By inverting the expression above, we have

$$n(p; H) \approx \sqrt{2H \ln \frac{1}{1-p}}.$$



The Euclidean algorithm in pseudocode

#### ALGORITHM 1 The Euclidean Algorithm.

```
procedure gcd(a, b): positive integers)

x := a

y := b

while y \neq 0

r := x \mod y

x := y

y := r

return x\{\gcd(a, b) \text{ is } x\}
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The number of divisions required to find gcd(a, b) is  $O(\log b)$ , where  $a \ge b$ . (this will be proved later.)



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#### Why?



Key steps in the Euclidean algorithm

```
egin{array}{lll} r_0 &= r_1 q_1 + r_2 & 0 \leq r_2 < r_1, \\ r_1 &= r_2 q_2 + r_3 & 0 \leq r_3 < r_2, \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ r_{n-2} &= r_{n-1} q_{n-1} + r_n & 0 \leq r_n < r_{n-1}, \\ r_{n-1} &= r_n q_n \ . \end{array}
```

Key steps in the Euclidean algorithm

```
r_0 = r_1q_1 + r_2 0 \le r_2 < r_1, r_1 = r_2q_2 + r_3 0 \le r_3 < r_2, 0 \le r_3 < r_3, 0 \le r_3 < r_2, 0 \le r_3 < r_3, 0 \le r_3 < r_3
```

#### **Observation:**

$$r_{i+2} = r_i \mod r_{i+1}$$

Key steps in the Euclidean algorithm

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We claim that  $r_{i+2} < \frac{1}{2}r_i$ 

Key steps in the Euclidean algorithm

$$r_0 = r_1q_1 + r_2$$
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Case (i): 
$$r_{i+1} \leq \frac{1}{2}r_i$$
:  $r_{i+2} < r_{i+1} \leq \frac{1}{2}r_i$ .

Case (ii): 
$$r_{i+1} > \frac{1}{2}r_i$$
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#### Next Lecture

solving linear recurrence ...

