CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Cartesian Product

- Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, the *Cartesian product* $A \times B$ is the set of pairs

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Cartesian product defines a set of all ordered arrangements of elements in the two sets.

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  Example: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$

  ◇ Is $R = \{(a, 1), (b, 2), (c, 2)\}$ a relation from A to B?

  ◇ Is $Q = \{(1, a), (2, b)\}$ a relation from A to B?

  ◇ Is $P = \{(a, a), (b, c), (b, a)\}$ a relation from A to A?

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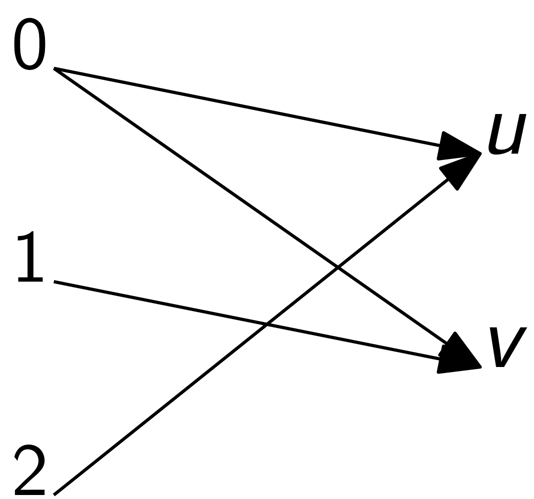

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R	u	v
0	×	×
1	×	
2		×

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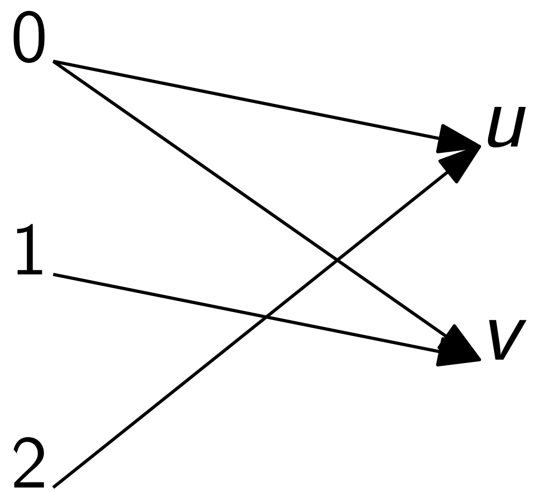

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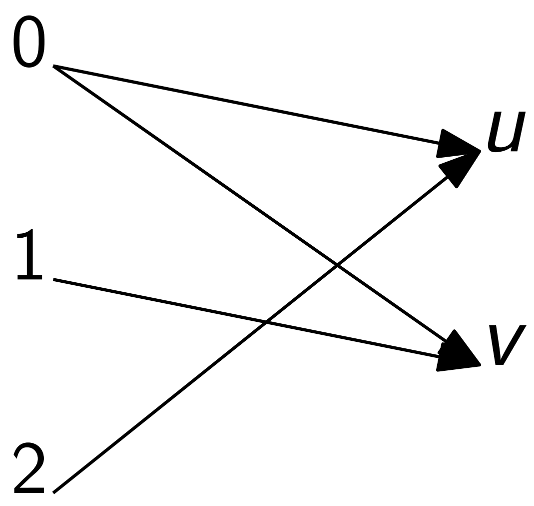

What is the difference between a relation and a function from A to B?

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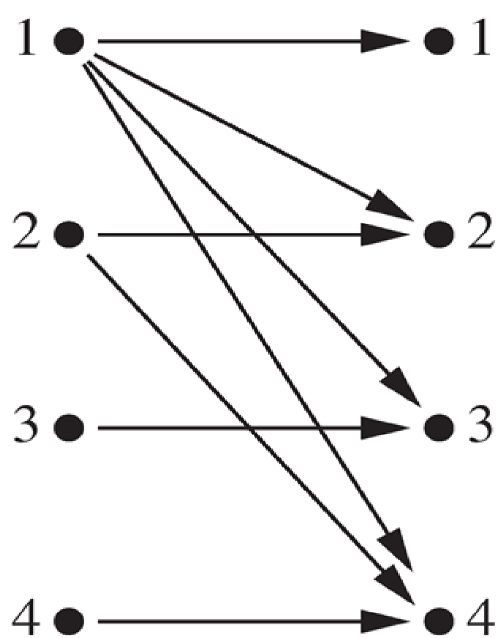

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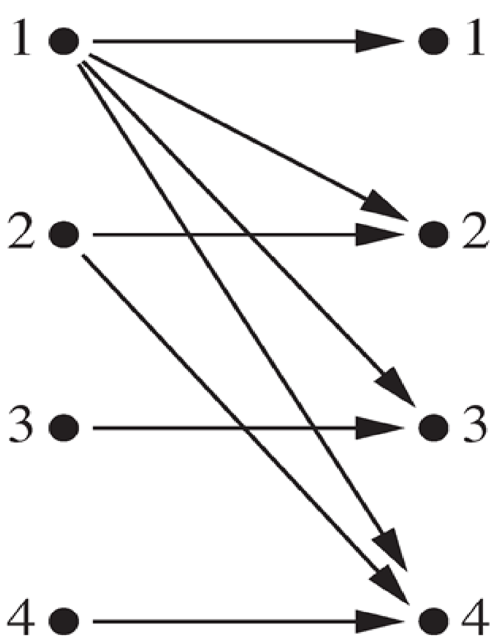

R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Number of Binary Relations

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  The number of subsets of a set with k elements is 2^k

Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

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  $R_{div} = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$.

$$MR_{div} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

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A relation R is reflexive if and only if MR has 1 in every position on its main diagonal.

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  No. $(1,1) \notin R$

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  No. $(1,2),(2,1) \in R_{\neq}$ but $(1,1) \notin R_{\neq}$.

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Combining Relations

- **Definition**: Let A and B be two sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

  Let $R \subseteq A \times B$ denote R is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

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  Set operations: union, intersection, difference, etc.

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  We may also combine relations by matrix operations.

Composite of Relations

- **Definition**: Let R be a relation from a set A to a set B and S be a relation from B to C. The *composite of R and S* is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

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"only if" part: by induction.

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  How many subsets on $n(n-1)$ elements are there?

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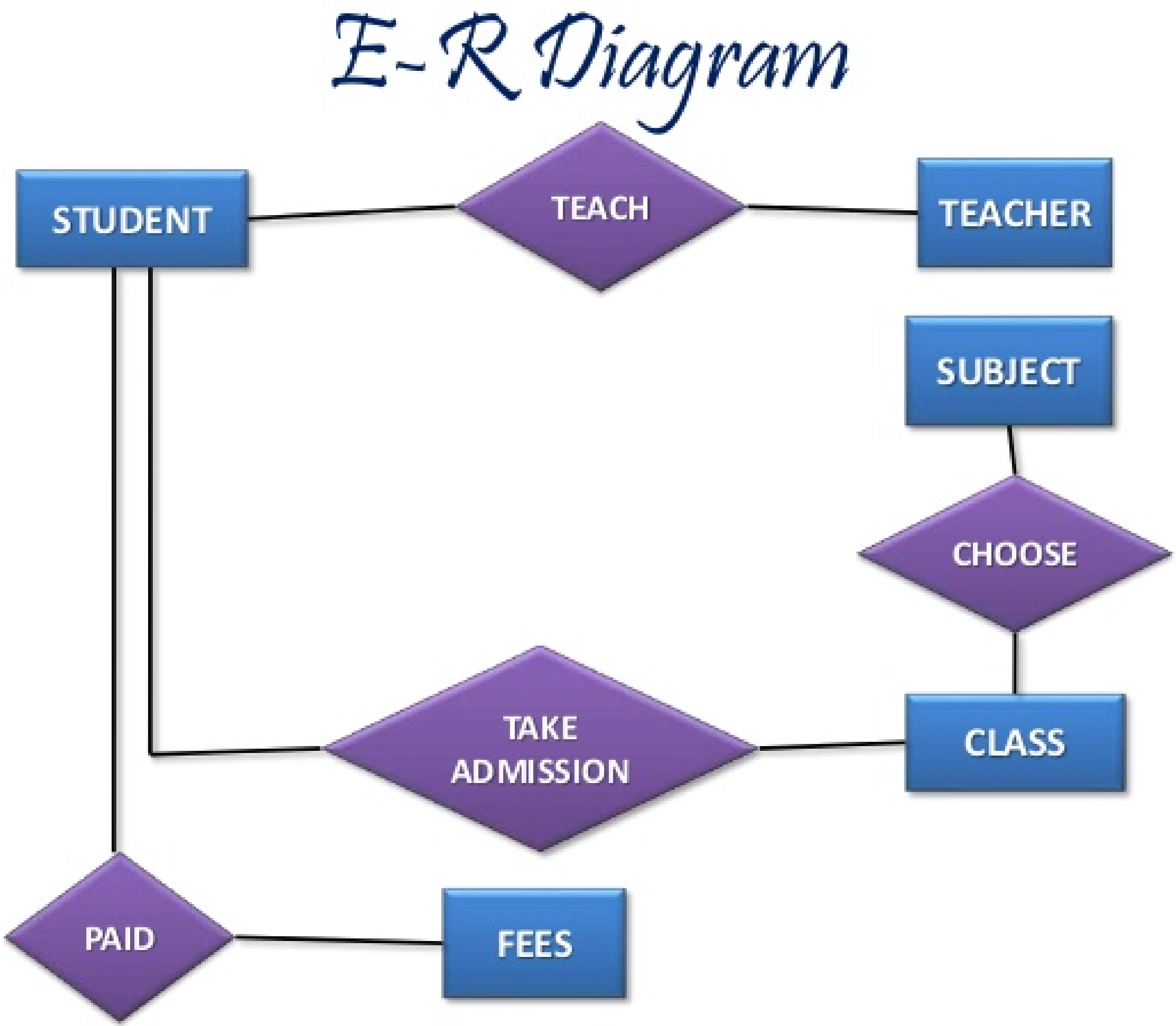
- A domain A_i is a *primary key* for the database if the relation R is functional in A_i.
- A *composite key* for the database is a set of domains $\{A_i, A_j, \cdots\}$ such that R contains at most 1 n-tuple $(\cdots, a_i, \cdots, a_j, \cdots)$ for each composite value $(a_i, a_j, \cdots) \in A_i \times A_j \times \cdots$.

Relational Databases



Selection Operators

- Let A be any *n-ary domain* $A = A_1 \times \cdots \times A_n$, and let $C : A \to \{T, F\}$ be any *condition* (predicate) on elements (n-tuples) of A.

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  – $\forall R \subseteq A$,

$$\begin{aligned} s_C(R) &= R \cap \{a \in A \mid s_C(a) = T\} \\ &= \{a \in R \mid s_C(a) = T\}. \end{aligned}$$

Selection Operator Example

- Suppose that we have a domain

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- Then, $s_{UpperLevel}$ is the selection operator that takes any relation R on A (database of students) and produces a relation consisting of just the upper-level classes (juniors and seniors).

Projection Operators

- Let $A = A_1 \times \cdots \times A_n$ be any n-ary domain, and let $\{i_k\} = (i_1, \ldots, i_m)$ be a sequence of indices all falling in the range 1 to n.

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- Then the *projection operator* on n-tuples

  $$P_{\{i_k\}} : A \to A_{i_1} \times \cdots \times A_{i_m}$$

  is defined by

  $$P_{\{i_k\}}(a_1, \cdots, a_n) = (a_{i_1}, \cdots, a_{i_m})$$

Projection Example

- Suppose that we have a tenary domain

  $\color{blue}{Cars = Model \times Year \times Color}$ $(n = 3)$

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- This operator can be usefully applied to a whole relation $R \subseteq Cars$ (database of cars) to obtain a list of $model/color$ combinations available.

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- A, B, C can also be sequences of elements rather that single elements.

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- Suppose that R_1 is a teaching assignment table, relating *Professors* to *Courses*.

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- Then $J(R_1, R_2)$ is like your class schedule, listing $(professor, course, room, time)$.

Next Lecture

- relation II …