CS201: Discrete Math for Computer Science
2020 Fall Semester Written Assignment # 5
Due: Dec. 22nd, 2020, please submit at the beginning of class

Q.1 Let S be the set of all strings of English letters. Determine whether these relations are *reflexive, irreflexive, symmetric, antisymmetric*, and/or *transitive*.

(1) $R_1 = \{(a, b)|a$ and b have no letters in common$\}$

(2) $R_2 = \{(a, b)|a$ and b are not the same length$\}$

(3) $R_3 = \{(a, b)|a$ is longer than $b\}$

Q.2 Show that a subset of an *antisymmetric* relation is also *antisymmetric*.

Q.3 How many relations are there on a set with n elements that are

(a) symmetric?

(b) antisymmetric?

(c) irreflexive?

(d) both reflexive and symmetric?

(e) neither reflexive nor irreflexive?

(f) both reflexive and antisymmetric?

(g) symmetric, antisymmetric and transitive?

Q.4 Prove or give a counterexample to the following: For a set A and a binary relation R on A, if R is reflexive and symmetric, then R must be transitive as well.

Q.5 Give an examples of a relation R such that its transitive closure R^* satisfies $R^* = R \cup R^2 \cup R^3$, but $R^* \neq R \cup R^2$.

Q.6 Let R be a reflexive relation on a set A. Show that $R \subseteq R^2$.

Q.7 Suppose that R_1 and R_2 are both *reflexive* relations on a set A.

(1) Show that $R_1 \oplus R_2$ is *irreflexive*.

(2) Is $R_1 \cap R_2$ also *reflexive*? Explain your answer.

(3) Is $R_1 \cup R_2$ also *reflexive*? Explain your answer.

Q.8 Suppose that R is a *symmetric* relation on a set A. Is $\overline{R}$ also symmetric? Explain your answer.

Q.9

(1) Give an example to show that the transitive closure of the symmetric closure of a relation is not necessarily the same as the symmetric closure of the transitive closure of this relation.

(2) Show that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

Q.10 Show that the relation R on $\mathbb{Z} \times \mathbb{Z}$ defined on $(a, b)\mathbb{R}(c, d)$ if and only if $a + d = b + c$ is an *equivalence* relation.

Q.11 Which of the following are equivalence relations on the set of all people?

(1) $\{(x, y)|x$ and y have the same sign of the zodiac$\}$

(2) $\{(x, y)|x$ and y were born in the smae year$\}$

(3) $\{(x, y)|x$ and y have been in the same city$\}$

Q.12 How many different equivalence relations with exactly three different equivalence classes are there on a set with five elements?

Q.13 Show that $\{(x, y)|x - y \in \mathbb{Q}\}$ is an equivalence relation on the set of real numbers, where $\mathbb{Q}$ denotes the set of rational numbers. What are $[1]$, $[\frac{1}{2}]$, and $[\pi]$?

Q.14 Which of these are posets?

(a) $(\mathbf{R}, =)$

(b) $(\mathbf{R}, <)$

(c) $(\mathbf{R}, \leq)$

(d) $(\mathbf{R}, \neq)$

Q.15 Consider a relation $\propto$ on the set of functions from $\mathbb{N}^+$ to $\mathbb{R}$, such that $f \propto g$ if and only if $f = O(g)$.

(a) Is $\propto$ an equivalence relation?

(b) Is $\propto$ a partial ordering?

(c) Is $\propto$ a total ordering?

Q.16 Let $\mathbf{R}(S)$ be the set of all relations on a set S. Define the relation $\preceq$ on $\mathbf{R}(S)$ by $R_1 \preceq R_2$ if $R_1 \subseteq R_2$, where R_1 and R_2 are relations on S. Show that $\mathbf{R}(s), \preceq)$ is a poset.

Q.17 For two positive integers, we write $m \preceq n$ if the sum of the (distinct) prime factors of the first is less than or equal to the product of the (distinct) prime factors of the second. For instance $75 \preceq 14$, because $3 + 5 \leq 2 \cdot 7$.

(a) Is this relation reflexive? Explain.

(b) Is this relation antisymmetric? Explain.

(c) Is this relation transitive? Explain.

Q.18 Answer these questions for the partial order represented by this Hasse diagram.

(a) Find the maximal elements.

(b) Find the minimal elements.

(c) Is there a greatest element?

(d) Is there a least element?

(e) Find all upper bounds of $\{a, b, c\}$.

(f) Find the least upper bound of $\{a, b, c\}$, if it exists.

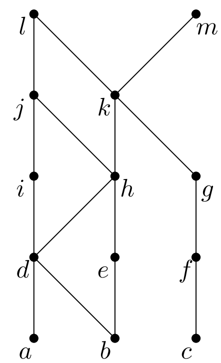

Figure 1: Q.18

(g) Find all lower bounds of $\{f, g, h\}$.

(h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.

Q.19 Let G be a simple graph. Show that the relation R on the set of vertices of G such that rRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G.

Q.20 A simple graph G is called *self-complementary* if G and $\overline{G}$ are isomorphic. Show that if G is a self-complementary simple graph with v vertices, then $v \equiv 0$ or $1 \pmod 4$.

Q.21 Let G be a *simple* graph with n vertices.

(a) What is the *maximum* number of edges G can have?

(b) If G is connected, what is the *minimum* number of edges G can have?

(c) Show that if the minimum degree of any vertex of G is greater than or equal to $(n-1)/2$, then G must be connected.

Q.22 Let $n \geq 5$ be an integer. Consider the graph G_n whose vertices are the sets $\{a, b\}$, where $a, b \in \{1, \ldots, n\}$ and $a \neq b$, and whose adjacency rule is *disjointness*, that is, $\{a, b\}$ is adjacent to $\{a', b'\}$ whenever $\{a, b\} \cap \{a', b'\} = \emptyset$.

(a) Draw G_5.

(b) Find the degree of each vertex in G_n.

Q.23 Let $G = (V, E)$ be a graph on n vertices. Construct a new graph, $G' = (V', E')$ as follows: the vertices of G' are the edges of G (i.e., $V' = E$), and two distinct edges in G are adjacent in G' if they share an endpoint.

(a) Draw G' for $G = K_4$.

(b) Find a formula for the number of edges of G' in terms of the vertex degrees of G.

Q.24 Let $G = (V, E)$ be an undirected graph and let $A \subseteq V$ and $B \subseteq V$. Show that

(1) $N(A \cup B) = N(A) \cup N(B)$.

(2) $N(A \cap B) \subseteq N(A) \cap N(B)$, and give an example where $N(A \cap B) \neq N(A) \cap N(B)$.

Q.25 Let G be a connected graph, with the vertex set V. The *distance* between two vertices u and v, denoted by $dist(u, v)$, is defined as the *minimal* length of a path from u to v. Show that $dist(u, v)$ is a metric, i.e., the following properties hold for any $u, v, w \in V$:

(i) $dist(u, v) \geq 0$ and $dist(u, v) = 0$ if and only if $u = v$.

(ii) $dist(u, v) = dist(v, u)$.

(iii) $dist(u, v) \leq dist(u, w) + dist(w, v)$.

Q.26 Show that if G is bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

Q.27 Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.

Q.28 Show that isomorphism of simple graphs is an equivalence relation.

Q.29 Suppose that G_1 and H_1 are isomorphic and that G_1 and H_2 are isomorphic. Prove or disprove that $G_1 \cup G_2$ and $H_1 \cup H_2$ are isomorphic.

Q.30 How can the adjacency matrix of $\overline{G}$ be found from the adjacency matrix of G, where G is a simple graph?

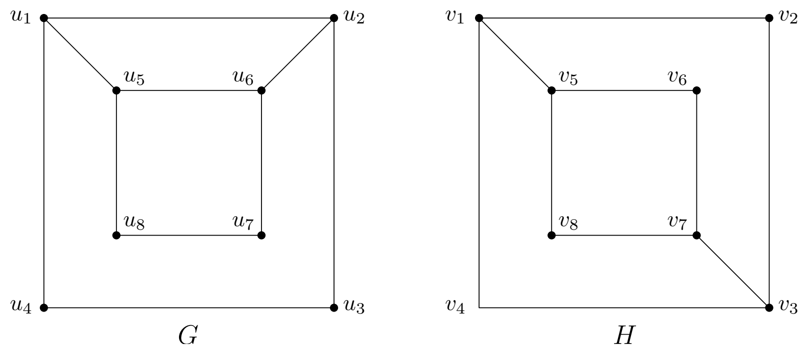

Figure 2: Q.27

Q.31 Show that if G is simple graph with at least 11 vertices, then either G or $\overline{G}$, the complement of G, is nonplanar.

Q.32 Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)v - (10/3)$ if $v \geq 4$.

Q.33 The **distance** between two distinct vertices v_1 and v_2 of a connected simple graph is the length (number of edges) of the shortest path between v_1 and v_2. The **radius** of a graph is the *minimum* over all vertices v of the maximum distance from v to another vertex. The **diameter** of a graph is the maximum distance between two distinct vertices. Find the radius and diameter of

(1) K_6

(2) $K_{4,5}$

(3) Q_3

(4) C_6

Q.34 Let n be a positive integer. Construct a **connected** graph with $2n$ vertices, such that there are *exactly* **two** vertices of degree i for each $i = 1, 2, \ldots, n$. (You can sketch some pictures, but your graph has to be described by a concise adjacency rule. Remember to prove that your graph is connected.)

Q.35

An n-cube is a cube in n dimensions, denoted by Q_n. The 1-cube, 2-cube, 3-cube are a line segment, a square, a normal cube, respectively, as shown below. In general, you can construct the $(n+1)$-cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels differing only in the first bit. Show that every n-cube has a Hamilton circuit.

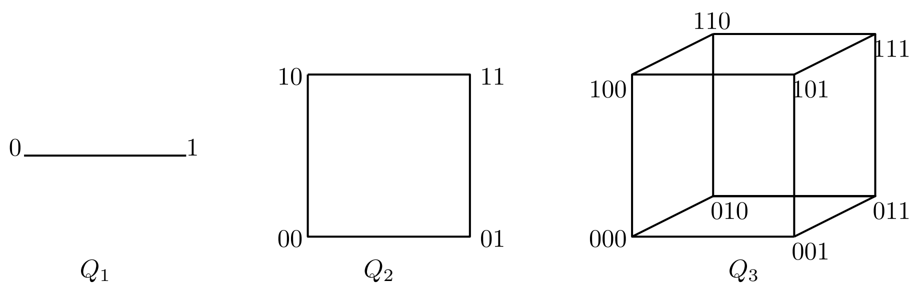

Figure 3: Q.35

Q.36 Consider the two graphs G and H. Answer the following three questions, and explain your answers.

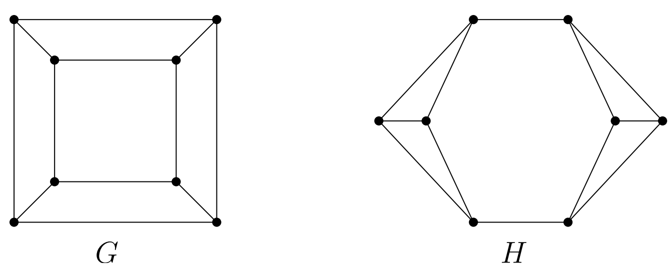

Figure 4: Q.36

(1) Which of the two graphs is/are *bipartite*?

(2) Are the two graphs *isomorphic* to each other?

(3) Which of the two graphs has/have an *Euler circuit*?

Q.37 There are 17 students who communicates with each other discussing problems in discrete math. They are only 3 possible problems, and each pair of students discuss one of these three 3 problems. Prove that there are at least 3 students who are all pairwise discussing the same problem.

Q.38

The **rooted Fibonacci trees** T_n are defined recursively in the following way. T_1 and T_2 are both the rooted tree consisting of a single vertex, and for $n = 3, 4, \ldots$, the rooted tree T_n is constructed from a root with T_{n-1} as its left subtree and T_{n-2} as its right subtree. How many vertices, leaves, and internal vertices does the rooted Fibonacci tree T_n have, where n is a positive integer? What is its height?

Q.39

What is the value of each of these postfix expressions?

(a) $5\ 2\ 1\ -\ -\ 3\ 1\ 4\ +\ +\ *$

(b) $9\ 3\ /\ 5\ +\ 7\ 2\ -\ *$

(c) $3\ 2\ *\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ *\ -$

Q.40

Use Prim's algorithm to find a minimum spanning tree for the given weighted graph.

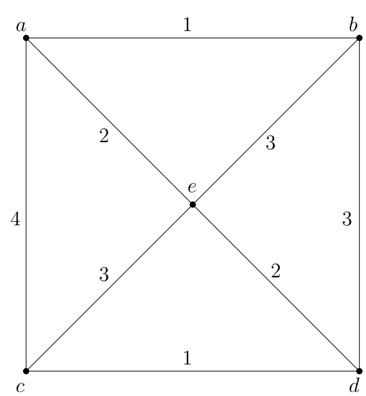

Figure 5: Q.40

Q.41

Use Kruskal's algorithm to find a minimum spanning tree for the weighted graph in Q.40.