CS201: Discrete Math for Computer Science
2020 Fall Semester Written Assignment #1
Due: Oct. 13th, 2020, please submit at the beginning of class

Q.1 Let p, q be the propositions
p: You get 100 marks on the final.
q: You get an A in this course.
Write these propositions using p and q and logical connectives (including negations).

(a) You do not get 100 marks on the final.

(b) You get 100 marks on the final, but you do not get an A in this course.

(c) You will get an A in this course if you get 100 marks on the final.

(d) If you do not get 100 marks on the final, then you will not get an A in this course.

(e) Getting 100 marks on the final is sufficient for getting an A in this course.

(f) You get an A in this course, but you do not get 100 marks on the final.

(g) Whenever you get an A in this course, you got 100 marks on the final.

Solution:

(a) $\neg p$

(b) $p \wedge \neg q$

(c) $p \to q$

(d) $\neg p \to \neg q$

(e) $p \to q$

(f) $q \wedge \neg p$

(g) $q \to p$

$\square$

Q.2 Construct a truth table for each of these compound propositions.

(a) $p \oplus \neg p$

(b) $\neg p \oplus \neg q$

(c) $(p \oplus q) \wedge (p \oplus \neg q)$

Solution:

(a)

p	$\neg p$	$(p \oplus \neg p)$
T	F	T
F	T	T

(b)

p	q	$\neg p$	$\neg q$	$\neg p \oplus \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	F

(c)

p	q	$p \oplus q$	$p \oplus \neg q$	$(p \oplus q) \wedge (p \oplus \neg q)$
T	T	F	T	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	F

$\square$

Q.3 Use truth tables to decide whether or not the following two propositions are equivalent.

(a) $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$

(b) $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$

(c) $(p \vee q) \to r$ and $(p \to r) \wedge (q \to r)$

(d) $(p \to \neg q) \leftrightarrow (r \to (p \vee \neg q))$ and $q \vee (\neg p \wedge \neg r)$

Solution:

(a) The combined truth table is:

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$p \wedge q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
F	F	T	T	T	T	F	T
F	T	F	T	F	F	F	F
T	F	F	F	T	F	F	F
T	T	T	F	F	F	T	T

By comparing the third and last columns, we have that they are equivalent.

(b) The truth table for $p \to (q \vee r)$ is :

p	q	r	$q \vee r$	$p \to (q \vee r)$
F	F	F	F	T
F	F	T	T	T
F	T	F	T	T
F	T	T	T	T
T	F	F	F	F
T	F	T	T	T
T	T	F	T	T
T	T	T	T	T

The truth table for $(p \to q) \vee (p \to r)$ is

p	q	r	$p \to q$	$p \to r$	$(p \to q) \vee (p \to r)$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	T	T
F	T	T	T	T	T
T	F	F	F	F	F
T	F	T	F	T	T
T	T	F	T	F	T
T	T	T	T	T	T

Since the final columns are the same in both truth tables, we know that these two propositions are equivalent.

(c) The truth table for $(p \vee q) \to r$ is :

p	q	r	$p \vee q$	$(p \vee q) \to r$
F	F	F	F	T
F	F	T	F	T
F	T	F	T	F
F	T	T	T	T
T	F	F	T	F
T	F	T	T	T
T	T	F	T	F
T	T	T	T	T

The truth table for $(p \to r) \wedge (q \to r)$ is

p	q	r	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	F	F
F	T	T	T	T	T
T	F	F	F	T	F
T	F	T	T	T	T
T	T	F	F	F	F
T	T	T	T	T	T

Since the final columns are the same in both truth tables, we know that these two propositions are equivalent.

(d) The two truth tables are:

p	q	r	$p \to \neg q$	$(p \vee \neg q)$	$r \to (p \vee \neg q)$	$(p \to \neg q) \leftrightarrow (r \to (p \vee \neg q))$
F	F	F	T	T	T	T
F	F	T	T	T	T	T
F	T	F	T	F	T	T
F	T	T	T	F	F	F
T	F	F	T	T	T	T
T	F	T	T	T	T	T
T	T	F	F	T	T	F
T	T	T	F	T	T	F

p	q	r	$\neg p \wedge \neg r$	$q \vee (\neg p \wedge \neg r)$
F	F	F	T	T
F	F	T	F	F
F	T	F	T	T
F	T	T	F	T
T	F	F	F	F
T	F	T	F	F
T	T	F	F	T
T	T	T	F	T

Since the final columns are not the same in both truth tables, we know that these two propositions are not equivalent.

$\square$

Q.4 Use logical equivalences to prove the following statements.

(a) $(p \wedge \neg q) \to r$ and $p \to (q \vee r)$ are equivalent.

(b) $(p \to q) \to ((r \to p) \to (r \to q))$ is a tautology.

Solution:

(a) We have

$$
\begin{aligned}
&(p \wedge \neg q) \to r \\
&\equiv \neg(p \wedge \neg q) \vee r \quad \text{Useful} \\
&\equiv (\neg p \vee q) \vee r \quad \text{De Morgan} \\
&\equiv \neg p \vee (q \vee r) \quad \text{Associative} \\
&\equiv p \to (q \vee r) \quad \text{Useful}
\end{aligned}
$$

Therefore, they are equivalent.

(b) We have

$$\begin{aligned}
&(p \to q) \to ((r \to p) \to (r \to q)) \\
&\equiv \neg(\neg p \vee q) \vee (\neg(\neg r \vee p) \vee (\neg r \vee q)) \quad \text{Useful} \\
&\equiv \neg(\neg p \vee q) \vee ((r \wedge \neg p) \vee (\neg r \vee q)) \quad \text{De Morgan} \\
&\equiv \neg(\neg p \vee q) \vee ((r \vee (\neg r \vee q)) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Distributive} \\
&\equiv \neg(\neg p \vee q) \vee (((r \vee \neg r) \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Associative} \\
&\equiv \neg(\neg p \vee q) \vee ((T \vee q) \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Complement} \\
&\equiv \neg(\neg p \vee q) \vee (T \wedge (\neg p \vee (\neg r \vee q))) \quad \text{Identity} \\
&\equiv \neg(\neg p \vee q) \vee (\neg p \vee (\neg r \vee q)) \quad \text{Identity} \\
&\equiv \neg(\neg p \vee q) \vee ((\neg p \vee q) \vee \neg r) \quad \text{Associative} \\
&\equiv (\neg(\neg p \vee q) \vee (\neg p \vee q)) \vee \neg r \quad \text{Associative} \\
&\equiv T \vee \neg r \quad \text{Complement} \\
&\equiv T \quad \text{Identity.}
\end{aligned}$$

Thus, it is a tautology.

□

Q.5

(a) Use a truth table to decide whether or not the following implication is a tautology:

$$(p \wedge (p \to q)) \to q.$$

(b) Using rules of logical equivalences, prove that

$$\neg p \to (q \to r) \equiv q \to (p \vee r).$$

Solution:

(a) The truth table for $(p \wedge (p \to q)) \to q$ is

p	q	$p \to q$	$p \wedge (p \to q)$	$(p \wedge (p \to q)) \to q$
F	F	T	F	T
F	T	T	F	T
T	F	F	F	T
T	T	T	T	T

Note that final column is all True because for it to be false we would need to have that $p \wedge (p \to q)$ is True and q is False. Therefore, it is a tautology.

(b) We have

$$
\begin{aligned}
&\neg p \to (q \to r \\
&\equiv \quad p \vee (q \to r) \quad \text{Definition} \\
&\equiv \quad p \vee (\neg q \vee r) \quad \text{Definition} \\
&\equiv \quad (p \vee \neg q) \vee r \quad \text{Associative} \\
&\equiv \quad (\neg q \vee p) \vee r \quad \text{Commutative} \\
&\equiv \quad \neg q \vee (p \vee r) \quad \text{Associative} \\
&\equiv \quad q \to (p \vee r) \quad \text{Useful}
\end{aligned}
$$

□

Q.6 Show that $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent.
Solution: It suffices to give a counterexample. When p, q and r are all false, $(p \to q) \to r$ is false, but $p \to (q \to r)$ is true.

□

Q.7 Explain, without using a truth table, why $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true, when p, q, and r have the same truth value and it is false otherwise.
Solution: The statement is true if and only if all the three clauses, $p \vee \neg q$, $q \vee \neg r$, and $r \vee \neg p$ are true. Suppose that p, q and r are all true, or all false, it is checked that each clause is true, and the statement is true. On the other hand, if one of the variables is true, and the other two false, then the clause containing the negation of that variable will be false, making the entire conjunction false. Similarly, if one of the variable is false and the other two true, then the clause containing that variable unnegated will be false, again making the entire statement false.

□

Q.8 Prove that if $p \wedge q$, $p \to \neg(q \wedge r)$, $s \to r$, then $\neg s$.
Solution:

(1)	$p \wedge q$	Premise
(2)	p	Simplication of (1)
(3)	$p \to \neg(q \wedge r)$	Premise
(4)	$\neg(q \wedge r)$	Modens ponens (2) (3)
(5)	$\neq q \vee \neg r$	De Morgan
(6)	q	Simplication of (1)
(7)	$\neg r$	Disjunctive syllogism (6) (7)
(8)	$s \to r$	Premise
(9)	$\neg s$	Modus tollens (7) (8)

$\square$

Q.9 Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++". Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

(a) There is a student at your school who can speak Russian and who knows C++.

(b) There is a student at your school who can speak Russian but who doesn't know C++.

(c) Every student at your school either can speak Russian or knows C++.

(d) No student at your school can speak Russian or knows C++.

Solution:

(a) $\exists x(P(x) \wedge Q(x))$

(b) $\exists x(P(x) \wedge \neg Q(x))$

(c) $\forall x(P(x) \vee Q(x))$

(d) $\forall x \neg(P(x) \vee Q(x))$

$\square$

Q.10 Let $F(x, y)$ be the statement "x can fool y", where the domain consists of all people in the world. Use quantifiers to express each of these statement.

(a) Everybody can fool Fred.

(b) Evelyn can fool everybody.

(c) Everybody can fool somebody.

(d) There is no one who can fool everybody.

(e) Everyone can be fooled by somebody.

(f) No one can fool both Fred and Jerry.

(g) Nancy can fool exactly two people.

(h) There is exactly one person whom everybody can fool.

(i) No one can fool himself or herself.

(j) There is someone who can fool exactly one person besides himself or herself.

Solution:

(a) $\forall x F(x, \text{Fred})$

(b) $\forall y F(\text{Evelyn}, y)$

(c) $\forall x \exists y F(x, y)$

(d) $\neg\exists x \forall y F(x, y)$

(e) $\forall y \exists x F(x, y)$

(f) $\neg\exists x(F(x, \text{Fred}) \land F(x, \text{Jerry}))$

(g) $\exists y_1 \exists y_2 (F(\text{Nancy}, y_1) \land F(\text{Nancy}, y_2) \land y_1 \neq y_2 \land \forall y (F(\text{Nancy}, y) \to (y = y_1 \lor y = y_2)))$

(h) $\exists y(\forall x F(x, y) \land \forall z(\forall x F(x, z) \to z = y))$

(i) $\neg\exists x F(x,x)$

(j) $\exists x\exists y(x \neq y \wedge F(x,y) \wedge \forall z((F(x,z) \wedge z \neq x) \rightarrow z = y))$

$\square$

Q.11 Express the negations of each of these statements so that all negation symbols immediately precede predicates.

(a) $\exists x\exists y(Q(x,y) \leftrightarrow Q(y,x))$

(b) $\forall y\exists x\exists z(T(x,y,z) \vee Q(x,y))$

(c) $\forall x\exists y\forall z T(x,y,z)$

(d) $\forall x\exists y(P(x,y) \rightarrow Q(x,y))$

Solution:

(a)

$$\begin{aligned}
\neg\exists x\exists y(Q(x,y) \leftrightarrow Q(y,x)) &\equiv \forall x\neg\exists y(Q(x,y) \leftrightarrow Q(y,x)) \\
&\equiv \forall x\forall y\neg(Q(x,y) \leftrightarrow Q(y,x)) \\
&\equiv \forall x\forall y(\neg Q(x,y) \leftrightarrow Q(y,x))
\end{aligned}$$

(b)

$$\begin{aligned}
\neg\forall y\exists x\exists z(T(x,y,z) \vee Q(x,y)) &\equiv \exists y\neg\exists x\exists z(T(x,y,z) \vee Q(x,y)) \\
&\equiv \exists y\forall x\neg\exists z(T(x,y,z) \vee Q(x,y)) \\
&\equiv \exists y\forall x\forall z\neg(T(x,y,z) \vee Q(x,y)) \\
&\equiv \exists y\forall x\forall z(\neg T(x,y,z) \wedge \neg Q(x,y))
\end{aligned}$$

(c)

$$\begin{aligned}
\neg\forall x\exists y\forall z T(x,y,z) &\equiv \exists x\neg\exists y\forall z T(x,y,z) \\
&\equiv \exists x\forall y\neg\forall y T(x,y,z) \\
&\equiv \exists x\forall y\exists y\neg T(x,y,z).
\end{aligned}$$

(d)

$$\begin{aligned}\neg\forall x\exists y(P(x,y)\to Q(x,y)) &\equiv \exists x\neg\exists y(P(x,y)\to Q(x,y))\\ &\equiv \exists x\forall y\neg(\neg P(x,y)\vee Q(x,y))\\ &\equiv \exists x\forall y(P(x,y)\wedge\neg Q(x,y)).\end{aligned}$$

□

Q.12

(a) Give the negation of the statement

$$\forall n\in\mathbb{N}\ (n^3+6n+5 \text{ is odd} \Rightarrow n \text{ is even}).$$

(b) Either the original statement in (a) or its negation is true. Which one is it and explain why?

Solution:

(a) The negation is

$$\exists x\in\mathbb{N}\ (n^3+6n+5 \text{ is odd} \wedge n \text{ is odd}).$$

(b) If n is odd then n^3+6n+5 is even because n^3 is then odd and $6n$ is then even. Therefore, the original statement is true.

□

Q.13 For the following argument, explain which rules of inference are used for each step.

"Each of the 93 students in this class owns a personal computer. Everyone who owns a personal computer can use a word processing program. Therefore, Zeke, a student in this class, can use a word processing program."

Solution: Let $c(x)$ be "x is in this class," $p(x)$ be "x owns a PC", and $w(x)$ be "x can use a word-processing program". The premises are $c(Zeke)$, $\forall x(c(x)\to p(x))$, and $\forall x(p(x)\to w(x))$. Using the second premise and universal instantiation and modus ponens, $c(Zeke)\to p(Zeke)$ follows. Using the first premise and modus ponens, $p(Zeke)$ follows. Using the third premise and universal instantiation, $p(Zeke)\to w(Zeke)$ follows. Finally, using modus ponens, $w(Zeke)$, the desired conclusion follows.

$\square$

Q.14 Prove the **triangle inequality**, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$.

Solution: We prove by four cases.

Case 1: $x \geq 0$ and $y \geq 0$. Then $|x| + |y| = x + y = |x + y|$.

Case 2: $x < 0$ and $y < 0$. Then $|x| + |y| = -x + (-y) = -(x + y) = |x + y|$.

Case 3: $x \geq 0$ and $y < 0$. Then $|x| + |y| = x + (-y)$. If $x \geq -y$, then $|x + y| = x + y$. But because $y < 0$, $-y > y$, so $|x| + |y| = x + (-y) > x + y = |x + y|$. If $x < -y$, then $|x + y| = -(x + y)$. But because $x < 0$, $x \geq -x$, so $|x| + |y| = x + (-y) \geq -x + (-y) = |x + y|$.

Case 4: $x < 0$ and $y \geq 0$. Similar to Case 3.

$\square$

Q.15 Prove or disprove that there is a rational number x and an irrational number y such that x^y is irrational.

Solution: Let $x = 2$ and $y = \sqrt{2}$. If $x^y = 2^{\sqrt{2}}$ is irrational, we are done. If not, let $x = 2^{\sqrt{2}}$ and $y = \sqrt{2}/4$. Then $x^y = (2^{\sqrt{2}})^{\sqrt{2}/4} = 2^{\sqrt{2}\cdot(\sqrt{2})/4} = \sqrt{2}$.

$\square$

Q.16 Prove or disprove that if a and b are rational numbers, then a^b is also rational. .

Solution: Take $a = 2$ and $b = 1/2$. Then $a^b = 2^{1/2} = \sqrt{2}$, and this number is not rational.

$\square$

Q.17 Prove that $\sqrt[3]{2}$ is irrational.

Solution: Suppose that $\sqrt[3]{2}$ is the rational number p/q, where p and q are positive integers with no common factors. Cubing both sides, we have $2 = p^3/q^3$, or $p^3 = 2q^3$. Thus p^3 is even. Since the product of odd number is odd, this means that p is even, so we can write $p = 2k$ for some integer k. We then have $q^3 = 4k^3$. Since q^3 is even, q must be even. We have now seen that both p and q are even, a contradiction.

$\square$

Q.18 Give a direct proof that: Let a and b be integers. If $a^2 + b^2$ is even, then $a + b$ is even.

Solution: Observe that $a^2 + b^2 = (a+b)^2 - 2ab$. Thus, $(a+b)^2$ has the same parity as $a^2 + b^2$. So $(a + b)^2$ is even. Then $a + b$ is also even.

$\square$