南方科技大學
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Irreflexive Relation: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

  Symmetric Relation: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

  Antisymmetric Relation: A relation R on a set A is called *antisymmetric* if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

  Transitive Relation: A relation R on a set A is called *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

Connectivity

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

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Theorem: The transitive closure of a relation R equals the connectivity relation R^*.

Recall Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

Equivalence Relation

- **Definition** A relation R on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

- **Definition** Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a, denoted by $[a]_R$. When only one relation is considered, we use the notation $[a]$.

$$[a]_R = \{b : \ (a, b) \in R\}$$

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Theorem Let $\{A_1, A_2, \ldots, A_i, \ldots\}$ be a partition of S. Then there is an equivalence relation R on S, that has the sets A_i as its equivalence classes.

Partial Ordering

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  $2, 4$ are comparable, $3, 5$ are incomparable.

Total Ordering

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Lexicographic Ordering

- **Definition** Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), i.e., $(a_1, a_2) \prec (b_1, b_2)$, either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ then $a_2 \prec_2 b_2$.

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  - ◇ $discreet \prec discrete$
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Hasse Diagram

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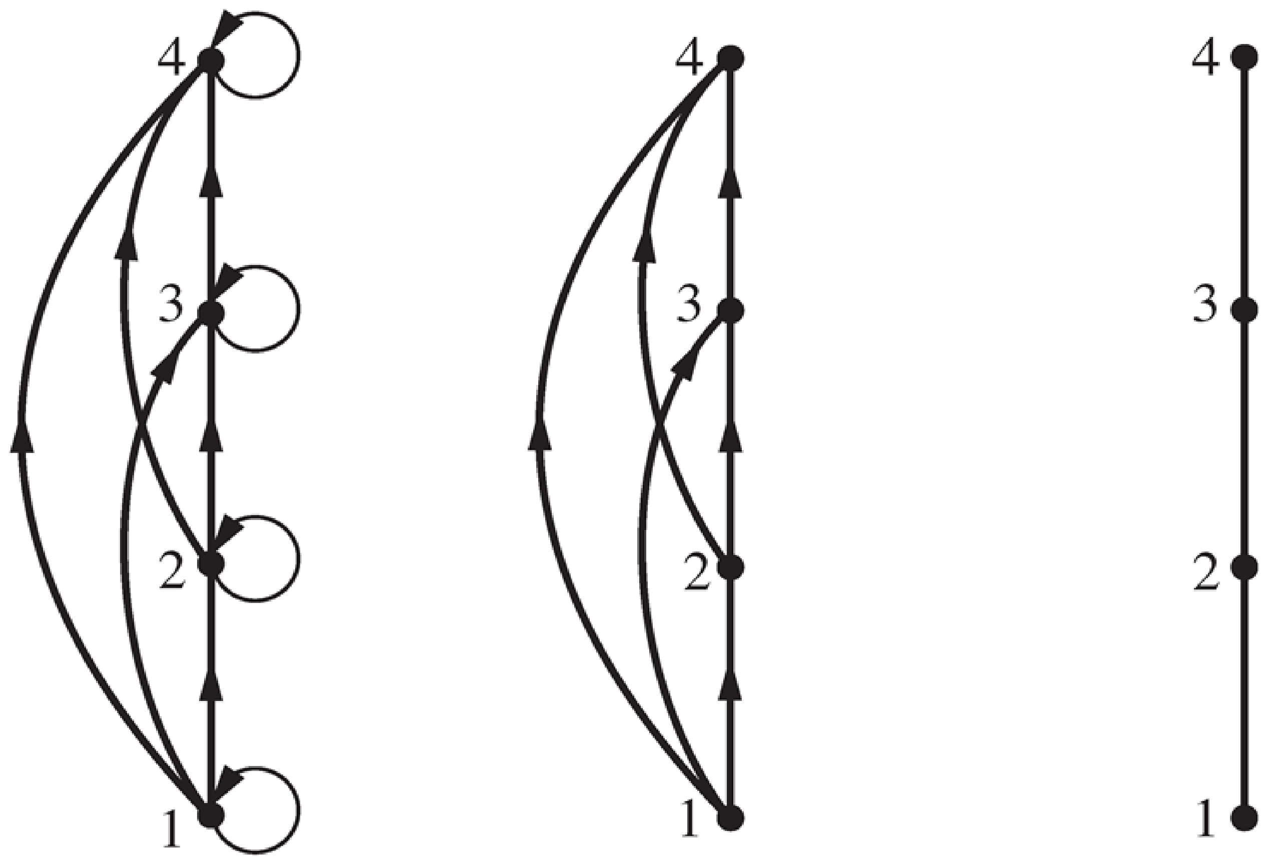
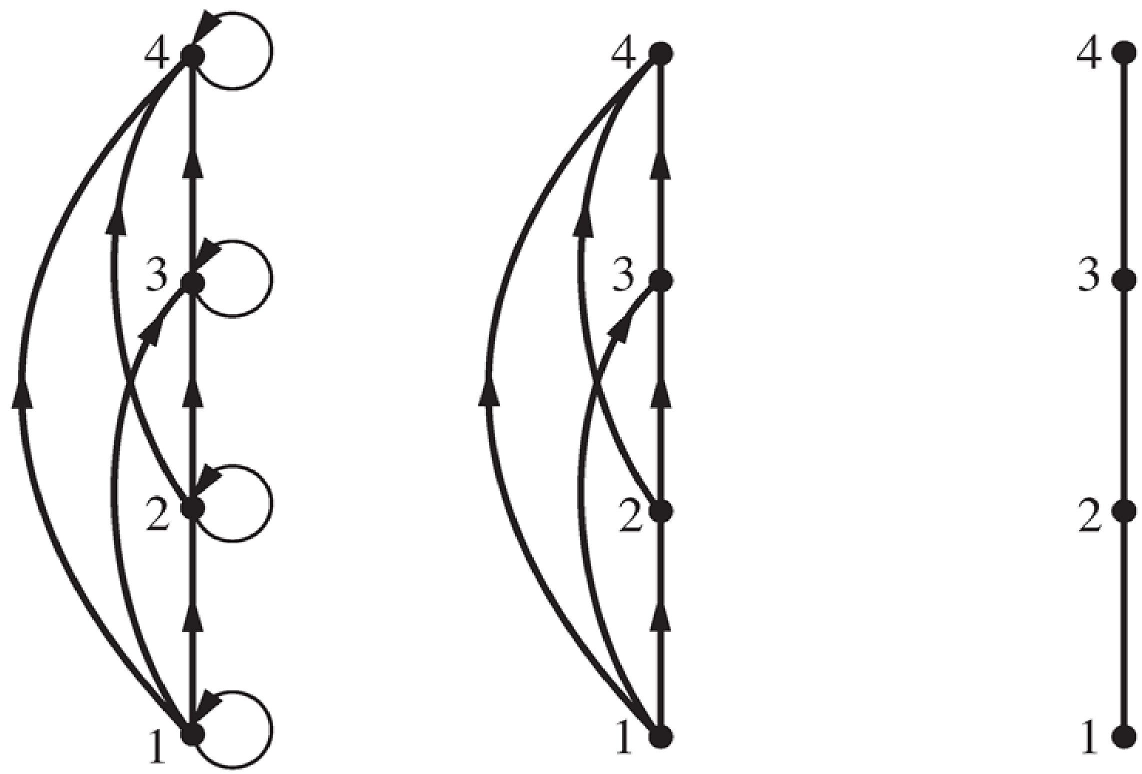


Hasse Diagram

- (a) A partial ordering. The loops are due to the reflexive property

  (b) The edges that must be present due to the transitive property are deleted

  (c) The Hasse diagram for the partial ordering (a)



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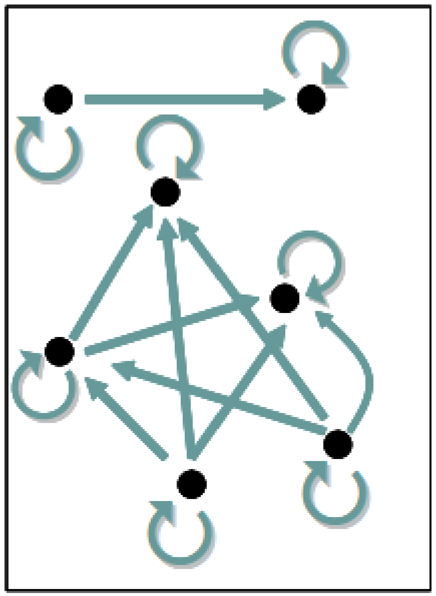
  - $\diamond$ Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

Hasse Diagram Example



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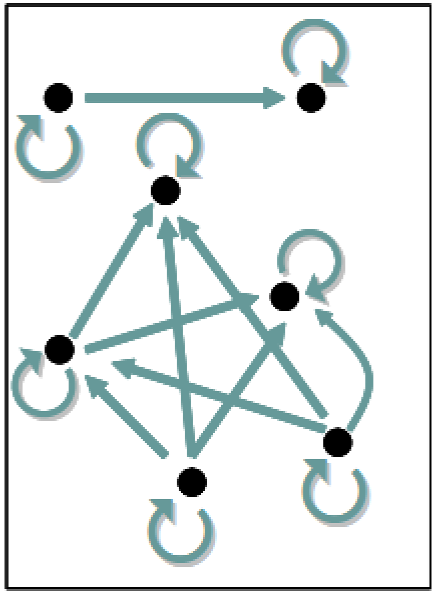

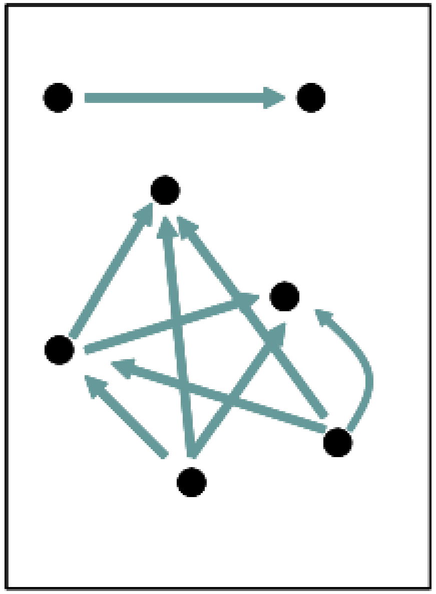

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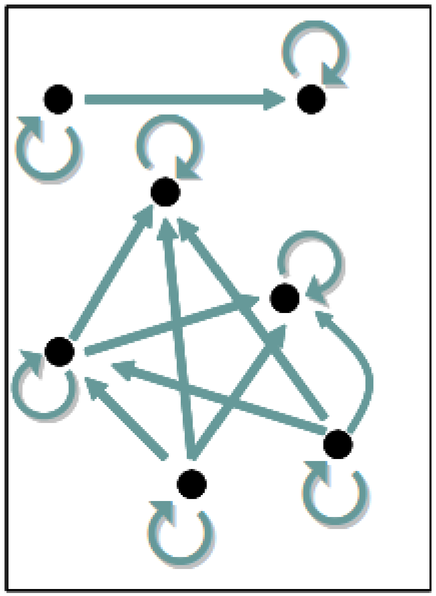

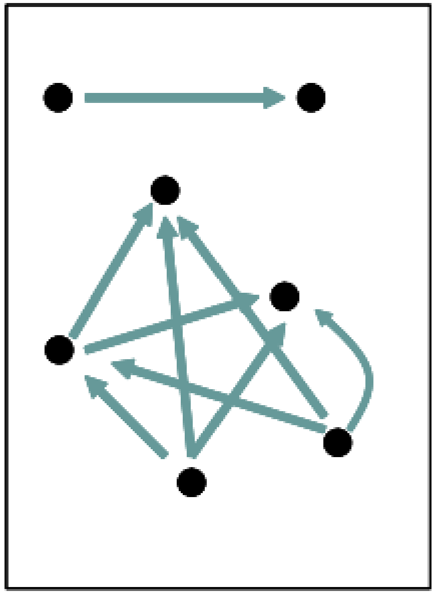

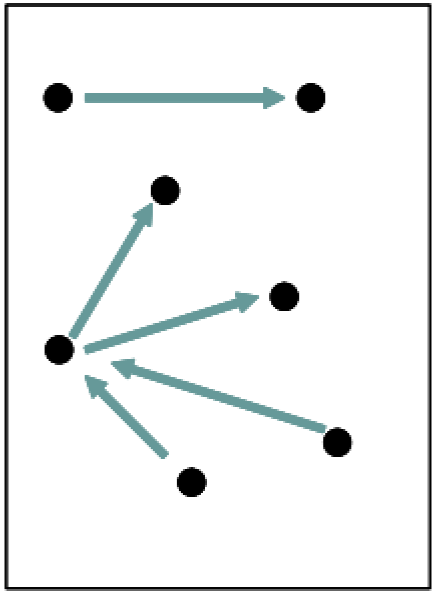

Maximal and Minimal Elements

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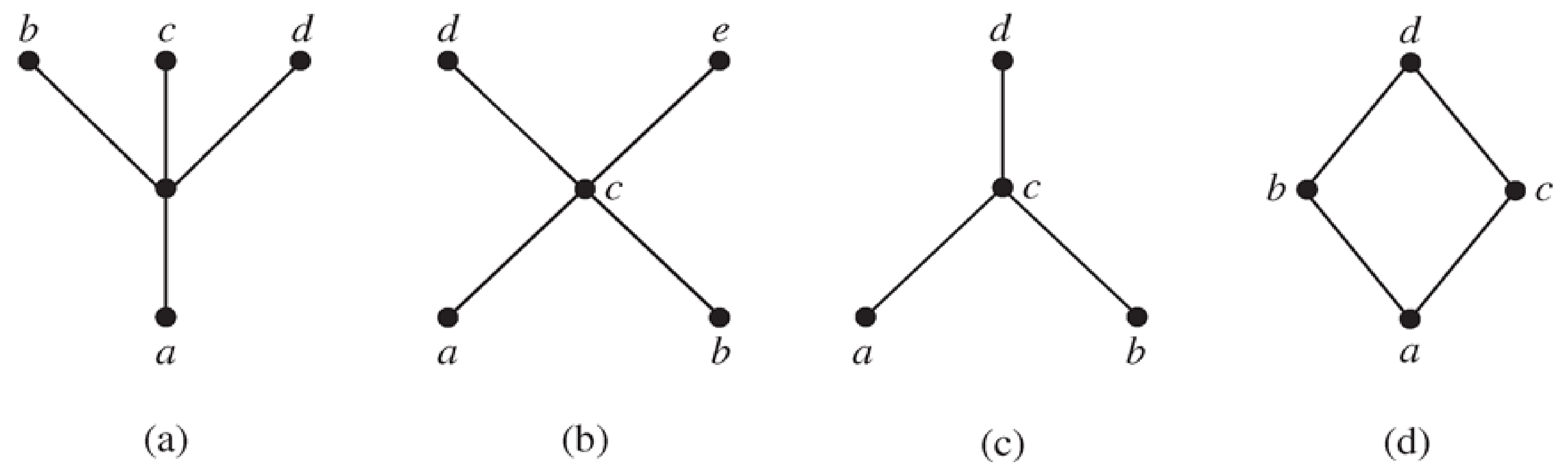

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Example Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

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p. 620, Theorem 1

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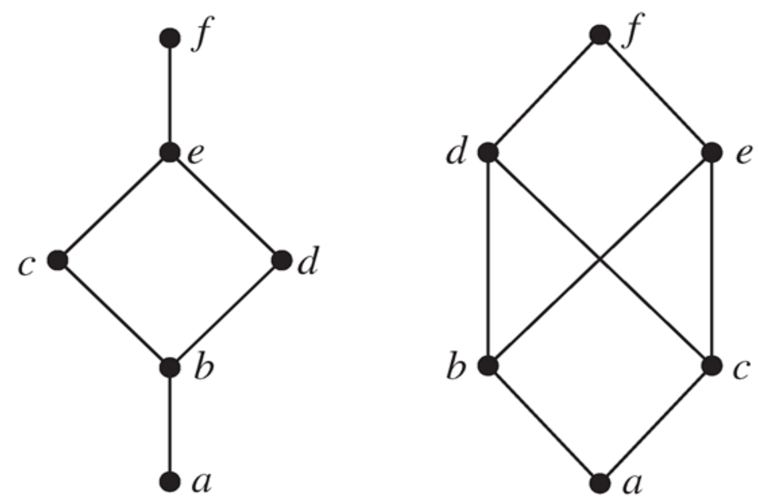

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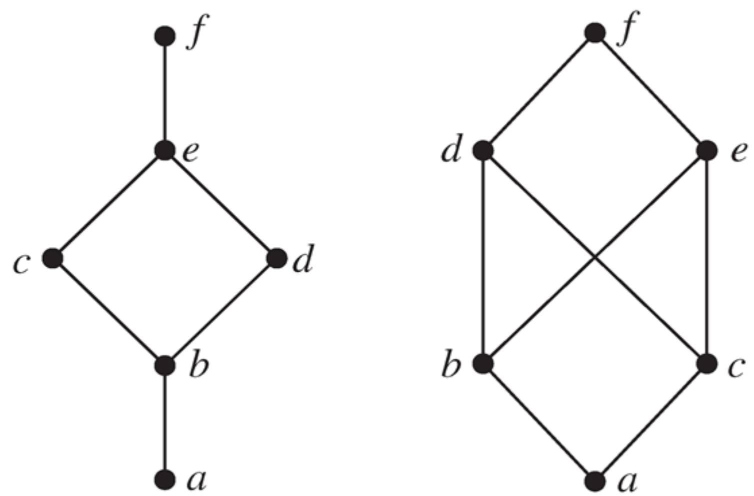

Example Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

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Topological sorting: Given a partial ordering R, find a total ordering $\preccurlyeq$ such that $a \preccurlyeq b$ whenever $a\ R\ b$. $\preccurlyeq$ is said *compatible with R*.

Topological Sorting for Finite Posets

procedure topological_sort (S: finite poset)

$k := 1$;
while $S \neq \emptyset$
  $a_k :=$ a minimal element of S
  $S := S \setminus \{a_k\}$
  $k := k + 1$
end while
// $\{a_1, a_2, \ldots, a_n\}$ is a compatible total ordering of S

Next Lecture

- graph theory I …