



# CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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# Solving Linear Recurrence Relations

- **Definition** A *linear homogeneous relation of degree  $k$*  with *constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

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By induction, such a recurrence relation is uniquely determined by this recurrence relation, and  $k$  initial conditions  $a_0, a_1, \dots, a_{k-1}$ .



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## Examples

$$P_n = (1.11)P_{n-1}$$

$$f_n = f_{n-1} + f_{n-2}$$

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$H_n = 2H_{n-1} + 1$  NOT homogeneous

$B_n = nB_{n-1}$  coefficients are not constants



# Solving Linear Recurrence Relations

- **Example** Consider the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2},$$

Which of the following are solutions?

◇  $a_n = 3n$ :

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- ◇ The solutions to the *characteristic equation* can yield an explicit formula for the sequence.

$$(r^k - c_1 r^{k-1} - \cdots - c_k) = 0$$



# Recall: Problem IV

## ■ Fibonacci number

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2$$



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◇ What is the closed-form expression of  $F_n$ ?

Consider  $x^n = x^{n-1} + x^{n-2}$ , with  $x \neq 0$ . There are two different roots

$$\phi = \frac{1 + \sqrt{5}}{2}, \quad \psi = \frac{1 - \sqrt{5}}{2}$$

Then  $F_n$  can be the form of  $a\phi^n + b\psi^n$ . By  $F_0 = 0$  and  $F_1 = 1$ , we have  $a + b = 0$  and  $\phi a + \psi b = 1$ , leading to  $a = \frac{1}{\sqrt{5}}$ ,  $b = -a$ . Therefore,

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$$



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**Theorem** If this CE has 2 roots  $r_1 \neq r_2$ , then the sequence  $\{a_n\}$  is a solution of the recurrence relation **if and only if**  $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$  for  $n \geq 0$  and constants  $\alpha_1, \alpha_2$ .



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See [Theorem 1 p. 515].



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**Example**  $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$



# The Case of Degenerate Roots

- **Theorem** If the CE  $r^2 - c_1 r - c_2 = 0$  has **only 1** root  $r_0$ , then

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**Proof?**

Exercise.



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# The Case of Degenerate Roots in General

- **Theorem** [Theorem 4, p.519] Suppose that there are  $t$  roots  $r_1, \dots, r_t$  with multiplicities  $m_1, \dots, m_t$ . Then

$$a_n = \sum_{i=1}^t \left( \sum_{j=0}^{m_i-1} \alpha_{i,j} n^j \right) r_i^n,$$

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## Example

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3} \text{ with } a_0 = 1, a_1 = -2, \\ a_2 = -1$$



# Linear Nonhomogeneous Recurrence Relations

- **Definition** A *linear nonhomogeneous relation* with constant coefficients may contain some terms  $F(n)$  that depend only on  $n$

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n).$$

The recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$  is called the *associated homogeneous recurrence relation*.



# Linear Nonhomogeneous Recurrence Relations

- **Theorem** If  $a_n = p(n)$  is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

Then all its solutions are of the form

$$a_n = p(n) + h(n),$$

where  $a_n = h(n)$  is any solution to the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$



# Solving Linear Nonhomogeneous Recurrence Relations

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We get  $c = -1$  and  $d = -3/2$ . Thus,

$$p(n) = -n - 3/2$$



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**Definition** The *generating function* for the sequence  $a_0, a_1, \dots, a_k, \dots$  of real numbers is the infinite series

$$G(x) = a_0 + a_1x + \dots + a_kx^k + \dots = \sum_{k=0}^{\infty} a_kx^k$$





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$$G(x) = C(m, 0) + \dots + C(m, m)x^m = (1 + x)^m$$



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$$1, 2, 3, 4, 5, \dots$$



# Operations of Generating Functions

- **Theorem** Let  $f(x) = \sum_{k=0}^{\infty} a_k x^k$ , and  $g(x) = \sum_{k=0}^{\infty} b_k x^k$ .  
Then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

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# Useful Generating Functions

$$(1 + x)^n = \sum_{k=0}^n C(n, k) x^k$$

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$$\frac{1 - x^{n+1}}{1 - x} = \sum_{k=0}^n x^k = 1 + x + x^2 + \dots + x^n$$

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$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$



# Counting and Generating Functions

- **Problem 1** How many solutions are there to the equation

$$x_1 + x_2 + x_3 = 17,$$

where  $x_1, x_2, x_3$  are **nonnegative** integers?



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$$C(n + r - 1, r - 1) = C(19, 2)$$



# Counting and Generating Functions

- **Problem 2** Find the number of solutions of

$$x_1 + x_2 + x_3 = 17,$$

where  $x_1, x_2, x_3$  are **nonnegative** integers with  $2 \leq x_1 \leq 5$ ,  
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Using *generating functions*, the number is the **coefficient** of  $x^{17}$  in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7)$$



# Counting and Generating Functions

- **Problem 3** In how many ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?



# Counting and Generating Functions

- **Problem 3** In how many ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?

The coefficient of  $x^8$  in the expansion

$$(x^2 + x^3 + x^4)^3$$



# Counting and Generating Functions

- **Problem 4** Use generating functions to find the number of  $k$ -combinations of a set with  $n$  elements,  $C(n, k)$ .



# Counting and Generating Functions

- **Problem 4** Use **generating functions** to find the number of  **$k$ -combinations of a set with  $n$  elements**,  $C(n, k)$ .

Each of the  $n$  elements in the set contributes the term  $(1 + x)$  to the generating function  $f(x) = \sum_{k=0}^n a^k x^k$ .  
Hence,  $f(x) = (1 + x)^n$ .

Then by the **binomial theorem**, we have  $a_k = \binom{n}{k}$ .



# Next Lecture

- relation ...

