南方科技大學
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Number Theory and Cryptography

- Division, Primes
- Congruence
- Greatest Common Divisor (GCD)
- Euler's Theorem / Fermart's Little Theorem

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- Greatest Common Divisor (GCD)

Find the GCD of 286 and 503.

$$\begin{aligned}
&\gcd(503, 286) && 503 = 1 \cdot 286 + 217 \\
&= \gcd(286, 217) && 286 = 1 \cdot 217 + 69 \\
&= \gcd(217, 69) && 217 = 3 \cdot 69 + 10 \\
&= \gcd(69, 10) && 69 = 6 \cdot 10 + 9 \\
&= \gcd(10, 9) && 10 = 1 \cdot 9 + 1 \\
&= 1 && 9 = 9 \cdot 1
\end{aligned}$$

$$\begin{aligned}
1 &= 10 - 1 \cdot 9 \\
1 &= 7 \cdot 10 - 1 \cdot 69 \\
1 &= 7 \cdot 217 - 22 \cdot 69 \\
1 &= 29 \cdot 217 - 22 \cdot 286 \\
1 &= 29 \cdot 503 - 51 \cdot 286
\end{aligned}$$

- E

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  find the modular inverse
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  $x^{\phi(n)} \equiv 1 \bmod n$ if $\gcd(x, n) = 1$
  $x^{p-1} \equiv 1 \bmod p$ if $x \not\equiv 0 \bmod p$

RSA Variant

$\mathcal{Q}$: Consider the RSA system. Let (e, d) be a key pair for the RSA. Define

$$\lambda(n) = \mathrm{lcm}(p-1, q-1)$$

and compute $d' = e^{-1} \bmod \lambda(n)$. Will decryption using d' instead of d still work? (prove $C^{d'} \bmod n = M$)

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Case I: $\gcd(M, n) = 1$

$$\begin{aligned} C^{d'} \bmod n &= M^{ed'} \bmod n = M^{k\lambda(n)+1} \bmod n \\ &= (M^{k\lambda(n)} \bmod n) M \bmod n \\ &= \left(M^{(p-1)(q-1)/\gcd(p-1,q-1)} \bmod n\right)^k M \bmod n \end{aligned}$$

By Fermat's theorem, $M^{(p-1)(q-1)/\gcd(p-1,q-1)} \bmod p = \left(M^{(q-1)/\gcd(p-1,q-1)}\right)^{p-1} \bmod p = 1$ and $M^{(p-1)(q-1)/\gcd(p-1,q-1)} \bmod q = 1$. Then by Chinese Remainder Theorem, we have $C^{d'} \bmod n = M$.

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Case II: $\gcd(M, n) = p$

$M = tp$ for some integer $0 < t < q$. We have $\gcd(M, q) = 1$ and $ed' = k\lambda(n) + 1$ for some integer k. By Fermat's theorem, we have

$(M^{k\lambda(n)} - 1) \bmod q = (M^{k(p-1)(q-1)/\gcd(p-1,q-1)} - 1) \bmod q = 0.$

Then

$$\begin{aligned}
(M^{ed'} - M) \bmod n &= M(M^{ed'-1} - 1) \bmod n \\
&= tp(M^{k\lambda(n)} - 1) \bmod pq \\
&= 0
\end{aligned}$$

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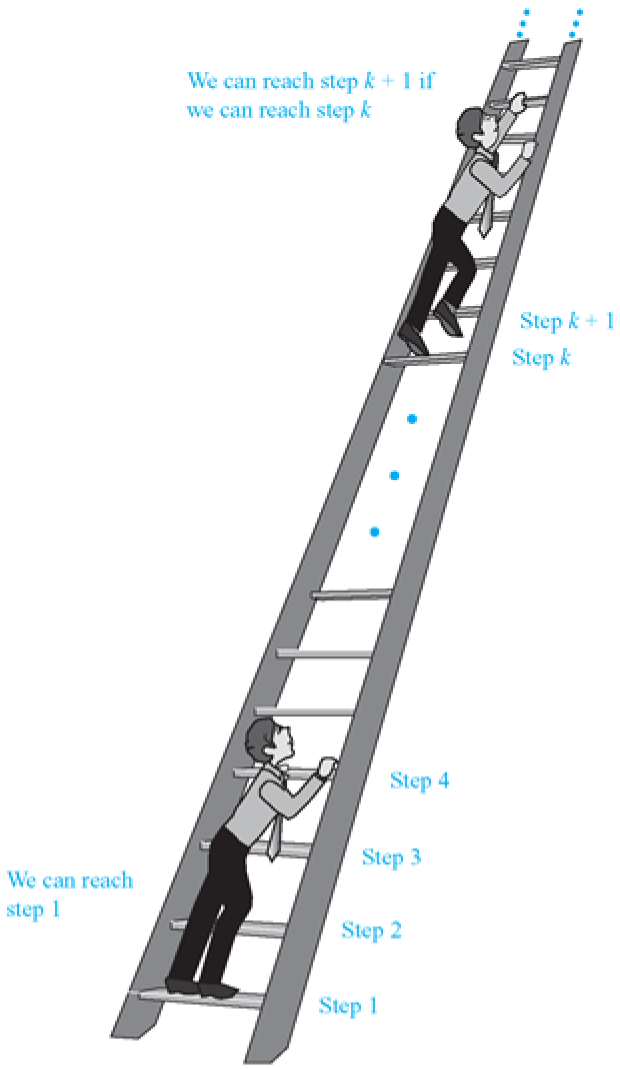
<u>Case III:</u> $\gcd(M, n) = q$

Similar to Case II.

<u>Case IV:</u> $\gcd(M, n) = pq$

Trivial.

Mathematical Induction



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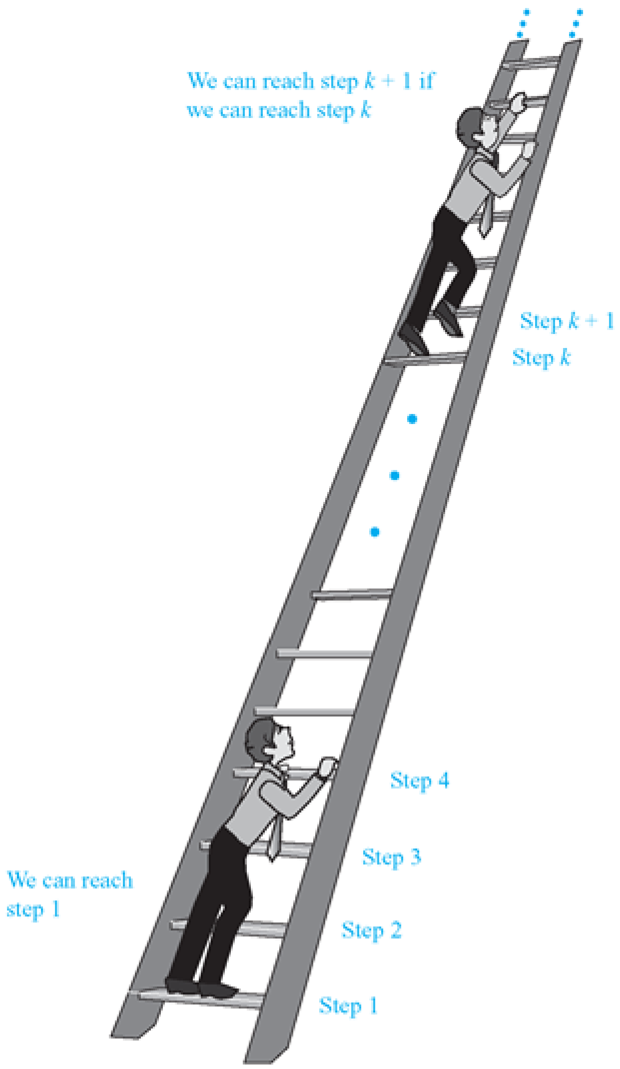

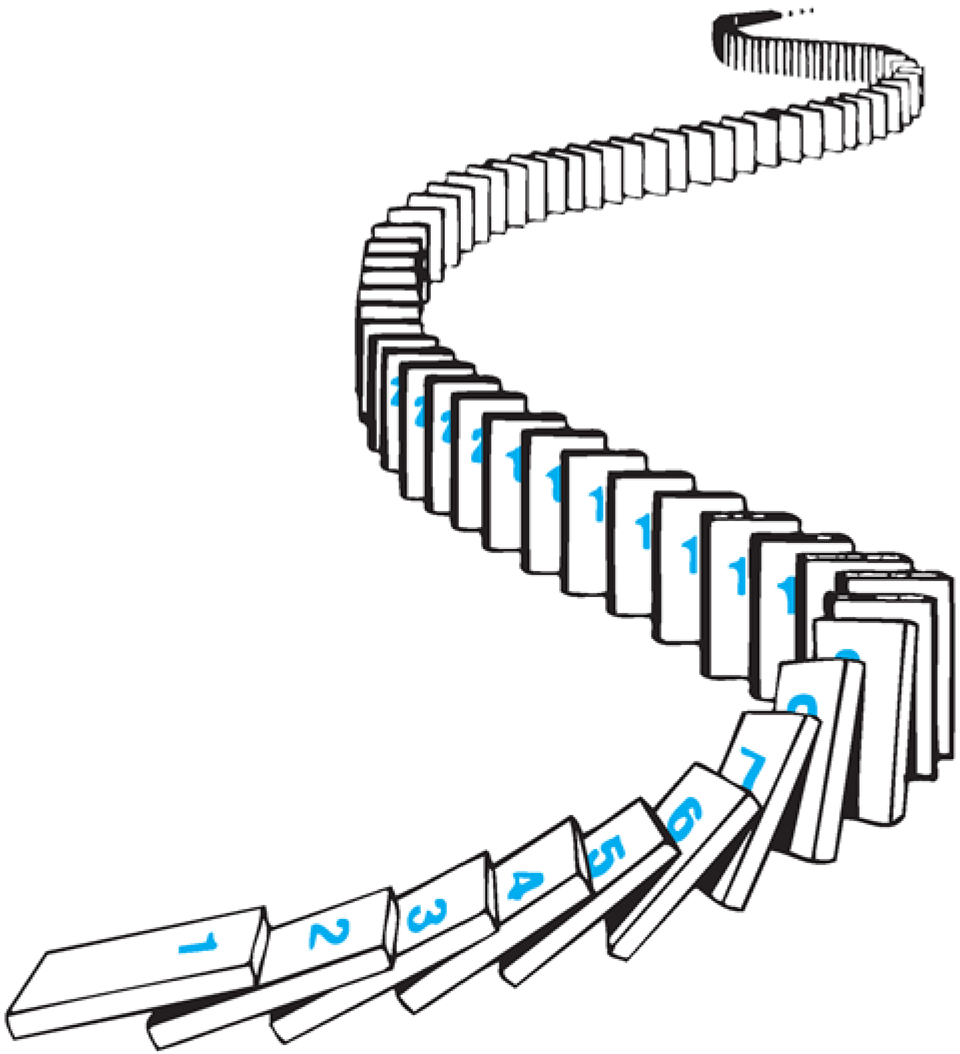

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- We conclude by distinguishing between the *weak principle* of mathematical induction and the *strong principle* of mathematical induction.

  The *strong principle* can actually be derived from the *weak principle*.

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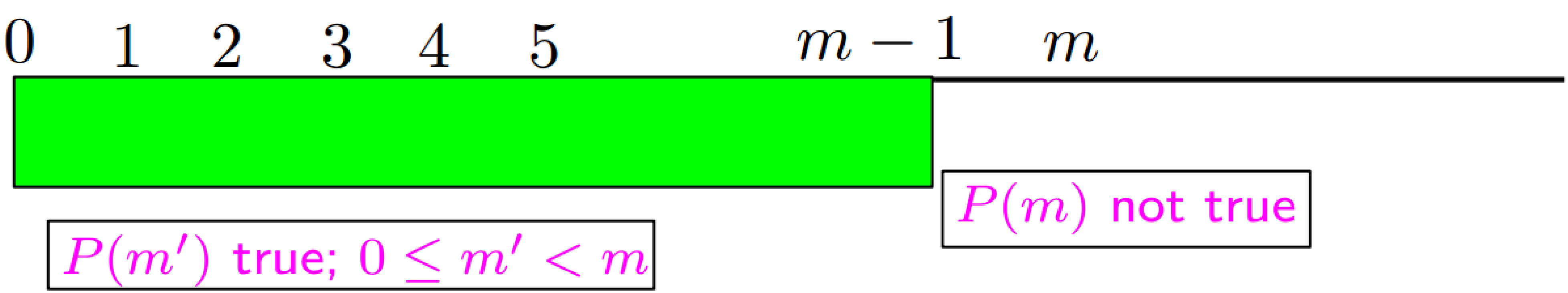

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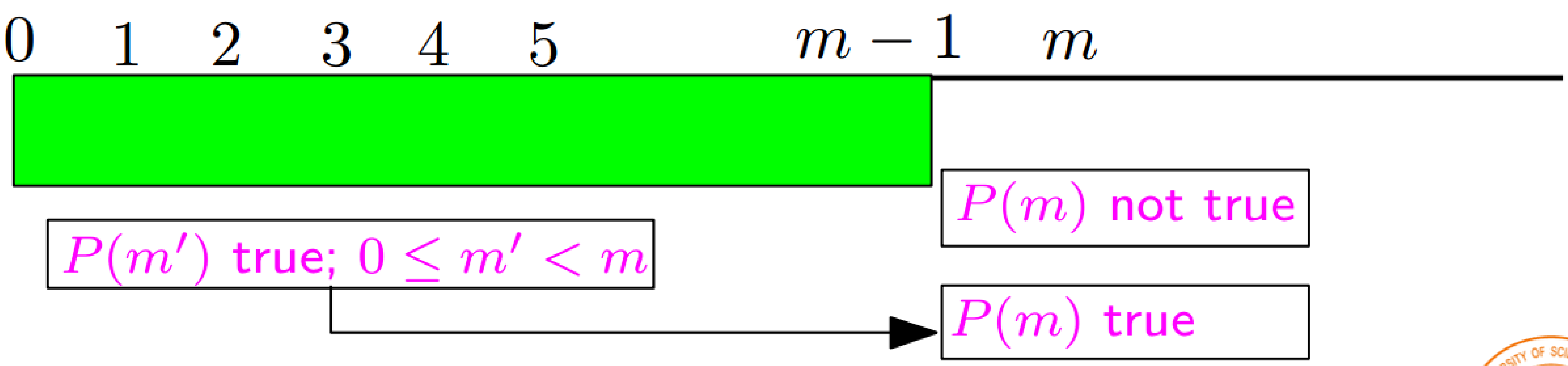

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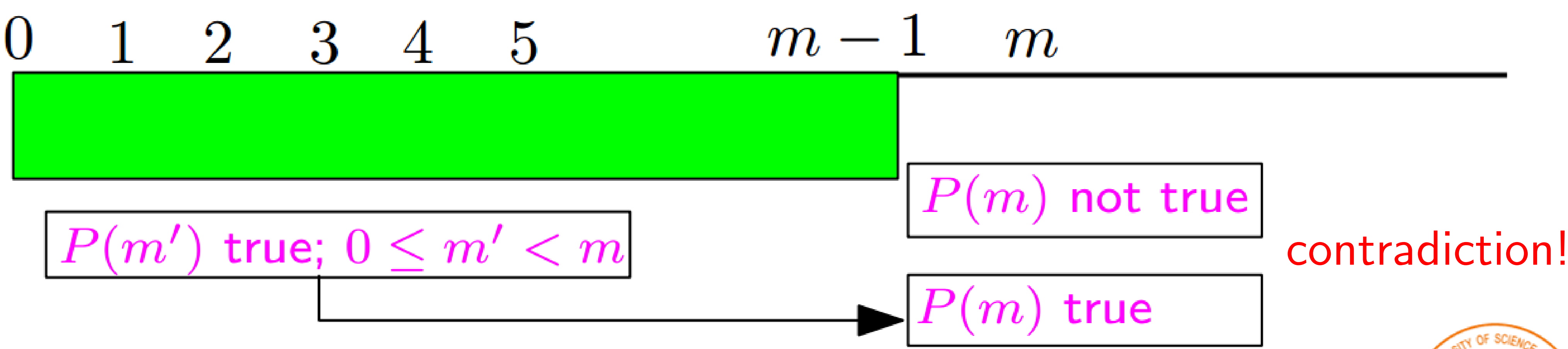

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◇ The smallest counterexample n is larger than 0

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  - ◇ Therefore, $(*)$ holds for all positive integers n.

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  The key step was proving that

$$P(n-1) \to P(n)$$

  where $P(n)$ is the statement

$$1+2+\cdots+n = \frac{n(n+1)}{2}$$

Example 2

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  Let $P(n)$ – $2^{n+1} \geq n^2 + 2$. We start by assuming that the statement
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is false.

When a for all quantifier is false, there must be some n for which it is false. Let n be the smallest nonnegative integer for which $2^{n+1} \ngeq n^2 + 2$.

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  Then setting $i = n - 1$ gives
  $$2^{(n-1)+1} \geq (n-1)^2 + 2.$$

  or
  $$(*) \quad 2^n \geq n^2 - 2n + 1 + 2 = n^2 - 2n + 3$$

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  Thus, we write
  $$\begin{aligned} 2^{n+1} &\geq 2n^2 - 4n + 6 \\ &= (n^2 + 2) + (n^2 - 4n + 4) \\ &= n^2 + 2 + (n - 2)^2 \\ &\geq n^2 + 2. \end{aligned}$$

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- Let $P(n)$ – $2^{n+1} \geq n^2 + 2$

  We just showed that

  (a) $P(0)$ is true

  (b) if $n > 0$, then $P(n-1) \rightarrow P(n)$

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  ◇ Thus, $P(n)$ is true for all $n \in N$.

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- What did we really do?

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Since $P(n-1) \rightarrow P(n)$, we see that

$P(0)$ implies $P(1)$, $P(1)$ implies $P(2)$, ...

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  This is actually **equivalent** to the *principle of mathematical induction*.

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  (a) – *Basic Step* *Inductive Hypothesis*

  (b) – *Inductive Step* *Inductive Conclusion*

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\begin{aligned}
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Hence, we've just prove that for $n > 2$, $P(n-1) \rightarrow P(n)$.

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Inductive Conclusion

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  - ◇ Iterating gives us a proof of $P(n)$ for all n

Strong Induction

- **Principle** (*The Strong Principle of Mathematical Induction*)

  (a) If the statement $P(b)$ is true

  (b) for all $n > b$, the statement
  $P(b) \wedge P(b+1) \wedge \cdots \wedge P(n-1) \rightarrow P(n)$ is true.

  then $P(n)$ is true for all integers $n \geq b$.

Example

- Prove that every positive integer is a power of a prime or the product of powers of primes.

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  ◇ Thus, by the strong principle of mathematical induction, every positive integer is a power of a prime or a product of powers of primes.

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Summary

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3. We conclude on the basis of the principle of mathematical induction that $P(n)$ is true for all $n \geq b$.

Recursion

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- A classical example of *recursion* is the **Towers of Hanoi** Problem.

Towers of Hanoi

Towers of Hanoi

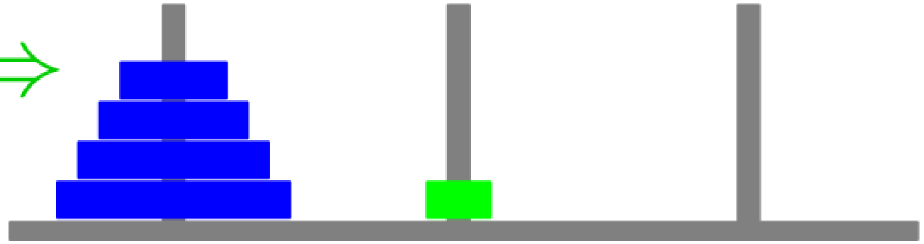
- 3 pegs; n disks of different sizes
- A *legal move* takes a disk from one peg and moves it onto another peg so that it is not on top of a smaller disk
- **Problem**: Find a (efficient) way to move all of the disks from one peg to another

Towers of Hanoi

Towers of Hanoi

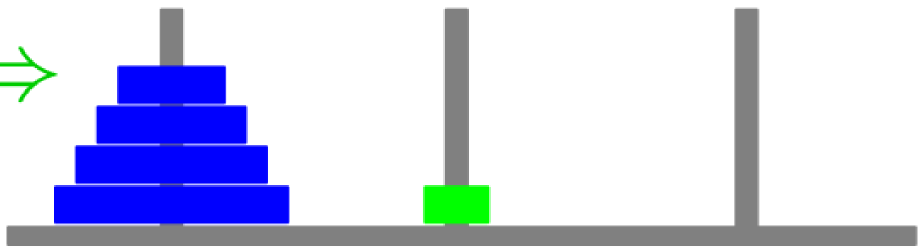
legal move

⇒

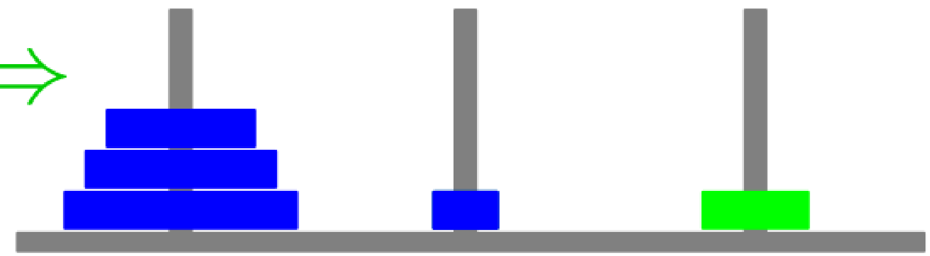


Towers of Hanoi

legal move

$\Rightarrow$ 

legal move

$\Rightarrow$ 

Towers of Hanoi

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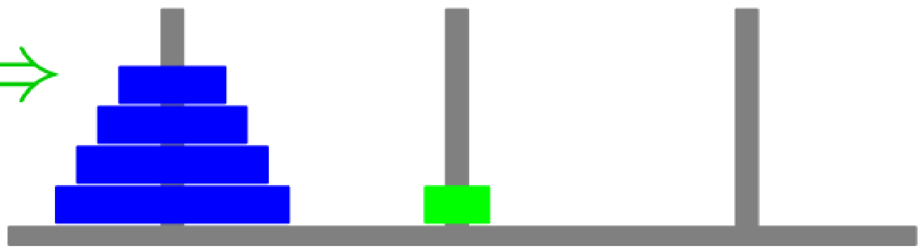

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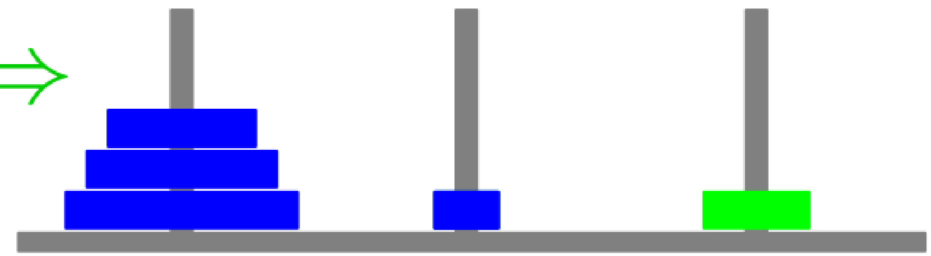

not legal

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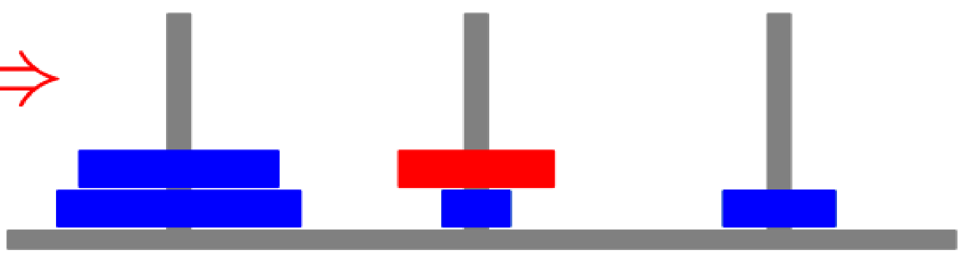

Towers of Hanoi

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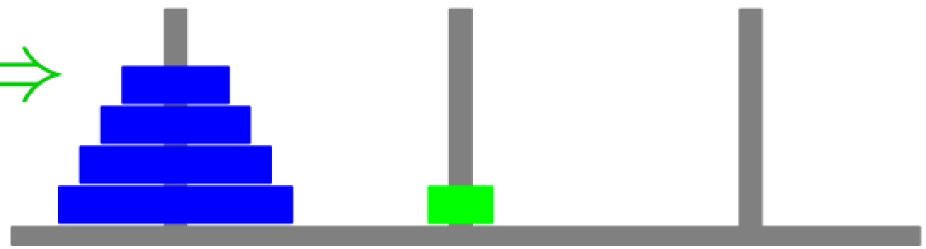

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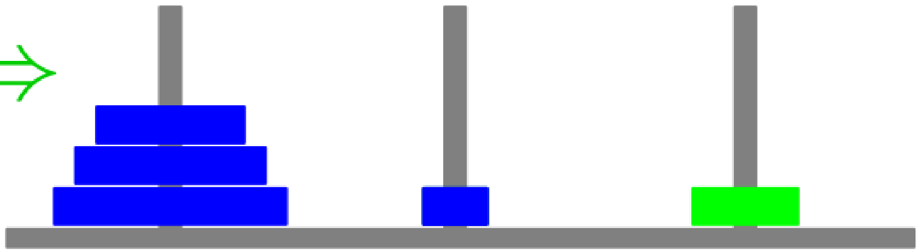

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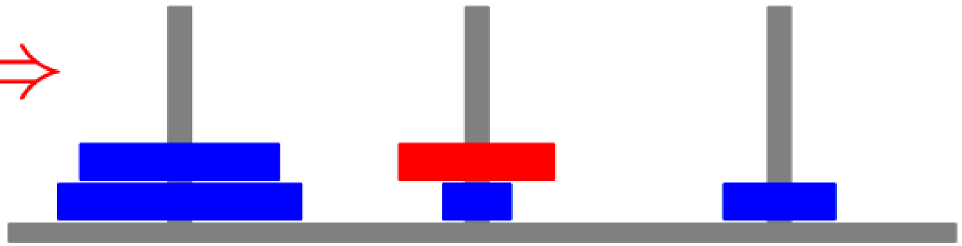

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Towers of Hanoi

- **Problem:** Start with n disks on leftmost peg

Towers of Hanoi

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Towers of Hanoi

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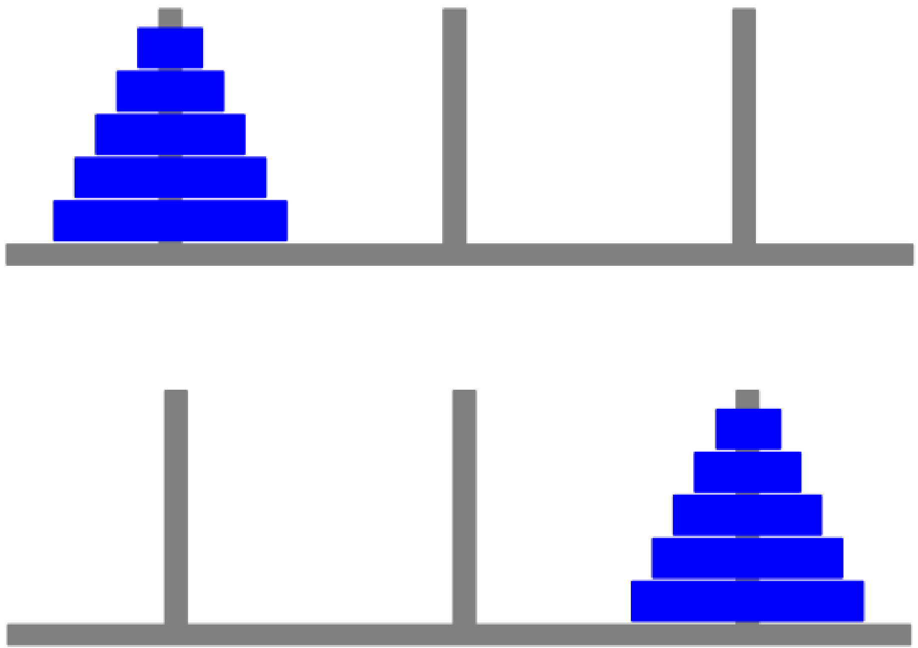

Towers of Hanoi

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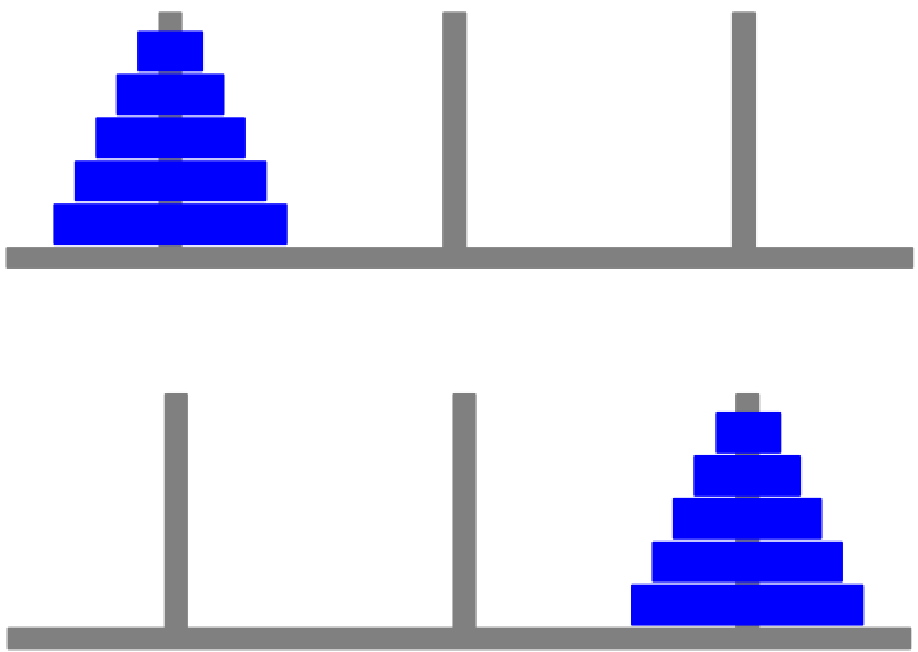

Given $i, j \in \{1,2,3\}$, let $\overline{\{i,j\}} = \{1,2,3\} - \{i\} - \{j\}$,
i.e., $\overline{\{1,2\}} = \{3\}$, $\overline{\{1,3\}} = \{2\}$, $\overline{\{2,3\}} = \{1\}$.

Towers of Hanoi

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  If $n = 1$, moving one disk from i to j is easy. Just move it.

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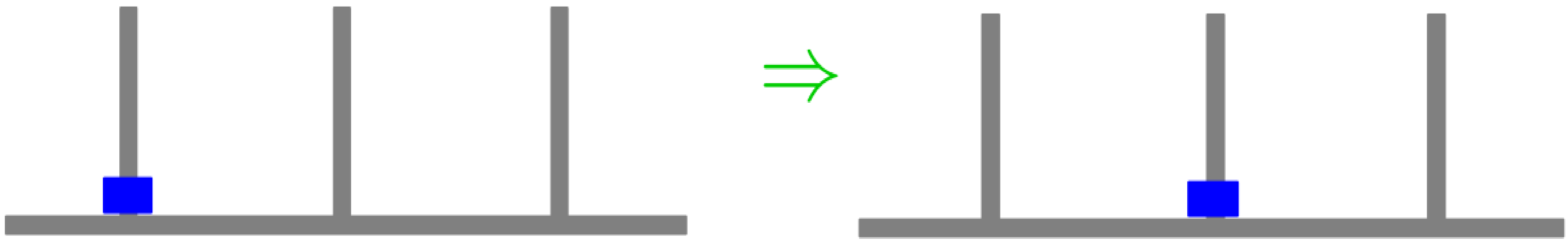

Towers of Hanoi

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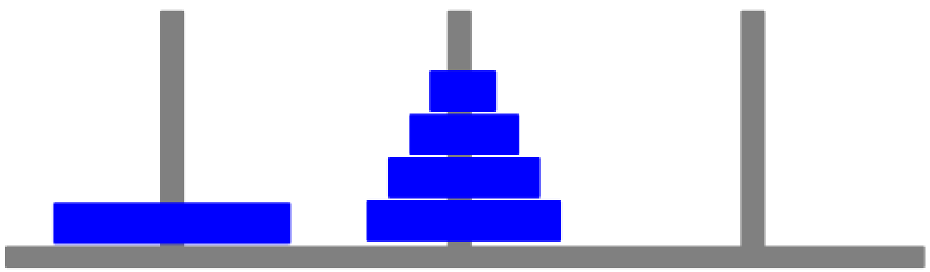

move top $n-1$ disks from i to $\overline{\{i,j\}}$

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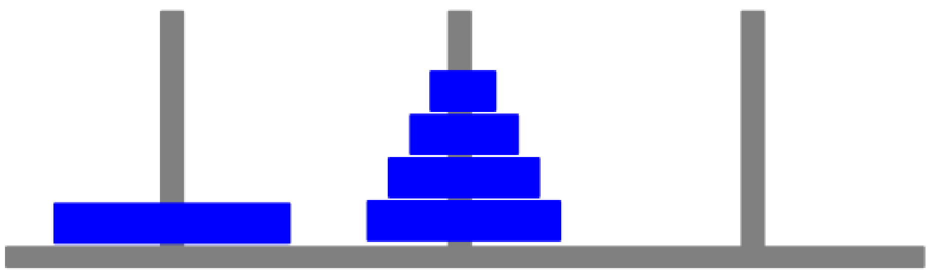

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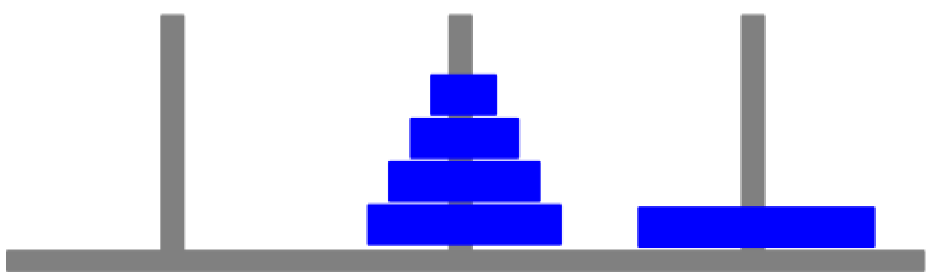

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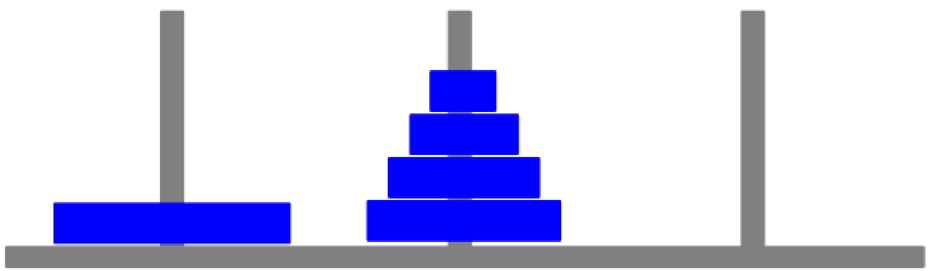
1) 

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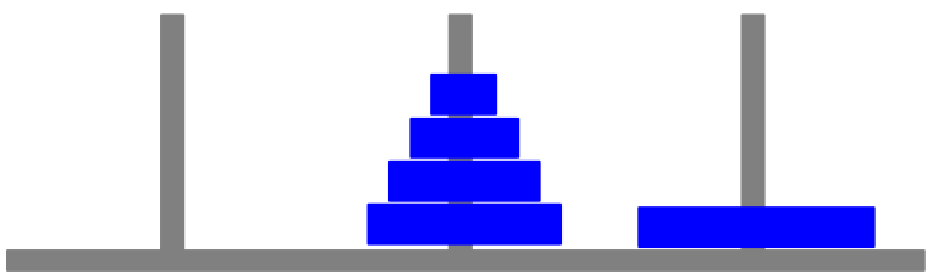
2) 

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3) 

move top $n-1$ disks from $\overline{\{i,j\}}$ to j

Towers of Hanoi

```java
public class Hanoi
{

    public void move(int n, char a, char b, char c)
    {
        if (n == 1)
            System.out.println("plate " + n + " from " + a + " to " + c);
        else
        {
            move(n-1,a,c,b);
            System.out.println("plate " + n + " from " + a + " to " + c);
            move(n-1,b,a,c);
        }

    }

```

Towers of Hanoi

To move n disks from i to j

i) move top $n-1$ disks from i to $\overline{\{i,j\}}$

ii) move largest disk from i to j

iii) move top $n-1$ disks from $\overline{\{i,j\}}$ to j

Towers of Hanoi

- To prove Correctness of solution, we are implicitly using induction

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i) move top $n-1$ disks from i to $\overline{\{i,j\}}$

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- $p(1)$ is statement that algorithm works for $n=1$ disks, which is obviously true
- $p(n-1) \rightarrow p(n)$ is *recursion* statement that

  if our algorithm works for $n-1$ disks, then we can build a correct solution for n disks

Towers of Hanoi

- Running time

  $M(n)$ is number of disk moves needed for n disks

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$M(1) = 1$

if $n > 1$, then $M(n) = 2M(n-1) + 1$

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- Iterating the recurrence gives

$$M(1) = 1,\ M(2) = 3,\ M(3) = 7,$$
$$M(4) = 15,\ M(5) = 31,\ \ldots$$

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- Iterating the recurrence gives

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Towers of Hanoi

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  Later, we'll also see how to solve without guessing

Towers of Hanoi

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Then $M(n) = 2M(n-1) + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 1$

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Towers of Hanoi

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- The second time was to derive the closed form solution $M(n) = 2^n - 1$ of the recurrence.

Next Lecture

- recurrence ...