CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

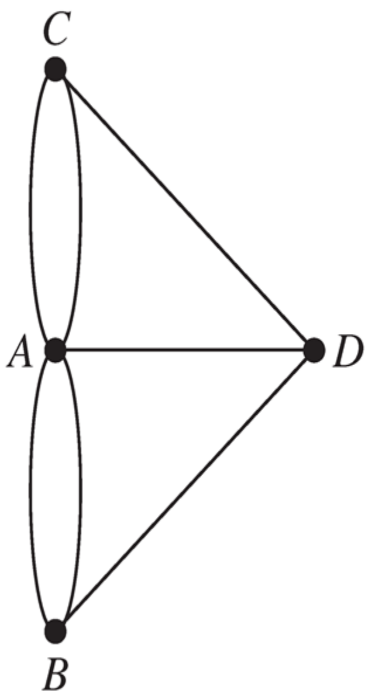
Dr. Qi Wang

Department of Computer Science and Engineering
Office: Room903, Nanshan iPark A7 Building
Email: wangqi@sustech.edu.cn

Euler Circuits and Euler Paths

- **Theorem** A connected multigraph with at least two vertices has an *Euler circuit* if and only if each of its vertices has **even** degree.

  Theorem A connected multigraph has an *Euler path* but not an *Euler circuit* if and only if it has exactly two vertices of odd degree.



No Euler circuit

Shortest Path Problems

- Using graphs with weights assigned to their edges

Shortest Path Problems

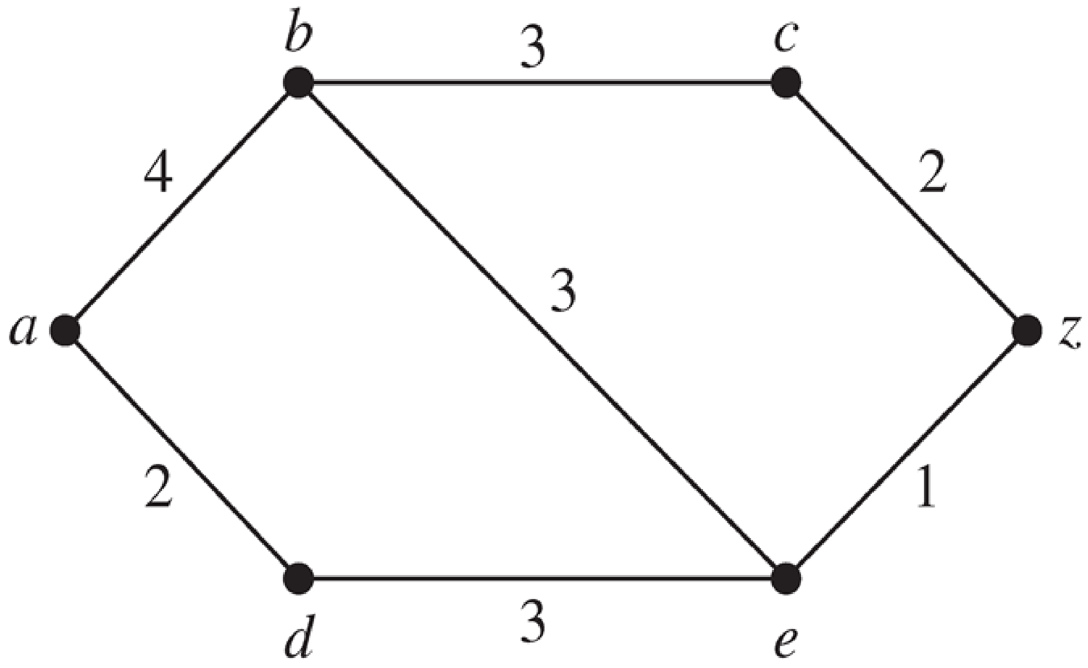
- Using graphs with weights assigned to their edges

  Such graphs are called *weighted graphs* and can model lots of questions involving distance, time consuming, fares, etc.

Shortest Path Problems

- Using graphs with weights assigned to their edges

  Such graphs are called *weighted graphs* and can model lots of questions involving distance, time consuming, fares, etc.



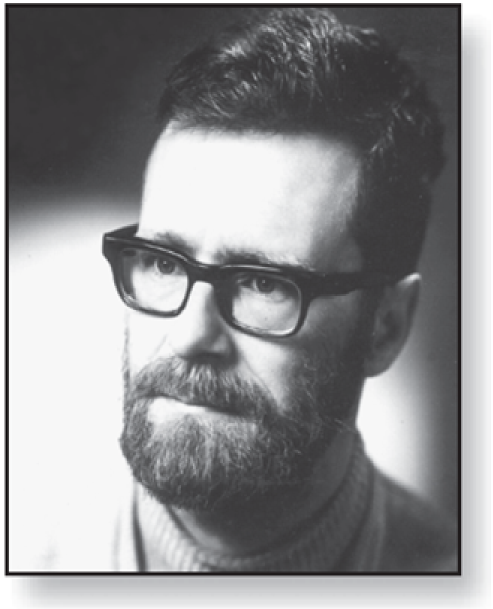
Shortest Path Problems

- **Definition** Let G^{α} be an weighted graph, with a weight function $\alpha : E \to \mathbf{R}$ on its edges. If $P = e_1 e_2 \cdots e_k$ is a path, then its weight is $\alpha(P) = \sum_{i=1}^{k} \alpha(e_i)$. The *minimum weighted distance* between two vertices is
$$d(u, v) = \min\{\alpha(P) | P : u \to v\}$$

Shortest Path Problems

- **Definition** Let G^α be an weighted graph, with a weight function $\alpha : E \to \mathbf{R}$ on its edges. If $P = e_1 e_2 \cdots e_k$ is a path, then its weight is $\alpha(P) = \sum_{i=1}^{k} \alpha(e_i)$. The *minimum weighted distance* between two vertices is

$$d(u, v) = \min\{\alpha(P) | P : u \to v\}$$



Edsger Wybe Dijkstra

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

  (ii) while $S \neq V$

  let $v \notin S$ be the vertex with the least value $d(v)$,

  $S = S \cup \{v\}$ for each $u \notin S$, replace $d(u)$ by

  $$\min\{d(u), d(v) + \alpha(u, v)\}$$

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

  (ii) while $S \neq V$

  let $v \notin S$ be the vertex with the least value $d(v)$,

  $S = S \cup \{v\}$ for each $u \notin S$, replace $d(u)$ by

  $\min\{d(u), d(v) + \alpha(u, v)\}$

  (iii) return all $d(v)$'s

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

  (ii) while $S \neq V$

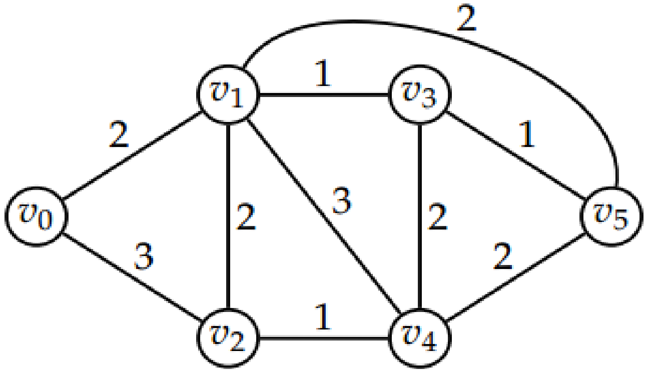
  let $v \notin S$ be the vertex with the least value $d(v)$,

  $S = S \cup \{v\}$ for each $u \notin S$, replace $d(u)$ by

  $$\min\{d(u), d(v) + \alpha(u, v)\}$$

  (iii) return all $d(v)$'s

  Example



Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

  (ii) while $S \neq V$

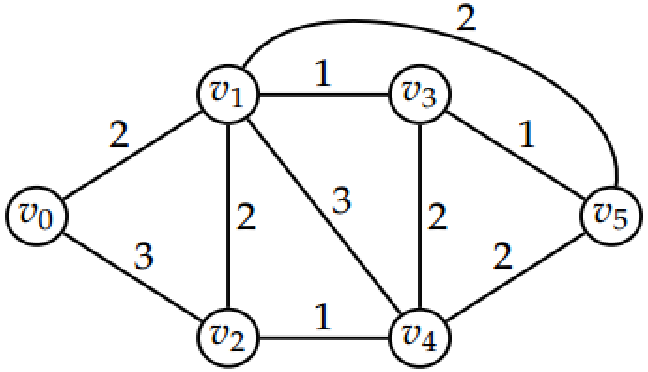
  let $v \notin S$ be the vertex with the least value $d(v)$,

  $S = S \cup \{v\}$ for each $u \notin S$, replace $d(u)$ by

  $$\min\{d(u), d(v) + \alpha(u, v)\}$$

  (iii) return all $d(v)$'s

  Example



$d(v_0) = 0$, all other $d(v) = \infty$

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

  (ii) while $S \neq V$

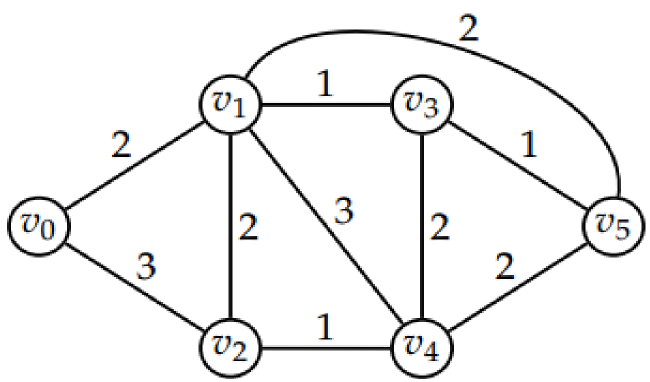
  let $v \notin S$ be the vertex with the least value $d(v)$,

  $S = S \cup \{v\}$ for each $u \notin S$, replace $d(u)$ by

  $\min\{d(u), d(v) + \alpha(u, v)\}$

  (iii) return all $d(v)$'s

  Example



v_0	v_1	v_2	v_3	v_4	v_5
0	∞	∞	∞	∞	∞

$d(v_0) = 0$, all other $d(v) = \infty$

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

  (ii) while $S \neq V$

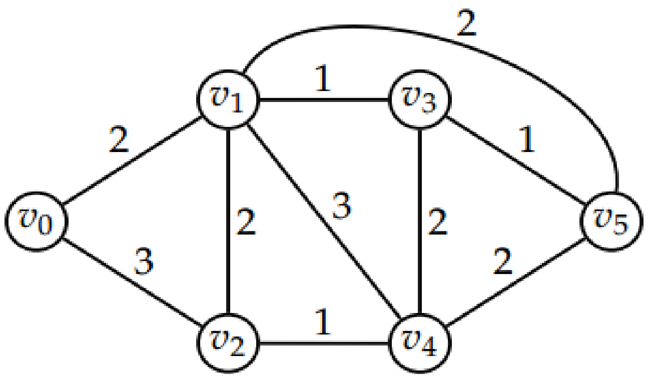
  let $v \notin S$ be the vertex with the least value $d(v)$,

  $S = S \cup \{v\}$ for each $u \notin S$, replace $d(u)$ by

  $$\min\{d(u), d(v) + \alpha(u, v)\}$$

  (iii) return all $d(v)$'s

Example



v_0	v_1	v_2	v_3	v_4	v_5
0	∞	∞	∞	∞	∞

$i = 0$

$d(v_1) = \min\{\infty, 2\} = 2$, $d(v_2) = \min\{\infty, 3\} = 3$

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

  (ii) while $S \neq V$

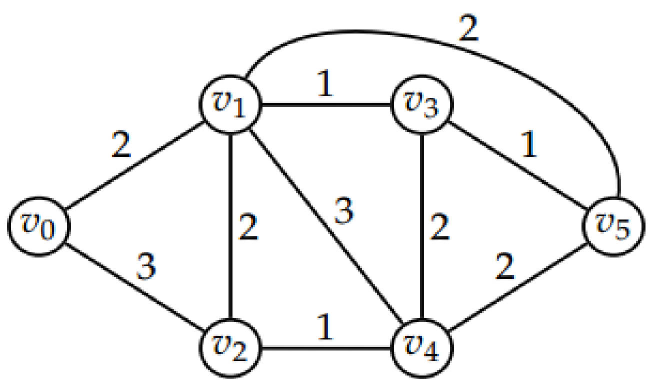
  let $v \notin S$ be the vertex with the least value $d(v)$,

  $S = S \cup \{v\}$ for each $u \notin S$, replace $d(u)$ by

  $$\min\{d(u), d(v) + \alpha(u, v)\}$$

  (iii) return all $d(v)$'s

Example



v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	∞	∞	∞

$i = 0$

$d(v_1) = \min\{\infty, 2\} = 2$, $d(v_2) = \min\{\infty, 3\} = 3$

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

  (ii) while $S \neq V$

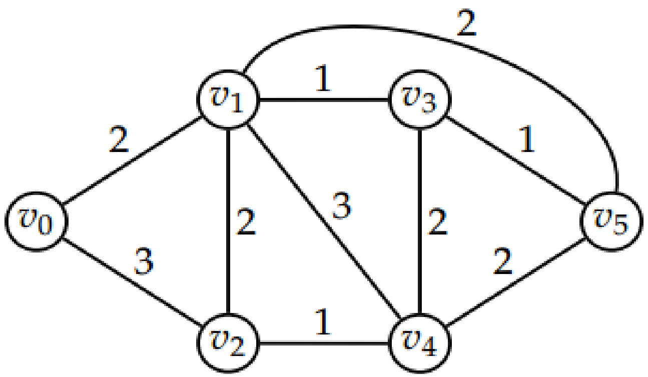
  let $v \notin S$ be the vertex with the least value $d(v)$,

  $S = S \cup \{v\}$ for each $u \notin S$, replace $d(u)$ by

  $$\min\{d(u), d(v) + \alpha(u, v)\}$$

  (iii) return all $d(v)$'s

  Example



v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	∞	∞	∞

$i = 1$

$d(v_2) = \min\{3, d(v_1) + \alpha(v_1 v_2)\} = \min\{3, 4\} = 3$,

$d(v_3) = 2 + 1 = 3$, $d(v_4) = 2 + 3 = 5$,

$d(v_5) = 2 + 2 = 4$

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

  (ii) while $S \neq V$

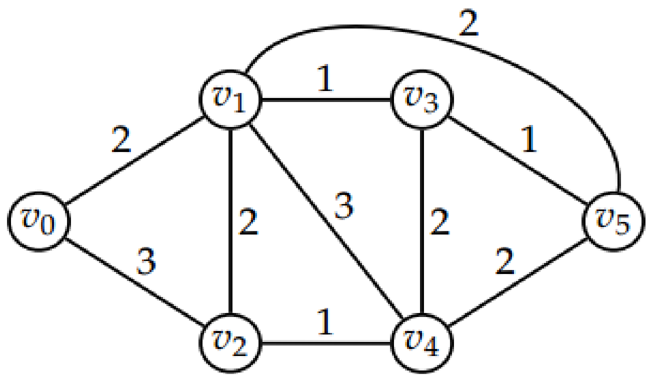
  let $v \notin S$ be the vertex with the least value $d(v)$,

  $S = S \cup \{v\}$ for each $u \notin S$, replace $d(u)$ by

  $$\min\{d(u), d(v) + \alpha(u, v)\}$$

  (iii) return all $d(v)$'s

Example



v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	3	5	4

$i = 1$

$d(v_2) = \min\{3, d(v_1) + \alpha(v_1 v_2)\} = \min\{3, 4\} = 3,$

$d(v_3) = 2 + 1 = 3$, $d(v_4) = 2 + 3 = 5,$

$d(v_5) = 2 + 2 = 4$

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

  (ii) while $S \neq V$

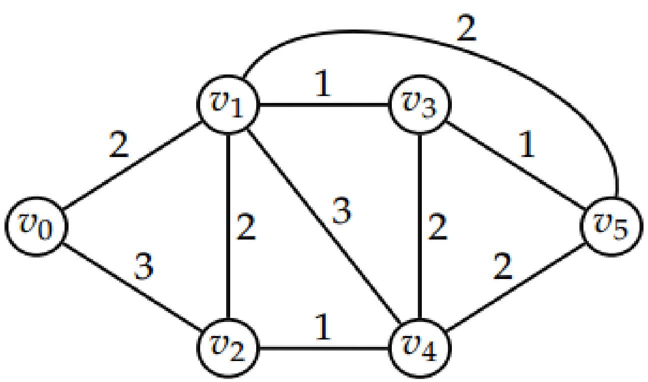
  let $v \notin S$ be the vertex with the least value $d(v)$,

  $S = S \cup \{v\}$ for each $u \notin S$, replace $d(u)$ by

  $$\min\{d(u), d(v) + \alpha(u, v)\}$$

  (iii) return all $d(v)$'s

  Example



v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	3	5	4

$i = 2$

$d(v_3) = \min\{3, \infty\} = 3$,

$d(v_4) = \min\{5, 3 + 1\} = 4$,

$d(v_5) = \min\{4, \infty\} = 4$

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

  (ii) while $S \neq V$

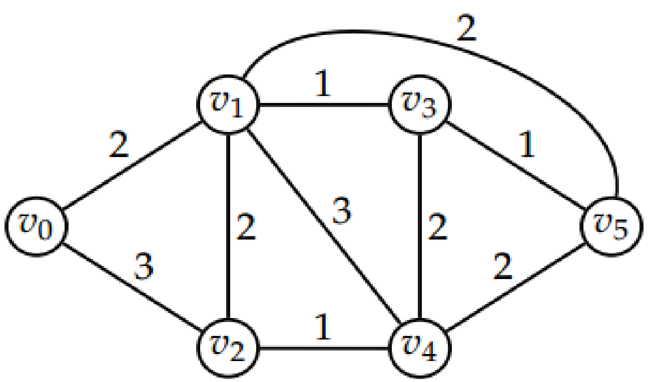
  let $v \notin S$ be the vertex with the least value $d(v)$,

  $S = S \cup \{v\}$ for each $u \notin S$, replace $d(u)$ by

  $$\min\{d(u), d(v) + \alpha(u, v)\}$$

  (iii) return all $d(v)$'s

Example



v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	3	4	4

$i = 2$

$d(v_3) = \min\{3, \infty\} = 3,$

$d(v_4) = \min\{5, 3 + 1\} = 4,$

$d(v_5) = \min\{4, \infty\} = 4$

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

  (ii) while $S \neq V$

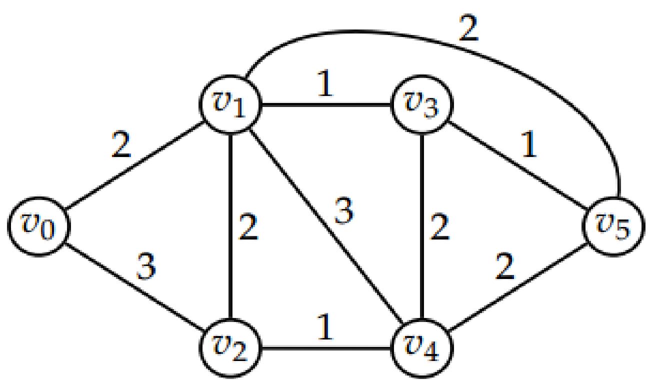
  let $v \notin S$ be the vertex with the least value $d(v)$,

  $S = S \cup \{v\}$ for each $u \notin S$, replace $d(u)$ by

  $$\min\{d(u), d(v) + \alpha(u, v)\}$$

  (iii) return all $d(v)$'s

  Example



v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	3	4	4

$i = 3$

$d(v_4) = \min\{4, 3 + 2\} = 4,$

$d(v_5) = \min\{4, 3 + 1\} = 4$

Dijkstra's Algorithm

- (i) Set $d(v_0) = 0$ and $d(v) = \infty$ for all $v \neq v_0$, $S = \emptyset$

  (ii) while $S \neq V$

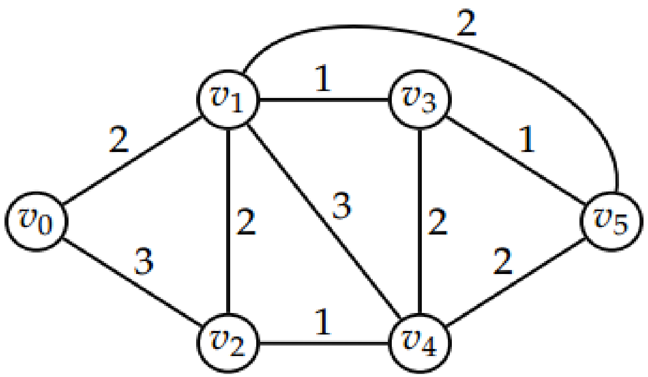
  let $v \notin S$ be the vertex with the least value $d(v)$,

  $S = S \cup \{v\}$ for each $u \notin S$, replace $d(u)$ by

  $$\min\{d(u), d(v) + \alpha(u, v)\}$$

  (iii) return all $d(v)$'s

Example



v_0	v_1	v_2	v_3	v_4	v_5
0	2	3	3	4	4

$i = 4$

$d(v_5) = \min\{4, 4 + 2\} = 4$

Dijkstra's Algorithm

- **Theorem** Dijkstra's algorithm finds the length of a shortest path between two vertices in a connnected simple undirected weighted graph.

Dijkstra's Algorithm

- **Theorem** Dijkstra's algorithm finds the length of a shortest path between two vertices in a connnected simple undirected weighted graph.

  Correctness

Dijkstra's Algorithm

- **Theorem** Dijkstra's algorithm finds the length of a shortest path between two vertices in a connnected simple undirected weighted graph.

  Correctness

  Theorem Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons) in a connected simple undirected weighted graph with n vertices.

Dijkstra's Algorithm

- **Theorem** Dijkstra's algorithm finds the length of a shortest path between two vertices in a connnected simple undirected weighted graph.

  Correctness

  Theorem Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons) in a connected simple undirected weighted graph with n vertices.

  Complexity

Dijkstra's Algorithm

- **Theorem** Dijkstra's algorithm finds the length of a shortest path between two vertices in a connnected simple undirected weighted graph.

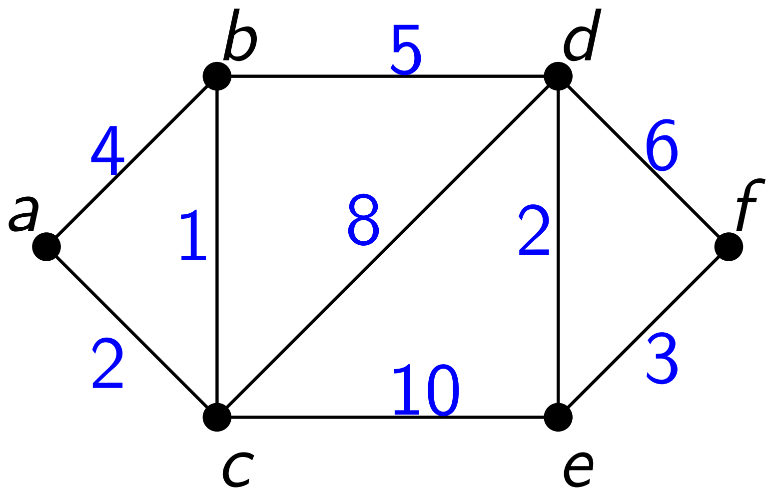
  Correctness

  Theorem Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons) in a connected simple undirected weighted graph with n vertices.

  Complexity

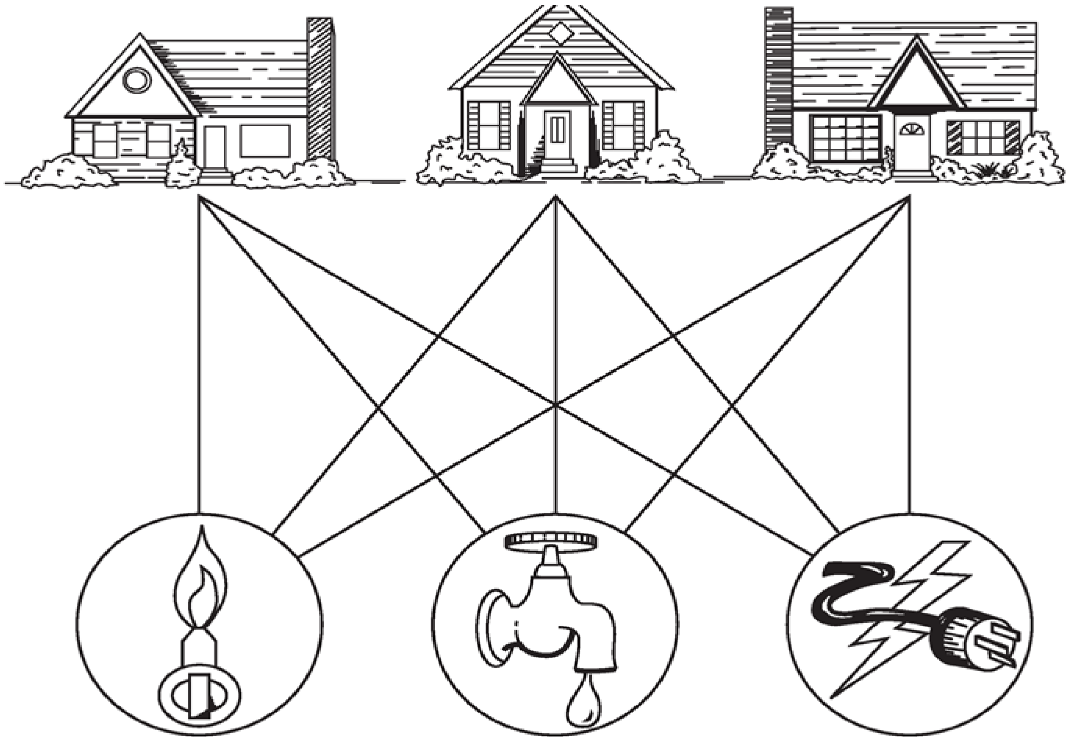
  read the Textbook p.712 – p.714

Another Example



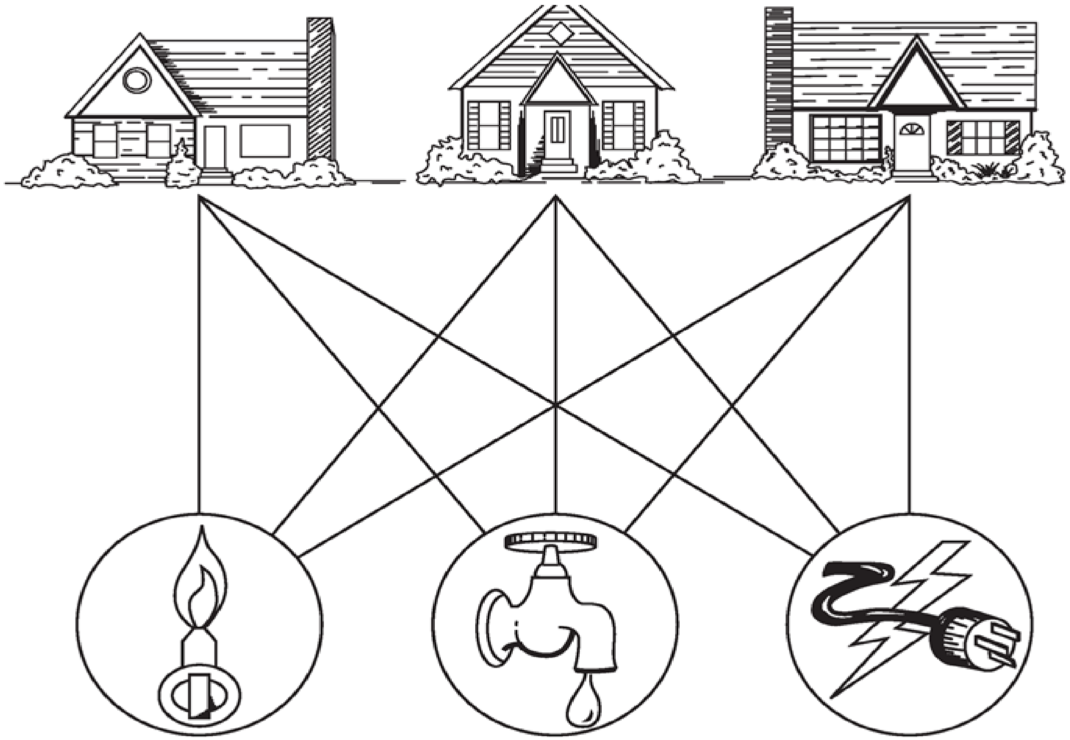
Planar Graphs

- Join three houses to each of three seperate utilities.



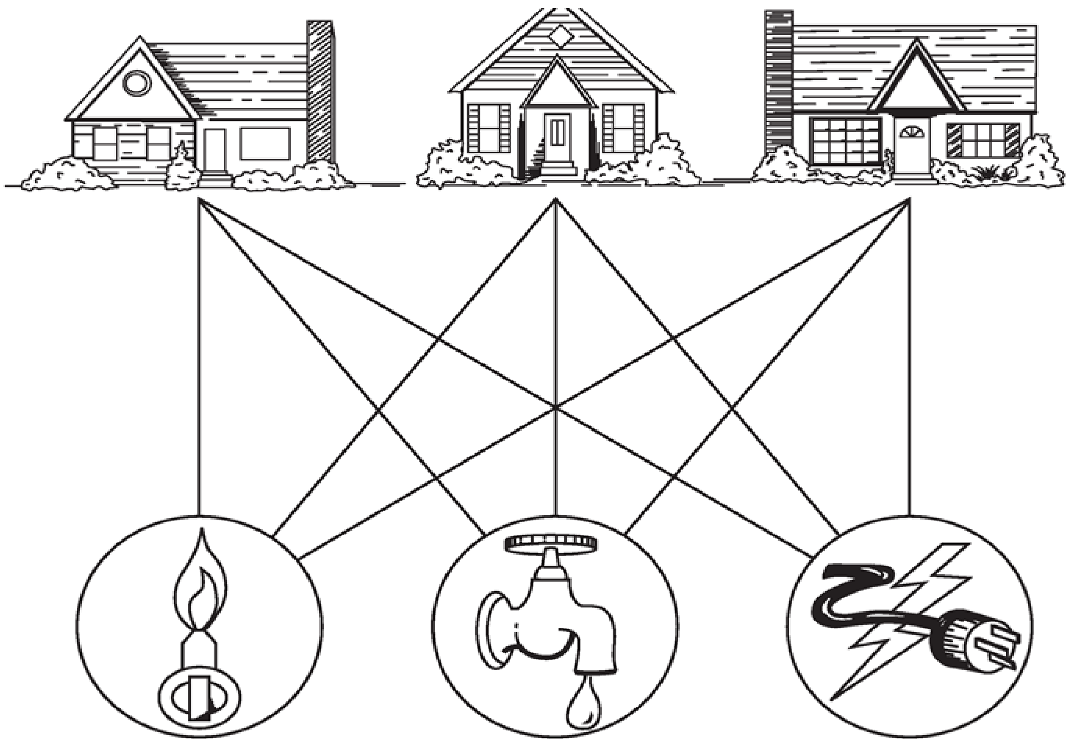
Planar Graphs

- Join three houses to each of three seperate utilities.



Can this graph be drawn in the plane s.t. no two of its edges cross?

- Join three houses to each of three seperate utilities.



Can this graph be drawn in the plane s.t. no two of its edges cross? $K_{3,3}$

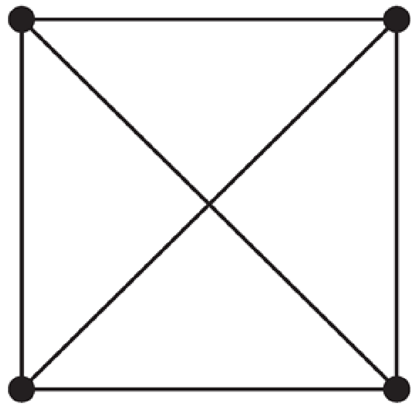
Planar Graphs

- **Definition** A graph is called *planar* if it can be drawn in the plane without any edges crossing. Such a drawing is called a *planar representation* of the graph.

Planar Graphs

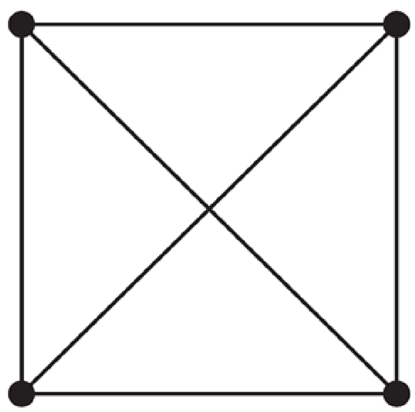
- **Definition** A graph is called *planar* if it can be drawn in the plane without any edges crossing. Such a drawing is called a *planar representation* of the graph.

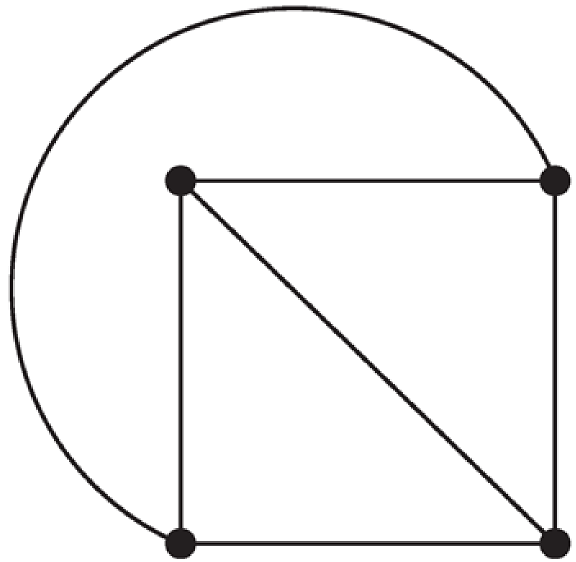
Example Is K_4 planar?



- **Definition** A graph is called *planar* if it can be drawn in the plane without any edges crossing. Such a drawing is called a *planar representation* of the graph.

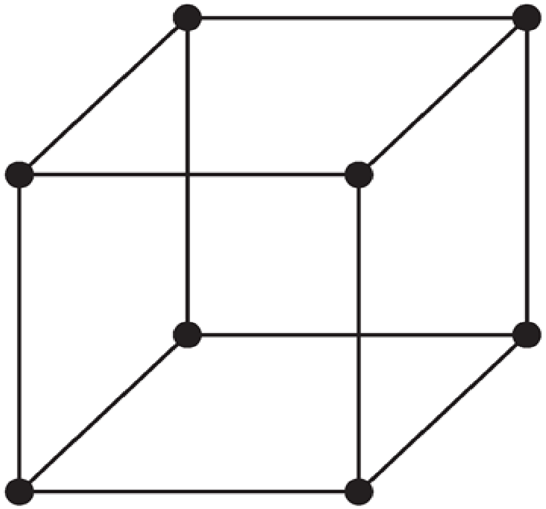
Example Is K_4 planar?





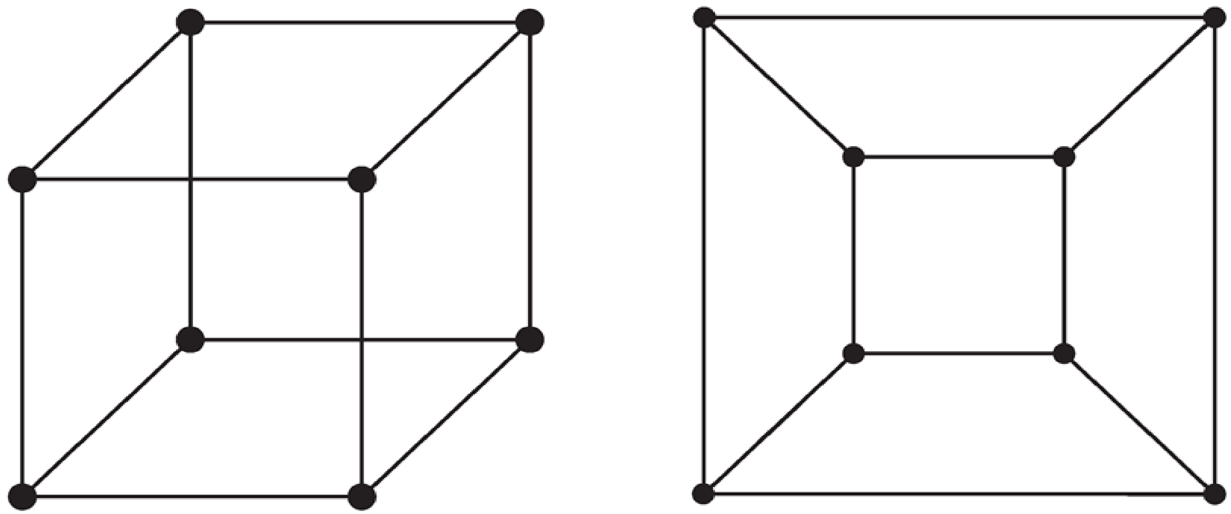
Planar Graphs

- **Example**



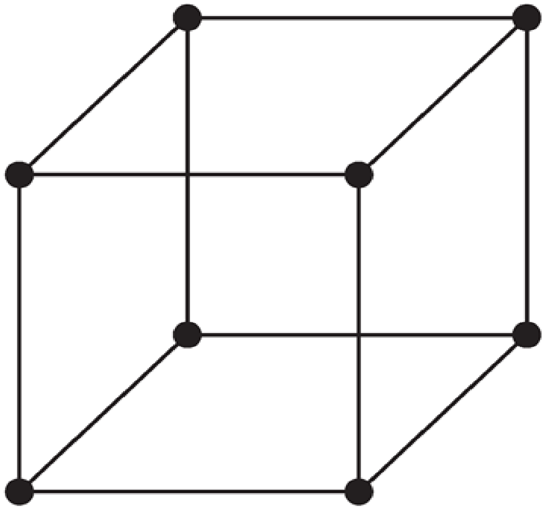
Planar Graphs

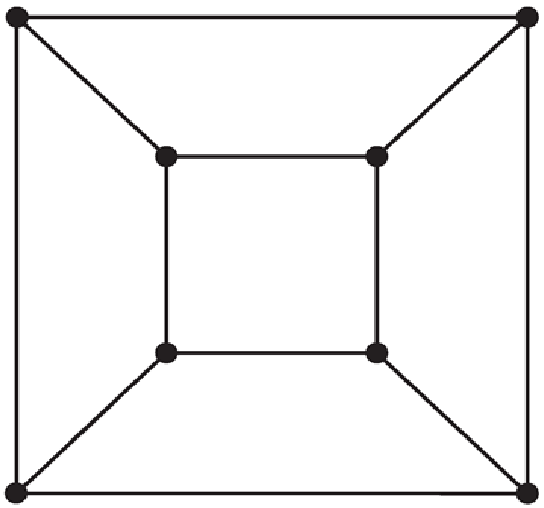
- **Example**

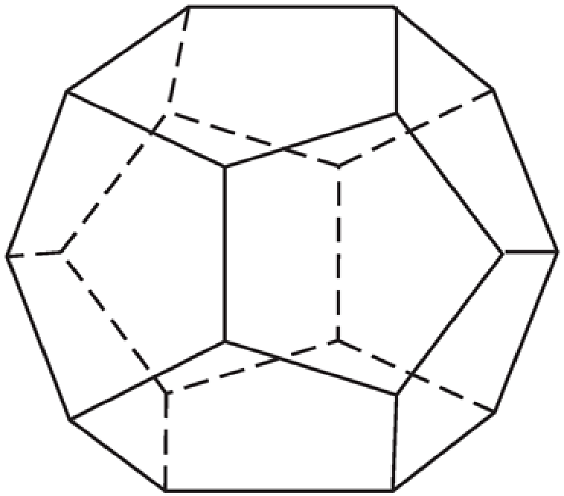


Planar Graphs

- **Example**

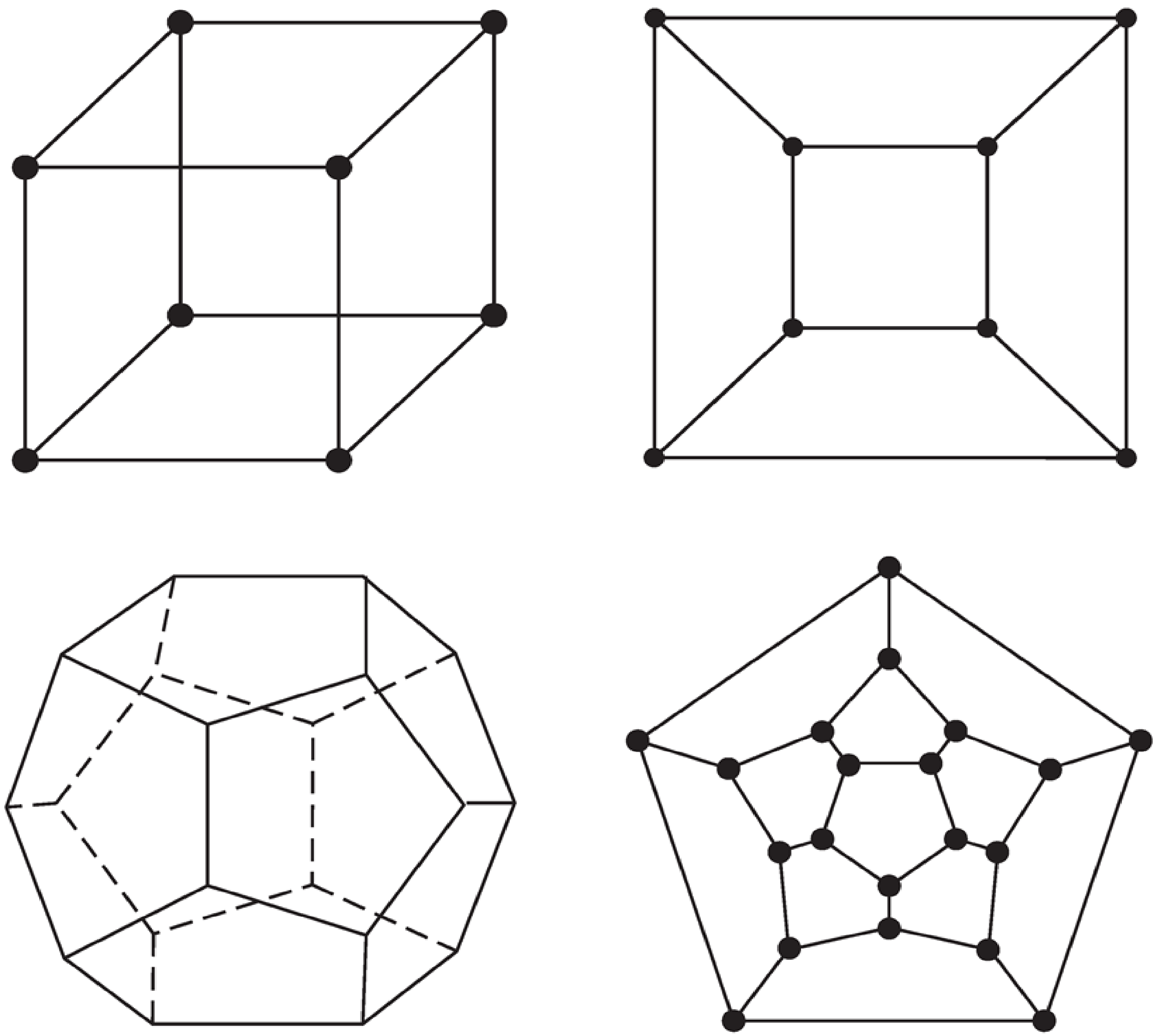






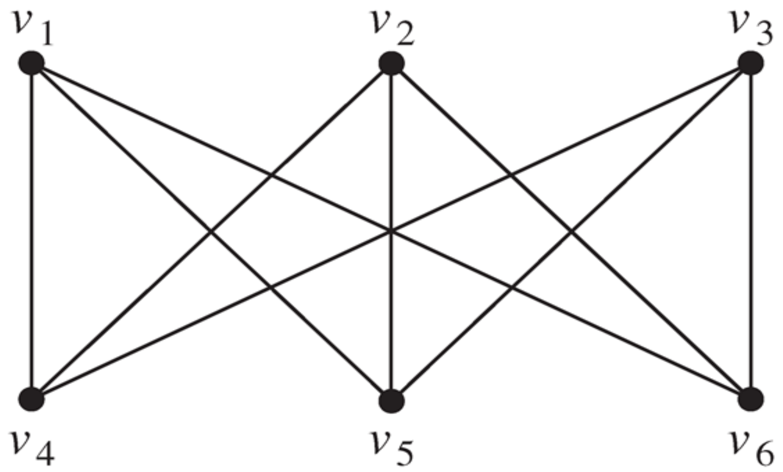
(a)

- **Example**



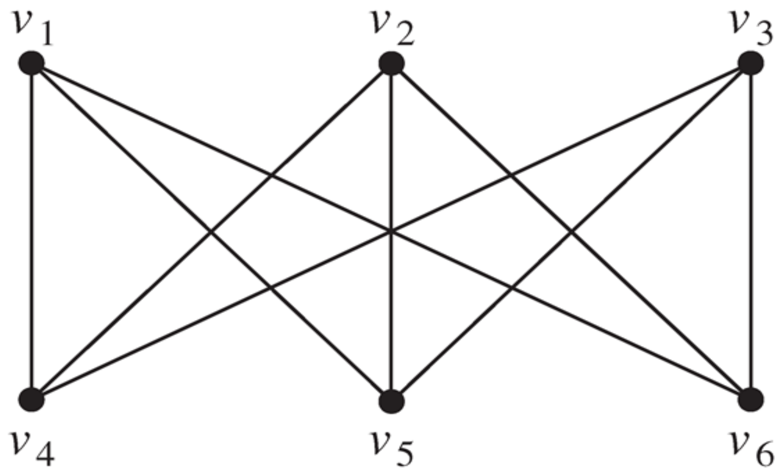
(a) (b)

- **Example**



Planar Graphs

- **Example**



Applications

- ◇ IC design
- ◇ design of road networks

Euler's Formula

- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

  Proof (by induction)

Euler's Formula

- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

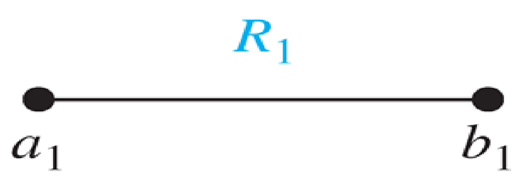
  Proof (by induction)

  Basic step:

- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

  Proof (by induction)

  Basic step:



Euler's Formula

- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

  Proof (by induction)

  Basic step:

  Inductive Hypothesis:

Euler's Formula

- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

  Proof (by induction)

  Basic step:

  Inductive Hypothesis:

$$r_k = e_k - v_k + 2$$

Euler's Formula

- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

  Proof (by induction)

  Basic step:

  Inductive Hypothesis:

  Inductive step:

Euler's Formula

- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

  Proof (**by induction**)

  Basic step:

  Inductive Hypothesis:

  Inductive step:
  Let $\{a_{k+1}, b_{k+1}\}$ be the edge that is added to G_k to obtain G_{k+1}.

Euler's Formula

- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

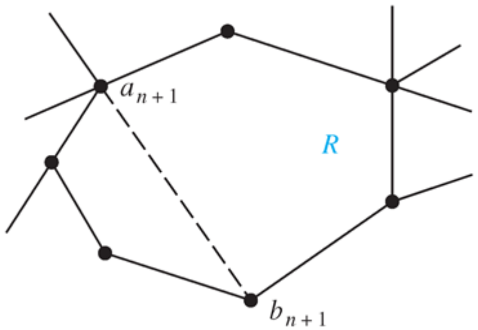
  Proof (by induction)

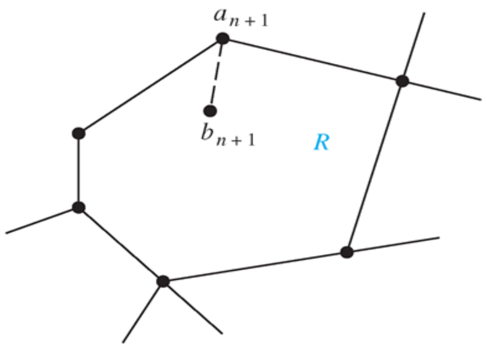
  Basic step:

  Inductive Hypothesis:

  Inductive step:

  Let $\{a_{k+1}, b_{k+1}\}$ be the edge that is added to G_k to obtain G_{k+1}.



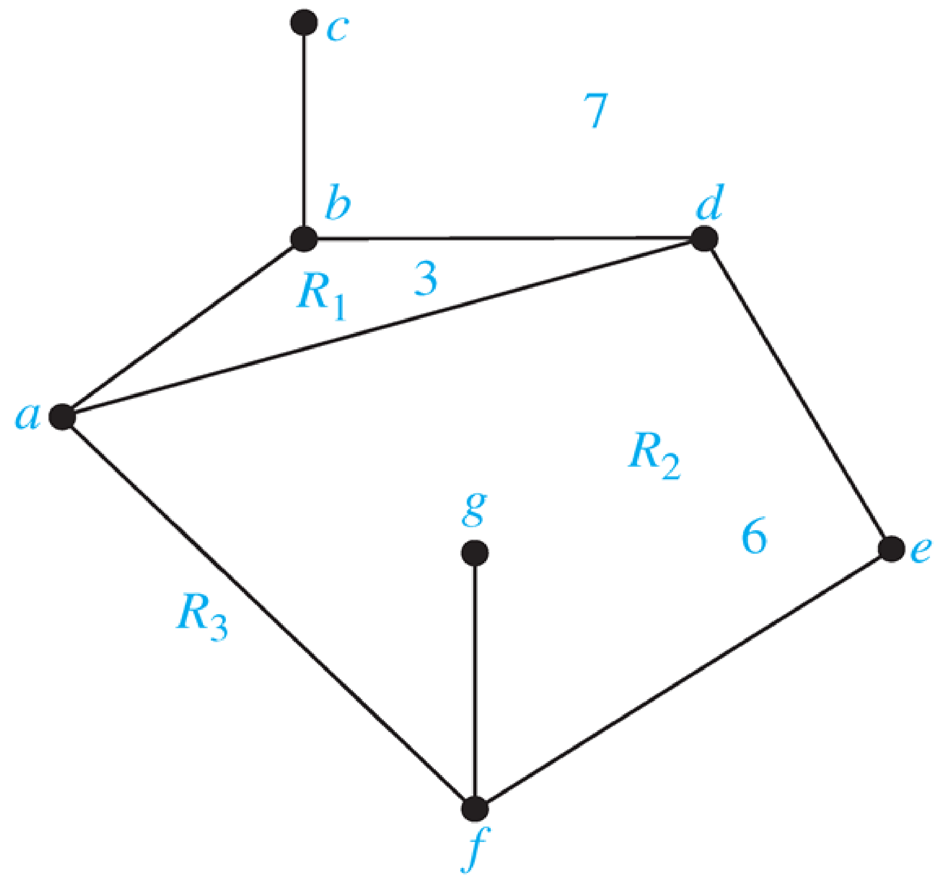


The Degree of Regions

- **Definition** The *degree* of a region is defined to be the number of edges on the boundary of this region. When an edge occurs twice on the boundary, it contributes two to the degree.

The Degree of Regions

- **Definition** The *degree* of a region is defined to be the number of edges on the boundary of this region. When an edge occurs twice on the boundary, it contributes two to the degree.



Corollaries

- **Corollary 1** If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Corollaries

- **Corollary 1** If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

  Proof The degree of every region is at least 3.

Corollaries

- **Corollary 1** If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

  Proof The degree of every region is at least 3.

  - $\diamond$ G is simple
  - $\diamond$ $v \geq 3$

Corollaries

- **Corollary 1** If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

  Proof The degree of every region is at least 3.

  - G is simple
  - $v \geq 3$

  The sum of the degrees of the regions is exactly twice the number of edges in the graph.

Corollaries

- **Corollary 1** If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

  Proof The degree of every region is at least 3.

  - G is simple
  - $v \geq 3$

  The sum of the degrees of the regions is exactly twice the number of edges in the graph.

$$2e = \sum_{\text{all regions } R} \deg(R) \geq 3r$$

Corollaries

- **Corollary 1** If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

  Proof The degree of every region is at least 3.

  - G is simple
  - $v \geq 3$

  The sum of the degrees of the regions is exactly twice the number of edges in the graph.

$$2e = \sum_{\text{all regions } R} \deg(R) \geq 3r$$

  By Euler's formula, the proof is completed.

Corollaries

- **Corollary 2** If G is a connected planar simple graph, then G has a vertex of degree not exceeding 5.

Corollaries

- **Corollary 2** If G is a connected planar simple graph, then G has a vertex of degree not exceeding 5.

  Proof

Corollaries

- **Corollary 2** If G is a connected planar simple graph, then G has a vertex of degree not exceeding 5.

  Proof

  (By contradiction)

  By Corollary 1 and the Handshaking Theorem.

- **Corollary 2** If G is a connected planar simple graph, then G has a vertex of degree not exceeding 5.

Proof

(By contradiction)

By Corollary 1 and the Handshaking Theorem.

Corollary 3 In a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Corollaries

- **Corollary 2** If G is a connected planar simple graph, then G has a vertex of degree not exceeding 5.

  Proof

  (By contradiction)

  By Corollary 1 and the Handshaking Theorem.

  Corollary 3 In a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

  Proof similar to that of Corollary 1.

Examples

- Show that K_5 is nonplanar.

Examples

- Show that K_5 is nonplanar.

  Using Corollary 1

Examples

- Show that K_5 is nonplanar.

  Using Corollary 1

  Show that $K_{3,3}$ is nonplanar.

Examples

- Show that K_5 is nonplanar.

  Using Corollary 1

  Show that $K_{3,3}$ is nonplanar.

  Using Corollary 3

Examples

- Show that K_5 is nonplanar.

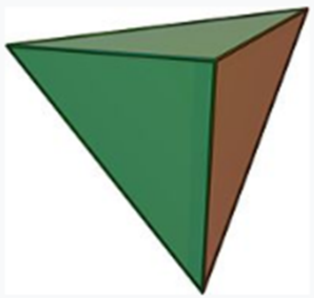
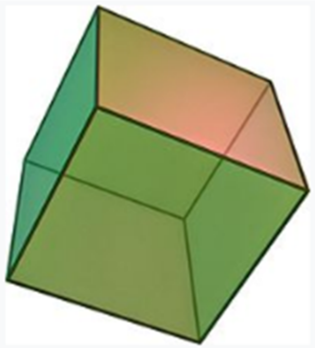
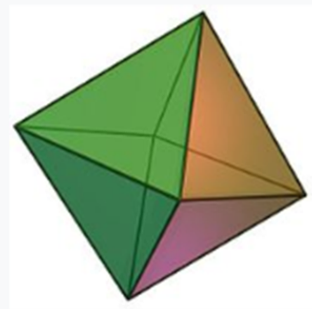
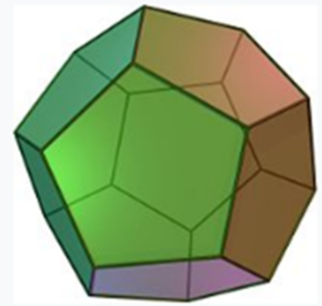
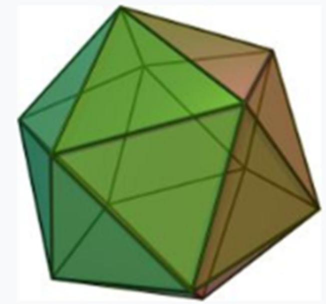
  Using Corollary 1

  Show that $K_{3,3}$ is nonplanar.

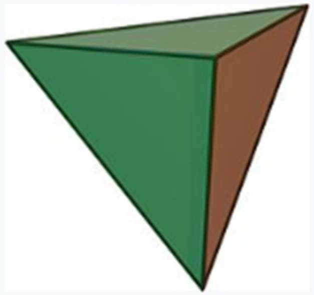
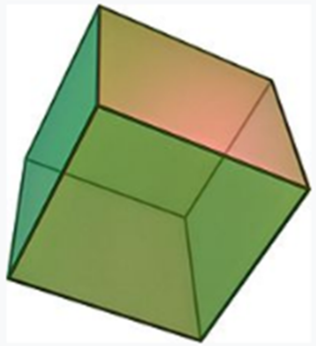
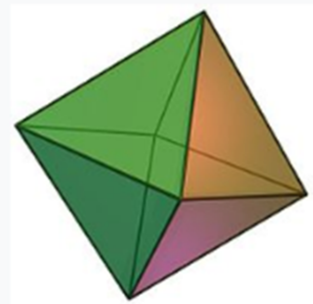
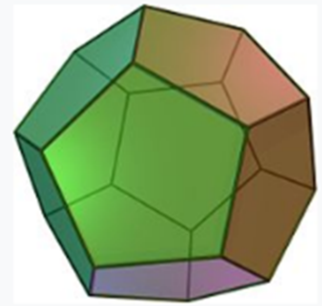
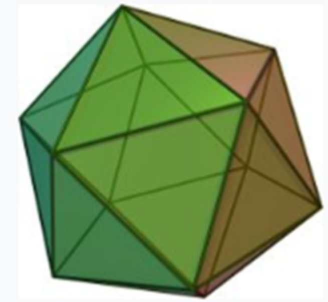
  Using Corollary 3

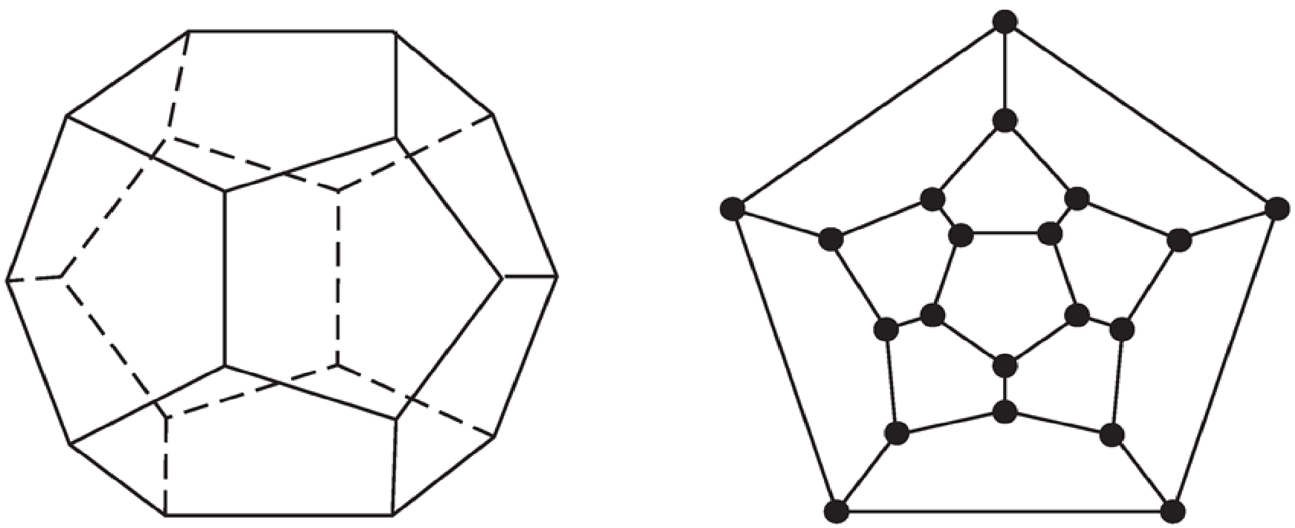
  Corollary 2 is used in the proof of Five Color Theorem.

Only 5 Platonic Solids

				
Tetrahedron {3, 3}	Cube {4, 3}	Octahedron {3, 4}	Dodecahedron {5, 3}	Icosahedron {3, 5}

Only 5 Platonic Solids

				
Tetrahedron {3, 3}	Cube {4, 3}	Octahedron {3, 4}	Dodecahedron {5, 3}	Icosahedron {3, 5}



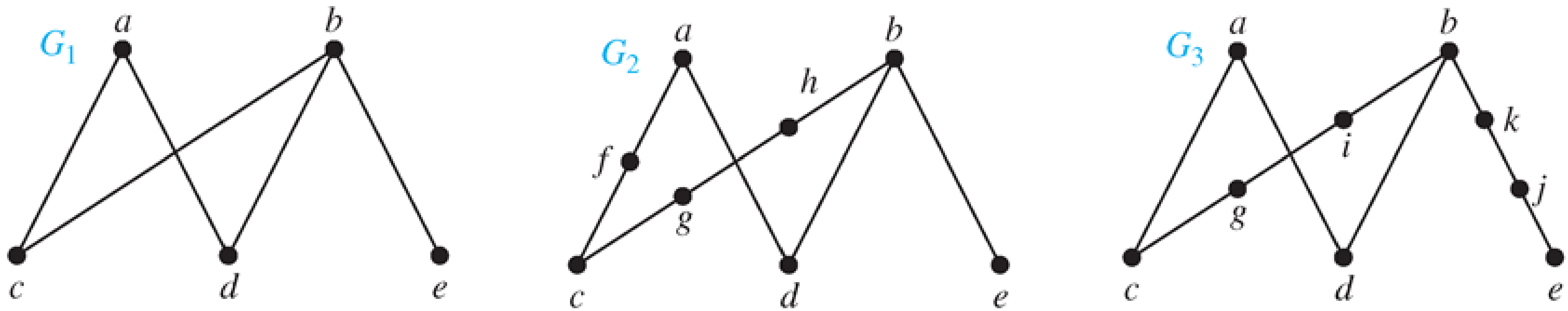
(a) (b)

Kuratowski's Theorem

- **Definition** If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an *elementary subdivision*. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *homomorphic* if they can be obtained from the same graph by a sequence of elementary subdivisions.

Kuratowski's Theorem

- **Definition** If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an *elementary subdivision*. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *homomorphic* if they can be obtained from the same graph by a sequence of elementary subdivisions.

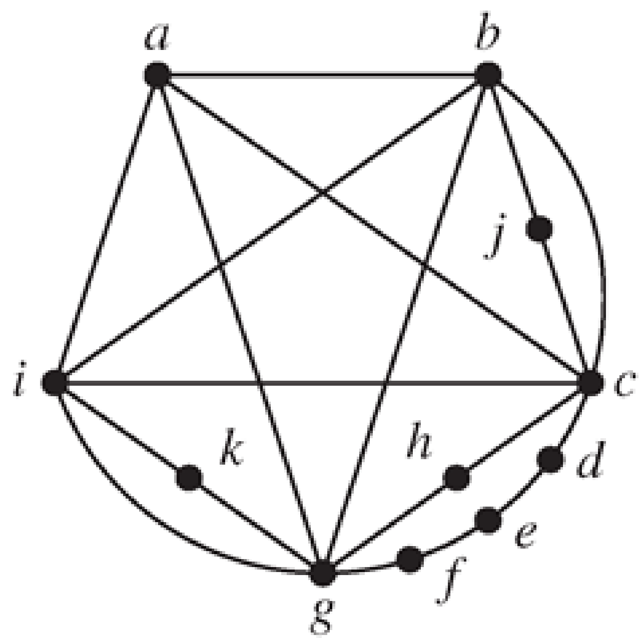


Kuratowski's Theorem

- **Definition** If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an *elementary subdivision*. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *homomorphic* if they can be obtained from the same graph by a sequence of elementary subdivisions.

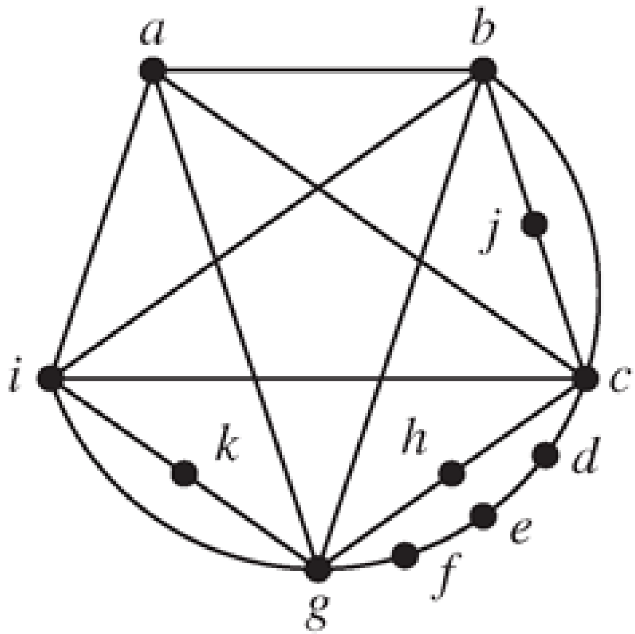
  Theorem A graph is nonplanar if and only if it contains a subgraph homomorphic to $K_{3,3}$ or K_5.

Examples

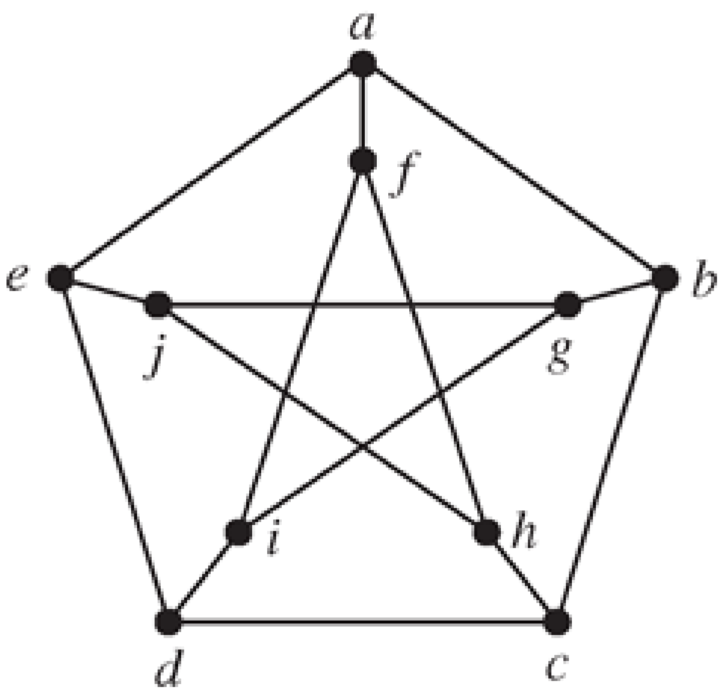


G

Examples

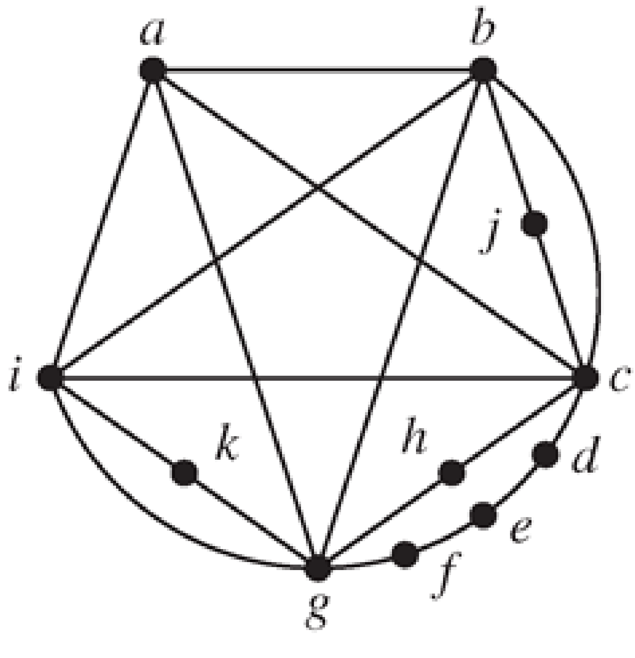


G

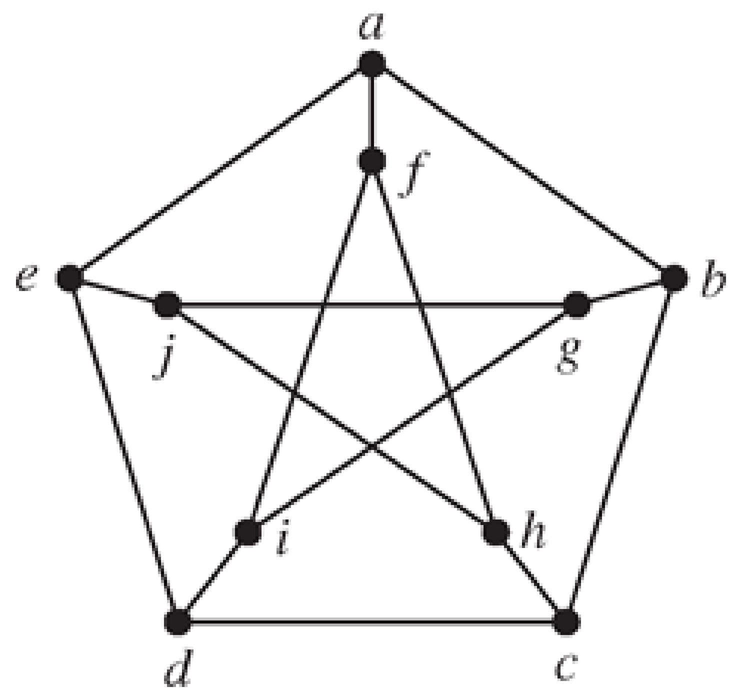


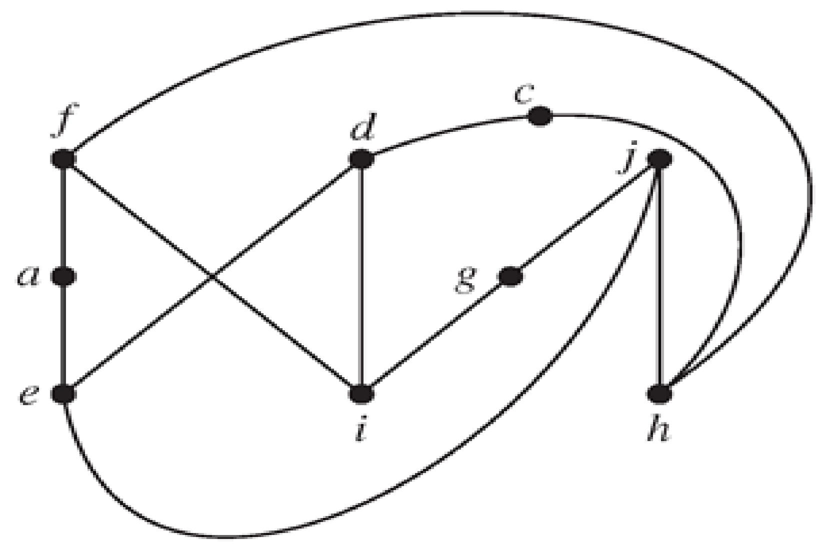
都不是平面图

Examples



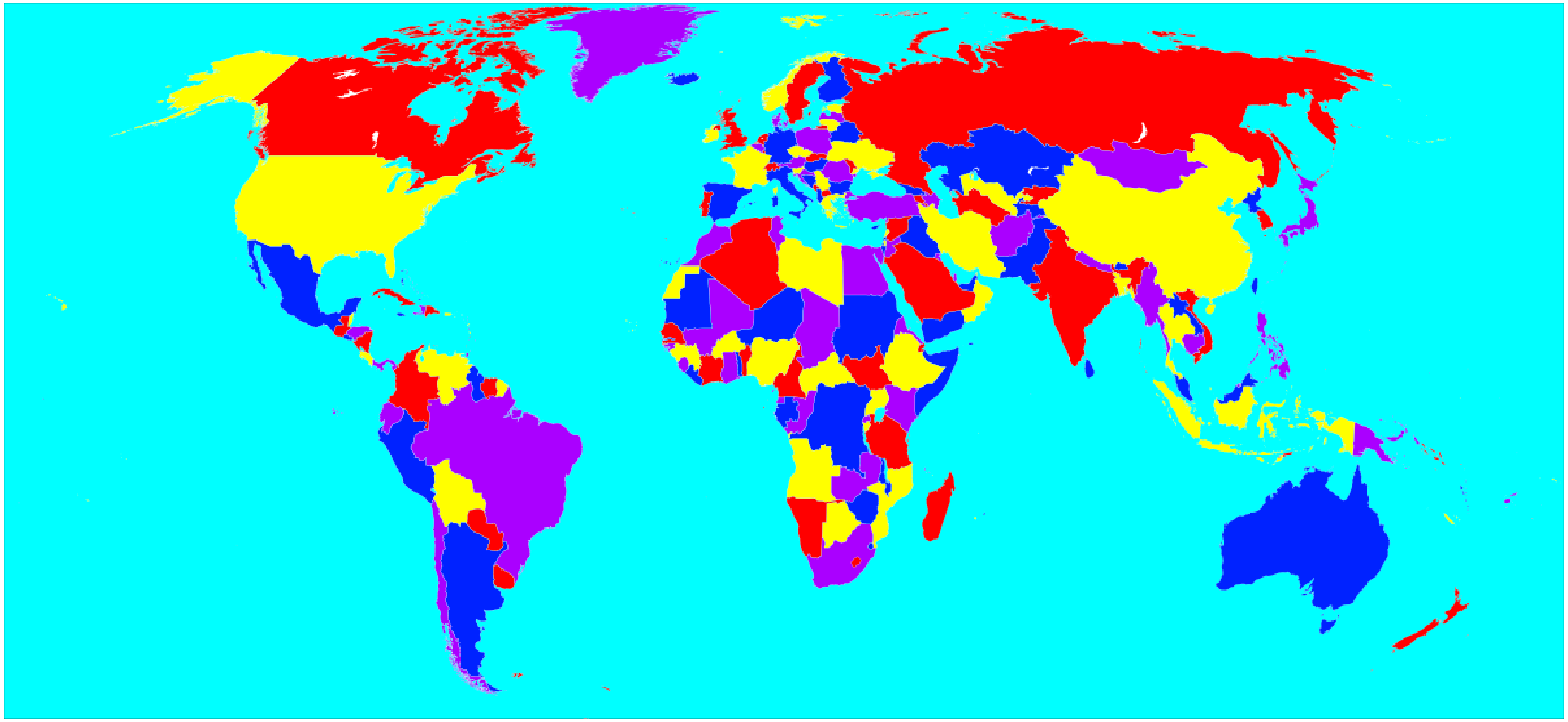
G





Graph Coloring

- **Four-color theorem** Given any separation of a plane into contiguous regions, producing a figure called a *map*, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.



Graph Coloring

Four-color theorem

◇ first proposed by Francis Guthrie in 1852

◇ his brother Frederick Guthrie told Augustus De Morgan

◇ De Morgan wrote to William Hamilton

◇ Alfred Kempe proved it incorrectly in 1879

◇ Percy Heawood found an error in 1890 and proved the *five-color theorem*

◇ Finally, Kenneth Appel and Wolfgang Haken proved it with case by case analysis by computer in 1976 (*the first computer-aided proof*)

◇ Kempe's incorrect proof serves as a basis

Graph Coloring

- A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Graph Coloring

- A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

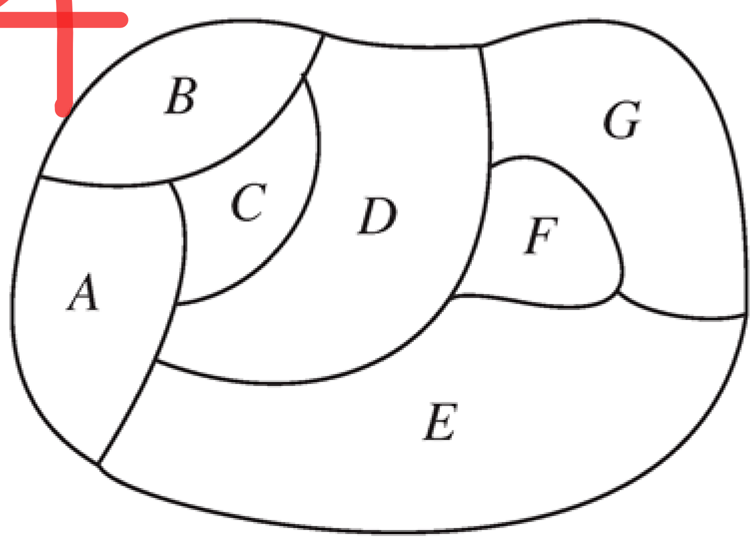
  The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph, denoted by $\chi(G)$.

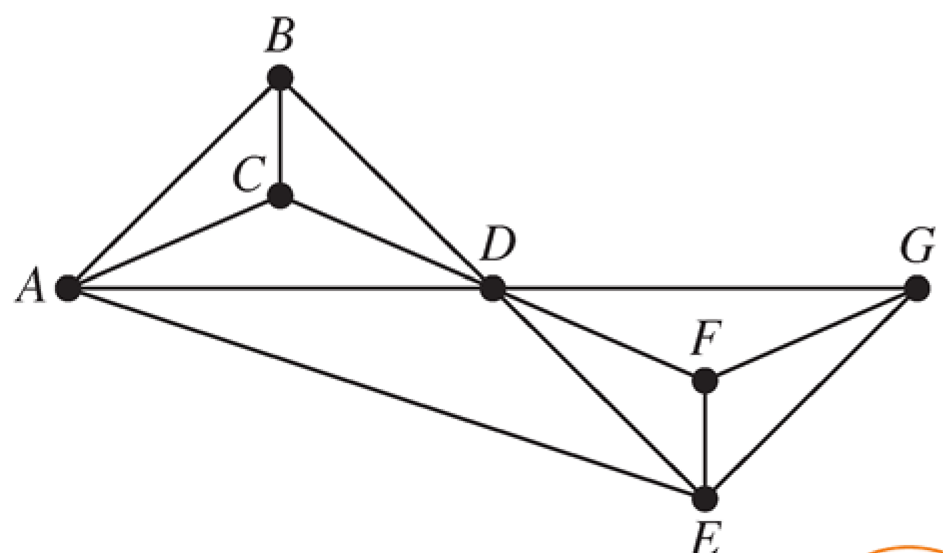
Graph Coloring

- A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

  The *chromatic number* of a graph is the least number of colors needed for a coloring of this graph, denoted by $\chi(G)$.

4



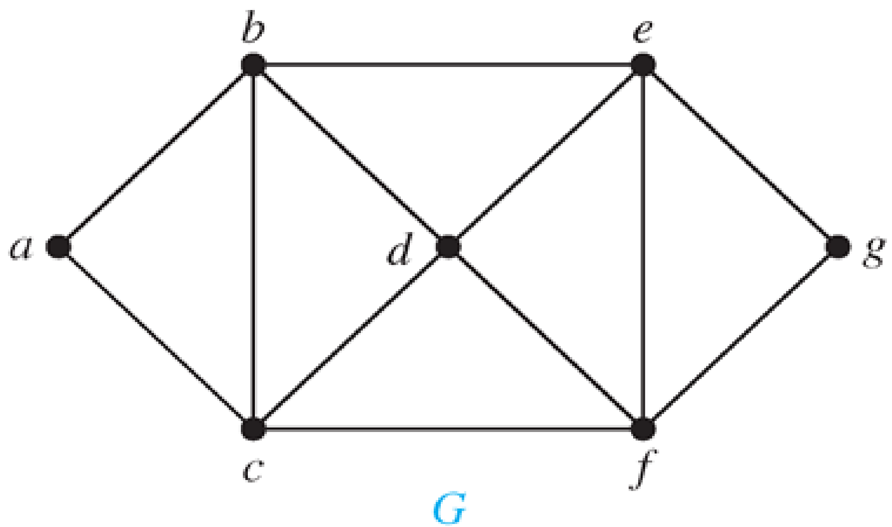


Graph Coloring

- **Theorem** (Four Color Theorem) The chromatic number of a planar graph is no greater than four.

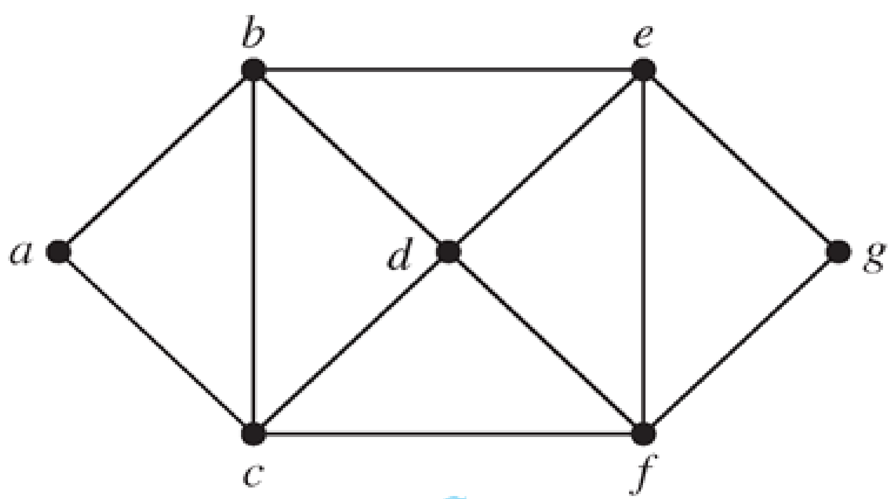
Graph Coloring

- **Theorem** (Four Color Theorem) The chromatic number of a planar graph is no greater than four.



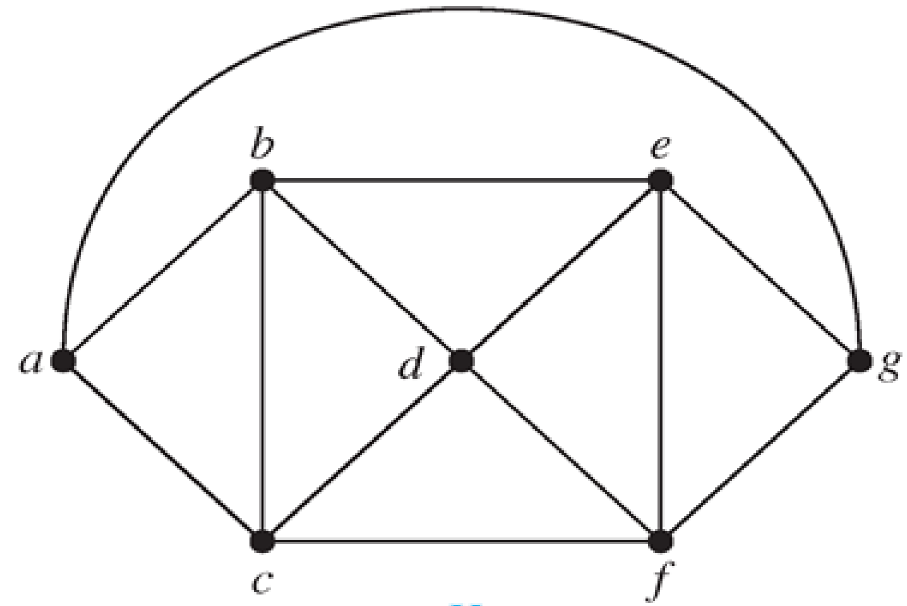
Graph Coloring

- **Theorem** (Four Color Theorem) The chromatic number of a planar graph is no greater than four.



G

3



H

Graph Coloring

- **Theorem** (Six Color Theorem) The chromatic number of a planar graph is no greater than six.

- **Theorem** (Six Color Theorem) The chromatic number of a planar graph is no greater than six.

  Proof (by induction on the number of vertices)
   w.l.o.g., assume that the graph is connected.

Graph Coloring

- **Theorem** (Six Color Theorem) The chromatic number of a planar graph is no greater than six.

  Proof (by induction on the number of vertices)
  w.l.o.g., assume that the graph is connected.

  Basic step: For one single vertex, pick an arbitrary color.

Graph Coloring

- **Theorem** (Six Color Theorem) The chromatic number of a planar graph is no greater than six.

  Proof (by induction on the number of vertices)
  w.l.o.g., assume that the graph is connected.

  Basic step: For one single vertex, pick an arbitrary color.
  Inductive hypothesis: Assume that every planar graph with $k \geq 1$ or fewer vertices can be 6-colored.

- **Theorem** (Six Color Theorem) The chromatic number of a planar graph is no greater than six.

  Proof (by induction on the number of vertices)
  w.l.o.g., assume that the graph is connected.

  Basic step: For one single vertex, pick an arbitrary color.
  Inductive hypothesis: Assume that every planar graph with $k \geq 1$ or fewer vertices can be 6-colored.
  Inductive step: Consider a planar graph with $k+1$ vertices.

Graph Coloring

- **Theorem** (Six Color Theorem) The chromatic number of a planar graph is no greater than six.

  Proof (by induction on the number of vertices)
  w.l.o.g., assume that the graph is connected.

  Basic step: For one single vertex, pick an arbitrary color.
  Inductive hypothesis: Assume that every planar graph with $k \geq 1$ or fewer vertices can be 6-colored.
  Inductive step: Consider a planar graph with $k+1$ vertices. Recall Corollary 2 (the graph has a vertex of degree 5 or fewer). Remove this vertex, by i.h., we can color the remaining graph with 6 colors. Put the vertex back in. Since there are at most 5 colors adjacent, so we have at least one color left.

Graph Coloring

- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

Graph Coloring

- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

  Proof (by induction on the number of vertices)
  w.l.o.g., assume that the graph is connected.

Graph Coloring

- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

  Proof (by induction on the number of vertices)
  w.l.o.g., assume that the graph is connected.

  Basic step: For one single vertex, pick an arbitrary color.

- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

  Proof (by induction on the number of vertices)
  w.l.o.g., assume that the graph is connected.

  Basic step: For one single vertex, pick an arbitrary color.
  Inductive hypothesis: Assume that every planar graph with $k \geq 1$ or fewer vertices can be 5-colored.

Graph Coloring

- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

  Proof (by induction on the number of vertices)
  w.l.o.g., assume that the graph is connected.

  Basic step: For one single vertex, pick an arbitrary color.
  Inductive hypothesis: Assume that every planar graph with $k \geq 1$ or fewer vertices can be 5-colored.
  Inductive step: Consider a planar graph with $k+1$ vertices.

Graph Coloring

- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

  Proof (by induction on the number of vertices)
  w.l.o.g., assume that the graph is connected.

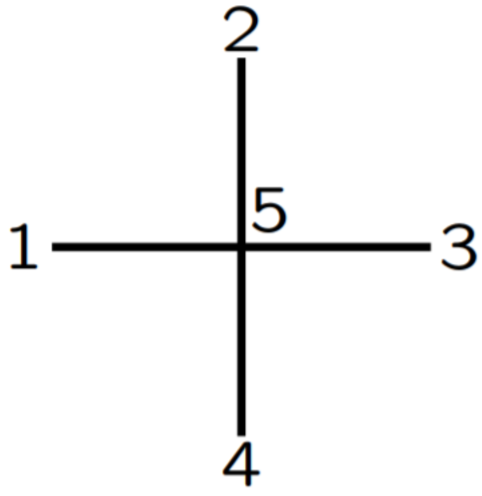
  Basic step: For one single vertex, pick an arbitrary color.
  Inductive hypothesis: Assume that every planar graph with $k \geq 1$ or fewer vertices can be 5-colored.
  Inductive step: Consider a planar graph with $k + 1$ vertices. Recall Corollary 2 (the graph has a vertex of degree 5 or fewer). Remove this vertex, by i.h., we can color the remaining graph with 5 colors. Put the vertex back in.

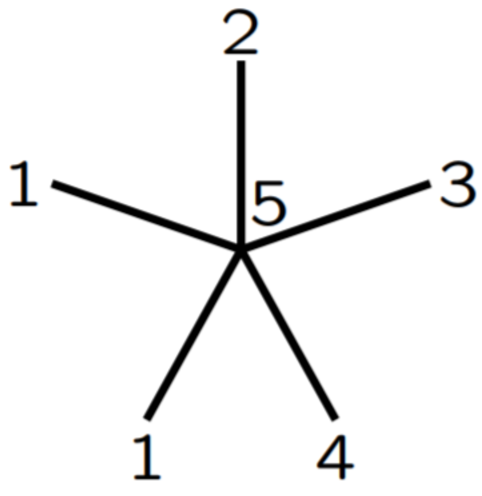
Graph Coloring

- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

  Proof (by induction on the number of vertices)
  w.l.o.g., assume that the graph is connected.

  If the vertex has degree less than 5, or if it has degree 5 and only ≤ 4 colors are used for vertices connected to it, we an pick an available color for it.



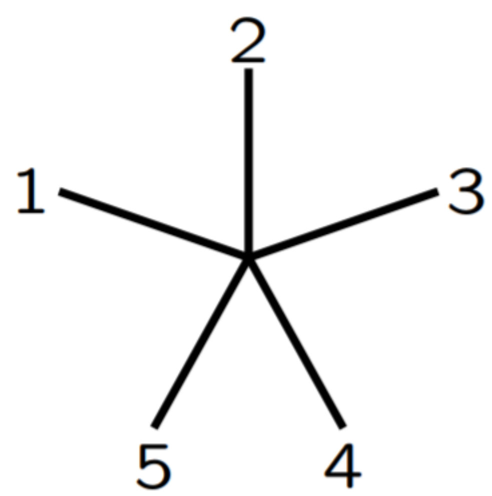


Graph Coloring

- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

  Proof (by induction on the number of vertices)

  If the vertex has degree 5, and all 5 colors are connected to it, we label the vertices adjacent to the “special” vertex (degree 5) 1 to 5 (in order).



Graph Coloring

- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

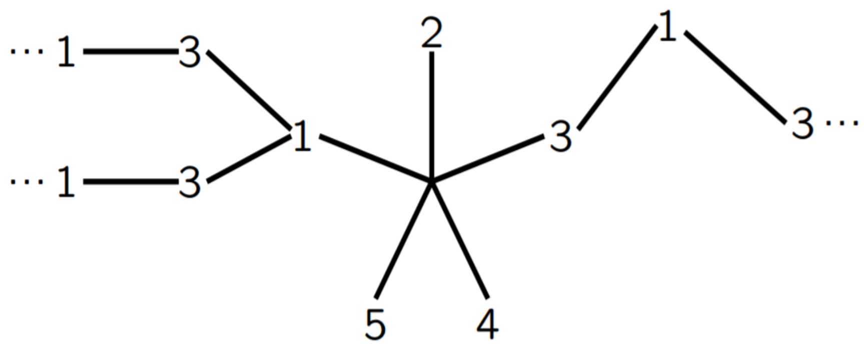
  Proof (by induction on the number of vertices)

  We make a subgraph out of all the vertices colored 1 or 3. If the adjacent vertex colored 1 and the adjacent vertex colored 3 are not connected by a path in the subgraph.

- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

  Proof (by induction on the number of vertices)

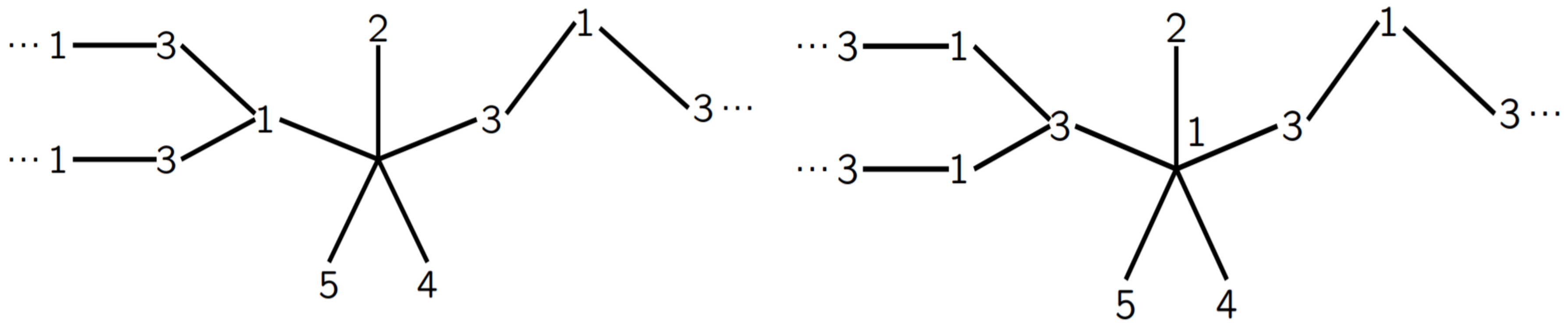
  We make a subgraph out of all the vertices colored 1 or 3. If the adjacent vertex colored 1 and the adjacent vertex colored 3 are not connected by a path in the subgraph.



- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

  Proof (by induction on the number of vertices)

  We make a subgraph out of all the vertices colored 1 or 3. If the adjacent vertex colored 1 and the adjacent vertex colored 3 are not connected by a path in the subgraph.



- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

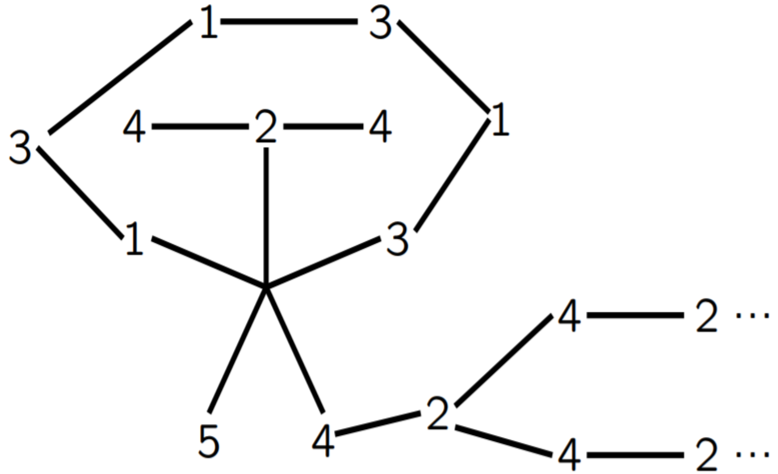
  Proof (by induction on the number of vertices)

  On the other hand, if the vertices colored 1 and 3 are connected via a path in the subgraph, we do the the same for the vertices colored 2 and 4. Note that this will be a disconnected pair of subgraphs, separated by a path connecting the vertices colored 1 and 3 (Why?)

- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

Proof (by induction on the number of vertices)

On the other hand, if the vertices colored 1 and 3 are connected via a path in the subgraph, we do the the same for the vertices colored 2 and 4. Note that this will be a disconnected pair of subgraphs, separated by a path connecting the vertices colored 1 and 3 (Why?)

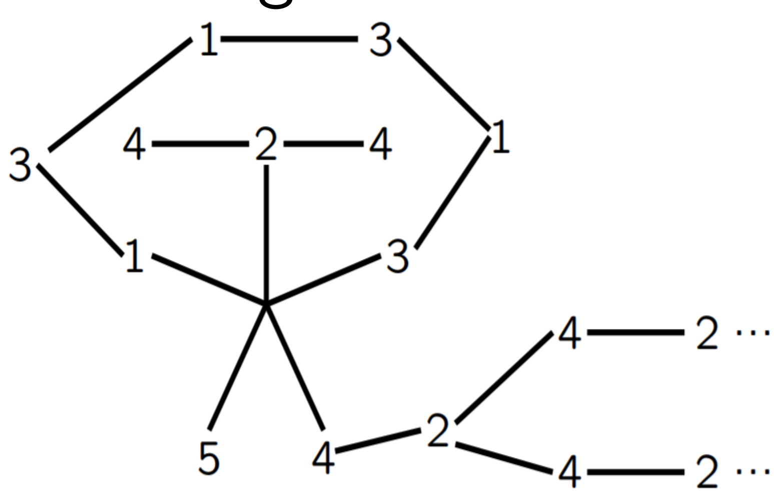


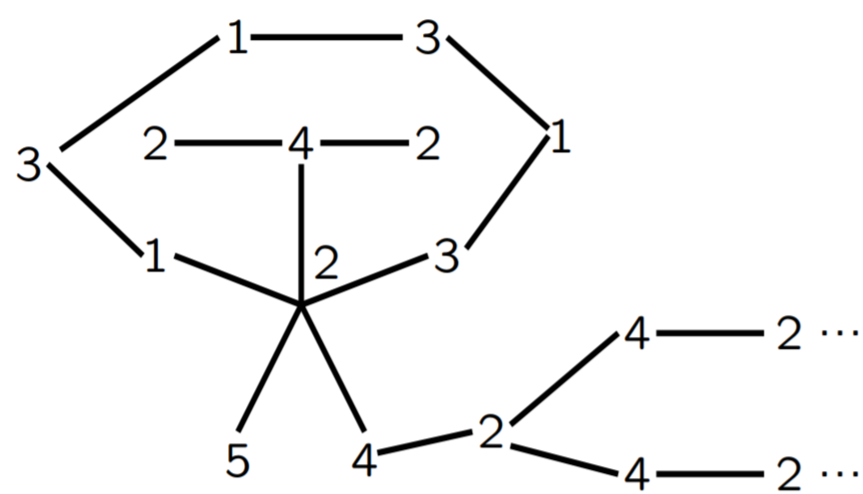
Graph Coloring

- **Theorem** (Five Color Theorem) The chromatic number of a planar graph is no greater than five.

  Proof (by induction on the number of vertices)

  On the other hand, if the vertices colored 1 and 3 are connected via a path in the subgraph, we do the the same for the vertices colored 2 and 4. Note that this will be a disconnected pair of subgraphs, separated by a path connecting the vertices colored 1 and 3 (Why?)



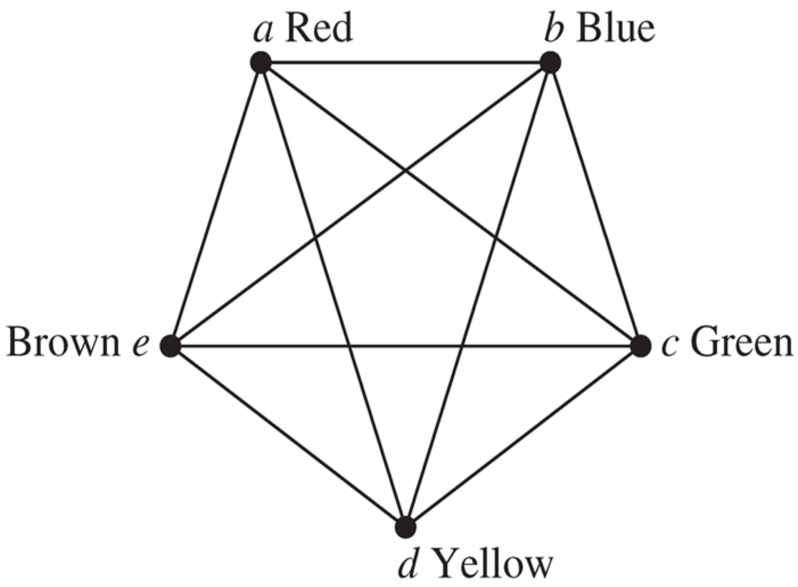


Examples

- What is the chromatic number of K_n, $K_{m,n}$, C_n?

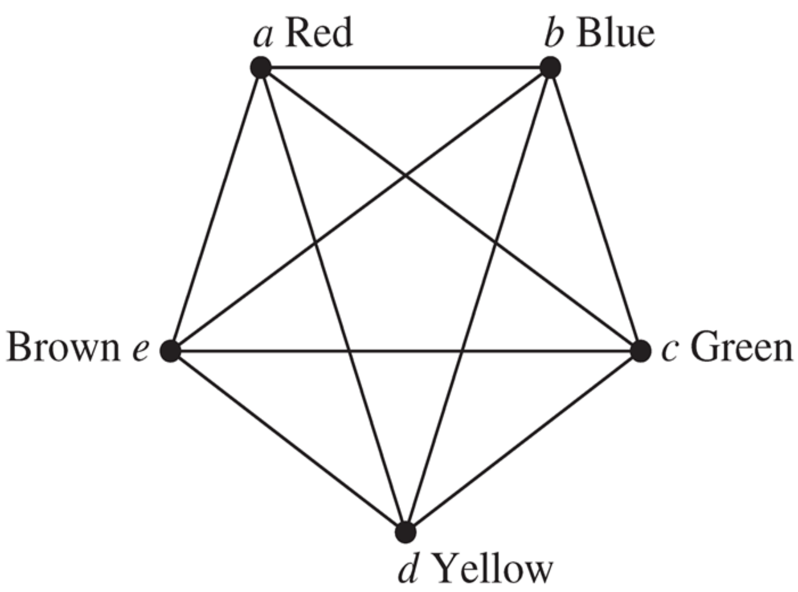
Examples

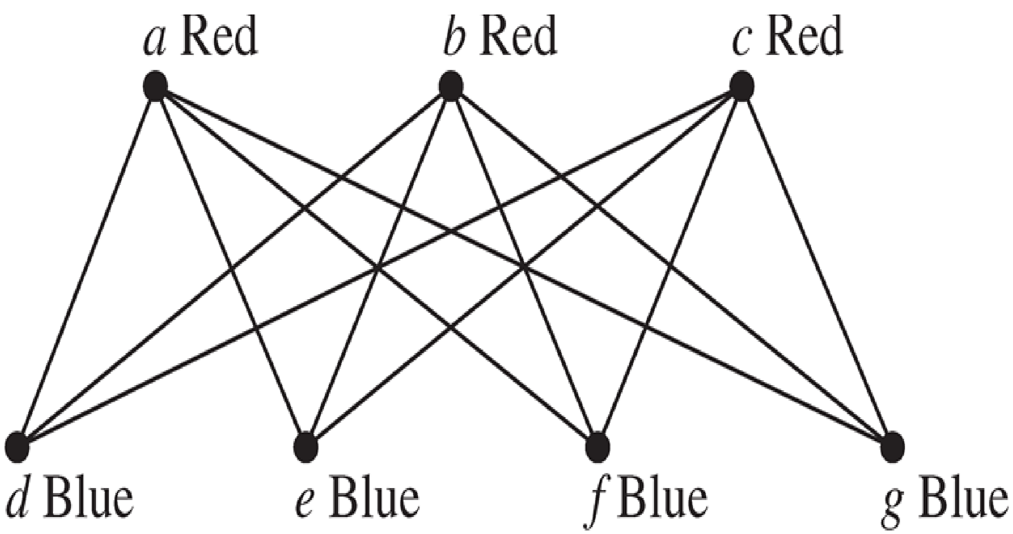
- What is the chromatic number of K_n, $K_{m,n}$, C_n?



Examples

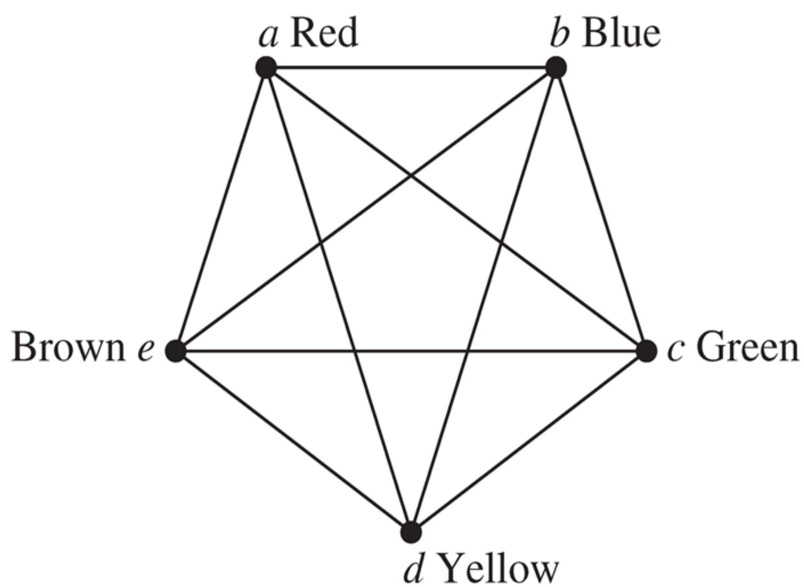
- What is the chromatic number of K_n, $K_{m,n}$, C_n?

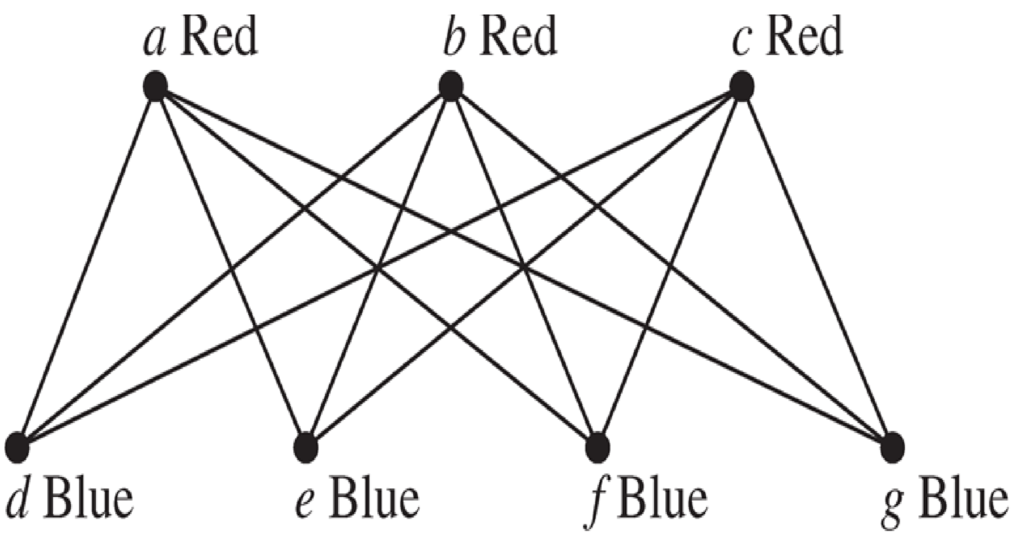


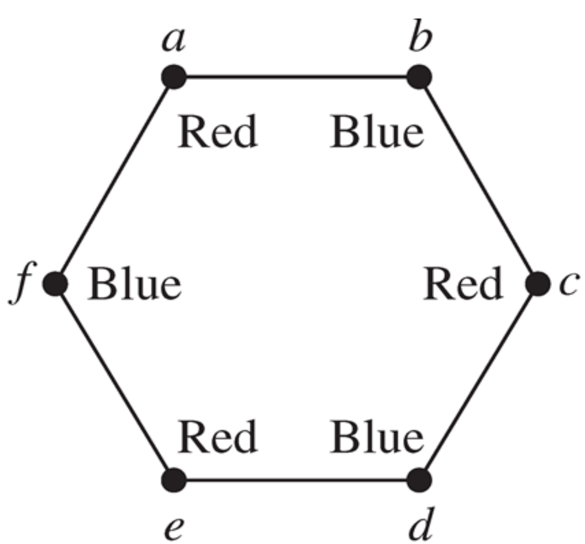


Examples

- What is the chromatic number of K_n, $K_{m,n}$, C_n?

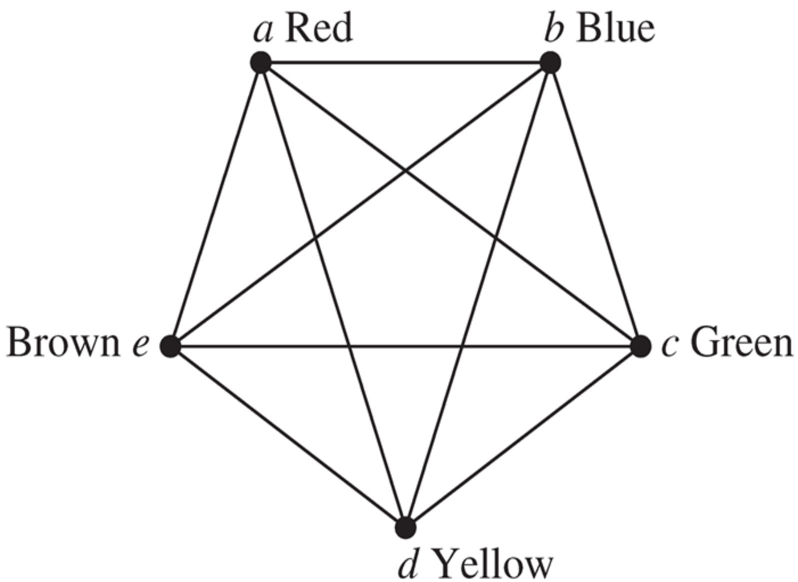


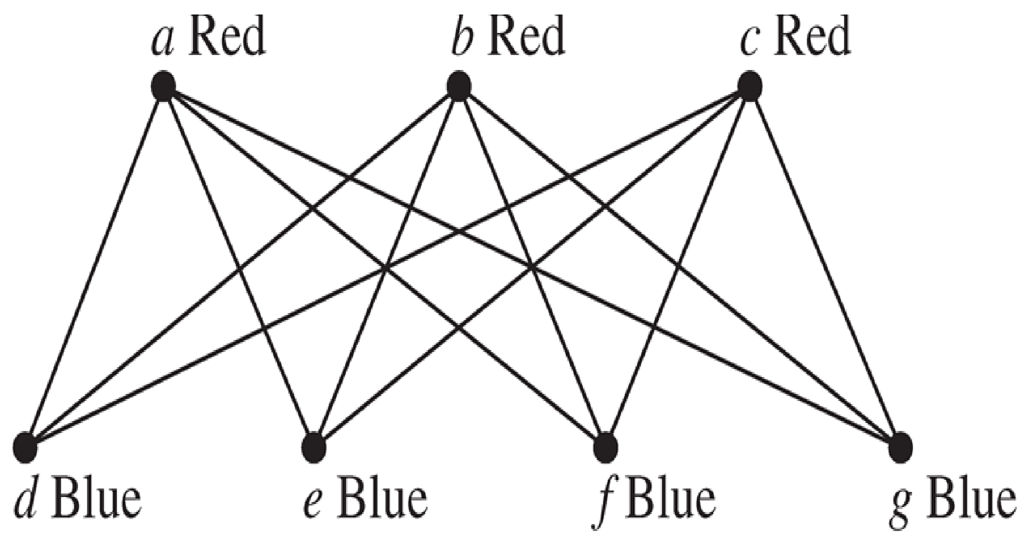


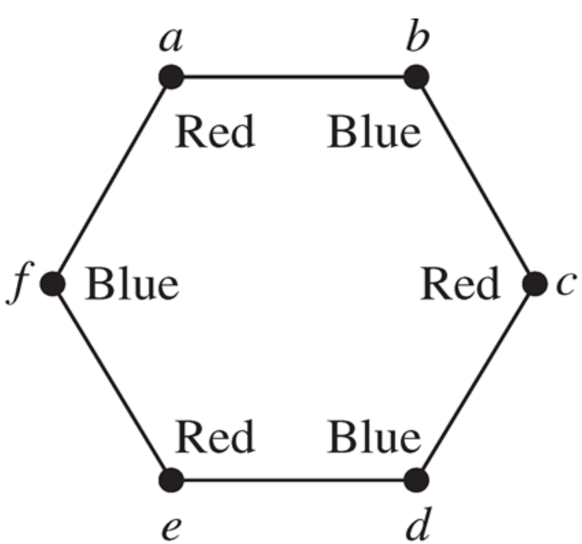


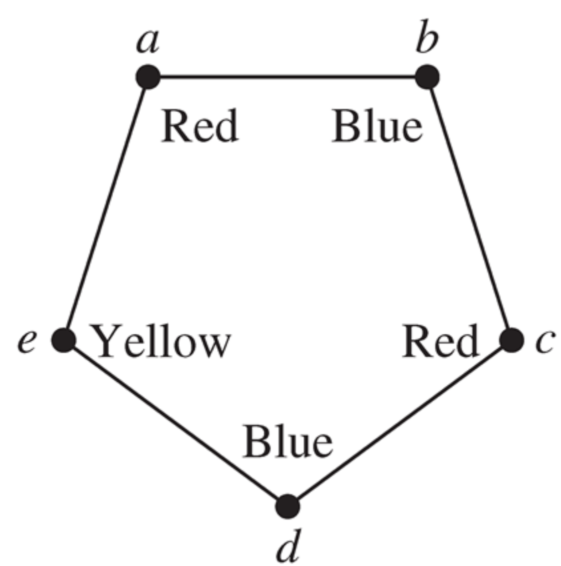
Examples

- What is the chromatic number of K_n, $K_{m,n}$, C_n?



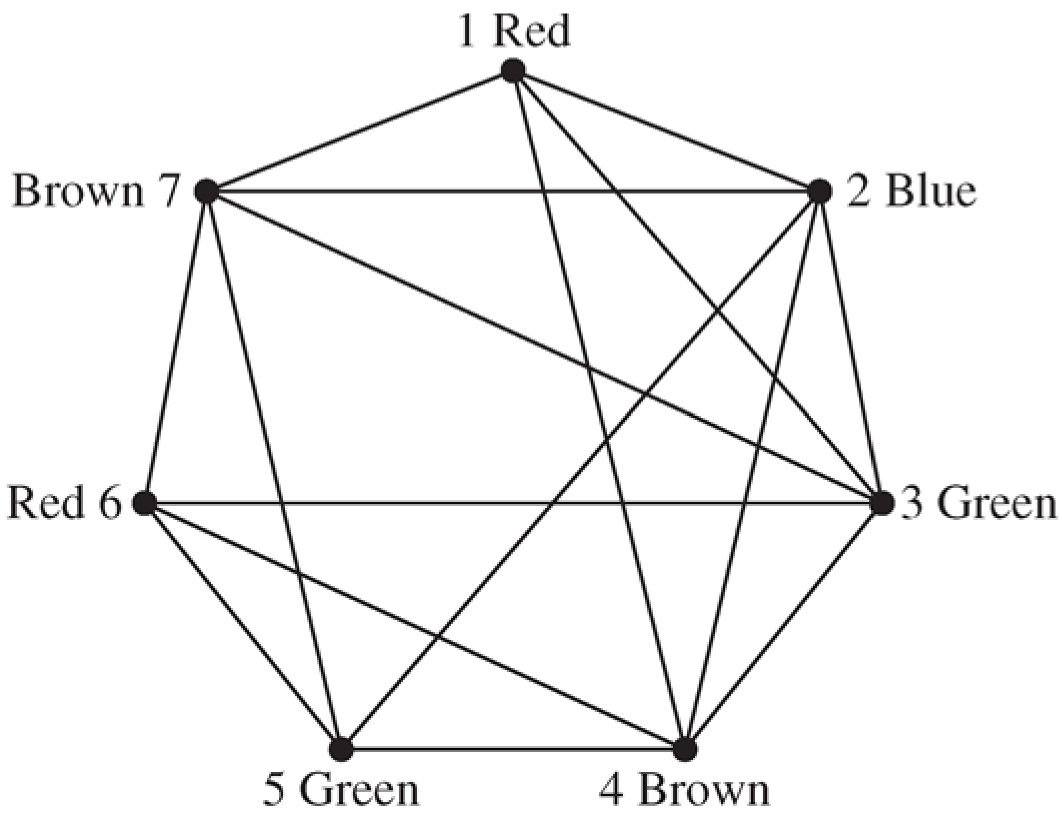






- Scheduling Final Exams

Vertices represent courses, and there is an edge between two vertices if there is a common student in the courses.



Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

Applications of Graph Coloring

- Channel Assignments

  Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel . How can the assignment of channels be modeled by graph coloring?

Applications of Graph Coloring

- Channel Assignments

  Television channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel . How can the assignment of channels be modeled by graph coloring?

Graph Coloring $\in$ NPC

Next Lecture

- tree ...