CS201: Discrete Math for Computer Science
2020 Fall Semester   Written Assignment # 2
Due: Oct. 27th, 2020, please submit at the beginning of class

Q.1 Suppose that A, B and C are three finite sets. For each of the following, determine whether or not it is true. Explain your answers.

(a) $(A - B = A) \rightarrow (B \subset A)$

(b) $(A - B = \emptyset) \rightarrow (A \cap B = B \cap A)$

(c) $(A \subseteq B) \rightarrow (|A \cup B| \geq 2|A|)$

(d) $\overline{(A - B)} \cap (B - A) = B$

Q.2 Let A, B and C be sets. Prove the following using set identities.

(1) $(B - A) \cup (C - A) = (B \cup C) - A$

(2) $(A \cap B) \cap \overline{(B \cap C)} \cap (A \cap C) = \emptyset$

Q.3 The *symmetric difference* of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

(a) Determine whether the symmetric difference is associative; that is, if A, B and C are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

(b) Suppose that A, B and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

Q.4 For each set defined below, determine whether the set is *countable* or *uncountable*. Explain your answers. Recall that $\mathbb{N}$ is the set of natural numbers and $\mathbb{R}$ denotes the set of real numbers.

(a) The set of all subsets of students in CS201

(b) $\{(a, b) | a,\ b \in \mathbb{N}\}$

(c) $\{(a, b) | a \in \mathbb{N},\ b \in \mathbb{R}\}$

Q.5 Give an example of two uncountable sets A and B such that the intersection $A \cap B$ is

(a) finite,

(b) countably infinite,

(c) uncountable.

Q.6 For each of the following mappings, indicate what type of function they are (if any), not a function, one-to-one, onto, neither or both. Explain your answers.

(a) The mapping f from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(x) = |2x|$.

(b) The mapping f from $\{1, 3\}$ to $\{2, 4\}$ defined by $f(x) = 2x$.

(c) The mapping f from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = 8 - 2x$.

(d) The mapping f from $\mathbb{R}$ to $\mathbb{Z}$ defined by $f(x) = \lfloor x + 1 \rfloor$.

(e) The mapping f from $\mathbb{R}^+$ to $\mathbb{R}^+$ defined by $f(x) = x - 1$.

(f) The mapping f from $\mathbb{Z}^+$ to $\mathbb{Z}^+$ defined by $f(x) = x + 1$.

Q.7 Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

Q.8 For each set A, the *identity function* $1_A : A \to A$ is defined by $1_A(x) = x$ for all x in A. Let $f : A \to B$ and $g : B \to A$ be the functions such that $g \circ f = 1_A$. Show that f is one-to-one and g is onto.

Q.9 Suppose that two functions $g : A \to B$ and $f : B \to C$ and $f \circ g$ denotes the *composition* function.

(a) If $f \circ g$ is one-to-one and g is one-to-one, must f be one-to-one? Explain your answer.

(b) If $f \circ g$ is one-to-one and f is one-to-one, must g be one-to-one? Explain your answer.

(c) If $f \circ g$ is one-to-one, must g be one-to-one? Explain your answer.

(d) If $f \circ g$ is onto, must f be onto? Explain your answer.

(e) If $f \circ g$ is onto, must g be onto? Explain your answer.

Q.10 Let x be a real number. Show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Q.11 Derive the formula for $\sum_{k=1}^{n} k^2$.

Q.12 Derive the formula for $\sum_{k=1}^{n} k^3$.

Q.13 Find a formula for $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$, when m is a positive integer.

Q.14 Show that a subset of a countable set is also countable.

Q.15 Assume that $|S|$ denotes the cardinality of the set S. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Q.16 If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?

Q.17 The *binary insertion sort* is a variantion of the insertion sort that uses a binary search technique rather than a linear search technique to insert the ith element in the correct place among the previously sorted elements. Express the binary insertion sort in pseudocode.

Q.18 If $f_1(x)$ and $f_1(x)$ are functions from the set of positive integers to the set of positive real numbers and $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$, is $(f_1 - f_2)(x)$ also $\Theta(g(x))$? Either prove that it is or give a counter example.

Q.19 Show that if $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_{n-1}$, and a_n are real numbers and $a_n \neq 0$, then $f(x)$ is $\Theta(x^n)$.

Q.20 Prove that $n \log n = \Theta(\log n!)$ for all positive integers n.

Q.21 Prove that for any $a > 1$, $\Theta(\log_a n) = \Theta(\log_2 n)$.

Q.22 The conventional algorithm for evaluating a polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$ at $x = c$ can be expressed in pseudocode by where the final value of y is the value of the polynomial at $x = c$. Exactly how many multiplications and additions are used to evaluate a polynomial of degree n at $x = c$? (Do not count additions used to increment the loop variable).

Q.23 There is a more efficient algorithm (in terms of the number of multi-

Algorithm 1 polynomial ($c, a_0, a_1, \ldots, a_n$: real numbers)

```
power := 1
y := a_0
for i := 1 to n do
    power := power * c
    y := y + a_i * power
end for
return y {y = a_n c^n + a_{n-1} c^{n-1} + ··· + a_1 c + a_0}
```

plications and additions used) for evaluating polynomials than the conventional algorithm described in the previous exercise. It is called **Horner's method**. This pseudocode shows how to use this method to find the value of $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ at $x = c$.

Algorithm 2 Horner ($c, a_0, a_1, \ldots, a_n$: real numbers)

```
y := a_n
for i := 1 to n do
    y := y * c + a_{n-i}
end for
return y {y = a_n c^n + a_{n-1} c^{n-1} + ··· + a_1 c + a_0}
```

Exactly how many multiplications and additions are used by this algorithm to evaluate a polynomial of degree n at $x = c$? (Do not count additions used to increment the loop variable.)