Chapter 3

Arithmetic for Computers

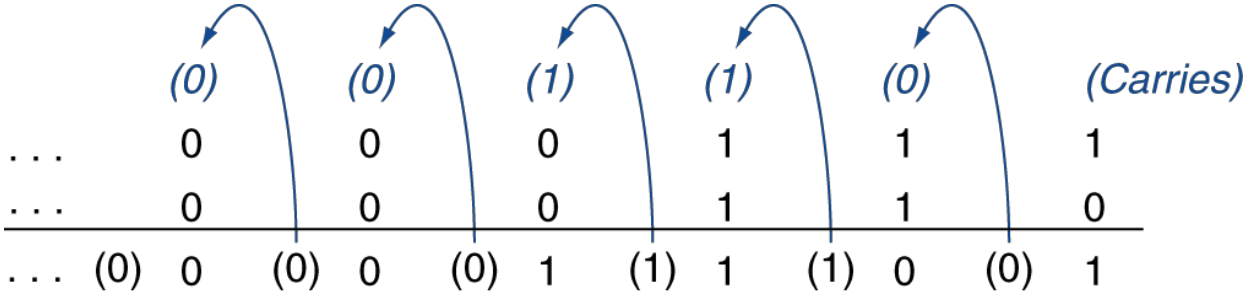
Arithmetic for Computers

- Operations on integers
  - Addition and subtraction
  - Multiplication and division
  - Dealing with overflow
- Floating-point real numbers
  - Representation and operations

Integer Addition

- Example: 7 + 6

2's complement

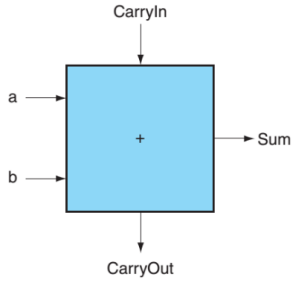


- Overflow if result out of range
  - Adding +ve and –ve operands, no overflow
  - Adding two +ve operands
    - Overflow if result sign is 1
  - Adding two –ve operands
    - Overflow if result sign is 0

进入到符号位的进位

⊕符号位输出的进位

1-bit adder

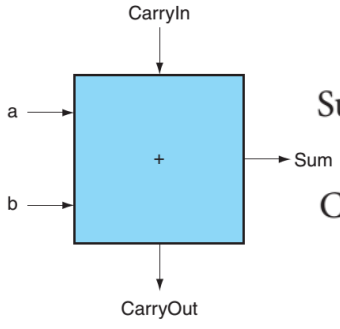


$$\text{Sum} = (a \cdot \overline{b} \cdot \overline{\text{CarryIn}}) + (\overline{a} \cdot b \cdot \overline{\text{CarryIn}}) + (\overline{a} \cdot \overline{b} \cdot \text{CarryIn}) + (a \cdot b \cdot \text{CarryIn})$$

$$\text{CarryOut} = (b \cdot \text{CarryIn}) + (a \cdot \text{CarryIn}) + (a \cdot b)$$

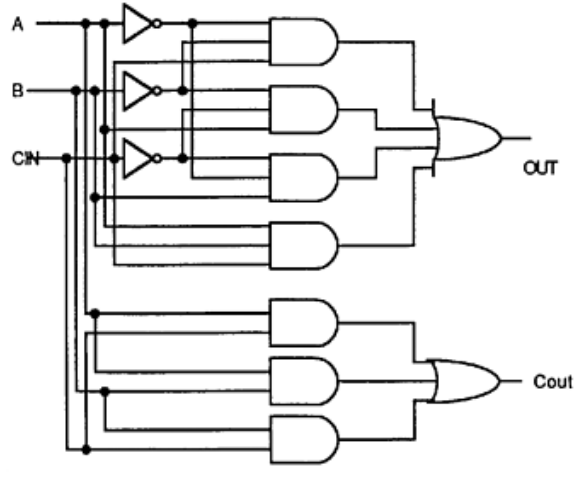
Inputs			Outputs		
a	b	CarryIn	CarryOut	Sum	Comments
0	0	0	0	0	$0 + 0 + 0 = 00_{two}$
0	0	1	0	1	$0 + 0 + 1 = 01_{two}$
0	1	0	0	1	$0 + 1 + 0 = 01_{two}$
0	1	1	1	0	$0 + 1 + 1 = 10_{two}$
1	0	0	0	1	$1 + 0 + 0 = 01_{two}$
1	0	1	1	0	$1 + 0 + 1 = 10_{two}$
1	1	0	1	0	$1 + 1 + 0 = 10_{two}$
1	1	1	1	1	$1 + 1 + 1 = 11_{two}$

1-bit Adder

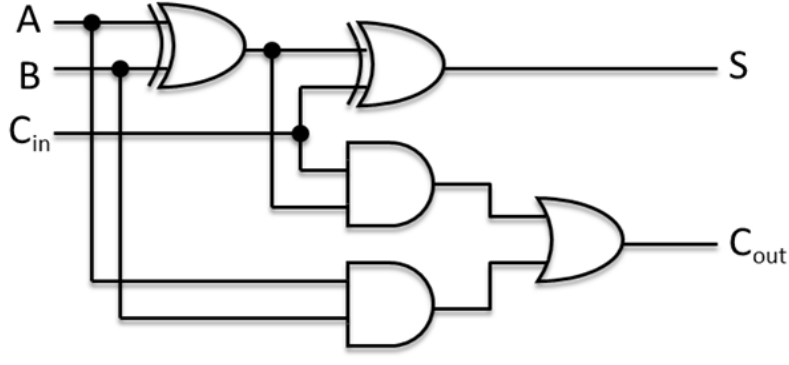


$$\text{Sum} = (a \cdot \overline{b} \cdot \overline{\text{CarryIn}}) + (\overline{a} \cdot b \cdot \overline{\text{CarryIn}}) + (\overline{a} \cdot \overline{b} \cdot \text{CarryIn}) + (a \cdot b \cdot \text{CarryIn})$$

$$\text{CarryOut} = (b \cdot \text{CarryIn}) + (a \cdot \text{CarryIn}) + (a \cdot b)$$



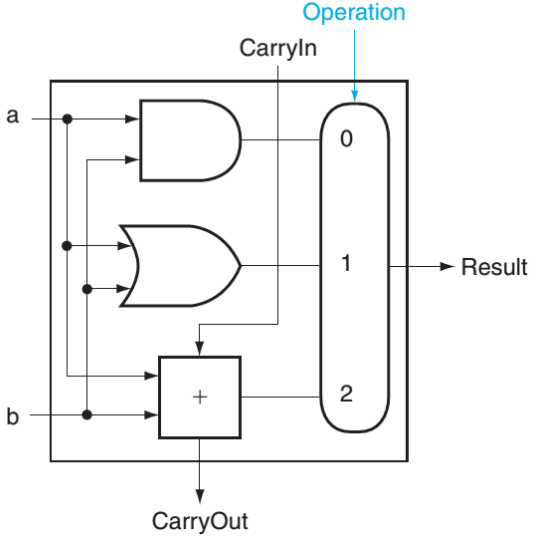
1-bit adder – version 1

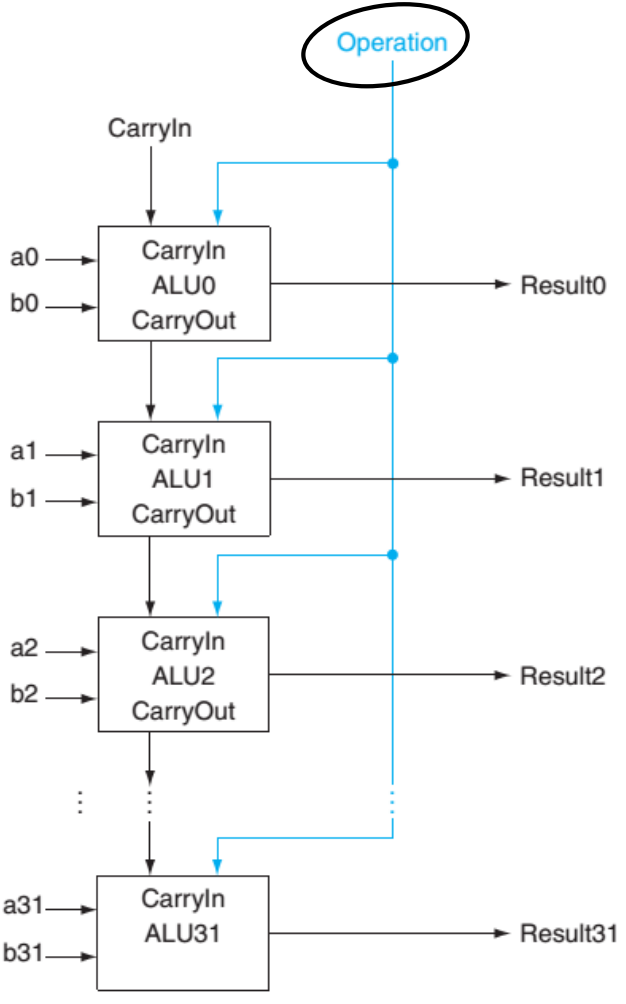


1-bit adder – version 2

1-bit ALU

- ALU: arithmetic logical unit
- 1-bit ALU and 32-bit ALU
  - If op = 0, o = a & b (and)
  - If op = 1, o = a | b (or)
  - If op = 2, o = a + b (add)





Integer Subtraction

- Add negation of second operand
- Example: 7 – 6 = 7 + (–6)

+7:	0000 0000 … 0000 0111
–6:	1111 1111 … 1111,1010
+1:	0000 0000 … 0000 0001

- Overflow if result out of range
  - Subtracting two +ve or two –ve operands, no overflow
  - Subtracting +ve from –ve operand
    - Overflow if result sign is 0
  - Subtracting –ve from +ve operand
    - Overflow if result sign is 1

Dealing with Overflow

- Some languages (e.g., C) ignore overflow
  - Use MIPS `addu, addiu, subu` instructions
- Other languages (e.g., Ada, Fortran) require raising an exception
  - Use MIPS `add, addi, sub` instructions
  - On overflow, invoke exception handler
    - Save PC in exception program counter (EPC) register
    - Jump to predefined handler address
    - `mfc0` (move from coprocessor reg) instruction can retrieve EPC value, to return after corrective action
- Note: addiu: "u" means it doesn't generate overflow exception, but the immediate can be a signed number

Arithmetic for Multimedia

- Graphics and media processing operates on vectors of 8-bit and 16-bit data
  - Use 64-bit adder, with partitioned carry chain
    - Operate on 8×8-bit, 4×16-bit, or 2×32-bit vectors
  - SIMD (single-instruction, multiple-data)
- Saturating operations 饱和计算
  - On overflow, result is largest representable value
    - Instead of 2s-complement modulo arithmetic
  - E.g., change the volume and brightness in audio or video

240 + 20 → 260

11111111

Multiplication

- Start with long-multiplication approach

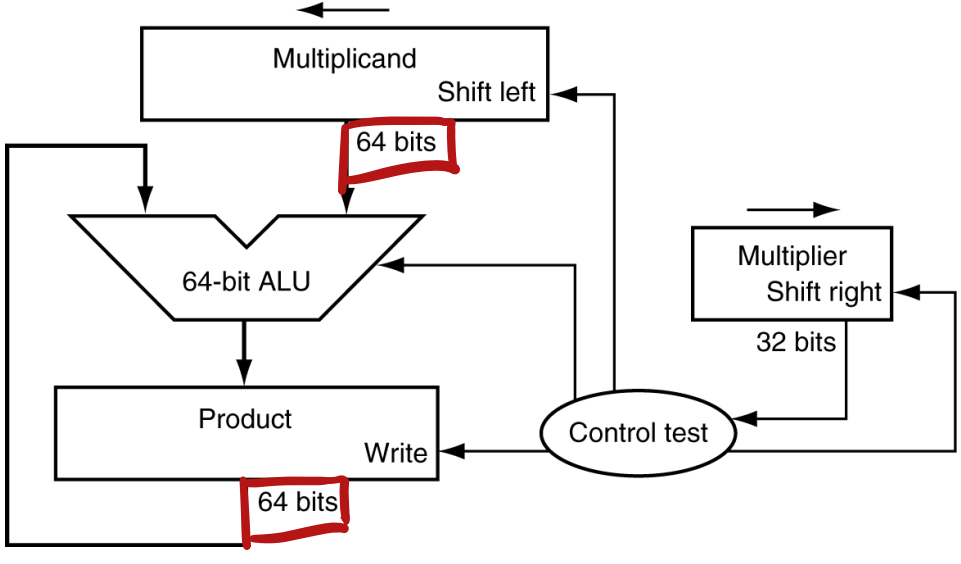
multiplicand

multiplier

product

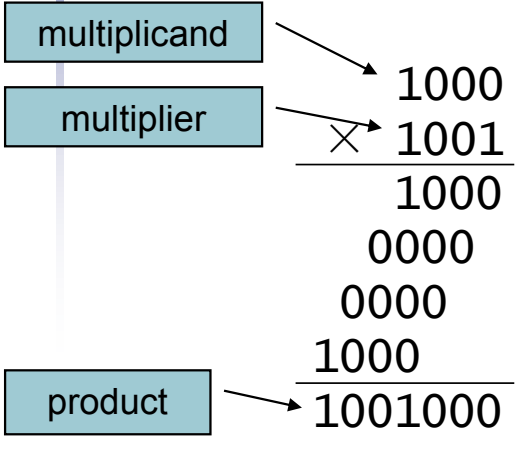
```
  1000
× 1001
------
  1000
 0000
0000
1000
-------
1001000
```

Length of product is the sum of operand lengths

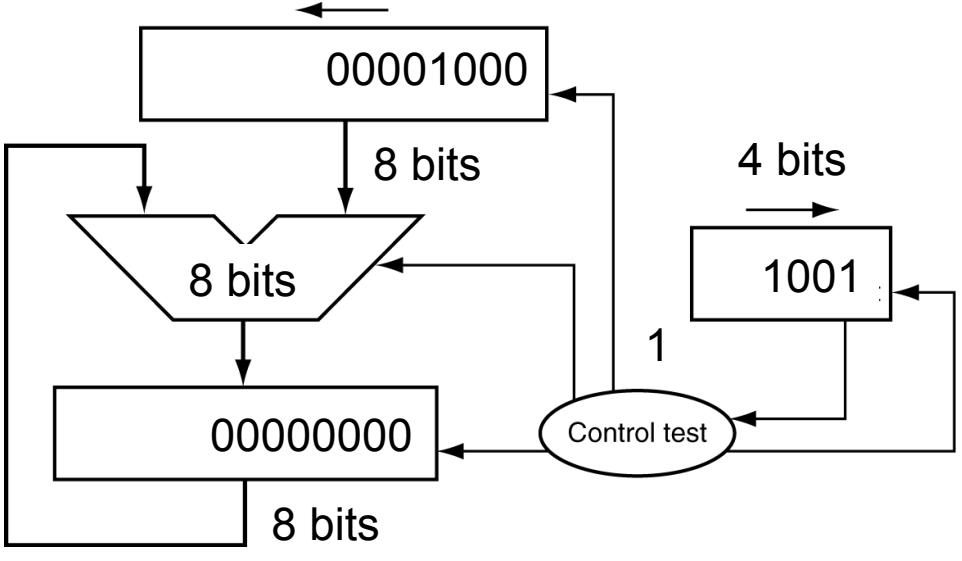


Multiplication

- Start with long-multiplication approach



Length of product is the sum of operand lengths



Multiplication

- Start with long-multiplication approach

multiplicand

multiplier

product

```
  1000
× 1001
------
  1000
 0000
0000
1000
------
1001000
```

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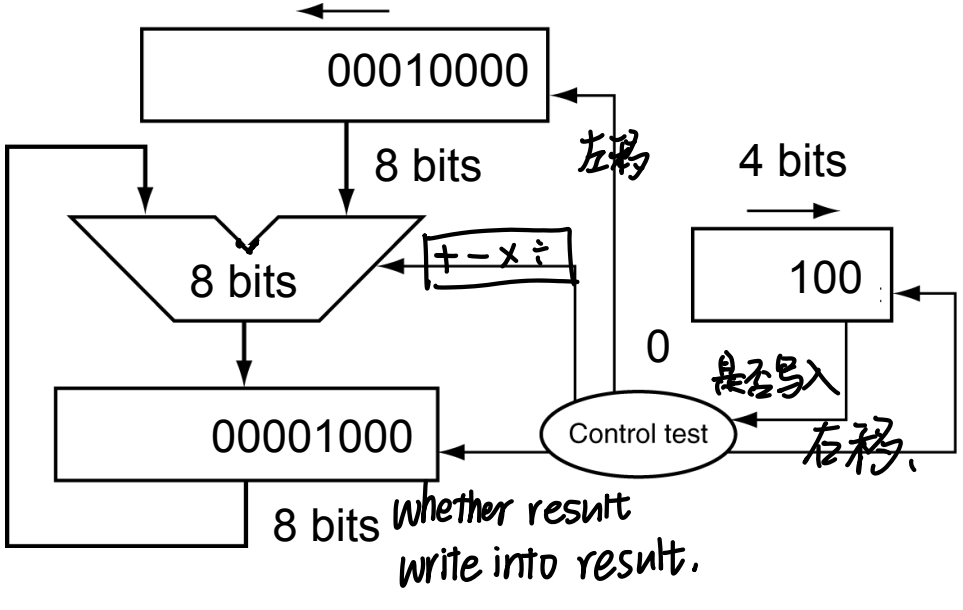

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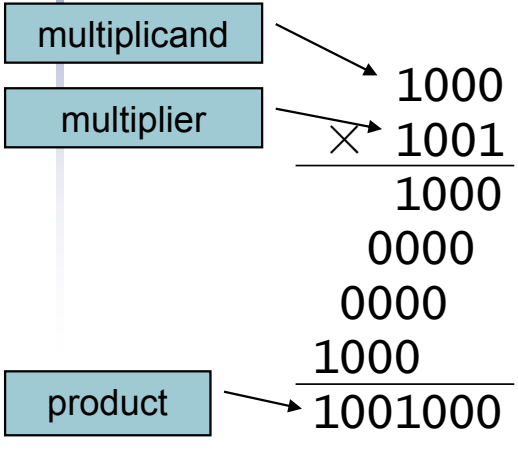
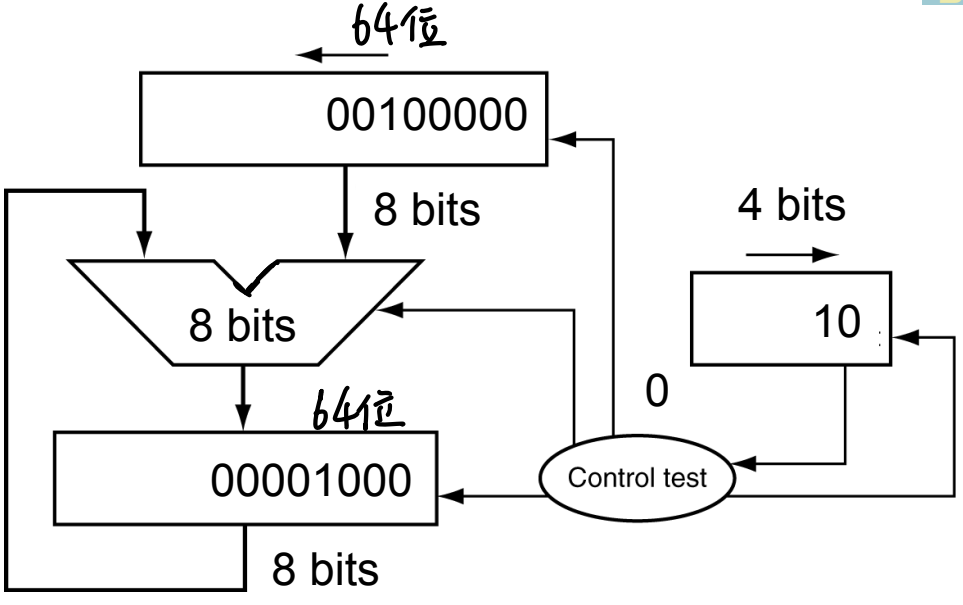
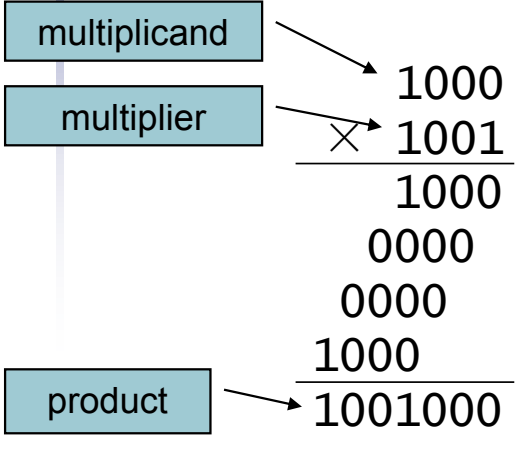


Length of product is the sum of operand lengths

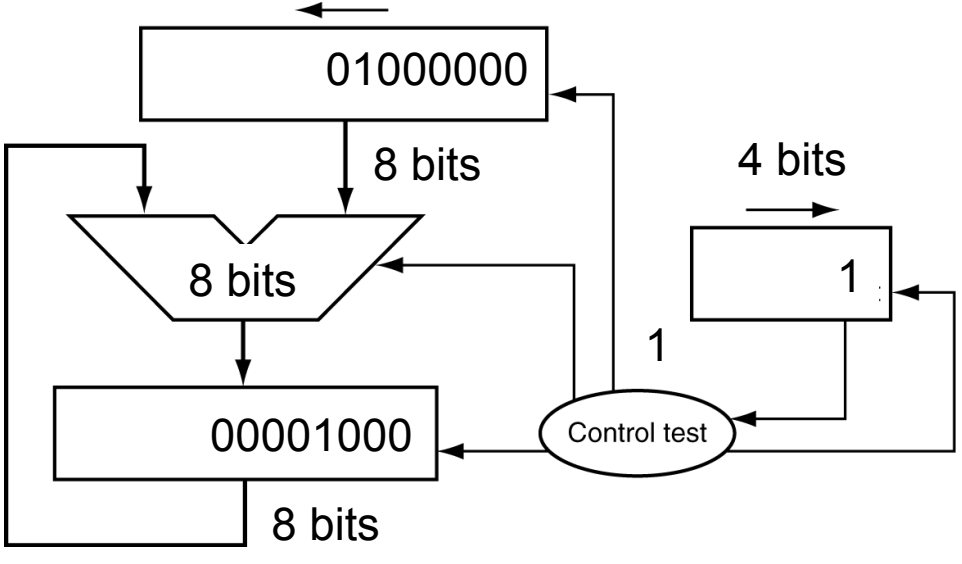


Multiplication

- Start with long-multiplication approach

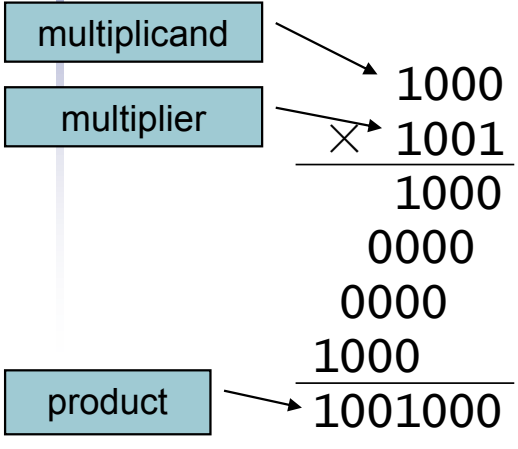


Length of product is the sum of operand lengths

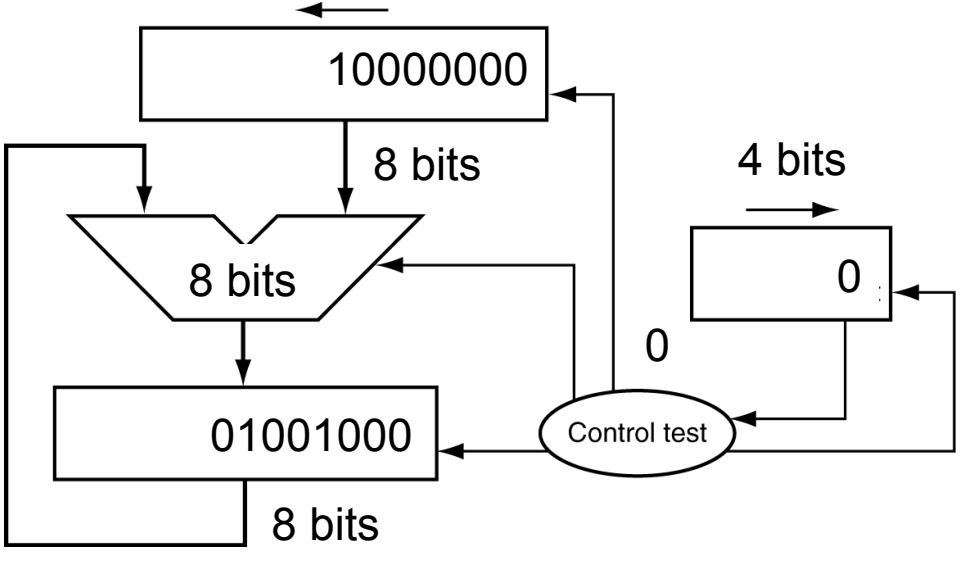


Multiplication

- Start with long-multiplication approach

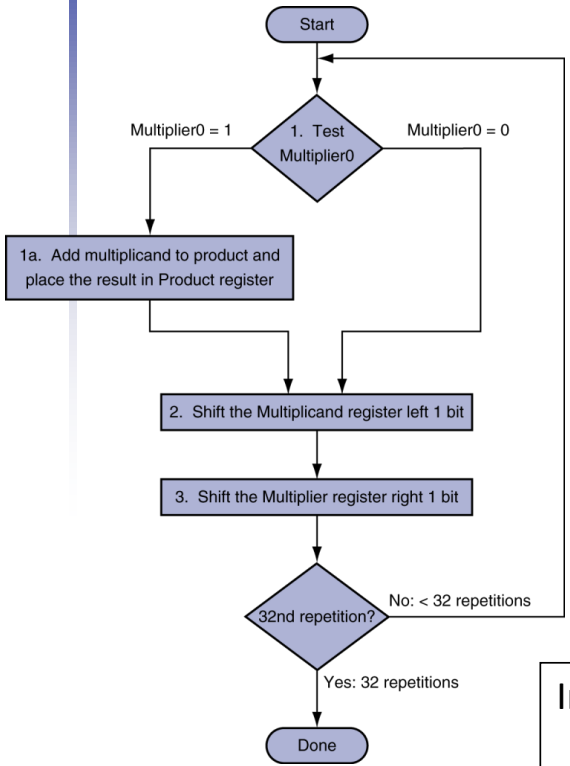


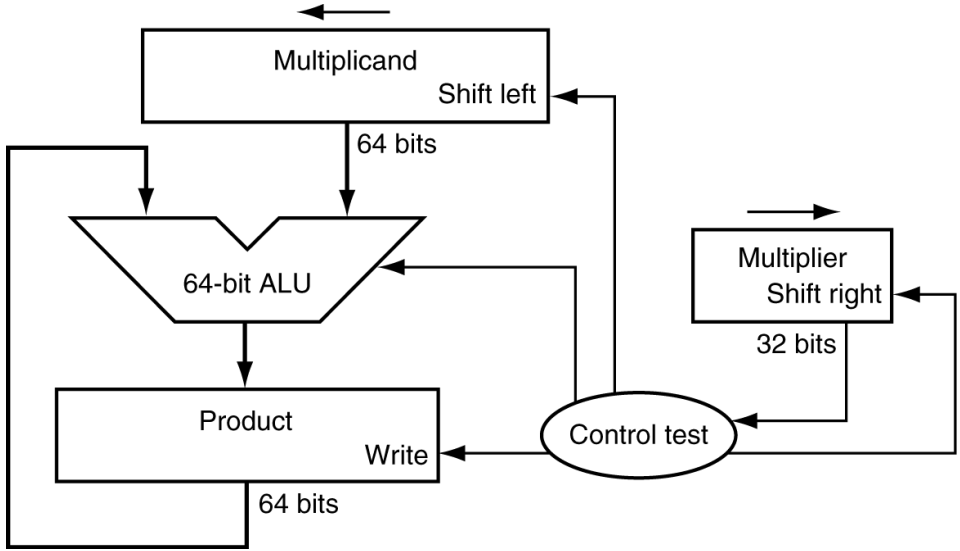
Length of product is the sum of operand lengths



Finish!

Multiplication Hardware





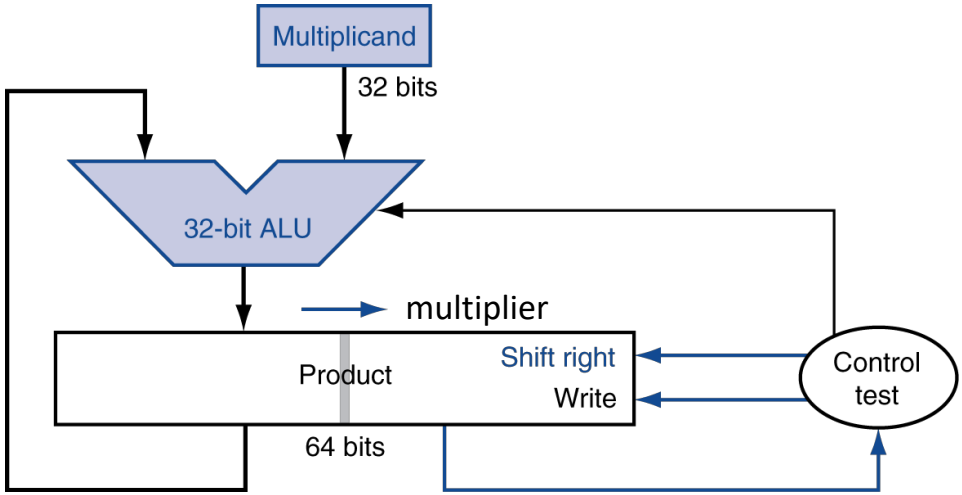
In every step:

- ✓ multiplicand is shifted
- ✓ next bit of multiplier is examined (also a shifting step)
- ✓ if this bit is 1, shifted multiplicand is added to the product

Optimized Multiplier

save regester / bit-width.

- Perform steps in parallel: add/shift

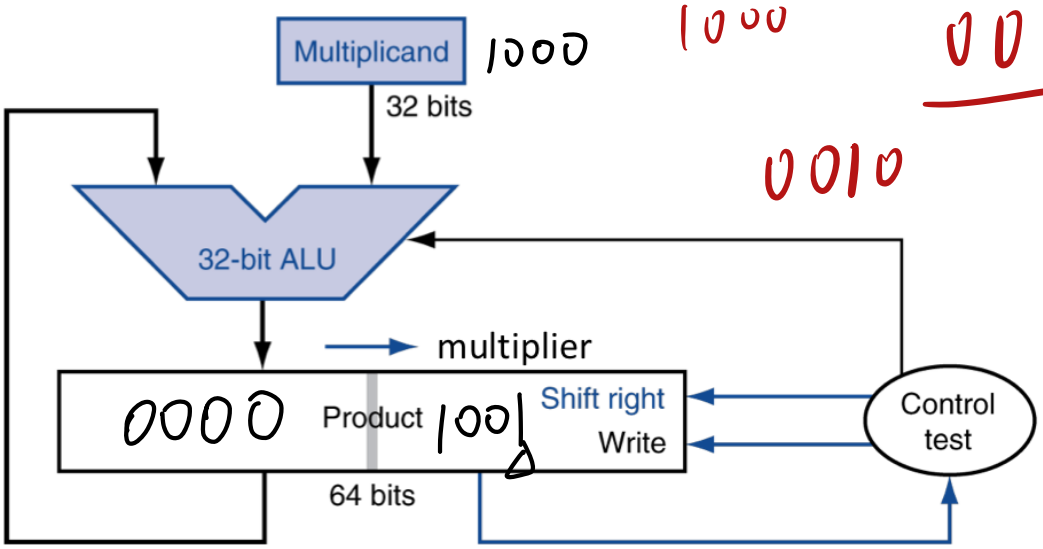


- ✓ The multiplier is initially stored in the right half of product register
- ✓ check the 0th bit in Product register, if 1, add left half of product with multiplicand
- ✓ the sum keeps shifting right
- ✓ at every step, number of bits in product + multiplier = 64, hence, they share a single 64-bit register
- ✓ for signed multiplication, it also works

8+4+2

-14

1001 = -7×2



1000

0010

0010

1110.

10000

1000 1001

01000100

00100010

00010001 +1000

10001

10001000

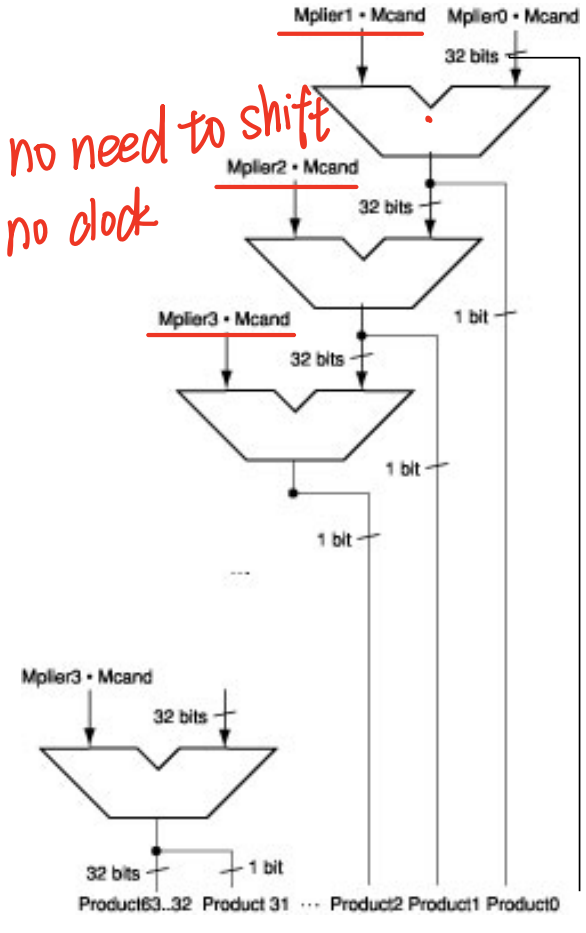
01001000

$$\begin{array}{r} 1000 \\ \times\ 1001 \\ \hline 1000 \\ 0000 \\ 0000 \\ 1000 \\ \hline \end{array}$$

有符号|无符号

8x(-9) sign extention.

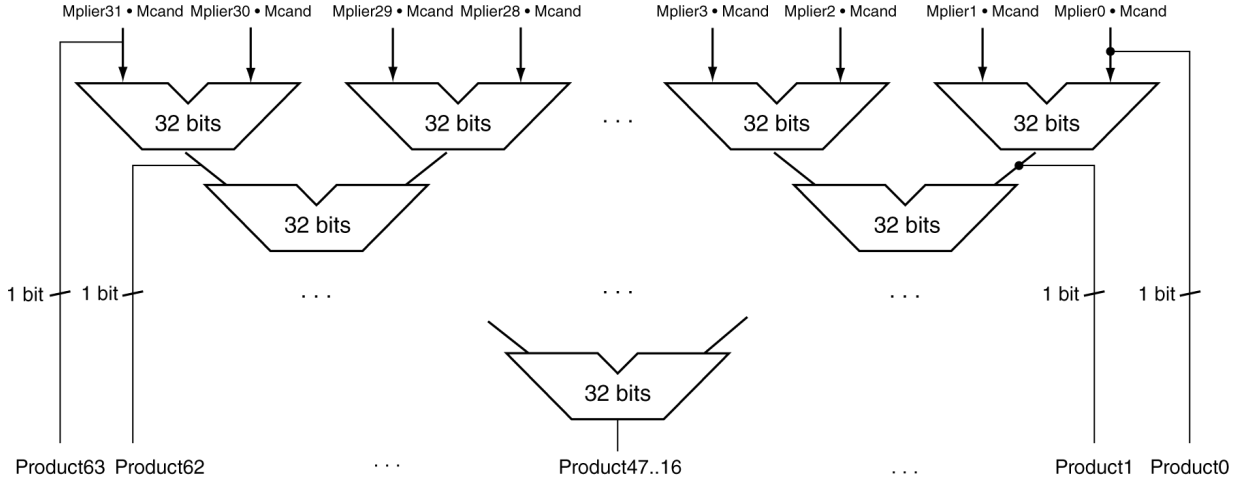
Faster Multiplier



- The previous algorithm requires a clock to ensure that the earlier addition has completed before shifting
- This algorithm can quickly set up most inputs – it then has to wait for the result of each add to propagate down – faster because no clock is involved
- high transistor cost

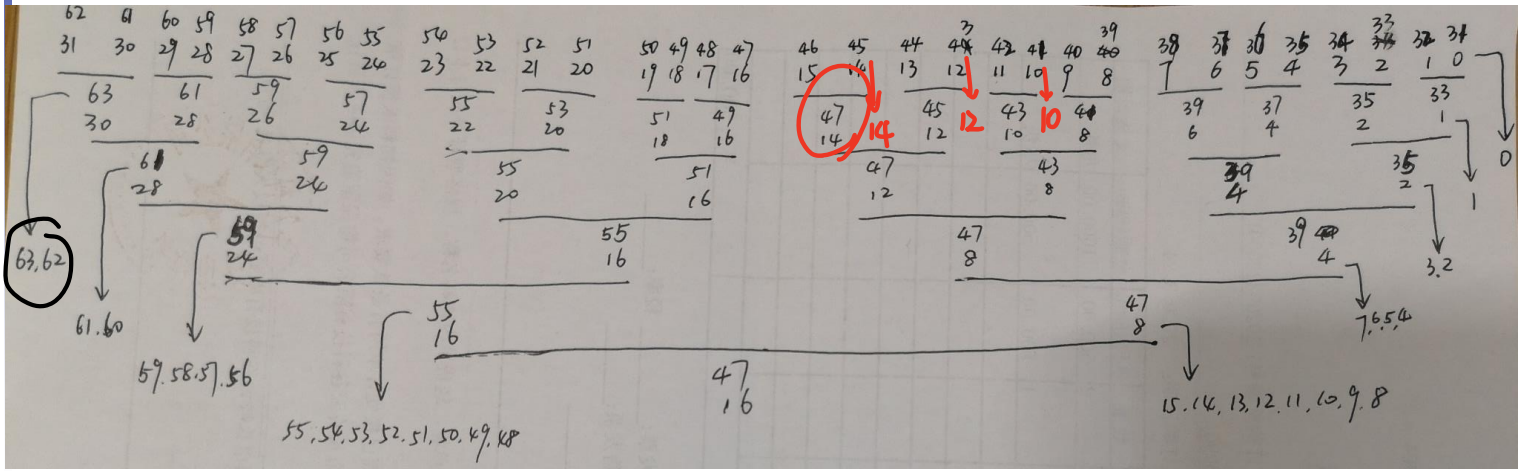
Faster Multiplier(先取绝对值)

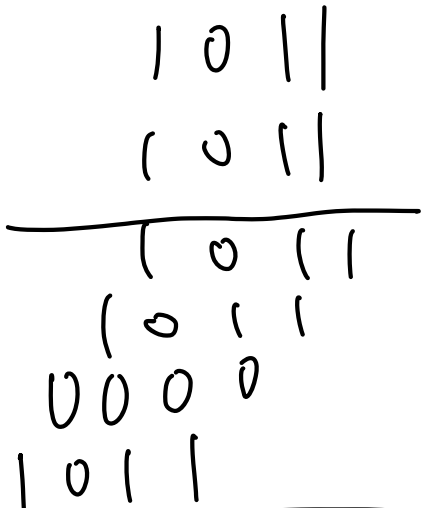
- Uses multiple pipelined adders
  - Cost/performance tradeoff



- Can be pipelined
  - Several multiplication performed in parallel

32-bit Faster Multiplier assume 有符号数

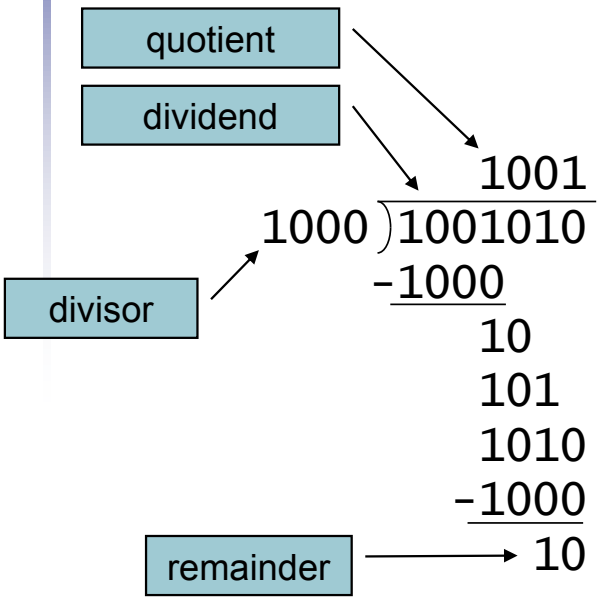




MIPS Multiplication

- Two 32-bit registers for product
  - HI: most-significant 32 bits
  - LO: least-significant 32-bits
- Instructions
  - `mult rs, rt  /  multu rs, rt`
    - 64-bit product in HI/LO
  - `mfhi rd  /  mflo rd`
    - Move from HI/LO to rd
    - Can test HI value to see if product overflows 32 bits
  - `mul rd, rs, rt`
    - Least-significant 32 bits of product –> rd

Division



n-bit operands yield *n*-bit quotient and remainder

- Check for 0 divisor
- Long division approach
  - If divisor ≤ dividend bits
    - 1 bit in quotient, subtract
  - Otherwise
    - 0 bit in quotient, bring down next dividend bit
- Restoring division
  - Do the subtract, and if remainder goes < 0, add divisor back
- Signed division
  - Divide using absolute values
  - Adjust sign of quotient and remainder as required

Divide Example

- Divide 7_{dec} ($0000\ 0111_{bin}$) by 2_{dec} (0010_{bin})

Iter	Step	Quot	Divisor	Remainder
0	Initial values	0000	0010 0000	0000 0111
1	Rem = Rem – Div	0000	0010 0000	1110 0111
	Rem < 0 ➔ +Div, shift 0 into Q	0000	0010 0000	0000 0111
	Shift Div right	0000	0001 0000	0000 0111
2	Same steps as 1	0000	0001 0000	1111 0111
		0000	0001 0000	0000 0111
		0000	0000 1000	0000 0111
3	Same steps as 1	0000	0000 0100	0000 0111
4	Rem = Rem – Div	0000	0000 0100	0000 0011
	Rem >= 0 ➔ shift 1 into Q	0001	0000 0100	0000 0011
	Shift Div right	0001	0000 0010	0000 0011
5	Same steps as 4	0011	0000 0001	0000 0001

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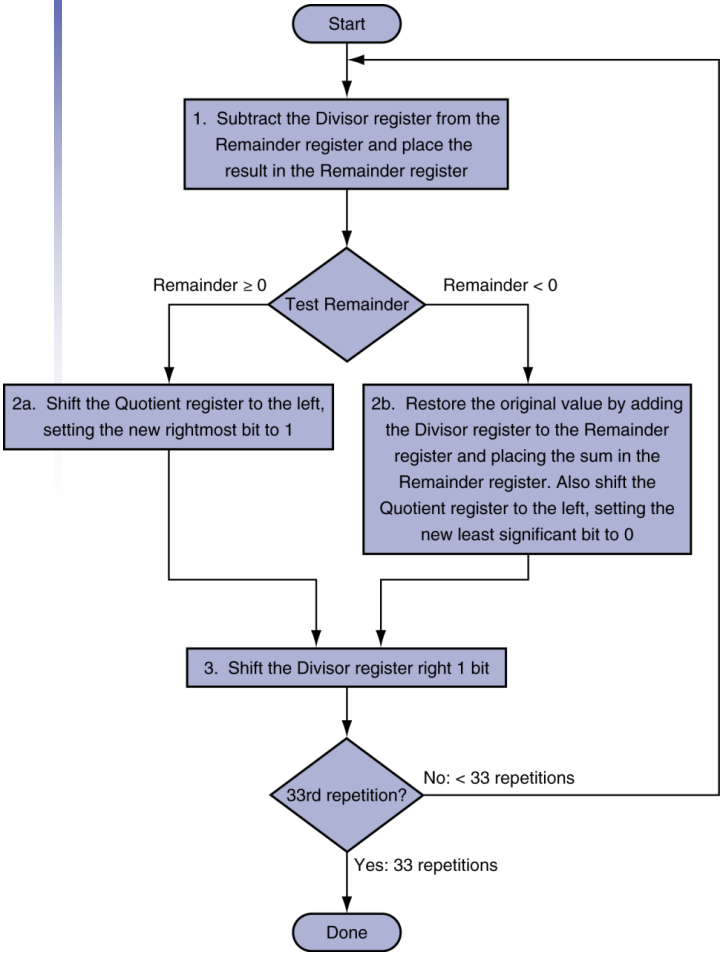
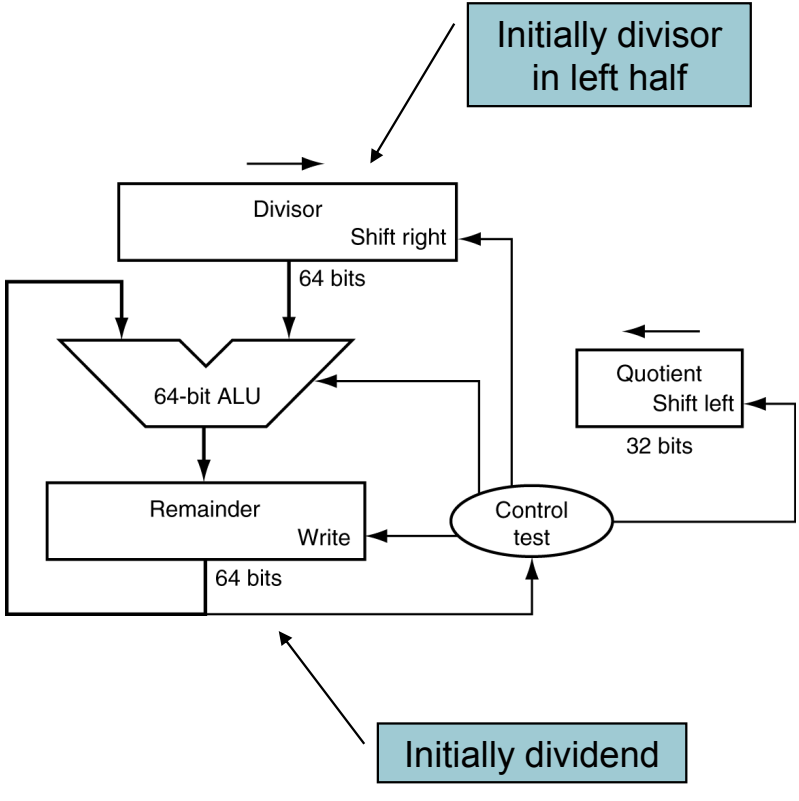
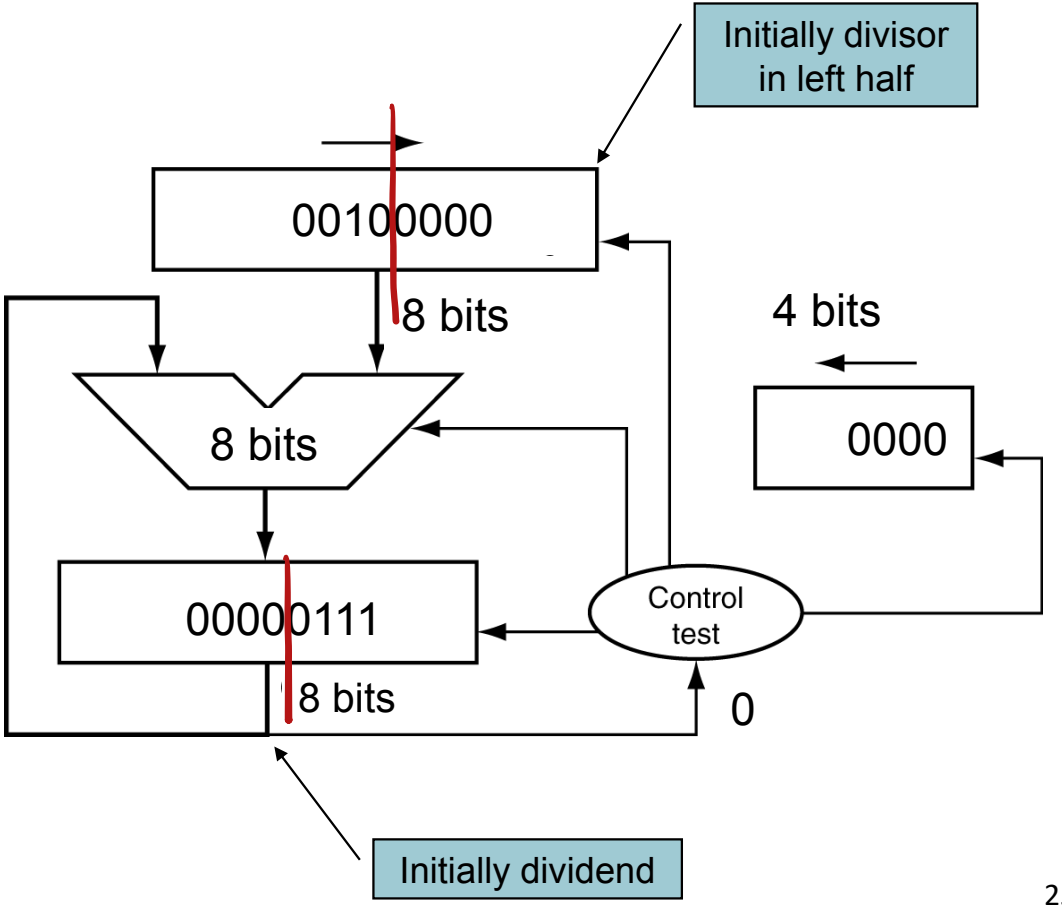
Division Hardware





Division Hardware



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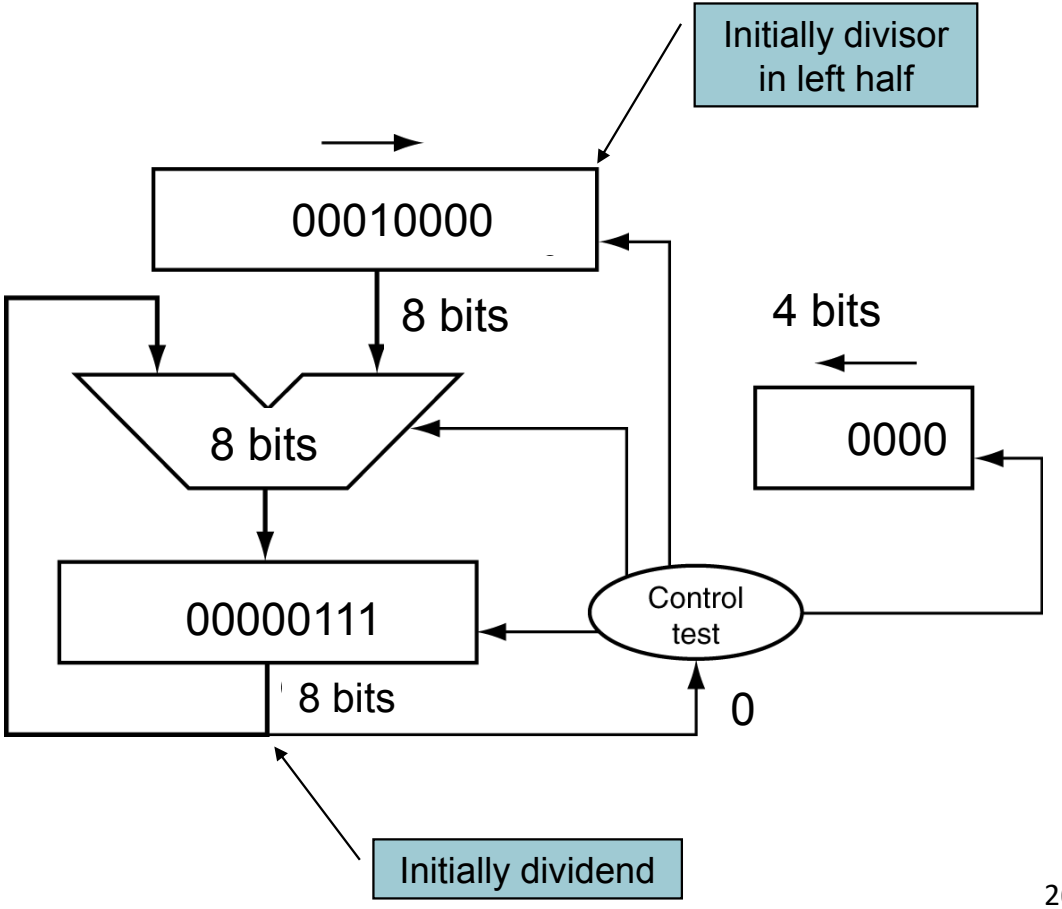
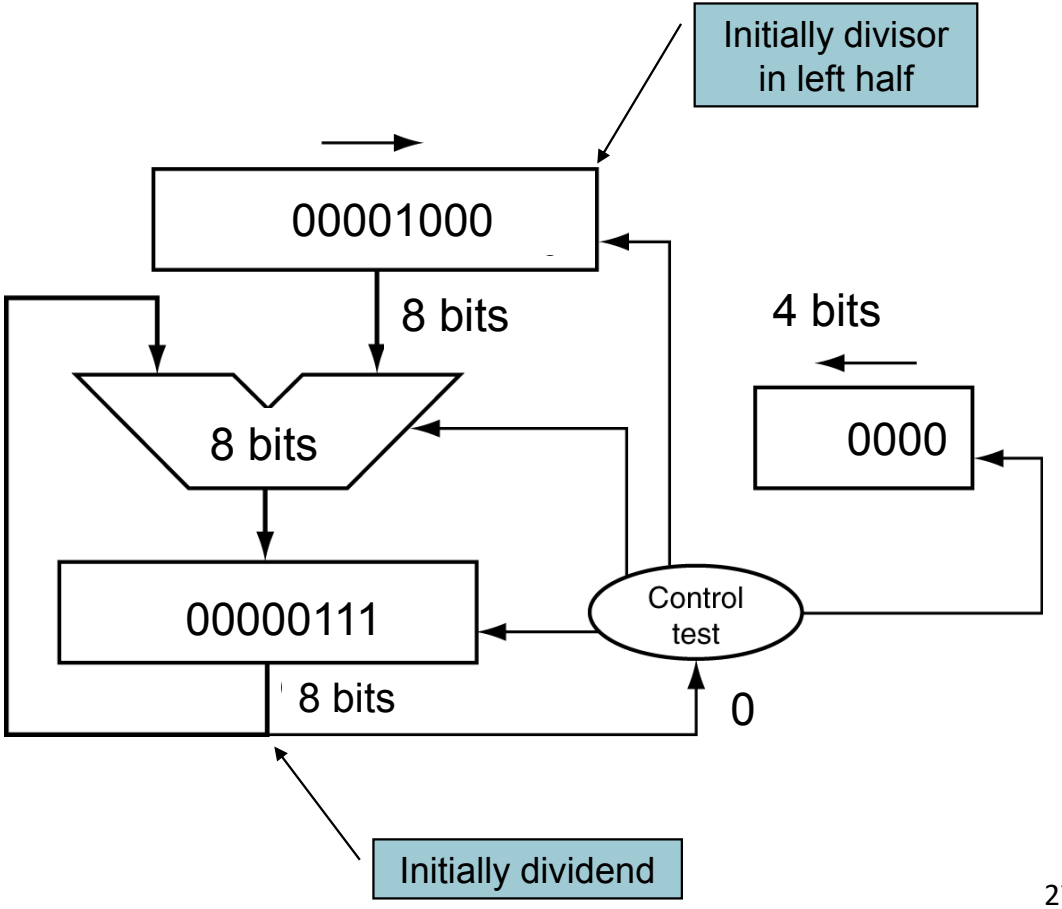
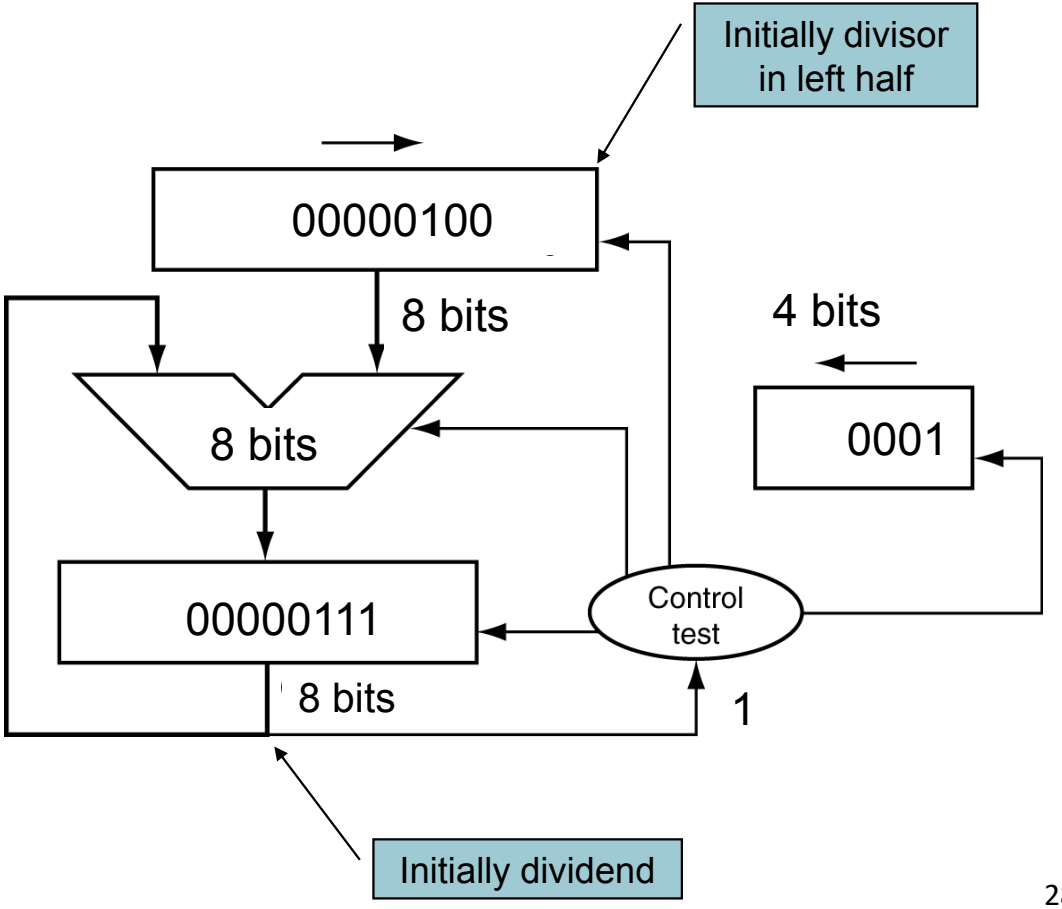


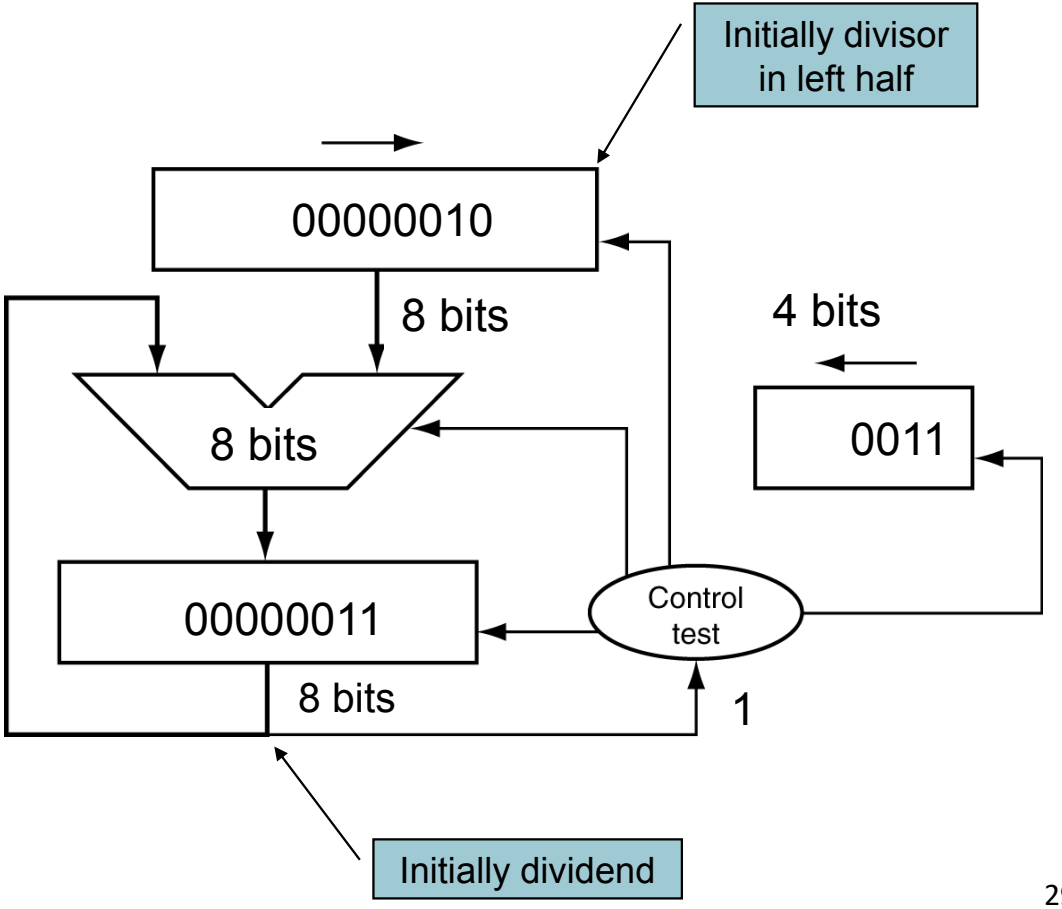
Division Hardware



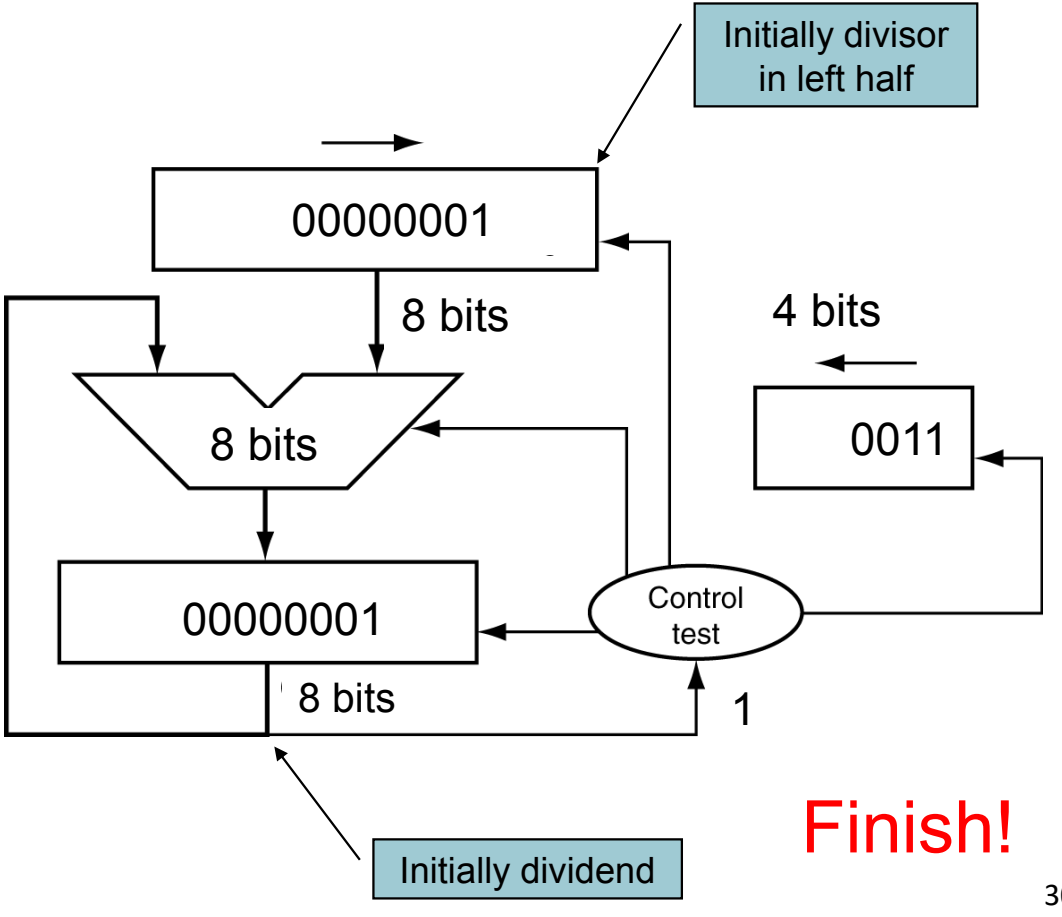
Division Hardware



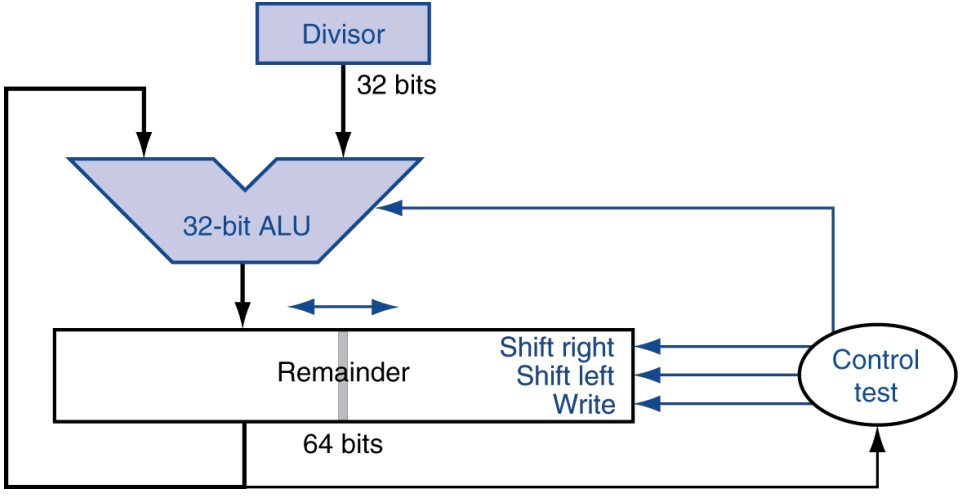
Division Hardware



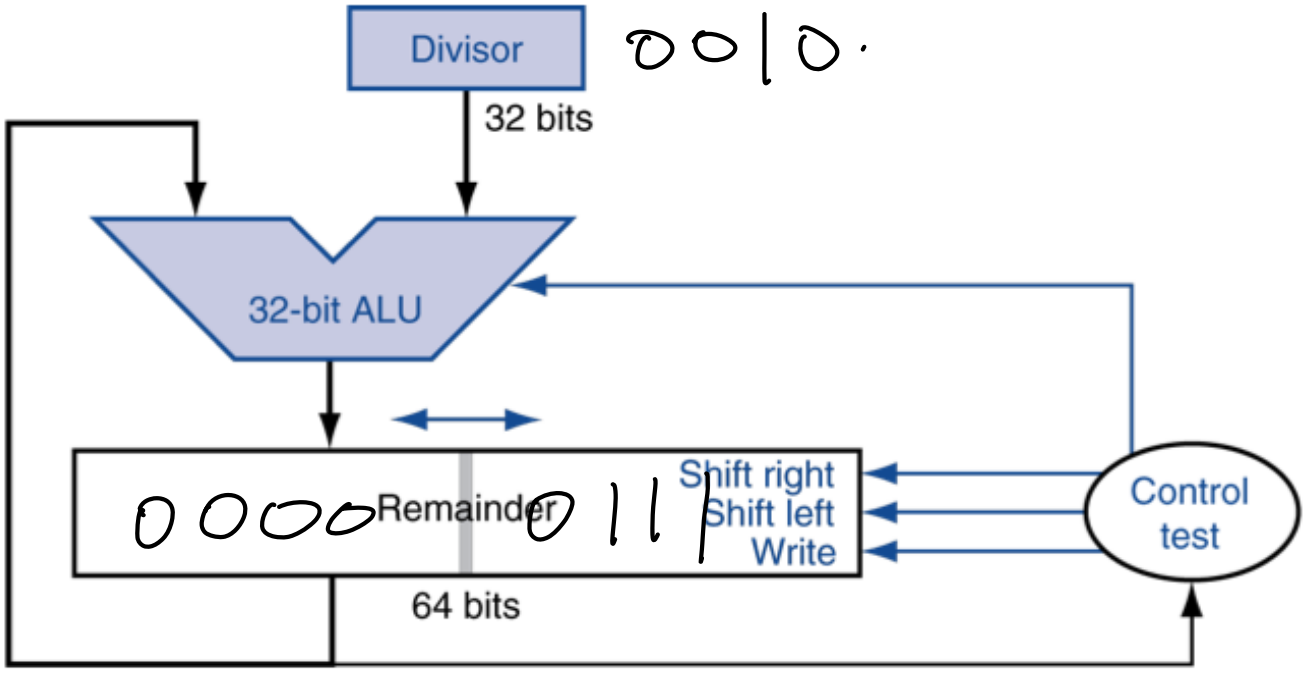
Division Hardware



Optimized Divider



- One cycle per partial-remainder subtraction
- Looks a lot like a multiplier!
  - Same hardware can be used for both



0010.

0000 0111 | 0

0001 1100

0011 1000

10

0001 1000

0011 0001

10

0001 0001

Signed Division

- (+7) ÷ (-2) = (-3) ⋯ (+1)
- (-7) ÷ (-2) = (+3) ⋯ (-1)
- The quotient is +, if the signs of divisor and dividend agrees, otherwise, quotient is –
- The sign of the remainder matches that of the dividend.

Faster Division

- Can’t use parallel hardware as in multiplier
  - Subtraction is conditional on sign of remainder
- Faster dividers (e.g. SRT devision) generate multiple quotient bits per step
  - Still require multiple steps

MIPS Division

- Use HI/LO registers for result
  - HI: 32-bit remainder
  - LO: 32-bit quotient
- Instructions
  - `div rs, rt` / `divu rs, rt`
  - No overflow or divide-by-0 checking
    - Software must perform checks if required
  - Use `mfhi`, `mflo` to access result

Homework

- Chapter3: 3.9 3.10 3.11 3.13 3.16 3.18