CS201: Discrete Math for Computer Science
2020 Fall Semester Written Assignment # 4
Due: Dec. 1st, 2020, please submit at the beginning of class

Q.1 Prove that if $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ whenever n is a nonnegative integer.

Q.2 Use induction to prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

Q.3 Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$\begin{aligned} & (A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \\ = \; & (A_1 \cap A_2 \cap \cdots \cap A_n) - B. \end{aligned}$$

Q.4 Prove that if $h > -1$, then $1 + nh \le (1+h)^n$ for all nonnegative integers n. This is called **Bernoulli's inequality**.

Q.5 Suppose that a and b are real numbers with $0 < b < a$. Prove that if n is a positive integer, then $a^n - b^n \le na^{n-1}(a-b)$.

Q.6 A store gives out gift certificates in the amounts of \$10 and \$25. What amounts of money can you make using gift certificates from the store? Prove your answer using strong induction.

Q.7 Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

Q.8 Suppose that the function f satisfies the recurrence relation $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square greater than 1 and $f(2) = 1$.

(a) Find $f(16)$

(b) Find a big-O estimate for $f(n)$. [Hint: make the substitution $m = \log n$.]

Q.9 Find $f(n)$ when $n = 4^k$, where f satisfies the recurrence relation $f(n) = 5f(n/4) + 6n$, with $f(1) = 1$.

Q.10 The running time of an algorithm A is described by the following recurrence relation:

$$S(n) = \begin{cases} b & n = 1 \\ 9S(n/2) + n^2 & n > 1 \end{cases}$$

where b is a positive constant and n is a power of 2. The running time of a competing algorithm B is described by the following recurrence relation:

$$T(n) = \begin{cases} c & n = 1 \\ aT(n/4) + n^2 & n > 1 \end{cases}$$

where a and c are positive constants and n is a power of 4. For the rest of this problem, you may assume that n is always a power of 4. You should also assume that $a > 16$. (Hint: you may use the equation $a^{\log_2 n} = n^{\log_2 a}$)

(a) Find a solution for $S(n)$. Your solution should be in *closed form* (in terms of b if necessary) and should *not* use summation.

(b) Find a solution for $T(n)$. Your solution should be in *closed form* (in terms of a and c if necessary) and should *not* use summation.

(c) For what range of values of $a > 16$ is Algorithm B at least as efficient as Algorithm A asymptotically ($T(n) = O(S(n))$)?

Q.11 Suppose that $n \geq 1$ is an integer.

(a) How many functions are there from the set $\{1, 2, \ldots, n\}$ to the set $\{1, 2, 3\}$?

(b) How many of the functions in part (a) are one-to-one functions?

(c) How many of the functions in part (a) are onto functions?

Q.12 How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

Q.13 How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$

(a) that are one-to-one?

(b) that assign 0 to both 1 and n?

(c) that assign 1 to exactly one of the positive integers less than n?

Q.14 Suppose that p and q are prime numbers and that $n = pq$. Use the principle of inclusion-exclusion to find the number of positive integers not exceeding n that are relatively prime to n, i.e., the Euler function $\phi(n)$.

Q.15 How many bit strings of length 6 have <u>at least</u> one of the following properties:

- start with 010
- start with 11
- end with 00

State clearly how you count and get your answer.

Q.16 Prove that the binomial coefficient

$$\binom{240}{120}$$

is divisible by $242 = 2 \cdot 121$.

Q.17 Consider all permutations of the letters A, B, C, D, E, F, G.

(a) How many of these permutations contains the strings ABC and DE (each as consecutive substring)?

(b) In how many permutations does A precede B? (not necessary immediately)

Q.18 Let (x_i, y_i), $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integers coordinates.

Q.19 Show that if p is a prime and k is an integer such that $1 \leq k \leq p - 1$, then p divides $\binom{p}{k}$.

Q.20 Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.

Q.21 Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$ with initial conditions $a_0 = 1$ and $a_1 = 0$.

Q.22

(a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.

(b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 4$.

Q.23 Let $S_n = \{1, 2, \ldots, n\}$ and let a_n denote the number of *non-empty* subsets of S_n that contain **no** two consecutive integers. Find a recurrence relation for a_n. Note that $a_0 = 0$ and $a_1 = 1$.

Q.24 Let $\mathbf{A}_n$ be the $n \times n$ matrix with 2's on its main diagonal, 1's in all positions next to a diagonal element, and 0's everywhere else. Find a recurrence relation for d_n, the determinant of $\mathbf{A}_n$. Solve this recurrence relation to find a formula for d_n.