CS201: Discrete Math for Computer Science
2020 Fall Semester   Written Assignment # 2
Due: Oct. 27th, 2020, please submit at the beginning of class

Q.1 Suppose that A, B and C are three finite sets. For each of the following, determine whether or not it is true. Explain your answers.

(a) $(A - B = A) \to (B \subset A)$

(b) $(A - B = \emptyset) \to (A \cap B = B \cap A)$

(c) $(A \subseteq B) \to (|A \cup B| \geq 2|A|)$

(d) $\overline{(A - B)} \cap (B - A) = B$

Solution:

(a) False. As an counterexample, let $A = \{1\}$, and $B = \{2\}$. Then $A = B = A$, but B is not a subset of A.

(b) True. $A \cap B = B \cap A$ is always true. This is a trivial proof.

(c) False. Let $A = B = \{1\}$. Then, $A \subseteq B$ is true, but $|A \cup B| = 1 < 2 = 2|A|$, which is false.

(d) False. Let $A = B = \{1\}$. Then, $\overline{A - B} \cap (B - A) = U \cap \emptyset \neq B = \{1\}$.

□

Q.2 Let A, B and C be sets. Prove the following using set identities.

(1) $(B - A) \cup (C - A) = (B \cup C) - A$

(2) $(A \cap B) \cap \overline{(B \cap C)} \cap (A \cap C) = \emptyset$

Solution:

(1) We have

$$\begin{aligned}
(B - A) \cup (C - A) &= (B \cap \overline{A}) \cup (C \cap \overline{A}) && \text{by definition} \\
&= \overline{A} \cap (B \cup C) && \text{ditributive law} \\
&= (B \cup C) - A && \text{by definition}
\end{aligned}$$

(2) We have

$$
\begin{aligned}
&(A \cap B) \cap \overline{(B \cap C)} \cap (A \cap C) \\
&\quad = \quad (A \cap B) \cap (A \cap C) \cap \overline{(B \cap C)} \quad \text{commutative law} \\
&\quad = \quad (A \cap B \cap C) \cap \overline{(B \cap C)} \qquad \text{associative law} \\
&\quad = \quad (A \cap B \cap C) \cap (\overline{B} \cup \overline{C}) \qquad \text{De Morgan} \\
&\quad = \quad ((A \cap B \cap C) \cap \overline{B}) \cup ((A \cap B \cap C) \cap \overline{C}) \quad \text{distributive law} \\
&\quad = \quad \emptyset \cup \emptyset \qquad \text{Complement} \\
&\quad = \quad \emptyset.
\end{aligned}
$$

□

Q.3 The *symmetric difference* of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

(a) Determine whether the symmetric difference is associative; that is, if A, B and C are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

(b) Suppose that A, B and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

Solution:

(a) Using membership table, one can show that each side consists of the elements that are in an odd number of the sets A, B and C. Thus, it follows.

(b) Yes. We prove that for every element $x \in A$, we have $x \in B$ and vice versa.

First, for elements $x \in A$ and $x \notin C$, since $A \oplus C = B \oplus C$, we know that $x \in A \oplus C$ and thus $x \in B \oplus C$. Since $x \notin C$, we must have $x \in B$. For elements $x \in A$ and $x \in C$, we have $x \notin A \oplus C$. Thus, $x \notin B \oplus C$. Since $x \in C$, we must have $x \in B$.

The proof of the other way around is similar.

□

Q.4 For each set defined below, determine whether the set is *countable* or *uncountable*. Explain your answers. Recall that $\mathbb{N}$ is the set of natural numbers and $\mathbb{R}$ denotes the set of real numbers.

(a) The set of all subsets of students in CS201

(b) $\{(a,b)|a,\ b \in \mathbb{N}\}$

(c) $\{(a,b)|a \in \mathbb{N},\ b \in \mathbb{R}\}$

Solution:

(a) Countable. The number of students in CS201 is finite, so the size of its power set is also finite. All finite sets are countable.

(b) Countable. The set is the same as $\mathbb{N} \times \mathbb{N}$. We now show a bijections between $\mathbb{Z}^+$ and the set:

$$(0,0),(0,1),(1,0),(1,1),(1,2),\ldots.$$

(c) Uncountable. Since $\mathbb{R}$ is uncountable, any sequence that includes an element from $\mathbb{R}$ must also be uncountable.

$\square$

Q.5 Give an example of two uncountable sets A and B such that the intersection $A \cap B$ is

(a) finite,

(b) countably infinite,

(c) uncountable.

Solution:

(a) $A = \{x \in \mathbb{R}|x \geq 0\}$, $B = \{x \in \mathbb{R}|x \leq 0\}$

(b) $A = \{x \in \mathbb{R}|0 < x < 1\} \cup \mathbb{N}$, $B = \{x \in \mathbb{R}|1 < x < 2\} \cup \mathbb{N}$

(c) $A = \{X \in \mathbb{R}|0 < x < 1\}$, $B = \{x \in \mathbb{R}|0 < x < 2\}$.

□

Q.6 For each of the following mappings, indicate what type of function they are (if any), not a function, one-to-one, onto, neither or both. Explain your answers.

(a) The mapping f from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(x) = |2x|$.

(b) The mapping f from $\{1, 3\}$ to $\{2, 4\}$ defined by $f(x) = 2x$.

(c) The mapping f from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = 8 - 2x$.

(d) The mapping f from $\mathbb{R}$ to $\mathbb{Z}$ defined by $f(x) = \lfloor x + 1 \rfloor$.

(e) The mapping f from $\mathbb{R}^+$ to $\mathbb{R}^+$ defined by $f(x) = x - 1$.

(f) The mapping f from $\mathbb{Z}^+$ to $\mathbb{Z}^+$ defined by $f(x) = x + 1$.

Solution:

(a) A function which is neither one-to-one nor onto.

(b) Not a function.

(c) A function which is both one-to-one and onto.

(d) A function which is onto but not one-to-one. .

(e) Not a function.

(f) A function which is one-to-one but not onto.

□

Q.7 Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.
Solution: We need to show that every element of $A \times B$ is also an element of $C \times D$. By definition, a typical element of $A \times B$ is a pair (a, b) where $a \in A$ and $b \in B$. Because $A \subseteq C$, we know that $a \in C$; similarly, $b \in D$. Therefore, we have $(a, b) \in C \times D$.

□

Q.8 For each set A, the *identity function* $1_A : A \to A$ is defined by $1_A(x) = x$ for all x in A. Let $f : A \to B$ and $g : B \to A$ be the functions such that $g \circ f = 1_A$. Show that f is one-to-one and g is onto.

Solution: First, let's show that f is one-to-one. Let x, y be two elements of A such that $f(x) = f(y)$. Then $x = 1_A(x) = g(f(x)) = 1_A(y) = y$.

Next, let's show that g is onto. Let x be any element of A. Then $f(x)$ is an element of B such that $g(f(x)) = 1_A(x) = x$, this means for any element in A, $f(x)$ is its preimage in the set B.

$\square$

Q.9 Suppose that two functions $g : A \to B$ and $f : B \to C$ and $f \circ g$ denotes the *composition* function.

(a) If $f \circ g$ is one-to-one and g is one-to-one, must f be one-to-one? Explain your answer.

(b) If $f \circ g$ is one-to-one and f is one-to-one, must g be one-to-one? Explain your answer.

(c) If $f \circ g$ is one-to-one, must g be one-to-one? Explain your answer.

(d) If $f \circ g$ is onto, must f be onto? Explain your answer.

(e) If $f \circ g$ is onto, must g be onto? Explain your answer.

Solution:

(a) No. We prove this by giving a counterexample. Let $A = \{1, 2\}$, $B = \{a, b, c\}$, and $C = A$. Define the function g by $g(1) = a$ and $g(2) = b$, and define the function f by $f(a) = 1$, and $f(b) = f(c) = 2$. Then it is easily verified that $f \circ g$ is one-to-one and g is one-to-one. But f is not one-to-one.

(b) Yes. For any two elements $x, y \in A$ with $x \neq y$, assume to the contrary that $g(x) = g(y)$. On one hand, since $f \circ g$ is one-to-one, we have $f \circ g(x) \neq f \circ g(y)$. One the other hand, $f \circ g(x) = f(g(x)) = f(g(y)) = f \circ g(y)$. This leads to a contradiction. Thus, $g(x) \neq g(y)$, which means that g must be one-to-one.

(c) Yes. Similar to (b), the condition that f is one-to-one is in fact not used.

(d) Yes. Since $f \circ g$ is onto, we know that $f \circ g(A) = C$, which means that $f(g(A)) = C$. Note that $g(A)$ is a subset of B, thus, $f(B)$ must also be C. This means that f is also onto.

(e) No. A counterexample is the same as that in (a).

$\square$

Q.10 Let x be a real number. Show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.
Solution:

Certainly every real number x lies in an interval $[n, n+1)$ for some integer n; indeed $n = \lfloor x \rfloor$.

- if $x \in [n, n + \frac{1}{3})$, then $3x$ lies in the interval $[3n, 3n + 1)$, so $\lfloor 3x \rfloor = 3n$. Moreover in this case $x + \frac{1}{3}$ is still less than $n + 1$, and $x + \frac{2}{3}$ is still less than $n + 1$. So, $\lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = n + n + n = 3n$ as well.
- if $x \in [n + \frac{1}{3}, n + \frac{2}{3})$, then $3x \in [3n + 1, 3n + 2)$, so $\lfloor 3x \rfloor = 3n + 1$. Moreover in this case $x + \frac{1}{3}$ is in $[n + \frac{2}{3}, n+1)$, and $x + \frac{2}{3}$ is in $[n+1, n+\frac{4}{3})$, so $\lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = n + n + (n + 1) = 3n + 1$ as well.
- if $x \in [n + \frac{2}{3}, n + 1)$, similar and both sides equal $3n + 2$.

$\square$

Q.11 Derive the formula for $\sum_{k=1}^{n} k^2$.
Solution: First we note that $k^3 - (k-1)^3 = 3k^2 - 3k + 1$. Then we sum this equation for all values of k from 1 to n. On the left, because of telescoping, we have just n^3; on the right we have

$$3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 3\sum_{k=1}^{n} k^2 - \frac{3n(n+1)}{2} + n.$$

Equating the two sides and solving for $\sum_{k=1}^{n} k^2$, we obtain

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= \frac{1}{3}\left(n^3 + \frac{3n(n+1)}{2} - n\right) \\
&= \frac{n}{3}\left(\frac{2n^2+3n+3-2}{2}\right) \\
&= \frac{n}{3}\left(\frac{2n^2+3n+1}{2}\right) \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

$\square$

Q.12 Derive the formula for $\sum_{k=1}^{n} k^3$.
Solution: Again, we use "telescoping" to derive this formula. Since $k^4 - (k-1)^4 = k^4 - (k^4 - 4k^3 + 6k^2 - 4k + 1) = 4k^3 - 6k^2 + 4k - 1$, we have

$$\begin{aligned}
\sum_{k=1}^{n}[k^4 - (k-1)^4] &= 4\sum_{k=1}^{n} k^3 - 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \\
&= 4\sum_{k=1}^{n} k^3 - 6n(n+1)(2n+1)/6 + 4n(n+1)/2 - n \\
&= 4\sum_{k=1}^{n} k^3 - n(n+1)(2n+1) + 2n(n+1) - n \\
&= n^4.
\end{aligned}$$

Thus, it then follows that

$$\begin{aligned}
4\sum_{k=1}^{n} k^3 &= n^4 + n(n+1)(2n+1) - 2n(n+1) + n \\
&= n^2(n+1)^2.
\end{aligned}$$

Then we get the formula $\sum_{k=1}^{n} k^3 = n^2(n+1)^2/4$.

$\square$

Q.13 Find a formula for $\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor$, when m is a positive integer.
Solution:

By the definition of the floor function, there are $2n+1$ n's in the summation. Let $n = \lfloor\sqrt{m}\rfloor - 1$. Then

$$\begin{aligned}
&\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor \\
&= \sum_{i=1}^{n}(2i^2+i) + (n+1)(m-(n+1)^2+1) \\
&= 2\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i + (n+1)(m-(n+1)^2+1) \\
&= \frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2} + (n+1)(m-(n+1)^2+1)
\end{aligned}$$

□

Q.14 Show that a subset of a countable set is also countable.
Solution: If a set A is countable, then we can list its elements, $a_1, a_2, a_3, \ldots, a_n, \ldots$ (possibly ending after a finite number of terms). Every subset of A consists of some (or none or all) of the items in this sequence, and we can list them in the same order in which they appear in the sequence. This gives us a sequence (again, infinite or finite) listing all the elements of the subset. Thus the subset is also countable.

□

Q.15 Assume that $|S|$ denotes the cardinality of the set S. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.
Solution:

By definition, we have one-to-one and onto functions $f : A \to B$ and $g : B \to C$. Then $g \circ f$ is a one-to-one and onto function from A to C, so we have $|A| = |C|$.

□

Q.16 If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?
Solution: Since $A = (A-B) \cup (A \cap B)$, if $A-B$ is countable, the elements of A can be listed in a sequence by alternating elements of $A - B$ and elements of $A \cap B$. This contradicts the uncountability of A.

□

Q.17 The *binary insertion sort* is a variantion of the insertion sort that uses a binary search technique rather than a linear search technique to insert the ith element in the correct place among the previously sorted elements. Express the binary insertion sort in pseudocode.
Solution:

□

Q.18 If $f_1(x)$ and $f_1(x)$ are functions from the set of positive integers to the set of positive real numbers and $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$, is $(f_1 - f_2)(x)$ also $\Theta(g(x))$? Either prove that it is or give a counter example.
Solution: This is false. Let $f_1 = 2x^2 + 3x$, $f_2 = 2x^2 + 2x$ and $g(x) = x^2$.

□

Q.19 Show that if $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_{n-1}$, and a_n are real numbers and $a_n \neq 0$, then $f(x)$ is $\Theta(x^n)$.
Solution:

We need to show inequalities in both ways. First, we show that $|f(x)| \leq Cx^n$ for all $x \geq 1$ in the following. Noting that $x^i \leq x^n$ for such values of x whenever $i < n$. We have the following inequalities, where M is the largest of the absolute values of the coefficients and $C = (n+1)M$:

$$\begin{aligned}
|f(x)| &= |a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0| \\
&\leq |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x + |a_0| \\
&\leq |a_n| x^n + |a_{n-1}| x^n + \cdots + |a_1| x^n + |a_0| x^n \\
&\leq Mx^n + Mx^n + \cdots + Mx^n \\
&= Cx^n.
\end{aligned}$$

Algorithm 1 binary insertion sort ($a_1, a_2, \ldots, a_n$: real numbers with $n \geq 2$)

```
for j := 2 to n do
  left := 1
  right := j − 1
  while left < right do
      middle := ⌊(left + right)/2⌋
    if a_j > a_middle then
      left := middle + 1
    else
      right := middle
    end if
  end while
  if a_j < a_left then
    {insert a_j in location i by moving a_i through a_{j−1} towards back of list}
    i := left
  else
    i := left + 1
  end if
  m := a_j
  for k := 0 to j − i − 1 do
    a_{j−k} := a_{j−k−1}
  end for
  a_i := m
end for
```

For the other direction, let k be chosen larger than 1 and larger than $2nm/|a_n|$, where m is the largest of the absolute values of the a_i's for $i < n$. Then each a_{n-i}/x^i will be smaller than $|a_n|/2n$ in absolute value for all $x > k$. Now we have for all $x > k$,

$$\begin{aligned} |f(x)| &= |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0| \\ &= x^n \left| a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right| \\ &\geq x^n |a_n/2|. \end{aligned}$$

□

Q.20 Prove that $n \log n = \Theta(\log n!)$ for all positive integers n.
Solution: We first prove that $n \log n = \Omega(\log n!)$. Since $n^n \geq 1 \cdot 2 \cdot \cdots \cdot n = n!$, we have $n \log n \geq \log n!$ for all positive integers n.

We now prove that $n \log n = O(\log n!)$. It is easy to check that $(n-i)(i+1) \geq n$ for $i = 0, 1, \ldots, n-1$. Thus, $(n!)^2 = (n \cdot 1)((n-1) \cdot 2)((n-2) \cdot 3) \cdots (2 \cdot (n-1))(1 \cdot n) \geq n^n$. Therefore, $2 \log n! \geq n \log n$.

□

Q.21 Prove that for any $a > 1$, $\Theta(\log_a n) = \Theta(\log_2 n)$.
Solution: We must show that there exist constants C_1, C_2 and n_0 such that $\log_a n \leq C_1 \log_2 n$ and $\log_2 n \leq C_2 \log_a n$ for all $n \geq n_0$. By the change of bases formula we have

$$\log_a n = \frac{\log_2 n}{\log_2 a}.$$

Now, let $C_1 = \frac{1}{\log_2 a}$, $C_2 = \log_2 a$, and $n_0 = 1$.

□

Q.22 The conventional algorithm for evaluating a polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ at $x = c$ can be expressed in pseudocode by where the final value of y is the value of the polynomial at $x = c$. Exactly how many multiplications and additions are used to evaluate a polynomial of degree n at $x = c$? (Do not count additions used to increment the loop variable).
Solution: $2n$ multiplications and n additions.

Algorithm 2 polynomial ($c, a_0, a_1, \ldots, a_n$: real numbers)

```
power := 1
y := a_0
for i := 1 to n do
    power := power * c
    y := y + a_i * power
end for
return y {y = a_n c^n + a_{n-1} c^{n-1} + ... + a_1 c + a_0}
```

$\square$

Q.23 There is a more efficient algorithm (in terms of the number of multiplications and additions used) for evaluating polynomials than the conventional algorithm described in the previous exercise. It is called **Horner's method**. This pseudocode shows how to use this method to find the value of $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ at $x = c$.

Algorithm 3 Horner ($c, a_0, a_1, \ldots, a_n$: real numbers)

```
y := a_n
for i := 1 to n do
    y := y * c + a_{n-i}
end for
return y {y = a_n c^n + a_{n-1} c^{n-1} + ... + a_1 c + a_0}
```

Exactly how many multiplications and additions are used by this algorithm to evaluate a polynomial of degree n at $x = c$? (Do not count additions used to increment the loop variable.)

Solution:

n multiplications and n additions.

$\square$