CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

Dr. QI WANG

Department of Computer Science and Engineering
Office: Room903, Nanshan iPark A7 Building
Email: wangqi@sustech.edu.cn

Binary Relations

- Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, the *Cartesian product* $A \times B$ is the set of pairs

$$\{(a_1, b_1), (a_2, b_2), \ldots, (a_1, b_n), \ldots, (a_m, b_n)\}$$

Definition: Let A and B be two sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

Let $R \subseteq A \times B$ denote R is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

Binary Relations

- Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, the *Cartesian product* $A \times B$ is the set of pairs
$$\{(a_1, b_1), (a_2, b_2), \ldots, (a_1, b_n), \ldots, (a_m, b_n)\}$$

Definition: Let A and B be two sets. A *binary relation from A to B* is a subset of a Cartesian product $A \times B$.

Let $R \subseteq A \times B$ denote R is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

Definition: A relation on the set A is a relation from A to itself.

Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Irreflexive Relation: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Irreflexive Relation: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

  Symmetric Relation: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Irreflexive Relation: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

  Symmetric Relation: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

  Antisymmetric Relation: A relation R on a set A is called *antisymmetric* if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Irreflexive Relation: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

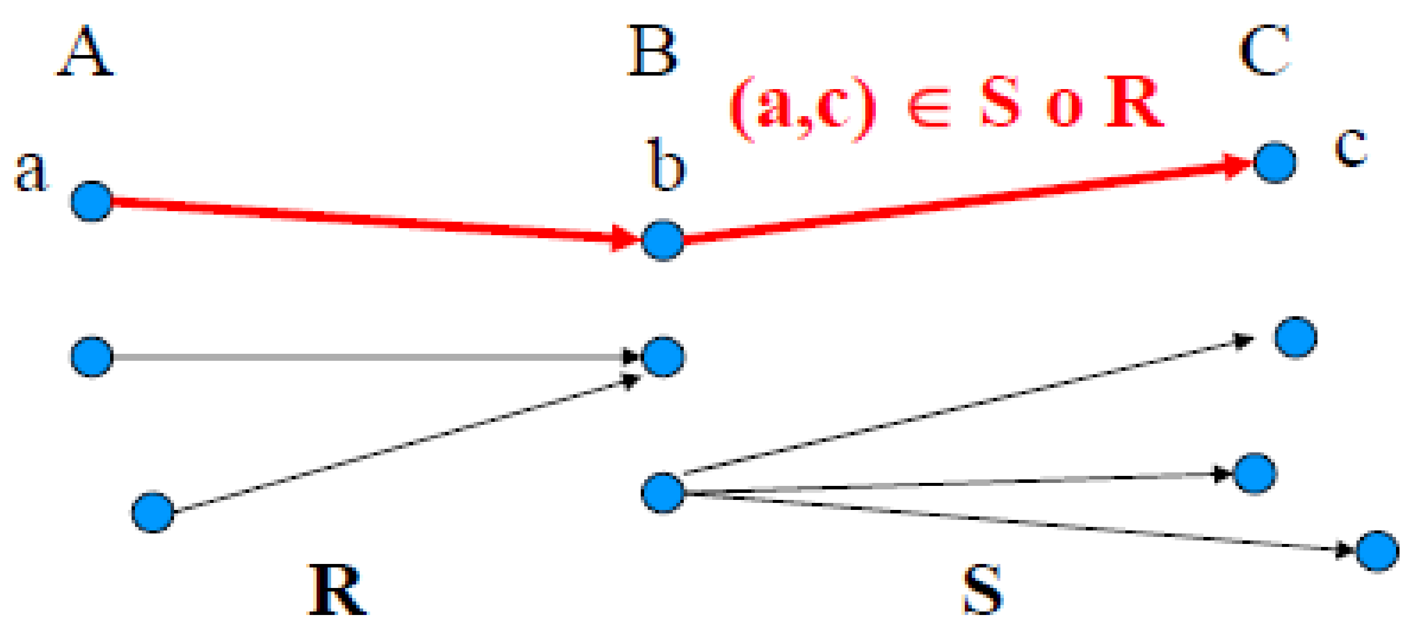
  Symmetric Relation: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

  Antisymmetric Relation: A relation R on a set A is called *antisymmetric* if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

  Transitive Relation: A relation R on a set A is called *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

Composite of Relations

- **Definition**: Let R be a relation from a set A to a set B and S be a relation from B to C. The *composite of R and S* is the relation consisting of the ordered pairs (a, c) where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.



Powers of R

- **Definition** Let R be a relation on A. The *powers* R^n, for $n = 1, 2, 3, \ldots$, is defined inductively by

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

Powers of R

- **Definition** Let R be a relation on A. The *powers* R^n, for $n = 1, 2, 3, \ldots$, is defined inductively by

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

Theorem The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Representing Relations

- Some ways to represent *n-ary* relations:
  - with an *explicit list* or *table* of its tuples
  - with a *function* from the domain to $\{T, F\}$

Representing Relations

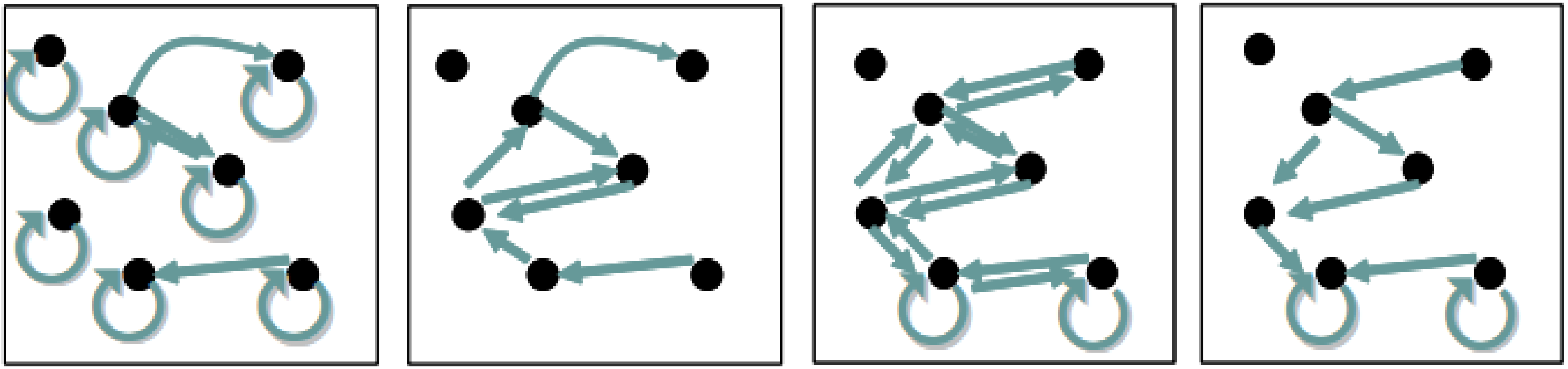
- Some ways to represent *n-ary* relations:
  - with an *explicit list* or *table* of its tuples
  - with a *function* from the domain to $\{T, F\}$

- Some special ways to represent binary relations:
  - with a *zero-one matrix*
  - with a *directed graph*

Representing Relations

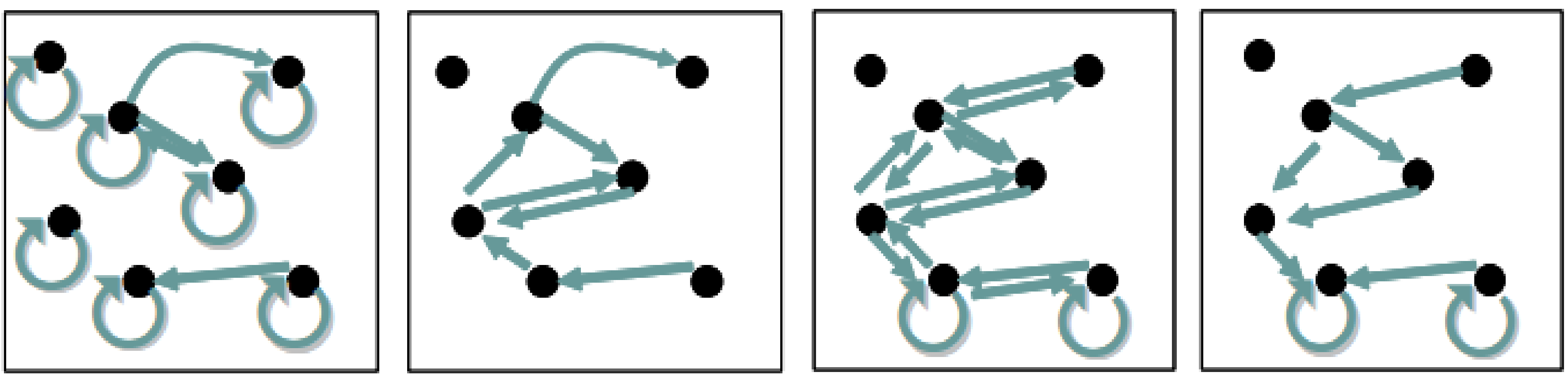
- Some ways to represent *n*-ary relations:
  - with an *explicit list* or *table* of its tuples
  - with a *function* from the domain to $\{T, F\}$

- Some special ways to represent binary relations:
  - with a *zero-one matrix*
  - with a *directed graph*



Representing Relations

- Some ways to represent *n-ary* relations:
  - with an *explicit list* or *table* of its tuples
  - with a *function* from the domain to $\{T, F\}$
- Some special ways to represent binary relations:
  - with a *zero-one matrix*
  - with a *directed graph*



reflexive irreflexive symmetric antisymmetric

Closures of Relations

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

Closures of Relations

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

Closures of Relations

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

  No. Why?

Closures of Relations

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

  No. Why? $(2,2)$ and $(3,3)$ is not in R.

Closures of Relations

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

  No. Why? $(2,2)$ and $(3,3)$ is not in R.

  The question is what is the minimal relation $S \supseteq R$ that is reflexive?

Closures of Relations

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

  No. Why? $(2,2)$ and $(3,3)$ is not in R.

  The question is what is the minimal relation $S \supseteq R$ that is reflexive?

  How to make R reflexive by minimum number of additions?

Closures of Relations

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

  No. Why? $(2,2)$ and $(3,3)$ is not in R.

  The question is what is the minimal relation $S \supseteq R$ that is reflexive?

  How to make R reflexive by minimum number of additions?

  Add $(2,2)$ and $(3,3)$

  Then $S = \{(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)\} \supseteq R$

Closures of Relations

- Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on $A = \{1,2,3\}$.

  Is this relation R reflexive?

  No. Why? $(2,2)$ and $(3,3)$ is not in R.

  The question is what is the minimal relation $S \supseteq R$ that is reflexive?

  How to make R reflexive by minimum number of additions?

  Add $(2,2)$ and $(3,3)$

  Then $S = \{(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)\} \supseteq R$

  The minimal set $S \supseteq R$ is called *the reflexive closure of R*.

Reflexive Closure

- The set S is called *the reflexive closure of* R if it:

Reflexive Closure

- The set S is called *the reflexive closure of* R if it:

  ◇ contains R

  ◇ is reflexive

  ◇ is minimal (is contained in every reflexive relation Q that contains R ($R \subseteq Q$), i.e., $S \subseteq Q$)

Closures on Relations

- Relations can have different properties:
  - reflexive
  - symmetric
  - transitive

Closures on Relations

- Relations can have different properties:
  - reflexive
  - symmetric
  - transitive

  We define:
  - reflexive closures
  - symmetric closures
  - transitive closures

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

  S is the **minimal set** containing R satisfying the property P.

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

  S is the **minimal set** containing R satisfying the property P.

  Example: (symmetric closure)

  $R = \{(1,2),(1,3),(2,2)\}$ on $A = \{1,2,3\}$

  What is the symmetric closure S of R?

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

  S is the **minimal set** containing R satisfying the property P.

  Example: (symmetric closure)

  $R = \{(1,2),(1,3),(2,2)\}$ on $A = \{1,2,3\}$

  What is the symmetric closure S of R?

  $R = \{(1,2),(1,3),(2,2)\} \cup \{(2,1),(3,1)\}$

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

  S is the **minimal set** containing R satisfying the property P.

  Example: (transitive closure)

  $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

  Is R transitive?

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

  S is the **minimal set** containing R satisfying the property P.

  Example: (transitive closure)

  $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

  Is R transitive? No

  How to make it transitive?

Closures

- **Definition** Let R be a relation on a set A. A relation S on A with property P is called *the closure of R with respect to P* if S is subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

  S is the **minimal set** containing R satisfying the property P.

  Example: (transitive closure)

  $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

  Is R transitive? No

  How to make it transitive?

  $S = \{(1,2),(2,2),(2,3)\} \cup \{(1,3)\}$

Transitive Closure

- We can represent the relation on the graph.

Transitive Closure

- We can represent the relation on the graph.

  Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

Transitive Closure

- We can represent the relation on the graph.

  Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

  Example:

  $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

  Transitive closure: $S = \{(1,2),(2,2),(2,3),(1,3)\}$

Transitive Closure

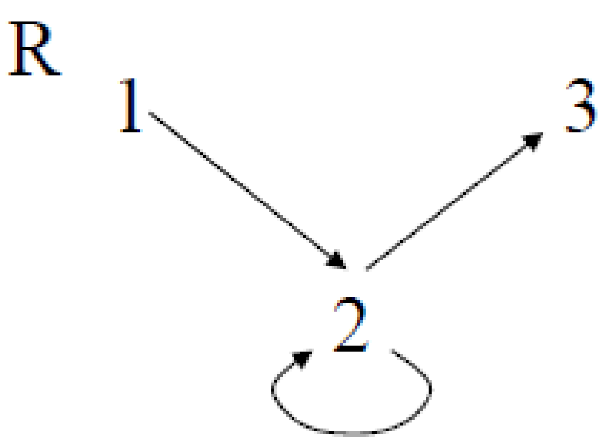
- We can represent the relation on the graph.

  Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

  Example:

  $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

  Transitive closure: $S = \{(1,2),(2,2),(2,3),(1,3)\}$



Transitive Closure

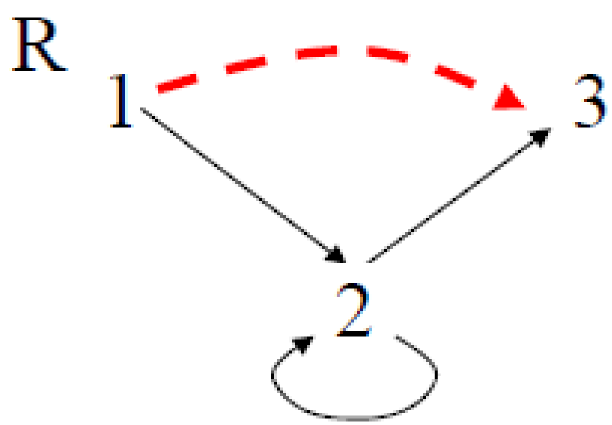
- We can represent the relation on the graph.

  Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

  Example:

  $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

  Transitive closure: $S = \{(1,2),(2,2),(2,3),(1,3)\}$



Transitive Closure

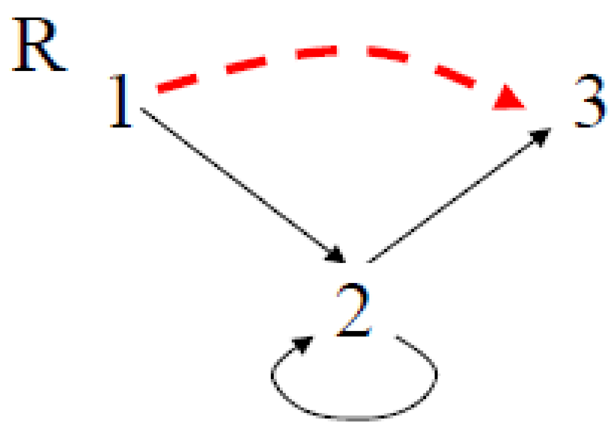
- We can represent the relation on the graph.

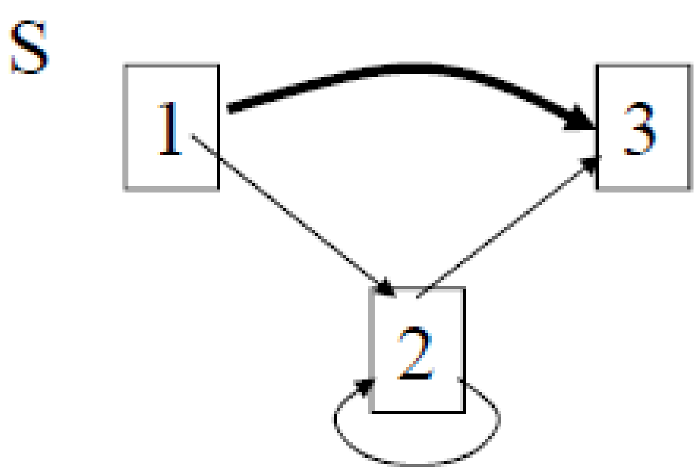
  Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

  Example:

  $R = \{(1,2),(2,2),(2,3)\}$ on $A = \{1,2,3\}$

  Transitive closure: $S = \{(1,2),(2,2),(2,3),(1,3)\}$





Paths in Directed Graphs

- **Definition** A *path* from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ in G, where n is nonnegative and $x_0 = a$ and $x_n = b$. A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

Paths in Directed Graphs

- **Definition** A *path* from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ in G, where n is nonnegative and $x_0 = a$ and $x_n = b$. A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

  Theorem Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

Path Length

- **Theorem** Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

  Proof (by induction)

- **Theorem** Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

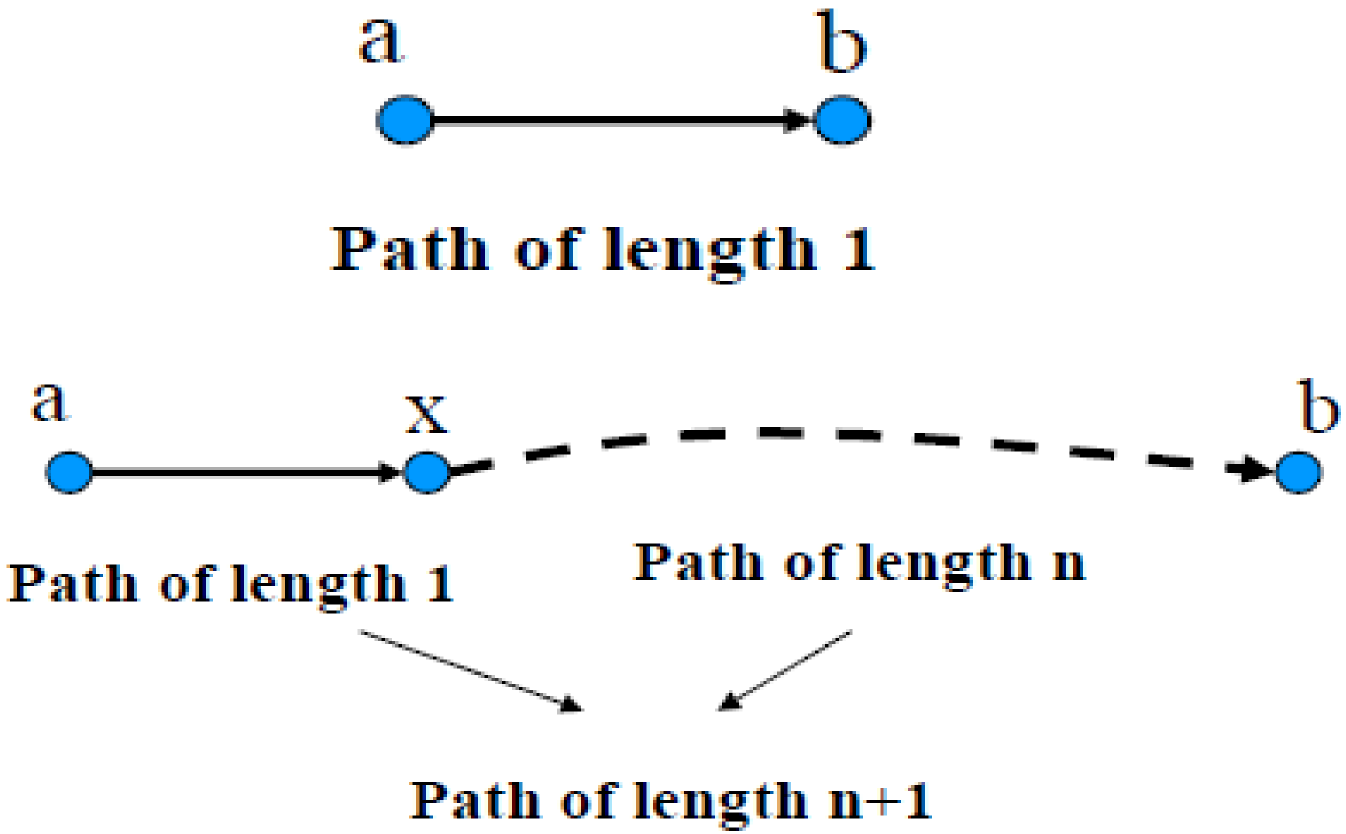
Proof (by induction)

Path of length 1

Path Length

- **Theorem** Let R be relation on a set A. There is a path of length n from a to b if and only if $(a, b) \in R^n$.

  Proof (by induction)



Connectivity Relation

- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

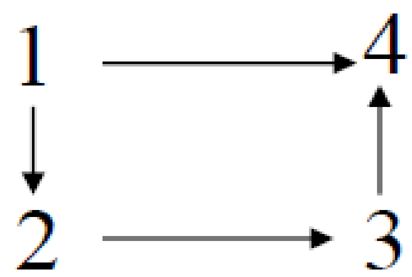
Connectivity Relation

- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

Example:

$$A = \{1, 2, 3, 4\}$$
$$R = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$$

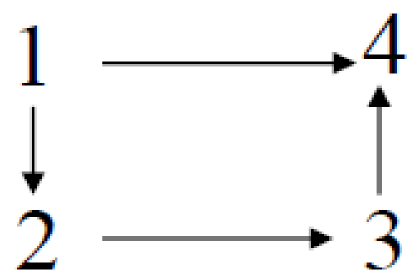


Connectivity Relation

- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

Example:



$$A = \{1, 2, 3, 4\}$$
$$R = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$$
$$R^2 = \{(1, 3), (2, 4)\}$$

Connectivity Relation

- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

Example:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 4 \\ 2 \arrow[r] & 3 \arrow[u] \end{tikzcd}$$

$$A = \{1, 2, 3, 4\}$$
$$R = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$$
$$R^2 = \{(1, 3), (2, 4)\}$$
$$R^3 = \{(1, 4)\}$$

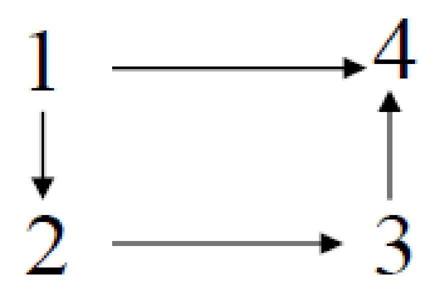
Connectivity Relation

- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

Example:

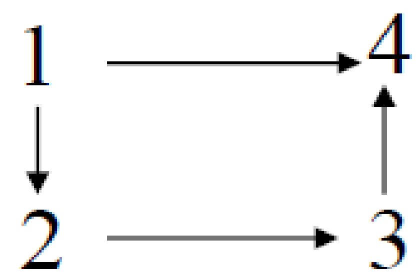
$$\begin{aligned} A &= \{1, 2, 3, 4\} \\ R &= \{(1,2), (1,4), (2,3), (3,4)\} \\ R^2 &= \{(1,3), (2,4)\} \\ R^3 &= \{(1,4)\} \\ R^4 &= \emptyset \end{aligned}$$



- **Definition** Let R be a relation on a set A. The *connectivity relation* R^* consists of all pairs (a, b) s.t. there is a path (of any length) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

Example:



$$A = \{1, 2, 3, 4\}$$
$$R = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$$
$$R^2 = \{(1, 3), (2, 4)\}$$
$$R^3 = \{(1, 4)\}$$
$$R^4 = \emptyset$$

$$R^* = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

Connectivity

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

Connectivity

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

  Proof (by intuition)

Connectivity

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

  Proof (by intuition)

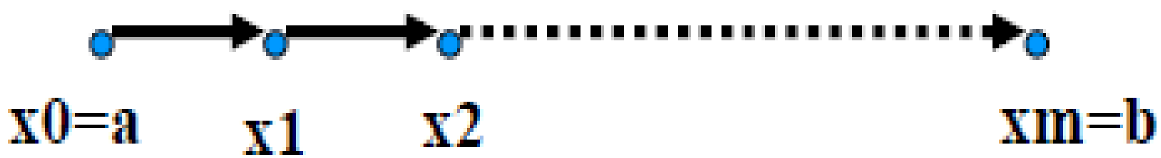
  ◇ There are at most n different elements we can visit on a path if the path does not have loops

Connectivity

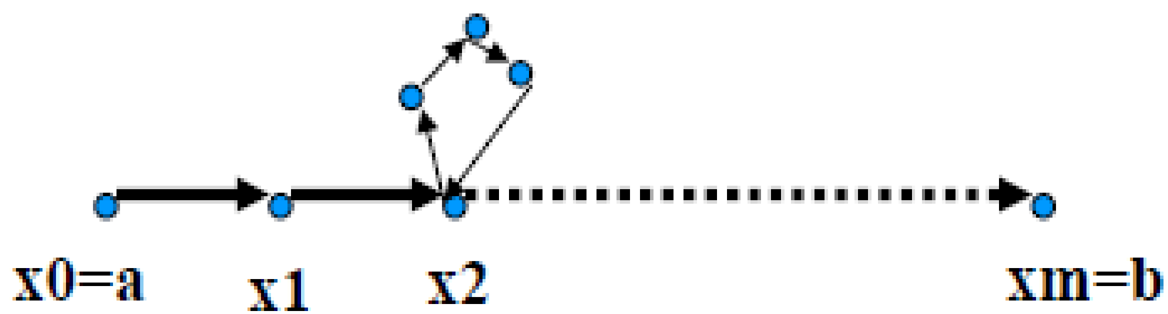
- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n - 1$.

  Proof (by intuition)

  ◇ There are at most n different elements we can visit on a path if the path does not have loops



  ◇ Loops may increase the length but the same node is visited more than once

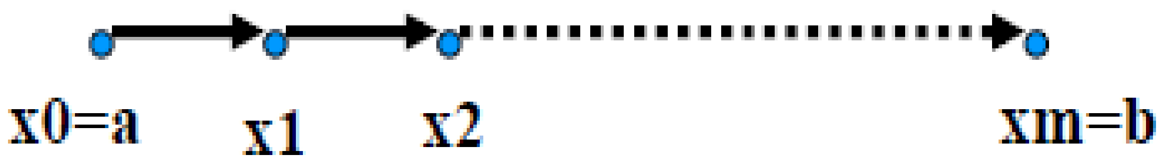


Connectivity

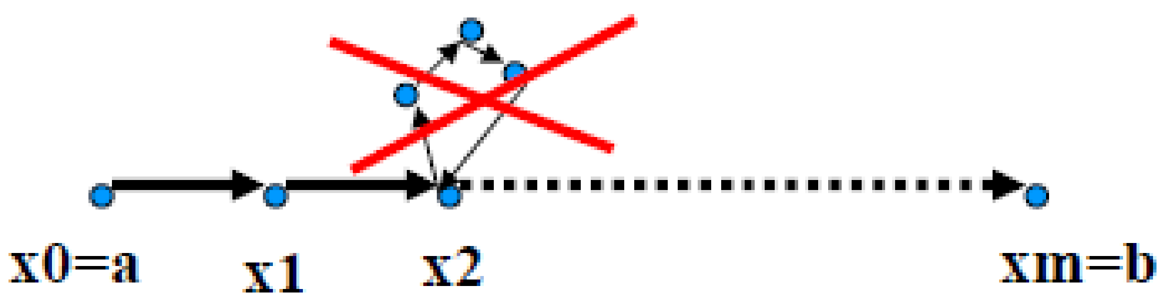
- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

  Proof (by intuition)

  ◇ There are at most n different elements we can visit on a path if the path does not have loops



  ◇ Loops may increase the length but the same node is visited more than once



Connectivity

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n - 1$.

$$R^* = \bigcup_{k=1}^{n} R^k$$

Connectivity

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

$$R^* = \bigcup_{k=1}^{n} R^k$$

Theorem: The transitive closure of a relation R equals the connectivity relation R^*.

Connectivity

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

$$R^* = \bigcup_{k=1}^{n} R^k$$

Theorem: The transitive closure of a relation R equals the connectivity relation R^*.

Recall Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

Connectivity

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

Connectivity

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

  Proof

Connectivity

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

Proof

1. R^* is transitive
2. $R^* \subseteq S$ whenever S is a transitive relation containing R

Connectivity

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

Proof

1. R^* is transitive
2. $R^* \subseteq S$ whenever S is a transitive relation containing R

1. If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c in R. Thus, there is a path from a to c in R. This means that $(a, c) \in R^*$.

Connectivity

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

Proof

1. R^* is transitive
2. $R^* \subseteq S$ whenever S is a transitive relation containing R

2. Suppose that S is a transitive relation containing R.

Connectivity

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

Proof

1. R^* is transitive
2. $R^* \subseteq S$ whenever S is a transitive relation containing R

2. Suppose that S is a transitive relation containing R.

Then S^n is also transitive and $S^n \subseteq S$. Why?

Connectivity

- **Theorem**: The transitive closure of a relation R equals the connectivity relation R^*.

Proof

1. R^* is transitive
2. $R^* \subseteq S$ whenever S is a transitive relation containing R

2. Suppose that S is a transitive relation containing R.

Then S^n is also transitive and $S^n \subseteq S$. Why?

We have $S^* \subseteq S$. Thus, $R^* \subseteq S^* \subseteq S$

Find Transitive Closure

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

Find Transitive Closure

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \mathbf{M}_R^{[n]}$$

Find Transitive Closure

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \mathbf{M}_R^{[n]}$$

Example

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{M}_{R^*} = ?$$

Simple Transitive Closure Algorithm

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

procedure transClosure ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$A := B := \mathbf{M}_R$;
for $i := 2$ to n
  $A := A \odot \mathbf{M}_R$
  $B := B \vee A$
return B
// B is the zero-one matrix for R^*

Simple Transitive Closure Algorithm

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

procedure transClosure ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$A := B := \mathbf{M}_R$;
for $i := 2$ to n
  $A := A \odot \mathbf{M}_R$
  $B := B \vee A$
return B
// B is the zero-one matrix for R^*

This algorithm takes $\Theta(n^4)$ time.

Simple Transitive Closure Algorithm

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

  procedure transClosure ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
  // computes R^* with zero-one matrices
  $A := B := \mathbf{M}_R$;
  for $i := 2$ to n
  $\quad A := A \odot \mathbf{M}_R$
  $\quad B := B \vee A$
  return B
  // B is the zero-one matrix for R^*

  This algorithm takes $\Theta(n^4)$ time. Why?

Roy-Warshall Algorithm

procedure Warshall ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$W := \mathbf{M}_R$;
for $k := 1$ to n
  for $i := 1$ to n
    for $j := 1$ to n
      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return W
// W is the zero-one matrix for R^*

Roy-Warshall Algorithm

procedure Warshall ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$W := \mathbf{M}_R$;
for $k := 1$ to n
  for $i := 1$ to n
    for $j := 1$ to n
      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return W
// W is the zero-one matrix for R^*

$w_{ij} = 1$ means there is a path from i to j going only through nodes $\leq k$.

$$W_{ij}^{[k]} = W_{ij}^{[k-1]} \vee \left(W_{ik}^{[k-1]} \wedge W_{kj}^{[k-1]} \right)$$

Roy-Warshall Algorithm

procedure Warshall ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes R^* with zero-one matrices
$W := \mathbf{M}_R$;
for $k := 1$ to n
  for $i := 1$ to n
    for $j := 1$ to n
        $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return W
// W is the zero-one matrix for R^*

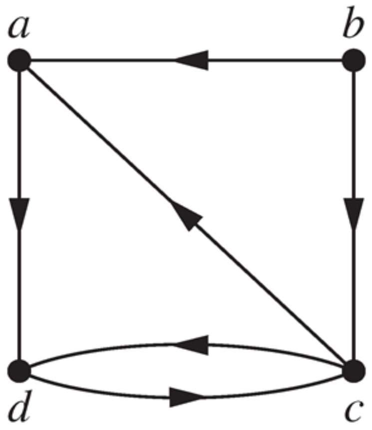
$w_{ij} = 1$ means there is a path from i to j going only through nodes $\leq k$.

$$W_{ij}^{[k]} = W_{ij}^{[k-1]} \vee \left(W_{ik}^{[k-1]} \wedge W_{kj}^{[k-1]} \right)$$

This algorithm takes $\Theta(n^3)$ time.

Example

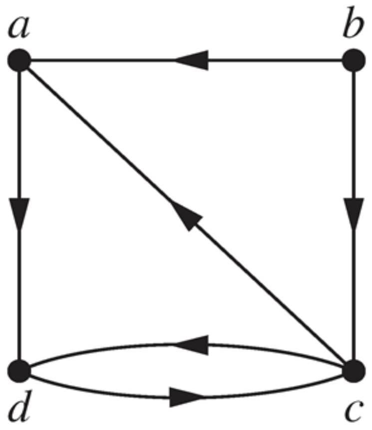
Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



Let $v_1 = a$, $v_2 = b$, $v_3 = c$, $v_4 = d$.

Example

Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.

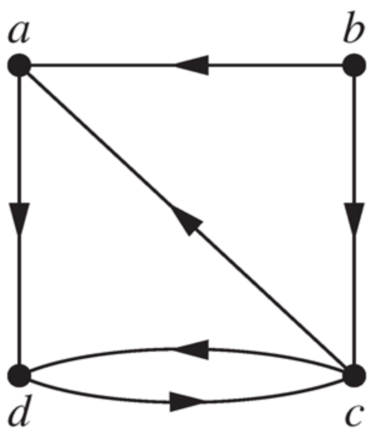


Let $v_1 = a$, $v_2 = b$, $v_3 = c$, $v_4 = d$.

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Example

Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



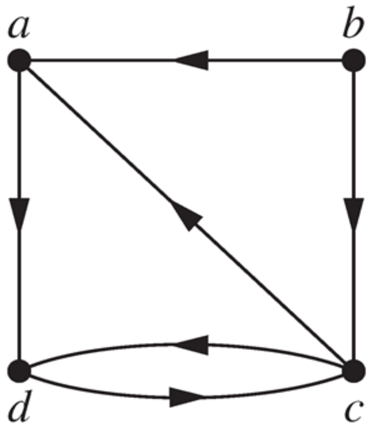
Let $v_1 = a$, $v_2 = b$, $v_3 = c$, $v_4 = d$.

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_2 = W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Example

Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



Let $v_1 = a$, $v_2 = b$, $v_3 = c$, $v_4 = d$.

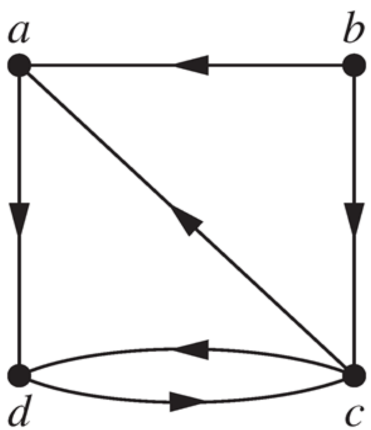
$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_2 = W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & \color{red}{1} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & \color{red}{1} \\ 1 & 0 & 0 & 1 \\ \color{red}{1} & 0 & 1 & \color{red}{1} \end{bmatrix}$$

Example

Find the matrices W_0, W_1, W_2, W_3, and W_4. The matrix W_4 is the transitive closure of R.



Let $v_1 = a$, $v_2 = b$, $v_3 = c$, $v_4 = d$.

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_2 = W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Equivalence Relation

- **Definition** A relation R on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Equivalence Relation

- **Definition** A relation R on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

  Example:

  $$A = \{0, 1, 2, 3, 4, 5, 6\}$$
  $$R = \{(a, b) : a \equiv b \bmod 3\}$$

Equivalence Relation

- **Definition** A relation R on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

  Example:

  $$A = \{0, 1, 2, 3, 4, 5, 6\}$$
  $$R = \{(a, b) : a \equiv b \bmod 3\}$$

  R has the following pairs:

  $\bullet(0, 0), (0, 3), (3, 0), (0, 6), (6, 0), (3, 3), (3, 6), (6, 3), (6, 6)$
  $\bullet(1, 1), (1, 4), (4, 1), (4, 4)$
  $\bullet(2, 2), (2, 5), (5, 2), (5, 5)$

Equivalence Relation

- Relation R on $A = \{0, 1, 2, 3, 4, 5, 6\}$ has the pairs:
  - $(0,0), (0,3), (3,0), (0,6), (6,0), (3,3), (3,6), (6,3), (6,6)$
  - $(1,1), (1,4), (4,1), (4,4)$
  - $(2,2), (2,5), (5,2), (5,5)$

Equivalence Relation

- Relation R on $A = \{0, 1, 2, 3, 4, 5, 6\}$ has the pairs:

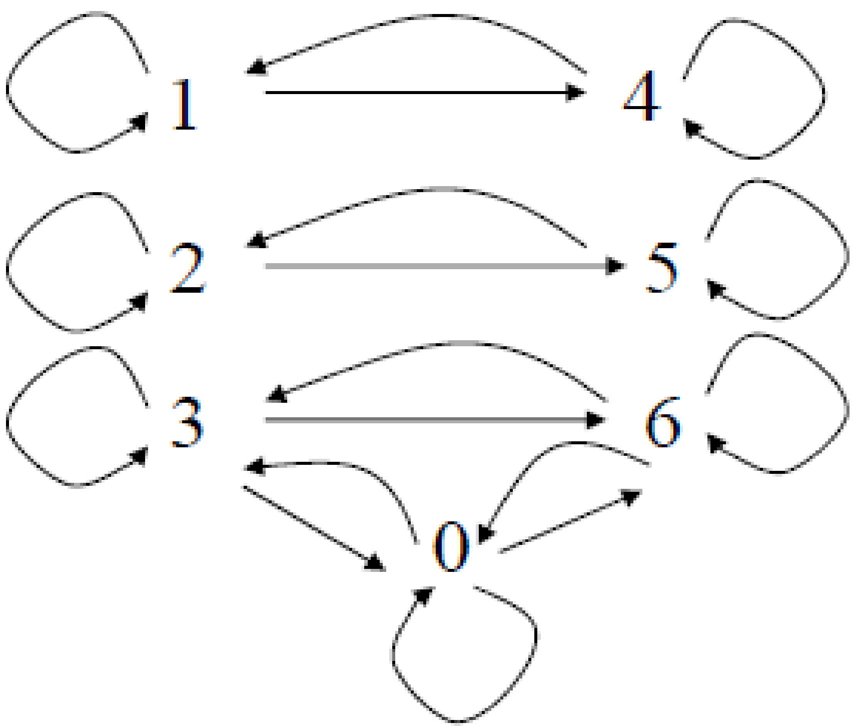
  $\bullet(0,0),(0,3),(3,0),(0,6),(6,0),(3,3),(3,6),(6,3),(6,6)$
  $\bullet(1,1),(1,4),(4,1),(4,4)$
  $\bullet(2,2),(2,5),(5,2),(5,5)$

  Is R reflexive?

Equivalence Relation

- Relation R on $A = \{0, 1, 2, 3, 4, 5, 6\}$ has the pairs:
  - $(0,0), (0,3), (3,0), (0,6), (6,0), (3,3), (3,6), (6,3), (6,6)$
  - $(1,1), (1,4), (4,1), (4,4)$
  - $(2,2), (2,5), (5,2), (5,5)$

Is R reflexive?

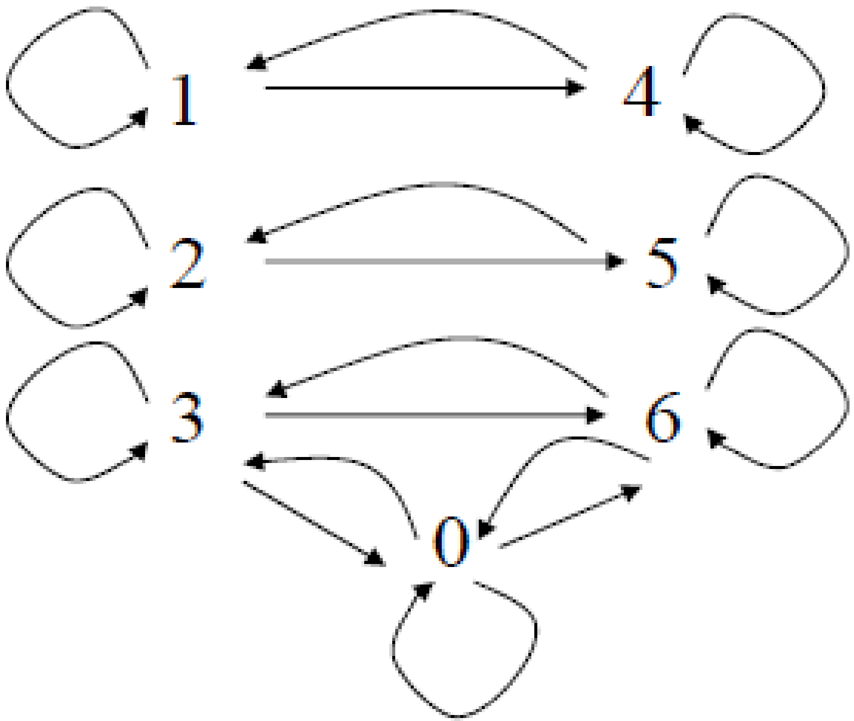


Equivalence Relation

- Relation R on $A = \{0, 1, 2, 3, 4, 5, 6\}$ has the pairs:

  $\bullet (0, 0), (0, 3), (3, 0), (0, 6), (6, 0), (3, 3), (3, 6), (6, 3), (6, 6)$
  $\bullet (1, 1), (1, 4), (4, 1), (4, 4)$
  $\bullet (2, 2), (2, 5), (5, 2), (5, 5)$

Is R reflexive? Yes

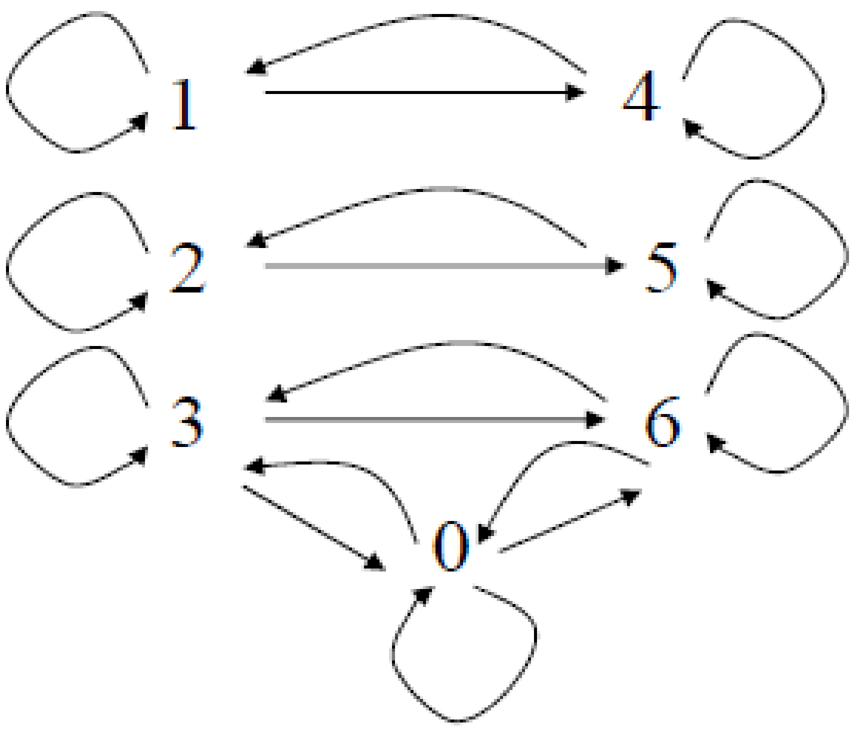


Equivalence Relation

- Relation R on $A = \{0, 1, 2, 3, 4, 5, 6\}$ has the pairs:
  - $(0,0), (0,3), (3,0), (0,6), (6,0), (3,3), (3,6), (6,3), (6,6)$
  - $(1,1), (1,4), (4,1), (4,4)$
  - $(2,2), (2,5), (5,2), (5,5)$

Is R reflexive? Yes

Is R symmetric?

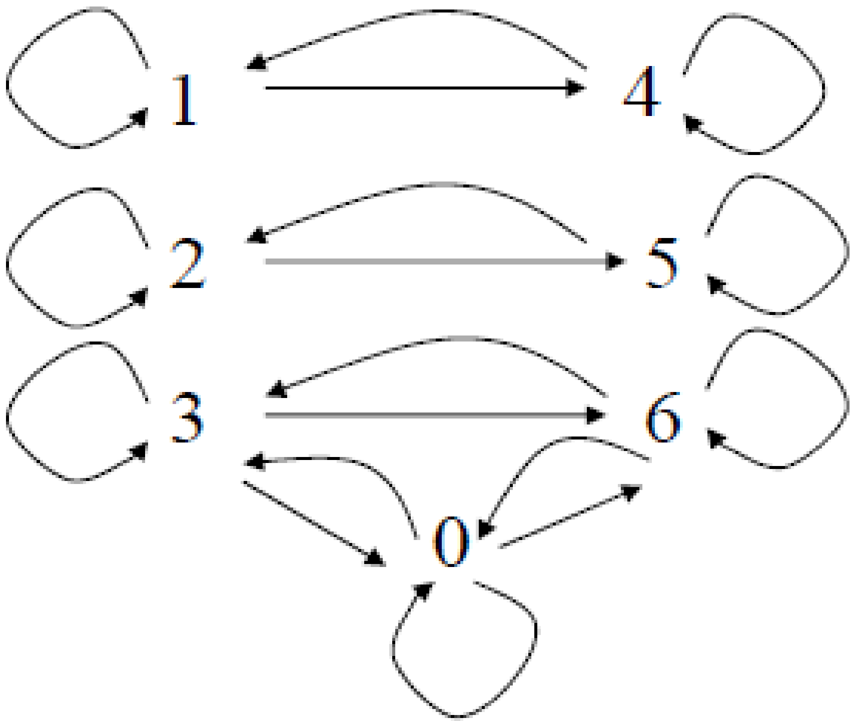


Equivalence Relation

- Relation R on $A = \{0, 1, 2, 3, 4, 5, 6\}$ has the pairs:
    - $(0, 0), (0, 3), (3, 0), (0, 6), (6, 0), (3, 3), (3, 6), (6, 3), (6, 6)$
    - $(1, 1), (1, 4), (4, 1), (4, 4)$
    - $(2, 2), (2, 5), (5, 2), (5, 5)$

Is R reflexive? Yes

Is R symmetric? Yes



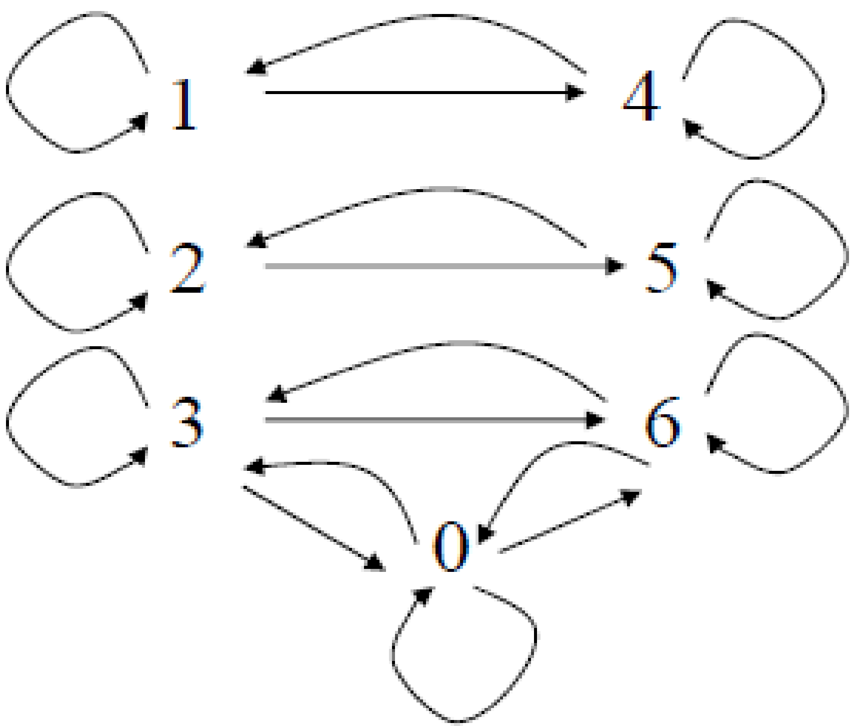
Equivalence Relation

- Relation R on $A = \{0, 1, 2, 3, 4, 5, 6\}$ has the pairs:
  - $(0,0), (0,3), (3,0), (0,6), (6,0), (3,3), (3,6), (6,3), (6,6)$
  - $(1,1), (1,4), (4,1), (4,4)$
  - $(2,2), (2,5), (5,2), (5,5)$

Is R reflexive? Yes

Is R symmetric? Yes

Is R transitive?



Equivalence Relation

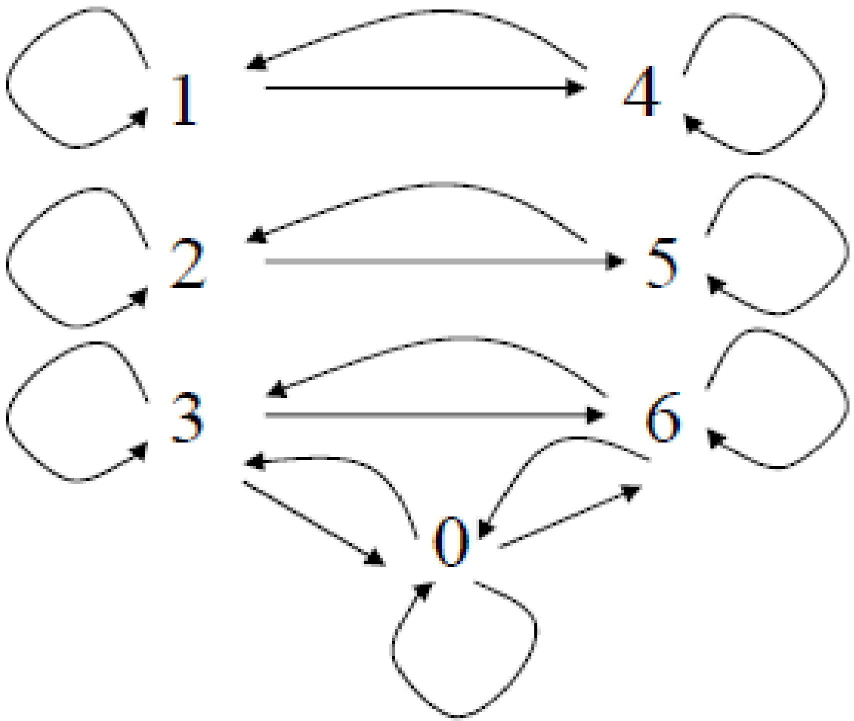
- Relation R on $A = \{0, 1, 2, 3, 4, 5, 6\}$ has the pairs:

  $\bullet(0, 0), (0, 3), (3, 0), (0, 6), (6, 0), (3, 3), (3, 6), (6, 3), (6, 6)$
  $\bullet(1, 1), (1, 4), (4, 1), (4, 4)$
  $\bullet(2, 2), (2, 5), (5, 2), (5, 5)$

Is R reflexive? Yes

Is R symmetric? Yes

Is R transitive? Yes



Equivalence Relation

- Relation R on $A = \{0, 1, 2, 3, 4, 5, 6\}$ has the pairs:

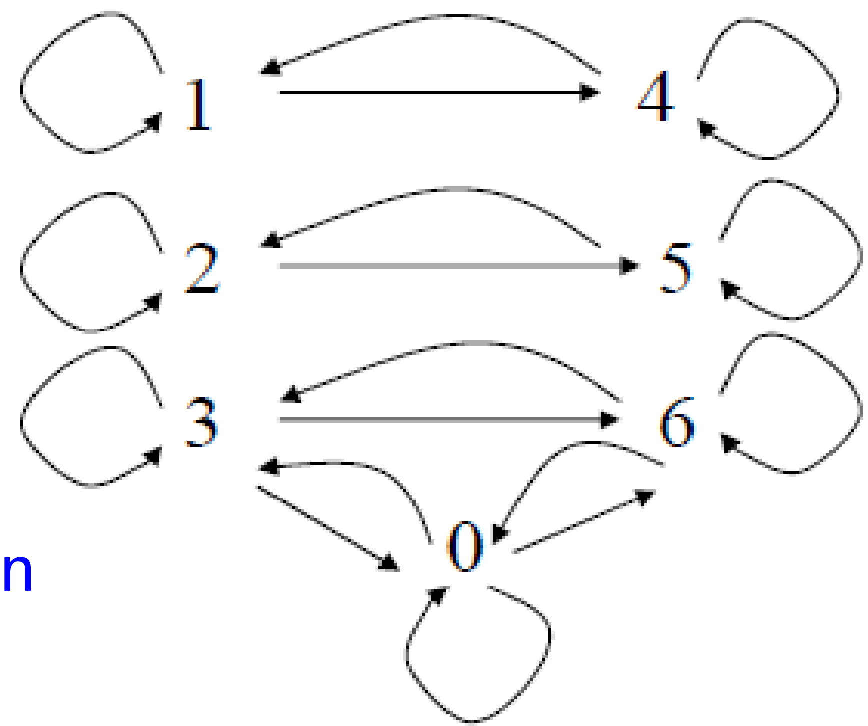
  $\bullet (0, 0), (0, 3), (3, 0), (0, 6), (6, 0), (3, 3), (3, 6), (6, 3), (6, 6)$
  $\bullet (1, 1), (1, 4), (4, 1), (4, 4)$
  $\bullet (2, 2), (2, 5), (5, 2), (5, 5)$

Is R reflexive? Yes

Is R symmetric? Yes

Is R transitive? Yes

R is an equivalence relation



Examples of Equivalence Relations

- **Examples**

  "Strings a and b have the same length."

  "Integers a and b have the same absolute value."

  "Real numbers a and b have the same fractional part (i.e., $a - b \in \mathbf{Z}$)."

Examples of Equivalence Relations

- **Examples**

  "Strings a and b have the same length."

  "Integers a and b have the same absolute value."

  "Real numbers a and b have the same fractional part (i.e., $a - b \in \mathbf{Z}$)."

  "The relation $\geq$ between real numbers."

  "has a common factor greater than 1 between natural numbers."

Equivalence Class

- **Definition** Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a, denoted by $[a]_R$. When only one relation is considered, we use the notation $[a]$.

Equivalence Class

- **Definition** Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a, denoted by $[a]_R$. When only one relation is considered, we use the notation $[a]$.

$$[a]_R = \{b : (a, b) \in R\}$$

Equivalence Class

- **Definition** Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a, denoted by $[a]_R$. When only one relation is considered, we use the notation $[a]$.

$$[a]_R = \{b : (a, b) \in R\}$$

$A = \{0, 1, 2, 3, 4, 5, 6\}$
$R = \{(a, b) : a \equiv b \bmod 3\}$

Equivalence Class

- **Definition** Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a, denoted by $[a]_R$. When only one relation is considered, we use the notation $[a]$.

$$[a]_R = \{b : (a, b) \in R\}$$

$$A = \{0, 1, 2, 3, 4, 5, 6\}$$
$$R = \{(a, b) : a \equiv b \bmod 3\}$$

$$[0] = [3] = [6] = \{0, 3, 6\}$$
$$[1] = [4] = \{1, 4\}$$
$$[2] = [5] = \{2, 5\}$$

Examples of Equivalence Classes

- **Examples**

  "Strings a and b have the same length."

  "Integers a and b have the same absolute value."

  "Real numbers a and b have the same fractional part (i.e., $a - b \in \mathbf{Z}$)."

Examples of Equivalence Classes

- **Examples**

  "Strings a and b have the same length."

  $[a] =$ the set of all strings of the same length as a

  "Integers a and b have the same absolute value."

  $[a] =$ the set $\{a, -a\}$

  "Real numbers a and b have the same fractional part (i.e., $a - b \in \mathbf{Z}$)."

  $[a] =$ the set $\{\ldots, a-2, a-1, a, a+1, a+2, \ldots\}$

Equivalence Class

- **Theorem** Let R be an equivalence relation on a set A. The following statements are equivalent:

  (i) $a\ R\ b$
  (ii) $[a] = [b]$
  (iii) $[a] \cap [b] \neq \emptyset$

Equivalence Class

- **Theorem** Let R be an equivalence relation on a set A. The following statements are equivalent:

  (i) $a\ R\ b$
  (ii) $[a] = [b]$
  (iii) $[a] \cap [b] \neq \emptyset$

  Proof

  $(i) \to (ii)$:

Equivalence Class

- **Theorem** Let R be an equivalence relation on a set A. The following statements are equivalent:

  (i) $a \; R \; b$
  (ii) $[a] = [b]$
  (iii) $[a] \cap [b] \neq \emptyset$

  Proof

  $(i) \to (ii)$: prove $[a] \subseteq [b]$ and $[b] \subseteq [a]$

Equivalence Class

- **Theorem** Let R be an equivalence relation on a set A. The following statements are equivalent:

  (i) $a \; R \; b$
  (ii) $[a] = [b]$
  (iii) $[a] \cap [b] \neq \emptyset$

  Proof

  $(i) \rightarrow (ii)$: prove $[a] \subseteq [b]$ and $[b] \subseteq [a]$

  $(ii) \rightarrow (iii)$:

Equivalence Class

- **Theorem** Let R be an equivalence relation on a set A. The following statements are equivalent:

  (i) $a \; R \; b$
  (ii) $[a] = [b]$
  (iii) $[a] \cap [b] \neq \emptyset$

  Proof

  $(i) \rightarrow (ii)$: prove $[a] \subseteq [b]$ and $[b] \subseteq [a]$

  $(ii) \rightarrow (iii)$: $[a]$ is not empty (R reflexive)

Equivalence Class

- **Theorem** Let R be an equivalence relation on a set A. The following statements are equivalent:

  (i) $a \; R \; b$
  (ii) $[a] = [b]$
  (iii) $[a] \cap [b] \neq \emptyset$

  Proof

  $(i) \rightarrow (ii)$: prove $[a] \subseteq [b]$ and $[b] \subseteq [a]$

  $(ii) \rightarrow (iii)$: $[a]$ is not empty (R reflexive)

  $(iii) \rightarrow (i)$:

Equivalence Class

- **Theorem** Let R be an equivalence relation on a set A. The following statements are equivalent:

  (i) $a\ R\ b$
  (ii) $[a] = [b]$
  (iii) $[a] \cap [b] \neq \emptyset$

  Proof

  $(i) \to (ii)$: prove $[a] \subseteq [b]$ and $[b] \subseteq [a]$

  $(ii) \to (iii)$: $[a]$ is not empty (R reflexive)

  $(iii) \to (i)$: there exists a c s.t. $c \in [a]$ and $c \in [b]$

Partition of a Set S

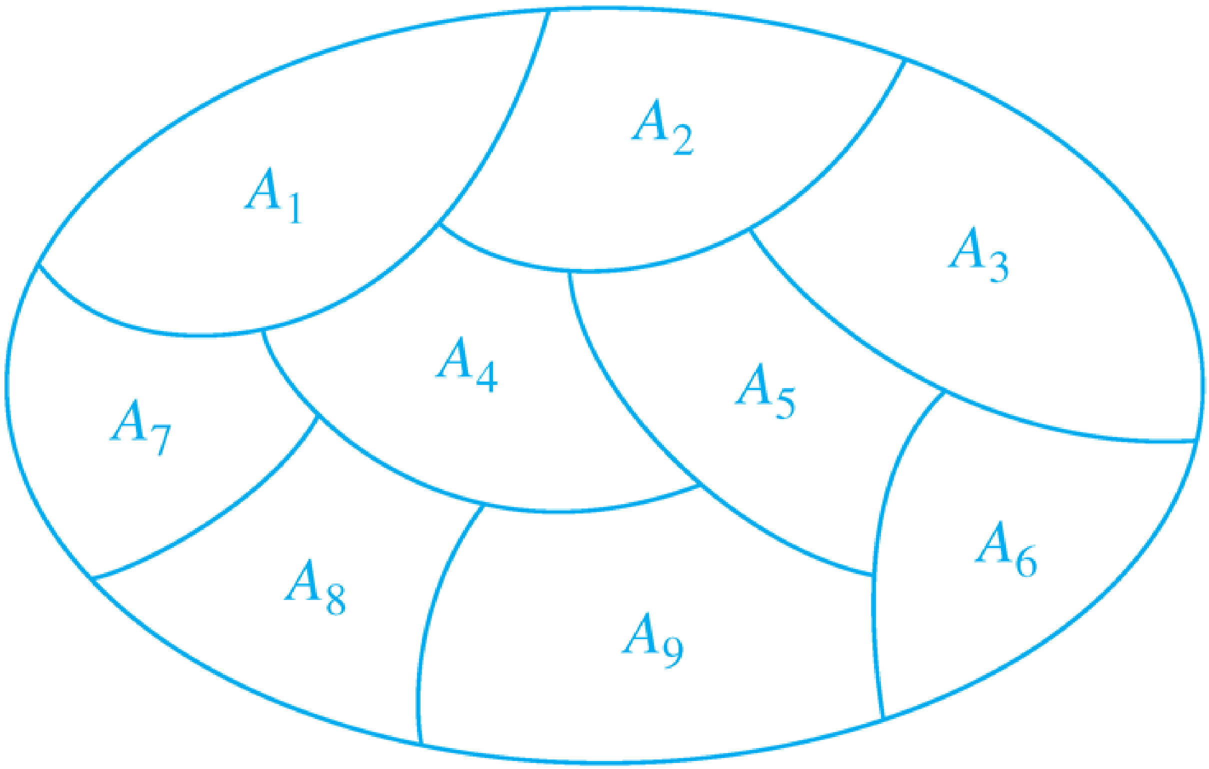
- **Definition** Let S be a set. A collection of nonempty subsets of S $A_1, A_2, \ldots, A_k$ is called *a partition of S* if:

$$A_i \cap A_j = \emptyset, \ i \neq j \text{ and } S = \bigcup_{i=1}^{k} A_i$$

Partition of a Set S

- **Definition** Let S be a set. A collection of nonempty subsets of S $A_1, A_2, \ldots, A_k$ is called *a partition of S* if:

$$A_i \cap A_j = \emptyset, \; i \neq j \text{ and } S = \bigcup_{i=1}^{k} A_i$$



Partition of a Set S

- **Definition** Let S be a set. A collection of nonempty subsets of S $A_1, A_2, \ldots, A_k$ is called *a partition of S* if:

$$A_i \cap A_j = \emptyset, \ i \neq j \text{ and } S = \bigcup_{i=1}^{k} A_i$$

Example:

$$A = \{0, 1, 2, 3, 4, 5, 6\}$$

$$A_1 = \{0, 3, 6\},\ A_2 = \{1, 4\},\ A_3 = \{2, 5\}$$

Partition of a Set S

- **Definition** Let S be a set. A collection of nonempty subsets of S $A_1, A_2, \ldots, A_k$ is called *a partition of S* if:

$$A_i \cap A_j = \emptyset, \;\; i \neq j \text{ and } S = \bigcup_{i=1}^{k} A_i$$

Example:

$$A = \{0, 1, 2, 3, 4, 5, 6\}$$

$$A_1 = \{0, 3, 6\},\; A_2 = \{1, 4\},\; A_3 = \{2, 5\}$$

Is A_1, A_2, A_3 a partition of S?

Equivalence Classes and Partitions

- **Theorem** Let R be an equivalence relation on a set A. Then union of all the equivalence classes of R is A:

$$A = \bigcup_{a \in A} [a]_R$$

Equivalence Classes and Partitions

- **Theorem** Let R be an equivalence relation on a set A. Then union of all the equivalence classes of R is A:

$$A = \bigcup_{a \in A} [a]_R$$

Theorem The equivalence classes form a partition of A.

Equivalence Classes and Partitions

- **Theorem** Let R be an equivalence relation on a set A. Then union of all the equivalence classes of R is A:

$$A = \bigcup_{a \in A} [a]_R$$

Theorem The equivalence classes form a partition of A.

Theorem Let $\{A_1, A_2, \ldots, A_i, \ldots\}$ be a partition of S. Then there is an equivalence relation R on S, that has the sets A_i as its equivalence classes.

Next Lecture

- relation, graph ...