南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

CS201 DISCRETE MATHEMATICS FOR COMPUTER SCIENCE

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Applications of Number Theory in Cryptography

- Introduction
- Symmetric cryptography
- Asymmetric cryptography
- RSA Cryptosystem
- DLP and El Gamal cryptography
- Diffie-Hellman key exchange protocol
- Crytocurrency, e.g., bitcoin

Fermat's Little Theorem

- **Theorem (Fermat's little theorem)** : Let p be a prime, and let x be an integer such that $x \not\equiv 0 \bmod p$. Then

$$x^{p-1} \equiv 1 \pmod{p}.$$

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$$\{1, 2, \ldots, p-1\} = \{x, 2x, \ldots, x(p-1) \pmod{p}\}$$

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  Theorem * There is a primitive root modulo n if and only if $n = 2, 4, p^e$ or $2p^e$, where p is an odd prime.

  $\mathcal{Q}$: proof? The number of primitive roots? $*$

Cryptography

- History of almost 4000 years (from 1900 B.C.)

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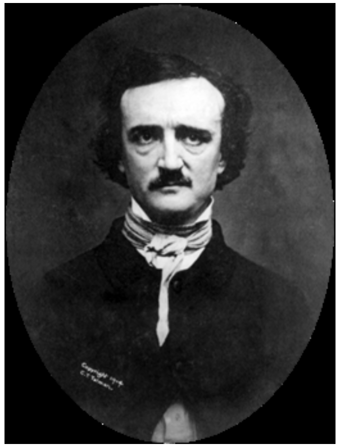
Cryptography = kryptos + graphos

(secret) (writing)

The term was first used in *The Gold-Bug*, by Edgar Allan Poe (1809 - 1849).



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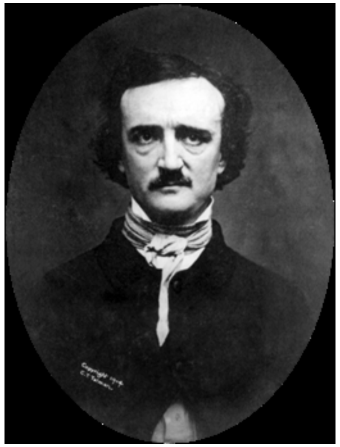


"Human ingenuity cannot concoct a cipher which human ingenuity cannot resolve." – 1941

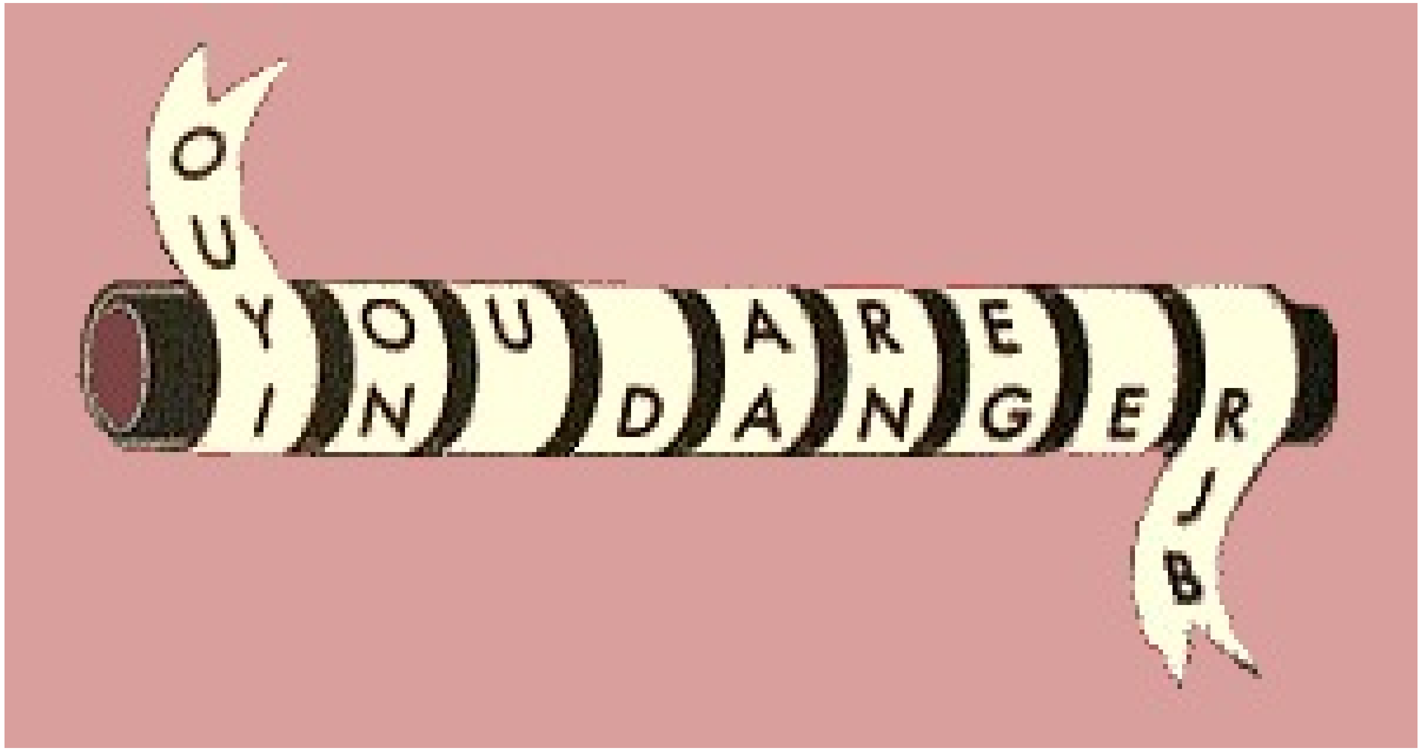
Cryptography

- One-sentence definition:

  "Cryptography is the practice and study of techniques for secure communication in the presence of third parties called *adversaries*." – Ronald L. Rivest

Some Examples

- In 405 BC, the Greek general LYSANDER OF SPARTA was sent a coded message written on the inside of a servant's belt.



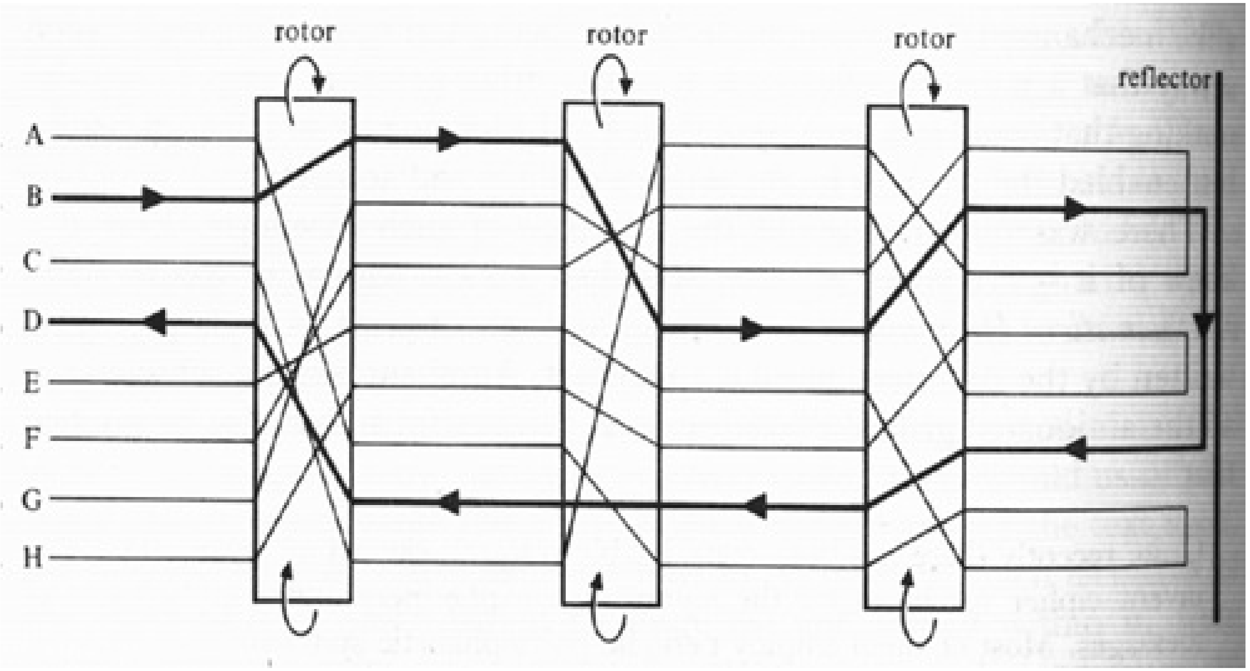
Some Examples

- The Greeks also invented a cipher which changed letters to numbers. A form of this code was still being used during *World War I*.

	1	2	3	4	5
1	A	B	C	D	E
2	F	G	H	I/J	K
3	L	M	N	O	P
4	Q	R	S	T	U
5	V	W	X	Y	Z

Some Examples

- Enigma, Germany coding machine in *World War II.*



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$\mathcal{Q}$: What if Bob could send Alice a "special key" useful only for encryption but no help for decryption?

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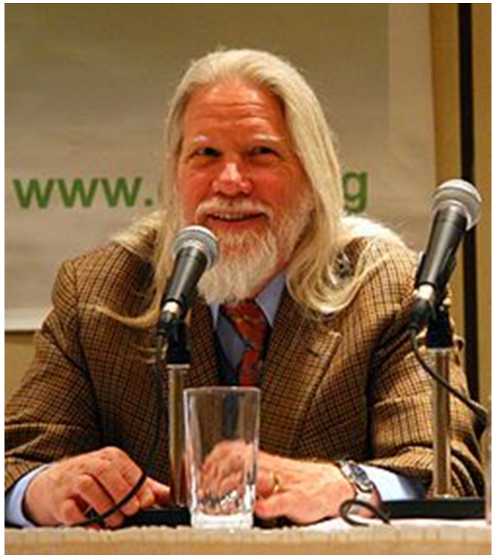
- History (from 1976)
  - ◇ W. Diffie, M. Hellman, "New direction in cryptography", *IEEE Transactions on Information Theory, vol. 22, pp. 644-654, 1976.*

"We stand today on the brink of a revolution in cryptography."



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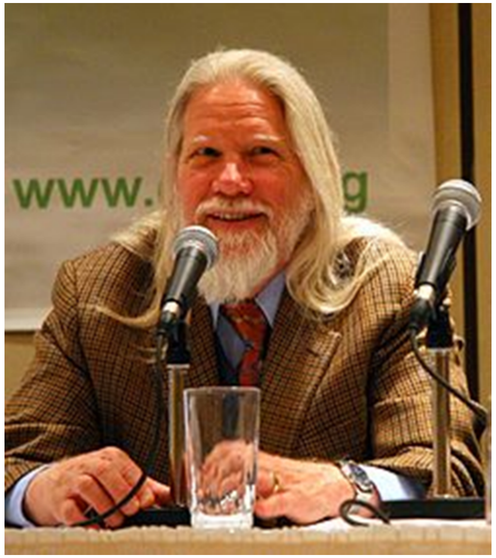
2015 **Turing Award**



Bailey W. Diffie

Martin E. Hellman

2015	Martin E. Hellman Whitfield Diffie	For fundamental contributions to **modern cryptography**. Diffie and Hellman's groundbreaking 1976 paper, "New Directions in Cryptography,"[39] introduced the ideas of public-key cryptography and digital signatures, which are the foundation for most regularly-used security protocols on the internet today.[40]

Public Key Cryptography

- Alice wants to send a message to Bob

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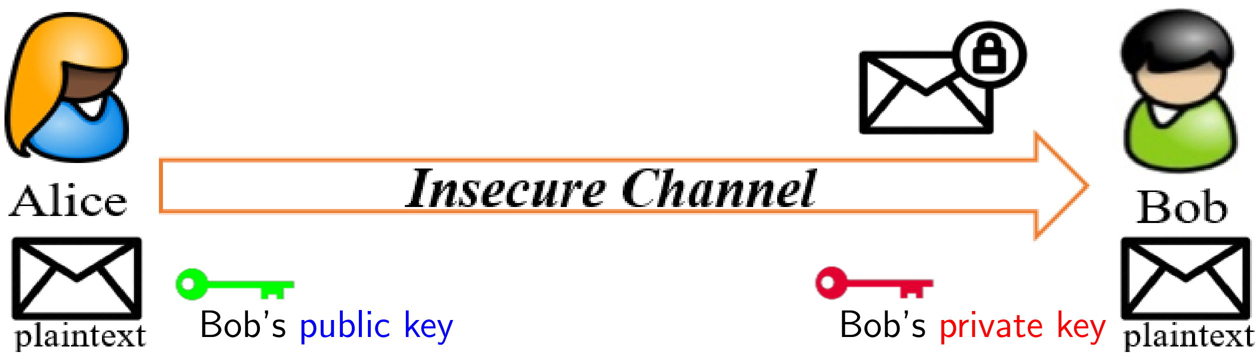

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Ronald L. Rivest


Adi Shamir


Leonard M. Adleman

R. Rivest, A. Shamir, L. Adleman, "A method for obtaining digital signatures and public-key cryptosystems", *Communications of the ACM*, vol. 21-2, pages 120-126, 1978.

RSA Public Key Cryptosystem

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Pick two large primes, p and q. Let $n = pq$, then $\phi(n) = (p-1)(q-1)$. Encryption and decryption keys e and d are selected such that

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- $ed \equiv 1 \pmod{\phi(n)}$

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Theorem (*Correctness*) : Let p and q be two odd primes, and define $n = pq$. Let e be relatively prime to $\phi(n)$ and let d be the multiplicative inverse of e modulo $\phi(n)$. For each integer x such that $0 \leq x < n$,

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RSA Public Key Cryptosystem: Example

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Encryption: $M = 28, C = M^7 \bmod 55 = 52$

Decryption: $M = C^{23} \bmod 55 = 28$

RSA Public Key Cryptosystem: Parameters

Parameters: $p \quad q \quad n \quad \phi(n) \quad e \quad d$

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$\mathcal{Q}$: Consider the RSA system, where $n = pq$ is the modulus. Let (e, d) be a key pair for the RSA. Define

$$\lambda(n) = \text{lcm}(p - 1, q - 1)$$

and compute $d' = e^{-1} \bmod \lambda(n)$. Will decryption using d' instead of d still work?

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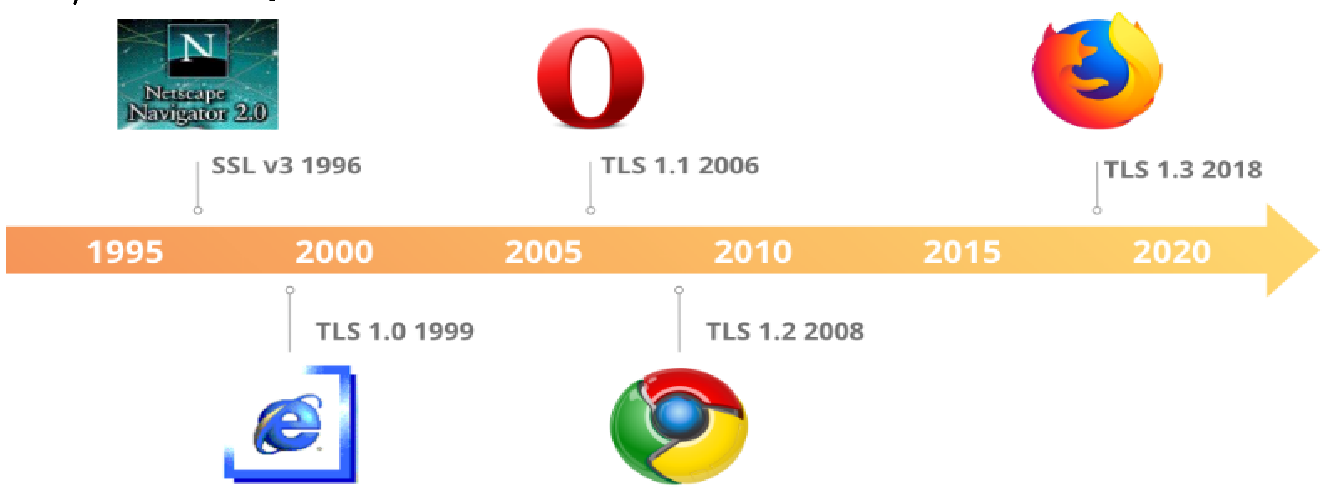

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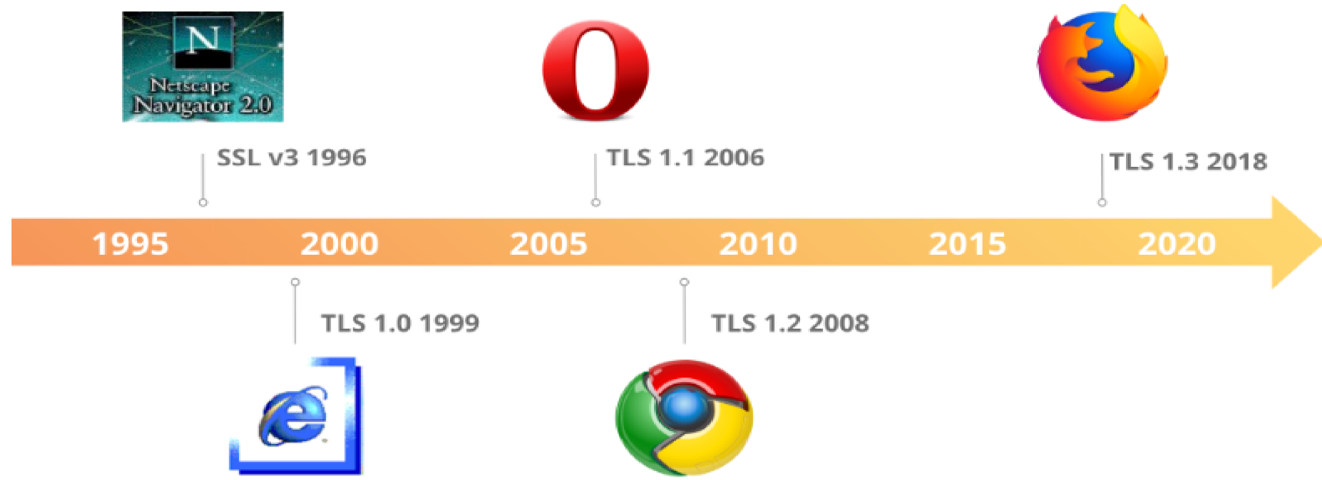

Key exchange/agreement and authentication

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DH-RSA	No	Yes	Yes	Yes	Yes	No
DHE-RSA (forward secrecy)	No	Yes	Yes	Yes	Yes	Yes
ECDH-RSA	No	No	Yes	Yes	Yes	No
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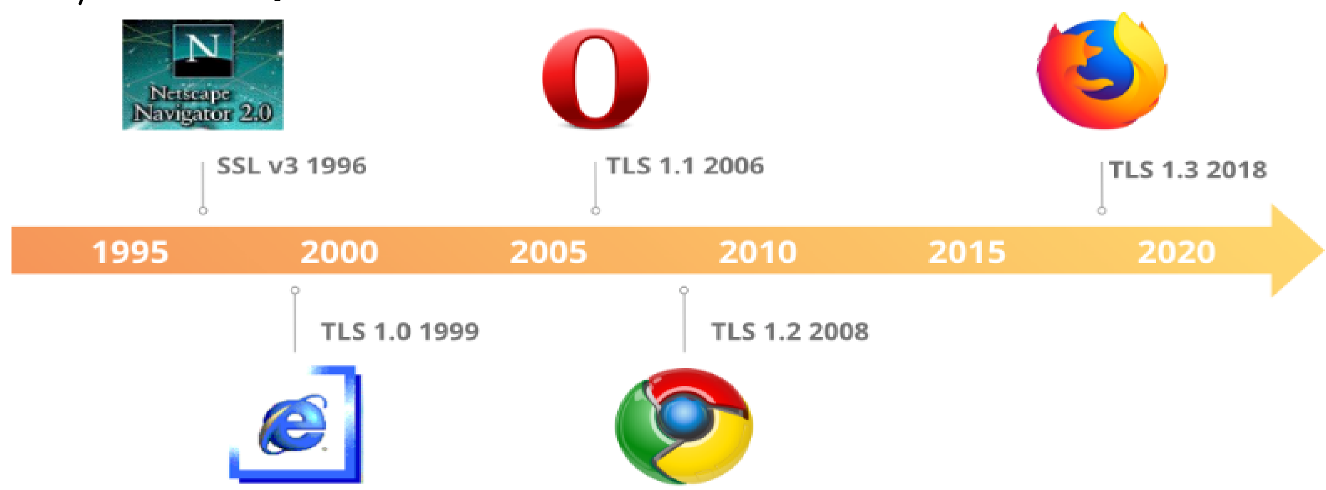

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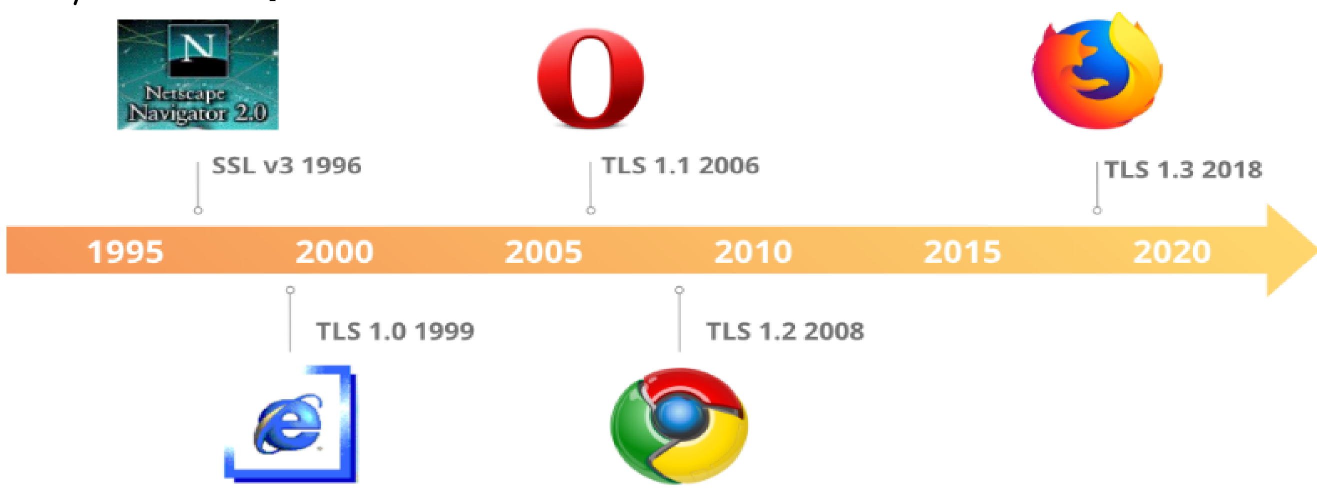

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CS 305 – Computer Networks
CS 403 – Cryptography and Network Security

Using RSA for Digital Signature

$S = M^d \bmod n$ (RSA **signature**)

$M = S^e \bmod n$ (RSA **verification**)

Why?

The Discrete Logrithm

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This is very hard!

El Gamal Encryption

- **Setup** Let p be a prime, and g be a generator of $\mathbb{Z}_p$. The private key x is an integer with $1 < x < p - 2$. Let $y = g^x \bmod p$. The public key for *El Gamal encryption* is (p, g, y).

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  El Gamal Encryption: Pick a random integer k from $\mathbb{Z}_{p-1}$,

$$a = g^k \bmod p$$
$$b = My^k \bmod p$$

  The ciphertext C consists of the pair (a, b).

  El Gamal Decryption:

$$M = b(a^x)^{-1} \bmod p$$

Using El Gamal for Digital Signature

$$a = g^k \bmod p$$
$$b = k^{-1}(M - xa) \bmod (p-1)$$

(El Gamal **signature**)

$$y^a a^b \equiv g^M \pmod{p}$$

(El Gamal **verification**)

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$\mathcal{Q}$: How to verify it?

An Example

Choose $p = 2579$, $g = 2$, and $x = 765$. Hence $y = 2^{765} \bmod 2579 = 949$.

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Encryption: Let $M = 1299$ and choose a random $k = 853$,

$$\begin{aligned} (a, b) &= (g^k \bmod p, My^k \bmod p) \\ &= (2^{853} \bmod 2579, 1299 \cdot 949^{853} \bmod 2579) \\ &= (435, 2396). \end{aligned}$$

Decryption:

$$M = b(a^x)^{-1} \bmod p = 2396 \times (435^{765})^{-1} \bmod 2579 = 1299.$$

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Attack 1: Use $M = by^{-k}$. However, k is randomly picked.

Attack 2: Use $M = b(a^x)^{-1} \bmod p$, but x is secret.

Diffie-Hellman Key Exchange Protocol

User A

Generate random
$X_A < p$
calculate
$Y_A = \alpha^{X_A} \bmod p$

Calculate
$k = (Y_B)^{X_A} \bmod p$

$$\xrightarrow{Y_A}$$

$$\xleftarrow[Y_B]{}$$

User B

Generate random
$X_B < p$
Calculate
$Y_B = \alpha^{X_B} \bmod p$

Calculate
$k = (Y_A)^{X_B} \bmod p$

Next Lecture

- induction ...