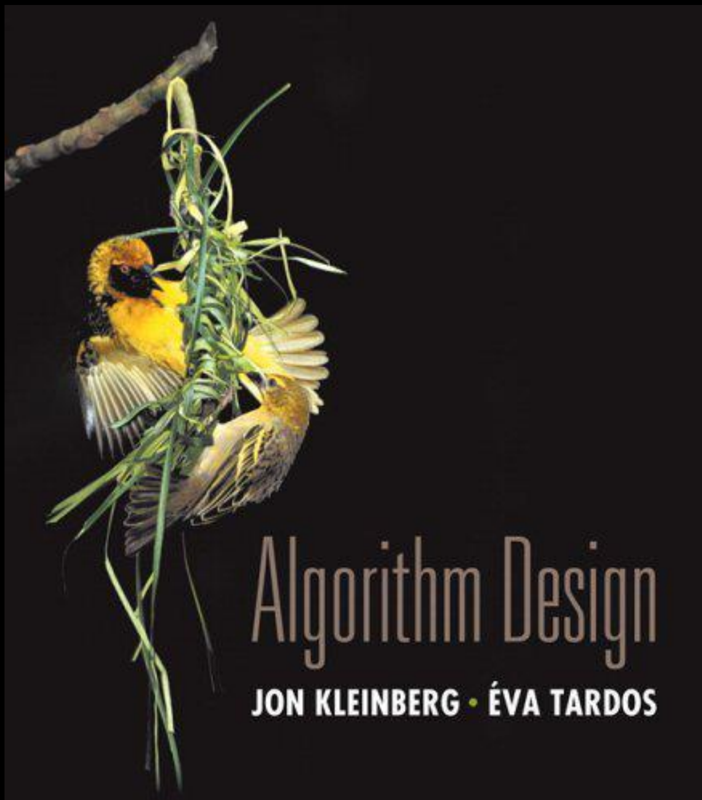


Chapter 7

Network Flow

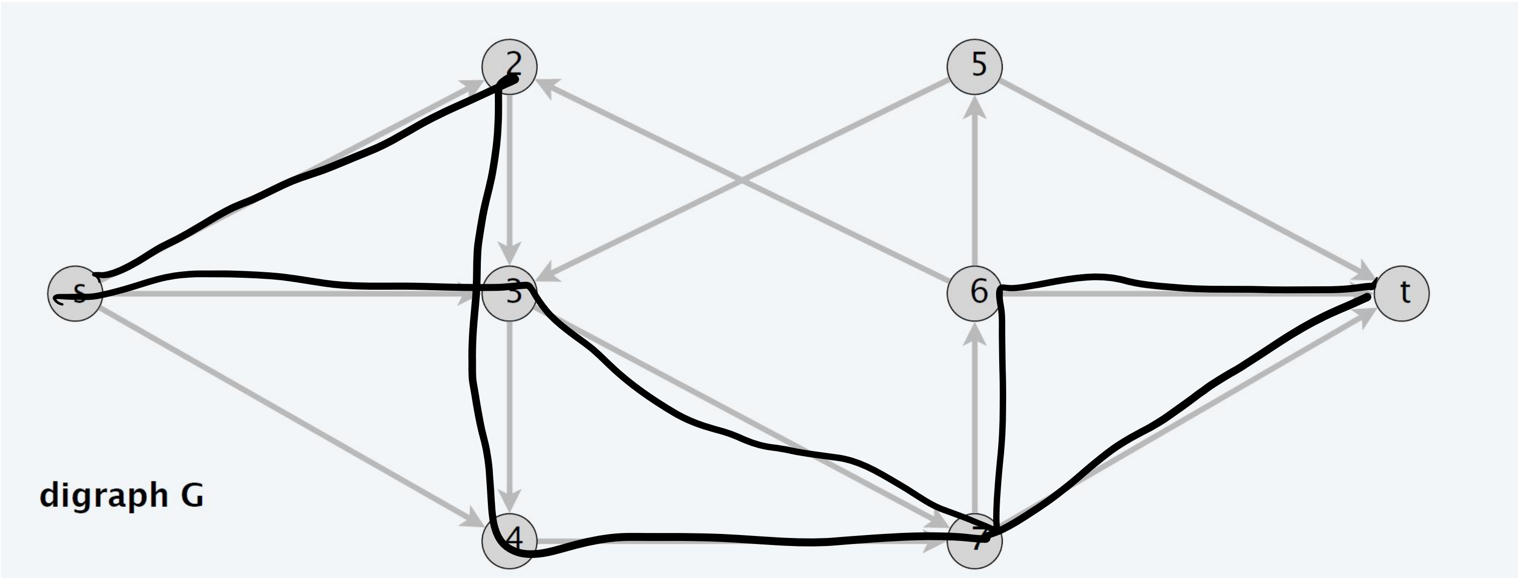
7.6 Disjoint Paths in Directed and Undirected Graphs

Edge-disjoint paths

Def. Two paths are edge-disjoint if they have no edge in common.

Edge-disjoint paths problem. Given a digraph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint $s \rightsquigarrow t$ paths.

Ex. Communication networks.



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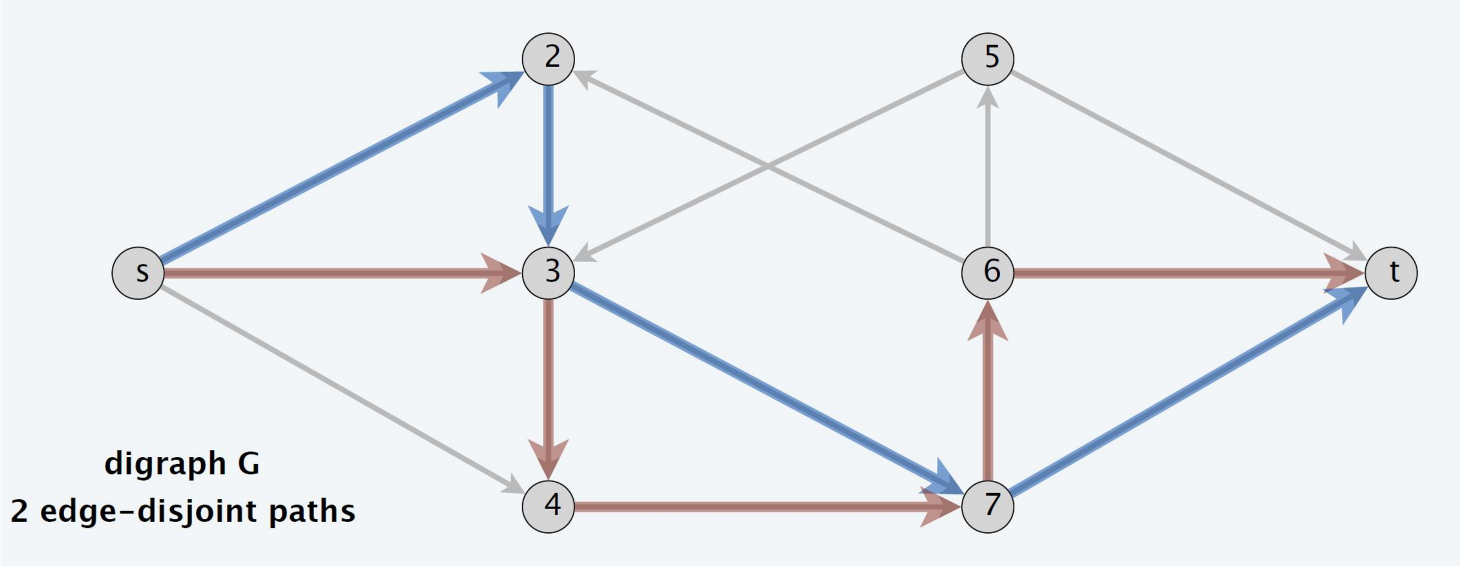


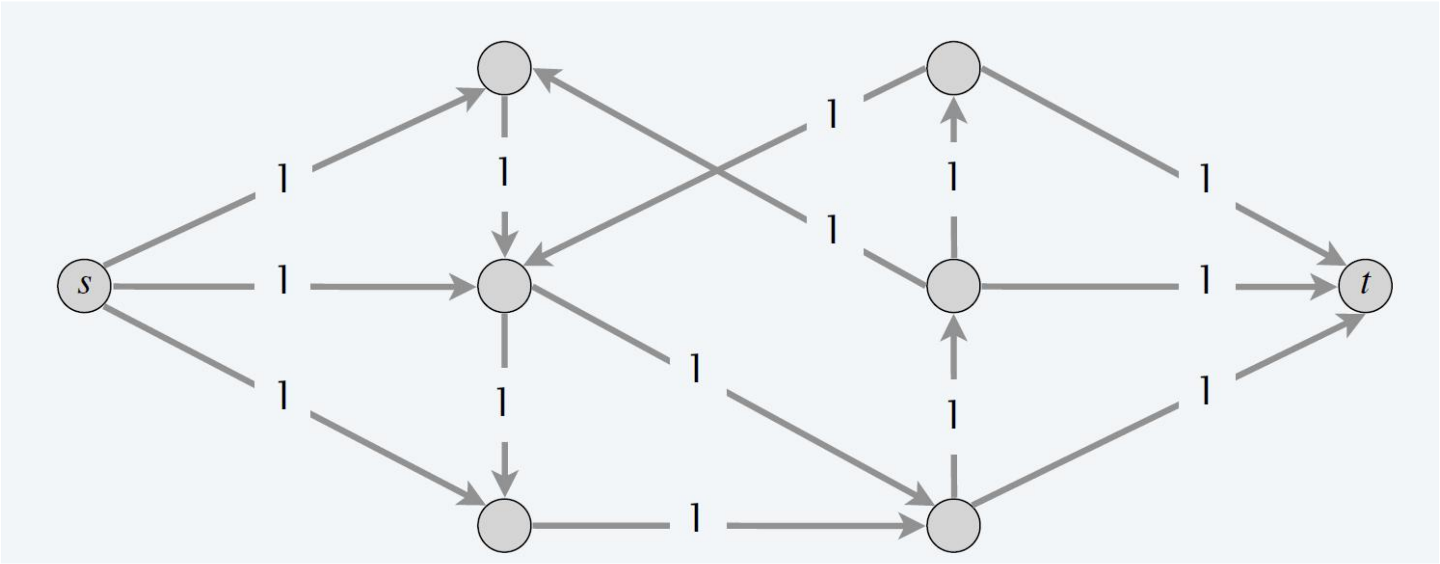
Edge-disjoint paths

Max-flow formulation. Assign unit capacity to every edge.

Theorem. Max number of edge-disjoint $s \leadsto t$ paths = value of max flow.

Pf. $\geq$

- Suppose there are k edge-disjoint $s \leadsto t$ paths $P1, \ldots, Pk$.
- Set $f(e) = 1$ if e participates in some path Pj ; else set $f(e) = 0$.
- Since paths are edge-disjoint, f is a flow of value k. ▪



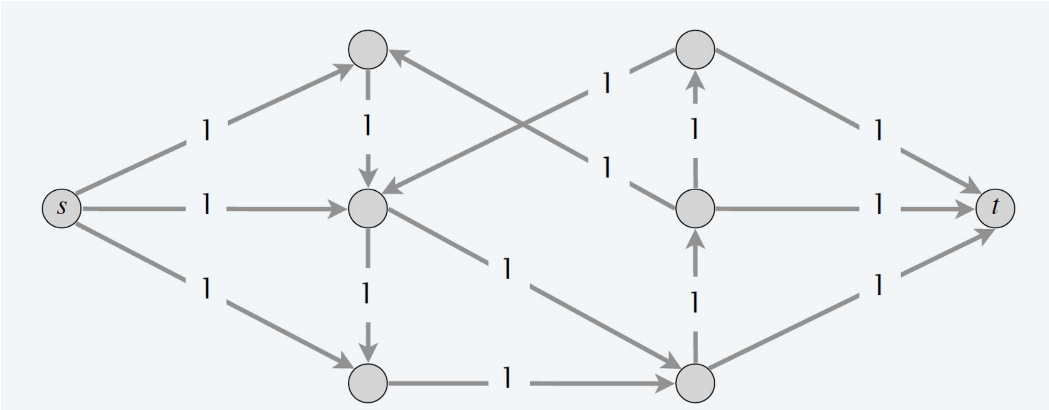
Edge-disjoint paths

Max-flow formulation. Assign unit capacity to every edge.

Theorem. Max number of edge-disjoint $s \rightsquigarrow t$ paths = value of max flow.

Pf. $\le$

- Suppose max flow value is k.
- Integrality theorem $\Rightarrow$ there exists 0–1 flow f of value k.
- Consider edge (s, u) with $f(s, u) = 1$.
  - by flow conservation, there exists an edge (u, v) with $f(u, v) = 1$
  - continue until reach t, always choosing a new edge
- Produces k (not necessarily simple) edge-disjoint paths

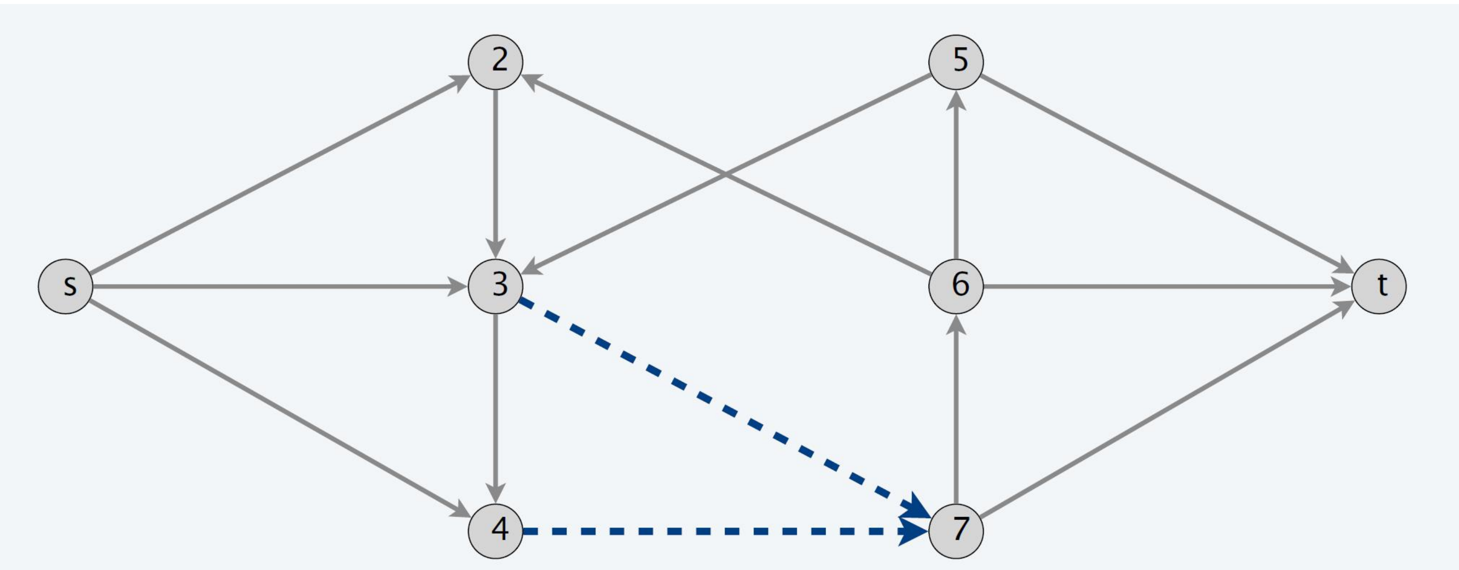


can eliminate cycles
to get simple paths
in $O(mn)$ time if desired
(flow decomposition)

Network connectivity

Def. A set of edges $F \subseteq E$ **disconnects** t from s if every $s \leadsto t$ path uses at least one edge in F.

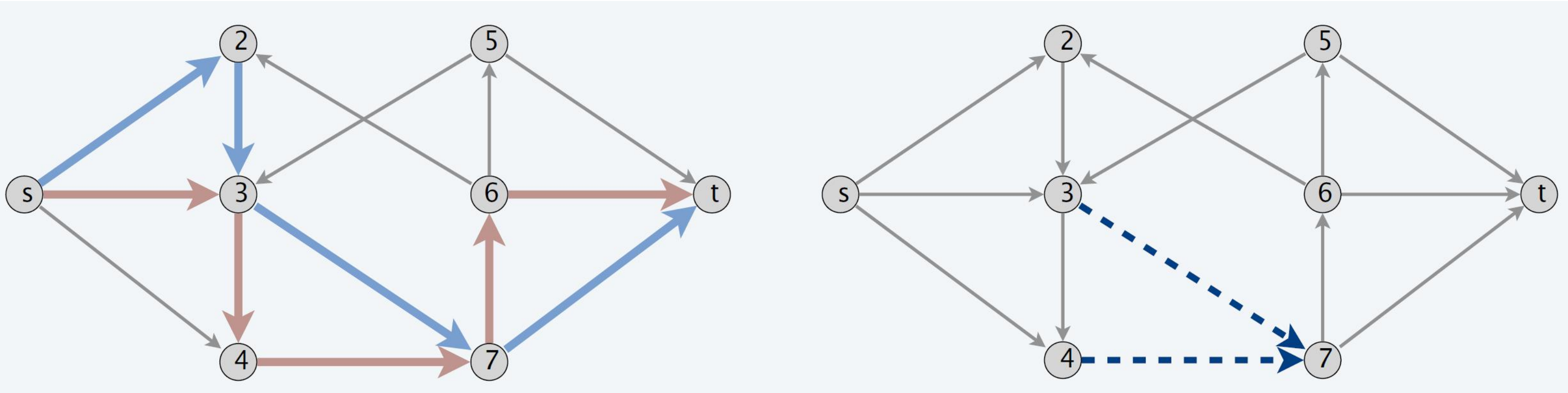
Network connectivity. Given a digraph $G = (V, E)$ and two nodes s and t, find min number of edges whose removal disconnects t from s.



Theorem. [Menger 1927] The max number of edge-disjoint $s \rightsquigarrow t$ paths equals the min number of edges whose removal disconnects t from s.

Pf. $\leq$

- Suppose the removal of $F \subseteq E$ disconnects t from s, and $|F| = k$.
- Every $s \rightsquigarrow t$ path uses at least one edge in F.
- Hence, the number of edge-disjoint paths is $\leq k$.

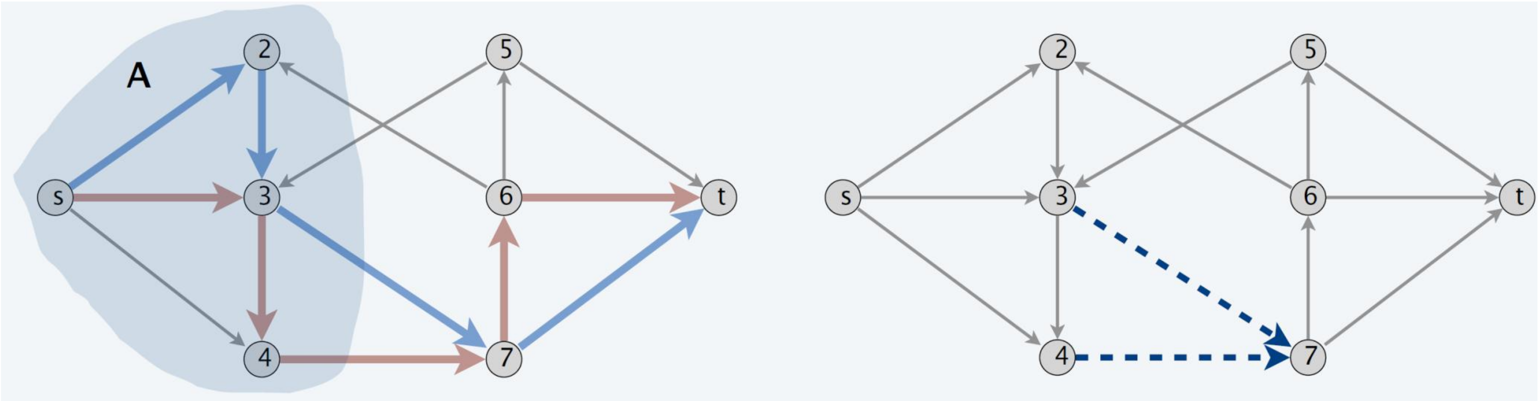


Menger's theorem

Theorem. [Menger 1927] The max number of edge-disjoint $s \rightsquigarrow t$ paths equals the min number of edges whose removal disconnects t from s.

Pf. $\geq$

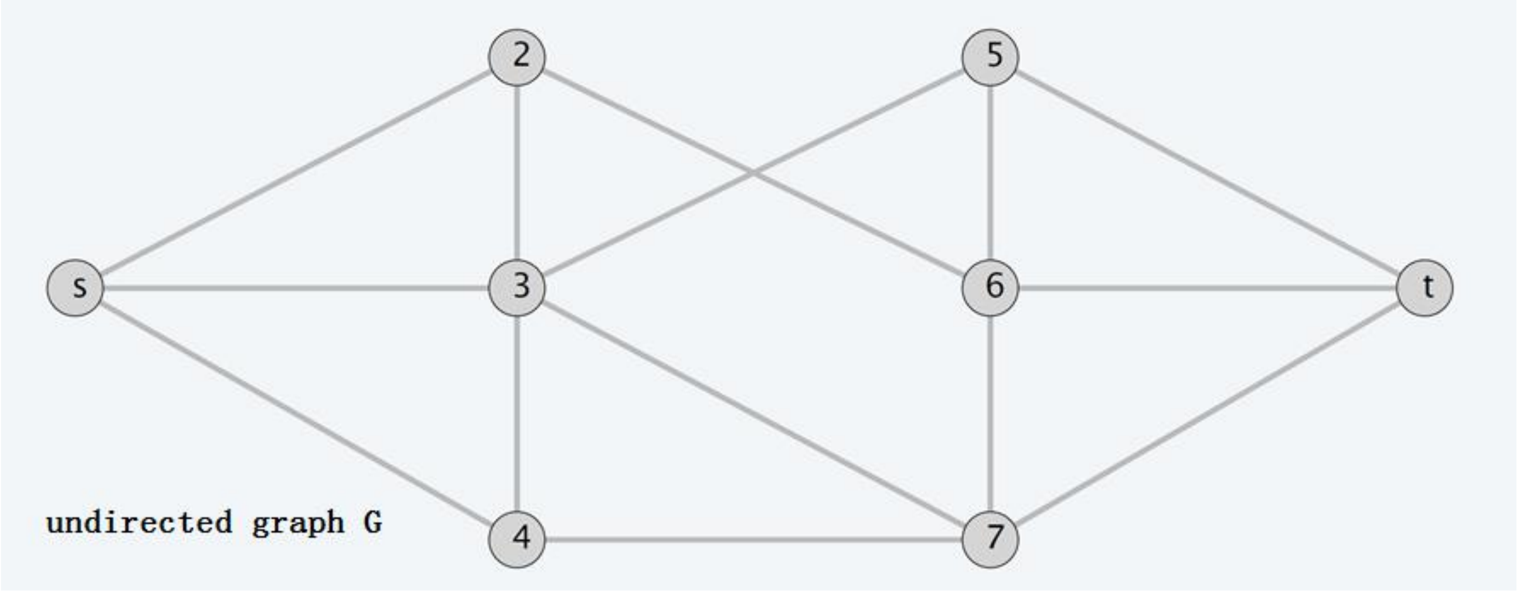
- Suppose max number of edge-disjoint paths is k.
- Then value of max flow = k.
- Max-flow min-cut theorem $\Rightarrow$ there exists a cut (A, B) of capacity k.
- Let F be set of edges going from A to B.
- $|F| = k$ and disconnects t from s.



Edge-disjoint paths in undirected graphs

Def. Two paths are edge-disjoint if they have no edge in common.

Edge-disjoint paths problem in undirected graphs. Given a graph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint s–t paths.



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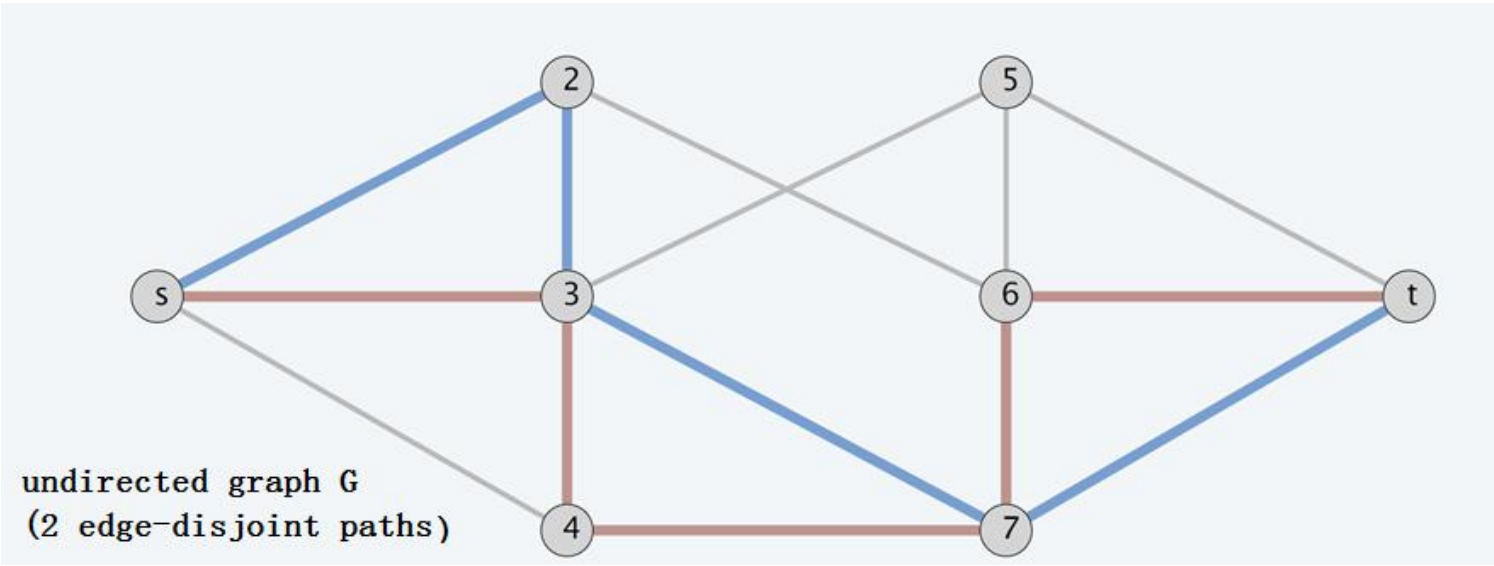

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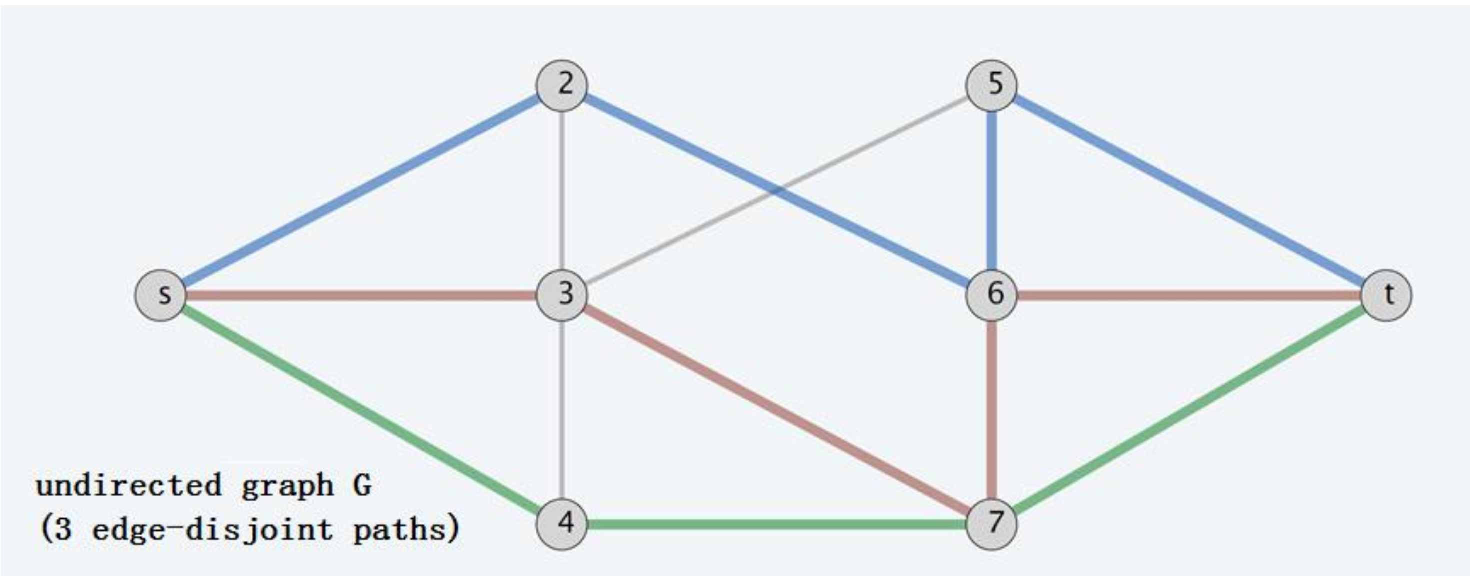
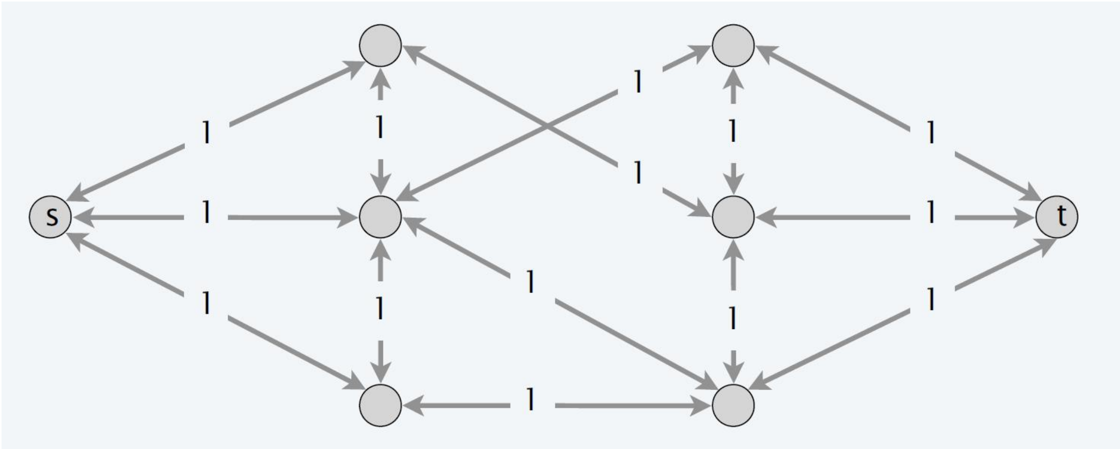


Edge-disjoint paths in undirected graphs

Max-flow formulation. Replace each edge with two antiparallel edges and assign unit capacity to every edge.

Observation. Two paths $P1$ and $P2$ may be edge-disjoint in the digraph but not edge-disjoint in the undirected graph.

if $P1$ uses edge (u, v)
and $P2$ uses its antiparallel edge (v, u)



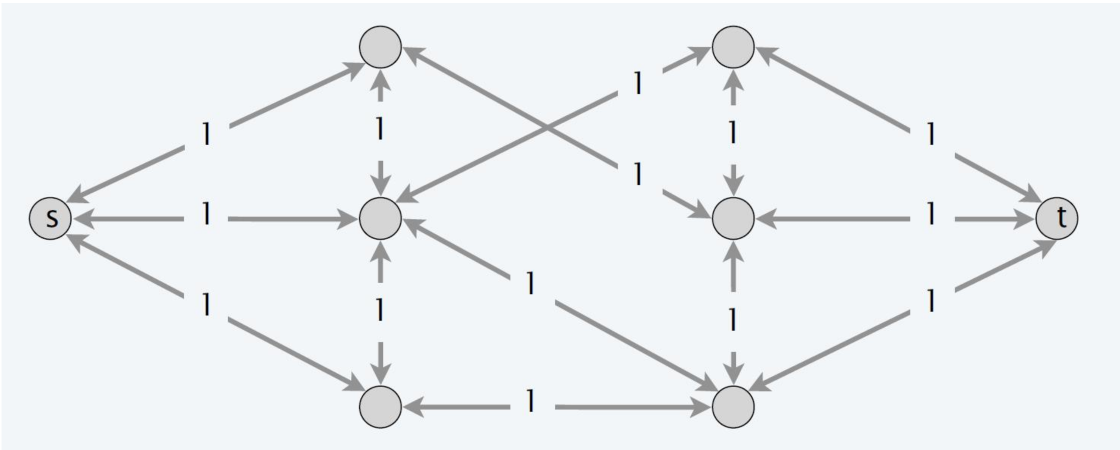
Edge-disjoint paths in undirected graphs

Max-flow formulation. Replace each edge with two antiparallel edges and assign unit capacity to every edge.

Lemma. In any flow network, there exists a maximum flow f in which for each pair of antiparallel edges e and e' : either $f(e) = 0$ or $f(e') = 0$ or both. Moreover, integrality theorem still holds.

Pf. [by induction on number of such pairs]

- Suppose $f(e) > 0$ and $f(e') > 0$ for a pair of antiparallel edges e and e'.
- Set $f(e) = f(e) - \delta$ and $f(e') = f(e') - \delta$, where $\delta = \min \{ f(e), f(e') \}$.
- f is still a flow of the same value but has one fewer such pair



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Lemma. In any flow network, there exists a maximum flow f in which for each pair of antiparallel edges e and e' : either $f(e) = 0$ or $f(e') = 0$ or both. Moreover, integrality theorem still holds.

Theorem. Max number of edge-disjoint $s \rightsquigarrow t$ paths = value of max flow.
Pf. Similar to proof in digraphs; use lemma.