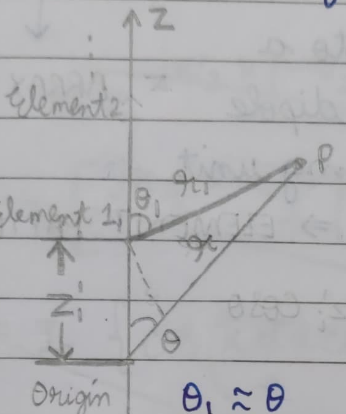
UNIT 3

5/3/24

ANTENNA ARRAYS

used for

Beam steering — Enhance gain (or directivity)



$\theta_1 \approx \theta$

$r_1 \approx r - z_1'\cos\theta$ (phase term)

$r_1 \approx r$ (amplitude term)

Distant electric field due to a z-directed Hertzian dipole placed at origin

$$E_{\theta rad} = \eta H_{\phi rad} = \frac{j\eta k I dl \sin\theta}{4\pi r} e^{-jkr}$$

Distant electric field due to a z-directed Hertzian dipole carrying a current I_1 placed at distance z_1' on z-axis is

$$E_{\theta rad_1} = \frac{j\eta k I_1 dl \sin\theta_1}{4\pi r_1} e^{-jkr_1}$$

$$E_{\theta rad_1} = \frac{j\eta k I_1 dl \sin\theta}{4\pi r} e^{-jk(r - z_1'\cos\theta)}$$

Similarly, distant electric field due to z-directed Hertzian dipole carrying a current I_2 placed at z_2' on z-axis is

$$E_{\theta rad_2} = \frac{j\eta k I_2 \sin\theta dl}{4\pi r} e^{-jk(r - z_2'\cos\theta)}$$

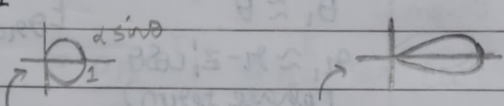
Net distant electric field due to N elements placed on z-axis, carrying currents $I_1, I_2, \ldots, I_N$, placed at distances $z_1', z_2', z_3', \ldots, z_N'$ respectively is given by

$$E_{\theta\,rad\,net} = \underbrace{\frac{j\eta k\,dl \sin\theta}{4\pi r} e^{-jkr}}_{\downarrow} \underbrace{\left\{ I_1 e^{jkz_1'\cos\theta} + I_2 e^{jkz_2'\cos\theta} + \dots + I_N e^{jkz_N'\cos\theta} \right\}}_{\downarrow}$$

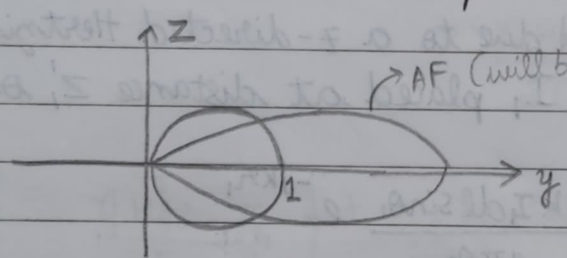
Distant Electric field due to a z-directed Hertzian dipole placed at origin, carrying unit Amperes of current ⇒ ELEMENT PATTERN

ARRAY FACTOR

ARRAY FACTOR $AF = \sum_{i=1}^{N} I_i e^{jkz_i'\cos\theta}$

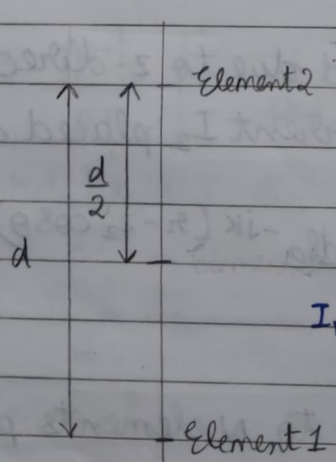


Radiation Pattern = Element pattern × AF



Simplest of arrays → has 2 elements {1 element ⇒ 1 antenna, not an array}

TWO-ELEMENT ARRAY



$I_2 = I_0 e^{j\frac{\alpha}{2}}$

$I_1 = I_0 e^{-j\alpha/2}$

relative
$\alpha \rightarrow$ phase difference

Consider the two-element array placed along z-axis
→ Element 1 placed at $z_1' = -\frac{d}{2}$ carries current

$I_1 = I_0 e^{-j\alpha/2}$

→ Element 2 placed at $z_1' = \frac{d}{2}$ carries a current

$I_2 = I_0 e^{j\alpha/2}$

$$AF = \sum_{i=1}^{N} I_i \, e^{jkz_i' \cos\theta}$$

$$= I_1 e^{jkz_1' \cos\theta} + I_2 e^{jkz_2' \cos\theta}$$

$$\cancel{= I_0 e^{j\alpha/2 \, k(-d/2)\cos\theta} + I_0 e^{-j\alpha/2 \, k\left(\frac{d}{2}\right)\cos\theta}}$$

$$\cancel{= I_0 \left[e^{jk\frac{\alpha}{4} d\cos\theta} + \left[e^{-jk\frac{\alpha}{4} d\cos\theta} \right) \right]}$$

$$\cancel{= I_0 \left\{ \cos\left(\frac{k\alpha d\cos\theta}{4}\right) + j\sin\left(\frac{k\alpha d\cos\theta}{4}\right) \right.}$$

$$\cancel{+ \cos(k\alpha}$$

$$AF = I_0 e^{-j\frac{\alpha}{2}} e^{jk\left(\frac{d}{2}\right)\cos\theta} + I_0 e^{+j\frac{\alpha}{2}} e^{jk\left(\frac{d}{2}\right)\cos\theta}$$

$$= I_0 \left[e^{-j\left(\frac{\alpha}{2} + k\frac{d}{2}\cos\theta\right)} + e^{j\left(\frac{\alpha}{2} + k\frac{d}{2}\cos\theta\right)} \right]$$

$$= I_0 \left[\cos\left(\frac{\alpha}{2} + k\frac{d}{2}\cos\theta\right) \mp j\sin\left(\frac{\alpha}{2} + k\frac{d}{2}\cos\theta\right) \right.$$

$$\left. + \cos\left(\frac{\alpha}{2} + k\frac{d}{2}\cos\theta\right) + j\sin\left(\frac{\alpha}{2} + k\frac{d}{2}\cos\theta\right) \right]$$

$$\boxed{AF = 2I_0 \cos\left(\frac{\alpha}{2} + \frac{kd\cos\theta}{2}\right)}$$

AF for 2 element array in which elements have a phase difference α

CASE 1 $\alpha = 0$ (phase difference between currents is zero, i.e. same phase)

$$AF = 2I_0 \cos\left(\frac{kd\cos\theta}{2}\right)$$

Maxima of $|AF|$

$$\frac{kd\cos\theta_m}{2} = \pm m\pi \qquad \text{maxima}$$

$m = 0, 1, 2, \ldots$

$$\cos\theta_m = \pm\frac{2m\pi}{kd} = \pm\frac{2m\pi}{\frac{2\pi}{\lambda}d}$$

$$\cos\theta_m = \pm\left(\frac{m\lambda}{d}\right) \quad \text{+ve value}$$

$\cos\theta_m = -\frac{m\lambda}{d} \geq -1$ $\cos\theta_m = \frac{m\lambda}{d} \leq 1$

$\Rightarrow |\cos\theta_m| \leq 1$

$\Rightarrow \boxed{\frac{m\lambda}{d} \leq 1}$

Not writing $\left|\frac{m\lambda}{d}\right|$

$\because \frac{m\lambda}{d}$ always ≥ 0

$m = 0, 1, \dots$

λ always +ve

d distance cannot be negative

★ For $d = \frac{\lambda}{2}$

$\frac{m\lambda}{d} = \frac{m\lambda}{\lambda/2} = 2m \leq 1$

$m \leq \frac{1}{2}$ m is only integer values

$\Rightarrow m = 0$ satisfies the given inequality

$\cos\theta_m = 0$

$\theta_m = \cos^{-1}(0)$

$\boxed{\theta_m = \frac{\pi}{2}}$ (plotting in 0 to π range)

$\therefore$ Maxima occurs at $\theta_m = \frac{\pi}{2}$

★ For $d = \frac{3\lambda}{4}$

$\frac{m\lambda}{d} = \frac{m\lambda}{3\lambda/4} = \frac{4m}{3} \leq 1$

$m \leq \frac{3}{4} \Rightarrow m = 0$

$\Rightarrow \boxed{\theta_m = \frac{\pi}{2}}$

★ For $d = \lambda$

$\frac{m\lambda}{d} = m \leq 1 \Rightarrow m = 0, 1$

$\cos\theta_m = \pm m$

$\cos\theta_m = 0$

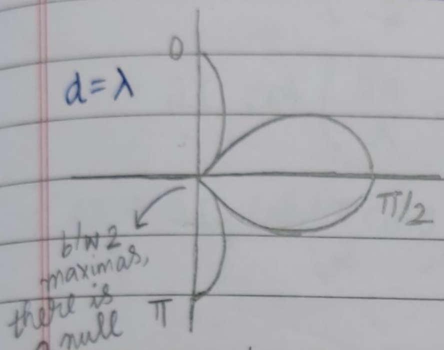
$\theta_m = \frac{\pi}{2}$

$\cos\theta_m = \pm 1$

$\theta_m = 0, \pi$

$$\boxed{\theta_m = 0, \frac{\pi}{2}, \pi}$$ 3 maximas

d = λ



Nulls of two-element array ($\alpha = 0$)

$$AF = 2I_0 \cos\left(\frac{kd\cos\theta}{2}\right)$$

$$\frac{kd\cos\theta_n}{2} = \pm \text{ odd multiples of } \frac{\pi}{2}$$

$$= \pm(2n-1)\frac{\pi}{2}$$

$$\cos\theta_n = \pm(2n-1)\frac{\pi}{kd} = \pm(2n-1)\frac{\pi}{\frac{2\pi}{\lambda}d} = \pm(2n-1)\frac{\lambda}{2d}$$

$$\boxed{\cos\theta_n = \pm(2n-1)\frac{\lambda}{2d}}$$

$$\cos\theta_n = \left(\frac{2n-1}{2}\right)\frac{\lambda}{d} \qquad \cos\theta_n = -\left(\frac{2n-1}{2}\right)\frac{\lambda}{d}$$

$n = 1, 2, 3, \ldots$

$$-1 \le \cos\theta_n \le 1$$

$$-1 \le \pm\left(\frac{2n-1}{2}\right)\frac{\lambda}{d} \le 1$$

$$-1 \le \left(n-\frac{1}{2}\right)\frac{\lambda}{d} \qquad \left(n-\frac{1}{2}\right)\frac{\lambda}{d} \le 1 \Rightarrow n - \frac{1}{2} \le \frac{d}{\lambda} \Rightarrow \boxed{n \le \frac{d}{\lambda} + \frac{1}{2}}$$

$$-1 \le -\left(n-\frac{1}{2}\right)\frac{\lambda}{d} \qquad -\left(n-\frac{1}{2}\right)\frac{\lambda}{d} \le 1$$

when $d < \frac{\lambda}{2}$

$n < 1$, hence there is no null in AF ($\because n = 1, 2, \ldots$)

$$|\cos\theta_n| = \left(\frac{2n-1}{2}\right)\frac{\lambda}{d} \leq 1$$

$$\left(n - \frac{1}{2}\right)\frac{\lambda}{d} \leq 1$$

$$n - \frac{1}{2} \leq \frac{d}{\lambda} \Rightarrow \boxed{n \leq \frac{d}{\lambda} + 1}$$

at $d = \frac{\lambda}{2}$

$n \leq 1$ There is one null in AF at $n = 1$

One maxima in AF at $m = 0$

$$\cos\theta_n = \pm\left(\frac{2(1)-1}{2}\right)\frac{\lambda}{d} = \pm\frac{1}{2} \times \frac{\lambda}{\lambda/2} = \pm 1$$

$$\theta_n = \cos^{-1}(1)$$

$$\theta_n = 0, \pi$$

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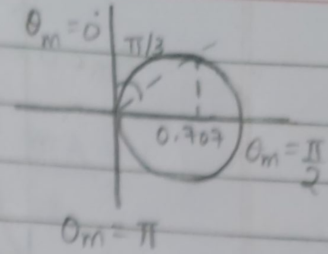
MAXIMA (for given d)	NULLS (for given d)
Using $m \leq \frac{d}{\lambda}$ we find all possible values of m ($m = 0, 1, 2, \ldots$)	Using $n \leq \frac{d}{\lambda} + \frac{1}{2}$, we find all possible values of n ($n = 1, 2, 3, \ldots$)
Using $\cos\theta_m = \pm\frac{m\lambda}{d}$, we find all θ_m for all m values	Using $\cos\theta_n = \pm\frac{(2n-1)\lambda}{2d}$ we find all θ_n for all n values

★ For $d = \frac{\lambda}{2}$

$m = 0$ $\qquad$ $n = 1$

$\cos\theta_m = 0$ $\qquad$ $\cos\theta_n = \pm 1$

$\theta_m = \frac{\pi}{2}$ $\qquad$ $\theta_n = 0, \pi$



$$AF = 2I_0 \cos\left(\frac{Kd}{2}\cos\theta\right)$$

$$= 2I_0 \cos\left(\frac{2\pi}{\lambda} \times \frac{\lambda}{2} \times \frac{1}{2}\cos\theta\right)$$

$$AF = 2I_0 \cos\left(\frac{\pi}{2}\cos\theta\right)$$

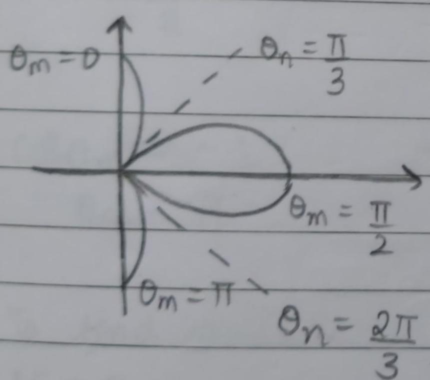
★ For $d = \lambda$

$\frac{m\lambda}{d} = \frac{m\lambda}{\lambda} = m \leq 1$ $\qquad$ $n \leq \frac{d}{\lambda} + \frac{1}{2}$

$\cos\theta_m = \pm m$ $\qquad$ $n \leq 1 + \frac{1}{2}$

$\cos\theta_m = 0$ $\qquad$ $\cos\theta_m = \pm 1$ $\qquad$ $n \leq 1.5$

$\theta_m = \frac{\pi}{2}$ $\qquad$ $\theta_m = 0, \pi$ $\qquad$ $n = 1$



$$\cos\theta_n = \pm \frac{(2(1)-1)}{2}\frac{\lambda}{\lambda}$$

$$\cos\theta_n = \pm\frac{1}{2}$$

$$\theta_n = \frac{\pi}{3}, \frac{2\pi}{3}$$

★ For $d = 2\lambda$

$\frac{m\lambda}{d} = \frac{m\lambda}{2\lambda} = \frac{m}{2} \leq 1$ $\qquad$ $n \leq \frac{d}{2\lambda} + \frac{1}{2} \Rightarrow n \leq 2 + \frac{1}{2}$

$n \leq 2.5$

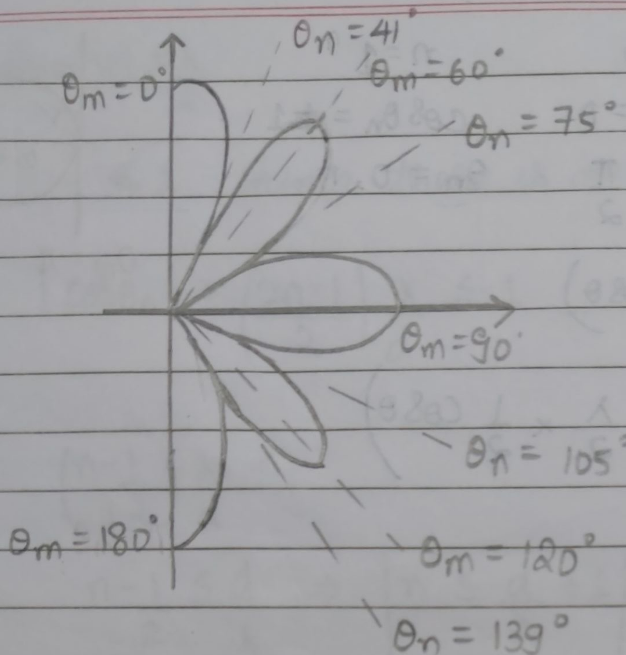
$m \leq 2$ $\qquad$ $n = 1, 2$

$\cos\theta_m = \pm\frac{m\lambda}{2\lambda} = \pm\frac{m}{2}$ $\qquad$ $\cos\theta_n = \frac{\pm(2n-1)}{4}$

$\cos\theta_m = 0$ $\qquad$ $\cos\theta_m = \pm\frac{1}{2}$ $\qquad$ $\cos\theta_m = \pm 1$ $\qquad$ $\cos\theta_n = \pm\frac{1}{4}$ $\qquad$ $\cos\theta_n = \pm\frac{3}{4}$

$\theta_m = \frac{\pi}{2}$ $\qquad$ $\theta_m = \frac{\pi}{3}, \frac{2\pi}{3}$ $\qquad$ $\theta_m = 0, \pi$ $\qquad$ $\theta_n = 75°, 105°$ $\qquad$ $\theta_n = 41°, 139°$

$\theta_m = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$



★ Maximum at $\frac{\pi}{2}$ → <u>BROADSIDE ARRAY</u>

<u>CASE 2</u> $\alpha \neq 0$

$$AF = 2I_0 \cos\left(\frac{\alpha}{2} + \frac{kd\cos\theta}{2}\right)$$

Let $\alpha = kd$ (α becomes notorious :p)

$$AF = 2I_0 \cos\left(\frac{kd}{2} + \frac{kd\cos\theta}{2}\right)$$

$$AF = 2I_0 \cos\left(\frac{kd}{2}(1+\cos\theta)\right)$$

<u>To find maxima</u>

$$\left|\cos\left[\frac{kd}{2}(1+\cos\theta_m)\right]\right| = 1$$

$$\frac{kd}{2}(1+\cos\theta_m) = 0, \pi, \dots = \pm m\pi \qquad m = 0, 1, 2, 3, \dots$$

$$\frac{\cancel{2}\cancel{\pi}}{\lambda}\frac{d}{\cancel{2}}(1+\cos\theta_m) = \pm m\cancel{\pi}$$

$$1+\cos\theta_m = \pm\frac{m\lambda}{d}$$

$$\cos\theta_m = \pm\frac{m\lambda}{d} - 1$$

$$\cos\theta_m = -\left(\frac{m\lambda}{d} + 1\right) \quad \text{OR} \quad \cos\theta_m = \frac{m\lambda}{d} - 1$$

Invalid
Since $|\cos\theta| \le 1$

$$\boxed{\cos\theta_m = \frac{m\lambda}{d} - 1}$$

$$|\cos\theta_m| \le 1$$

$$\frac{m\lambda}{d} - 1 \le 1$$

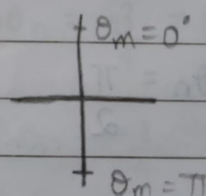
$$\frac{m\lambda}{d} \le 2 \Rightarrow \boxed{m \le \frac{2d}{\lambda}}$$

* For $d = \frac{\lambda}{2}$ $\quad m \le \frac{2d}{\lambda} \Rightarrow m \le \frac{2}{\lambda}\frac{\lambda}{2} \Rightarrow m \le 1$

$$m = 0, 1$$

$\cos\theta_m = -1$ $\quad\quad$ $\cos\theta_m = 1$

$\theta_m = \pi$ $\quad\quad$ $\theta_m = 0$



<u>To find nulls</u>

AF = 0

$$\cancel{\cos}\left|\cos\left[\frac{kd}{2}(1+\cos\theta_n)\right]\right| = 0$$

$$\frac{2\pi d}{2\lambda}(1+\cos\theta_n) = \cancel{\frac{\pi}{2}} \pm (2n-1)\frac{\pi}{2} \qquad n = 1, 2, 3, \ldots$$

$$1+\cos\theta_n = \pm\frac{(2n-1)\lambda}{2d} \Rightarrow \cos\theta_n = -1 \pm \left(\frac{2n-1}{2}\right)\frac{\lambda}{d}$$

$$\cos\theta_n = -\left[1 + \left(\frac{2n-1}{2}\right)\frac{\lambda}{d}\right] \quad \text{OR} \quad \cos\theta_n = -1 + \left(\frac{2n-1}{2}\right)\frac{\lambda}{d}$$

Invalid $\because$ $|\cos\theta_n|$ cannot exceed 1

$$\boxed{\cos\theta_n = -1 + \left(\frac{2n-1}{2}\right)\frac{\lambda}{d}}$$

$$-1 + \left(\frac{2n-1}{2}\right)\frac{\lambda}{d} \le 1$$

$$\left(\frac{2n-1}{2}\right)\frac{\lambda}{d} \le 2$$

$$2n-1 \le \frac{4d}{\lambda}$$

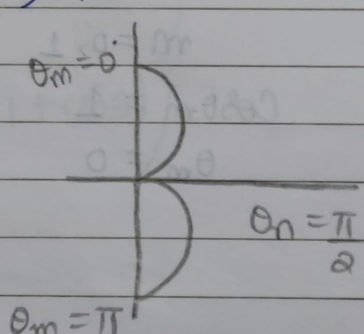
$$2n \le \frac{4d}{\lambda} + 1 \Rightarrow \boxed{n \le \frac{2d}{\lambda} + \frac{1}{2}}$$

★ For $d = \frac{\lambda}{2}$ $\quad n \le \frac{2}{\lambda}\frac{\lambda}{2} + \frac{1}{2} \Rightarrow n \le 1.5$

$n = 1$

$$\cos\theta_n = -1 + \left(\frac{2(1)-1}{2}\right)\frac{2\lambda}{\lambda} = -1 + 1 = 0$$

$\theta_n = \frac{\pi}{2}$



★ For $d = \lambda$

$m \le \frac{2d}{\lambda} \Rightarrow m \le 2$

$m = 0, 1, 2$

$\cos\theta_m = m - 1$

$\cos\theta_m = -1 \qquad \cos\theta_m = 1$

$\theta_m = \pi \qquad \theta_m = 0$

$\cos\theta_m = 0$

$\theta_m = \frac{\pi}{2}$

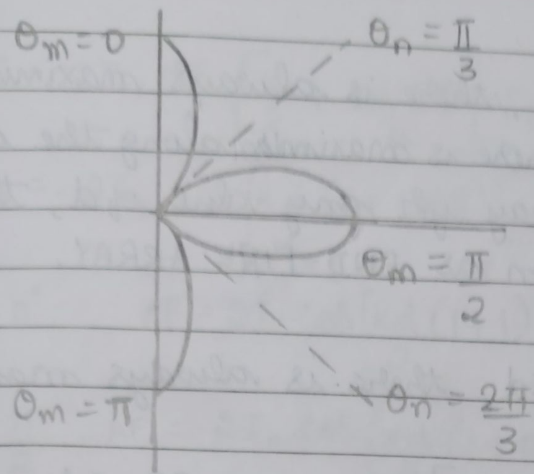
$n \le \frac{2\lambda}{\lambda} + \frac{1}{2} \Rightarrow n \le 2.5$

$n = 1, 2$

$\cos\theta_n = \left(\frac{2n-1}{2}\right) - 1$

$\cos\theta_n = -1/2 \qquad \cos\theta_n = 1/2$

$\theta_n = \frac{2\pi}{3} \qquad \theta_n = \frac{\pi}{3}$



★ For $d = 2\lambda$

$m \le \frac{2(2\lambda)}{\lambda} \Rightarrow m \le 4$

$m = 0, 1, 2, 3, 4$

$\cos\theta_m = \frac{m}{2} - 1$

$m=0 \quad \cos\theta_m = -1 \Rightarrow \theta_m = \pi$

$m=1 \quad \cos\theta_m = -1/2 \Rightarrow \theta_m = \frac{2\pi}{3}$

$m=3 \quad \cos\theta_m = 1/2 \Rightarrow \theta_m = \frac{\pi}{3}$

$m=2 \quad \cos\theta_m = 0 \Rightarrow \theta_m = \pi/2$

$m=4 \quad \cancel{\cos\theta_m} = \cos\theta_m = 1 \Rightarrow \theta_m = 0$

$\theta_m = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$

$n \le \frac{2(2\lambda)}{\lambda} + \frac{1}{2} = n \le 4.5$

$n = 1, 2, 3, 4$

$\cos\theta_n = \left(\frac{2n-1}{4}\right) - 1$

~~$n=0 \quad \cos\theta_n = \frac{-5}{4} \Rightarrow \theta_n =$~~

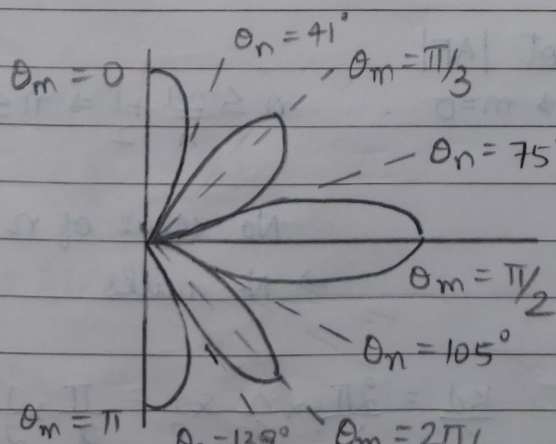
$n=1 \quad \cos\theta_n = \frac{-3}{4} \Rightarrow \theta_n = \cancel{135°}\ 139°$

$n=2 \quad \cos\theta_n = -\frac{1}{4} \Rightarrow \theta_n = 105°$

$n=3 \quad \cos\theta_n = 1/4 \Rightarrow \theta_n = 75°$

$n=4 \quad \cos\theta_n = \frac{3}{4} \Rightarrow \theta_n = 41°$

$\theta_n = 41°, 75°, 105°, 139°$



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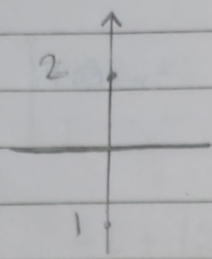


$\alpha = kd$

as $\theta = \pi$, there is always maximum

Where there is maximum along the axis of the array for any value of d, the array is known as END-FIRE ARRAY.

* Prove that when $\alpha = -kd$, there is always maximum at $\theta = 0$

$\Rightarrow$ $AF = 2I_0 \cos\left[\frac{\alpha}{2} + \frac{kd\cos\theta}{2}\right]$ $\qquad 2I_0 = 1$

$= \cos\left[\frac{-kd}{2} + \frac{kd\cos\theta}{2}\right]$

$= \cos\left[\frac{kd}{2}(\cos\theta - 1)\right]$

$\cos\left[\frac{kd(\cos\theta - 1)}{2}\right] = 1$ Maximum

$\Rightarrow \frac{kd}{2}(\cos\theta - 1) = 0$

$\Rightarrow \cos\theta - 1 = 0$

$\cos\theta = 1$

$\theta = 0^\circ$

* For a 2-element array with $\alpha = kd$ and $d = \frac{\lambda}{6}$, determine max and nulls and plot $|AF|$

$\Rightarrow$ $m \le \frac{2d}{\lambda} \Rightarrow m \le 1/3 \Rightarrow m = 0$

$\cos\theta_m = \frac{m\lambda}{d} - 1$

$\cos\theta_m = 0 - 1$

$\cos\theta_m = -1$

$\theta_m = \pi$

$n \le \frac{2d}{\lambda} + \frac{1}{2} \Rightarrow n \le 0.833$

No value of n

$\Rightarrow$ No nulls

$\frac{kd}{2} = \frac{2\pi}{\lambda} \times \frac{\lambda}{6} \times \frac{1}{2} = \frac{\pi}{3} \times \frac{1}{2}$

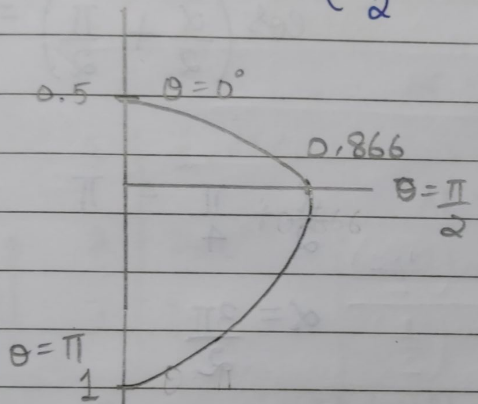
$$AF = 2I_0 \cos\left[\frac{\alpha}{2} + \frac{kd\cos\theta}{2}\right] = 2I_0 \cos\left[\frac{kd}{2} + \frac{kd\cos\theta}{2}\right]$$

$$AF = 2I_0 \cos\left[\frac{kd(1+\cos\theta)}{2}\right]$$

$\theta = 0°$ $\quad AF = \cancel{2I_0}\cos\left[\frac{kd(1+1)}{2}\right] = \cos\frac{\pi}{3} = \frac{1}{2} = 0.5$ $\quad (2I_0 = 1)$

$\theta = \frac{\pi}{2}$ $\quad AF = 2I_0 \cos\left(\frac{kd}{2}(1+0)\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} = 0.866$

$\theta = \pi$ $\quad AF = 2I_0 \cos\left(\frac{kd}{2}(1-1)\right) = \cos 0 = 1$



★ $d = \frac{\lambda}{4}$ $\quad \alpha = kd = \frac{2\pi}{\lambda} \times \frac{\lambda}{4} = \frac{\pi}{2}$

$m \leq \frac{2\lambda}{\lambda 4} \Rightarrow m \leq \frac{1}{2} \Rightarrow m = 0$

$\boxed{m = 0}$

$\cos\theta_m = \frac{m\lambda}{d} - 1$

$\cos\theta_m = 0 - 1$

$\boxed{\theta_m = \pi}$

$n \leq \frac{2}{\lambda}\frac{\lambda}{4} + \frac{1}{2} \Rightarrow n \leq 1$

$\boxed{n = 1}$

$\cos\theta_n = \left(\frac{2n-1}{2}\right)\frac{4\lambda}{\lambda} - 1$

$= \frac{4}{2} - 1 = 2 - 1$

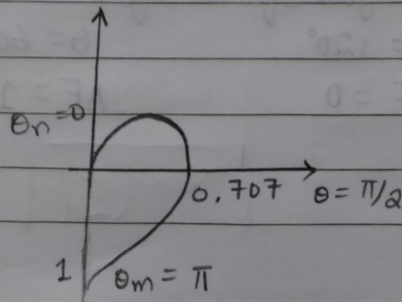
$\cos\theta_n = 1 \Rightarrow \boxed{\theta_n = 0}$

$$AF = 2I_0 \cos\left(\frac{\pi}{4}(1+\cos\theta)\right)$$

$\theta = 0°$ $\quad AF = \cos\frac{\pi}{2} = 0$

$\theta = \frac{\pi}{2}$ $\quad AF = \cos\pi/4 = 0.707$

$\theta = \pi$ $\quad AF = \cos 0 = 1$



Example Determine the phase difference α to obtain a maximum at $\theta = \frac{\pi}{3}$ for a 2-element array with $d = \frac{\lambda}{2}$

$$AF = 2I_0 \cos\left(\frac{\alpha}{2} + \frac{kd\cos\theta}{2}\right) \qquad kd = \frac{2\pi}{\lambda} \times \frac{\lambda}{2} = \pi$$

$$|AF|_{normalised} = \left|\cos\left(\frac{\alpha}{2} + \frac{kd\cos\theta}{2}\right)\right| \qquad \theta_m = \frac{\pi}{3} \quad \cos\theta_m = \frac{1}{2}$$

at maximum $|AF|_{nor} = 1$ $\quad \theta = \theta_m = \frac{\pi}{3}$

$$\cos\left(\frac{\alpha}{2} + \frac{\pi}{2}\cos\theta_m\right) = 1 \qquad \cos\left(\frac{\alpha}{2} + \frac{\pi}{2}\right) = -1$$

$$\cos\left(\frac{\alpha}{2} + \frac{\pi}{4}\right) = 1 \qquad \frac{\alpha}{2} + \frac{\pi}{4} = \pi$$

$$\frac{\alpha}{2} + \frac{\pi}{4} = 0 \qquad \alpha = \frac{3\pi}{2}$$

$$\alpha = -\frac{\pi}{2}$$

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When $\alpha = -\frac{\pi}{2}$

$$AF = 2I_0 \cos\left[-\frac{\pi}{4} + \frac{\pi}{2}\cos\theta\right]$$

$\theta = 0 \qquad AF = \frac{1}{\sqrt{2}} = 0.707$

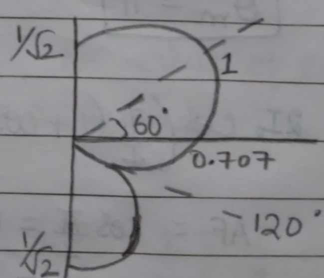
$\theta = \frac{\pi}{2} \qquad AF = \frac{1}{\sqrt{2}} = 0.707$

$\theta = \pi \qquad AF = 0.707$

When you find for

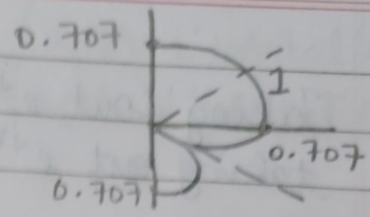
$\theta = 120^\circ \qquad \theta = 60^\circ$

$AF = 0 \qquad AF = 1$



When $\alpha = \frac{3\pi}{2}$

$$AF = \cancel{2I_0}^{=1} \cos\left(\frac{3\pi}{4} + \frac{\pi}{2}\cos\theta\right)$$



When $\theta = 0^\circ$, $AF = -1/\sqrt{2}$

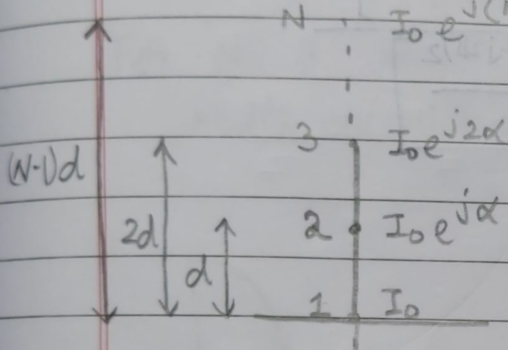
$\theta = \pi/2$, $AF = 1/\sqrt{2}$

$\theta = 2\pi/3$, $AF = 0$

$\theta = \pi$, $AF = 1/\sqrt{2}$

$\theta = \frac{\pi}{3}$, $AF = -1$

UNIFORM LINEAR ARRAY OF N-ELEMENTS



$$AF = \sum_{i=1}^{N} I_i e^{jkz_i\cos\theta}$$

$$AF = I_0 + I_0 e^{j\alpha} e^{jkd\cos\theta} + I_0 e^{j2\alpha} e^{jk(2d)\cos\theta} + I_0 e^{j3\alpha} e^{jk3d\cos\theta} + \dots + I_0 e^{j(N-1)\alpha} e^{jk(N-1)d\cos\theta}$$

$$AF = I_0 \left\{1 + e^{j(\alpha + kd\cos\theta)} + e^{j2(\alpha + kd\cos\theta)} + e^{j3(\alpha + kd\cos\theta)} + \dots + e^{j(N-1)(\alpha + kd\cos\theta)}\right\}$$

Let $\psi = \alpha + kd\cos\theta$

$$AF = I_0 \left[1 + e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \dots + e^{j(N-1)\psi}\right] \longrightarrow (1)$$

$$(AF)e^{j\psi} = I_0 \left[e^{j\psi} + e^{j2\psi} + e^{j3\psi} + e^{j4\psi} + \dots + e^{jN\psi}\right] \longrightarrow (2)$$

$(2) - (1) \Rightarrow$ $AF\left[e^{j\psi} - 1\right] = I_0\left[e^{jN\psi} - 1\right]$

$$\boxed{AF = \frac{I_0\left[e^{jN\psi} - 1\right]}{e^{j\psi} - 1}}$$

Taking out $e^{jN\psi/2}$ from the numerator and taking out $e^{j\psi/2}$ from the denominator

$$AF = I_0 \frac{e^{jN\psi/2}}{e^{j\psi/2}} \frac{[e^{jN\psi/2} - e^{-jN\psi/2}]}{[e^{j\psi/2} - e^{-j\psi/2}]}$$

Multiply and divide by 2j

$$AF = I_0 e^{j(N-1)\psi/2} \frac{\left[\frac{e^{jN\psi/2} - e^{-jN\psi/2}}{2j}\right]}{\left[\frac{e^{j\psi/2} - e^{-j\psi/2}}{2j}\right]}$$

$$\boxed{AF = I_0 e^{j(N-1)\frac{\psi}{2}} \frac{\sin\left(\frac{N\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}}$$

$$|AF|_{normalised} = \left|\frac{\sin\left(\frac{N\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}\right|$$

Using L'Hospital's Rule,

$$|AF|_{norm} = \frac{\cos\left(\frac{N\psi}{2}\right)\frac{N}{2}}{\cos\left(\frac{\psi}{2}\right)\frac{1}{2}} = \frac{N\cos\left(\frac{N\psi}{2}\right)}{\cos\left(\frac{\psi}{2}\right)}$$

For $\psi = 0$, $|AF|_{norm} = N$, which is the maximum value

$$|AF|_{norm} = \left|\frac{\sin\left(\frac{N\psi}{2}\right)}{N\sin\left(\frac{\psi}{2}\right)}\right|$$

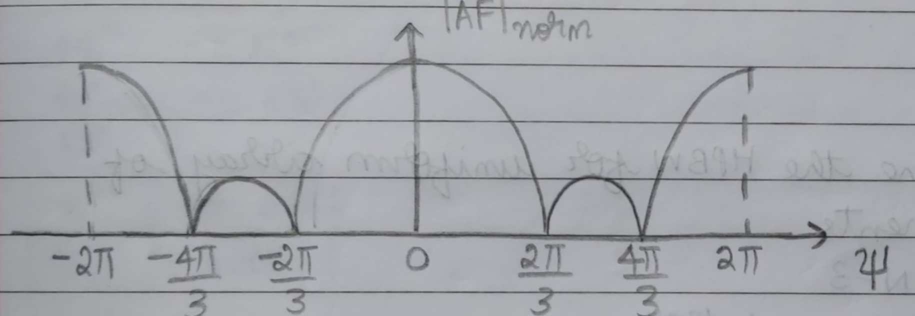
For $\psi = 2\pi$, $|AF|_{norm} = 1$ {after L'Hospital's}
For $\psi = -2\pi$, $|AF|_{norm} = 1$ {after L'Hospital's}

To know where the nulls are,
AF should be zero

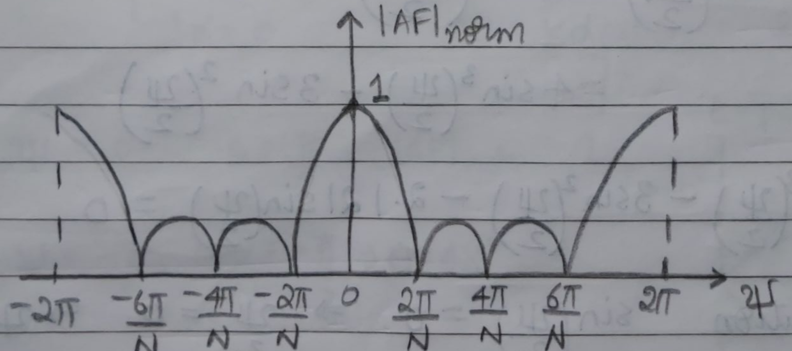
$\sin\left(\frac{N\psi}{2}\right) = 0 \Rightarrow \frac{N\psi}{2} = \pm n\pi$

Nulls occur at $\boxed{\psi = \pm \frac{2n\pi}{N}}$ $n \neq 0$ ($\because$ max)
$n \neq$ multiple of N
($\because$ maxima)

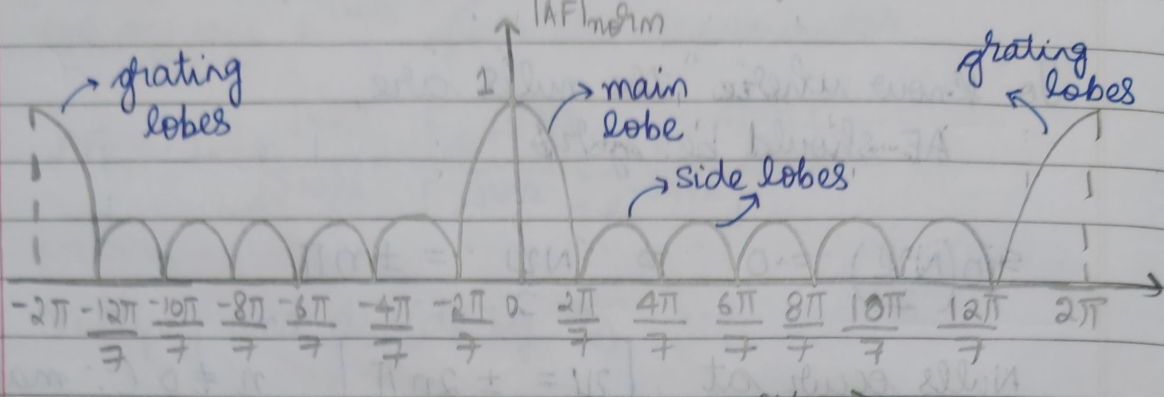
For N = 3
Nulls occurs at $\pm \frac{2n\pi}{3} \Rightarrow \pm \frac{2\pi}{3}, \pm \frac{4\pi}{3}$



~~For N = 4~~
~~Nulls occur at~~ $\cancel{\pm \frac{2n\pi}{4} \Rightarrow}$



For $N = 7$



Nulls occur at $\pm \frac{2n\pi}{7} \Rightarrow \pm\frac{2\pi}{7}, \pm\frac{4\pi}{7}, \pm\frac{6\pi}{7},$

$\pm\frac{8\pi}{7}, \pm\frac{10\pi}{7}, \pm\frac{12\pi}{7}$

Q1.

(a) Determine the HPBW for uniform array of 3 elements

$N = 3$

$$0.707 = \frac{\sin\left(\frac{3\psi}{2}\right)}{3\sin\left(\frac{\psi}{2}\right)}$$

$$2.121\sin\left(\frac{\psi}{2}\right) = \sin\left(\frac{3\psi}{2}\right)$$

$$= 4\sin^3\left(\frac{\psi}{2}\right) - 3\sin^2\left(\frac{\psi}{2}\right)$$

$$4\sin^3\left(\frac{\psi}{2}\right) - 3\sin^2\left(\frac{\psi}{2}\right) - 2.121\sin\left(\frac{\psi}{2}\right) = 0$$

One solution $\sin\left(\frac{\psi}{2}\right) = 0 \Rightarrow \frac{\psi}{2} = 0 \Rightarrow \psi = 0$

$$4\sin^2\left(\frac{\psi}{2}\right) - 3\sin\left(\frac{\psi}{2}\right) - 2.121 = 0$$

$\sin\left(\frac{\psi}{2}\right) = 1.194$ $\qquad \sin\left(\frac{\psi}{2}\right) = 0.444\angle 180^\circ$

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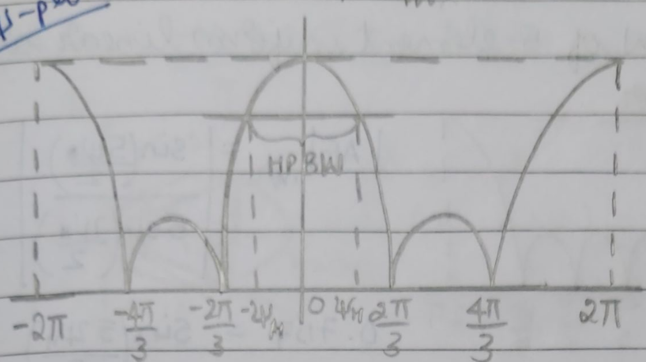
ψ-plot



$$|AF|_{max} = \left| \frac{\sin\left(\frac{3\psi}{2}\right)}{3\sin\left(\frac{\psi}{2}\right)} \right|$$

$$0.707 = \frac{\sin\left(\frac{3\psi_H}{2}\right)}{3\sin\left(\frac{\psi_H}{2}\right)}$$

$$\psi_H = \frac{2\pi}{9} = 40^\circ \hookrightarrow \frac{1}{3}\left(\frac{2\pi}{3}\right)$$

$$3 \times 707 \sin\left(\frac{\psi_H}{2}\right) = \sin\left(\frac{3\psi_H}{2}\right)$$

$\psi_H = 40^\circ$ $\quad 2.121 \sin(20^\circ) - \sin(60^\circ) = \text{Error} = -0.1406$

ψ_H	Error
40°	-0.1406
44°	-0.119
48°	-0.088
52°	-0.048
55°	-0.0012
56°	0.001

$$HPBW = 2\psi_H = 2(56^\circ) = \underline{112^\circ}$$

(b) If element separation $d = \frac{\lambda}{2}$, draw the plot of $|AF|_{nor}$ vs. θ for broadside array

$\hookrightarrow \alpha = 0$

$$\psi = \alpha + kd\cos\theta$$

$$\psi = kd\cos\theta = \frac{2\pi}{\lambda} \times \frac{\lambda}{2}\cos\theta$$

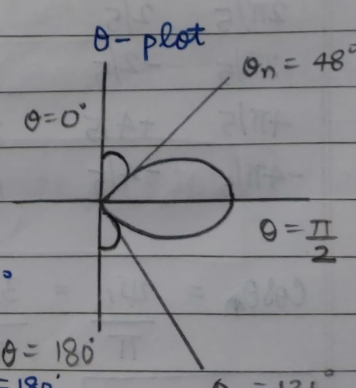
$$\boxed{\psi = \pi\cos\theta}$$

When $\psi = 0$ $\quad \theta = \frac{\pi}{2}$ $\quad |AF|_{nor} = 1$

$\psi = 2\pi/3$ $\quad \cos\theta = +\frac{2}{3} \Rightarrow \theta = 48^\circ$

$\psi = -\frac{2\pi}{3}$ $\quad \cos\theta = \frac{-2}{3} \Rightarrow \theta = 131.8^\circ$

θ-plot



$$|AF|_{nor} = \left| \frac{\sin\left(\frac{3\pi}{2}\cos\theta\right)}{3\sin\left(\frac{\pi}{2}\cos\theta\right)} \right|$$

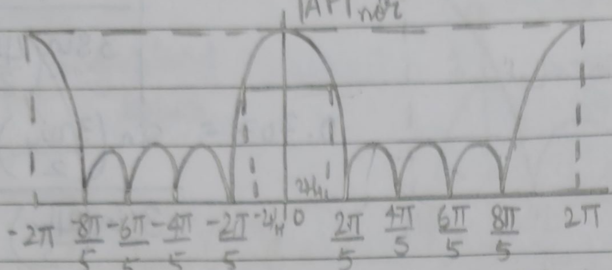
$\theta = 0^\circ$ $\quad |AF|_{nor} = \frac{1}{3}$

$\theta = 180^\circ$ $\quad |AF|_{nor} = \frac{1}{3}$

Q2.

a) Determine the HPBW of 5-element uniform linear array

ψ-plot



$$|AF|_{nor} = \left| \frac{\sin\left(\frac{5\psi}{2}\right)}{5\sin\left(\frac{\psi}{2}\right)} \right|$$

$$0.707 = \frac{\sin\left(\frac{5\psi_H}{2}\right)}{5\sin(\psi_H/2)}$$

$$\text{Error} = 5 \times 0.707 \sin\left(\frac{\psi_H}{2}\right) - \sin\left(\frac{5\psi_H}{2}\right)$$

$$\psi_H = \frac{1}{3}\left(\frac{2\pi}{5}\right) = \frac{2\pi}{15} = 24^\circ$$

ψ_H	Error
24°	-0.131057
28°	-0.084498
32°	-0.010429 (least error)
36°	0.09237
33°	0.012549
32.5°	0.000834

$\boxed{\psi_H = 32.5^\circ}$

$\boxed{HPBW = 2\psi_H = 65^\circ}$

b) Element separation $d = \frac{\lambda}{2}$. Draw the plot of $|AF|_{nor}$ vs. θ for broadside array $\alpha = 0$

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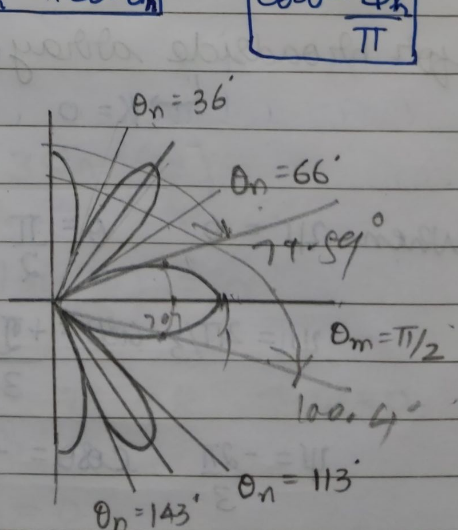
$$\psi = \alpha + kd\cos\theta = 0 + \frac{2\pi}{\lambda} \times \frac{\lambda}{2}\cos\theta$$

$\boxed{\psi_h = \pi\cos\theta_h}$ $\boxed{\cos\theta = \frac{\psi_h}{\pi}}$

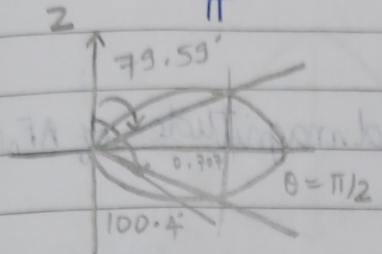
ψ	$\cos\theta$	θ
0	0	$\pi/2$
$2\pi/5$	$2/5$	66.42°
$-2\pi/5$	$-2/5$	113.578°
$4\pi/5$	$+4/5$	36.869°
$-4\pi/5$	$-4/5$	143.13°



$$\cos\theta_h = \frac{\psi_h}{\pi} = \frac{32.5^\circ \times \pi/180^\circ}{\pi}$$

$\boxed{\theta_n = 79.59^\circ}$

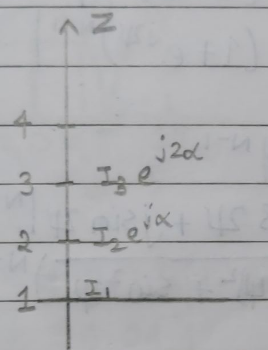
$\cos\theta_n = \frac{-32.5^\circ \times \pi/180^\circ}{\pi} \Rightarrow \boxed{\theta_n = 100.4^\circ}$



θ-plot

$HPBW = 100.4^\circ - 79.59^\circ$
$= 20.81^\circ$

LINEAR ARRAY WITH NON-UNIFORM EXCITATION



α - progressive phase difference

α is maintained the same and only the amplitude of current changes

$$AF = I_1 + I_2 e^{j\psi} + I_3 e^{j2\psi} + \dots + I_N e^{j(N-1)\psi}$$

$$AF = I_1 \left(1 + a_1 e^{j\psi} + a_2 e^{j2\psi} + \dots + a_{N-1} e^{j(N-1)\psi}\right)$$

for non-uniform excitation

SPECIAL CASE

when magnitudes of the currents in the array have binomial distribution, the array is known as BINOMIAL ARRAY.

current magnitudes for the 3 elements

$$AF_3 = I_1 (1 + 2e^{j\psi} + 1e^{j2\psi}) = I_1 (1 + e^{j\psi})^{3-1} = I_1 (1 + e^{j\psi})^2$$

$$AF_4 = I_1 (1 + e^{j\psi})^{4-1} = I_1 (1 + 3e^{j\psi} + 3e^{j2\psi} + 1e^{j3\psi})$$

Let $z = e^{j\psi}$

$\psi = \alpha + kd\cos\theta$

$$\boxed{AF_N = I_1 (1+z)^{N-1}}$$

$$AF_6 = I_1(1+z)^5$$
$$AF_6 = I_1(1+5z+10z^2+5z^3+10z^4+1z^5)$$

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To determine $|AF_N|_{nor}$ (normalised magnitude of AF_N)

$$|AF_N|_{nor} = \frac{|AF_N|}{|AF_N|_{max}}$$

* Magnitude of AF_N $\Rightarrow$ $|AF_N| = |I_1(1+z)^{N-1}|$
$$= I_1 |(1+e^{j\psi})^{N-1}|$$

$$|a^N| = |a|^N$$

$$|AF_N| = I_1 |(1+e^{j\psi})|^{N-1}$$
$$= I_1 |1+\cos\psi + j\sin\psi|^{N-1}$$
$$= I_1\left(\sqrt{(1+\cos\psi)^2+\sin^2\psi}\right)^{N-1}$$
$$= I_1(1+\cos^2\psi+2\cos\psi+\sin^2\psi)^{\frac{N-1}{2}}$$
$$= I_1(2+2\cos\psi)^{\frac{N-1}{2}}$$
$$= I_1[2(1+\cos\psi)]^{\frac{N-1}{2}}$$

$$\boxed{|AF_N| = I_1[2(1+\cos\psi)]^{\frac{N-1}{2}}}$$

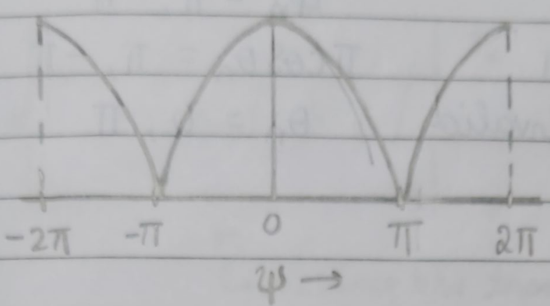
max when $\cos\psi = 1$

$$\boxed{|AF_N|_{max} = I_1(4)^{\frac{N-1}{2}}}$$

$$|AF_N|_{nor} = \frac{|AF_N|}{|AF_N|_{max}} = \frac{I_1[2(1+\cos\psi)]^{\frac{N-1}{2}}}{I_1(4)^{\frac{N-1}{2}}}$$

$$\boxed{|AF_N|_{nor} = \left(\frac{1+\cos\psi}{2}\right)^{\frac{N-1}{2}}}$$

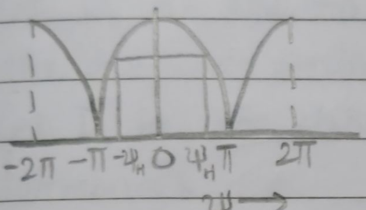
MAXIMUM POINTS OF BINOMIAL ARRAY
NULL POINTS



$\psi_{max} = 0, 2\pi, -2\pi$

$\psi_{min} = \pi, -\pi = \psi_n$

Example a) Determine HPBW in ψ-plot for a 5-element binomal array



$$0.707 = \left(\frac{1+\cos\psi_H}{2}\right)^{\frac{5-1}{2}}$$

$$0.707 = \left(\frac{1+\cos\psi_H}{2}\right)^{2}$$

$$0.84083 = \frac{1+\cos\psi_H}{2}$$

$$\cos\psi_H = 1.68166 - 1$$

$$\psi_H = \cos^{-1}(0.68166)$$

$$\boxed{\psi_H = 47.02^\circ}$$

$HPBW = 2\psi_H = 2(47.02^\circ)$

$\boxed{HPBW = 94.04^\circ}$

b) Find HPBW in θ-plot if element separation $d = \lambda/2$ and $\alpha = 0$ (broadside array)

$$\psi = \alpha + kd\cos\theta = 0 + \frac{2\pi}{\lambda} \times \frac{\lambda}{2}\cos\theta$$

$\boxed{\psi = \pi\cos\theta}$

$\psi_H = \pi\cos\theta_{H_1}$

$\cos\theta_{H_1} = \frac{\psi_H}{\pi}$

$= \frac{47.02^\circ}{180^\circ}$

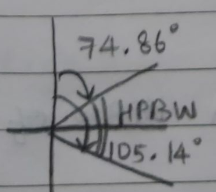
$\theta_{H_1} = 74.86^\circ$

$-\psi_H = \pi\cos\theta_{H_2}$

$\cos\theta_{H_2} = \frac{-\psi_H}{\pi}$

$= \frac{-47.02^\circ}{180^\circ}$

$\theta_{H_2} = 105.14^\circ$



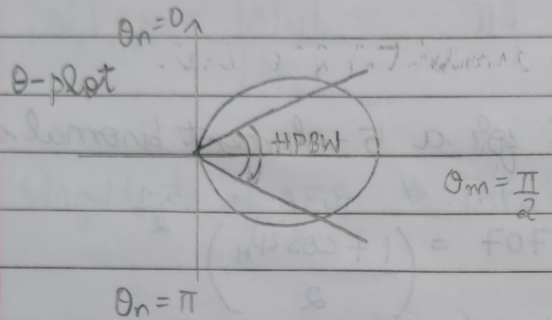
$HPBW = \theta_{H_2} - \theta_{H_1} \Rightarrow \boxed{(HPBW)_\theta = 30.28^\circ}$

c) Draw the θ-plot

$\psi_{max} = 0, 2\pi, -2\pi$ $\qquad$ $\psi_n = \pi, -\pi$

$\pi\cos\theta_m = 0, 2\pi, -2\pi$ $\qquad$ $\pi\cos\theta_n = \pi, -\pi$

$\theta_m = \frac{\pi}{2}$, invalid, invalid $\qquad$ $\theta_n = 0, \pi$



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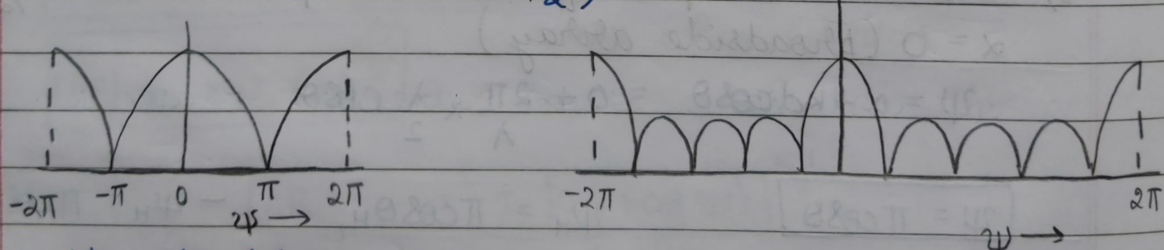
Compare the 5-element binomial and 5-element uniform array HPBWs

BINOMIAL $\qquad$ **UNIFORM**

$(HPBW)_\psi = 94^\circ$ $\qquad$ $(HPBW)_\psi = 65^\circ$

$(HPBW)_\theta = 30.28^\circ$ $\qquad$ $(HPBW)_\theta = 20.81^\circ$

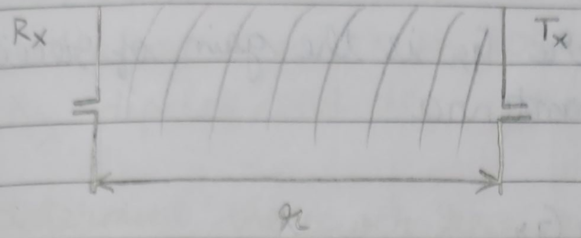
$(\alpha = 0, \; d = \lambda/2)$



No side lobes

* HPBW of binomial array is broader than HPBW of uniform array

FRIIS TRANSMISSION FORMULA



↳ distance b/w transmitter & receiver

- Let the power transmitted by the T_x antenna be P_t
- Let U_t be the radiation intensity due to T_x antenna, at the receiver antenna
- Let D_t be the directivity of the T_x antenna at R_x antenna terminals

$$D_t = \frac{U_t}{U_0} = \frac{U_t}{P_t/4\pi} = \frac{4\pi U_t}{P_t}$$

For a lossless T_x antenna $D_t = G_t$, where G_t is the gain of the T_x antenna

$$G_t = \frac{4\pi U_t}{P_t}$$

If S_t is the power density due to T_x antenna at R_x antenna terminals,

$$U_t = S_t r^2$$

Hence $$G_t = \frac{4\pi r^2 S_t}{P_t}$$

Aperture $A_r = \frac{P_r}{S_t} \Rightarrow S_t = \frac{P_r}{A_r}$ where P_r is power received by the R_x antenna and A_r is the effective aperture of the R_x antenna

In G_t, $G_t = \frac{4\pi r^2}{P_t}\frac{P_r}{A_r}$

$$P_r = \frac{P_t G_t A_r}{4\pi r^2}$$

$A_r = \left(\frac{\lambda^2}{4\pi}\right) G_r$ where G_r is the gain of receiver antenna

In P_r, $P_r = \frac{P_t G_t}{4\pi r^2}\left(\frac{\lambda^2}{4\pi}\right) G_r$

$$\boxed{P_r = P_t G_t G_r \left(\frac{\lambda}{4\pi r}\right)^2}$$

$$10 \log_{10} P_r = 10 \log_{10}\left(P_t G_t G_r \left(\frac{\lambda}{4\pi r}\right)^2\right)$$

$$= 10 \log_{10} P_t + 10 \log_{10} G_t + 10 \log_{10} G_r + 20 \log_{10}\left(\frac{\lambda}{4\pi r}\right)$$

$$\boxed{P_r (dBm) = P_t (dBm) + G_t (dB) + G_r (dB) + \text{Path loss } (dB)}$$

$\Downarrow$
power in terms of 1mW

$$\underset{(dB)}{\text{Path loss}} = 20 \log_{10}\left(\frac{\lambda}{4\pi r}\right)$$

Example Let power received be -20 dBm, calculate this power in Watts.

$$-20 = 10 \log_{10}\left(\frac{P_r}{1\times10^{-3}}\right)$$

$$-2 = \log\left(\frac{P_r}{10^{-3}}\right) \Rightarrow 10^{-2} = \frac{P_r}{10^{-3}}$$

$$P_r = 10^{-2} \times 10^{-3}$$

$$\boxed{P_r = 10^{-5}\ W}$$

Example Two antennas (one Tx and one Rx) are placed facing each other 5 km distance apart. Tx antenna is radiating a power of 100 Watts. If the antennas are $\lambda/2$ dipoles and the operating frequency is 300 MHz, determine the power received in watts, as well as in dBm.

$r = 5\,km = 5 \times 10^3\,m$

$P_t = 100\,W$

$f = 300\,MHz = 300 \times 10^6\,Hz$

$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{300 \times 10^6} = 1\,m$

Assuming lossless antennas with max directivity

$G = eD = D$

$D_{max}\left(\frac{\lambda}{2}\right) = 1.642$

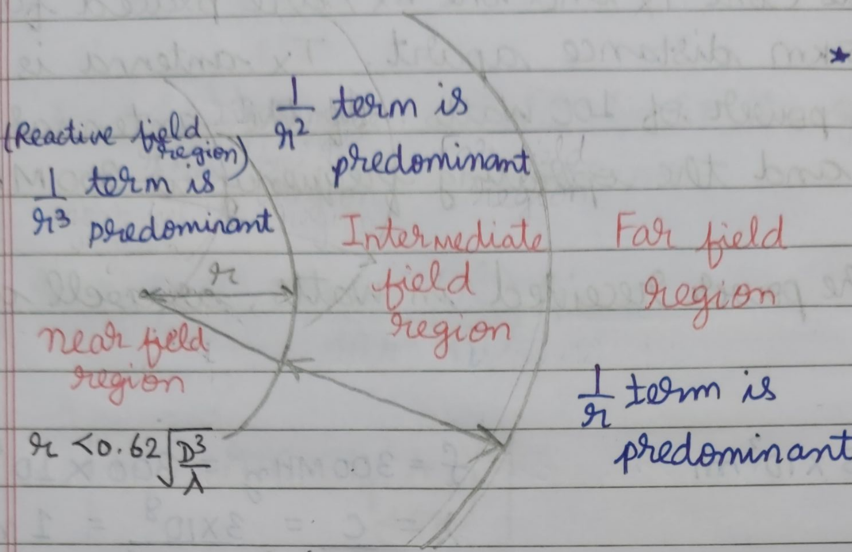
$G_t = G_r = 1.642$

Path loss $= 20 \log_{10}\left(\frac{\lambda}{4\pi r}\right) = -95.9636$

$P_r = P_t G_t G_r \left(\frac{\lambda}{4\pi r}\right)^2 \Rightarrow \boxed{P_r = 6.83 \times 10^{-8}\,W}$

$P_{r(dBm)} = 10 \log_{10}\left(\frac{P_r}{10^{-3}}\right) \Rightarrow \boxed{P_{r(dBm)} = -41.65\,dBm}$

FIELD REGIONS



* D is the maximum dimension of the antenna

$$0.62\frac{\sqrt{D^3}}{\sqrt{\lambda}} < r < \frac{2D^2}{\lambda}$$

$$r > \frac{2D^2}{\lambda}$$