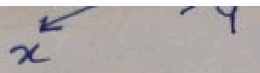


COORDINATE SYSTEMS

① Cartesian (Rectangular) x, y, z

② Cylindrical ~~r, θ, z~~ ρ, ϕ, z → $0 \le \rho < \infty$; $0 \le \phi < 2\pi$ azimuthal angle; $-\infty < z < \infty$

③ Spherical r, θ, ϕ → $0 \le \phi < 2\pi$; $0 \le r < \infty$; $0 \le \theta < \pi$ (coaltitude angle)

VECTOR CALCULUS

Vector $A \rightarrow \bar{A}$ (Bold A) : $A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z$

A_x → magnitude; $\bar{a}_x$ → unit vector

Differential length (displacement) $\int$ $\quad d\vec{l} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$

Differential area $\iint$ $\quad dA$

Differential volume $\iiint$ $\quad dV$

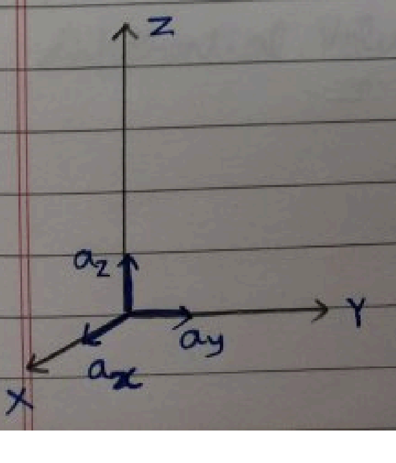
Gradient of a scalar → gives a vector

Curl of a vector → gives a vector
↳ cross product

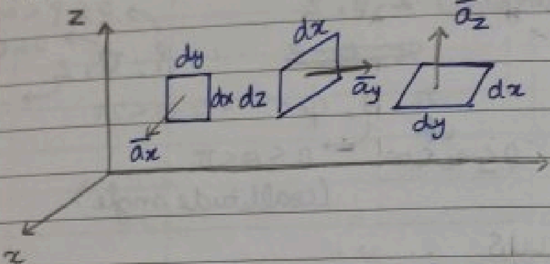
LINEAR TIME-INVARIANT SYSTEM (LTI) → System's performance doesn't change with time

↓

The relation b/w i/p & o/p satisfies homogeneity & additivity



DIFFERENTIAL AREA



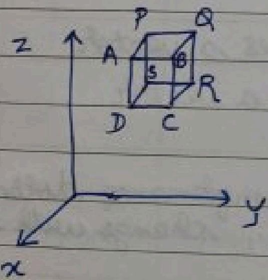
Vector Coming out of the Surface

Differential Normal Surface Area (Vector)

$$d\vec{S} = \begin{cases} dy\,dz\ \vec{a}_x \\ dx\,dz\ \vec{a}_y \\ dx\,dy\ \vec{a}_z \end{cases}$$

Differential Volume (scalar)

$$dV = dx\,dy\,dz$$



P to Q   +ve y-direction

$$d\vec{l} = dy\,\vec{a}_y$$

S to Q $d\vec{l} = dy\,\vec{a}_y + dz\,\vec{a}_z$

D to Q $d\vec{l} = -dx\,\vec{a}_x + dy\,\vec{a}_y + dz\,\vec{a}_z$

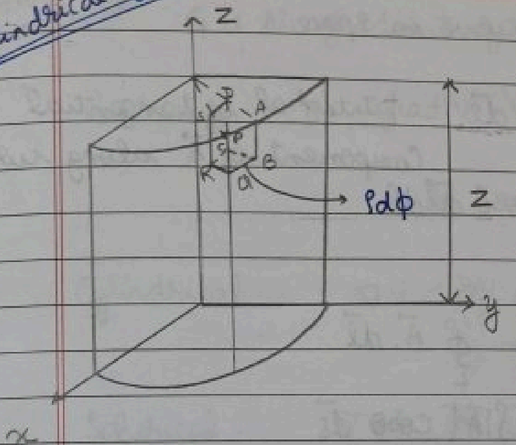
Surface ABCD

$$d\vec{S} = dy\,dz\ \vec{a}_x$$

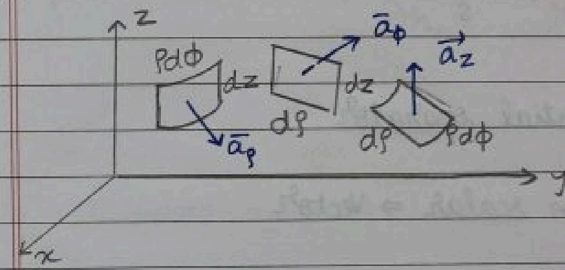
Surface PQRS

$$d\vec{S} = -dy\,dz\ \vec{a}_x$$

Cylindrical Systems



$\rho d\phi \Rightarrow$ differential length



Differential displacement

$$d\vec{l} = d\rho\,\vec{a}_\rho + \rho\,d\phi\,\vec{a}_\phi + dz\,\vec{a}_z$$

Spherical Systems

Differential displacement

$$d\vec{l} = dr\,\vec{a}_r + r\,d\theta\,\vec{a}_\theta + r\sin\theta\,d\phi\,\vec{a}_\phi$$

Differential Normal Surface Area

$$d\vec{S} = \begin{cases} r^2\sin\theta\,d\theta\,d\phi\,\vec{a}_r \\ r\sin\theta\,dr\,d\phi\,\vec{a}_\theta \\ r\,dr\,d\theta\,\vec{a}_\phi \end{cases}$$

Differential Volume $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

Line: Path along a curve in space

Line integral ⇒ $\int_L \vec{A} \cdot d\vec{l}$ Integral of tangential component of $\vec{A}$ along curve

$= \int_a^b |\vec{A}| \cos\theta \, dl$

Closed Contour Integral $\oint_L \vec{A} \cdot d\vec{l}$

Surface Integral $\vec{\psi} = \int_S |\vec{A}| \cos\theta \, d\vec{s}$

Del operator ∇

Vector differential operator

- not a vector
- ∇ operates on scalar ⇒ Vector

Gradient of a scalar S ∇S

Divergence of a vector $\vec{A}$ $\nabla \cdot \vec{A}$

Curl of a vector $\vec{A}$ $\nabla \times \vec{A}$

Laplacian of a scalar V $\nabla^2 V$

Cartesian Coordinates $\nabla = \frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z$

Cylindrical Coordinates $\nabla = \vec{a}_\rho \frac{\partial}{\partial \rho} + \vec{a}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \vec{a}_z \frac{\partial}{\partial z}$

Gradient of a Scalar

G is the gradient of V

Cartesian : $\text{grad } V = \nabla V = \frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z$

Cylindrical : $\nabla V = \frac{\partial V}{\partial \rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{a}_\phi + \frac{\partial V}{\partial z}\vec{a}_z$

Spherical : $\nabla V = \frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi$

ELECTROSTATICS

concepts applicable to static (or time invariant) electric fields in free space (or vacuum)

COULOMB's LAW (1785)

The law states that the force $\vec{F}$ between 2 point charges Q_1 and Q_2 is

- along the line joining them
- $\propto Q_1 Q_2$
- $\propto \frac{1}{R^2}$ where R is the distance between the charges

$$\underset{N}{F} = \overset{\text{constant of proportionality}}{K}\frac{\overset{C}{Q_1 Q_2}}{\underset{m}{R^2}} \qquad K = \frac{1}{4\pi\varepsilon_0} \approx 9\times10^{9}\ m/F$$

$\varepsilon_0 = 8.854\times10^{-12}$ F/m

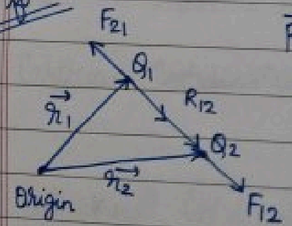
permittivity of free space

$\varepsilon_0 \approx \frac{10^{-9}}{36\pi}$ F/m

Scalar form

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}$$

Vector form



$\vec{R}_{12} + \vec{r}_1 = \vec{r}_2$

$\Rightarrow \vec{R}_{12} = \vec{r}_2 - \vec{r}_1$

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\vec{a}_{12}$$

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^3}\vec{R}_{12} = \frac{Q_1 Q_2(\vec{r}_2 - \vec{r}_1)}{4\pi\varepsilon_0 (|\vec{r}_2 - \vec{r}_1|)^3}$$

$$\vec{F}_{21} = |\vec{F}_{12}|\,\vec{a}_{R_{21}} = |\vec{F}_{12}|\,(-\vec{a}_{R_{12}}) = -\vec{F}_{12}$$

Like charges repel each other

Unlike charges attract each other

Principle of Superposition

N charges $Q_1, Q_2, \ldots, Q_N$

Position vectors w.r.t origin $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$

Resultant force $\vec{F}$ on a charge Q located at a point with its position vector $\vec{r}$, is the vector sum of the forces exerted on Q by each of the charges $Q_1, Q_2, \ldots, Q_N$

$$\vec{F} = \frac{QQ_1(\vec{r}-\vec{r}_1)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_1|^3} + \frac{QQ_2(\vec{r}-\vec{r}_2)}{4\pi\varepsilon_0|\vec{r}-\vec{r}_2|^3} + \cdots$$

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{Q_k(\vec{r}-\vec{r}_k)}{|\vec{r}-\vec{r}_k|^3}$$

Electric Field ~~Intensity~~ Strength $\vec{E} = \frac{\vec{F}}{Q}$ (N/C or V/m)

The force that a unit positive charge charge experiences when placed in an electric field

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\vec{a}_R = \frac{Q(\vec{r}-\vec{r}')}{4\pi\varepsilon_0|\vec{r}-\vec{r}'|^3} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a}_R$$

Superposition Principle : $\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{Q_k(\vec{r}-\vec{r}_k)}{|\vec{r}-\vec{r}_k|^3}$

Example 1 Point charges 1mC and −2mC are located at (3, 2, −1) and (−1, −1, 4) respectively. Calculate the electric force on a 10nC charge placed at (0, 3, 1) and also the electric field intensity at that point.

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^3}\vec{R}_{12}$$

$$\vec{F}_x = \frac{(1\times10^{-3})\times(-2\times10^{-3})}{(3-0)^3}$$

$$\bar{F} = \frac{Q}{4\pi\varepsilon_0}\left\{\sum_{k=1,2} \frac{Q_k(\bar{r}-\bar{r}_k)}{|\bar{r}-\bar{r}_k|^3}\right\}$$

$$= \frac{10\times10^{-9}}{\frac{4\pi \times 10^{-9}}{36\pi}}\left\{\frac{10^{-3}[(0,3,1)-(3,2,-1)]}{|(0,3,1)-(3,2,-1)|^3} + \frac{(-2\times10^{-3})[(0,3,1)-(-1,-1,4)]}{|(0,3,1)-(-1,-1,4)|^3}\right\}$$

$$= 90\times10^{-3}\left\{\frac{(-3,1,2)}{|(-3,1,2)|^3} - \frac{2[(+1,4,-3)]}{|(+1,4,-3)|^3}\right\}$$

$$= 90\times10^{-3}\left\{\frac{[(-3,1,2)]}{14\sqrt{14}} - \frac{2[(+1,4,-3)]}{26\sqrt{26}}\right\}$$

$$= 90\times10^{-3}\left\{-0.07235\,\bar{a}_x - 0.04125\,\bar{a}_y + 0.08343\,\bar{a}_z\right\}$$

$$\boxed{\bar{F} = (-6.512\,\bar{a}_x - 3.712\,\bar{a}_y + 7.51\,\bar{a}_z)\,mN}$$

$$\bar{E} = \frac{\bar{F}}{Q} = \frac{\bar{F}}{10\times10^{-9}} = (-65.12\,\bar{a}_x - 37.12\,\bar{a}_y + 75.1\,\bar{a}_z)\,kV/m$$

~~Homework~~ HW Point charges 5nC and −2nC are located at (2, 0, 4) and (−3, 0, 5) respectively. Determine the force on a 1nC point charge located at (1, −3, 7). Evaluate the electric field intensity at this point as well.

$$\bar{F} = \frac{Q}{4\pi\varepsilon_0}\left\{\sum_{k=1,2} \frac{Q_k(\bar{r}-\bar{r}_k)}{|\bar{r}-\bar{r}_k|^3}\right\}$$

$$= \frac{10^{-9}}{\frac{4\pi\times10^{-9}}{36\pi}}\left\{\frac{5\times10^{-9}[(1,-3,7)-(2,0,4)]}{|(1,-3,7)-(2,0,4)|^3} + \frac{-2\times10^{-9}[(1,-3,7)-(-3,0,5)]}{|(1,-3,7)-(-3,0,5)|^3}\right\}$$

$$= 9\times10^{-9}\left\{\frac{5[(-1,-3,3)]}{|(-1,-3,3)|^3} - \frac{2[(4,-3,2)]}{|(4,-3,2)|^3}\right\}$$

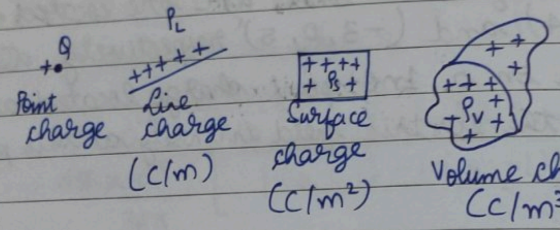
$$= 9\times10^{-9}\left\{\frac{(-5,-15,15)}{19\sqrt{19}} + \frac{(-8,6,-4)}{29\sqrt{29}}\right\}$$

$$= 9\times10^{-9}\left\{-0.1116\,\bar{a}_x - 0.1427\,\bar{a}_y + 0.1555\,\bar{a}_z\right\}$$

$$\boxed{\bar{F} = (-1.0044\,\bar{a}_x - 1.2843\,\bar{a}_y + 1.3995\,\bar{a}_z)\,nN}$$

$$\bar{E} = \frac{\bar{F}}{Q} = \frac{\bar{F}}{10^{-9}} = (-1.0044\,\bar{a}_x - 1.2843\,\bar{a}_y + 1.3995\,\bar{a}_z)\,V/m$$

Electric Field Due to Continuous Charge Distributions



Charge element dQ
Total charge Q

$dQ = \rho_L\, dl \rightarrow Q = \int_L \rho_L\, dl$ (Line charge)

$dS = \rho_S\, dS \rightarrow Q = \int_S \rho_S\, dS$ (Surface charge)

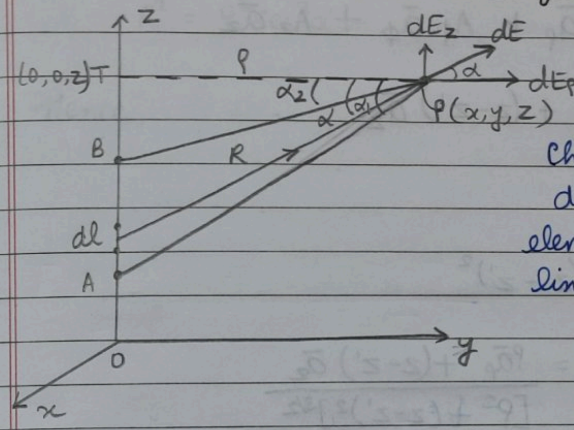
$dV = \rho_V\, dV \rightarrow Q = \int_V \rho_V\, dV$ (Volume charge)

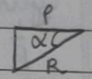
$\vec{E} = \int_L \frac{\rho_L\, dl}{4\pi\varepsilon_0 R^2}\vec{a}_R$ Line charge

$\bar{E} = \int_S \frac{\rho_S\, dS}{4\pi\varepsilon_0 R^2}\bar{a}_R$ Surface Charge

$\bar{E} = \int_V \frac{\rho_V\, dv}{4\pi\varepsilon_0 R^2}\bar{a}_R$ Volume charge

Electric Field Due to a Line Charge





$\cos\alpha = \frac{\rho}{R} \Rightarrow R = \frac{\rho}{\cos\alpha} = \rho\sec\alpha$

Charge element dQ associated with element $dl = dz$ of the line is

$dQ = \rho_L\, dl$
$= \rho_L\, dz$

$\therefore$ Total charge $Q = \int_{z_A}^{z_B} \rho_L\, dz$

Electric field intensity $\bar{E} = \int_L \frac{\rho_L\, dl}{4\pi\varepsilon_0 R^2}\bar{a}_R = \int_L \frac{\rho_L\, dz}{4\pi\varepsilon_0 R^2}\bar{a}_R$ — (1)

Field point (x, y, z) Source point (x', y', z')
$= (0, 0, z')$ $\because$ it lies on z-axis

$dl = dz'$

$\bar{R} = (x, y, z) - (x', y', z') = \underbrace{x}_{A_x}\bar{a}_x + \underbrace{y}_{A_y}\bar{a}_y + \underbrace{(z-z')}_{A_z}\bar{a}_z$

Identities $\rho = \sqrt{x^2+y^2}$ $x = \rho\cos\phi$

$\phi = \tan^{-1}(y/x)$ $y = \rho\sin\phi$

$z = z$ $z = z$

Vector to Vector Transformation $A_\rho = \cos\phi\, A_x + \sin\phi\, A_y$
$= x\cos\phi + y\sin\phi$
$= \rho\cos^2\phi + \rho\sin^2\phi$
$A_\rho = \rho$

$A_\phi = -\sin\phi\, A_x + \cos\phi\, A_y = -x\sin\phi + y\cos\phi$
$= -\rho\cos\phi\sin\phi + \rho\sin\phi\cos\phi = 0$

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Cylindrical

$\bar{R} = A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z$

$\bar{R} = \rho \bar{a}_\rho + (z-z') \bar{a}_z$

10/8/23

$R^2 = \rho^2 + (z-z')^2$

$$\frac{\bar{a}_R}{R^2} = \frac{\bar{R}}{|\bar{R}|^3} = \frac{\rho \bar{a}_\rho + (z-z')\bar{a}_z}{[\rho^2 + (z-z')^2]^{3/2}}$$

Substituting in ①

$$\bar{E} = \int_L \frac{\rho_L dl}{4\pi\varepsilon_0 R^2} \bar{a}_R = \frac{\rho_L}{4\pi\varepsilon_0} \int \frac{\rho \bar{a}_\rho + (z-z')\bar{a}_z}{[\rho^2 + (z-z')^2]^{3/2}} dz'$$

The integral is evaluated using the angles α, α_1 & α_2

$R = [\rho^2 + (z-z')^2]^{1/2} = \rho \sec\alpha$

$z' = OT - \rho \tan\alpha$

$\Rightarrow dz' = -\rho \sec^2\alpha \, d\alpha$

$$\bar{E} = \frac{+\rho_L}{4\pi\varepsilon_0} \int_L \frac{\bar{a}_R}{R^2} dz' = \frac{\rho_L}{4\pi\varepsilon_0} \int_L \frac{\bar{R}}{R^2 . R} dz'$$

$$= \frac{\rho_L}{4\pi\varepsilon_0} \int_L \frac{\rho \bar{a}_\rho + (z-z')\bar{a}_z}{R^2 . R} dz'$$

$\rho = R\cos\alpha \qquad z - z' = R\sin\alpha \quad \Rightarrow dz' = -\rho^2 \sec\alpha \, d\alpha$

$$\bar{E} = \frac{\rho_L}{4\pi\varepsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{[R\cos\alpha \, \bar{a}_\rho + R\sin\alpha \, \bar{a}_z]}{R^2 R} (-\rho\sec^2\alpha) d\alpha$$

$$= \frac{-\rho_L}{4\pi\varepsilon_0} \int^{\alpha_2} (\cos\alpha \, \bar{a}_\rho + \sin\alpha \, \bar{a}_z) (\rho \sec^2\alpha) \, d\alpha$$

Since $R = [\rho^2 + (z-z')^2]^{0.5} = \rho\sec\alpha$

$R^2 = \rho^2 \sec^2\alpha$

Hence,

$$\bar{E} = \frac{-\rho_L}{4\pi\varepsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{\rho\sec^2\alpha [\cos\alpha \, \bar{a}_\rho + \sin\alpha \, \bar{a}_z]}{\rho^2 \sec^2\alpha}$$

$$\Rightarrow \bar{E} = \frac{-\rho_L}{4\pi\varepsilon_0 \rho} \int_{\alpha_1}^{\alpha_2} [\cos\alpha \, \bar{a}_\rho + \sin\alpha \, \bar{a}_z] d\alpha$$

* For finite line charges (with α_1 & α_2 known)

$$\bar{E} = \frac{\rho_L}{4\pi\varepsilon_0 \rho} [-(\sin\alpha_2 - \sin\alpha_1)\bar{a}_\rho + (\cos\alpha_2 - \cos\alpha_1)\bar{a}_z]$$

* For an infinite line charge A $(0,0,-\infty)$
B $(0,0,\infty)$

$\alpha_1 = \frac{\pi}{2}$ z-component vanishes since $\cos\frac{\pi}{2} = 0$

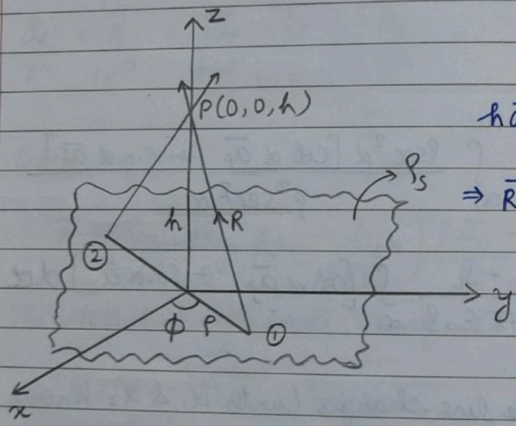
$\alpha_2 = -\frac{\pi}{2}$

$$\bar{E} = \frac{\rho_L}{4\pi\varepsilon_0\rho} [-(-1-1)\bar{a}_\rho] = \frac{\rho_L}{2\pi\varepsilon_0\rho} \bar{a}_\rho$$

* If the line is not along the z-axis, ρ is perpendicular distance from this line to the point of interest

* $\bar{a}_\rho$ is a unit vector along that displacement directed from the line charge (source) to the 'field point'.

Electric Field Due to a Surface Charge



$h\bar{a}_z = \rho\,\bar{a}_\rho + \bar{R}$

$\Rightarrow \bar{R} = -\rho\bar{a}_\rho + h\bar{a}_z$

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

$R = |\bar{R}| = \sqrt{\rho^2 + h^2}$ $\qquad \bar{a}_R = \frac{\bar{R}}{R}$

$dQ = \rho_s\, dS = \rho_s\, \rho\, d\phi\, d\rho$, normal vector is $\bar{a}_z$

$$d\bar{E} = \frac{\rho_s dS}{4\pi\varepsilon_0 R^2}\,\frac{\bar{R}}{R} = \frac{\rho_s\,\rho\, d\phi\, d\rho}{4\pi\varepsilon_0 R^3}\,\bar{R} = \frac{\rho_s\,\rho\, d\phi\, d\rho\,[-\rho\bar{a}_\rho + h\bar{a}_z]}{4\pi\varepsilon_0\,[\rho^2 + h^2]^{3/2}}$$

NOTE Owing to the symmetry of charge distribution, for every element 1, there is a corresponding element 2

whose contribution along $\bar{a}_\rho$ cancels that of element 1.

Thus the contributions to E_ρ add up to zero so that $\bar{E}$ has only z-component.

Hence,
$$\bar{E} = \int_S d\bar{E}_z = \frac{\rho_s}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{h\rho\, d\phi\, d\rho}{[\rho^2 + h^2]^{3/2}}\,\bar{a}_z$$

$$\bar{E} = \frac{\rho_s}{2\varepsilon_0}\,\bar{a}_z$$

$$\bar{E} = \int_S d\bar{E}_z = \frac{\rho_s}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{h\rho\, d\phi\, d\rho}{[\rho^2 + h^2]^{3/2}}\,\bar{a}_z$$

$$= \frac{\rho_s h}{4\pi\varepsilon_0}(2\pi)\int_{\rho=0}^{\infty}\frac{\rho\, d\rho}{[\rho^2 + h^2]^{3/2}}\,\bar{a}_z = \frac{\rho_s h}{2\varepsilon_0}\int_{\rho=0}^{\infty}\frac{\rho\, d\rho}{[\rho^2 + h^2]^{3/2}}\,\bar{a}_z$$

Let $u = \rho^2 + h^2$

$du = 2\rho\, d\rho$

$\rho\, d\rho = \frac{du}{2}$

When $\rho = 0$, $u = h^2$

$\rho = \infty$, $u = \infty$

$$\bar{E} = \frac{\rho_s h}{2\varepsilon_0}\left(\frac{1}{2}\right)\int_{u=h^2}^{\infty} u^{-3/2}\, du\, \bar{a}_z$$

$$\boxed{\vec{E} = \frac{\rho_s}{2\varepsilon_0}\,\bar{a}_z}$$

valid for $h > 0$

$\vec{E}$ has only z-component if the charge is in the xy-plane

If $h < 0$ $\quad \vec{a}_z$ should be replaced by $(-\bar{a}_z)$.

Infinite sheet of charge

$$\bar{E} = \frac{\rho_s}{2\varepsilon_0}\bar{a}_n$$

$\bar{a}_n \hookrightarrow$ unit vector normal to the sheet

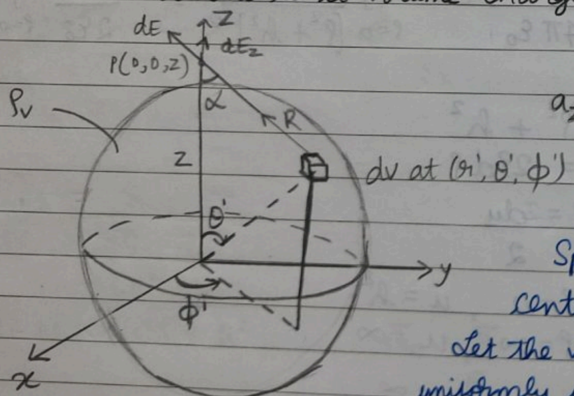
* $\bar{E}$ is normal to the sheet and it is surprisingly independent of the distance between the sheet & the point of observation P.

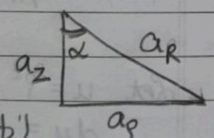
Parallel plate capacitor

$$\bar{E} = \frac{\rho_s}{2\varepsilon_0}\vec{a}_n + \frac{(-\rho_s)(-\vec{a}_n)}{2\varepsilon_0}$$

$$\boxed{\bar{E} = \frac{\rho_s}{\varepsilon_0}\vec{a}_n}$$

Electric Field Due to a Volume Charge





Sphere radius a centered at origin

Let the volume be filled uniformly with a volume charge density ρ_v (C/cubic m)

Charge dQ associated with elemental volume dv chosen at (r', θ', ϕ') is $dQ = \rho_v dv$

Total charge in the sphere $Q = \int_v \rho_v \, dv = \rho_v \int_v dv = \rho_v \frac{4}{3}\pi a^3$

$$\bar{E} = \int_v \frac{\rho_v dv}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

$$\cos\alpha\,\bar{a}_z = \sin\alpha\,\bar{a}_\rho + \bar{a}_R$$

$$\bar{a}_R = \cos\alpha\,\bar{a}_z - \sin\bar{a}_\rho$$

* Owing to symmetry of charge distribution, $\bar{E}_x = \bar{E}_y = 0$

$$E_z = \bar{E}\cdot\bar{a}_z = \int_v dE\cos\alpha = \frac{\rho_v}{4\pi\varepsilon_0}\int_v \frac{dv\cos\alpha}{R^2}$$

$$dv = dr'\, r'd\theta'\, r'\sin\theta' d\phi'$$

$$\Rightarrow dv = (r')^2\sin\theta'\, dr' d\theta' d\phi'$$

Trigonometric Cosine Rule

$$R^2 = z^2 + (r')^2 - 2zr'\cos\theta'$$

and $(r')^2 = z^2 + R^2 - 2zR\cos\alpha$

$$\cos\theta' = \frac{z^2 + (r')^2 - R^2}{2zr'} \qquad \cos\alpha = \frac{z^2 + R^2 - (r')^2}{2zR}$$

θ' varies from 0 to π

R varies from $(z - r')$ to $(z + r')$ ⇒ P outside sphere

$$E_z = \frac{\rho_v}{4\pi\varepsilon_0}\int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} (r')^2 \frac{R\,dR}{zr'}\,dr'\,\frac{z^2+R^2-(r')^2}{2zR}\frac{1}{R^2}$$

$$E_z = \frac{1}{4\pi\varepsilon_0}\frac{1}{z^2}\left\{\rho_v \frac{4}{3}\pi a^3\right\}$$

$$\because Q = \rho_v \frac{4}{3}\pi a^3$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 z^2}\vec{a}_z$$

$$E_z = \frac{\rho_v}{4\pi\varepsilon_0}\int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} (r')^2 \frac{R\,dR}{2r'}\, dr' \, \frac{z^2+R^2-(r')^2}{2zR}$$

$$= \frac{\rho_v (2\pi)}{8\pi\varepsilon_0 z^2}\int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r' \left\{1+\frac{z^2-(r')^2}{R^2}\right\} dR\, dr'$$

$$= \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2}\int_{r'=0}^{a} r' \left[R - \frac{(z^2-(r')^2)}{R}\right]_{z-r'}^{z+r'} dr'$$

$$= \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2}\int_{r'=0}^{a} 4(r')^2 dr' = \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2}\,\frac{4a^3}{3}$$

Re-arranging

$$E_z = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{z^2}\left\{\frac{4}{3}\pi a^3 \rho_v\right\}$$

Since $Q = \rho_v \frac{4}{3}\pi a^3$

$$\boxed{\vec{E} = \frac{Q}{4\pi\varepsilon_0 z^2}\,\bar{a}_z} \quad \text{for } P(0,0,z)$$

For $P(r,\theta,\phi)$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\vec{a}_r$$

It is identical to $\vec{E}$ at some point due to point charge Q

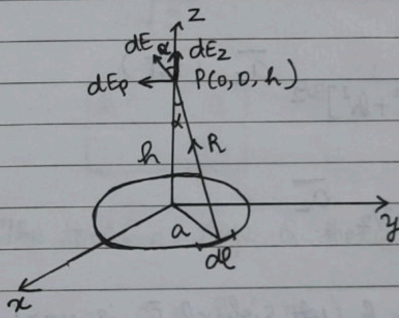
14/8/23

Q. A circular ring of radius 'a' carries a uniform charge ρ_L C/m and is placed on the XY plane with the axis same as z-axis.

a) Show that electric field intensity $\vec{E}(0,0,h) = \frac{\rho_L a h}{2\varepsilon_0 (h^2+a^2)^{3/2}}\bar{a}_z$ V/m

b) what are the values of 'h' for which $\bar{E}$ is max?

c) If the total charge on the ring is Q, find $\bar{E}$ as $a \to 0$



a) General form $\bar{E} = \int \frac{\rho_L\, dl}{4\pi\varepsilon_0 R^2}\bar{a}_R$

$dl = a\,d\phi$ (a → radius) (cartesian system, but lying on a circle)

$\bar{R} = -a\bar{a}_\rho + h\bar{a}_z$

$|\bar{R}|^2 = R = (a^2+h^2)^{1/2}$

$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{\bar{R}}{R}$

$\frac{\bar{a}_R}{R^2} = \frac{\bar{R}}{R^3} = \frac{-a\bar{a}_\rho + h\bar{a}_z}{(a^2+h^2)^{3/2}}$

$$\therefore \bar{E} = \frac{\rho_L}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi} \frac{(-a\bar{a}_\rho + h\bar{a}_z)(a\,d\phi)}{(a^2+h^2)^{3/2}}$$

NOTE: By symmetry, the contributions along $\bar{a}_\rho$ add up to zero.

This is evident from the fact that for every element dl, there is a corresponding element diametrically opposite that gives an equal but opposite dE_ρ so that the two contributions cancel each other.
Thus, we are left with z- component only.

Hence

$$\bar{E} = \frac{\rho_L\, ha}{4\pi\varepsilon_0[a^2+h^2]^{3/2}}\,\bar{a_z}\int_0^{2\pi} d\phi$$

$$= \frac{\rho_L\, ha}{4\pi\varepsilon_0[a^2+h^2]^{3/2}}\,\bar{a_z}\,(2\pi)$$

$$\bar{E} = \frac{\rho_L\, ha}{2\varepsilon_0(a^2+h^2)^{3/2}}\,\bar{a_z}$$

b) To get the values of h for which $\bar{E}$ is maximum, we need to differentiate the magnitude of $\bar{E}$ ($|\bar{E}|$) w.r.t h

$$|\bar{E}| = \frac{\rho_L\, ha}{2\varepsilon_0(a^2+h^2)^{3/2}}$$

$$\frac{d|\bar{E}|}{dh} = \frac{d}{dh}\left[\frac{\rho_L\, h\, a}{2\varepsilon_0(a^2+h^2)^{3/2}}\right]$$

$$= \frac{\rho_L\, a}{2\varepsilon_0}\left[\frac{3/2\,(a^2+h^2)^{1/2}(2h)(h)-(a^2+h^2)^{3/2}(1)}{(a^2+h^2)^{3/2\times 2}}\right]$$

$$\frac{d|\bar{E}|}{dh} = \frac{\rho_L\, a}{2\varepsilon_0}\left[\frac{3h^2(a^2+h^2)^{1/2}-(a^2+h^2)^{3/2}}{(a^2+h^2)^3}\right]$$

$$\frac{d|\bar{E}|}{dh} = 0 = \frac{\rho_L\, a}{2\varepsilon_0}\left[\frac{3h^2(a^2+h^2)^{1/2}-(a^2+h^2)^{3/2}}{(a^2+h^2)^3}\right]$$

$$\rho_L\, a\left[3h^2(a^2+h^2)^{1/2}-(a^2+h^2)^{3/2}\right] = 0$$

$$3h^2(a^2+h^2)^{1/2} = (a^2+h^2)^{3/2}$$

$$3h^2 = a^2+h^2$$

$$2h^2 = a^2$$

$$h = \sqrt{\frac{a^2}{2}}$$

$$\therefore\ \boxed{h = \pm\frac{a}{\sqrt{2}}}$$

c) The total charge Q held by the ring

$$Q = \rho_L(2\pi a)$$

$$\rho_L = \frac{Q}{2\pi a}$$

Substituting for ρ_L in the expression for $\bar{E}$

$$\bar{E} = \frac{Q}{2\pi a}\,\frac{a\,h}{2\varepsilon_0(a^2+h^2)^{3/2}}\,\bar{a_z}$$

$$\boxed{\bar{E} = \frac{Q\,h}{4\pi\varepsilon_0(a^2+h^2)^{3/2}}\,\bar{a_z}}$$

As $a \to 0$, the expression for $\bar{E}$ reduces to

$$\boxed{\bar{E} = \frac{Q}{4\pi\varepsilon_0 h^2}\,\bar{a_z}}$$

This expression is similar to the one for $\bar{E}$ having a charge Q at a distance r

HW Q2. A circular disc of radius 'a' is uniformly charged with charge density ρ_s C/m^2. The disc lies on $z=0$ plane, with its axis along the z-axis.

a) Show that at a point $(0,0,h)$ with $h \neq 0$
$$\bar{E} = \frac{\rho_s}{2\varepsilon_0}\left\{1 - \frac{h}{(h^2+a^2)^{1/2}}\right\}\bar{a}_z$$

b) From the above result, derive the $\bar{E}$ field due to an infinite sheet of charge on the $z=0$ plane.

c) If $a \ll h$, show that $\bar{E}$ is similar to the field due to a point charge.

Q3. A finite sheet with $0 \le x \le 1$, $0 \le y \le 1$ on $z=0$ plane has a charge density $\rho_s = xy(x^2+y^2+25)^{3/2}$ $\frac{nC}{m^2}$

a) Evaluate the total charge on the sheet.
b) Evaluate the electric field intensity at $(0,0,5)$.
c) The force experienced by a ~~-5~~ $-1mC$ charge located at $(0,0,5)$.

a) $Q = \int_S \rho_s \, dS$

$$Q = \int_0^1\int_0^1 xy(x^2+y^2+25)^{3/2}\,dx\,dy \quad nC$$

~~$u = xy$~~ ~~$dv = (x^2+y^2+25)^{3/2}$~~

~~$v = \frac{(x^2+y^2+25)^{5/2}}{5/2}$~~

$u = (x^2+y^2+25)^{3/2}$ $\qquad dv = xy$

$du = \frac{3}{2}(x^2+y^2+25)^{1/2}(2x)$ $\qquad v = \frac{x^2}{2}y$

$du = 3x(x^2+y^2+25)^{1/2}$

$$Q = \int_0^1\left[\frac{x^2}{2}y(x^2+y^2+25)^{3/2}\Big|_0^1 - \int_0^1 \frac{x^2}{2}y\left(3x(x^2+y^2+25)^{1/2}\right)dx\right]dy$$

$x\,dx = \frac{1}{2}d(x^2)$ $\qquad y\,dy = \frac{1}{2}d(y^2)$

$$Q = \frac{1}{2}\int_0^1 y\,dy\int_0^1 (x^2+y^2+25)^{3/2}\,d(x^2) \quad nC$$

$$= \frac{1}{2}\int_0^1 y\,dy\left[\frac{(x^2+y^2+25)^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_0^1$$

$$= \frac{1}{2}\int_0^1 y\,dy\left[\frac{(y^2+26)^{5/2}}{5/2} - \frac{(y^2+25)^{5/2}}{5/2}\right]$$

$$= \frac{2}{5}\times\frac{1}{2}\int_0^1 y\,dy\left[(y^2+26)^{5/2} - (y^2+25)^{5/2}\right]$$

$$= \frac{1}{5}\int_0^1\left[y(y^2+26)^{5/2} - y(y^2+25)^{5/2}\right]dy$$

$y\,dy = \frac{1}{2}d(y^2)$

$$Q = \frac{1}{10}\left[\int_0^1 (y^2+26)^{5/2}\,d(y^2) - \int_0^1 (y^2+25)^{5/2}\,d(y^2)\right]$$

$$= \frac{1}{10}\left[\frac{(y^2+26)^{5/2+1}}{\frac{5}{2}+1} - \frac{(y^2+25)^{\frac{5}{2}+1}}{\frac{5}{2}+1}\right]_0^1$$

$$= \frac{1}{10}\times\frac{2}{7}\left[(y^2+26)^{7/2} - (y^2+25)^{7/2}\right]_0^1$$

$$= \frac{1}{35}\left[\left\{(27)^{7/2} - (26)^{7/2}\right\} - \left\{(26)^{7/2} - (25)^{7/2}\right\}\right]$$

$$= \frac{1}{35}\left[(27)^{7/2} - 2(26)^{7/2} + (25)^{7/2}\right]$$

$$\boxed{Q = 33.1467\,nC}$$

b) $\bar{E} = \int_S \frac{\rho_s \, dS}{4\pi\varepsilon_0 R^2}\bar{a}_R = \int_S \frac{\rho_s \, dS}{4\pi\varepsilon_0}\frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}$

$dS = dx\,dy$

$\bar{r}-\bar{r}' = (0,0,5)-(x,y,0) = (-x,-y,5)$

$= -x\bar{a}_x - y\bar{a}_y + 5\bar{a}_z$

$$\bar{E} = \int_0^1\int_0^1 \frac{xy(x^2+y^2+25)^{3/2}\times 10^{-9})(-x\bar{a}_x - y\bar{a}_y + 5\bar{a}_z)\,dxdy}{4\pi\cdot\frac{10^{-9}}{36\pi}(x^2+y^2+25)^{3/2}}$$

$$= 9\int_0^1\int_0^1 xy(-x\bar{a}_x - y\bar{a}_y + 5\bar{a}_z)\,dxdy$$

$$= 9\left\{-\left[\int_0^1 x^2dx\int_0^1 y\,dy\right]\bar{a}_x - \left[\int_0^1 x\,dx\int_0^1 y^2dy\right]\bar{a}_y + 5\left[\int_0^1 x\,dx\int_0^1 y\,dy\right]\bar{a}_z\right\}$$

$$= 9\left\{-\left[\frac{x^3}{3}\Big|_0^1 \frac{y^2}{2}\Big|_0^1\right]\bar{a}_x - \left[\frac{x^2}{2}\Big|_0^1 \frac{y^3}{3}\Big|_0^1\right]\bar{a}_y + 5\left[\frac{x^2}{2}\Big|_0^1 \frac{y^2}{2}\Big|_0^1\right]\bar{a}_z\right\}$$

$$= 9\left\{-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\bar{a}_x - \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\bar{a}_y + 5\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\bar{a}_z\right\}$$

$$= 9\left\{\frac{-1}{6}\bar{a}_x - \frac{1}{6}\bar{a}_y + \frac{5}{4}\bar{a}_z\right\}$$

$$\bar{E} = \left(\frac{-3}{2}\bar{a}_x - \frac{3}{2}\bar{a}_y + \frac{45}{4}\bar{a}_z\right)\frac{V}{m} = (-1.5\bar{a}_x - 1.5\bar{a}_y + 11.25\bar{a}_z)\ V/m$$

c)

$\bar{F} = \bar{E}q = (-1.5\bar{a}_x - 1.5\bar{a}_y + 11.25\bar{a}_z)(-1m)$

$\bar{F} = (1.5\bar{a}_x + 1.5\bar{a}_y - 11.25\bar{a}_z)$ mN

Imp

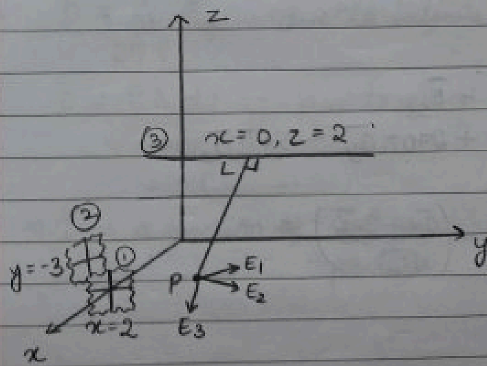
Q4. A square plate defined by $-2 \le x \le 2$ & $-2 \le y \le 2$ and $z = 0$ carries a charge $12|y|$ mC/m². Find the ~~$\bar{E}$ at (0,0,~~ total charge on the plate and $\bar{E}$ at (0, 0, 10)

Q = 192 mC

$\bar{E} = 16.6\,\bar{a}_z$ MV/m or $17.28\,\bar{a}_z$ MV/m

Q5. Planes $x = 2$ and $y = -3$ respectively carry charges 10 nC/m² and 15 nC/m². If the line $x = 0$ and $z = 2$ carries a charge 10π nC/m, calculate $\bar{E}$ at (1, 1, −1) due to these three charge distributions.

Let $\bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3$, where $\bar{E}_1, \bar{E}_2, \bar{E}_3$ are respectively the contributions to the $\bar{E}$ at the field point (1, 1, −1) due to the infinite sheet ① ($x = 2$), ~~and~~ infinite sheet ② ($y = -3$) and the infinite line charge ③ ($x = 0$, $z = 2$)



$\bar{E}_L = \frac{\rho_L}{2\pi\varepsilon_0 \rho} \bar{a}_\rho$ $\bar{E}_S = \frac{\rho_S}{2\varepsilon_0} \bar{a}_n$

$$\bar{E}_1 = \frac{\rho_S}{2\varepsilon_0}(-\bar{a}_x) = \frac{10\times10^{-9}}{2\frac{10^{-9}}{36\pi}} = -180\pi\,\bar{a}_x \ \frac{V}{m}$$

$$\bar{E}_2 = \frac{\rho_S}{2\varepsilon_0}(+\bar{a}_y) = \frac{15\times10^{-9}}{2\times\frac{10^{-9}}{36\pi}}\bar{a}_y = 270\pi\,\bar{a}_y \ V/m$$

$$\bar{E}_3 = \frac{\rho_L}{2\pi\varepsilon_0\rho}\bar{a}_\rho = \frac{\rho_L}{2\pi\varepsilon_0\rho}\frac{\bar{R}}{|\bar{R}|}$$

NOTE: In this case, $\bar{a}_\rho$ has a different meaning. It is to be taken as a unit vector along LP, perpendicular to the line charge and ρ is the length of LP. If we choose the point L on the line charge to be (0, 1, 2), this joining of L and P, where P is (1,1,-1) will result in the line LP being perpendicular to line 3.

$$\bar{R} = (1-0)\bar{a}_x + (1-1)\bar{a}_y + (-1-2)\bar{a}_z$$

$$\bar{R} = 1\bar{a}_x - 3\bar{a}_z \qquad |\bar{R}| = \sqrt{1+9} = \sqrt{10}$$

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$$\bar{E} = \bar{E}_{S_1} + \bar{E}_{S_2} + \bar{E}_{S_3}$$
$$= -162\pi\,\bar{a}_x + 270\pi\,\bar{a}_y - 54\pi\,\bar{a}_z$$

$$\bar{E}_3 = \frac{10\pi\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}\times\sqrt{10}}\times\left(\frac{\bar{a}_x - 3\bar{a}_z}{\sqrt{10}}\right) =$$

18

HW

Q6. In the previous question, line $x=0$, $z=2$ is rotated by $90°$ about the point (0, 2, 2) so that it becomes $x=0, y=2$. Find $\bar{E}$ at (1, 1, -1).

Electric Flux Density

Flux due to an electric field $\bar{E}$

$\psi = \int_S \bar{E}\cdot d\bar{s} \Rightarrow$ scalar

in coulombs

$\bar{E}$ is dependent on the medium in which the charge is placed.

Let $\bar{D} = \varepsilon_0 \bar{E}$ ELECTRIC FLUX DENSITY (C/m^2)
ELECTRIC DISPLACEMENT

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

$\Rightarrow \bar{D} = \frac{Q}{4\pi R^2}\bar{a}_R$ due to a point charge

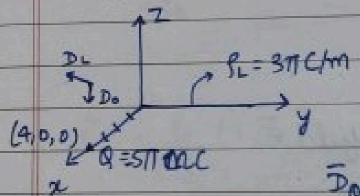
$\bar{D} = \frac{\rho_S}{2}\bar{a}_n$ due to infinite surface charge

$\bar{D} = \frac{\rho_L}{2\pi\rho}\bar{a}_\rho$ due to infinite line charge

$\bar{D} = \int_V \frac{\rho_v\,dv}{4\pi R^2}\bar{a}_R$ due to volume charge distribution

$\Rightarrow \bar{D}$ is a function of - charge (distribution)
- position

Q1. Evaluate $\bar{D}$ at $(4,0,3)$ if there is a point charge -5π mC at $(4,0,0)$ and a line charge 3π mC/m along y-axis.



$$\bar{D}_Q = \frac{Q}{4\pi R^2}\bar{a}_R$$

$$\bar{D} = \bar{D}_Q + \bar{D}_L$$

$$\bar{D}_Q = \frac{Q}{4\pi}\frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}$$

$$\bar{r}-\bar{r}' = (4,0,3)-(4,0,0) = (0,0,3)$$
$$|\bar{r}-\bar{r}'| = \sqrt{0^2+0^2+3^2} = \sqrt{9} = 3$$
$$|\bar{r}-\bar{r}'|^3 = 3^3 = 27$$

$$\bar{D}_Q = \frac{Q}{4\pi}\frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3} = \frac{-5\pi\times10^{-3}}{4\pi}\times\frac{3}{27}\bar{a}_z$$

$$\bar{D}_Q = -0.1388\,\bar{a}_z \text{ mC/m}^2 \quad — ①$$

In this case, the field point $(4,0,3)$ is in the x-z plane with $y=0$. A line drawn from any point in this plane to the origin is perpendicular to y-axis.

The given infinite line charge (source) lies on the Y-axis.

∴ Using the concept of displacement vector

$$\bar{a}_\rho = \frac{(4,0,3)-(0,0,0)}{|(4,0,3)-(0,0,0)|} = \frac{1}{5}(4,0,3)$$

$$\rho = |(4,0,3)-(0,0,0)| = \sqrt{4^2+0^2+3^2} = 5$$

$$\bar{D}_L = \frac{3\pi\times10^{-3}}{2\pi\times5}\left(\frac{1}{5}\right)(4\bar{a}_x+3\bar{a}_z)$$

$$\bar{D}_L = 0.24\,\bar{a}_x + 0.18\,\bar{a}_z \text{ mC/m}^2 \quad — ②$$

$$\bar{D} = \bar{D}_Q + \bar{D}_L \qquad ①+②$$

$$\boxed{\bar{D} = 0.24\bar{a}_x + 0.0412\,\bar{a}_z \text{ mC/m}^2}$$

HW

Q2. A point charge Q of magnitude 30nC is located at the origin, while a plane $y=3$ carries a charge of 10 nC/m^2. Find the electric flux density $\bar{D}$ at $(0,4,3)$.

Divergence

CARTESIAN $\quad \nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

CYLINDRICAL $\quad \nabla\cdot\bar{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$

SPHERICAL $\quad \nabla\cdot\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

PROPERTIES

→ Produces a scalar field due to dot product
⇒ $\nabla\cdot(\bar{A}+\bar{B}) = \nabla\cdot A + \nabla\cdot B$
⇒ $\nabla\cdot(V\bar{A}) = V\nabla\cdot\bar{A} + \bar{A}\cdot\nabla V$

$$\nabla \cdot \bar{A} = \lim_{\Delta V \to 0} \oint_S \frac{\bar{A} \cdot d\bar{s}}{\Delta V}$$

For symmetrical charge distribution

DIVERGENCE / GAUSS-OSTROGRADSKY THEOREM

$$\oint_S \bar{A} \cdot d\bar{s} = \oint_V \nabla \cdot \bar{A}\, dv$$

The total outward flux of a vector field $\bar{A}$ through the closed surface $\bar{S}$ is the same as the volume integral of the divergence of $\bar{A}$.

GAUSS' LAW The total ~~out~~ electric flux Ψ through any closed surface is equal to the total charge enclosed by the surface.

$$\Psi = Q_{enclosed}$$

LHS

$$\Psi = \oint_S d\Psi = \oint_S \bar{D} \cdot d\bar{s}$$

RHS

$$Q = \int_V \rho_v \, dV$$

$$Q = \int_S \bar{D} \cdot d\bar{s} = \int_V \rho_v \, dV = \Psi \longrightarrow (1)$$

Integral form of Gauss Law

Applying Divergence Theorem

$$\oint_S \bar{D} \cdot d\bar{s} = \int_V \nabla \cdot \bar{D} \, dv \longrightarrow (2)$$

Comparing (1) & (2)

$$\rho_v = \nabla \cdot \bar{D}$$

Point/Differential form of Gauss' Law

MAXWELL'S FIRST EQUATION

Volume charge density is the same as divergence of electric flux density

A continuous charge distribution has

⇒ Rectangular Symmetry if it depends only on x (or y or z)

⇒ Cylindrical Symmetry if it depends only on ρ

⇒ Spherical symmetry if it depends on r (independent of θ & ϕ)

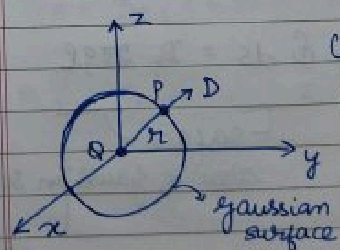
* Cannot make use of Gauss' Law to determine $\bar{E}$ or $\bar{D}$ if charge distribution is not symmetric
Use Coulomb's Law instead

Preliminaries

- Check for symmetry
- If exists, construct closed gaussian surface
- Choose $\bar{D}$ such that it is normal or tangential to the surface
- $\bar{D} \cdot d\bar{s} = 0$ if D is tangential to surface
  $= |\bar{D} d\bar{s}|$ if D is normal to surface

Applications

POINT CHARGE



Consider a point charge Q located at the origin

$\bar{D}$ is normal everywhere in the sphere

$$\bar{D} = D_r \, \bar{a}_r$$

Gauss' Law $\displaystyle Q = \oint_S \bar{D} \cdot d\bar{s} = D_r \oint ds = D_r \underbrace{(4\pi r^2)}_{\text{Surface area of sphere}}$

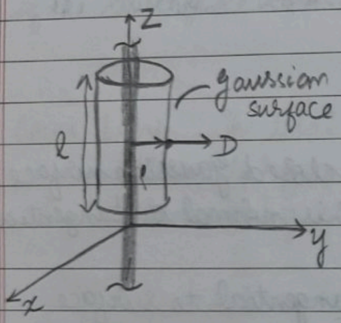
$$D_r = \frac{Q}{4\pi r^2}$$

$$\vec{D} = \frac{Q}{4\pi r^2}\vec{a_r}$$ → Proved using Coulomb's Law

$$\bar{E} = \frac{\bar{D}}{\varepsilon_0}$$

$$\Rightarrow \bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\bar{a_r}$$

INFINITE LINE CHARGE



Suppose an infinite line of uniform charge ~~the~~ ρ_L C/m lies on the z-axis

$\bar{D}$ is constant on & normal to the cylindrical gaussian surface

$$\bar{D} = D_\rho \bar{a}_\rho$$

Applying Gauss' Law to any arbitrary length l of the line

$$\rho_L l = Q = \oint_S \bar{D}\cdot d\bar{s} = D_\rho \underbrace{\oint_S ds}_{\text{surface}} = D_\rho \underbrace{2\pi\rho l}_{\text{area of gaussian surface}}$$

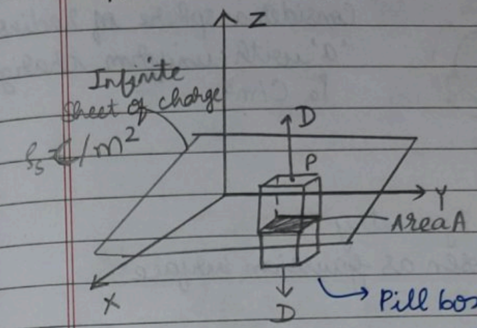
$\oint_S \bar{D}\cdot d\bar{s}$ evaluated at top & bottom $= 0$ $\because$ $\bar{D}$ has no z-component ($\bar{D}$ is tangential to these surfaces)

$$\rho_L l = D_\rho 2\pi\rho l \Rightarrow D_\rho = \frac{\rho_L l}{2\pi\rho l} = \frac{\rho_L}{2\pi\rho}$$

$$\boxed{\bar{D} = \frac{\rho_L}{2\pi\rho}\bar{a_\rho}}$$

21/8/23

Applications of Gauss' Law ⇒ **INFINITE SHEET CHARGE**



↳ Pill box ⇒ cut symmetrically by sheet charge & has 2 faces

Consider an infinite sheet of uniform charge ρ_s C/m² lying on $z=0$ plane

$\bar{D}$ is normal to the sheet

$\bar{D} = D_z \bar{a_z}$ in the z-direction

Gauss' Law

$$\rho_s \int_S ds = Q = \oint_S \bar{D}\cdot d\bar{s} = D_z\left\{\int_{top} ds + \int_{bottom} ds\right\}$$

$\bar{D}$ has no component along a_x & a_y
$\bar{D}$ is tangential to the sides of the box
} $\bar{D}\cdot d\bar{s}$ evaluated on sides of box $= 0$

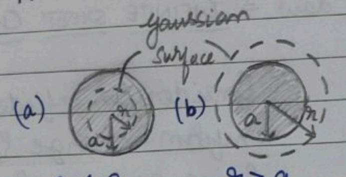
Bottom & top surfaces → area A each

$$\rho_s A = D_z(A+A) = 2D_z A \text{ or } D = \frac{\rho_s}{2}$$

$$\Rightarrow \vec{D} = D_z\vec{a_z} = \frac{\rho_s}{2}\vec{a_z}$$

$$\boxed{\vec{E} = \frac{\rho_s}{2\varepsilon_0}\vec{a_z}}$$

Applications of Gauss' Law - UNIFORMLY CHARGED SPHERE



Consider a sphere of radius 'a' with uniform charge $\rho_0 \; C/m^3$

Charge has spherical symmetry
↳ Spherical surface chosen as Gaussian surface

(a) $r \le a$ Total charge enclosed by spherical surface of radius r:

$$Q_{enc} = \int_V \rho_0 \, dV = \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$Q_{enc} = \rho_0 \frac{4}{3} \pi r^3$$

$$\Psi = \oint_S \vec{D} \cdot d\vec{S} = D_r \oint_S dS = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$\Psi = D_r \, 4\pi r^2$$

$$\Psi = Q_{enc}$$

$$D_r \, 4\pi r^2 = \rho_0 \frac{4}{3} \pi r^3$$

$$\Rightarrow D_r = \frac{r}{3} \rho_0$$

As $\vec{D} = D_r \vec{a}_r$

$$\vec{D} = \frac{r}{3} \rho_0 \vec{a}_r \quad \text{for} \quad 0 < r \le a$$

(b) $r \ge a$ The charge enclosed by the surface is the entire charge

$$Q_{enc} = \int_V \rho_0 \, dV = \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$Q_{enc} = \rho_0 \frac{4}{3} \pi a^3$$

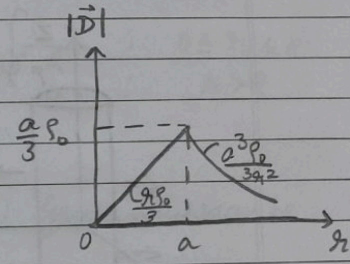
$$\Psi = D_r 4\pi r^2$$

$$\Psi = Q_{enc} \Rightarrow D_r 4\pi r^2 = \rho_0 \frac{4}{3} \pi a^3$$

$$D_r = \frac{a^3}{3r^2} \rho_0$$

$$\vec{D} = \frac{a^3}{3r^2} \rho_0 \vec{a}_r \quad \text{for} \quad r \ge a$$

$$\vec{D} = \begin{cases} \frac{r}{3} \rho_0 \vec{a}_r & \text{for } 0 < r \le a \\ \frac{a^3}{3r^2} \rho_0 \vec{a}_r & \text{for } r \ge a \end{cases}$$



Q1. Given that $\vec{D} = z\rho \cos^2\phi \, \bar{a}_z \; C/m^2$, calculate the charge density at $(1, \frac{\pi}{4}, 3)$ and the total charge enclosed by the cylinder of radius 1m with z varying from −2 to 2 m.

From point form of Gauss' Law

$$\rho_v = \nabla \cdot \vec{D} = \frac{\partial D_z}{\partial z} \qquad \because \frac{\partial D_\rho}{\partial \rho} = \frac{\partial D_\phi}{\partial \phi} = 0$$

$$\rho_v = \rho \cos^2\phi$$

At $(1, \frac{\pi}{4}, 3)$ $\quad \rho_v = 1 \left(\cos\frac{\pi}{4}\right)^2 = 1 \times \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

$$\boxed{\rho_v = 0.5 \; C/m^3}$$

METHOD 1 : Based on the definition of volume charge

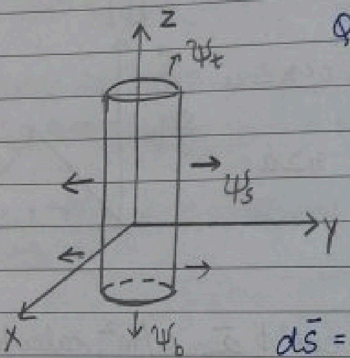
$$Q = \int_V \rho_v \, dv = \int_V \rho \cos^2\phi \, (d\rho)(\rho \, d\phi)(dz)$$

$$= \int_{z=-2}^{2} dz \int_{\phi=0}^{2\pi} \cos^2\phi \, d\phi \int_{\rho=0}^{1} \rho^2 \, d\rho = z\Big|_{-2}^{2} \int_{\phi=0}^{2\pi} \frac{\cos 2\phi - 1}{2} d\phi \; \frac{\rho^3}{3}\Big|_0^1$$

$$= \cancel{z\Big|_{-2}^{2} \int_{\phi=0}^{2\pi} \frac{\cos - \sin(\phi/2)}{2} d\phi \; \frac{\rho^3}{3}\Big|_0^1} =$$

$$\boxed{Q = \frac{4}{3}\pi \ C}$$

METHOD 2: Using Gauss' Law



$$Q = \Psi = \oint_S \vec{D} \cdot d\vec{s}$$

$$= \Psi_s + \Psi_t + \Psi_b$$

$\Psi_s = 0 \quad \because \vec{D}$ is tangential to the surfaces

$$d\bar{s} = d\rho \, \rho \, d\phi \, \vec{a}_z$$

$$\bar{D} \cdot d\bar{s} = (z\rho\cos^2\phi)(d\rho \, \rho \, d\phi)$$

$$= z\rho^2\cos^2\phi \, d\phi \, d\rho$$

Top $\quad \Psi_t = \int_{\rho=0}^{1}\int_{\phi=0}^{2\pi} z\rho^2\cos^2\phi \, d\phi \, d\rho \Big|_{z=2}$

$$= 2\int_{\rho=0}^{1} \rho^2 d\rho \int_{\phi=0}^{2\pi} \cos^2\phi \, d\phi$$

$$= \frac{2}{3}\rho^3\Big|_0^1 \pi \Rightarrow \Psi_t = \frac{2\pi}{3} \ C$$

Bottom $\quad \Psi_b \Rightarrow d\bar{s} = -\rho \, d\phi \, d\rho \, \bar{a}_z$ (-ve sign)

$$\Psi_b = -\int_{\rho=0}^{1}\int_{\phi=0}^{2\pi} (z\rho\cos^2\phi)(\rho \, d\rho \, d\phi)\Big|_{z=-2}$$

$$= -(-2)\int_0^1 \rho^2 d\rho \int_{\phi=0}^{2\pi} \cos^2\phi \, d\phi$$

$$\Psi_b = \frac{2\pi}{3} \ C$$

$$Q = \Psi = \Psi_t + \Psi_b = \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3} \ C$$

Q2. A charged distribution with spherical symmetry has density as defined by

$$\rho_v = \begin{cases} \frac{\rho_0 r}{R} & ; \ 0 \le r \le R \\ 0 & ; \ r > R \end{cases}$$

Determine $\bar{E}$ everywhere.

Since a symmetry exists (sphere), Gauss' Law can be applied to determine $\bar{E}$.

$$\varepsilon_0 \oint_S \bar{E} \cdot d\bar{s} = Q_{enc} = \int_V \rho_v \, dV \qquad \because \bar{D} = \varepsilon_0 \bar{E}$$

For $r < R$

$$\varepsilon_0 E_r \, 4\pi r^2 = Q_{enc} = \int_{r=0}^{r}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \rho_v \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{\rho_0}{R}\int_{r=0}^{r} r^3 dr \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{\rho_0}{R} \frac{r^4}{4} (2)(2\pi)$$

$$\varepsilon_0 E_r \, 4\pi r^2 = \frac{\rho_0 r^4 \pi}{R}$$

$$E_r = \frac{\rho_0 r^2}{4\varepsilon_0 R}$$

$$\bar{E} = E_r \bar{a}_r = \frac{\rho_0 r^2}{4\varepsilon_0 R} \bar{a}_r \quad V/m$$

For $r > R$

$$\varepsilon_0 E_r \, 4\pi r^2 = Q_{enc} = \int_0^R \frac{\rho_0 r}{R}(4\pi r^2) \, dr + \int_R^r (0) \, 4\pi r^2 dr$$

$$E_r = \frac{\rho_0 R^4}{4\varepsilon_0 \pi r^2} \times \frac{4\pi}{R}$$

$$\varepsilon_0 E_r 4\pi r^2 = \frac{\rho_0}{R} \times 4\pi \int_0^R r^3\, dr = \frac{\rho_0}{R} \times 4\pi \times \frac{R^4}{4}$$

$$\varepsilon_0 E_r 4\pi r^2 = \rho_0 \pi R^3$$

$$E_r = \frac{\rho_0 R^3}{4 r^2 \varepsilon_0}$$

$$\bar{E} = E_r \bar{a}_r = \frac{\rho_0 R^3}{4 \varepsilon_0 r^2} \bar{a}_r \quad \text{V/m}$$

HW

Q3. A charge distribution in free space has $\rho_v = 2r$ nC/m^3 with r varying from 0 to 10 m and 0 elsewhere. Determine (i) $\bar{E}$ ($r = 2$m) (ii) $\bar{E}$ ($r = 12$m)

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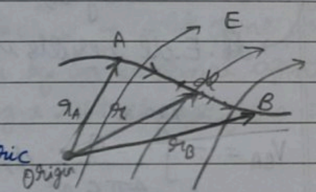
8 × 1-marks
2 × 2-marks
2 × 4-marks

ELECTRIC POTENTIAL

$\bar{E}$ can be evaluated
- → Using COULOMB'S LAW ⇒ charge distribution is known
- → Using GAUSS' LAW ⇒ charge distribution is symmetric

can also be ascertained from ↓

Scalar Electric Potential V



Point charge Q is to be moved from point A to B in an electric field $\bar{E}$

Force on Q $\quad \bar{F} = Q\bar{E}$ (from Coulomb's law)

Work done in displacing charge by $d\bar{l}$

$\to dW = -\bar{F}\cdot d\bar{l} = -Q\bar{E}\cdot d\bar{l}$

⇓ work is being done by an external agent

Total work done $\quad W = -Q\int_A^B \bar{E}\cdot d\bar{l}$ → Scalar

(Potential energy required in moving charge from A to B)

$\Rightarrow V_{BA} = \frac{W}{Q} = -\int_A^B \bar{E}\cdot d\bar{l}$ Potential difference between points A & B (A → initial, B → final)

V_{BA}
- → negative: Loss in potential energy (Work done by field)
- → positive: Gain in potential energy (external agent performs work)

↓
- path independent
- $\frac{\text{Joules}}{\text{Coulombs}}$ = Volts

$\bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r$ $\bar{E}$ due to point charge Q at origin

$$V_{BA} = -\int_A^B \bar{E}.d\bar{l} = -\int_A^B \frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r . dr\bar{a}_r$$

$d\bar{l} = dr\bar{a}_r + rd\theta\,\bar{a}_\theta + r\sin\theta\, d\phi\,\bar{a}_\phi$

$\bar{E}$ has only $\bar{a}_r$

$\therefore$ $\bar{E}.d\bar{l}$ results in only $\bar{a}_r$ component

$$V_{BA} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 r^2}dr = \frac{-Q}{4\pi\varepsilon_0}\int_{r_A}^{r_B}\frac{1}{r^2}dr$$

$$V_{BA} = \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{r_B} - \frac{1}{r_A}\right\} = \frac{Q}{4\pi\varepsilon_0 r_B} - \frac{Q}{4\pi\varepsilon_0 r_A} = V_B - V_A$$

$\Downarrow$ potential at B with reference to A

$\Downarrow$ absolute potentials

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Point charge considered

→ Reference : infinity

Potential at infinity = 0

at $r_A \to \infty$ $V_A = 0$

Potential at any point $r_B \to r$ due to a point charge Q located at origin :

$$V = \frac{Q}{4\pi\varepsilon_0 r} \Rightarrow \text{Scalar} \quad \text{— (1)}$$

* CONSERVATIVE Vectors whose line integral does not depend on path of integration

$\therefore$ $\bar{E}$ is conservative

ELECTRIC POTENTIAL : The potential at any point is the potential difference b/w that point and a chosen point (reference point) at which potential is zero

$$V = -\int_\infty^r \bar{E}.d\bar{l}$$

$V(\infty) = 0$ assumed

Potential at a distance 'r' from point charge ⇒ work done per unit charge by an external agent in transferring a test charge from ∞ to that point

$$V(\bar{r}) = \frac{Q}{4\pi\varepsilon_0|\bar{r}-\bar{r}'|}$$

SUPERPOSITION PRINCIPLE

For 'n' points charges $Q_1, Q_2, \ldots, Q_n$ located at points with position vectors $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n$, the potential at a point P with a position vector $\bar{r}$

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{K=1}^{n}\frac{Q_K}{|\bar{r}-\bar{r}_K|} \quad \text{POINT CHARGES}$$

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L(\bar{r}')\,dl'}{|\bar{r}-\bar{r}'|} \quad \text{LINE CHARGE}$$

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_S(\bar{r}')\,ds'}{|\bar{r}-\bar{r}'|} \quad \text{SURFACE CHARGE}$$

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_V(\bar{r}')\,dv'}{|\bar{r}-\bar{r}'|} \quad \text{VOLUME CHARGE}$$

Prime (') ⇒ source point

Arbitrary point chosen as reference (instead of ∞)

$$V = \frac{Q}{4\pi\varepsilon_0 r} + \underbrace{C}_{\text{constant}}$$

Potential at any point can be evaluated by

METHOD 1

Charge distribution is known

METHOD 2

$\bar{E}$ is known

$\Rightarrow V = -\int \bar{E}\cdot d\bar{l} + C$

$\Rightarrow V_{BA} = V_B - V_A = -\int_A^B \bar{E}\cdot d\bar{l}$

$= \frac{W}{Q}$

Q1. 2 point charges $-4\mu C$ and $+5\mu C$ are located at $(2,-1,3)$ and $(0,4,-2)$ respectively. Find the potential at $(1,0,1)$ assuming potential at infinity is zero.

By the Principle of Superposition,

$$V(\bar{r}) = \frac{Q_1}{4\pi\varepsilon_0|\bar{r}-\bar{r}_1|} + \frac{Q_2}{4\pi\varepsilon_0|\bar{r}-\bar{r}_2|}$$

$\bar{r}-\bar{r}_1 = (1,0,1)-(2,-1,3) = (-1,1,-2)$

$|\bar{r}-\bar{r}_1| = \sqrt{1+1+4} = \sqrt{6}$

$\bar{r}-\bar{r}_2 = (1,0,1)-(0,4,-2) = (1,-4,+3)$

$|\bar{r}-\bar{r}_2| = \sqrt{1+16+9} = \sqrt{26}$

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{-4}{\sqrt{6}} + \frac{5}{\sqrt{26}}\right]\mu V$$

$$V(\bar{r}) = \frac{1}{4\pi \frac{10^{-9}}{36\pi}} \times 10^{-6}\,[-0.65241]$$

$$\boxed{V(\bar{r}) = -5.872\ kV}$$

HW

Q2. If a point charge $3\mu C$ is located at origin in addition to the 2 charges given in Q1, find the potential at $(-1,5,2)$, assuming $V(\infty)=0$.

Q3. A point charge of $5nC$ is located at $(-3,4,0)$ while a line $y=1$, $z=1$ carries a uniform charge of $2nC/m$.

(a) If $V=0$ V at $O(0,0,0)$, find V at $A(5,0,1)$

(b) If $V=100$V at $B(1,2,1)$, find V at $C(-2,5,3)$

(c) If $V=-5$V at $O(0,0,0)$, find V_{CB}.

Let the potential V at any point be $\underbrace{V_Q}_{\text{potential due to the point charge}} + \underbrace{V_L}_{\text{potential due to the line charge}} = V$

$$V_Q = -\int \bar{E}\cdot d\bar{l} = -\int \frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r \cdot dr\,\bar{a}_r$$

$$= -\int \frac{Q}{4\pi\varepsilon_0 r^2}dr$$

$$V_Q = \frac{Q}{4\pi\varepsilon_0 r} + C_1$$

$$V_L = -\int \bar{E}\cdot d\bar{l} = -\int \frac{\rho_L}{2\varepsilon_0\pi\rho}\bar{a}_\rho \cdot d\rho\,\bar{a}_\rho$$

$$= -\int \frac{\rho_L}{2\pi\varepsilon_0\rho}d\rho$$

$$V_L = -\frac{\rho_L}{2\pi\varepsilon_0}\ln(\rho) + C_2$$

Hence, $V = V_Q + V_L = \frac{Q}{4\pi\varepsilon_0 r} + C_1 \frac{-\rho_L}{2\pi\varepsilon_0} \ln(\rho) + C_2$

$$V = \frac{1}{2\pi\varepsilon_0}\left[\frac{Q}{2r} - \rho_L \ln(\rho)\right] + C$$

From the general expression for V due to the point charge and line charge, ρ is the perpendicular distance from the line ($y=1, z=1 \Rightarrow$ parallel to x-axis) to the field point & r is the distance of point charge to field point.

a) If $V=0$ at $O(0,0,0)$ and V at $A(5,0,1)$ is to be determined, we need to evaluate the values of r and ρ at O and A.

Find r — Distance formula $d^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2$ — (1)

Find ρ — for any point (x,y,z), we utilize the fact that ρ is the perpendicular distance from (x,y,z) to the line charge.
Hence ρ is the distance between (x,y,z) and $(x,1,1)$

$$\rho = |(x,y,z) - (x,1,1)| = \sqrt{(y-1)^2 + (z-1)^2} \quad \text{— (2)}$$

Considering the points $O(0,0,0)$ & $A(5,0,1)$

$\rho_0 = |(0,0,0) - (0,1,1)| = \sqrt{2}$

~~$\rho_A =$~~

$r_0 = |(0,0,0) - (-3,4,0)| = 5$

$\rho_A = |(5,0,1) - (5,1,1)| = \cancel{\sqrt{26}}\ 1$

$r_A = |(5,0,1) - (-3,4,0)| = \sqrt{81} = 9$

Therefore

$$V_0 - V_A = \left\{\frac{-\rho_L}{2\pi\varepsilon_0}\ln(\rho_0) + \frac{Q}{4\pi\varepsilon_0 r_0} + C\right\} - \left\{\frac{-\rho_L}{2\pi\varepsilon_0}\ln(\rho_A) + \frac{Q}{4\pi\varepsilon_0 r_A} + C\right\}$$

$$= \frac{\rho_L}{2\pi\varepsilon_0}\left\{\ln(\rho_A) - \ln(\rho_0)\right\} + \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{r_0} - \frac{1}{r_A}\right\}$$

$$= \frac{2\times10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi}}\left\{\ln(1) - \ln(\sqrt{2})\right\} + \frac{Q}{4\pi \times \frac{10^{-9}}{36\pi}}\left\{\frac{1}{5} - \frac{1}{9}\right\}$$

$$= -12.476 + 9 \times \frac{4}{45} \times 5 = -12.476 + 4$$

$V_0 - V_A = -8.4766\,V$

$0 - V_A = -8.4766$

$\Rightarrow \boxed{V_A = 8.4766\,V}$

b) $V = 100V$ at $B(1,2,1)$
V at $C(-2,5,3) = ?$

$\rho_B = |(1,2,1) - (1,1,1)| = 1$

$r_B = |(1,2,1) - (-3,4,0)| = \sqrt{21}$

$\rho_C = |(-2,5,3) - (-2,1,1)| = \sqrt{20}$

$r_C = |(-2,5,3) - (-3,4,0)| = \sqrt{11}$

$$V_C - V_B = \frac{-\rho_L}{2\pi\varepsilon_0}\ln\left(\frac{\rho_C}{\rho_B}\right) + \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{r_C} - \frac{1}{r_B}\right\}$$

$$V_C - 100 = \frac{-2\times10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi}}\ln\left(\frac{\sqrt{20}}{1}\right) + \frac{5\times10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}}\left\{\frac{1}{\sqrt{11}} - \frac{1}{\sqrt{21}}\right\}$$

$V_C = -36 \times 1.49786 + 3.7482 + 100$

$= -53.9232 + 3.7482 + 100$

$\boxed{V_C = 49.825\ V}$

c) $V_{CB} = V_C - V_B = 49.825 - 100$

$\boxed{V_{CB} = -50.175\ V}$

HW

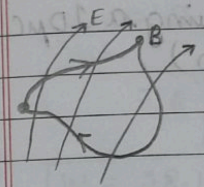
Q4. A point charge of 5nC is located at the origin. If V=2V at (0, 6, -8)

a) Find the potential at A(-3, 2, 6)
b) Find the potential at B(1, 5, 7)
c) V_{AB}

Relation between $\bar{E}$ and $\bar{V}$ ⇒ MAXWELL's EQUATION

$V_{BA} = -V_{AB}$ ⇒ $V_{BA} + V_{AB} = 0$

$\oint_L \bar{E} \cdot d\bar{l} = 0$ Closed line integral



↳ Line integral of $\bar{E}$ along any closed path = 0
i.e. no net work is done in moving a charge along a closed path in an electrostatic field

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Electric potential $V = -\int \bar{E} \cdot d\bar{l}$

$dV = -\bar{E} \cdot d\bar{l} = -E_x\,dx - E_y\,dy - E_z\,dz$ ——(i)

Calculus $dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$ ——(ii)

Comparing (i) & (ii) ⇒ $E_x = \frac{-\partial V}{\partial x}$ ⇒ $E_y = \frac{-\partial V}{\partial y}$

⇒ $E_z = \frac{-\partial V}{\partial z}$

$\bar{E} = -\nabla V$ Electric field is the gradient of V

↓ direction of $\bar{E}$ is opposite to the direction in which scalar field V increases

Q1. Given scalar potential $V = \frac{10}{r^2} \sin\theta \cos\phi$

(a) Find $\bar{D}$ at $\left(2, \frac{\pi}{2}, 0^\circ\right)$

(b) Compute the work done in moving a $10 \mu C$ from $A(1, 30^\circ, 120^\circ)$ to $B(4, 90^\circ, 60^\circ)$

(a) $\bar{D} = \varepsilon_0 \bar{E}$ $\qquad \bar{E} = -\nabla V$

$$\bar{E} = -\nabla V = -\left\{\frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi\right\}$$

$$= -\left\{\frac{-20}{r^3}\sin\theta\cos\phi\,\bar{a}_r + \frac{1}{r}\frac{10}{r^2}\cos\theta\cos\phi\,\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{10}{r^2}\sin\theta(-\sin\phi)\,\bar{a}_\phi\right\}$$

$$\bar{E} = \left(\frac{20}{r^3}\sin\theta\cos\phi\,\bar{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\bar{a}_\theta + \frac{10}{r^3}\sin\phi\,\bar{a}_\phi\right) \text{V/m}$$

$$\bar{D} = \frac{10^{-9}}{36\pi} \times \bar{E}$$

$$\bar{D}\left(2, \frac{\pi}{2}, 0^\circ\right) = \frac{10^{-9}}{36\pi}\left[\frac{20}{2^3}\underbrace{\sin\frac{\pi}{2}}_{=1}\underbrace{\cos 0^\circ}_{=1}\bar{a}_r - \frac{10}{2^3}\underbrace{\cos\frac{\pi}{2}}_{=0}\cos 0^\circ\,\bar{a}_\theta + \frac{10}{2^3}\underbrace{\sin 0^\circ}_{=0}\,\bar{a}_\phi\right]$$

$$= \varepsilon_0 \times \frac{20}{8}\bar{a}_r$$

$$= 2.5\,\varepsilon_0\,\bar{a}_r$$

$$\boxed{\bar{D} = 22.1\,\bar{a}_r\ \text{pC/m}^2}$$

(b) METHOD 1 Since V is known

$$W = -Q\int_A^B \bar{E}\cdot d\bar{l} = Q V_{BA} = Q(V_B - V_A)$$

$$= Q\left[\frac{10}{4^2}\sin 90^\circ \cos 60^\circ - \frac{10}{1^2}\sin 30^\circ \cos 120^\circ\right]$$

$$= Q\left[\frac{5}{16} - \left(\frac{-5}{2}\right)\right] = 10 \times 10^{-6} \times \frac{45}{16}$$

$$\boxed{W = 28.125\ \mu J}$$

METHOD 2 Since the electrostatic field is conservative, the path of integration $-\frac{W}{Q} = \int_L \bar{E}\cdot d\bar{l}$ is immaterial

$\therefore$ The work done in moving Q from $A(1, 30^\circ, 120^\circ)$ to $B(4, 90^\circ, 60^\circ)$ is the same as that moving from A to A', from A' to B', and finally from B' to B as shown :

$$\begin{array}{ccc}
A(1, 30^\circ, 120^\circ) & & B(4, 90^\circ, 60^\circ) \\
d\bar{l} = dr\,\bar{a}_r \downarrow & & \uparrow d\bar{l} = r\sin\theta\, d\phi\, \bar{a}_\phi \\
A'(4, 30^\circ, 120^\circ) & \xrightarrow[d\bar{l} = r\,d\theta\,\bar{a}_\theta]{} & B'(4, 90^\circ, 120^\circ)
\end{array}$$

Since the general expression for $\bar{E}$ is known:

$$-\frac{W}{Q} = \int_{r=1}^{4} \frac{20}{r^3}\sin\theta\cos\phi\, dr \Big|_{\theta=30^\circ,\ \phi=120^\circ}$$

$$+ \int_{\theta=30^\circ}^{90^\circ} \frac{-10}{r^3}\cos\theta\cos\phi\, r\, d\theta \Big|_{r=4,\ \phi=120^\circ}$$

$$+ \int_{\phi=120^\circ}^{60^\circ} \frac{10}{r^3}\sin\phi\, r\sin\theta\, d\phi \Big|_{r=4,\ \theta=90^\circ}$$

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$$-\frac{W}{Q} = 20 \sin 30^\circ \cos 120^\circ \left. \frac{r^{-3+1}}{-3+1} \right|_{r=1}^{4}$$

$$+ \left(\frac{-10}{4^2} \cos 120^\circ \right) \sin\theta \Big|_{\theta=30^\circ}^{90^\circ}$$

$$+ \frac{10}{4^2} \sin 90^\circ \left(-\cos\phi\right) \Big|_{\phi=120^\circ}^{60^\circ}$$

$$= \frac{-5}{-2}\left(\frac{1}{4^2} - \frac{1}{1^2}\right) + \frac{5}{16}\left(\sin 90^\circ - \sin 30^\circ\right)$$

$$+ \frac{5}{8}\left(\cos 120^\circ - \cos 60^\circ\right)$$

$$= \frac{5}{2} \times \left(\frac{-15}{16}\right) + \frac{5}{16}\left(\frac{1}{2}\right) + \frac{5}{8}\left(-1\right)$$

$$= -\frac{75}{32} + \frac{5}{32} - \frac{5}{8}$$

$$-\frac{W}{Q} = -\frac{45}{16}$$

$$\Rightarrow \boxed{W = 28.125\ \mu J}$$

HW

Q2. Given $\bar{E} = (3x^2 + y)\,\bar{a}_x + x\bar{a}_y$ kV/m, find the work done in moving a $-2\mu C$ charge from $(0,5,0)$ to $(2,-1,0)$ by taking the straight line path.

(a) $(0,5,0) \rightarrow (2,5,0) \rightarrow (2,-1,0)$

(b) $y = 5 - 3x$

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

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PRINCIPLE OF CHARGE CONSERVATION

The time rate of 'decrease' of charge within a given volume must be equal to the 'net outward' current flow through the surface of the volume.

$$I_{out} = \oint \bar{J} \cdot d\bar{s} = -\frac{dQ_{in}}{dt}$$

$\downarrow$ Current density

Using Divergence Theorem $\oint \bar{J} \cdot d\bar{s} = \int_v \nabla \cdot \bar{J}\, dv$

$$\frac{-dQ_{in}}{dt} = \frac{-d}{dt}\int_v \rho_v\, dv = -\int_v \frac{\partial \rho_v}{dt}\, dv$$

$$\int_v \nabla \cdot \bar{J}\, dv = -\int_v \frac{\partial \rho_v}{\partial t}\, dv$$

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$$

CONTINUITY EQUATION (Continuity of Current Equation)

↳ No accumulation of charges at any point when currents are flowing

* DC currents (steady)

$\frac{\partial \rho_v}{\partial t} = 0 \Rightarrow \nabla \cdot \bar{J} = 0$

$\downarrow$

total charge leaving a vol = total charge entering it

RELAXATION / REARRANGEMENT TIME

Ohm's Law $\bar{J} = \sigma \bar{E}$ (σ → conductivity (mhos/m or S/m))

Gauss' Law $\rho_v = \nabla \cdot \bar{D}$

$\bar{D} = \varepsilon \bar{E} \Rightarrow \nabla \cdot \bar{E} = \dfrac{\rho_v}{\varepsilon}$

$\nabla \cdot \bar{J} = -\dfrac{\partial \rho_v}{\partial t} \Rightarrow \nabla \cdot \sigma \bar{E} = \dfrac{\sigma \rho_v}{\varepsilon} = -\dfrac{\partial \rho_v}{\partial t}$

$\dfrac{\partial \rho_v}{\partial t} + \dfrac{\sigma}{\varepsilon}\rho_v = 0 \Rightarrow \dfrac{\partial \rho_v}{\partial t} = -\dfrac{\sigma}{\varepsilon}\rho_v$

$\ln(\rho_v) = -\dfrac{\sigma}{\varepsilon} t + \ln(\rho_{v_0})$

$\rho_v = \rho_{v_0} e^{-t/T_r}$

ρ_{v_0} ↳ initial volume charge density at $t=0$

Relaxation time $T_r = \dfrac{\varepsilon}{\sigma}$ (in seconds)

$\rho_v = \rho_{v_0} e^{-t/T_r} \Rightarrow$ Introduction of charge at some interior part of material results in decay of volume charge density ρ_v

$T_r \Rightarrow$ Time it takes a charge placed in the interior of a material to drop to e^{-1} (=36.8%) of its initial value

Example: Copper $\sigma = 5.87 \times 10^7$ S/m, $\varepsilon_r = 1$

$T_r = \dfrac{\varepsilon}{\sigma} = \dfrac{\varepsilon_0 \varepsilon_r}{\sigma} = \underline{1.5 \times 10^{-19} \text{ seconds}}$

Fused quartz $\sigma = 10^{-17}$ S/m, $\varepsilon_r = 5$

$T_r = \dfrac{\varepsilon}{\sigma} = \dfrac{\varepsilon_0 \varepsilon_r}{\sigma} = 4427000 = \underline{4.4 \times 10^6 \text{ seconds}}$

HOMOGENEOUS MEDIUM

- Permittivity ε does not change from point-to-point

ISOTROPIC MEDIUM

- Permittivity ε does not change with direction

LINEAR MEDIUM

Since $\bar{D} = \varepsilon \bar{E}$, a medium is said to be linear if the permittivity ε does not change with the applied $\bar{E}$ field

Consider a medium which is linear & isotropic.

Applying Gauss Law

$\nabla \cdot \bar{D} = \nabla \cdot \varepsilon \bar{E} = \rho_v$

Since $\bar{E} = -\nabla V$

$\boxed{\nabla \cdot (-\varepsilon \nabla V) = \rho_v}$ Valid for a non-homogeneous medium

In the case of homogeneous medium (ε remaining constant)

$\boxed{\nabla \cdot (\nabla V) = -\dfrac{\rho_v}{\varepsilon} = \nabla^2 V}$ Poisson's Equation

When $\rho_v = 0$ (Charge-free region)

$\nabla \cdot (\nabla V) = 0$

$\boxed{\nabla^2 V = 0}$ Laplace Equation (for any scalar field)

$\nabla^2 \Rightarrow$ Laplacian operator

* $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$ Cartesian

* $\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left\{\rho \dfrac{\partial V}{\partial \rho}\right\} + \dfrac{1}{\rho^2}\dfrac{\partial^2 V}{\partial \phi^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$

* $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left\{r^2 \dfrac{\partial V}{\partial r}\right\} + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta \dfrac{\partial V}{\partial \theta}\right) + \dfrac{1}{r^2 \sin^2\theta}\dfrac{\partial^2 V}{\partial \phi^2} = 0$

General Procedure to Solve Laplace / Poisson's Equation

1) Solve the Laplace / Poisson's Equation using
   a) direct integration ⇒ when V is a function of one variable
   b) variables separable method ⇒ when V is a function of one than one variable

2) The solution at this point is not unique, it is expressed in terms of unknown constants of integration.

3) Apply the boundary condition to determine a unique solution for V. This the unique solution.

4) Having obtained the potential V, it is easy to compute $\bar{E}$, $\bar{D}$ and $\bar{J}$.

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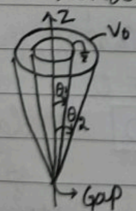
Q1. Two conducting cones with $\theta_1 = \pi/10$ and $\theta_2 = \pi/6$ of infinite extent separated by an infinitesimal gap at $r = 0$. If $V(\theta_1 = \frac{\pi}{10}) = 0$ and $V(\theta_2 = \frac{\pi}{6}) = 50V$, compute V and $\bar{E}$.



Since V depends on θ, use Laplace equation in its spherical form

$$\nabla^2 V = 0$$

$$\underbrace{\cancel{\text{in terms of } r}}_{=0} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \underbrace{\cancel{\text{in terms of } \phi}}_{=0} = 0$$

$$\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0$$

$$\Downarrow$$

$$\frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) = 0$$

Integrating once

$$\sin\theta \frac{dV}{d\theta} = C_1$$

$$\frac{d(\operatorname{cosec}\theta)}{d\theta} = -\operatorname{cosec}\theta \cot\theta$$

$$dV = C_1 \frac{d\theta}{\sin\theta}$$

$$\int dV = \int \frac{C_1\, d\theta}{\sin\theta} = \int \frac{C_1\, d\theta}{2\sin\theta/2 \cos\theta/2}$$

Divide numerator and denominator by $\cos(\theta/2)$

$$\int dV = \int \frac{\frac{C_1\, d\theta}{\cos\theta/2}}{\frac{2\sin\theta/2 \cos\theta/2}{\cos\theta/2}} = C_1 \int \frac{\frac{1}{2}\sec^2\theta/2\, d\theta}{\tan\theta/2}$$

$$V = C_1 \ln\left(\tan\frac{\theta}{2}\right) + C_2$$

Condition 1 $\quad V_{\theta = \theta_1 = \frac{\pi}{10}} = 0$

$$V_{\theta_1} = C_1 \ln\left(\tan\frac{\pi}{20}\right) + C_2 = 0$$

$$\boxed{C_2 = -C_1 \ln\left(\tan\frac{\pi}{20}\right)}$$

$$V = C_1 \ln\left(\tan\frac{\theta}{2}\right) - C_1 \ln\left(\tan\frac{\theta_1}{2}\right)$$

$$\boxed{V = C_1 \ln\left\{\frac{\tan(\theta/2)}{\tan(\theta_1/2)}\right\}}$$

Condition 2 $V = V_0$ when $\theta = \theta_2$

$$V_0 = C_1 \ln\left\{\frac{\tan(\theta_2/2)}{\tan(\theta_1/2)}\right\}$$

$$C_1 = \frac{V_0}{\ln\left\{\frac{\tan(\theta_2/2)}{\tan(\theta_1/2)}\right\}}$$

$$V = \frac{V_0}{\ln\left\{\frac{\tan(\theta_2/2)}{\tan(\theta_1/2)}\right\}} \times \ln\left\{\frac{\tan(\theta/2)}{\tan(\theta_1/2)}\right\}$$

$\theta_1 = \frac{\pi}{10}$ $\quad \theta_2 = \frac{\pi}{6}$ $\quad V_0 = 50V$

$$V = \frac{50}{\ln\left\{\frac{\tan(\pi/12)}{\tan(\pi/20)}\right\}} \ln\left\{\frac{\tan(\theta/2)}{\tan(\pi/20)}\right\}$$

$$V = 95.1 \ln\left\{\frac{\tan(\theta/2)}{0.1584}\right\} \text{ Volts}$$

$\bar{E} = -\nabla V$

$\bar{E} = -\frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta$ $\quad$ Only in terms of θ

$\Rightarrow \bar{E} = -\frac{1}{r}\frac{dV}{d\theta}\bar{a}_\theta$

$$= -\frac{1}{r}\frac{d}{d\theta}\left[95.1 \ln\left\{\frac{\tan\theta/2}{0.1584}\right\}\right]\bar{a}_\theta$$

$$= -\frac{95.1}{r}\frac{d}{d\theta}\left[\ln(\tan\theta/2)\right]\bar{a}_\theta$$

$$= -\frac{95.1}{r}\frac{1}{\tan\theta/2}\cdot\frac{\sec^2\theta}{2}\cdot\frac{1}{2}\bar{a}_\theta \qquad \frac{\cos\theta/2}{\sin\theta/2}\cdot\frac{1}{\cos^2\theta/2}$$

$$\bar{E} = -\frac{95.1}{r}\frac{\bar{a}_\theta}{2\sin\theta/2\cos\theta/2} = \frac{-95.1}{r\sin\theta}\bar{a}_\theta$$

$$\bar{E} = -\nabla V = -\frac{1}{r}\frac{d}{d\theta}\left[\frac{V_0}{\ln\left\{\frac{\tan(\theta_2/2)}{\tan(\theta_1/2)}\right\}} \times \left\{\ln\frac{\tan(\theta/2)}{\tan(\theta_1/2)}\right\}\right]\bar{a}_\theta$$

$$= \frac{-V_0}{r\ln\left\{\frac{\tan(\theta_2/2)}{\tan(\theta_1/2)}\right\}}\frac{d}{d\theta}\left[\ln\left(\tan\frac{\theta}{2}\right) - \ln\tan\left(\frac{\theta_1}{2}\right)\right]\bar{a}_\theta$$

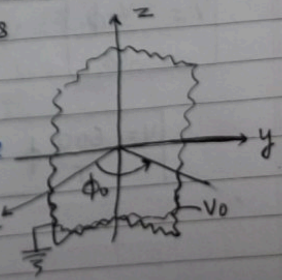
$$= \frac{-V_0}{r\ln\left\{\frac{\tan(\theta_2/2)}{\tan(\theta_1/2)}\right\}}\frac{1}{\tan(\theta/2)}\sec^2\left(\frac{\theta}{2}\right)\cdot\frac{1}{2}\bar{a}_\theta$$

$$\boxed{\bar{E} = \frac{-V_0}{r\sin\theta\ln\left\{\frac{\tan(\theta_2/2)}{\tan(\theta_1/2)}\right\}}\bar{a}_\theta}$$

$V_0 = 50V$ $\quad \theta_1 = \frac{\pi}{10}$ $\quad \theta_2 = \frac{\pi}{6}$

$$\boxed{\bar{E} = \frac{-95.1}{r\sin\theta}\bar{a}_\theta \text{ V/m}}$$

Q2. Semi infinite conducting planes at $\phi = 0$ and $\phi = \frac{\pi}{6}$ are separated by a very very small insulating gap as shown. If $V|_{\phi=0} = 0V$ and $V|_{\phi=\frac{\pi}{6}} = 100V$, calculate the potential V and the electric field intensity between the plates



From the given problem statement, we infer that we need to use the Laplace equation in its cylindrical form.

This is due to the fact that the boundary conditions depend on ϕ.

Using the ϕ-component from the Laplace form

$$\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$ $\rho \neq 0$ measured from zero to infinity

$$\Rightarrow \frac{\partial^2 V}{\partial \phi^2} = 0$$

$$\frac{d^2 V}{d\phi^2} = 0$$ Integrating once

$$\frac{dV}{d\phi} = C_1$$ Integrating again

$$\boxed{V = C_1\phi + C_2}$$

$V(\phi = 0) = 0V$

$0 = C_1(0) + C_2 \Rightarrow \boxed{C_2 = 0}$

$$\boxed{V = C_1\phi}$$

$V(\phi = \pi/6 = \phi_0) = \cancel{100}\, V_0$

$\cancel{C_1 \frac{\pi}{6} = 100\, V_0}$ $C_1\phi_0 = V_0$

$$\boxed{C_1 = \frac{V_0}{\phi_0}}$$

$$\boxed{V = \frac{V_0\phi}{\phi_0}}$$ Volts

$$\boxed{V = \frac{600}{\pi}\phi \text{ Volts}}$$

$$\bar{E} = -\nabla V = -\frac{1}{\rho}\frac{\partial V}{\partial \phi}\bar{a}_\phi = \frac{-1}{\rho}\frac{dV}{d\phi}\bar{a}_\phi$$

$$= \frac{-1}{\rho}\frac{d}{d\phi}\left(\frac{V_0\phi}{\phi_0}\right)\bar{a}_\phi$$

$$\bar{E} = -\frac{V_0}{\rho\phi_0}\bar{a}_\phi \Rightarrow \boxed{\bar{E} = \frac{-600}{\pi\rho}\bar{a}_\phi \text{ V/m}}$$

Electric Boundary Conditions

1) If the field exists in a region consisting of 2 different media (permittivity), the conditions that the field must satisfy at the interface separating the media are called BOUNDARY CONDITIONS.

2) The conditions are helpful in determining the field on one side of the boundary if the field on the other side is known.

3 cases are considered:

(i) Dielectric (ε_{r_1}) - Dielectric (ε_{r_2})
(ii) Conductor - Dielectric
(iii) Conductor - Free space

Maxwell's Equations: $\oint_L \bar{E}\cdot d\bar{l} = 0$

$\oint_S \bar{D}\cdot d\bar{S} = Q_{enc}$

$\bar{E} = \bar{E}_t + \bar{E}_n$ $\bar{D} = \bar{D}_t + \bar{D}_n$

CASE (i) DIELECTRIC (ε_{r_1}) - DIELECTRIC (ε_{r_2}) BOUNDARY CONDITIONS

Fig 1
$\bar{E}$ field

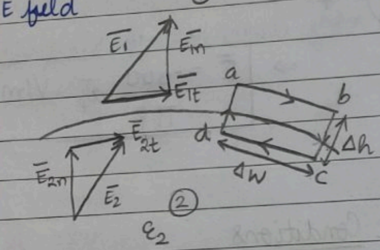
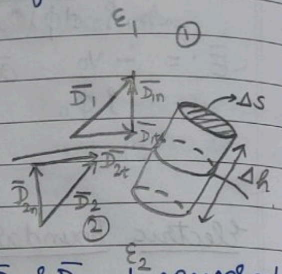


Fig 2
$\bar{D}$ field



$\bar{E_1}$ & $\bar{E_2}$ decomposed into $\bar{E_t}$ & $\bar{E_n}$ components

$\bar{D_1}$ & $\bar{D_2}$ decomposed into $\bar{D_t}$ & $\bar{D_n}$ components

$\varepsilon_1 = \varepsilon_0 \varepsilon_{r_1}$; $\varepsilon_2 = \varepsilon_0 \varepsilon_{r_2}$
$\bar{E_1} = \bar{E_{1t}} + \bar{E_{1n}}$
$\bar{E_2} = \bar{E_{2t}} + \bar{E_{2n}}$

$\varepsilon_1 = \varepsilon_0 \varepsilon_{r_1}$; $\varepsilon_2 = \varepsilon_0 \varepsilon_{r_2}$
$\bar{D_1} = \bar{D_{1t}} + \bar{D_{1n}}$
$\bar{D_2} = \bar{D_{2t}} + \bar{D_{2n}}$

While applying the equation $\oint_L \bar{E}\cdot d\bar{l} = 0$ to the closed path abcda as shown, assuming the path to be very small w.r.t spatial variations, we have

opposite to direction of $\vec{E_{1n}}$

$$0 = \underbrace{E_{1t}\,\Delta W}_{\text{path ab}} \underbrace{- E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2}}_{\text{path bc}} \underbrace{- E_{2t}\,\Delta W}_{\text{path cd}}$$
$$\underbrace{+ E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2}}_{\text{path da}}$$

$E_{1t}\,\Delta W - E_{2t}\,\Delta W = 0$
$[E_{1t} - E_{2t}]\,\Delta W = 0$ $\Delta W \neq 0$

$E_{1t} - E_{2t} = 0$

$\Rightarrow \boxed{E_{1t} = E_{2t}}$ Boundary condition ①
No change in tangential component of $\bar{E}$ field

INFERENCE
- The tangential components of $\bar{E}$ are same on both sides of the interface or boundary.
- $\bar{E_t}$ undergoes no change at the boundary and it is said to be continuous across the boundary

$\bar{D} = \varepsilon_0 \bar{E} = \bar{D_t} + \bar{D_n}$

$\bar{E_{1t}} = \bar{E_{2t}}$

$\Rightarrow \boxed{\dfrac{\bar{D_{1t}}}{\varepsilon_1} = \dfrac{\bar{D_{2t}}}{\varepsilon_2}}$ Boundary condition ②

INFERENCE
- The tangential components of $\bar{D}$ field does undergo a change
  $\therefore$ $\bar{D_t}$ is said to be discontinuous across the interface

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Referring to figure 2, the dot product $\bar{D}\cdot d\bar{s}$ due to the sides = 0

$\oint_S \bar{D}\cdot d\bar{s} = Q_{enc}$

$\Delta Q = \rho_s \Delta S = \underbrace{D_{1n}\,\Delta S}_{\text{region 1}} \underbrace{- D_{2n}\,\Delta S}_{\text{region 2}}$

Opp to direction of D_{2n}

$\boxed{D_{1n} - D_{2n} = \rho_s}$ Boundary condition ③ (General case)

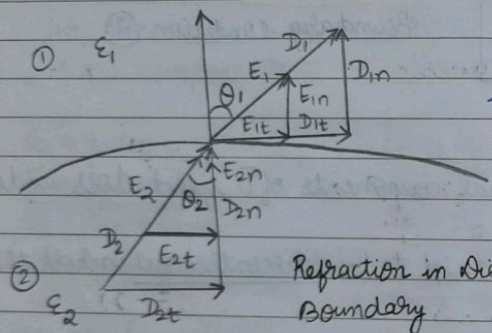
Here, ρ_s is defined as the free charge density placed deliberately at the interface or boundary.

If the boundary or interface is devoid of any charge density ($\rho_s = 0$), then the boundary condition becomes

$\boxed{D_{1n} = D_{2n}}$ with $\rho_s = 0$ Boundary condition ③

$\boxed{\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}}$ with $\rho_s = 0$ Boundary condition ④

$\boxed{\varepsilon_{r1} E_{1n} = \varepsilon_{r2} E_{2n}}$



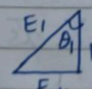
θ_1 and θ_2 are the angles made by the vectors with a common normal that is considered

Refraction in Dielectric-Dielectric Boundary

Using Boundary Condition ①

$E_{1t} = E_{2t}$

$E_1 \sin\theta_1 = E_{1t} = E_{2t} = E_2 \sin\theta_2$



$\sin\theta_1 = \frac{E_{1t}}{E_1}$

$E_{1t} = E_1 \sin\theta_1$

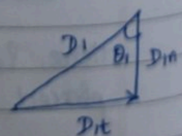
$\boxed{E_1 \sin\theta_1 = E_2 \sin\theta_2}$ — ①

Assuming $\rho_s = 0$

$D_{1n} = D_{2n}$ (Boundary condition ③)

$\varepsilon_1 E_1 \cos\theta_1 = D_{1n} = D_{2n} = \varepsilon_2 E_2 \cos\theta_2$

$\boxed{\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2}$ — ②



$\cos\theta_1 = \frac{D_{1n}}{D_1}$

$D_{1n} = D_1 \cos\theta_1$
$= \varepsilon_1 E_1 \cos\theta_1$

* Good conductor $\bar{E}$ field $= 0$

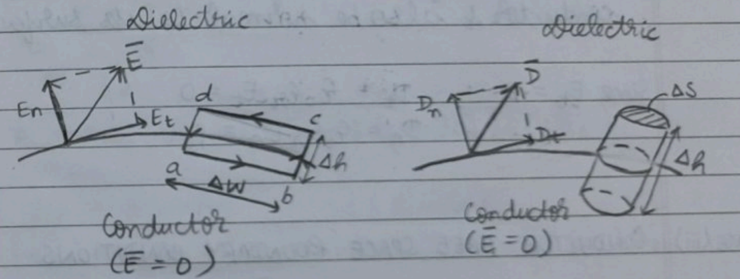
From ① & ② $\frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2}$

$\Downarrow$

Law of Refraction of $\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}}$ $\because \varepsilon = \varepsilon_0 \varepsilon_r$

Electric field at the boundaries or interface with charge free boundary ($\rho_s = 0$)

CASE (ii) CONDUCTOR - DIELECTRIC BOUNDARY CONDITIONS



Applying $\oint_L \bar{E} \cdot d\bar{l} = 0$ to the path a-b-c-d-a

$$\underbrace{(0)\Delta w}_{\substack{\text{path ab} \\ \bar{E}=0}} + \underbrace{(0)\frac{\Delta h}{2} + E_n\frac{\Delta h}{2}}_{\text{path bc}} \underbrace{- E_t \Delta w}_{\text{path cd}} \underbrace{- E_n \frac{\Delta h}{2} + (0)\frac{\Delta h}{2}}_{\text{path da}} = 0$$

$-E_t \Delta w = 0$ $\Delta w \neq 0$

$\boxed{E_t = 0}$ Boundary condition ①

$\oint_S \bar{D} \cdot d\bar{S} = Q_{enc}$

$\Delta Q = D_n \Delta S - (0)\Delta S = \rho_s \Delta S$

$\boxed{D_n = \rho_s}$ Boundary condition ②

Under static conditions, the following can be inferred for a perfect conductor:

(i) No electric field may exist within the conductor $\Rightarrow \bar{E} = 0$ and $\rho_v = 0$

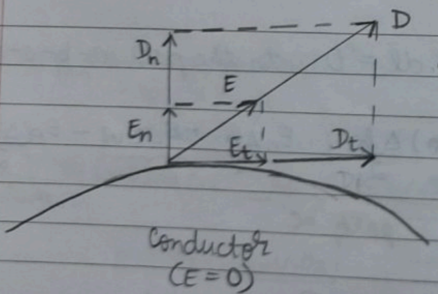
(ii) Since $\bar{E} = -\nabla V = 0$, there can be no potential difference between any 2 points in the conductor or a conductor is an equipotential body

(iii) An electric field $\bar{E}$ must be external to the conductor & also be normal to its surface.

Since $E_t = 0$ $\quad D_t = \varepsilon_0 \varepsilon_r E_t = 0$

$D_n = \varepsilon_0 \varepsilon_r E_n = \rho_s$

CASE (iii) CONDUCTOR – FREE SPACE BOUNDARY CONDITIONS



free space
$\varepsilon = \varepsilon_0 \varepsilon_r$
$= \varepsilon_0 \quad \because \varepsilon_r = 1$ in free space

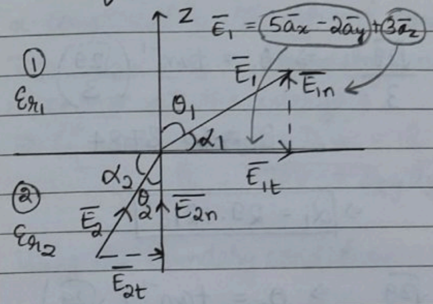
$E_t = 0 \qquad D_n = \rho_s$

$D_t = \varepsilon_0 E_t = 0 \qquad \varepsilon_0 E_n = \rho_s$

Q1. Two extensive homogeneous, isotropic dielectrics meet on a plane $z=0$. For $z>0$ $\varepsilon_{r_1} = 4$

For $z<0$ $\varepsilon_{r_2} = 3$

A uniform electric field $\bar{E}_1 = 5\bar{a}_x - 2\bar{a}_y + 3\bar{a}_z$ kV/m exists for $z \geq 0$



a) Find $\bar{E}_2$

b) Evaluate the angles $\bar{E}_1$ and $\bar{E}_2$ make with the interface

a) $E_{1n} = \bar{E}_1 \cdot \bar{a}_n = (5\bar{a}_x - 2\bar{a}_y + 3\bar{a}_z)\cdot(1\bar{a}_z)$

$E_{1n} = 3$ kV/m

$\boxed{\bar{E}_{1n} = 3\bar{a}_z \text{ kV/m}}$

Similarly $\bar{E}_{2n} = (\bar{E}_2 \cdot \bar{a}_z)\bar{a}_z$

$\bar{E}_1 = \bar{E}_{1t} + \bar{E}_{1n}$

$\Rightarrow \bar{E}_{1t} = \bar{E}_1 - \bar{E}_{1n} = (5\bar{a}_x - 2\bar{a}_y)$ kV/m

From Boundary condition ①,

$\bar{E}_{1t} = \bar{E}_{2t}$

$\Rightarrow \boxed{\bar{E}_{2t} = (5\bar{a}_x - 2\bar{a}_y) \text{ kV/m}}$

Since $\rho_s = 0$ (or not specified in the given question)

$\bar{D}_{1n} = \bar{D}_{2n}$

$\varepsilon_{r_1} \bar{E}_{1n} = \varepsilon_{r_2} \bar{E}_{2n}$

$\Rightarrow \bar{E}_{2n} = \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} \bar{E}_{1n} = \frac{4}{3} \times 3\bar{a}_z = 4\bar{a}_z$ kV/m

$$\therefore \bar{E}_2 = \bar{E}_{2t} + \bar{E}_{2n} = \underline{(5\bar{a}_x - 2\bar{a}_y + 4\bar{a}_z)\ \text{kV/m}}$$

b) To compute α_1 and α_2

$\alpha_1 = 90^\circ - \theta_1$ $\qquad$ $\alpha_2 = 90^\circ - \theta_2$

$$\tan\theta_1 = \frac{|E_{1t}|}{|E_{1n}|} = \frac{\sqrt{29}}{3} \Rightarrow \theta_1 = \tan^{-1}\left(\frac{\sqrt{29}}{3}\right)$$

$$\theta_1 = 60.8784^\circ$$

$$\Rightarrow \boxed{\alpha_1 = 29.1216^\circ}$$

$$\tan\theta_2 = \frac{|E_{2t}|}{|E_{2n}|} = \frac{\sqrt{29}}{4} \Rightarrow \theta_2 = \tan^{-1}\left(\frac{\sqrt{29}}{4}\right)$$

$$\theta_2 = 53.3957^\circ$$

$$\Rightarrow \boxed{\alpha_2 = 36.6043^\circ}$$

HW

Q2. A homogeneous dielectric with $\varepsilon_r = 2.5$ fills region 1 where $x<0$ while region 2 with $x>0$ is free space.

a) If $\bar{D}_1 = (12\bar{a}_x - 10\bar{a}_y + 4\bar{a}_z)\ \text{nC/m}^2$, find $\bar{D}_2$ and θ_2

b) If $E_2 = 12$ V/m and $\theta_2 = 60^\circ$, find E_1 and θ_1

HW

Q3. It is found that $\bar{E} = (60\bar{a}_x + 20\bar{a}_y - 30\bar{a}_z)$ mV/m at a particular point on the interface between air and a conducting surface. Evaluate $\bar{D}$ and ρ_s at that point.

Ans 2:

$x>0$ ② Free space $\varepsilon_r = 1$ $\quad \underline{E_2} = 12$ V/m $\quad D_2 \quad \theta_2 = 60^\circ$

$x<0$ ① Dielectric $\varepsilon_r = 2.5$ $\quad \bar{D}_1 = (12\bar{a}_x - 10\bar{a}_y + 4\bar{a}_z)\ \text{nC/m}^2$

$\varepsilon_2 = \varepsilon_0\varepsilon_r = \varepsilon_0$

$\varepsilon_1 = \varepsilon_0\varepsilon_r = 2.5\varepsilon_0$

(a) x-component is the normal component

$D_{2n} - D_{1n} = \rho_s$ $\quad$ Boundary condition

No charge at the interface

$\Rightarrow \rho_s = 0 \quad \Rightarrow D_{2n} = D_{1n} = 12$

$$\bar{D}_2 = 12\bar{a}_x + D_{2y}\bar{a}_y + D_{2z}\bar{a}_z$$

Using the boundary condition $E_{1t} = E_{2t}$

$$\Rightarrow \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

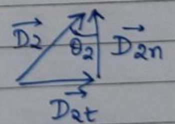
$$\frac{-10\bar{a}_y + 4\bar{a}_z}{2.5\varepsilon_0} = \frac{D_{2y}\bar{a}_y + D_{2z}\bar{a}_z}{\varepsilon_0}$$

$D_{2y} = \dfrac{-10}{2.5}$ $\qquad$ $D_{2z} = \dfrac{4}{2.5}$

$D_{2y} = -4$ $\qquad$ $D_{2z} = 1.6$

$$\therefore \boxed{\bar{D}_2 = 12\bar{a}_x - 4\bar{a}_y + 1.6\bar{a}_z\ \text{nC/m}^2}$$

$D_{2n} = |\bar{D}_2| \cos\theta_2$



$12 = \sqrt{12^2 + (-4)^2 + (1.6)^2} \times \cos\theta_2$

$$\cos\theta_2 = \frac{12}{\sqrt{162.56}} \Rightarrow \theta_2 = \cos^{-1}(0.9411837)$$

$$\boxed{\theta_2 = 19.748^\circ}$$

(b) $E_2 = 12\, V/m \qquad \theta_2 = 60^\circ$

$E_1 \sin\theta_1 = E_2 \sin\theta_2$

$E_1 \sin\theta_1 = 6\sqrt{3}$

$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2$

$2.5\varepsilon_0 E_1 \cos\theta_1 = \varepsilon_0 \times 12 \cos 60^\circ$

$E_1 \cos\theta_1 = \frac{6}{2.5} = \frac{12}{5}$

$\frac{E_1 \sin\theta_1}{E_1 \cos\theta_1} = \frac{6\sqrt{3}}{12/5}$

$\tan\theta_1 = \frac{5\sqrt{3}}{2} \Rightarrow \boxed{\theta_1 = 76.996^\circ}$

$E_1 = \frac{6\sqrt{3}}{\sin(76.996^\circ)} \Rightarrow \boxed{E_1 = 10.6658\ V/m}$

Ans 3:

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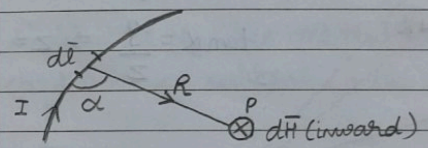
Unit 2

MAGNETOSTATICS

BIOT-SAVART'S LAW

The differential magnetic field intensity $d\bar{H}$ produced at a point P by the differential current element $Id\bar{l}$ is proportional to the product $Id\bar{l}$ and the sine of the angle α between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element.



* RIGHT-HAND THUMB RULE used to find the direction of $\bar{H}$

$\otimes$ ~~dots~~ inward $\qquad \odot$ outward

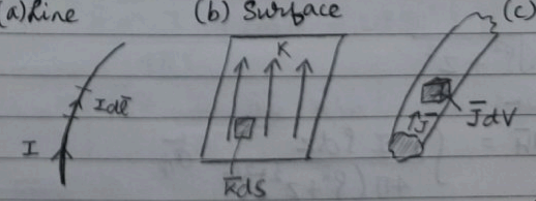
Magnitude form $dH = \frac{I dl \sin\alpha}{4\pi R^2}$ UNIT: A/m

Cross product $\bar{A} \times \bar{B} = AB\sin\theta$

Vectorial form $d\bar{H} = \frac{Id\bar{l} \times \bar{a}_R}{4\pi R^2} = \frac{Id\bar{l} \times \bar{R}}{4\pi R^3}$

$R = |\bar{R}| \qquad \bar{a}_R = \frac{\bar{R}}{|R|}$

(a) Line (b) Surface (c) Volume



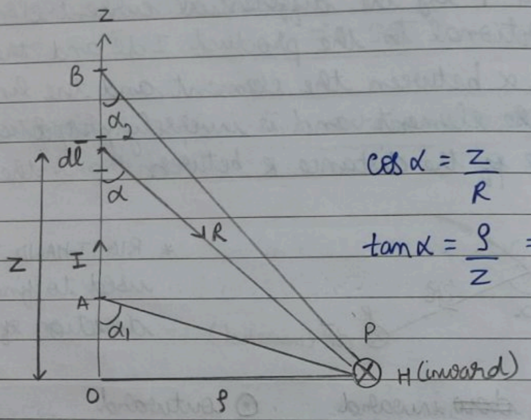
$\boxed{Id\bar{l} \equiv \bar{K}ds \equiv \bar{J}dV}$

$H = \int_L \frac{Id\bar{l} \times \bar{a}_R}{4\pi R^2} \qquad H = \int_S \frac{\bar{K}ds \times \bar{a}_R}{4\pi R^2} \qquad H = \int_V \frac{\bar{J}dV \times \bar{a}_R}{4\pi R^2}$

$\bar{K}$ = Surface ~~charge~~ current density (A/m)

$\bar{J}$ = Volume ~~current~~ current density (A/m^2)

REMARK The direction of $d\bar{H}$ can be determined by the RIGHT HAND RULE with the right hand thumb pointing in the direction of the current and right hand fingers encircling the wire in the direction of $d\bar{H}$.



$\cos\alpha = \frac{z}{R}$ $\sin\alpha = \frac{\rho}{R}$

$\tan\alpha = \frac{\rho}{z} \Rightarrow z = \rho\cot\alpha$

$d\bar{H} = \frac{I d\bar{l} \times \bar{R}}{4\pi R^3}$ $d\bar{l} = dz\,\bar{a}_z$

$\rho\bar{a}_\rho = z\bar{a}_z + \bar{R}$ (from $\Delta OPd\bar{l}$)

$\Rightarrow \bar{R} = \rho\bar{a}_\rho - z\bar{a}_z$

$$d\bar{l} \times \bar{R} = \begin{vmatrix} \bar{a}_\rho & \bar{a}_\phi & \bar{a}_z \\ 0 & 0 & dz \\ \rho & 0 & -z \end{vmatrix} = \rho\, dz\, \bar{a}_\phi$$

$|\bar{R}| = R = \sqrt{\rho^2 + z^2}$

$$\bar{H} = \int d\bar{H} = \int \frac{I\rho\, dz}{4\pi(\rho^2+z^2)^{3/2}}\,\bar{a}_\phi$$

$z = \rho\cot\alpha \Rightarrow dz = -\rho\,\text{cosec}^2\alpha\, d\alpha$

$(\rho^2+z^2)^{3/2} = (\rho^2 + \rho^2\cot^2\alpha)^{3/2} = \rho^3(1+\cot^2\alpha)^{3/2}$

$= \rho^3(\text{cosec}^2\alpha)^{3/2} = \rho^3\,\text{cosec}^3\alpha$

$$\bar{H} = \frac{-I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\rho^2\,\text{cosec}^2\alpha}{\rho^3\,\text{cosec}^3\alpha}\, d\alpha\, \bar{a}_\phi$$

$$= \frac{-I}{4\pi}\int_{\alpha_1}^{\alpha_2} \sin\alpha\, d\alpha\, \bar{a}_\phi$$

$$\boxed{\bar{H} = \frac{I}{4\pi\rho}\{\cos\alpha_2 - \cos\alpha_1\}\,\bar{a}_\phi \quad A/m}$$

valid for any orientation of current-carrying straight conductor

Angles α_1 & α_2 are measured from along the direction of current in the conductor to the vector direction.

REMARKS (i) The above expression is generally applicable for any straight filamentary conductor. The conductor need not lie on Z-axis, however it should be straight.

(ii) $\bar{H}$ is always along the unit vector $\bar{a}_\phi$ (along concentric circular path) irrespective of the length of the wire or field point (point of interest)

SPECIAL CASE 1 The conductor is of semi-infinite length w.r.t P so that the point A is now at the origin and B is at $(0,0,\infty)$. In this case, $\alpha_1 = 90°$, $\alpha_2 = 0°$

$$\bar{H} = \frac{I}{4\pi\rho}\{\cos 0° - \cos 90°\}\,\bar{a}_\phi$$

$$\Rightarrow \boxed{\bar{H} = \frac{I}{4\pi\rho}\,\bar{a}_\phi \quad A/m}$$ Semi-infinite conductor

SPECIAL CASE 2

A $(0,0,\infty)$ $\alpha_1 = 180°$ $\alpha_2 = 0°$

B $(0,0,-\infty)$

$$\bar{H} = \frac{I}{4\pi\rho}\{\cos 0° - \cos 180°\}\bar{a}_\phi = \frac{2I}{4\pi\rho}\bar{a}_\phi$$

$$\boxed{\bar{H} = \frac{I}{2\pi\rho}\bar{a}_\phi \text{ A/m}}$$

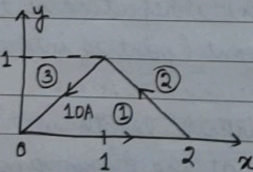
To compute $\bar{a}_\phi$ in the expression of $\bar{H}$ is not always that easy. A simple approach is to determine $\bar{a}_\phi$ from the relation

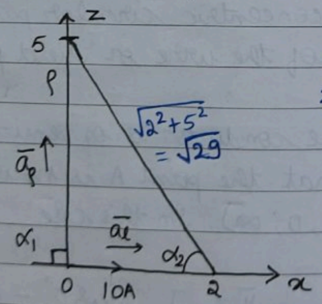
$$\boxed{\bar{a}_\phi = \bar{a}_l \times \bar{a}_\rho}$$

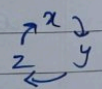
$\bar{a}_\rho$ → unit vector along the $\perp^r$ drawn from the line current to the field point

$\bar{a}_l$ → the unit vector along the line current or conductor

Q1. The conducting triangular loop shown in the figure carries a current of 10A. Find the magnetic field intensity $\bar{H}(0,0,5)$ due to side 1 of the loop.







$$\bar{H} = \frac{I}{4\pi\rho}\{\cos\alpha_2 - \cos\alpha_1\}\bar{a}_\phi$$

~~$\cos\alpha_2 =$~~

$\tan\alpha_2 = \frac{5}{2}$

$\alpha_1 = 90°$ $\cos\alpha_2 = \frac{2}{\sqrt{29}}$ $\rho = 5$

$\cos\alpha_1 = 0$

$\bar{a}_\phi = \bar{a}_l \times \bar{a}_\rho$ $\bar{a}_l = \bar{a}_x$

$\bar{a}_\phi = \bar{a}_x \times \bar{a}_z$ $\bar{a}_\rho = \bar{a}_z$

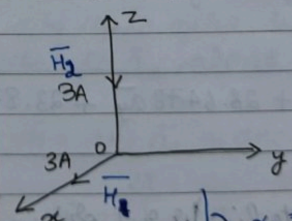
$= -\bar{a}_y$

$$\bar{H} = \frac{10}{4\pi(5)}\left\{\frac{2}{\sqrt{29}} - 0\right\}(-\bar{a}_y)$$

$$\boxed{\bar{H} = -0.0591\,\bar{a}_y \text{ A/m}} \Rightarrow \boxed{\bar{H} = -59.10986\,\bar{a}_y \text{ mA/m}}$$

Q2. Evaluate $\bar{H}(0,0,5)$ due to side 3 of the triangular loop.

Q3. Find $\bar{H}(-3,4,0)$ due to the given filament current.



$\bar{H} = \bar{H}_1 + \bar{H}_2$ Superposition

$$\bar{H} = \frac{I}{4\pi\rho}\{\cos\alpha_2 - \cos\alpha_1\}\bar{a}_\phi$$

① $\alpha_1 = 180°$ $\alpha_2 = 90°$

$\rho = \sqrt{(-3)^2+4^2} = 5$

$\bar{a}_\phi = \bar{a}_l \times \bar{a}_\rho \Rightarrow \bar{a}_l = -\bar{a}_z$ $\bar{a}_\rho = -\frac{3}{5}\bar{a}_x + \frac{4}{5}\bar{a}_y$

$$\bar{a}_\phi = (-\bar{a}_z)\times\left(-\frac{3}{5}\bar{a}_x + \frac{4}{5}\bar{a}_y\right) = +\frac{3}{5}\bar{a}_y + \frac{4}{5}\bar{a}_x$$

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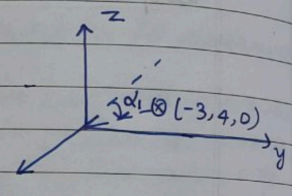
$$\bar{H}_2 = \frac{I}{4\pi\rho}\left\{\cos\alpha_2 - \cos\alpha_1\right\}\bar{a}_\phi$$

$$= \frac{3}{4\pi(5)}\left\{\cos 90^\circ - \cos 180^\circ\right\}\left[\frac{4}{5}\bar{a}_x + \frac{3}{5}\bar{a}_y\right]$$

$$\bar{H}_2 = 0.0382\,\bar{a}_x + 0.02865\,\bar{a}_y \quad A/m$$

$$\Rightarrow \bar{H}_2 = \left(38.1971\,\bar{a}_x + 28.6478\,\bar{a}_y\right)\frac{mA}{m}$$

To compute $\bar{H}_1$



$\alpha_1 = \tan^{-1}\left(\frac{4}{3}\right)$ $\qquad \alpha_2 = 0^\circ$

$= 53.13^\circ$ $\qquad \rho = 4$

$\bar{a}_\ell = \bar{a}_x \qquad \bar{a}_\rho = \bar{a}_y \qquad \bar{a}_\phi = \bar{a}_\ell \times \bar{a}_\rho = \bar{a}_x \times \bar{a}_y = \bar{a}_z$

$$\bar{H}_1 = \frac{I}{4\pi\rho}\left\{\cos\alpha_2 - \cos\alpha_1\right\}\bar{a}_\phi$$

$$= \frac{3}{4\pi\times 4}\left\{\cos 0^\circ - \cos(53.13^\circ)\right\}\bar{a}_z$$

$$= 0.023873\,\bar{a}_z \quad A/m$$

$$\bar{H}_1 = 23.8732\,\bar{a}_z \quad mA/m$$

$$\boxed{\bar{H} = \bar{H}_1 + \bar{H}_2 = \left(38.1971\,\bar{a}_x + 28.6478\,\bar{a}_y + 23.8732\,\bar{a}_z\right)\frac{mA}{m}}$$

HW

Q4. A positive Y-axis (semi-infinite line w.r.t origin) carries a filamentary current of 2A in $-\bar{a}_y$ direction. Find $\bar{H}$ at A(2,3,0) and B(3,12,−4)

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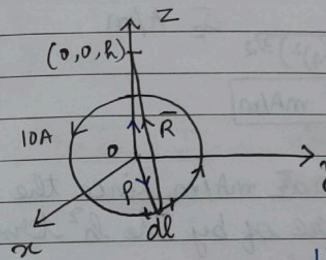
Q5. A circular loop or ring $x^2+y^2=9$ with $z=0$ carries a DC of 10A along the $\bar{a}_\phi$ direction. Determine the magnetic field intensity $\bar{H}$ at (0,0,4) and (0,0,−4).



$$d\bar{H} = \frac{I\,d\bar{\ell}\times\bar{R}}{4\pi R^3}$$

$$d\bar{\ell} = \rho\,d\phi\,\bar{a}_\phi$$

$\rho\bar{a}_\rho + \cancel{h\bar{a}_z}\ \bar{R} = h\bar{a}_z$

$\Rightarrow \bar{R} = h\bar{a}_z - \rho\bar{a}_\rho$

$$|\bar{R}| = R = (\rho^2+h^2)^{1/2}$$

$$R^3 = (\rho^2+h^2)^{3/2}$$

$$d\bar{\ell}\times\bar{R} = \begin{vmatrix} \bar{a}_\rho & \bar{a}_\phi & \bar{a}_z \\ 0 & \rho\,d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h\,d\phi\,\bar{a}_\rho + \rho^2 d\phi\,\bar{a}_z$$

$$d\bar{H} = \frac{I\,d\bar{\ell}\times\bar{R}}{4\pi R^3} = \frac{\cancel{10}\ I}{4\pi(\rho^2+h^2)^{3/2}}\left\{\rho h\,d\phi\,\bar{a}_\rho + \rho^2 d\phi\,\bar{a}_z\right\}$$

Due to the symmetry, the contributions along $\bar{a}_\rho$ will add up to zero. We are left with the contributions due to $\bar{a}_z$ alone.

$$d\bar{H} = \underbrace{dH_\rho}_{=0}\,\bar{a}_\rho + dH_z\,\bar{a}_z$$

$$\Rightarrow d\bar{H} = \int_{\phi=0}^{2\pi}\frac{I}{4\pi(\rho^2+h^2)^{3/2}}\,\rho^2\,d\phi\Big\}\,\bar{a}_z$$

$$= \frac{I\rho^2}{4\pi(\rho^2+h^2)^{3/2}}\,\phi\Big|_0^{2\pi}\,\bar{a}_z$$

$$d\bar{H} = \frac{I\rho^2}{2(\rho^2+h^2)^{3/2}}\,\bar{a}_z \quad A/m$$

$\rho = 3$

$\bar{H}(0,0,4) = \dfrac{10 \times 3^2}{2(3^2+4^2)^{3/2}} \, \bar{a}_z$ A/m

$\boxed{\bar{H}(0,0,4) = 360 \, \bar{a}_z \text{ mA/m}}$

$\bar{H}(0,0,-4) = \dfrac{10 \times 3^2}{2(3^2+(-4)^2)^{3/2}} \, \bar{a}_z$ A/m

$\boxed{\bar{H}(0,0,-4) = 360 \, \bar{a}_z \text{ mA/m}}$

$\bar{H}(0,0,-4)$ is also $360 \bar{a}_z$ mA/m since the negative sign of h is taken care of by the h^2 term in the denominator.

AMPERE's CIRCUIT LAW

⇒ The line integral of $\bar{H}$ around a closed path is the same as the net current I_{enc} enclosed by the path

⇒ $\oint_L \bar{H} \cdot d\bar{l} = I_{enc}$ Integral form of MAXWELL's THIRD EQUATION

→ can be used easily to determine $\bar{H}$ if the current distribution is symmetrical

→ special case of Biot-Savart's Law

STOKE's THEOREM

⇒ The circulation of a vector field $\bar{A}$ around a (closed) path L is equal to the surface integral of the curl of $\bar{A}$ over the open surface S bounded by L, provided $\bar{A}$ & $\nabla \times \bar{A}$ are continuous on S.

$$\oint_L \bar{A} \cdot d\bar{l} = \int_S \underbrace{(\nabla \times \bar{A})}_{\text{curl of } A} \cdot d\bar{s}$$

$\oint_L \bar{H} \cdot d\bar{l} = I_{enc}$ Applying Stoke's Theorem

$I_{enc} = \oint_L \bar{H} \cdot d\bar{l} = \int_S (\nabla \times \bar{H}) \cdot d\bar{s}$

↓

$\int_S \bar{J} \cdot d\bar{s}$

$\nabla \times \bar{H} = \bar{J}$ Differential or Point form of Maxwell's Third Equation

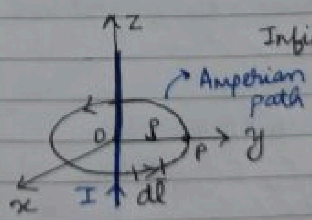
$\nabla \times \bar{H} = \bar{J} \neq 0$

↳ magnetostatic field is <u>not conservative</u>

Applications of Ampere's Circuit Law

1. Used to determine $\bar{H}$ for symmetric current distribution
2. The relation $\oint \bar{H} \cdot d\bar{l} = I_{enc}$ is made use of to compute $\bar{H}$.
3. For symmetrical current distribution, $\bar{H}$ is either parallel or perpendicular to $d\bar{l}$.
4. When $\bar{H}$ is parallel to $d\bar{l}$, $|\bar{H}|$ is constant.

Q. Infinite Filamentary Line Current



• Consider an infinitely long filamentary conductor carrying a current I along the z-axis. To determine $\bar{H}$ at a field point P, we make a closed path to pass the point P

• This path on which the Ampere's law is to be applied is known as the AMPERIAN PATH.

· In the present case, we choose a concentric circle as the Amperian Path

$\cancel{\bar{H} = \frac{I}{2\pi\rho}}$ $\quad \oint_L \bar{H} \cdot d\bar{l} = I_{encl}$

<u>LHS</u> $\quad d\bar{l} = \rho \, d\phi \, \bar{a}_\phi$

$$\bar{H} = H_\rho \bar{a}_\rho + H_\phi \bar{a}_\phi + H_z \bar{a}_z$$

$$\bar{H} \cdot d\bar{l} = H_\phi \rho \, d\phi$$

$$\int_{\phi=0}^{2\pi} H_\phi \rho \, d\phi = 2\pi\rho H_\phi$$

<u>RHS</u> $\quad I_{encl} = I$

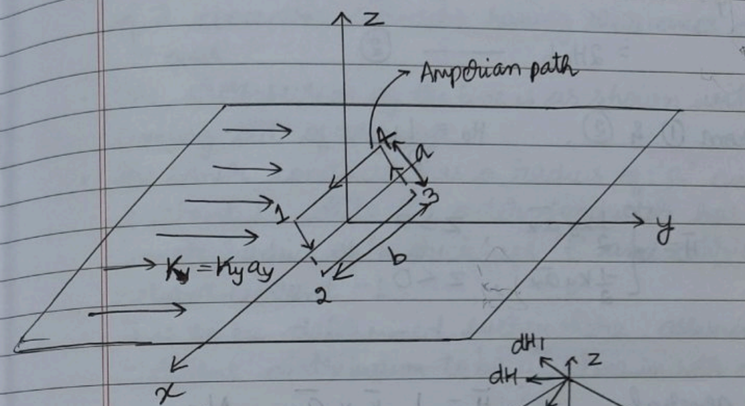
Equating LHS and RHS

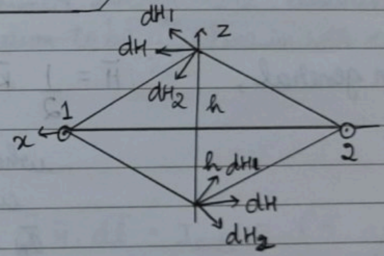
$$2\pi\rho H_\phi = I$$

$$H_\phi = \frac{I}{2\pi\rho} \quad \Rightarrow \bar{H} = H_\phi \bar{a}_\phi$$

$$\Rightarrow \boxed{\bar{H} = \frac{I}{2\pi\rho} \bar{a}_\phi \text{ A/m}}$$

Infinite Sheet of Current





Consider an infinite current sheet in the $z=0$ plane. The sheet has a uniform current density $\bar{K} = K_y \bar{a}_y$ A/m. The amperian path is chosen with the dimensions as shown.

Now, looking at this relation $\oint_L \bar{H} \cdot d\bar{l} = I_{encl}$

Considering the RHS

$$I_{encl} = K_y b \quad \text{—— (1)}$$

LHS

$$\bar{a}_\phi = \bar{a}_l \times \bar{a}_\rho = \bar{a}_y \times \bar{a}_z = \bar{a}_x$$

$$\bar{H} = \begin{cases} H_0 \bar{a}_x & \forall \; z>0 \\ -H_0 \bar{a}_x & \forall \; z<0 \end{cases}$$

$$\oint \bar{H} \cdot d\bar{l} = \left\{ \int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right\} \bar{H} \cdot d\bar{l}$$

$$\oint \bar{H}\cdot d\bar{l} = 0(-a) + (-H_0)(-b) + 0(a) + (H_0)(b)$$

$$= 2H_0 b \quad \text{—— (2)}$$

From (1) & (2), $H_0 = \frac{1}{2} K_y$

$$\bar{H} = \begin{cases} \frac{1}{2}K_y \bar{a_x}, & z>0 \\ -\frac{1}{2}K_y \bar{a_x}, & z<0 \end{cases}$$

In general, $\bar{H} = \frac{1}{2}\bar{K} \times \bar{a_N}$ A/m

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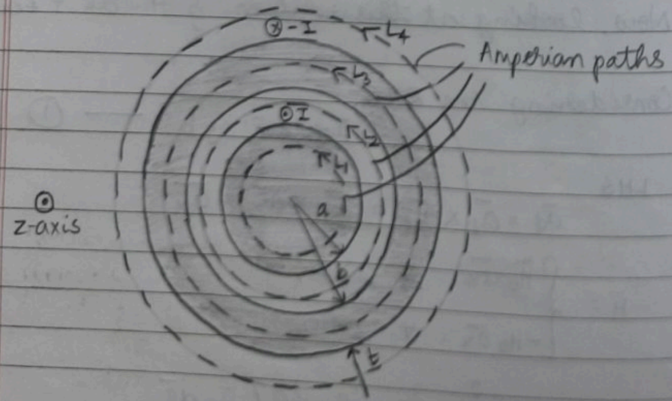
where $\bar{K}$ is the surface current density (A/m)
$\bar{a_N}$ is the unit vector along the perpendicular drawn from the sheet current to the field/observation point

Infinitely Long Co-axial Transmission Line



- Consider an infinitely long transmission line consisting of 2 concentric cylinders having their axes along the z-axis.
- The cross-section of the line is as shown, with z-axis coming out of the page
- The inner conductor has a radius of 'a' and carries current I, while the outer conductor has an inner radius of 'b', thickness 't' and carries the return current −I.
- $\bar{H}$ is to be determined everywhere, assuming the current distribution to be uniform in both conductors.

$$\oint_L \bar{H}\cdot d\bar{l} = I_{encl}$$

(i) $0 \le \rho < a$ $\quad \oint_{L_1} \bar{H}\cdot d\bar{l} = I_{encl} = \int \bar{J}\cdot d\bar{s}$

RHS : $\int \bar{J}\cdot d\bar{s} = \iint \left\{ \frac{I}{\pi a^2} \bar{a_z} \right\} \cdot \{ \rho\, d\rho\, d\phi\, \bar{a_z} \}$ acting normal to the surface

RHS : $\int \bar{J}\cdot d\bar{s} = \frac{I}{\pi a^2} \int_{\rho=0}^{\rho} \rho\, d\rho \int_{\phi=0}^{2\pi} d\phi$

$$= \frac{I}{\pi a^2} \left.\frac{\rho^2}{2}\right|_0^{\rho} \left.\phi\right|_0^{2\pi}$$

$$= \frac{I\rho^2}{a^2}$$

LHS : $\oint_{L_1} \bar{H}\cdot d\bar{l}$

$= H_\phi \oint d\bar{l} = H_\phi\,(2\pi\rho)$ circumference

$\bar{H}$ can have H_ρ, H_ϕ, H_z components
$d\bar{l}$ has only component along $\bar{a_\phi}$ since amperian path is circular

Equating LHS & RHS

$$H_\phi(2\pi\rho) = \frac{I\rho^2}{a^2}$$

$$\boxed{H_\phi = \frac{I\rho}{2\pi a^2}} \quad \text{—— (1)}$$

(ii) $a \le \rho < b$ $\quad \oint_{L_2} \bar{H}\cdot d\bar{l} = I_{encl}$

LHS : $H_\phi(2\pi\rho)$

RHS : current enclosed by amperian path = I

$\Rightarrow H_\phi(2\pi\rho) = I$

$$\boxed{H_\phi = \frac{I}{2\pi\rho}} \quad \text{—— (2)}$$

(iii) $b \le \rho < b+t$ $\quad \oint_{L_3} \bar{H}\cdot d\bar{l} = I_{encl}$

LHS : $H_\phi(2\pi\rho)$

RHS : $I_{encl} = I + \int \bar{J}\cdot d\bar{s}$

$$\bar{J} = \frac{-I}{\pi\{(b+t)^2-(b)^2\}}\bar{a}_z \qquad d\bar{s} = \rho\, d\rho\, d\phi\, \bar{a}_z$$

$$I_{encl} = I - \frac{I}{\pi\{(b+t)^2-(b)^2\}}\int_{\rho=b}^{\rho}\rho\, d\rho \int_{\phi=0}^{2\pi} d\phi$$

$$= I\left\{1 - \frac{1}{\pi\{b^2+t^2+2bt-b^2\}}\frac{\rho^2-b^2}{2}(2\pi)\right\}$$

$$I_{encl} = I\left\{1 - \frac{(\rho^2-b^2)}{t^2+2bt}\right\}$$

Equating LHS & RHS

$$H_\phi(2\pi\rho) = I\left\{1 - \frac{(\rho^2-b^2)}{t^2+2bt}\right\}$$

Wave guide - Hollow metallic tubes used for high freq & high power operations

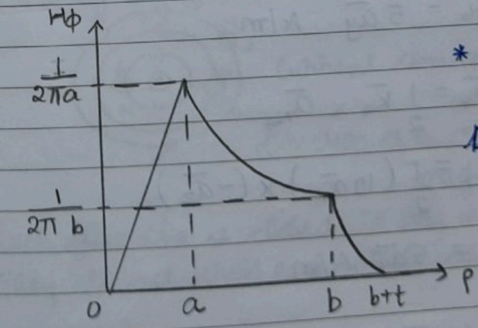
$$\boxed{H_\phi = \frac{I}{2\pi\rho}\left\{1 - \frac{(\rho^2-b^2)}{t^2+2bt}\right\}} \quad \text{—— (3)}$$

(iv) $\rho \ge b+t$

$\oint_{L_4} \bar{H}\cdot d\bar{l} = I_{encl} = I + (-I) = 0$

$\Rightarrow \boxed{H_\phi = 0}$ —— (4)

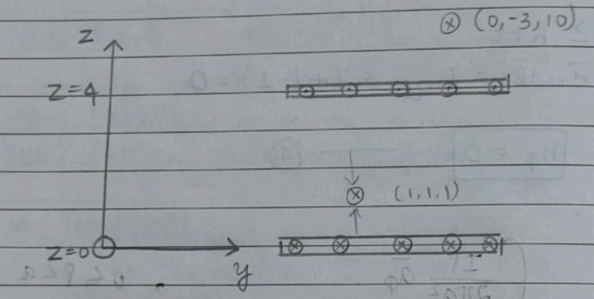
$$\bar{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\bar{a}_\phi, & 0 \le \rho < a \\ \dfrac{I}{2\pi\rho}\bar{a}_\phi, & a \le \rho < b \\ \dfrac{I}{2\pi\rho}\left[1 - \dfrac{(\rho^2-b^2)}{t^2+2bt}\right]\bar{a}_\phi, & b \le \rho < b+t \\ 0, & \rho \ge b+t \end{cases}$$



* No magnetic flux leakage in coaxial cables ⇒ cannot be taped

* Frequency of operation in Giga Hertz range
  ↳ introduces heavy losses
* Frequency of operation in lower Hz range
  ↳ expensive

Example 1 Planes $z=0$ and $z=4$ carry a current of $\bar{K} = -10\,\bar{a}_x$ A/m and $\bar{K} = +10\bar{a}_x$ A/m respectively. Determine $\bar{H}$ at $(1,1,1)$ and $(0,-3,10)$.



$\bar{H} = \bar{H}_0 + \bar{H}_4$ $\qquad$ $\bar{H} = \frac{1}{2}\bar{K} \times \bar{a}_N$

$\bar{H}(1,1,1)$ $\quad \bar{H}_0 = \frac{1}{2}\bar{K}_0 \times \bar{a}_{N0}$

$= \frac{1}{2}(-10\bar{a}_x) \times (\bar{a}_z)$

$\bar{H}_0 = 5\,\bar{a}_y$ A/m

$\bar{H}_4 = \frac{1}{2}\bar{K}_4 \times \bar{a}_{N4}$

$= \frac{1}{2}(10\bar{a}_x) \times (-\bar{a}_z)$

$\bar{H}_4 = 5\bar{a}_y$ A/m

$\therefore \bar{H}(1,1,1) = \underline{10\,\bar{a}_y \text{ A/m}}$

$\bar{H}(0,-3,10)$ $\quad \bar{H} = \bar{H}_0 + \bar{H}_4$

$= \frac{1}{2}(-10\bar{a}_x) \times (\bar{a}_z) + \frac{1}{2}(10\bar{a}_x) \times (\bar{a}_z)$

$\bar{H} = \underline{\underline{0 \text{ A/m}}}$ $\quad$ Outside the sheets, having opposite current densities

Magnetic scalar vector potential (not part of syllabus) $\mu = \mu_0\mu_r$

MAGNETIC FLUX DENSITY ($\bar{B}$)

$\bar{B}$ similar to $\bar{D} = \varepsilon_0\bar{E}$ (in free space)

$\bar{B} = \mu_0\,\bar{H}$

μ_0 → Permeability of free space

→ $\mu_0 = 4\pi \times 10^{-7}$ H/m

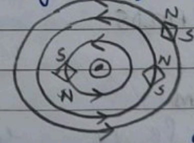
The magnetic flux through a surface is given by

$$\psi = \int_S \bar{B} \cdot d\bar{s}$$

ψ → Webers (Wb); $\bar{B}$ → Wb/m² or Tesla (T)

The magnetic flux line is a path to which $\bar{B}$ is tangential at every point on the line. It is the line along which the needle of a magnetic compass will orient itself if placed in the presence of a magnetic field.

The magnetic flux lines due to a straight long wire

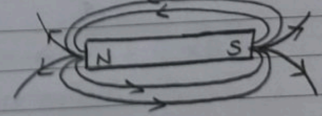


current coming out

direction of $\bar{B}$ → indicated by north of magnetic ~~sp~~ compass

* Each flux line is closed, it has no beginning or end. They do not cross each other as well.

* Unlike electric flux lines, magnetic flux lines always close upon themselves



* It is not possible to have isolated magnetic poles or magnetic charges

$$\oint_S \bar{B} \cdot d\bar{s} = 0$$

Applying Divergence Theorem $\oint_S \bar{B} \cdot d\bar{s} = \int_V (\nabla \cdot B)\, dv = 0$

$\rightarrow \nabla \cdot \bar{B} = 0$ Maxwell's 4^{th} equation

NOTE: Magnetostatic field is not conservative while magnetic flux is conservative

INFERENCES: (i) Magnetostatic fields have no sources or sinks
(ii) Magnetic field lines are always conservative.

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MAXWELL's EQUATIONS FOR STATIC FIELDS

(POINT or) DIFFERENTIAL FORM	INTEGRAL FORM	
$\nabla \cdot \bar{D} = \rho_v$	$\oint_S \bar{D} \cdot d\bar{s} = \int_V \rho_v \, dv$	Gauss' Law
$\nabla \cdot \bar{B} = 0$	$\oint_S \bar{B} \cdot d\bar{s} = 0$	Non-existence of magnetic monopole
$\nabla \times \bar{E} = 0$	$\oint_L \bar{E} \cdot d\bar{l} = 0$	Conservative nature of electrostatic field
$\nabla \times \bar{H} = \bar{J}$	$\oint_L \bar{H} \cdot d\bar{l} = \int_S \bar{J} \cdot d\bar{s}$	Ampere's Law

FORCES DUE TO E-M FIELDS

(A) Force on a charged particle

* The electric force $\bar{F}_e$ on a stationary or moving electric charge Q in an electric field is given by the COULOMB's LAW. It is related to the electric field $\bar{E}$ as:

$$\bar{F}_e = Q\bar{E} \quad \text{--- (1)}$$

* A magnetic field can exert force only on a moving charge.
* From experiments, the magnetic force $\bar{F}_m$ experienced by a charge Q moving with a velocity $\bar{u}$ in a magnetic field $\bar{B}$ is given by:

$$\bar{F}_m = Q\vec{u} \times \vec{B} \quad \text{--- (2)}$$

$\bar{F}_m$ is always perpendicular to the velocity $\bar{u}$ and the magnetic flux density $\bar{B}$

REMARKS (i) $\bar{F}_e$ is independent of the velocity $\bar{u}$ of the charge Q
(ii) $\bar{F}_e$ can perform work on the charge and change its kinetic energy
(iii) Unlike $\bar{F}_e$, $\bar{F}_m$ depends on $\bar{u}$ and is normal to it. Therefore, $\bar{F}_m$ cannot perform work on the charge Q
(iv) $\bar{F}_m$ does not increase the kinetic energy of the charge
(v) $\bar{F}_m$ in general is very small compared to $\bar{F}_e$ except at high velocities.

$$\bar{F} = \bar{F}_e + \bar{F}_m = Q\{\bar{E} + \bar{u} \times \bar{B}\} \quad \text{--- (3)}$$

$\rightarrow$ Lorentz Force equation

* If the mass of the charged particle moving in $\bar{E}$ and $\bar{B}$ fields is m, by Newton's Second Law of Motion,

$$\bar{F} = m\frac{d\bar{u}}{dt} = Q\{\bar{E} \times \bar{u} \times \bar{B}\} \quad \text{—— (4)}$$

→ In such situations, the energy can only be transferred by the electric field

STATE OF THE PARTICLE (CHARGE)	$\bar{E}$ FIELD ONLY	$\bar{B}$ FIELD ONLY	BOTH $\bar{E}$ & $\bar{B}$ FIELDS
Stationary	$Q\bar{E}$	—	$Q\bar{E}$
Moving	$Q\bar{E}$	$Q\bar{u} \times \bar{B}$	$Q\{\bar{E} + \bar{u} \times \bar{B}\}$

(B) Force on a Current element

To determine the force on a current element $Id\bar{l}$ of a current carrying conductor due to a magnetic field $\bar{B}$, we can modify the equation (2) using the fact for convection current,

convection current density

$$\bar{J} = \rho_v \bar{u} \quad \text{—— (5)}$$

volume charge density (ρ_v), velocity ($\bar{u}$)

Recall

$$Id\bar{l} \equiv \bar{K}ds \equiv \bar{J}dv \quad \text{—— (6)}$$

Combining equations (5) and (6),

$$Id\bar{l} = \rho_v \bar{u}\, dv = dQ\bar{u}$$

$$Id\bar{l} = \frac{dQ}{dt}d\bar{l} = dQ\frac{d\bar{l}}{dt} = dQ\bar{u}$$

mechanical electrical

$$\boxed{Id\bar{l} = dQ\bar{u}} \quad \text{—— (7)}$$

- From equation (7), an ~~incremental~~ elemental charge dQ moving with the velocity $\bar{u}$ (thereby producing convection current element $dQ\bar{u}$) is equivalent to a conduction current element $Id\bar{l}$.
- Thus the force on a current element $Id\bar{l}$ in a magnetic field $\bar{B}$ is found from equation (2) by merely replacing $Q\bar{u}$ from equation (7).

$$\Rightarrow d\bar{F} = Id\bar{l} \times \bar{B} \quad \text{—— (8)} \quad **$$

If the current is through a closed path L or a circuit, the force on the circuit is given by:

$$\bar{F} = \oint_L Id\bar{l} \times \bar{B} \quad \text{—— (9)}$$

<u>IMPORTANT</u> <u>REMARKS</u>:

(i) Magnetic field produced by the current element $Id\bar{l}$ does not exert force on the element itself. Just as a point charge does not exert force on itself.

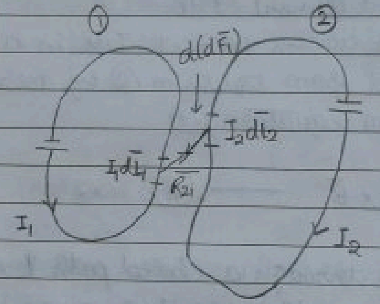
(ii) The $\bar{B}$ field that exerts force on the current element $Id\bar{l}$ must be due to another element or external to $Id\bar{l}$.

$$d\bar{F} = \bar{K}ds \times \bar{B} \;/\; \bar{J}dv \times \bar{B}$$

$$\bar{F} = \int_S \bar{K}ds \times \bar{B} \;/\; \int_V \bar{J}dv \times \bar{B}$$

The magnetic flux density $\bar{B}$ is defined as force per unit current element (equation 8)

(c) Force between two current elements



Consider the force between 2 current elements $I_1 d\bar{l}_1$ and $I_2 d\bar{l}_2$ as shown.
According to Biot-Savart's Law, these 2 current elements produce their associated $\bar{B}$ fields.

From eq^n 8

$$d(d\bar{F}_1) = I_1 d\bar{l}_1 \times d\bar{B}_2 \quad \text{— (10)}$$

Also from Biot-Savart's Law,

$$d\bar{B}_2 = \mu_0 d\bar{H} = \mu_0 \left(\frac{I_2 d\bar{l}_2 \times \bar{a}_{R_{21}}}{4\pi R_{21}^2} \right) \quad \text{— (11)}$$

Therefore

$$d(d\bar{F}_1) = \frac{\mu_0 I_1 d\bar{l}_1 \times (I_2 d\bar{l}_2 \times \bar{a}_{R_{21}})}{4\pi R_{21}^2} \quad \text{— (12)}$$

$$\bar{F}_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\bar{l}_2 \times d\bar{l}_2 \times \bar{a}_{R_{21}}}{R_{21}^2} \quad \text{— (13)}$$

$$\boxed{\bar{F}_2 = -\bar{F}_1}$$

To compute $\bar{F}_2$ analytically, replace the subscripts in equation (13) $1 \to 2$ or $2 \to 1$.

Example 1

A charged particle moves with the uniform velocity of $4\bar{a}_x$ m/s in a region where $\bar{E} = 20\bar{a}_y$ V/m and $\bar{B} = B_0 \bar{a}_z$ Wb/m². Determine B_0 such that the velocity of the particle remains constant.

Since the velocity of the particle needs to be constant, its acceleration should be zero.
Alternatively, the particle should experience zero force.

$$\bar{F} = Q\{\bar{E} + \bar{u} \times \bar{B}\} = 0$$

$$\bar{E} + \bar{u} \times \bar{B} = 0$$

$$20\bar{a}_y + (4\bar{a}_x) \times (B_0 \bar{a}_z) = 0$$

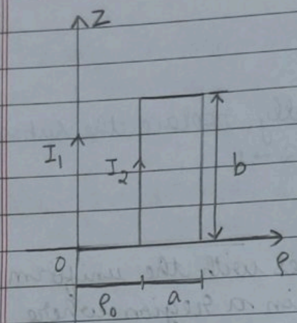
$$20\bar{a}_y - 4B_0 \bar{a}_y = 0$$

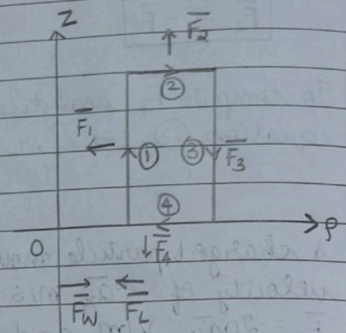
$$\Rightarrow \boxed{B_0 = 5 \text{ Wb/m}^2}$$

Used in VELOCITY FILTERS, where charged particles with a particular velocity are filtered out

$u = \frac{E}{B}$ Used in particle accelerators

Example 2





A rectangular loop carrying current I_2 is placed parallel to an infinitely long filamentary wire carrying current I_1 as shown. Show that the force experienced by the loop is given by

$$\bar{F}_L = \frac{-\mu_0 I_1 I_2 b}{2\pi}\left\{\frac{1}{\rho_0} - \frac{1}{\rho_0+a}\right\}\bar{a}_\rho \text{ N}$$

Let $\bar{F}_L$ and $\bar{F}_W$ be the forces experienced by the loop and the wire respectively.

Let $\bar{F}_L = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$

$$= I_2 \oint_L d\bar{L}_2 \times \bar{B}_1$$

$$\bar{B}_1 = \frac{\mu_0 I_1}{2\pi\rho_0}\bar{a}_\phi$$

$$\bar{F}_1 = I_2 \int_L d\bar{L}_2 \times \bar{B}_1$$

$$= I_2 \int_{z=0}^{b} dz\,\bar{a}_z \times \frac{\mu_0 I_1}{2\pi\rho_0}\bar{a}_\phi$$

$$= \frac{-\mu_0 I_1 I_2}{2\pi\rho_0}\int_{z=0}^{b} dz\,\bar{a}_\rho$$

$$\Rightarrow \bar{F}_1 = \frac{-\mu_0 I_1 I_2 (b)}{2\pi\rho_0}\bar{a}_\rho \quad \text{(attractive force)}$$

$$\bar{F}_2 = I_2 \int_{\rho=\rho_0}^{\rho_0+a} d\rho\,\bar{a}_\rho \times \frac{\mu_0 I_1}{2\pi\rho}\bar{a}_\phi$$

$$= \frac{\mu_0 I_1 I_2}{2\pi}\int_{\rho=\rho_0}^{\rho_0+a}\frac{d\rho}{\rho}\bar{a}_z$$

$$\Rightarrow \bar{F}_2 = \frac{\mu_0 I_1 I_2}{2\pi}\ln\left\{\frac{\rho_0+a}{\rho_0}\right\}\bar{a}_z$$

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$$\bar{F}_3 = I_2\int_{z=b}^{0}(+dz\,\bar{a}_z) \times \frac{\mu_0 I_1}{2\pi(\rho_0+a)}\bar{a}_\phi \qquad d\bar{L}_2 = -dz\,\bar{a}_z$$

$$= \frac{-I_2\mu_0 I_1}{2\pi(\rho_0+a)}(-b)\bar{a}_\rho$$

$$\Rightarrow \bar{F}_3 = \frac{\mu_0 I_1 I_2 b}{2\pi(\rho_0+a)}\bar{a}_\rho \quad \text{(repulsive)}$$

$$\bar{F}_4 = I_2\int_{\rho=\rho_0+a}^{\rho_0} d\rho\,\bar{a}_\rho \times \frac{\mu_0 I_1}{2\pi\rho}\bar{a}_\phi \qquad d\bar{L}_2 = -d\rho\,\bar{a}_\rho$$

$$\Rightarrow \bar{F}_4 = \frac{-\mu_0 I_1 I_2}{2\pi}\ln\left\{\frac{\rho_0+a}{\rho_0}\right\}\bar{a}_z$$

$$\bar{F}_L = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$$

$$= \frac{-\mu_0 I_1 I_2 b}{2\pi\rho_0}\bar{a}_\rho \cancel{+ \frac{\mu_0 I_1 I_2}{2\pi}\ln\left\{\frac{\rho_0+a}{\rho_0}\right\}\bar{a}_z} + \frac{\mu_0 I_1 I_2 b}{2\pi(\rho_0+a)}\bar{a}_\rho$$

$$\cancel{-\frac{\mu_0 I_1 I_2}{2\pi}\ln\left\{\frac{\rho_0+a}{\rho_0}\right\}\bar{a}_z}$$

$$\boxed{\bar{F}_L = \frac{-\mu_0 I_1 I_2 b}{2\pi}\left\{\frac{-1}{\rho_0+a} + \frac{1}{\rho_0}\right\}\bar{a}_\rho \text{ N}}$$

Force experienced by the wire due to the loop

$\bar{F}_W = -\bar{F}_L$

1. In static EM fields, the electric and magnetic fields are independent of each other
2. In dynamic EM fields, these fields are inter-dependent
3. Time-varying electric fields necessarily involves a corresponding time-varying magnetic fields.
4. Time-varying EM fields are generally expressed as $\bar{E}(x, y, z, t)$ and $\bar{H}(x, y, z, t)$ {Cartesian form}
5. Electrostatic fields are usually produced by static electric charges, while magnetostatic fields are due to motion of electric charges with uniform velocity (direct current) OR static magnetic charges (magnetic poles)
6. Time-varying fields or waves are usually due to accelerated charges or time-varying currents

I Electromotive forces based on Faraday's experiments

According to Faraday's experiments, a static magnetic field doesn't produce current. In a closed circuit, a time-varying field produces an induced voltage called ELECTROMOTIVE FORCE (EMF) that causes flow of current.

FARADAY'S LAW
The induced EMF (V_{emf}) in any closed circuit is equal to the time rate of change of the magnetic flux linkage of the circuit.

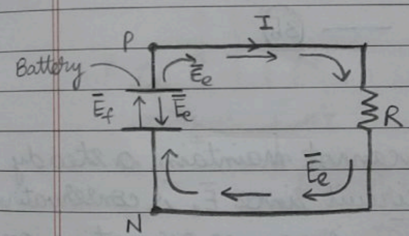
$$V_{emf} = -N \frac{d\psi}{dt} \quad \text{—— (1)}$$

N - number of turns of the circuit
ψ - magnetic flux

The negative sign indicates that the induced voltage acts in such a way as to oppose the flux producing it. This behaviour is in line with LENZ LAW.

LENZ LAW emphasises that the direction of current flow in the circuit is such that the induced magnetic field produced by the induced current will oppose the change in the original magnetic field.

* Electric field can be produced by:
(a) electric charges
(b) emf produced fields



The battery is the source of emf. The electrochemical reaction of the battery results in an emf produced field $\bar{E}_f$. Due to accumulation of charge at the battery terminals, an electrostatic field $\bar{E}_e$ also exists. The total electric field at any point is given by

$$\bar{E} = \bar{E}_f + \bar{E}_e \quad \text{—— (2)}$$

$\bar{E}_f = 0$ outside the battery.
$\bar{E}_f$ and $\bar{E}_e$ have opposite directions in the battery as shown.

The direction of $\bar{E}_e$ inside the battery is opposite to that outside it.

If we integrate equation ② over the closed circuit, we have

$$\oint_L \bar{E}\cdot d\bar{l} = \oint_L \bar{E}_f \cdot d\bar{l} + \oint_L \bar{E}_e \cdot d\bar{l}$$

$$= \int_N^P \bar{E}_f \cdot d\bar{l} + \underset{\Downarrow}{0} \quad \text{—— (3a)}$$

∵ Electrostatic field over a closed circuit = 0 (conservative)

The emf of the battery is the line integral of the emf produced field.

$$V_{emf} = \int_N^P \bar{E}_f \cdot d\bar{l} \quad \text{—— (3b)}$$

REMARKS

(i) An electrostatic field $\bar{E}_e$ cannot maintain a steady current in a closed circuit since $\bar{E}_e$ is conservative

(ii) An emf produced field $\bar{E}_f$ is non-conservative and therefore helps in the flow of current through the circuit

Consider a circuit with a single ~~term~~ turn $(N=1)$.
Equation ① becomes

$$V_{emf} = \frac{-d\psi}{dt} \quad \text{—— ④}$$

In terms of $\bar{B}$ and $\bar{E}$, equation ④ can be written as

$$V_{emf} = \oint_L \bar{E}\cdot d\bar{l} = \frac{-d}{dt}\int_S \bar{B}\cdot d\bar{s} \quad \text{—— ⑤}$$

REMARK It is clear from equation ⑤ that in a time-varying situation, both electric and magnetic fields are present and inter-related.

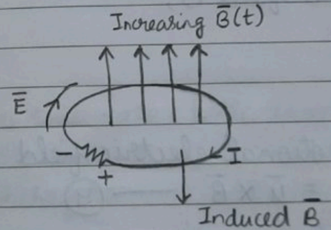
There are 3 ways of achieving variation in the magnetic flux w.r.t time such as:

(i) by having a stationary loop in a time-varying $\bar{B}$ field

(ii) by having a time-varying loop area in a static $\bar{B}$ field

(iii) by having a time-varying loop area in a time varying $\bar{B}$ field

CASE (i)



As integration over space and differentiation w.r.t time are independent operations, we have,

$$-\frac{d}{dt}\int_S \bar{B}\cdot d\bar{s} = -\int_S \frac{d}{dt}(\bar{B}\cdot d\bar{s})$$

Equation ⑤ now becomes

$$V_{emf} = \oint_L \bar{E}\cdot d\bar{l} = -\int_S \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s} \quad \text{—— ⑥}$$

partial derivative since $\bar{B}(x,y,z,t)$

This emf induced by the time-varying current (produced by the time-varying $\bar{B}$ field) in a stationary loop is referred as TRANSFORMER EMF.

By applying Stokes' Theorem to equation ⑥,

$$\int(\nabla\times\bar{E})\cdot d\bar{s} = -\int \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s} \quad \text{—— ⑦}$$

From equation (7),

$$\boxed{\nabla \times \bar{E} = \frac{-\partial \bar{B}}{\partial t}} \quad \text{--- (8)}$$

Curl equation has changed for a time-varying field

→ $\bar{E}$ is no more conservative

CASE (ii) When a conducting loop is moving in a static $\bar{B}$ field, an emf is induced in the loop. While recalling that the force on a charge moving with a uniform velocity $\bar{u}$ in a magnetic field $\bar{B}$,

$$\bar{F}_m = Q\bar{u} \times \bar{B}$$

We now define the motional electric field $\bar{E}_m$ as

$$\bar{E}_m = \frac{\bar{F}_m}{q} = \bar{u} \times \bar{B} \quad \text{--- (9)}$$

The expression for V_{emf} is given by

$$V_{emf} = \oint_L \bar{E}_m \cdot d\bar{l} = \oint_L (\bar{u} \times \bar{B}) \cdot d\bar{l} \quad \text{--- (10)}$$

This type of emf is called MOTIONAL EMF or FLUX-CUTTING EMF as it is due to motional action.

Applying Stokes' Theorem to equation (10),

$$\int_S (\nabla \times \bar{E}_m) \cdot d\bar{s} = \int_S (\nabla \times (\bar{u} \times \bar{B})) \cdot d\bar{s} \quad \text{--- (11)}$$

$$\boxed{\nabla \times \bar{E}_m = \nabla \times (\bar{u} \times \bar{B})} \quad \text{--- (12)}$$

In general, equation (10) is difficult to solve. Some precautions are required.

(i) The integral in equation (10) is zero along the portion of the loop where $\bar{u} = 0$. Thus, $d\bar{l}$ is taken along the portion of the loop that is cutting the $\bar{B}$ field where $\bar{u}$ has a non-zero value.

(ii) The direction of the induced current is same as that of $\bar{E}_m$ or $\bar{u} \times \bar{B}$. The limits of the integral in equation (10) are selected in a direction opposite to the induced current, thereby satisfying Lenz law

CASE (iii)

In this case, both transformer EMF and motional EMF are present.

Therefore,

$$V_{emf} = \oint_L \bar{E} \cdot d\bar{l}$$

$$= -\int_S \left(\frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}\right) + \oint_L (\bar{u} \times \bar{B}) \cdot d\bar{l} \quad \text{--- (13)}$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} + \nabla \times (\bar{u} \times \bar{B}) \quad \text{--- (14)}$$

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DISPLACEMENT CURRENT

For static EM fields, it is known that

$$\nabla \times \bar{H} = \bar{J} \quad \text{——} \quad (1)$$

Since the divergence of a curl of any vector field is equal to zero

$$\nabla \cdot (\nabla \times \bar{H}) = 0 = \nabla \cdot \bar{J} \quad \text{——} \quad (2)$$

From Continuity equation, we have

$$\nabla \cdot \bar{J} = \frac{-\partial \rho_v}{\partial t} \neq 0 \quad \text{——} \quad (3)$$

From the above, it is observed that equations (2) & (3) contradict each other. To take care of this issue, equation (1) is modified as

$$\nabla \times \bar{H} = \bar{J} + \underbrace{\bar{J}_d}_{\text{is to be defined \& computed}} \quad \text{——} \quad (4)$$

Applying divergence to equation (4),

$$\nabla \cdot (\nabla \times \bar{H}) = 0 = \nabla \cdot \bar{J} + \nabla \cdot \bar{J}_d \quad \text{——} \quad (5)$$

In order to make equations (3) & (5) to agree with each other, $\nabla \cdot \bar{J}_d = -\nabla \cdot \bar{J} = \frac{\partial \rho_v}{\partial t}$ is required

$$= \frac{\partial (\nabla \cdot \bar{D})}{\partial t}$$

$$\nabla \cdot \bar{J}_d = \nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

$$\Rightarrow \boxed{\bar{J}_d = \frac{\partial \bar{D}}{\partial t}} \quad \text{——} \quad (6)$$

Substituting equation (6) in (4)

$$\boxed{\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}} \quad \text{——} \quad (7)$$

The term $\bar{J}_d = \frac{\partial \bar{D}}{\partial t}$ is known as DISPLACEMENT CURRENT DENSITY

while $\bar{J}$ is known as CONDUCTION CURRENT DENSITY, $\bar{J} = \sigma \bar{E}$.

REMARKS (i) Introduction of $\bar{J}_d$ in equation (4) was one of the major contributions of Maxwell

(ii) Without the term $\bar{J}_d$, propagation of EM waves would be impossible.

(iii) At low frequencies, $\bar{J}_d$ is usually neglected when compared to $\bar{J}$. However at radio frequencies, both are comparable.

DISPLACEMENT CURRENT (I_d)

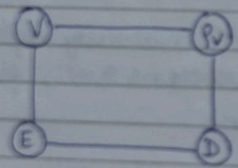
is defined as

$$I_d = \int_S \bar{J}_d \cdot d\bar{s} = \int_S \frac{\partial \bar{D}}{\partial t} \cdot d\bar{s}$$

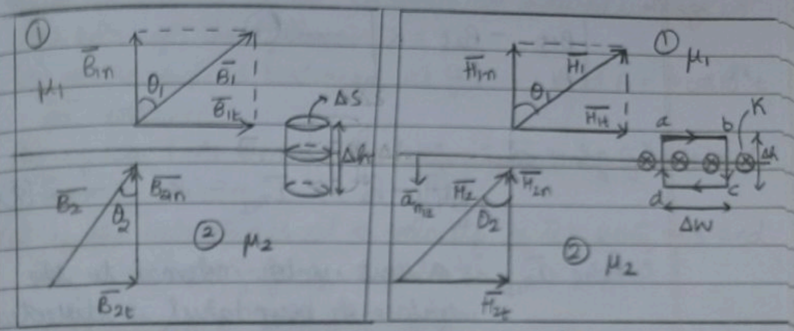
REMARKS (i) Displacement current is a result of time-varying electric field

(ii) Such types of currents are found in capacitors when alternating voltage is applied

Differential form	Integral form	Remarks
$\nabla \cdot \bar{D} = \rho_v$	$\oint_S \bar{D} \cdot d\bar{s} = \int_V \rho_v \, dv$	Gauss' Law
$\nabla \cdot \bar{B} = 0$	$\oint_S \bar{B} \cdot d\bar{s} = 0$	Non-existence of isolated magnetic charge
$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	$\oint_L \bar{E} \cdot d\bar{l} = -\frac{\partial}{\partial t} \int_S \bar{B} \cdot d\bar{s}$	Faraday's Law
$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$	$\oint_L \bar{H} \cdot d\bar{l} = \int_S \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right) \cdot d\bar{s}$	Ampere's Circuit Law



MAGNETIC FIELDS IN MATERIAL SPACE – BOUNDARY CONDITIONS



(a) (b)

Gauss' Law for Magnetostatics $\oint_S \bar{B} \cdot d\bar{s} = 0$ — ①

Ampere's Circuit Law $\oint_L \bar{H} \cdot d\bar{l} = I$ — ②

Applying ① to Fig(a) $\Rightarrow B_{1n} \Delta S - B_{2n} \Delta S = 0$ — ③

$\boxed{B_{1n} = B_{2n}}$ $\boxed{\mu_1 H_{1n} = \mu_2 H_{2n}}$ — ④

INFERENCES: (i) The normal components of $\bar{B}$ field do not undergo any change across the boundary or interface, i.e., it remains unchanged.

(ii) The normal components of $\bar{H}$ field undergo a change across the boundary or interface.

Applying equation ② to the Amperian path of Fig (b)

$$K\Delta W = H_{1t}\Delta W - H_{1n}\cancel{\frac{\Delta h}{2}} - H_{2n}\cancel{\frac{\Delta h}{2}} - H_{2t}\Delta W + H_{2n}\cancel{\frac{\Delta h}{2}} + H_{1n}\cancel{\frac{\Delta h}{2}} \quad — ⑤$$

$$K\Delta W = H_{1t}\Delta W - H_{2t}\Delta W = (H_{1t} - H_{2t})\Delta W$$

$$\boxed{H_{1t} - H_{2t} = K} \quad \text{— (6)}$$

$$\boxed{\frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = K} \quad \text{— (7)}$$

In general

$$(\bar{H}_1 - \bar{H}_2) \times \bar{a}_{n_{12}} = \bar{K} \quad \text{— (8)}$$

where $\bar{a}_{n_{12}}$ is a unit vector normal to the interface or boundary & is directed from medium (1) to medium (2)

Inferences from (6) & (7),

when the interface has a surface current density K, the tangential components of $\bar{H}$ field and $\bar{B}$ field undergo ~~the~~ a change.

Special case: $\bar{K} = 0$ or the interface doesn't contain surface charge

$$\boxed{H_{1t} = H_{2t}} \quad \text{— (9)} \qquad \boxed{\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}} \quad \text{— (10)}$$

Tangential components of $\bar{H}$ field remain unchanged

Tangential components of $\bar{B}$ field undergo a change

Assume the interface is devoid of surface current. The law of refraction of the magnetic flux line can be expressed as

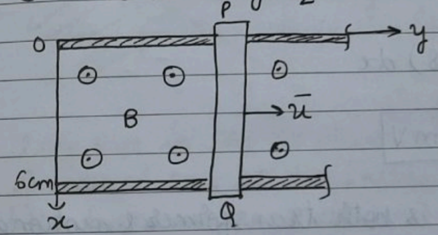
$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2} = \frac{\mu_{r_1}}{\mu_{r_2}}} \quad \text{— (11)}$$

Q1. A conducting bar can slide freely over 2 conducting rails as shown below. Calculate the induced voltage in the bar for the following:

(a) if the bar is stationed at $y = 8$ cm and $\bar{B} = 4\cos 10^6 t \, \bar{a}_z$ mWb/m^2

(b) if the bar slides at a velocity $\bar{u} = 20\,\bar{a}_y$ m/s and $\bar{B} = 4\bar{a}_z$ mWb/m^2

(c) if the bar slides at a velocity of $\bar{u} = 20\bar{a}_y$ m/s and $\bar{B} = 4\cos(10^6 t - y)\,\bar{a}_z$ mWb/m^2



(a) The induced emf is a transformer emf since the bar is stationary and the $\bar{B}$ field is varying.

$$V_{emf} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s} \qquad d\bar{s} = dx\,dy\,\bar{a}_z$$

$$\frac{\partial \bar{B}}{\partial t} = -4 \times 10^6 \sin(10^6 t) \times 10^{-3}\,\bar{a}_z$$

$$V_{emf} = -\int_{x=0}^{0.06}\int_{y=0}^{0.08} \left\{ -4 \times 10^6 (\sin 10^6 t) \times 10^{-3}\,\bar{a}_z \right\} dx\,dy\,\bar{a}_z$$

$$= 4 \times 10^3 \sin(10^6 t) \int_{x=0}^{0.06} dx \int_{y=0}^{0.08} dy$$

$$= 4 \times 10^3 \sin(10^6 t) \times 0.06 \times 0.08$$

$$\boxed{V_{emf} = 19.2 \sin(10^6 t) \text{ V}}$$

(b) The induced emf is motional emf since the bar moves with a velocity $\bar{u}$ and $\bar{B}$ field is constant.

$$V_{emf} = \oint_L (\bar{u} \times \bar{B}) \cdot d\bar{l}$$

$$d\bar{l} = dx\,\bar{a}_x$$

$$\bar{u} \times \bar{B} = (20\,\bar{a}_y) \times (4\bar{a}_z)\,10^{-3}$$
$$= 80 \times 10^{-3}\,\bar{a}_x$$
$$\bar{u} \times \bar{B} = 0.08\,\bar{a}_x$$

$$V_{emf} = \int_{x=0.06}^{0} (0.08)\,dx$$

$$\boxed{V_{emf} = -4.8\text{ mV}}$$

(c) The induced emf is both transformer & motional

$$V_{emf} = \frac{-d\Psi}{dt} \qquad d\bar{s} = dx\,dy\,\bar{a}_z$$

$$\Psi = \int_S \bar{B} \cdot d\bar{s} = \int_{x=0}^{0.06}\int_{y=0}^{y} 4\cos(10^6 t - y)\,dx\,dy \times 10^{-3}$$

$$= 4 \times 10^{-3} \int_{x=0}^{0.06} dx \int_{y=0}^{y} \cos(10^6 t - y)\,dy$$

$$= 4 \times 10^{-3} \times 0.06\left[-(+\sin(10^6 t - y))\right]\Big|_{y=0}^{y}$$

$$= 2.4 \times 10^{-4}\left[-\sin(10^6 t - y) + \sin(10^6 t)\right]$$

$$\Psi = -0.24\sin(10^6 t - y) + 0.24\sin(10^6 t)\text{ mWb}$$

$$\frac{dy}{dt} = u \Rightarrow y = ut = 20t \text{ (scalar)}$$

$$V_{emf} = \frac{-d\Psi}{dt} = +240\cos(10^6 t - y) - 240\cos(10^6 t)\text{ V}$$

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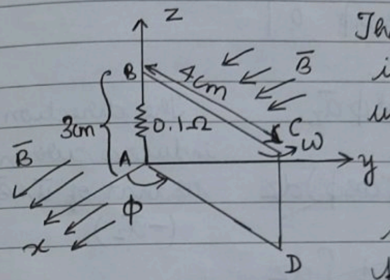
Q2.



The conducting loop shown in the figure is inside a uniform magnetic field $\bar{B} = 50\,\bar{a}_x$ mWb/m². If the side DC of the loop cuts the flux lines at a frequency of 50 Hz and the loop lies in the YZ plane at time t = 0, evaluate:

a) induced emf at t = 1 msec
b) induced current at t = 3 msec

From the given problem statement it can be inferred that the induced emf is a motional emf since $\bar{B}$ field is constant (time-invariant) and the loop is moving.

$$V_{emf} = \oint_L (\bar{u} \times \bar{B}) \cdot d\bar{l}$$

$$d\bar{l} = dz\,\bar{a}_z \quad (\because \text{DC is cutting flux lines})$$

angular velocity

$$\bar{u} = \frac{d\bar{l}'}{dt} = \frac{\rho\,d\phi}{dt}\bar{a}_\phi = \rho\left(\frac{d\phi}{dt}\right)\bar{a}_\phi$$
$$= \rho\omega\,\bar{a}_\phi$$

$$\cancel{\bar{u} \times \bar{B} = (\rho\omega\,\bar{a}_\phi) \times (50\,\bar{a}_x)\,10^{-3}}$$

Since $\bar{u}$ and $\bar{B}$ are in different coordinate systems, we need to convert one of them to perform the cross-product.

$$\boxed{\bar{B} = B_0\,\bar{a}_x = B_0\left(\cos\phi\,\bar{a}_\rho - \sin\phi\,\bar{a}_\phi\right)}$$

$B_0 = 50\times10^{-3}$

$$\bar{u}\times\bar{B} = \begin{vmatrix} \bar{a}_\rho & \bar{a}_\phi & \bar{a}_z \\ 0 & \rho\omega & 0 \\ B_0\cos\phi & -B_0\sin\phi & 0 \end{vmatrix}$$

$\bar{u}\times\bar{B} = -\rho\omega B_0\cos\phi\,\bar{a}_z$

The direction of the induced current is same as that of $\bar{u}\times\bar{B}$, i.e. $(-\bar{a}_z)$

$\cancel{V_{emf} = \int(-\rho\omega B_0\cos\phi)\,dz}$

$\cancel{= -\rho\omega B_0\cos\phi\int dz}$

Therefore, the limits while evaluating the line integral to compute V_{emf} should be in opposite direction of induced current $(\bar{a}_z)$.

$\omega = 2\pi f = 100\pi$

$\cancel{V_{emf} = \oint_L(\bar{u}\times\bar{B})\cdot d\bar{l}} = \quad (\bar{u}\times\bar{B})\cdot d\bar{l} = -\rho\omega B_0\cos\phi\,dz$

ρ distance of element from Z-axis

$\rho = 4cm = 0.04\,m$

$$V_{emf} = \int_{z=0}^{0.03} -\rho\omega B_0\cos\phi\,dz = -0.04\times50\times10^{-3}\omega\cos\phi\,z\Big|_0^{0.03}$$

$$= -0.04\times50\times10^{-3}\times100\pi\times0.03\times\cos\phi$$

$$\boxed{V_{emf} = -6\pi\cos\phi\ mV}$$

$\omega = \dfrac{d\phi}{dt} \Rightarrow \phi = \omega t + C$

From the problem statement, at $t=0$, the loop lies on the YZ plane $\Rightarrow \phi = \dfrac{\pi}{2}$ $\quad\phi$ is measured from X-axis

at $t=0$ $\quad \cancel{\phi = \omega}$ $\quad \dfrac{\pi}{2} = \omega\times0 + C$

$C = \dfrac{\pi}{2}$

$\therefore\ \phi = \omega t + \dfrac{\pi}{2}$

$\therefore\ V_{emf} = -6\pi\cos\left(\omega t + \dfrac{\pi}{2}\right)\ mV$

$V_{emf} = 6\pi\sin(\omega t)\ mV$

a) $t = 1ms$ $\quad V_{emf} = 6\pi\sin(100\pi\times1\times10^{-3})\ mV$

$\boxed{V_{emf} = 5.8248\ mV}$ $\quad$ (radians)

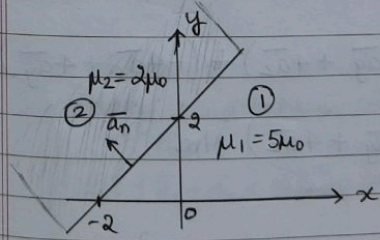
b) $t = 3ms$ $\quad I_{induced} = \dfrac{V_{emf}}{R}$

$V_{emf}(t=3ms) = 15.2496\ mV$

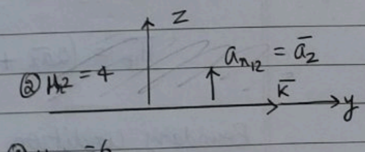
$I_{induced} = \dfrac{15.2496\times10^{-3}}{0.1}$

$\boxed{I_{induced} = 0.1525\ A}$

Q3.



Ex 1



Ex 2

Ex 1 $\quad$ Given that $\bar{H}_1 = -2\bar{a}_x + 6\bar{a}_y + 4\bar{a}_z$ A/m in a region defined by $\quad y - x - 2 \le 0$, where $\mu_1 = 5\mu_0$.

Compute (a) $\bar{B}_1$ $\quad$ (b) $\bar{H}_2$ and $\bar{B}_2$ in the region $y - x - 2 \ge 0$, where $\mu_2 = 2\mu_0$

Ex 1

(a) $\bar{B}_1 = \mu_1 \bar{H}_1$ $\mu_1 = 5\mu_0 = 5(4\pi \times 10^{-7}) = 2\pi \times 10^{-6}$

$\bar{B}_1 = 2\pi(-2\bar{a}_x + 6\bar{a}_y + 4\bar{a}_z)$ $\mu Wb/m^2$

$\bar{B}_1 = (-4\pi\bar{a}_x + 12\pi\bar{a}_y + 8\pi\bar{a}_z)$ $\mu Wb/m^2$

$\bar{B}_1 = (-12.566\bar{a}_x + 37.699\bar{a}_y + 25.1327\bar{a}_z)$ $\mu Wb/m^2$

(b) From fundamentals,

$$\bar{a}_n = \frac{\nabla f}{|\nabla f|}$$

$f(x,y) = y - x - 2$

$\nabla f = \frac{\partial f}{\partial x}\bar{a}_x + \frac{\partial f}{\partial y}\bar{a}_y = -\bar{a}_x + \bar{a}_y$

$|\nabla f| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

$\bar{a}_n = \frac{1}{\sqrt{2}}(\bar{a}_x + \bar{a}_y)$

$\bar{H}_{1n} = (\bar{H}_1 \cdot \bar{a}_n)\bar{a}_n = \left(\frac{+2}{\sqrt{2}} + \frac{6}{\sqrt{2}}\right)\bar{a}_n = 4\sqrt{2}\,\bar{a}_n$

$\bar{H}_{1n} = (-4\bar{a}_x + 4\bar{a}_y)$ A/m

$\bar{H}_1 = \bar{H}_{1n} + \bar{H}_{1t}$

$\bar{H}_{1t} = (-2\bar{a}_x + 6\bar{a}_y + 4\bar{a}_z) - (-4\bar{a}_x + 4\bar{a}_y)$

$\bar{H}_{1t} = (2\bar{a}_x + 2\bar{a}_y + 4\bar{a}_z)$ A/m

Boundary condition

Assuming no current in the interface

$\bar{H}_{1t} = \bar{H}_{2t} = (2\bar{a}_x + 2\bar{a}_y + 4\bar{a}_z)$ A/m

$\bar{B}_{1n} = \bar{B}_{2n} \Rightarrow \mu_1 \bar{H}_{1n} = \mu_2 \bar{H}_{2n}$

$\bar{H}_{2n} = \frac{\mu_1}{\mu_2}\bar{H}_{1n}$

$\bar{H}_{2n} = \frac{5\mu_0}{2\mu_0}\{-4\bar{a}_x + 4\bar{a}_y\}$

$\bar{H}_{2n} = (-10\bar{a}_x + 10\bar{a}_y)$ A/m

$\bar{H}_2 = \bar{H}_{2n} + \bar{H}_{2t} = (-8\bar{a}_x + 12\bar{a}_y + 4\bar{a}_z)$ A/m

$\bar{B}_2 = \mu_2 \bar{H}_2$ $\mu_2 = 2\mu_0 = 2(4\pi \times 10^{-7}) = \cancel{8\pi \times 10} = 0.8\pi \times 10^{-6}$

$\bar{B}_2 = (-20.106\bar{a}_x + 30.159\bar{a}_y + 10.05\bar{a}_z)$ $\mu Wb/m^2$

Ex2 The XY-plane serves as the interface between 2 different media.
Medium 1 ($z<0$) is filled with a material whose $\mu_r = 6$ and medium 2 ($z>0$) is filled with a material having $\mu_r = 4$.
If the interface carries a current $\left(\frac{1}{\mu_0}\right)\bar{a}_y$ mA/m and $\bar{B}_2 = 5\bar{a}_x + 8\bar{a}_z$ mWb/m², find $\bar{H}_1$ and $\bar{B}_1$.

Ex2 Since the interface carries a current, the following boundary condition is used:

(i) $\bar{B}_{1n} = \bar{B}_{2n} \Rightarrow \mu_1 \bar{H}_{1n} = \mu_2 \bar{H}_{2n}$

(ii) $(\bar{H}_1 - \bar{H}_2) \times \bar{a}_{n_{12}} = \bar{K}$

Let $\bar{B}_1 = (B_x, B_y, B_z)$ mWb/m²

$\bar{B}_{1n} = \bar{B}_{2n} = 8\bar{a}_z$ $\bar{B}_2 = \bar{B}_{2t} + \bar{B}_{2n}$ — 1

$= B_{2t}\bar{a}_x + B_{2n}\bar{a}_z$

$$\bar{H_2} = \frac{\bar{B_2}}{\mu_2} = \frac{1}{4\mu_0}(5\bar{a_x} + 8\bar{a_z}) \quad \text{—} \ (2)$$

$$\bar{H_1} = \frac{\bar{B_1}}{\mu_1} = \frac{1}{6\mu_0}\{B_x\bar{a_x} + B_y\bar{a_y} + B_z\bar{a_z}\} \quad \text{—} \ (3)$$

Having found the normal components, we can find the tangential components from the cross-product condition $(\bar{H_1} - \bar{H_2}) \times \bar{a}_{n_{12}} = \bar{K}$

$$\bar{H_1} \times \bar{a}_{n_{12}} = \bar{H_2} \times \bar{a}_{n_{12}} + \bar{K}$$

$$\frac{1}{6\mu_0}\{B_x\bar{a_x} + B_y\bar{a_y} + B_z\bar{a_z}\} \times \bar{a_z}$$

$$= \frac{1}{4\mu_0}(5\bar{a_x} + 8\bar{a_z}) \times \bar{a_z} + \frac{1}{\mu_0}\bar{a_y}$$

$$\frac{1}{6}(-B_x\bar{a_y} + B_y\bar{a_x}) = \frac{1}{4}(-5\bar{a_y}) + \bar{a_y} = -\frac{1}{4}\bar{a_y}$$

$$B_x = \frac{6}{4} = 1.5 \qquad B_y = 0 \qquad B_z = 8$$

$$\therefore \ \bar{B_1} = (1.5\bar{a_x} + 8\bar{a_z}) \ \text{mWb/m}^2$$

$$\bar{H_1} = \frac{\bar{B_1}}{\mu_1} = \frac{1}{6\mu_0}(1.5\bar{a_x} + 8\bar{a_z})$$

$$\bar{H_1} = (198.943\,\bar{a_x} + 1061.033\,\bar{a_z}) \ \text{A/m}$$

$$\bar{H_2} = (994.72\,\bar{a_x} + 1591.54\,\bar{a_z}) \ \text{A/m}$$

Thanks for checking out this space!!

Before every exam over the past two semesters & even now, we turned to Divya's running class notes for simple explanations to complex topics & as a useful collection of problems with solutions.
We wanted to share it with everyone, while giving her due credit.
Hope you found it as helpful as we did!

- Yours in B Section