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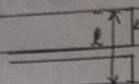
UNIT 2

HERTZIAN DIPOLE ANTENNA & RADIATION CHARACTERISTICS

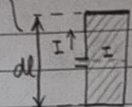
Hertzian dipole Antenna
- Simplest form of antenna



length $l \leq \frac{\lambda}{50}$

Area of plot = Idl

The wire antenna is called
- Hertzian dipole antenna
- Elemental dipole
- Infinitisimal dipole



$$\boxed{\bar{H} = \frac{1}{\mu} \nabla \times \bar{A}}$$

$$\nabla \times \bar{H} = j\omega\varepsilon\bar{E} + \bar{J}$$

$$\frac{1}{\mu}\nabla \times \nabla \times \bar{A} = j\omega\varepsilon\bar{E} + \bar{J}$$

$$\nabla(\nabla.\bar{A}) - \nabla^2\bar{A} = \mu(j\omega\varepsilon\bar{E} + \bar{J})$$

$$\boxed{\bar{E} = \frac{1}{j\omega\varepsilon\mu}\left[\nabla(\nabla.\bar{A}) - \nabla^2\bar{A} - \mu\bar{J}\right]}$$

For an operating frequency of f = 1GHz,

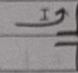
$$c = \lambda f \Rightarrow \lambda = \frac{c}{f} = \frac{3\times10^8}{1\times10^9} = 0.3\,m$$

$$\frac{\lambda}{50} = \frac{0.3}{50} = 6\,mm$$

∴ Length of antenna has to be ≤ 6mm to be called a ~~HERZTIAN~~ HERTZIAN DIPOLE

$$\bar{J}dv' = \bar{J}\underbrace{dx'dy'}_{\text{Area}}\bar{dz}'$$

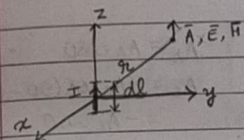
(J: current density)



$$= I\bar{dz}'$$

$$\tilde{I} = I\cos(\omega t)$$

constant phase throughout the length of the antenna



Replacing $J\,dv'$ in $\bar{A}$ expression

$$\bar{A} = \int_{-dl/2}^{dl/2} \frac{\mu}{4\pi r} e^{-jkr}\, I\bar{dz}'$$

↳ a vector ∵ I is a scalar

Scalar form $$A_z = \int_{z=-dl/2}^{dl/2} \frac{\mu I}{4\pi r} e^{-jkr}\, dz'$$

Assume I is constant throughout the length of the antenna
This approximation does not hold good for long antennas (Not constant with time)

$$A_z = \frac{\mu I e^{-jkr}}{4\pi r}\int_{-dl/2}^{dl/2} dz' = \frac{\mu I e^{-jkr}}{4\pi r}\left\{\frac{dl}{2} - \left(-\frac{dl}{2}\right)\right\}$$

$$= \frac{\mu(\overbrace{Idl}^{\text{current element}})}{4\pi r} e^{-jkr}$$

$$\boxed{A_z = \frac{\mu(Idl)}{4\pi r} e^{-jkr}}$$

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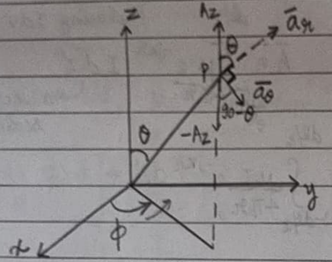
$$\bar{H} = \frac{1}{\mu}\nabla\times\bar{A} = \frac{1}{\mu}\times\frac{1}{r^2\sin\theta}\begin{vmatrix}\bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & rA_\theta & r\sin\theta\,A_\phi\end{vmatrix}$$



$A_r = A_z\cos\theta$

$A_\theta = -A_z\cos(90^\circ-\theta)$

$= -A_z\sin\theta$

$A_\phi = A_z\cos 90^\circ = 0$

$$\cancel{\bar{H} = \frac{1}{\mu}\times\frac{1}{r^2\sin\theta}\begin{vmatrix}\bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_z\cos\theta & (-A_z\sin\theta)r & 0\end{vmatrix}}$$

Dipole is placed symmetrical to z-axis

$\Rightarrow$ No ϕ variation

$\therefore \frac{\partial}{\partial\phi} = 0$

$$\cancel{= \frac{1}{\mu r^2\sin\theta}\times\left[\left\{\frac{\partial}{\partial r}(-A_z\sin\theta\, r) - \frac{\partial}{\partial\theta}(A_z\cos\theta)\right\}\right.}$$

$$\cancel{= \frac{1}{\mu}\times\frac{1}{r^2\sin\theta}\begin{vmatrix}\bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & 0 \\ A_z\cos\theta & -rA_z\sin\theta & 0\end{vmatrix}}$$

$$\cancel{= \frac{1}{\mu r^2\sin\theta}\times\left\{\left[\frac{\partial}{\partial r}(-rA_z\sin\theta) - \frac{\partial}{\partial\theta}(A_z\cos\theta)\right]\bar{a}_r - r\bar{a}_\theta\left[\frac{\partial}{\partial r}(0) - 0\right]\right\} + r\sin\theta\,\bar{a}_\phi\left[\frac{\partial}{\partial r}(-rA_z\sin\theta) - \frac{\partial}{\partial\theta}(A_z\cos\theta)\right]}$$

$$\cancel{= \frac{1}{\mu r^2\sin\theta}\left\{(-A_z\sin\theta + A_z\sin\theta)\bar{a}_r + 0 + r\sin\theta(-A_z\sin\theta + A_z\sin\theta)\bar{a}_\phi\right\}}$$

$\cancel{\boxed{\bar{H} = 0}}$

Dipole is placed symmetrical to z-axis

$\Rightarrow$ No ϕ-variation $\qquad \therefore \frac{\partial}{\partial\phi} = 0$

$$\bar{H} = \frac{1}{\mu}\times\frac{1}{r^2\sin\theta}\begin{vmatrix}\bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & 0 \\ A_z\cos\theta & -rA_z\sin\theta & 0\end{vmatrix}$$

$$= \frac{1}{\mu r^2\sin\theta}\left[\bar{a}_r\left\{\frac{\partial}{\partial\theta}(0) + 0\right\} - r\bar{a}_\theta\left\{\frac{\partial}{\partial r}(0) - 0\right\} + r\sin\theta\,\bar{a}_\phi\left\{\frac{\partial}{\partial r}(-rA_z\sin\theta) - \frac{\partial}{\partial\theta}(A_z\cos\theta)\right\}\right]$$

$$\cancel{= \frac{1}{\mu r^2\sin\theta}\, r\sin\theta\,(-A_z\sin\theta +}$$

$$\bar{H} = -\frac{\bar{a}_\phi}{\mu r}\left[\frac{\partial}{\partial r}(rA_z\sin\theta) - A_z\sin\theta\right]$$

$$= -\frac{\bar{a}_\phi\sin\theta}{\mu r}\left[\frac{\partial}{\partial r}(rA_z) - A_z\right]$$

$$\frac{\partial}{\partial r}(rA_z) = r\frac{\partial}{\partial r}A_z + A_z$$

$$= -\frac{\bar{a}_\phi\sin\theta}{\mu r}\left[r\frac{\partial}{\partial r}A_z + A_z - A_z\right]$$

$$\bar{H} = -\frac{\bar{a}_\phi\sin\theta}{\mu r}\left(r\frac{\partial}{\partial r}A_z\right)$$

$$\frac{\partial}{\partial r}A_z = \frac{\partial}{\partial r}\left\{\frac{\mu I dl}{4\pi r}e^{-jkr}\right\} = \frac{\mu I dl}{4\pi}\left[\frac{\partial}{\partial r}\left(\frac{e^{-jkr}}{r}\right)\right]$$

$$\frac{\partial A_z}{\partial r} = \frac{\mu I dl}{4\pi}\left[\frac{r(-jk)e^{-jkr} - e^{-jkr}}{r^2}\right]$$

$$r\frac{\partial A_z}{\partial r} = \frac{-\mu I dl}{4\pi}\left[jk + \frac{1}{r}\right]e^{-jkr}$$

$$\bar H = \frac{-\bar a_\phi \sin\theta}{\mu r}\left[\frac{-\mu I dl}{4\pi}\left(jk+\frac{1}{r}\right)e^{-jkr}\right]$$

$$\boxed{\bar H = \bar a_\phi \sin\theta \left(\frac{Idl}{4\pi}\right)\left(\frac{jk}{r}+\frac{1}{r^2}\right)e^{-jkr}}$$

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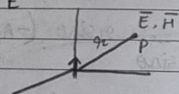
To determine electric field $\bar E$



$\nabla\times\bar H = j\omega\varepsilon\bar E + \bar J$

In the field region $\bar J = 0$

$\Rightarrow \nabla\times\bar H = j\omega\varepsilon\bar E \quad \Rightarrow \bar E = \frac{1}{j\omega\varepsilon}(\nabla\times\bar H)$

$$\bar E = \frac{1}{j\omega\varepsilon}\times\frac{1}{r^2\sin\theta}\begin{vmatrix} \bar a_r & r\bar a_\theta & r\sin\theta\,\bar a_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ H_r & rH_\theta & r\sin\theta H_\phi \end{vmatrix}$$

$\bar H$ has only $\bar a_\phi$ component $\Rightarrow H_r = 0$

$\Rightarrow H_\theta = 0$

Source placed symmetrical about z-axis → No variation in ϕ

$\Rightarrow \frac{\partial}{\partial\phi} = 0$

$\bar E = \bar a_r E_r + \bar a_\theta E_\theta$

$$\bar E = \frac{1}{j\omega\varepsilon}\times\frac{1}{r^2\sin\theta}\begin{vmatrix} \bar a_r & r\bar a_\theta & r\sin\theta\,\bar a_\phi \\ \partial/\partial r & \partial/\partial\theta & 0 \\ 0 & 0 & r\sin\theta H_\phi \end{vmatrix}$$

$$\bar E = \frac{1}{j\omega\varepsilon}\times\frac{1}{r^2\sin\theta}\left\{\bar a_r\frac{\partial}{\partial\theta}(r\sin\theta H_\phi) - r\bar a_\theta\frac{\partial}{\partial r}(r\sin\theta H_\phi)\right\}$$

$$E_r = \frac{1}{j\omega\varepsilon}\times\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta H_\phi) = \frac{1}{j\omega\varepsilon r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta H_\phi)$$

$$= \frac{1}{j\omega\varepsilon r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left[\frac{Idl\sin^2\theta}{4\pi}\left(\frac{jk}{r}+\frac{1}{r^2}\right)e^{-jkr}\right]\right\}$$

$$= \frac{1}{j\omega\varepsilon r\cancel{\sin\theta}}\times 2\cancel{\sin\theta}\cos\theta\left(\frac{Idl}{4\pi}\right)\left(\frac{jk}{r}+\frac{1}{r^2}\right)e^{-jkr}$$

$$\boxed{E_r = \frac{Idl\cos\theta}{j\omega\varepsilon 2\pi}\left(\frac{jk}{r^2}+\frac{1}{r^3}\right)e^{-jkr}}$$

$$E_\theta = \frac{-r}{j\omega\varepsilon r^2\sin\theta}\frac{\partial}{\partial r}(r\sin\theta H_\phi)$$

$$\cancel{E_\theta = \frac{j}{\omega\varepsilon r\sin\theta}\left[r\frac{\partial}{\partial r}(\sin\theta H_\phi) + \sin\theta H_\phi\right]}$$

$$= \frac{j}{\omega\varepsilon r}\left[H_\phi + r\frac{\partial}{\partial r}(H_\phi)\right]$$

$$\cancel{= \frac{j}{\omega\varepsilon r}\left[\left(\frac{Idl\sin\theta}{4\pi}\right)\left(\frac{jk}{r}+\frac{1}{r^2}\right)e^{-jkr} + r\frac{Idl\sin\theta}{4\pi}\right.}$$

$$E_\theta = \frac{j}{\omega\varepsilon r\sin\theta}\sin\theta\,\frac{\partial}{\partial r}(rH_\phi)$$

$$= \frac{j}{\omega\varepsilon r}\frac{\partial}{\partial r}(rH_\phi)$$

$$= \frac{j}{\omega\varepsilon r}\frac{\partial}{\partial r}\left[\frac{Idl\sin\theta}{4\pi}\left(jk+\frac{1}{r}\right)e^{-jkr}\right]$$

$$= \frac{jIdl\sin\theta}{4\pi\omega\varepsilon r}\left[\cancel{jk}\,\frac{\partial}{\partial r}\left(jk+\frac{1}{r}\right)\times e^{-jkr} + \left(jk+\frac{1}{r}\right)\frac{\partial}{\partial r}e^{-jkr}\right]$$

$$= \frac{jIdl\sin\theta}{4\pi\omega\varepsilon r}\left[\frac{-1}{r^2}e^{-jkr} + \left(jk+\frac{1}{r}\right)(-jk)e^{-jkr}\right]$$

$$= \frac{jIdl\sin\theta}{4\pi\omega\varepsilon r}\left[k^2 - \frac{jk}{r} - \frac{1}{r^2}\right]e^{-jkr}$$

$$= \frac{Idl\sin\theta}{4\pi\omega\varepsilon}\left[\frac{jk^2}{r} - j\left(\frac{jkr+1}{r^3}\right)\right]e^{-jkr}$$

$$= \frac{Idl\sin\theta}{4\pi\omega\varepsilon}\left[\frac{jk^2}{r} + \frac{kr - j}{r^3}\right]e^{-jkr}$$

$$\boxed{E_\theta = \frac{Idl\sin\theta}{4\pi\omega\varepsilon}\left[\frac{jk^2}{r} + \frac{k}{r^2} - \frac{j}{r^3}\right]e^{-jkr}}$$

* Fields which vary as $\frac{1}{r}$ are known as distant fields / far fields / radiation fields
* Fields which vary as $\frac{1}{r^2}$ are known as inductive fields
* Fields which vary as $\frac{1}{r^3}$ are known as capacitive fields / electrostatic fields

DISTANT FIELDS (FAR FIELDS) or RADIATION FIELDS

$$\boxed{H_{\phi rad} = \frac{jkIdl\sin\theta}{4\pi r}e^{-jkr}}$$

$$\boxed{E_{\theta rad} = \frac{jk^2 Idl\sin\theta}{4\pi\varepsilon\omega r}e^{-jkr}}$$

for Hertzian dipole

Ratio
$$\frac{E_{\theta rad}}{H_{\phi rad}} = \frac{\dfrac{jk^2Idl\sin\theta}{4\pi\varepsilon\omega r}e^{-jkr}}{\dfrac{jkIdl\sin\theta}{4\pi r}e^{-jkr}}$$

$$= \frac{k}{\omega\varepsilon} = \frac{\omega\sqrt{\mu\varepsilon}}{\omega\varepsilon} = \frac{\sqrt{\mu\varepsilon}}{\varepsilon}$$

$$\boxed{\frac{E_{\theta rad}}{H_{\phi rad}} = \sqrt{\frac{\mu}{\varepsilon}} = \eta}$$

η is the intrinsic impedance of the medium into which the antenna is radiating

* If medium is free space $\eta = \eta_0 = 120\pi$

POWER DENSITY $\bar{S} = \frac{1}{2}\mathrm{Re}\{\bar{E}\times\bar{H}^*\}$ $\bar{E}\to V/m$ $\bar{H}\to A/m$

$\Rightarrow \bar{S}\to W/m^2$

$\bar{E} = \bar{a}_\theta E_{\theta rad}$ $\bar{H} = \bar{a}_\phi H_{\phi rad}$

$\bar{H}^* = \bar{a}_\phi H^*_{\phi rad}$

r θ ϕ

$\theta\times\phi = r$

$$\bar{S} = \frac{1}{2}\mathrm{Re}\{(\bar{a}_\theta E_{\theta rad})\times(\bar{a}_\phi H^*_{\phi rad})\}$$

$$= \frac{1}{2}\mathrm{Re}\{E_{\theta rad}H^*_{\phi rad}\,\bar{a}_r\}$$

$$= \bar{a}_r\frac{1}{2}\mathrm{Re}\{\eta H_{\phi rad}H^*_{\phi rad}\}$$

$$\bar{S} = \bar{a}_r \frac{\eta}{2} Re\{ H_{\phi rad} H^*_{\phi rad}\}$$

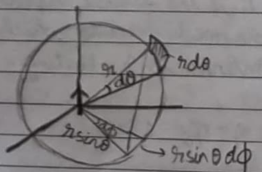
$$\boxed{\bar{S} = \bar{a}_r \frac{\eta}{2} |H_{\phi rad}|^2}$$

$$\bar{S} = \bar{a}_r \frac{\eta}{2} \left(\frac{K I dl \sin\theta}{4\pi r}\right)^2$$

$$\bar{S} = \bar{a}_r \frac{\eta}{2} \left(\frac{k^2 I^2 dl^2 \sin^2\theta}{16\pi^2 r^2}\right)$$

$$\boxed{\bar{S} = \bar{a}_r \frac{\eta k^2 I^2 dl^2 \sin^2\theta}{32\pi^2 r^2}}$$

TOTAL POWER RADIATED BY HERTZIAN DIPOLE



$$da = r^2 \sin\theta \, d\theta \, d\phi$$
$$\bar{da} = \bar{a}_r \, r^2 \sin\theta \, d\theta \, d\phi$$

$$P = \int_{Area} \bar{S} \cdot \bar{da} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} S r^2 \sin\theta \, d\theta \, d\phi$$

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left\{ (r^2 \sin\theta) \left(\frac{\eta k^2 I^2 dl^2 \sin^2\theta}{32\pi^2 r^2} \right) d\theta \, d\phi \right\}$$

$$= \frac{\eta k^2 I^2 dl^2}{32\pi^2} \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin^3\theta \, d\theta$$

$$P = \frac{\eta k^2 I^2 dl^2}{32\pi^2} \times 2\pi \int_{\theta=0}^{\pi} \sin^3\theta \, d\theta$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$P = \frac{\eta k^2 I^2 dl^2}{16\pi} \int_{\theta=0}^{\pi} \sin^3\theta \, d\theta$$

$$= \frac{\eta k^2 I^2 dl^2}{16\pi} \int_{\theta=0}^{\pi} \left(\frac{3\sin\theta - \sin 3\theta}{4} \right) d\theta$$

$$= \frac{\eta k^2 I^2 dl^2}{64\pi} \left[3\int_{\theta=0}^{\pi} \sin\theta \, d\theta - \int_{\theta=0}^{\pi} \sin 3\theta \, d\theta \right]$$

$$= \frac{\eta k^2 I^2 dl^2}{64\pi} \left[-3\cos\theta \Big|_0^{\pi} + \frac{1}{3} \cos 3\theta \Big|_0^{\pi} \right]$$

$$= \frac{\eta k^2 I^2 dl^2}{64\pi} \left[-3(\cos\pi - \cos 0) + \frac{1}{3}(\cos 3\pi - \cos 0) \right]$$

$$= \frac{\eta k^2 I^2 dl^2}{64\pi} \left[6 + \frac{1}{3}(-1-1) \right]$$

$$= \frac{\eta k^2 I^2 dl^2}{64\pi} \left[6 - \frac{2}{3} \right] = \frac{\eta k^2 I^2 dl^2}{64\pi} \times \frac{16}{3}$$

$$\boxed{P = \frac{\eta k^2 I^2 dl^2}{12\pi}}$$

Q. Determine the power radiated by a dipole of length $\frac{\lambda}{50}$ which carries a current of 5 A.

$c = \lambda f$ $\qquad \eta = 120\pi$

$f = \frac{c}{\lambda}$ $\qquad \omega = 2\pi f = \frac{2\pi c}{\lambda}$

$k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c} = \frac{2\pi c}{\lambda c} = \frac{2\pi}{\lambda}$

$$P = \frac{\eta k^2 I^2 dl^2}{12\pi} = \frac{\cancel{120}^{10}\cancel{\pi}\left(\frac{2\pi}{\lambda}\right)^2 (5)^2 \left(\frac{\lambda}{50}\right)^2}{\cancel{12\pi}}$$

$$= \frac{4\pi^2}{\cancel{\lambda^2}} \times \cancel{25} \times \frac{\cancel{\lambda^2}}{\cancel{2500}_{100}} \times 1\cancel{0}$$

$= 0.4\pi^2$

$\boxed{P = 3.95\ W}$

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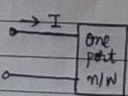
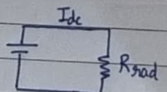
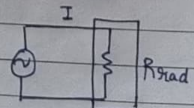
Consider the dipole to be a one-port network



Equivalent resistance of the network : R_{rad}



DC $\Rightarrow P = I_{dc}^2 R_{rad}$



AC $\quad P = \left(\frac{I}{\sqrt{2}}\right)^2 R_{rad}$

$i(t) = Re\{Ie^{j\omega t}\}$

$= I\cos\omega t$

$I_{rms} = I_{eff} = I_{dc} = \frac{I}{\sqrt{2}}$

$$P = \frac{I^2 R_{rad}}{2} = \frac{\eta k^2 I^2 dl^2}{12\pi}$$

$$R_{rad} = \frac{\eta k^2 \cancel{I^2} dl^2}{6\pi}$$

If the dipole is radiated into free space, obtain R_{rad} in terms of $\left(\frac{dl}{\lambda}\right)$

$\eta = \eta_0 = 120\pi$

$k = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c} = \frac{2\pi f}{\lambda f} = \frac{2\pi}{\lambda}$

$$R_{rad} = 120\pi\left(\frac{2\pi}{\lambda}\right)^2 dl^2 \times \frac{1}{6\pi} = \frac{480\pi^3}{6\pi}\left(\frac{dl}{\lambda}\right)^2$$

$$\boxed{R_{rad} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2}$$

Example: A Hertzian dipole of length 5 cm is radiating into free space at 100 MHz. Determine the power radiated and the radiation resistance of the dipole if the ~~instantantan~~ instantaneous current flowing into the dipole is $i(t) = 10\cos(2\pi \times 10^8 t)$

$\omega = 2\pi \times 10^8$ rad/s $\qquad dl = 5$ cm $\qquad \eta = \eta_0 = 120\pi$

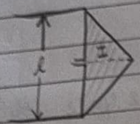
$f = 10^8$ Hz $= 100$ MHz $\qquad \lambda = \frac{c}{f} = \frac{3\times10^8}{10^8} = 3$ m

$$R_{rad} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2 = 80\pi^2\left(\frac{0.05}{3}\right)^2$$

$\boxed{R_{rad} = 0.219\ \Omega}$

$P = \frac{I^2 R_{rad}}{2} = \frac{(10)^2}{2} \times 0.219 \Rightarrow \boxed{P = 10.95\ W}$

SHORT / SMALL DIPOLE $\quad l \sim \frac{\lambda}{10}$



Area of the plot $= \frac{1}{2} I l$

$$H_{\phi rad} = \frac{jkIl\sin\theta}{8\pi r} e^{-jkr}$$

$$E_{\theta rad} = \eta H_{\phi rad}$$

$$P = \frac{\eta k^2 I^2 l^2}{48\pi} \qquad R_{rad} = 20\pi^2 \left(\frac{l}{\lambda}\right)^2$$

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Example 1: A short dipole of 15cm is operating at 200 MHz. Determine the power radiated and the radiation resistance of the dipole if the instantaneous current flowing in the dipole is $i(t) = 5\cos(\omega t)$. $\mathrm{Re}\{Ie^{j\omega t}\} = I\cos\omega t$

↳ Phasor current $\qquad \lambda = c/f = \frac{3\times10^8}{200\times10^6} = 1.5$

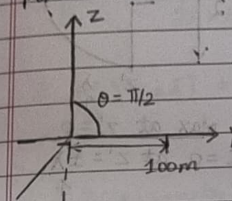
$$R_{rad} = 20\pi^2\left(\frac{l}{\lambda}\right)^2 = 20\pi^2\left(\frac{15}{150}\right)^2$$

$$\boxed{R_{rad} = 1.974\ \Omega}$$

$$P_{rad} = \frac{I^2 R_{rad}}{2} = \frac{5^2\times1.974}{2}$$

$$\boxed{P_{rad} = 24.67\ W}$$

Example 2: The distant magnetic field at a distance of 100 m on Y-axis due to a z-directed short dipole at origin is $H_{\phi rad} = j0.1e^{-j200\pi}$. Determine the ~~maximum~~ power density at a distance 100 m, power radiated and radiation resistance of the dipole.



* Power density depends on distance $(\alpha \frac{1}{r^2})$
* Power radiated is independent of distance r

$$H_{\phi rad} = \frac{jkIl\sin\theta}{8\pi r} e^{-jkr} \qquad l \sim \frac{\lambda}{10}$$

$$j(0.1)e^{-j200\pi} = \frac{jkIl\sin\theta}{8\pi r} e^{-jkr}$$

$r = 100\text{ m} \Rightarrow k = 2\pi$

$\theta = 90°$

$$\frac{j2\pi\, I \times \cancel{100}\,l \times 1}{4\cdot 8\pi\times100} e^{-j200\pi} = j(0.1)e^{-j200\pi}$$

$$\frac{Il}{400} = 0.1$$

$$Il = 40\ Am$$

$$\bar{S} = \frac{\eta}{2}|H_{\phi rad}|^2 \bar{a_y} = \frac{120\pi}{2}(0.1)^2 \bar{a_y}$$

$$\bar{S} = 1.885\ \bar{a_y}\ W/m^2$$

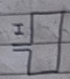
$$\bar{S} = \frac{\eta k^2 I^2 l^2 \sin^2\theta}{128\pi^2 r^2}\bar{a_y} = \frac{(120\pi)(2\pi)^2(40)^2(1)^2}{128\pi^2(100)^2}\bar{a_y}$$

$$\boxed{\bar{S} = 1.885\ \bar{a_y}\ W/m^2} \qquad \bar{a_r} = \bar{a_y} \because \text{on Y-axis}$$

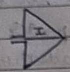
$$P = \frac{\eta k^2 I^2 l^2}{48\pi} = \frac{(120\pi)(2\pi)^2(40)^2}{48\pi} \Rightarrow \boxed{P = 157.914\ kW}$$

$$R_{rad} = 20\pi^2\left(\frac{l}{\lambda}\right)^2 = 20\pi^2\left(\frac{\lambda/10}{\lambda}\right)^2 = \frac{20\pi^2}{100} \Rightarrow \boxed{R_{rad} = 1.974\ \Omega}$$

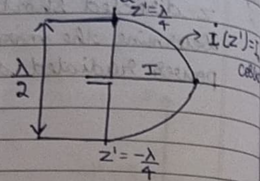
HERTZIAN DIPOLE



SHORT DIPOLE



$\frac{\lambda}{2}$ DIPOLE



I max at $z'=0$

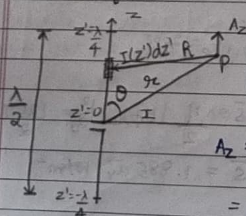
$I=0$ at $z'=\pm\frac{\lambda}{4}$

Sinusoidal distribution in between

$\frac{\lambda}{2}$ dipole

phase constant

$I(z') = I_0 \cos(kz')$



Point P is at farfield

@ $z'=0$ $\quad I(z')=I_0$

@ $z'=\pm\frac{\lambda}{4}$ $\quad k=\frac{2\pi}{\lambda}$

$kz' = \frac{2\pi}{\lambda}\left(\pm\frac{\lambda}{4}\right) = \pm\frac{\pi}{2}$

$\cos(\pi/2) = 0$

$I(z') = 0$

$$A_z = \frac{\mu}{4\pi}\int \frac{I(z')e^{-jkr}}{r}dz'$$

$$= \frac{\mu}{4\pi r}e^{-jkr}\int_{z'=-\lambda/4}^{z'=\lambda/4} I(z')\,dz'$$

$$= \frac{\mu}{4\pi r}e^{-jkr}\int_{z'=-\lambda/4}^{z'=\lambda/4} I_0\cos(kz')\,dz'$$

$$= \frac{\mu I_0}{4\pi r}e^{-jkr}\frac{1}{k}\sin(kz')\Big|_{-\lambda/4}^{\lambda/4}$$

$$A_z = \frac{\mu I_0}{4\pi r k}\left[1-(-1)\right]$$

$$\boxed{A_z = \frac{\mu I_0 e^{-jkr}}{2\pi k r}}$$ approximation

$$\bar{H} = \frac{1}{\mu}(\nabla\times\bar{A}) = \frac{1}{\mu}\times\frac{1}{r^2\sin\theta}\begin{vmatrix}\bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix}$$

due to symmetry $\frac{\partial}{\partial\phi}=0$ $\quad A_\phi = 0$

$$\bar{H} = \frac{1}{\mu}\times\frac{1}{r^2\sin\theta}\begin{vmatrix}\bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & 0 \\ A_r & rA_\theta & 0\end{vmatrix}$$

$A_r = A_z\cos\theta \qquad A_\theta = -A_z\sin\theta \qquad A_\phi = 0$

$$\bar{H} = \frac{1}{\mu}\times\frac{1}{r^2\sin\theta}\begin{vmatrix}\bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & 0 \\ A_z\cos\theta & -A_z\sin\theta\, r & 0\end{vmatrix}$$

$$= \frac{1}{\mu}\times\frac{1}{r^2\sin\theta}\times r\sin\theta\,\bar{a}_\phi\left(\frac{\partial}{\partial r}(-A_z r\sin\theta) - \frac{\partial}{\partial\theta}(A_z\cos\theta)\right)$$

$$= \frac{1}{\mu}\times\frac{1}{r}\left\{-\sin\theta\frac{\partial}{\partial r}(rA_z) + \sin\theta\,\cancel{\frac{\partial}{\partial\theta}}(A_z)\right\}\bar{a}_\phi$$

$$\bar{H} = \frac{\sin\theta}{\mu r}\left\{A_z - \frac{\partial}{\partial r}(rA_z)\right\}\bar{a}_\phi = \frac{\sin\theta}{\mu r}\left\{A_z - A_z - r\frac{\partial}{\partial r}A_z\right\}\bar{a}_\phi$$

$$\bar{H} = -\frac{\bar{a}_\phi\sin\theta}{\mu r}\left\{r\frac{\partial A_z}{\partial r}\right\}$$

$$\frac{\partial A_z}{\partial r} = \frac{\partial}{\partial r}\left\{\frac{\mu I_0}{2\pi k r} e^{-jkr}\right\} = \frac{\mu I_0}{2\pi k}\frac{\partial}{\partial r}\left\{\frac{e^{-jkr}}{r}\right\}$$

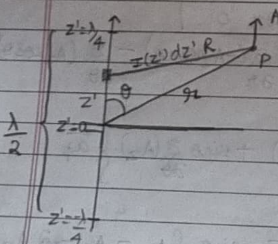
$$= \frac{\mu I_0}{2\pi k}\left[\frac{-e^{-jkr} + (-jk) r e^{-jkr}}{r^2}\right]$$

$$\frac{\partial A_z}{\partial r} = -\frac{\mu I_0}{2\pi k}\left[\frac{jk}{r} + \frac{1}{r^2}\right] e^{-jkr}$$

$$r\frac{\partial A_z}{\partial r} = \frac{-\mu I_0}{2\pi k}\left[jk + \frac{1}{r}\right] e^{-jkr}$$

$$\bar{H} = \frac{-\bar{a}_\phi \sin\theta}{\mu r}\left\{\frac{-\mu I_0}{2\pi k}\left[jk + \frac{1}{r}\right] e^{-jkr}\right\}$$

$$\boxed{\bar{H} = \frac{I_0 \sin\theta}{2\pi k}\left[\frac{jk}{r} + \frac{1}{r^2}\right] e^{-jkr}}$$



$$R \approx r - z'\cos\theta$$

$$A_z = \frac{\mu}{4\pi}\int_{z'=-\lambda/4}^{z'=\lambda/4} \frac{I(z')}{R} e^{-jkR}\, dz'$$

$$A_z = \frac{\mu}{4\pi}\int_{-\lambda/4}^{\lambda/4} \frac{I_0 \cos(kz')\, e^{-jk(r - z'\cos\theta)}}{r - z'\cos\theta}\, dz'$$

In the amplitude term $R \approx r$

$$kz'\Big|_{\lambda/4} = \frac{2\pi}{\lambda} \times \frac{\lambda}{4} = \frac{\pi}{2}$$

$$A_z = \frac{\mu I_0}{4\pi}\int_{-\lambda/4}^{\lambda/4} \frac{\cos(kz')}{r} e^{-jk(r - z'\cos\theta)}\, dz'$$

$$A_z = \frac{\mu I_0}{4\pi r} e^{-jkr}\int_{-\lambda/4}^{\lambda/4} \cos(kz')\left\{\cos(kz'\cos\theta) + j\sin(kz'\cos\theta)\right\} dz'$$

$$= \frac{\mu I_0}{4\pi r} e^{-jkr}\Bigg[\underbrace{\int_{-\lambda/4}^{\lambda/4} \cos(kz')\cos(kz'\cos\theta)\, dz'}_{\text{even function}} + j\underbrace{\int_{-\lambda/4}^{\lambda/4} \cos(kz')\sin(kz'\cos\theta)\, dz'}_{\text{odd function}}\Bigg]$$

$\Rightarrow$ symmetric integration $= 0$

$$= \frac{\mu I_0}{4\pi r} e^{-jkr}\int_{0}^{\lambda/4} 2\cos(kz')\cos(kz'\cos\theta)\, dz'$$

$$= \frac{\mu I_0}{2\pi r} e^{-jkr}\int_{0}^{\lambda/4} \frac{1}{2}\left[\cos(kz' + kz'\cos\theta) + \cos(kz' - kz'\cos\theta)\right] dz'$$

$$= \frac{\mu I_0}{4\pi r} e^{-jkr}\int_{0}^{\lambda/4} \cos(k(1+\cos\theta)z') + \cos(k(1-\cos\theta)z')\, dz'$$

$$= \frac{\mu I_0}{4\pi r} e^{-jkr}\left[\frac{1}{k(1+\cos\theta)}\sin(k(1+\cos\theta)z') + \frac{1}{k(1-\cos\theta)}\sin(k(1-\cos\theta)z')\right]_0^{\lambda/4}$$

$$= \frac{\mu I_0}{4\pi r} e^{-jkr}\left[\frac{\sin\left(\frac{\pi}{2}(1+\cos\theta)\right)}{k(1+\cos\theta)} + \frac{\sin\left(\frac{\pi}{2}(1-\cos\theta)\right)}{k(1-\cos\theta)}\right]$$

$$A_z = \frac{\mu I_0}{4\pi r k} e^{-jkr}\left[\frac{(1-\cos\theta)\sin\left[\frac{\pi}{2} + \frac{\pi}{2}\cos\theta\right] + (1+\cos\theta)\sin\left[\frac{\pi}{2} - \frac{\pi}{2}\cos\theta\right]}{\sin^2\theta}\right]$$

$$A_z = \frac{\mu I_0 e^{-jkr}}{4\pi r k \sin^2\theta}\left[(1-\cos\theta)\cos\left(\frac{\pi}{2}\cos\theta\right) + (1+\cos\theta)\cos\left(\frac{\pi}{2}\cos\theta\right)\right]$$

$$= \frac{\mu I_0 e^{-jkr}}{4\pi r k \sin^2\theta}\left\{2\,\cancel{\cos\theta}\cos\left(\frac{\pi}{2}\cos\theta\right)\right\}$$

$$\boxed{A_z = \left(\frac{\mu I_0}{2\pi r k}\right)\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\right)e^{-jkr}}$$

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$$\bar{H} = \frac{1}{\mu}(\nabla\times\bar{A}) = \frac{1}{\mu}\times\frac{1}{r^2\sin\theta}\begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\bar{a}_\phi\sin\theta \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

Symmetrically placed on z-axis $\frac{\partial}{\partial\phi} = 0 \quad A_\phi = 0$

$$A_r = A_z\cos\theta \qquad A_\theta = -A_z\sin\theta$$

$$\bar{H} = \frac{1}{\mu}\times\frac{1}{r^2\sin\theta}\begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\bar{a}_\phi\sin\theta \\ \partial/\partial r & \partial/\partial\theta & 0 \\ A_r & rA_\theta & 0 \end{vmatrix}$$

$$\bar{H}_\phi = \frac{1}{\mu r^2\sin\theta}\times r\sin\theta\left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}(A_r)\right\}\bar{a}_\phi$$

$$\cancel{= \frac{1}{\mu r}\left\{(A_\theta) + r\frac{\partial(A_\theta)}{\partial r} - \frac{\partial(A_r)}{\partial\theta}\right\}} \qquad \bar{H}_\phi = \left[\frac{1}{\mu r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{\mu r}\frac{\partial A_r}{\partial\theta}\right]$$

$$\cancel{\bar{H}_\phi = \frac{1}{\mu r}\left\{-A_z\sin\theta + r\frac{\partial}{\partial r}(-A_z\sin\theta) - \frac{\partial}{\partial\theta}(A_z\cos\theta)\right\}}$$

$$\cancel{r\frac{\partial(-A_z\sin\theta)}{\partial r} = r\frac{\partial}{\partial r}\left\{\left(\frac{-\mu I_0}{2\pi r k}\right)\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\right)e^{-jkr}\times\sin\theta\right\}}$$

$$\bar{H}_\phi = \frac{1}{\mu r}\frac{\partial}{\partial r}(r(-A_z\sin\theta)) = \frac{1}{\mu r}\frac{\partial}{\partial r}\left\{r\left(\frac{-\mu I_0}{2\pi r k}\right)\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\right)e^{-jkr}\sin\theta\right\}$$

$$= \frac{1}{\mu r}\left\{\frac{-\mu I_0}{2\pi k}\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)\frac{\partial}{\partial r}(e^{-jkr})\right\}$$

$$= \frac{-I_0}{2\pi r k}\,\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\,(-j\cancel{k})\,e^{-jkr}$$

$$\frac{1}{\mu r}\frac{\partial}{\partial r}(rA_\theta) = \frac{jI_0}{2\pi r}\left\{\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right\}e^{-jkr}.$$

$$\frac{-1}{\mu r}\frac{\partial}{\partial\theta}(A_z\cos\theta) = \frac{-1}{\mu r}\times\frac{\mu I_0}{2\pi r k}e^{-jkr}\frac{\partial}{\partial\theta}\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\cos\theta\right)$$

$$= \frac{-I_0}{2\pi k r^2}\frac{\partial}{\partial\theta}\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\right)\cos\theta$$

* This term varies with $\frac{1}{r^2}$ and hence does not contribute to radiation/distant field. Therefore, this term can be ignored while computing the expression for $\bar{H}_{\phi rad}$.

for $\frac{\lambda}{2}$ dipole, the expression for $\bar{H}_{\phi rad}$ is

$$\boxed{\bar{H}_{\phi rad} = \frac{jI_0}{2\pi r}\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)e^{-jkr}\,\bar{a}_\phi}$$

$$E_{\theta rad} = \eta H_{\phi rad}$$

$$\boxed{E_{\theta rad} = j\left(\frac{\eta I_0}{2\pi r}\right)\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)e^{-jkr}}$$

* Power density for $\frac{\lambda}{2}$ dipole

$$\bar{S} = \frac{1}{2}\mathrm{Re}\{\bar{E}\times\bar{H}^*\} = \frac{\bar{a}_r}{2}\mathrm{Re}\{E_{\theta rad}\cdot H^*_{\phi rad}\}$$

$$E_{\theta rad} = \eta H_{\phi rad}$$

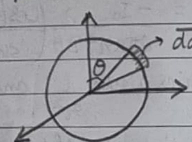
$$\bar{S} = \frac{\bar{a}_r}{2}\eta|H_{\phi rad}|^2$$

$$S = \frac{\eta}{2}\left[\left(\frac{I_0}{2\pi r}\right)\left\{\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right\}\right]^2$$

NOTE: Sometimes I_0 can be a complex quantity as it is phasor current

* Power radiated for $\frac{\lambda}{2}$ dipole

$$P = \int_A \bar{S}\cdot d\bar{a}$$



$$d\bar{a} = \bar{a}_r r^2 \sin\theta\, d\theta\, d\phi$$

$$P = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} S r^2 \sin\theta\, d\theta\, d\phi$$

$$P = r^2\int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi}\frac{\eta}{2}\left[\left(\frac{I_0}{2\pi r}\right)\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)\right]^2 \sin\theta\, d\theta$$

$$= \frac{\eta I_0^2}{8\pi^2}\times 2\pi\int_{\theta=0}^{\pi}\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\, d\theta$$

$$P = \frac{\eta I_0^2}{4\pi}\times(1.218)$$

$$|I_0|^2 \Rightarrow I_{eff} = \frac{I_0}{\sqrt{2}} \Rightarrow I_0 = \sqrt{2}\, I_{eff}$$

$$I_0^2 = 2 I_{eff}^2$$

$$P = \frac{\eta I_{eff}^2}{4\pi}\times 2\times 1.218$$

$$\boxed{P = \frac{1.218\eta}{2\pi} I_{eff}^2}$$

In free space $\eta = \eta_0 = 120\pi$

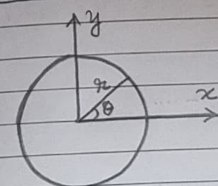
$$\frac{1.218\times 120\pi}{2\pi}\times I_{eff}^2 = 73.08\, I_{eff}^2$$

$$\boxed{P_{free\ space} \approx 73\, I_{eff}^2}$$

* Radiation resistance

$$P_{rad} = R_{rad}\, I_{eff}^2 \Rightarrow \boxed{R_{rad} = 73\,\Omega}$$

Some more antenna parameters:



Circumference $= 2\pi r$

A complete arc of a circle subtends an angle of 2π at the centre.

$2\pi^c \rightarrow$ arc of $2\pi r$ length

$1^c \rightarrow$ arc of r length

* In 3D, angle is measured in terms of STERADIAN

A spherical surface of $4\pi r^2$ subtends a solid angle of 4π steradian.

$4\pi r^2$ surface area $\longrightarrow$ covered by 4π steradian

$\Rightarrow$ 1 steradian covers a surface area of r^2.

Power radiated in 1 steradian = Power radiated through surface area of r^2

= Power density $\times\, r^2$

Radiation intensity $U = Sr^2$

$$\boxed{U = Sr^2}$$

* Types of radiators (or antennas)

(1) ISOTROPIC RADIATOR Radiates uniformly in all directions

(2) DIRECTIONAL RADIATOR (antenna) → In certain directions, it has more radiation compared to that in other directions

→ A sub-class of this radiator is the

OMNI-DIRECTIONAL RADIATOR

→ Radiates uniformly in a particular plane. ~~oth~~

→ While in plane orthogonal to this plane, the radiation is directional.

ISOTROPIC ANTENNA

P is power radiated

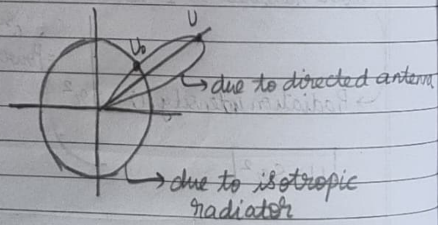
Power density $\boxed{S = \dfrac{P}{4\pi r^2}}$

Radiation intensity $\boxed{U_0 = Sr^2 = \dfrac{P}{4\pi}}$ (U_0 notation is used only for Isotropic antenna)

* DIRECTIVITY OF AN ANTENNA

The ratio of radiation intensity in a particular direction to the radiation intensity due to isotropic radiators.

$\boxed{D = \dfrac{U}{U_0}}$



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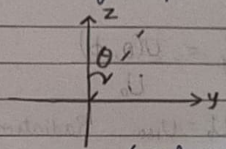
RADIATION PATTERN

It is a graphical representation of some parameters of an antenna

* E-field
* H-field
* U
* P
* Plane

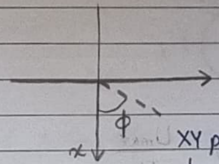
Radiation pattern is plotted along 2 orthogonal planes

*



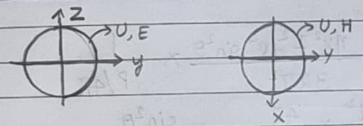
YZ plane (E-plane or θ-plane) plot

*



XY plane (H-plane or ϕ-plane) plot

* Isotropic antenna



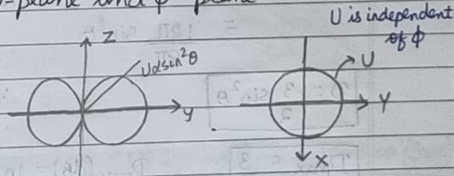
Example: Determine the radiation intensity of a Hertzian dipole and plot it along θ-plane and ϕ-plane

$U = S r^2$

$S = \dfrac{\eta k^2 I^2 dl^2 \sin^2\theta}{32\pi^2 r^2}$

$U = \dfrac{\eta k^2 I^2 dl^2 \sin^2\theta}{32\pi^2}$

$\Rightarrow U \propto \sin^2\theta$



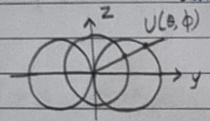
OMNIDIRECTIONAL ANTENNA

* No antenna radiates uniformly
Isotropic antenna is just used as a mathematical model to define Directivity
It is fictitious.

DIRECTIVITY

$$\text{Directivity} = \frac{U(\theta, \phi)}{U_0}$$

$U_0 = U_{iso}$ Radiation intensity of an isotropic antenna which radiates the same power



$$\text{Max Directivity} = \frac{U_{max}}{U_0}$$

HERTZIAN DIPOLE

$$\text{Directivity} = \frac{U}{U_0} = \frac{\eta k^2 I^2 dl^2 \sin^2\theta}{32\pi^2} \times \frac{1}{P/4\pi}$$

$$= \frac{\eta k^2 I^2 dl^2}{32\pi^2 \, 8\pi} \times \frac{4\pi}{\frac{\eta k^2 I^2 dl^2}{12\pi}} \sin^2\theta$$

$$= \frac{12\pi}{8\pi} \sin^2\theta$$

$$\boxed{D = \frac{3}{2}\sin^2\theta}$$

$$\boxed{D_{max} = \frac{3}{2}}$$

$$D_{max}(dB) = 10 \log_{10}(D_{max})$$

$$\boxed{D_{max}(dB) = 1.76 \text{ dB}}$$

SHORT DIPOLE

$$\text{Directivity} = \frac{U}{U_0} = \frac{\eta k^2 I^2 dl^2}{128\pi^2}\sin^2\theta \times \frac{4\pi}{\frac{\eta k^2 I^2 l^2}{48\pi}}$$

$$\boxed{D = \frac{3}{2}\sin^2\theta}$$

$$\boxed{D_{max} = \frac{3}{2}}$$

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$\frac{\lambda}{2}$ DIPOLE

$$\text{Directivity } D = \frac{4\pi U}{P} = \frac{4\pi S r^2}{P} = \frac{4\pi \times \frac{\eta}{2}\left(\frac{I_0}{2\pi r}\right)^2 \left(\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}\right) r^2}{\frac{\eta I_0^2}{4\pi} \times (1.218)}$$

$$= \frac{\cancel{4\pi} \times \frac{\cancel{\eta}}{\cancel{2}} \frac{\cancel{I_0^2}}{\cancel{4\pi^2} \cancel{r^2}} \times \cancel{r^2} \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \times \cancel{4\pi}}{\cancel{\eta} \cancel{I_0^2} \times 1.218}$$

$$= \frac{2}{1.218} \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$

$$\boxed{D = 1.642 \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}}$$

$$\frac{dD}{d\theta} = \frac{\sin^2\theta \left\{2\cos\left(\frac{\pi}{2}\cos\theta\right)\left(-\sin\left(\frac{\pi}{2}\cos\theta\right)\right)\frac{\pi}{2}(-\sin\theta)\right\} - 2\sin\theta\cos\theta\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^4\theta}$$

$$\frac{\partial D}{\partial \theta} = 0 \Rightarrow \pi \cos\left(\frac{\pi}{2}\cos\theta\right)\sin\left(\frac{\pi}{2}\cos\theta\right)\sin^3\theta - 2\sin\theta\cos\theta\cos^2\left(\frac{\pi}{2}\cos\theta\right) =$$

D will be max at $\theta = 90^\circ$

$$\boxed{D_{max} = 1.642}$$

$U \propto \dfrac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$ @ $\theta = 0^\circ$ $\frac{0}{0}$ form obtained

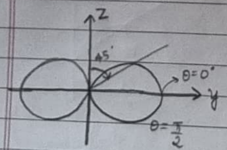
$\Rightarrow$ Apply L'Hospital's Rule

$$\lim_{\theta\to 0}\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} = \lim_{\theta\to 0}\frac{\cancel{2}\cos(\frac{\pi}{2}\cos\theta)\sin(\frac{\pi}{2}\cos\theta)\cancel{\sin\theta}\,\frac{\pi}{2}}{\cancel{2}\cancel{\sin\theta}\cos\theta}$$

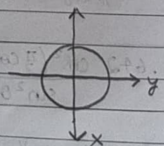
$$= \frac{\frac{\pi}{2}\cos(\frac{\pi}{2})\sin(\frac{\pi}{2})}{\cos\theta}$$

$\therefore U = 0$ @ $\theta = 0^\circ$

U @ $\theta = 45^\circ \Rightarrow \dfrac{\cos^2(\frac{\pi}{2}\times\frac{1}{\sqrt{2}})}{(\frac{1}{\sqrt{2}})^2} = 0.3943$



θ-plane



φ-plane

Omni-directional

$\cos 2\theta = 1 - 2\sin^2\theta$

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

Example: The radiation intensity of an antenna is given by

$$U(\theta,\phi) = \sin\theta\sin\phi, \quad 0 \le \theta \le \pi, \; 0 \le \phi \le \pi$$
$$= 0, \quad \text{elsewhere}$$

Determine the total power radiated and the directivity of the antenna.

$da = r^2 \sin\theta\, d\theta\, d\phi$

$U = S r^2$

$S = \dfrac{U}{r^2} = \dfrac{1}{r^2}\sin\theta\sin\phi$

$$P = \int_{Area} S \cdot da$$
$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{\pi} \frac{1}{r^2}\sin\theta\sin\phi \; r^2\sin\theta \, d\theta\, d\phi$$
$$= \frac{r^2}{r^2}\int_{\theta=0}^{\pi}\sin^2\theta\, d\theta \int_{\phi=0}^{\pi}\sin\phi\, d\phi$$
$$= -\cos\phi\Big|_0^{\pi} \int_{\theta=0}^{\pi}\frac{1-\cos 2\theta}{2}\, d\phi$$
$$P = (1-(-1))\frac{1}{2}\left[\phi - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$
$$= 2\times\frac{1}{2}\left[(\pi - 0) - \frac{1}{2}(\cancel{\sin 2\pi}^{=0} - \sin 0)\right]$$

$$\boxed{P = \pi \text{ Watts}}$$

$$D = \frac{4\pi U}{P}$$
$$= \frac{4\pi \times \sin\theta\sin\phi}{\pi}$$

$$\boxed{\begin{aligned} D &= 4\sin\theta\sin\phi, \quad 0\le\theta\le\pi,\; 0\le\phi\le\pi \\ &= 0 \quad \text{elsewhere}\end{aligned}}$$

$\boxed{D_{max} = 4}$ at $\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2}$

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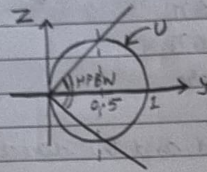
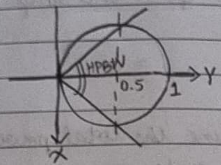
HALF-POWER BEAM WIDTH (HPBW)





Example 1 Determine HPBW in θ-plane and ϕ-plane of an antenna whose radiation intensity is

$U(\theta,\phi) = \sin^2\theta \sin\phi \qquad 0 \le \theta \le \pi ;\ 0 \le \phi \le \pi$

$= 0 \qquad$ otherwise

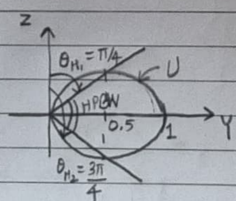
In θ-plane (YZ-plane or E-plane)

$U\left(\theta, \frac{\pi}{2}\right) = \sin^2\theta \qquad 0 \le \theta \le \pi ,\ 0 \le \phi \le \pi$

$\frac{1}{2} = 0.5 = \sin^2(\theta_H)$

$\sin(\theta_H) = \frac{1}{\sqrt{2}} \Rightarrow \theta_H = \frac{\pi}{4}, \frac{3\pi}{4}$

$HPBW(\theta\text{-plane}) = \theta_{H_2} - \theta_{H_1} = \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}$



In ϕ-plane (XY-plane or H-plane)

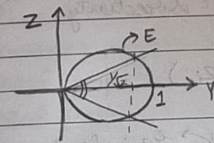
$U\left(\frac{\pi}{2}, \phi\right) = \sin\phi \qquad 0 \le \theta \le \pi ,\ 0 \le \phi \le \pi$

$0.5 = \sin(\phi_H)$

$\phi_H = \frac{\pi}{6}, \frac{5\pi}{6}$

$HPBW(\phi\text{-plane}) = \phi_{H_2} - \phi_{H_1} = \frac{5\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{3}$

Example 2 Determine HPBW in θ and ϕ plane if the radiation pattern of an antenna is $E(\theta,\phi) = \sin\theta\sin\phi \qquad 0 \le \theta \le \pi,\ 0 \le \phi \le \pi$



$S \propto E^2$

$P \propto E^2$

$E \propto \sqrt{P}$

In θ-plane $\quad E\left(\theta, \frac{\pi}{2}\right) = \sin\theta$

$\frac{1}{\sqrt{2}} = \sin\theta_H \Rightarrow \theta_H = \frac{\pi}{4}, \frac{3\pi}{4}$

$HPBW(\theta\text{-plane}) = \theta_{H_2} - \theta_{H_1} = \frac{\pi}{2}$

In ϕ-plane $\quad E\left(\frac{\pi}{2}, \phi\right) = \sin\phi$

$\frac{1}{\sqrt{2}} = \sin\phi_H \Rightarrow \phi_H = \frac{\pi}{4}, \frac{3\pi}{4}$

$HPBW(\phi\text{-plane}) = \phi_{H_2} - \phi_{H_1} = \frac{\pi}{2}$

ANTENNA EFFICIENCY (e)

$$e = \frac{\text{Radiated power}}{\text{Input power}} = \frac{P}{P_{in}} = \frac{I_{eff}^2 R_{rad}}{I_{eff}^2 (R_{loss} + R_{rad})}$$

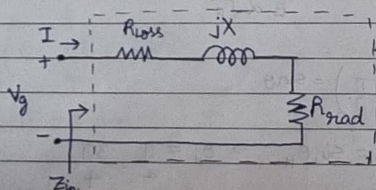
$$\boxed{e = \frac{R_{rad}}{R_{loss} + R_{rad}}}$$

R_{loss} is the resistance of the antenna wire

ANTENNA GAIN (G)

$\boxed{G = eD}$ If $R_{loss} \simeq 0$, $e = 1 = 100\%$.
$\Rightarrow$ Gain = directivity

INPUT IMPEDANCE OF AN ANTENNA (Z_{in})



equivalent circuit of an antenna

$$Z_{in} = R_{loss} + jX + R_{rad}$$

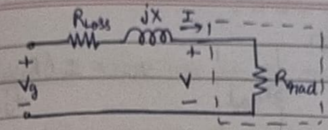
$$P_{in} = \frac{1}{2} Re\{V_g I^*\}$$ where V_g and I are phasors

$$P_{in} = \frac{1}{2} Re\{I Z_{in} I^*\}$$

$$= \frac{1}{2} |I|^2 Re\{Z_{in}\}$$

$$P_{in} = \frac{1}{2} |I|^2 (R_{loss} + R_{rad})$$

$$P_{in} = I_{eff}^2 (R_{loss} + R_{rad})$$



$$P_{rad} = \frac{1}{2} Re\{V I^*\}$$

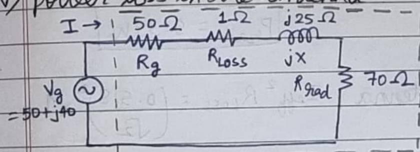
$$P_{rad} = \frac{1}{2} Re\{I R_{rad} I^*\}$$

$$= \frac{1}{2} |I|^2 R_{rad}$$

$$P_{rad} = I_{eff}^2 R_{rad}$$

Example: A voltage source $V_g = (50 + j40)\,V$ with source impedance of $50\,\Omega$ is connected to an antenna with $R_{rad} = 70\,\Omega$, $R_{LOSS} = 1\,\Omega$ and reactance $jX = j25\,\Omega$. Calculate

i) antenna efficiency
ii) real power delivered by the source
iii) real power input to the antenna
iv) power radiated by the antenna
v) power loss in the antenna



i) antenna efficiency

$$e = \frac{R_{rad}}{R_{rad} + R_{Loss}} = \frac{70}{70+1}$$

$$= 0.98591$$

$$\boxed{e = 98.6\%}$$

ii) real (or average) power delivered by the source

$$P_g = \frac{1}{2} Re\{V_g I^*\}$$

$$I = \frac{V_g}{R_g + R_{Loss} + R_{rad} + jX} = \frac{50 + j40}{121 + j25}$$

$$I = 0.5182 \angle 26.98^\circ \text{ A}$$

$$I^* = 0.5182 \angle -26.98^\circ \text{ A}$$

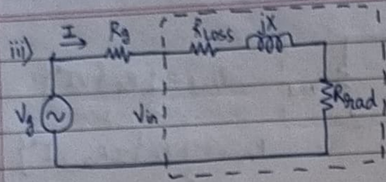
$$= \frac{1}{2} Re\{(10\sqrt{41} \angle 38.659^\circ)(0.5182 \angle -26.98^\circ)\}$$

$$= \frac{1}{2} Re\{32.494 + 6.7167\}$$

$$= \frac{1}{2} [32.494]$$

$$\boxed{P_g = 16.247\,W}$$

iii)



$V_{in} = I(R_{Loss} + R_{rad} + jX)$

Real power (average power) input to the antenna

$= \frac{1}{2} Re\{V_{in} I^*\} = \frac{1}{2} Re\{I(R_{Loss} + R_{rad} + jX) I^*\}$

$= \frac{1}{2} |I|^2 (R_{Loss} + R_{rad})$

$= \frac{1}{2} (0.5182)^2 (1 + 70)$

$\boxed{P_{in} = 9.525\,W}$

iv) power radiated by antenna $P_{rad} = I_{eff}^2 R_{rad}$

$= \left(\frac{0.518}{\sqrt{2}}\right)^2 \times 70$

$\boxed{P_{rad} = 9.39\,W}$

v) power loss in antenna $I_{eff}^2 R_{Loss} = \left(\frac{0.518}{\sqrt{2}}\right)^2 \times 1$

$= \underline{0.134\,W}$

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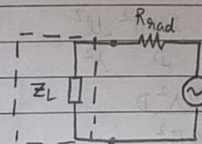
Effective aperture of an antenna

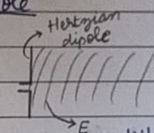
is the ratio of the power received (P_r) by the antenna to the power density (S) of the electromagnetic wave impinged (strike) on the antenna.

$$A_e = \frac{P_r}{S}$$

Effective aperture of a Hertzian dipole

$A_e = \frac{P_r}{S}$





$V = -\int \bar{E}.dl$

$|V| = |E|\,dl$

When the antenna is matched $Z_L = R_{rad}$ $\quad I = \frac{V_a}{2R_{rad}}$

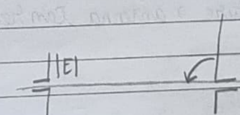
$P_r = \frac{1}{2} Re\{VI^*\} = \frac{1}{2} Re\{R_{rad} I I^*\}$

$= \frac{1}{2}|I|^2 R_{rad} = \frac{1}{2} \frac{V_a^2}{4R_{rad}^2} \times R_{rad}$

$\boxed{P_r = \frac{V_a^2}{8R_{rad}}}$ $\quad$ $\boxed{P_r = \frac{|E|^2 dl^2}{8R_{rad}}}$

$S = \frac{1}{2} Re\{\bar{E} \times \bar{H}^*\} = \frac{1}{2} E_\theta H_\phi^* = \frac{1}{2} \eta |H_\phi|^2 = \frac{1}{2} \frac{E_\theta E_\theta^*}{\eta}$

$\boxed{S = \frac{|E_\theta|^2}{2\eta}}$



$\boxed{S = \frac{|E|^2}{2\eta}}$

$$A_e = \frac{P_r}{S} = \frac{|E|^2 dl^2}{8R_{rad} \times \frac{|E|^2}{2\eta}}$$

$$A_e = \frac{\eta\, dl^2}{4R_{rad}}$$

$$R_{rad} = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2$$

$$D = \frac{3}{2}$$

$$R_{rad} D = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2 \times \frac{3}{2}$$

$$R_{rad} = \frac{120\pi^2}{D} \frac{dl^2}{\lambda^2}$$

$$A_e = \frac{\eta\, dl^2}{4 \times \frac{120\pi^2}{D} \times \frac{dl^2}{\lambda^2}} = \frac{\eta \lambda^2 D}{480\pi^2}$$

In free space $\eta = \eta_0 = 120\pi$ $\quad A_e = \frac{120\pi \lambda^2 D}{480\pi^2} = \left(\frac{\lambda^2}{4\pi}\right) D$

$G = eD$ $\quad R_{Loss} = 0 \Rightarrow e = 1$

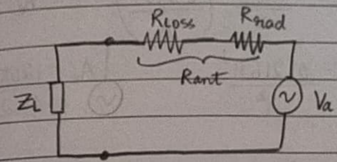
$G = D$

$$\boxed{A_e = \left(\frac{\lambda^2}{4\pi}\right) G} \; m^2$$

Aperture is essentially the area
↳ larger aperture ⇒ antenna can receive more power

Consider lossy Hertzian Dipole $\quad R_{Loss} \neq 0$

Equivalent circuit of receiver antenna



$R_{ant} = R_{rad} + R_{Loss}$

Antenna is matched
$\Rightarrow Z_L = R_{ant}$

$$I = \frac{V_a}{2R_{ant}}$$

$$P_r = \frac{1}{2} Re\{V_a I^*\} = \frac{1}{2} Re\{R_{ant} I I^*\}$$

$$= \frac{1}{2} |I|^2 R_{ant} = \frac{1}{2} \frac{|V_a|^2}{4R_{ant}^2} R_{ant}$$

$$\boxed{P_r = \frac{|V_a|^2}{8R_{ant}}}$$

$$S = \frac{|E|^2}{2\eta} \qquad A_e = \frac{P_r}{S} = \frac{|V_a|^2\, 2\eta}{8R_{ant} |E|^2} = \frac{\eta |E|^2 dl^2}{4R_{ant} |E|^2}$$

$$A_e = \frac{\eta\, dl^2}{4R_{ant}}$$

$R_{ant} = R_{Loss} + R_{rad}$

$$e = \frac{R_{rad}}{R_{Loss} + R_{rad}} = \frac{R_{rad}}{R_{ant}}$$

$$R_{ant} = \frac{R_{rad}}{e}$$

$$A_e = \frac{\eta\, dl^2}{4R_{rad}} \times e = \frac{\eta\, dl^2 e}{4 \times \frac{120\pi^2}{D} \frac{dl^2}{\lambda^2}}$$

In free space $\quad \frac{120\pi e D \lambda^2}{4 \times 120\pi^2} = \left(\frac{\lambda^2}{4\pi}\right) eD \Rightarrow \boxed{A_e = \left(\frac{\lambda^2}{4\pi}\right) G}$

Example 1: Determine the effective aperture of a Hertzian dipole of length 6 cm and radiation resistance of 2 Ω.

$$R_{rad} = \frac{120\pi^2}{D}\left(\frac{dl}{\lambda}\right)^2 \Rightarrow D = \frac{120\pi^2}{2}\left(\frac{6\times10^{-2}}{\lambda}\right)^2 \qquad A_e = \frac{\eta\, dl^2}{4R_{rad}}$$

$$D = \frac{0.216\pi^2}{\lambda^2} \qquad A_e = \frac{120\pi}{4\times2}(6\times10^{-2})^2$$

$e = 1 \quad G = D$

$$A_e = \left(\frac{\lambda^2}{4\pi}\right)G = \frac{\lambda^2}{4\pi}\times\frac{0.216\pi^2}{\lambda^2} = 0.216\pi \qquad \boxed{A_e = 0.1696\,m^2}$$

$$\boxed{A_e = 0.16964\,m^2}$$

$$R_{rad} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2$$

$$\lambda^2 = \frac{80\pi^2 dl^2}{R_{rad}} = \frac{80\pi^2(6\times10^{-2})^2}{2} = 0.144\pi^2$$

$G = D_{max} = \frac{3}{2}$ with $e = 1$

$$A_e = \left(\frac{\lambda^2}{4\pi}\right)G = \frac{0.144\pi^2}{4\pi}\times\frac{3}{2}$$

$$\boxed{A_e = 0.16964\,m^2}$$

Example 2: Determine the effective aperture of a Hertzian dipole received a power of 1 mW when an electric field of 1 V/m is impinged on it.

$$S = \frac{|E|^2}{2\eta} = \frac{1^2}{2\times120\pi} = 1.3263\times10^{-3}$$

$$A_e = \frac{P_r}{S} = \frac{1\times10^{-3}}{1.3263\times10^{-3}} = 753.982\times10^{-3} \Rightarrow \boxed{A_e = 0.7539\,m^2}$$

POLARISATION

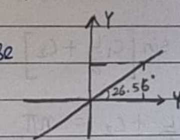
Polarization of an antenna is the polarization of electromagnetic wave it has radiated in the farfield.

POLARIZATION is the locus of tip of instantaneous electric field on the plane perpendicular to the direction of wave propagation.

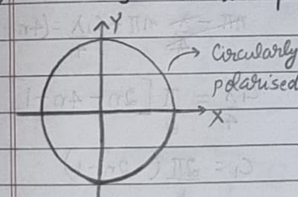
(1) when only E_x or E_y present → Wave is said to be LINEARLY POLARISED

(2) when $E_x = E_y$ but with same phase → Wave is LINEARLY POLARISED

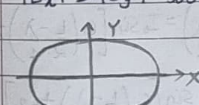
i) $E_x = 2E_y$ but with same phase



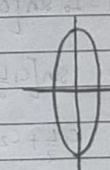
ii) $|E_x| = |E_y|$ but with phase difference of $\frac{\pi}{2}$



iii) $|E_x| > |E_y|$ with phase difference $\frac{\pi}{2}$



$|E_x| < |E_y|$



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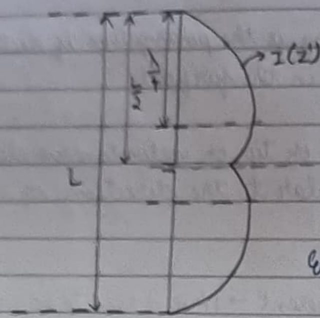
FINITE LENGTH DIPOLE



$$I(z') = I_0 \sin[C_1 z' + C_2]$$

$I(z')$ at $z' = \frac{L}{2} \Rightarrow I(z') = 0$

$I(z')$ at $z' = \frac{L}{2} - \frac{\lambda}{4}$ is I_0 (max value)

Even function $I(z') = I(-z')$

$$I\left(\frac{L}{2}\right) = I_0 \sin\left[C_1\frac{L}{2} + C_2\right] = 0$$

$I_0 \neq 0 \quad \sin\left[C_1\frac{L}{2} + C_2\right] = 0$

general case

$$C_1\frac{L}{2} + C_2 = n\pi$$

$$\frac{2\pi}{\lambda}(-2n-1)\frac{L}{2} + C_2 = n\pi$$

$$C_2 = n\pi + \frac{\pi L}{\lambda}(2n+1)$$

$$I\left(\frac{L}{2} - \frac{\lambda}{4}\right) = I_0 \sin\left[C_1\left(\frac{L}{2} - \frac{\lambda}{4}\right) + C_2\right] = I_0$$

$$\sin\left[C_1\left(\frac{L}{2} - \frac{\lambda}{4}\right) + C_2\right] = 1$$

$$C_1\left(\frac{L}{2} - \frac{\lambda}{4}\right) + C_2 = (4n+1)\frac{\pi}{2}$$

$$\cancel{n\pi - \frac{\lambda}{4}}\; n\pi - C_1\frac{\lambda}{4} = (4n+1)\frac{\pi}{2}$$

$$C_1\frac{\lambda}{4} = \frac{\pi}{2}[2n - 4n - 1]$$

$$C_1 = \frac{2\pi}{\lambda}(-2n-1)$$

$$I\left(\frac{L}{2}\right) = I_0 \sin\left[C_1\frac{L}{2} + C_2\right] = 0$$

$I_0 \neq 0 \quad \sin\left[C_1\frac{L}{2} + C_2\right] = 0$

$$C_1\frac{L}{2} + C_2 = 0$$

$$C_2 = -C_1\frac{L}{2}$$

$$\boxed{C_2 = \frac{\pi L}{\lambda}}$$

$$I\left(\frac{L}{2} - \frac{\lambda}{2}\right) = I_0 \sin\left[C_1\left(\frac{L}{2} - \frac{\lambda}{4}\right) + C_2\right] = I_0$$

$$\sin\left[C_1\left(\frac{L}{2} - \frac{\lambda}{4}\right) + C_2\right] = 1$$

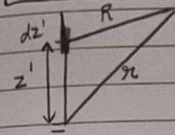
$$C_1\frac{L}{2} - C_1\frac{\lambda}{4} + C_2 = \frac{\pi}{2}$$

$$\boxed{C_1 = -\frac{2\pi}{\lambda}}$$

$$\boxed{C_1 = -\frac{2\pi}{\lambda} = -k} \qquad \boxed{C_2 = \frac{\pi L}{\lambda} = \frac{kL}{2}}$$

$$I(z') = I_0 \sin\left[-kz' + \frac{kL}{2}\right]$$

$$\boxed{I(z') = I_0 \sin\left[k\left(\frac{L}{2} - z'\right)\right]}$$ for top arm (positive z')



$$A_z = \frac{\mu}{4\pi}\int_{-L/2}^{L/2} \frac{I(z')\,dz'}{R} e^{-jkR}$$

In phase term $R \approx r - z'\cos\theta$

In amplitude term $R \approx r$

$$A_z = \frac{\mu}{4\pi}\int_{-L/2}^{L/2} \frac{I(z')\,dz'}{r} e^{-jk(r - z'\cos\theta)}$$

$$= \frac{\mu e^{-jkr}}{4\pi r}\int_{-L/2}^{L/2} I(z') e^{jkz'\cos\theta}\,dz'$$

for whole arm

$$= \frac{\mu}{4\pi r}e^{-jkr}\int_{-L/2}^{L/2} I(z')\left[\cos(kz'\cos\theta) + j\sin(kz'\cos\theta)\right]dz'$$

$$= \frac{\mu}{4\pi r}e^{-jkr}\left\{\int_{-L/2}^{L/2} \underbrace{I(z')\cos(kz'\cos\theta)}_{\text{even} \times \text{even} \Rightarrow \text{even}}\,dz' + j\int_{-L/2}^{L/2} \underbrace{\overbrace{I(z')}^{\text{even}}\,\overbrace{\sin(kz'\cos\theta)}^{\text{odd}}}_{\text{even} \times \text{odd} = 0}\,dz'\right\}$$

$$= \frac{\mu}{4\pi r}e^{-jkr}\, 2\int_0^{L/2} I(z')\cos(kz'\cos\theta)\,dz'$$

for top arm

$$= \frac{\mu I_0}{2\pi r}e^{-jkr}\int_0^{L/2} \sin\left[k\left(\frac{L}{2} - z'\right)\right]\cos(kz'\cos\theta)\,dz'$$

$$= \frac{\mu I_0}{4\pi r}e^{-jkr}\int_0^{L/2}\left\{\sin\left(\frac{kL}{2} - kz' + kz'\cos\theta\right) + \sin\left(\frac{kL}{2} - kz' - kz'\cos\theta\right)\right\}dz'$$

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$$A_z = \frac{\mu I_0}{4\pi r} e^{-jkr}\left[\underbrace{\int_0^{L/2} \sin\left[\frac{kL}{2} - kz'(1-\cos\theta)\right] dz'}_{I_1} + \underbrace{\int_0^{L/2} \sin\left[\frac{kL}{2} - kz'(1+\cos\theta)\right] dz'}_{I_2} \right]$$

$$I_1 = \int_{z'=0}^{L/2} \sin\left[\frac{kL}{2} - kz'(1-\cos\theta)\right] dz'$$

Let $\frac{kL}{2} - kz'(1-\cos\theta) = u$

$-k(1-\cos\theta)dz' = du$

$dz' = \frac{-du}{k(1-\cos\theta)}$

$$= \int_{z'=0}^{L/2} \frac{\sin u\,(-du)}{k(1-\cos\theta)}$$

$$= \frac{1}{k(1-\cos\theta)} \cos u \Big|_{z'=0}^{L/2}$$

$$= \frac{1}{k(1-\cos\theta)} \cos\left(\frac{kL}{2} - kz'(1-\cos\theta)\right)\Big|_0^{L/2}$$

$$= \frac{1}{k(1-\cos\theta)}\left\{\cos\left[\frac{kL}{2} - \frac{kL}{2} + \frac{kL\cos\theta}{2}\right] - \cos\left[\frac{kL}{2}\right]\right\}$$

$$I_1 = \frac{1}{k(1-\cos\theta)}\left\{\cos\left(\frac{kL\cos\theta}{2}\right) - \cos\left(\frac{kL}{2}\right)\right\}$$

$$I_2 = \int_{z'=0}^{L/2} \sin\left[\frac{kL}{2} - kz'(1+\cos\theta)\right] dz'$$

Let $\frac{kL}{2} - kz'(1+\cos\theta) = v$

$-k(1+\cos\theta)\,dz' = dv$

$dz' = \frac{-dv}{k(1+\cos\theta)}$

$$= \int_{z'=0}^{L/2} \sin v \left(\frac{-dv}{k(1+\cos\theta)}\right)$$

$$I_2 = \frac{1}{k(1+\cos\theta)} \cos v \Big|_{z'=0}^{L/2} = \frac{1}{k(1+\cos\theta)} \cos\left[\frac{kL}{2} - kz'(1+\cos\theta)\right]_0^{L/2}$$

$$= \frac{1}{k(1+\cos\theta)}\left\{\cos\left[\frac{kL}{2} - \frac{kL}{2} - \frac{kL\cos\theta}{2}\right] - \cos\left[\frac{kL}{2}\right]\right\}$$

$$I_2 = \frac{1}{k(1+\cos\theta)}\left\{\cos\left(\frac{kL\cos\theta}{2}\right) - \cos\left(\frac{kL}{2}\right)\right\}$$

$$A_z = \frac{\mu I_0}{4\pi r} e^{-jkr}\left\{\frac{1}{k(1-\cos\theta)}\left[\cos\left(\frac{kL\cos\theta}{2}\right) - \cos\frac{kL}{2}\right] + \frac{1}{k(1+\cos\theta)}\left[\cos\left(\frac{kL\cos\theta}{2}\right) - \cos\frac{kL}{2}\right]\right\}$$

$$= \frac{\mu I_0}{4\pi r k}\left[\cos\left(\frac{kL\cos\theta}{2}\right) - \cos\frac{kL}{2}\right]\left\{\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta}\right\} e^{-jkr}$$

$$= \frac{\mu I_0}{4\pi r k}\left(\frac{2\,\cancel{\cos\theta}}{\sin^2\theta}\right)\left[\cos\left(\frac{kL\cos\theta}{2}\right) - \cos\frac{kL}{2}\right] e^{-jkr}$$

$$A_z = \frac{\mu I_0}{2\pi r k \sin^2\theta}\left[\cos\left(\frac{kL\cos\theta}{2}\right) - \cos\frac{kL}{2}\right] e^{-jkr}$$

$$\boxed{A_z = \frac{\mu I_0}{2\pi r k} e^{-jkr}\left[\frac{\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\left(\frac{kL}{2}\right)}{\sin^2\theta}\right]}$$

General expression (can be used for any length of dipole)

$L = \frac{\lambda}{2}$ $\quad k = \frac{2\pi}{\lambda}$ $\quad \frac{kL}{2} = \frac{2\pi}{\lambda} \times \frac{\lambda}{2} \times \frac{1}{2} = \frac{\pi}{2}$

$$A_z = \frac{\mu I_0}{2\pi r k} e^{-jkr}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right) - \cos\left(\frac{\pi}{2}\right)}{\sin^2\theta}\right]$$

$$A_z = \frac{\mu I_0\, e^{-jkr}}{2\pi r k}\,\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$ for $\frac{\lambda}{2}$ dipole

To find fields

$$\bar{H} = \frac{1}{\mu}\nabla\times\bar{A} = \frac{1}{\mu r^2\sin\theta}\begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

Dipole is about z-axis ; ϕ symmetry (no variation)

$A_r = A_z\cos\theta$ $\quad A_\phi = 0$

$A_\theta = -A_z\sin\theta$

$$\bar{H} = \frac{1}{\mu r^2\sin\theta}\begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & 0 \\ A_r & rA_\theta & 0 \end{vmatrix}$$

$$= \frac{1}{\mu r^2\sin\theta}\left\{\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right] r\sin\theta\,\bar{a}_\phi\right\}$$

$$\bar{H} = \frac{1}{\mu r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right]\bar{a}_\phi$$

$$H_\phi = \frac{1}{\mu r}\left[\frac{\partial}{\partial r}\cancel{\left\{r\cos\theta\,\frac{\mu I_0}{2\pi r k}e^{-jkr}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\right.}\right.$$

$$H_\phi = \frac{1}{\mu r}\left[\frac{-\partial}{\partial r}(r\sin\theta A_z) - \frac{\partial}{\partial\theta}(A_z\cos\theta)\right]$$

$$= \frac{1}{\mu r}\left[-\sin\theta\left\{A_z + r\frac{\partial}{\partial r}A_z\right\} + \sin\theta\frac{\partial}{\partial\theta}A_z\right]$$

$$H_\phi = \frac{\sin\theta}{\mu r}\left[\frac{\partial}{\partial\theta}A_z - A_z - r\frac{\partial}{\partial r}A_z\right]$$

$$H_\phi = \left\{\frac{1}{\mu r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{\mu r}\frac{\partial}{\partial\theta}A_r\right\}$$

$$\frac{1}{\mu r}\frac{\partial}{\partial r}(rA_\theta) = \frac{1}{\mu r}\frac{\partial}{\partial r}(-r\sin\theta A_z)$$

$$= \frac{1}{\mu r}\frac{\partial}{\partial r}\left\{-\cancel{r}\cancel{\sin\theta}\times\frac{\mu I_0}{2\pi \cancel{r} k}e^{-jkr}\frac{\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\left(\frac{kL}{2}\right)}{\cancel{\sin^2\theta}\ \sin\theta}\right\}$$

$$= \frac{1}{\cancel{\mu} r}\times\frac{-\cancel{\mu}I_0}{2\pi k\sin\theta}\left\{\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\frac{kL}{2}\right\}\frac{\partial}{\partial r}\left(e^{-jkr}\right)$$

$$= \frac{-I_0}{2\pi \cancel{k} r\sin\theta}\left(\cos\frac{kL}{2}\cos\theta - \cos\frac{kL}{2}\right)(-j\cancel{k})e^{-jkr}$$

$$\frac{1}{\mu r}\frac{\partial}{\partial r}(rA_\theta) = \frac{jI_0}{2\pi r}\left(\frac{\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\left(\frac{kL}{2}\right)}{\sin\theta}\right)e^{-jkr}$$

$$\frac{-1}{\mu r}\frac{\partial}{\partial\theta}(A_r) = \frac{-1}{\mu r}\frac{\partial}{\partial\theta}(A_z\cos\theta)$$

$$= \frac{-1}{\mu r}\frac{\partial}{\partial\theta}\left\{\frac{\mu I_0}{2\pi r k}e^{-jkr}\left(\frac{\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\left(\frac{kL}{2}\right)}{\sin^2\theta}\right)\cos\theta\right\}$$

$$= \frac{-I_0}{2\pi r^2 k}\frac{\partial}{\partial\theta}\left\{\left(\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\left(\frac{kL}{2}\right)\right)\left(\frac{\cos\theta}{\sin^2\theta}\right)\right\}$$

This term varies with $1/r^2$ and hence does not contribute to radiation / distant field and can be ignored while computing the expression for $H_{\phi rad}$

for finite length dipole

$$\boxed{H_{\phi rad} = \frac{jI_0}{2\pi r}e^{-jkr}\left\{\frac{\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\left(\frac{kL}{2}\right)}{\sin\theta}\right\}}$$

$$\boxed{E_{\theta rad} = \eta H_{\phi rad}}$$

$$S = \frac{1}{2} Re\left\{ E_{\theta rad} H^*_{\phi rad} \right\} = \frac{1}{2} Re\left\{ \eta H_{\phi rad} H^*_{\phi rad} \right\}$$

$$\boxed{S = \frac{\eta}{2} \left| H_{\phi rad} \right|^2}$$

Power density in W/m^2

$$P = \int_{Area} S.da = \int\oint_{Area} S r^2 \sin\theta \, d\theta \, d\phi$$

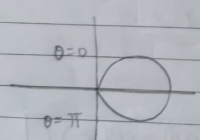
$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\eta}{2} \left\{ \left[\left(\frac{I_0}{2\pi r}\right) \left(\frac{\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\left(\frac{kL}{2}\right)}{\sin\theta} \right) \right]^2 \right\} r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{\eta}{2} \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \frac{I_0^2}{4\pi^2 \cancel{r^2}} \left[\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\left(\frac{kL}{2}\right) \right]^2 \frac{\cancel{r^2} \cancel{\sin\theta}}{\cancel{\sin^2\theta}\ \sin\theta} d\theta$$

$$= \frac{\eta}{2} \times \frac{I_0^2}{4\pi^2} \times 2\pi \int_{\theta=0}^{\pi} \frac{\left[\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\left(\frac{kL}{2}\right) \right]^2}{\sin\theta} d\theta$$

$$\boxed{P = \frac{\eta I_0^2}{4\pi} \int_0^{\pi} \frac{1}{\sin\theta} \left[\cos\left(\frac{kL}{2}\cos\theta\right) - \cos\left(\frac{kL}{2}\right) \right]^2 d\theta}$$

Power in Watts (can be found for a specific L)



$U \propto \sin^2\theta$

$U = 0 \qquad \theta = 0$ or π

Null to Null Beam Width $= \pi - 0 = \pi$
$= 180^\circ$

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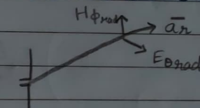
POYNTING VECTOR

The instantaneous Poynting vector is the cross-product of instantaneous electric field vector and instantaneous magnetic field vector

$$\tilde{\bar{S}} = \tilde{\bar{E}} \times \tilde{\bar{H}}$$

Direction of $\tilde{\bar{E}} \times \tilde{\bar{H}}$ indicates the direction of wave propagation (or the direction of power flow)

$$\bar{S} = \frac{1}{T} \int_0^T \tilde{\bar{S}} \, dt$$



For the Hertzian dipole

$$H_{\phi rad} = \frac{jk I dl \sin\theta}{4\pi r} e^{-jkr}$$

$$e^{j\pi/2} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = j$$

$$= \frac{k I dl \sin\theta}{4\pi r} e^{-jkr} e^{j\pi/2}$$

$$H_{\phi rad} = H_{\phi_0} e^{-j(kr - \pi/2)}$$

$$E_{\theta rad} = \eta H_{\phi rad} = \eta H_{\phi_0} e^{-j(kr-\pi/2)}$$

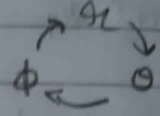
$$\tilde{H}_{\phi rad} = Re\left\{ H_{\phi_0} e^{-j(kr-\pi/2)} e^{j\omega t} \right\}$$

$$= H_{\phi_0} \cos(\omega t - kr + \pi/2)$$

$$\tilde{H}_{\phi rad} = -H_{\phi_0} \sin(\omega t - kr)$$

$$\tilde{\bar{H}}_{\phi rad} = -H_{\phi_0} \sin(\omega t - kr)\, \bar{a}_\phi$$

$$\tilde{E}_{\theta rad} = -\eta H_{\phi_0} \sin(\omega t - kr)$$



$$\tilde{E}_{\theta rad} = -\eta H_{\phi_0} \sin(\omega t - kr)\ \bar{a}_\theta$$

Instantaneous Poynting vector

$$\tilde{S} = \tilde{E} \times \tilde{H} = \eta H_{\phi_0}^2 \sin^2(\omega t - kr)\ \bar{a}_r$$

Time average of $\tilde{S}$ is $\bar{S} = \frac{1}{T}\int_0^T \tilde{S}\,dt$

$$= \frac{\bar{a}_r}{T}\int_0^T \eta H_{\phi_0}^2 \sin^2(\omega t - kr)\,dt$$

$$= \frac{\eta H_{\phi_0}^2 \bar{a}_r}{2T}\int_0^T \left(1 - \cos[2(\omega t - kr)]\right)dt$$

$$\bar{S} = \frac{\eta H_{\phi_0}^2}{2T}\bar{a}_r \left\{ t - \frac{1}{2(\omega)}\sin(2\omega t - 2kr) \right\}_0^T$$

$$= \frac{\eta H_{\phi_0}^2 \bar{a}_r}{2T}\left[(T-0) - \frac{1}{2\omega}\left(\sin\left(2 \times \frac{2\pi}{T} \times T - 2kr\right) - \sin(-2kr) \right) \right]$$

$$= \frac{\eta H_{\phi_0}^2}{2T}\left[T - \frac{1}{2\omega}\left(-\sin(2kr) + \sin(2kr)\right) \right] \bar{a}_r$$

$$= \frac{\eta H_{\phi_0}^2}{2T} \times T\ \bar{a}_r$$

$$\bar{S} = \frac{\eta H_{\phi_0}^2}{2}\bar{a}_r$$

$$\boxed{S = \frac{\eta H_{\phi_0}^2}{2}}$$

Power density The ~~insta~~ time average of instantaneous poynting vector gives power density