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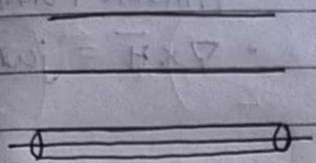
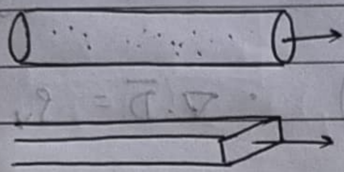


2-wired cable
Coaxial cable

- High attenuation losses when higher frequencies are involved
- can't handle high power

Circular
Rectangular



Waveguides
- hollow; can be any shape
- losses are less
- high power handling capability

HISTORY

LORD RAYLEIGH
- in 1897 → proposed that waves can travel in hollow pipes
  → gave a mathematical formulation
  → paper was neglected and forgotten, didn't garner much attention
- in 1936 → professors (scientists) from MIT and BELL re-discovered waveguides independently
  (MIT → Barrow; BELL → Southworth)
  → presented in the same conference on wave propagation in waveguides

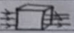
★ OLIVER HEAVYSIDE gave the Transmission line equation

MAXWELL'S EQUATIONS

TIME-VARYING FORM (time-varying)

- $\nabla \times \tilde{\bar{H}}$ (→ vector) $= \dfrac{\partial \tilde{\bar{D}}}{\partial t} + \tilde{\bar{J}}$
- $\nabla \times \tilde{\bar{E}} = -\dfrac{\partial \tilde{\bar{B}}}{\partial t}$
- $\nabla \cdot \tilde{\bar{D}} = \tilde{\rho}_v$ (time-varying scalar)
- $\nabla \cdot \tilde{\bar{B}} = 0$ ⇒ divergence is zero
  ⇒ no magnetic charges

PHASOR FORM

- $\nabla \times \bar{H} = j\omega\varepsilon\bar{E} + \bar{J}$
- $\nabla \times \bar{E} = -j\omega\mu\bar{H}$
- $\nabla \cdot \bar{D} = \rho_v$
- $\tilde{\bar{B}}$ 
- $\nabla \cdot \bar{B} = 0$

Time-varying form can be written in phasor form when :
- sinusoidal waveforms

WAVE PROPAGATION IN WAVEGUIDES

- Inside waveguides, there are no sources (source-free region)
  $\sigma = 0 \Rightarrow J = 0$, $\rho_v = 0$

$$\nabla \times \bar{H} = j\omega\varepsilon\bar{E} \longrightarrow (1)$$
$$\nabla \times \bar{E} = -j\omega\mu\bar{H} \longrightarrow (2)$$
$$\nabla \cdot \bar{D} = 0 \longrightarrow (3)$$
$$\nabla \cdot \bar{B} = 0 \longrightarrow (4)$$

$\bar{B} = \mu\bar{H}$
$\bar{D} = \varepsilon\bar{E}$

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Taking curl of (1) ⇒ $\nabla\times\nabla\times\bar{H} = j\omega\varepsilon\nabla\times\bar{E} = j\omega\varepsilon(-j\omega\mu\bar{H})$

$$\nabla(\nabla\cdot\bar{H}) - \nabla^2\bar{H} = \omega^2\mu\varepsilon\bar{H}$$

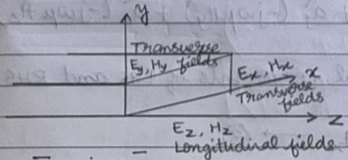
$$\boxed{\nabla^2\bar{H} + k^2\bar{H} = 0}$$
$$\boxed{\nabla^2\bar{E} + k^2\bar{E} = 0}$$

Similarly, taking curl of (2)

Homogeneous Helmholtz equations for source-free regions

where $k = \omega\sqrt{\mu\varepsilon}$ is wave number of EM wave propagating in an unbounded medium

FIELDS IN RECTANGULAR WAVEGUIDE



$$\bar{H} = \hat{a}_x H_x + \hat{a}_y H_y + \hat{a}_z H_z$$
$$\bar{E} = \hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z$$

(1) $\nabla\times\bar{H} = j\omega\varepsilon\bar{E}$

LHS: $\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix} = \hat{a}_x\left(\dfrac{\partial}{\partial y}H_z - \dfrac{\partial}{\partial z}H_y\right) + \hat{a}_y\left(\dfrac{\partial}{\partial z}H_x - \dfrac{\partial}{\partial x}H_z\right) + \hat{a}_z\left(\dfrac{\partial}{\partial x}H_y - \dfrac{\partial}{\partial y}H_x\right)$

RHS: $\hat{a}_x(j\omega\varepsilon E_x) + \hat{a}_y(j\omega\varepsilon E_y) + \hat{a}_z(j\omega\varepsilon E_z)$

Equating the individual components of LHS and RHS

$$\frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y = j\omega\varepsilon E_x \longrightarrow (5)$$

$$\frac{\partial}{\partial z}H_x - \frac{\partial}{\partial x}H_z = j\omega\varepsilon E_y \longrightarrow (6)$$

$$\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x = j\omega\varepsilon E_z \longrightarrow (7)$$

(2) $\nabla \times \bar{E} = -j\omega\mu \bar{H}$

LHS: $\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = \hat{a}_x\left(\frac{\partial}{\partial y}E_z - \frac{\partial}{\partial z}E_y\right) + \hat{a}_y\left(\frac{\partial}{\partial z}E_x - \frac{\partial}{\partial x}E_z\right) + \hat{a}_z\left(\frac{\partial}{\partial x}E_y - \frac{\partial}{\partial y}E_x\right)$

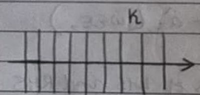
RHS: $\hat{a}_x(-j\omega\mu H_x) + \hat{a}_y(-j\omega\mu H_y) + \hat{a}_z(-j\omega\mu H_z)$

Equating the individual components of LHS and RHS

$$\frac{\partial}{\partial z}E_y - \frac{\partial}{\partial y}E_z = j\omega\mu H_x \longrightarrow (8)$$

$$\frac{\partial}{\partial x}E_z - \frac{\partial}{\partial z}E_x = j\omega\mu H_y \longrightarrow (9)$$

$$\frac{\partial}{\partial y}E_x - \frac{\partial}{\partial x}E_y = j\omega\mu H_z \longrightarrow (10)$$



$\bar{E}_z = \bar{E}_{z0}\, e^{-jkz}$
in unbounded free-space

$\bar{E}_x = \bar{E}_{x0}\, e^{-j\beta z}$ $\quad \bar{H}_x = \bar{H}_{x0}\, e^{-j\beta z}$

$\bar{E}_y = \bar{E}_{y0}\, e^{-j\beta z}$ $\quad \bar{H}_y = \bar{H}_{y0}\, e^{-j\beta z}$

$\bar{E}_z = \bar{E}_{z0}\, e^{-j\beta z}$ $\quad \bar{H}_z = \bar{H}_{z0}\, e^{-j\beta z}$

Wave propagating in +ve z direction

$$\frac{\partial}{\partial z}H_y = \frac{\partial}{\partial z}\underbrace{\left(\bar{H}_{y0}\, e^{-j\beta z}\right)}_{H_y} = -j\beta H_{y0} e^{-j\beta z}$$
$$= -j\beta H_y$$

$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\varepsilon E_x \longrightarrow (5a)$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \longrightarrow (6a)$$

$$\cancel{-j\beta}\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \longrightarrow (7) \quad \text{No change}$$

$$-j\beta E_y - \frac{\partial}{\partial y}E_z = j\omega\mu H_x \longrightarrow (8a)$$

$$\frac{\partial}{\partial x}E_z + j\beta E_x = j\omega\mu H_y \longrightarrow (9a)$$

$$\frac{\partial}{\partial y}E_x - \frac{\partial}{\partial x}E_y = j\omega\mu H_z \longrightarrow (10) \quad \text{No change}$$

* Express transverse fields (E_x, E_y, H_x, H_y) in terms of longitudinal fields (E_z, H_z)

Using (9a), eliminate H_y from (5a)

$$H_y = \frac{1}{j\omega\mu}\left(\frac{\partial}{\partial x}E_z + j\beta E_x\right)$$

in (5a) $\Rightarrow$ $\frac{\partial H_z}{\partial y} + j\beta\left\{\frac{1}{j\omega\mu}\left(\frac{\partial}{\partial x}E_z + j\beta E_x\right)\right\} = j\omega\varepsilon E_x$

$$\frac{\partial H_z}{\partial y} + \frac{\beta}{\omega\mu}\frac{\partial E_z}{\partial x} + \frac{j\beta^2}{\omega\mu}E_x = j\omega\varepsilon E_x$$

$$\frac{\partial}{\partial y}H_z + \frac{\beta}{\omega\mu}\frac{\partial}{\partial x}E_z = j\left(\frac{-\beta^2}{\omega\mu} + \omega\varepsilon\right)E_x$$

$$= j\left(\frac{\omega^2\mu\varepsilon - \beta^2}{\omega\mu}\right)E_x$$

$$E_x = \frac{-j\omega\mu}{\omega^2\mu\varepsilon - \beta^2}\left(\frac{\partial}{\partial y}H_z + \frac{\beta}{\omega\mu}\frac{\partial}{\partial x}E_z\right)$$

$$= \frac{j\omega\mu}{\beta^2 - \omega^2\mu\varepsilon}\left(\frac{\partial}{\partial y}H_z + \frac{\beta}{\omega\mu}\frac{\partial}{\partial x}E_z\right)$$

$$\boxed{k^2 - \beta^2 = k_c^2}$$

$$E_x = \frac{-j\omega\mu}{k_c^2}\left(\frac{\partial}{\partial y}H_z + \frac{\beta}{\omega\mu}\frac{\partial}{\partial x}E_z\right)$$

$$\boxed{E_x = \frac{-j\beta}{k_c^2}\left(\frac{\partial}{\partial x}E_z + \frac{\omega\mu}{\beta}\frac{\partial H_z}{\partial y}\right)}$$

$$H_y = \frac{1}{j\beta}\left(j\omega\varepsilon E_x - \frac{\partial}{\partial y}H_z\right)$$

Using (8a), eliminate H_x from (6a)

$$H_x = \frac{1}{j\omega\mu}\left(-j\beta E_y - \frac{\partial}{\partial y}E_z\right)$$

in (6a) $\Rightarrow$ $$-j\beta\left\{\frac{1}{j\omega\mu}\left(-j\beta E_y - \frac{\partial}{\partial y}E_z\right)\right\} - \frac{\partial}{\partial x}H_z = j\omega\varepsilon E_y$$

$$\frac{\beta}{\omega\mu}\left(j\beta E_y + \frac{\partial}{\partial y}E_z\right) - \frac{\partial}{\partial x}H_z = j\omega\varepsilon E_y$$

$$\frac{\beta}{\omega\mu}\frac{\partial}{\partial y}E_z - \frac{\partial}{\partial x}H_z = \left(j\omega\varepsilon - \frac{j\beta^2}{\omega\mu}\right)E_y = j\left(\frac{\omega^2\mu\varepsilon - \beta^2}{\mu\omega}\right)E_y$$

$$E_y = \frac{-j\omega\mu}{k_c^2}\left(\frac{\beta}{\omega\mu}\frac{\partial E_z}{\partial y} - \frac{\partial H_z}{\partial x}\right)$$

$$\boxed{E_y = \frac{-j\beta}{k_c^2}\left(\frac{\partial}{\partial y}E_z - \frac{\omega\mu}{\beta}\frac{\partial}{\partial x}H_z\right)}$$

~~Using the expression for E_y in (8a) to find H_x~~

$$\cancel{-j\beta\left(\frac{-j\beta}{k_c^2}\left(\frac{\partial}{\partial y}E_z - \frac{\omega\mu}{\beta}\frac{\partial}{\partial x}H_z\right)\right)} - \frac{\partial}{\partial y}E_z = j\omega\mu H_x$$

$$\cancel{\frac{-\beta^2}{k_c^2}}$$

Using (6a), eliminate ~~H~~ E_y from (8a)

$$E_y = \frac{1}{j\omega\varepsilon}\left(-j\beta H_x - \frac{\partial}{\partial x}H_z\right)$$

in (8a) $\Rightarrow$ $$\frac{-j\beta}{j\omega\varepsilon}\left(-j\beta H_x - \frac{\partial}{\partial x}H_z\right) - \frac{\partial}{\partial y}E_z = j\omega\mu H_x$$

$$\frac{j\beta^2}{\omega\varepsilon}H_x + \frac{(\beta)}{\omega\varepsilon}\frac{\partial}{\partial x}H_z - \frac{\partial}{\partial y}E_z = j\omega\mu H_x$$

$$\frac{\beta}{\omega\varepsilon}\frac{\partial}{\partial x}H_z - \frac{\partial}{\partial y}E_z = j\left(\omega\mu - \frac{\beta^2}{\omega\varepsilon}\right)H_x = j\left(\frac{\omega^2\mu\varepsilon - \beta^2}{\omega\varepsilon}\right)H_x$$

$$\frac{-j\omega\varepsilon}{k_c^2}\left(\frac{\beta}{\omega\varepsilon}\frac{\partial}{\partial x}H_z - \frac{\partial}{\partial y}E_z\right) = H_x$$

$$\boxed{H_x = \frac{-j\beta}{k_c^2}\left(\frac{\partial}{\partial x}H_z - \frac{\omega\varepsilon}{\beta}\frac{\partial E_z}{\partial y}\right)}$$

Using (5a), eliminate E_x from (9a)

$$E_x = \frac{1}{j\omega\varepsilon}\left\{\frac{\partial H_z}{\partial y} + j\beta H_y\right\}$$

in (9a) $\Rightarrow$ $$\frac{\partial}{\partial x}E_z + j\beta\left[\frac{1}{j\omega\varepsilon}\left\{\frac{\partial H_z}{\partial y} + j\beta H_y\right\}\right] = j\omega\mu H_y$$

$$\frac{\partial}{\partial x}E_z + \frac{\beta}{\omega\varepsilon}\frac{\partial H_z}{\partial y} + \frac{j\beta^2}{\omega\varepsilon}H_y = j\omega\mu H_y$$

$$\frac{\partial}{\partial x}E_z + \frac{\beta}{\omega\varepsilon}\frac{\partial H_z}{\partial y} = j\left(\omega\mu - \frac{\beta^2}{\omega\varepsilon}\right)H_y = j\left(\frac{\omega^2\mu\varepsilon - \beta^2}{\omega\varepsilon}\right)H_y$$

$$\frac{\partial}{\partial x}E_z + \frac{\beta}{\omega\varepsilon}\frac{\partial H_z}{\partial y} = j\frac{k_c^2}{\omega\varepsilon}H_y$$

$$H_y = \frac{-j\omega\varepsilon}{k_c^2}\left\{\frac{\partial}{\partial x}E_z + \frac{\beta}{\omega\varepsilon}\frac{\partial H_z}{\partial y}\right\}$$

$$\boxed{H_y = \frac{-j\beta}{k_c^2}\left(\frac{\partial H_z}{\partial y} + \frac{\omega\varepsilon}{\beta}\frac{\partial E_z}{\partial x}\right)}$$

TEM (Transverse Electromagnetic) Wave

↳ No longitudinal electric and magnetic waves

when $E_z = H_z = 0$
all transverse fields vanish which means there is no TEM wave propagation inside the waveguide

when $E_z = 0$ Transverse Electric (TE) Wave

when $H_z = 0$ Transverse Magnetic (TM) Wave

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TE Waves (Transverse Electric) $E_z = 0$ $H_z \neq 0$

$$E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y} \qquad H_x = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial x}$$

$$E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x} \qquad H_y = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial y}$$

Considering Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\hat{a}_x H_x + \hat{a}_y H_y + \hat{a}_z H_z)$$

$$+ k^2(\hat{a}_x H_x + \hat{a}_y H_y + \hat{a}_z H_z) = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)H_z + k^2 H_z = 0$$

$$H_z = H_{z0}e^{-j\beta z}$$

$$\cancel{\frac{\partial^2}{\partial z^2}H_z + k^2 H_z = 0}$$

$$\frac{\partial}{\partial z}\left\{-j\beta H_{z_0} e^{-j\beta z}\right\} \cancel{+ k^2 H_z} \cancel{= 0}$$

$$-\beta^2 H_{z_0} e^{-j\beta z} \cancel{+ k^2 H_z = 0}$$

$$\frac{\partial^2 H_z}{\partial z^2} = -\beta^2 H_z \cancel{+ k^2 H_z = 0}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) H_z + k^2 H_z = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2\right) H_{z_0} e^{-j\beta z} + k^2 H_{z_0} e^{-j\beta z} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2\right) H_{z_0} + k^2 H_{z_0} = 0$$

$$\frac{\partial^2 H_{z_0}}{\partial x^2} + \frac{\partial^2 H_{z_0}}{\partial y^2} + (k^2 - \beta^2) H_{z_0} = 0$$

$$\frac{\partial^2 H_{z_0}}{\partial x^2} + \frac{\partial^2 H_{z_0}}{\partial y^2} + k_c^2 H_{z_0} = 0$$

Let $H_{z_0} = XY$ $\qquad k_c^2 = k_x^2 + k_y^2$

X: function of x only; Y: function of y only

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} + k_c^2 (XY) = 0$$

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} + (k_x^2 + k_y^2)(XY) = 0$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + (k_x^2 + k_y^2)(XY) = 0$$

Divide throughout by XY

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + (k_x^2 + k_y^2) = 0$$

$$\left(\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + k_x^2\right) + \left(\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k_y^2\right) = 0$$

X is not dependent on Y and vice versa
The individual components have to be zero for their sum to be equal to zero.

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + k_x^2 = 0 \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k_y^2 = 0$$

$$\frac{\partial^2 X}{\partial x^2} + k_x^2 X = 0$$

$$X = A\sin(k_x x)$$

$$\frac{\partial X}{\partial x} = k_x A\cos(k_x x)$$

$$\frac{\partial^2 X}{\partial x^2} = -k_x^2 A\sin(k_x x)$$

$$-k_x^2 A\sin(k_x x) + k_x^2 A\sin(k_x x) = 0$$

$\therefore$ The solution $X = A\sin(k_x x)$ satisfies the equation

$X = B\cos(k_x x)$

$$\frac{\partial X}{\partial x} = -Bk_x \sin(k_x x)$$

$$\frac{\partial^2 X}{\partial x^2} = -Bk_x^2 \cos(k_x x)$$

$$-Bk_x^2 \cos(k_x x) + Bk_x^2 \cos(k_x x) = 0$$

$\therefore$ The solution $X = B\cos(k_x x)$ satisfies the ~~solution~~ equation

$$X = A\sin(k_x x) + B\cos(k_x x)$$

$$\frac{\partial X}{\partial x} = k_x A\cos(k_x x) - Bk_x \sin(k_x x)$$

$$\frac{\partial^2 X}{\partial x^2} = -Ak_x^2 \sin(k_x x) - Bk_x^2 \cos(k_x x)$$

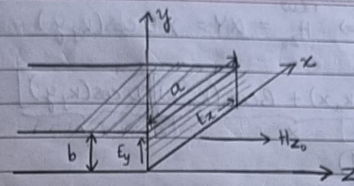
$$-Ak_x^2 \sin(k_x x) - Bk_x^2 \cos(k_x x) + Ak_x^2 \sin(k_x x) + Bk_x^2 \cos(k_x x) = 0$$

$\therefore$ The solution $X = A\sin(k_x x) + B\cos(k_x x)$ satisfies the equation.

$$X = A\sin(k_x x) + B\cos(k_x x)$$

Similarly

$$Y = C\sin(k_y y) + D\cos(k_y y)$$



$E_x = E_{x_0} e^{-j\beta z}$

$y = 0$ Bottom wall

$y = b$ Top wall

Boundary condition $E_x = 0$ at $y = 0$ (1), $y = b$ (2) } Tangential components

$$E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y} = \frac{-j\omega\mu}{k_c^2}\left(\frac{\partial H_{z_0} e^{-j\beta z}}{\partial y}\right)$$

$$E_x = \frac{-j\omega\mu}{k_c^2}\left(\frac{\partial H_{z_0}}{\partial y}\right) e^{-j\beta z}$$

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$$H_{z_0} = XY = [A\sin(k_x x) + B\cos(k_x x)][C\sin(k_y y) + D\cos(k_y y)]$$

$$E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial (XY)}{\partial y} e^{-j\beta z}$$

$$E_x = \frac{-j\omega\mu}{k_c^2} X e^{-j\beta z}\frac{\partial Y}{\partial y}$$

$Y = C\sin(k_y y) + D\cos(k_y y)$

$\frac{\partial Y}{\partial y} = Ck_y \cos(k_y y) - Dk_y \sin(k_y y)$

$$E_x = \frac{-j\omega\mu}{k_c^2} X e^{-j\beta z} k_y [C\cos(k_y y) - Dk_y \sin(k_y y)]$$

at $y = 0$, $E_x = 0$

$$0 = \frac{-j\omega\mu}{k_c^2} X e^{-j\beta z} k_y [C\cos 0 - D\underbrace{\sin 0}_{=0}]$$

$$\underbrace{\left[\frac{-j\omega\mu}{k_c^2} X e^{-j\beta z} k_y\right]}_{\neq 0} C = 0$$

$\therefore$ $\boxed{C = 0}$

$Y = D\cos(k_y y) \Rightarrow \text{new } H_{z_0} = XY = XD\cos(k_y y)$

$\text{new } H_{z_0} = \underbrace{[A\sin(k_x x) + B\cos(k_x x)]}_{X}\underbrace{[D\cos(k_y y)]}_{Y}$

after applying ①

$$E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y} = \frac{-j\omega\mu}{k_c^2}\frac{\partial}{\partial y}(H_{z_0}e^{-j\beta z})$$

$$= \frac{-j\omega\mu}{k_c^2}\frac{\partial}{\partial y}(XYe^{-j\beta z}) = \frac{-j\omega\mu}{k_c^2}Xe^{-j\beta z}\frac{\partial}{\partial y}[D\cos(k_y y)]$$

$$E_x = \frac{-j\omega\mu}{k_c^2}Xe^{-j\beta z}(-Dk_y\sin(k_y y))$$

Applying condition ② $E_x\big|_{y=b} = 0$

$$0 = \underbrace{\frac{+j\omega\mu}{k_c^2}Xe^{-j\beta z}}_{\neq 0 \text{ Not a function of } y}[Dk_y\sin(k_y b)]$$

$D \neq 0 \quad \because C = 0$, if $D = 0$ as well, Y will vanish

$\therefore \sin(k_y b) = 0$

$k_y b = n\pi$

$$\boxed{k_y = \frac{n\pi}{b}}$$

After applying ① & ②

$$H_{z_0} = XY = [A\sin(k_x x) + B\cos(k_x x)][D\cos(\tfrac{n\pi}{b}y)]$$

Boundary conditions ③ $E_y = 0$ at $x = 0$ (front wall)
④ $E_y = 0$ at $x = a$ (back wall)

$$E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x} = \frac{j\omega\mu}{k_c^2}\frac{\partial}{\partial x}(H_{z_0}e^{-j\beta z})$$

$$= \frac{j\omega\mu}{k_c^2}\frac{\partial}{\partial x}(XYe^{-j\beta z})$$

$$E_y = \frac{j\omega\mu}{k_c^2}Ye^{-j\beta z}\frac{\partial X}{\partial x}$$

$$E_y = \frac{j\omega\mu}{k_c^2}[D\cos(\tfrac{n\pi}{b}y)]e^{-j\beta z}\frac{\partial}{\partial x}[A\sin(k_x x) + B\cos(k_x x)]$$

$$\frac{\partial X}{\partial x} = \frac{\partial}{\partial x}[A\sin(k_x x) + B\cos(k_x x)] = Ak_x\cos(k_x x) - Bk_x\sin(k_x x)$$

$$E_y = \frac{j\omega\mu}{k_c^2}D\cos(\tfrac{n\pi}{b}y)e^{-j\beta z}k_x[A\cos(k_x x) - B\sin(k_x x)]$$

Applying condition ③ $E_y\big|_{x=0} = 0$

$$0 = \underbrace{\frac{j\omega\mu}{k_c^2}D\cos(\tfrac{n\pi}{b}y)e^{-j\beta z}k_x}_{\neq 0 \text{ Not a function of } x}A$$

$\therefore \boxed{A = 0}$

After applying ①, ②, ③, new $H_{z_0} = B\cos(k_x x)D\cos(\frac{n\pi}{b}y)$

$$E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x} = \frac{j\omega\mu}{K_c^2}\frac{\partial}{\partial x}\left(H_{z_0}e^{-j\beta z}\right) = \frac{j\omega\mu}{K_c^2} Y e^{-j\beta z}\frac{\partial X}{\partial x}$$

$$\frac{\partial X}{\partial x} = \frac{\partial}{\partial x}\left[B\cos(k_x x)\right] = -BK_x\sin(K_x x)$$

$$E_y = \frac{j\omega\mu}{k_c^2} D\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\left[-BK_x\sin(K_x x)\right]$$

applying ④ $E_y\big|_{x=a} = 0$

$$0 = \underbrace{\frac{j\omega\mu}{k_c^2} D\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}}_{\neq 0 \text{ not a function of } x}\left[-BK_x\sin(K_x a)\right]$$

$B \neq 0$ $\because A = 0$, if $B = 0$, X will vanish

$\therefore \sin(k_x a) = 0$

$k_x a = m\pi$

$$\boxed{K_x = \frac{m\pi}{a}} \qquad \therefore X = B\cos\left(\frac{m\pi}{a}x\right)$$

After applying conditions ①, ②, ③, ④

$$\boxed{H_{z_0} = B\cos\left(\frac{m\pi}{a}x\right) D\cos\left(\frac{n\pi}{b}y\right)}$$

Let $BD = A'$

→ amplitude of magnetic field H_{z_0}

→ depends on excitation sent through waveguide

$$\boxed{H_z = A'\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}}$$

$$E_x = \frac{-j\beta}{k_c^2}\left(\frac{\partial}{\partial x}E_z + \frac{\omega\mu}{\beta}\frac{\partial H_z}{\partial y}\right) \qquad E_z = 0 \text{ for TE wave}$$

$$\frac{\partial H_z}{\partial y} = \frac{\partial}{\partial y}\left\{A'\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\right\} = A'\cos\left(\frac{m\pi}{a}x\right)e^{-j\beta z}\frac{\partial}{\partial y}\cos\left(\frac{n\pi}{b}y\right)$$

$$= -A'e^{-j\beta z}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\times\frac{n\pi}{b}$$

$$E_x = \frac{-j\beta}{k_c^2}\frac{\omega\mu}{\beta}\left\{-A'e^{-j\beta z}\frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\right\}$$

$$\boxed{E_x = \frac{j\omega\mu}{k_c^2}A'e^{-j\beta z}\frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)}$$

$$E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}$$

$$\frac{\partial H_z}{\partial x} = \frac{\partial}{\partial x}\left\{A'\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\right\}$$

$$= A'e^{-j\beta z}\cos\left(\frac{n\pi}{b}y\right)\frac{\partial}{\partial x}\left\{\cos\left(\frac{m\pi}{a}x\right)\right\}$$

$$= -A'e^{-j\beta z}\frac{m\pi}{a}\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{a}x\right)$$

$$E_y = \frac{j\omega\mu}{k_c^2}\left\{-A'e^{-j\beta z}\frac{m\pi}{a}\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{a}x\right)\right\}$$

$$\boxed{E_y = \frac{-j\omega\mu}{k_c^2}A'e^{-j\beta z}\frac{m\pi}{a}\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{a}x\right)}$$

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$$H_x = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial x}$$

$$H_x = \frac{-j\beta}{k_c^2}\left\{-A'e^{-j\beta z}\frac{m\pi}{a}\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{a}x\right)\right\}$$

$$\boxed{H_x = \frac{j\beta}{k_c^2}A'e^{-j\beta z}\frac{m\pi}{a}\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{a}x\right)}$$

$$H_y = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial y}$$

$$H_y = \frac{-j\beta}{k_c^2}\left\{-A'e^{-j\beta z}\frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\right\}$$

$$\boxed{H_y = \frac{j\beta}{k_c^2}A'e^{-j\beta z}\frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)}$$

$$k_x = \frac{m\pi}{a} \qquad k_y = \frac{n\pi}{b}$$

$$k_c^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$k_c^2 = k^2 - \beta^2$$

$$k_c^2 = \omega^2\mu\varepsilon - \beta^2 \qquad k_c^2 = \omega_c^2\mu\varepsilon$$

$$\omega_c^2\mu\varepsilon = \omega^2\mu\varepsilon - \beta^2$$

$$\beta^2 = \omega^2\mu\varepsilon - \omega_c^2\mu\varepsilon = (\omega^2 - \omega_c^2)\mu\varepsilon = \omega^2\mu\varepsilon\left\{1 - \frac{\omega_c^2}{\omega^2}\right\}$$

$\cos(\omega t - \beta z)$ represents wave propagation
↳ varying with time as well as direction

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$$\omega = 2\pi f \qquad \omega_c = 2\pi f_c$$

μ, ε are the parameters of dielectric material inside waveguide.

$$\beta^2 = \omega^2\mu\varepsilon\left\{1 - \left(\frac{f_c}{f}\right)^2\right\}$$

$$\boxed{\beta = \omega\sqrt{\mu\varepsilon}\left\{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}\right\}}$$

$f_c > f \Rightarrow \frac{f_c}{f} > 1$

$\sqrt{1-\left(\frac{f_c}{f}\right)^2}$ < 0

↳ imaginary

$$E_x = E_{x_0}e^{-j\beta z}$$

$$\tilde{E}_x = Re\{E_{x_0}e^{+j\omega t}\} = Re\{E_{x_0}e^{-j\beta z}e^{j\omega t}\}$$

$$\tilde{E}_x = E_{x_0}\cos(\omega t - \beta z)$$

β has to be real

if β is imaginary, it'll be only attenuation; no wave propagation

$\Rightarrow \boxed{f > f_c} \Rightarrow \frac{f_c}{f} < 1$ for wave propagation

$$k_c^2 = \omega_c^2\mu\varepsilon = 4\pi f_c^2\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$2\pi f_c\sqrt{\mu\varepsilon} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$\frac{1}{\sqrt{\mu\varepsilon}} = \vartheta$ velocity

$$f_c = \frac{\vartheta}{2\pi}\times\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Cutoff frequency of the waveguide $\boxed{f_c = \frac{\vartheta}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$

Outside wave guide, it is always free space

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$\mu = \mu_r \mu_0$ μ_r relative
$\varepsilon = \varepsilon_r \varepsilon_0$ ε_r relative

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$= \frac{1}{2\sqrt{\mu_r\varepsilon_r}\sqrt{\mu_0\varepsilon_0}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$$

$$\boxed{f_c = \frac{c}{2\sqrt{\mu_r\varepsilon_r}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

Cut-off frequency

$$\beta = \omega\sqrt{\mu\varepsilon}\left\{\sqrt{1-\left(\frac{f_c}{f}\right)^2}\right\} \qquad f > f_c$$

$$= 2\pi f\sqrt{\mu\varepsilon}\left\{\sqrt{1-\left(\frac{f_c}{f}\right)^2}\right\}$$

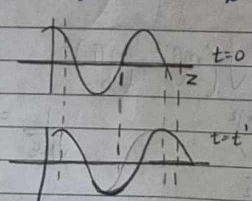
* β has to be real for wave propagation
* If β is imaginary, there is no wave propagation, only attenuation

$E_x = E_{x_0} e^{-j\beta z}$

$$\tilde{E}_x = Re\{E_x e^{j\omega t}\} = Re\{E_{x_0} e^{j(\omega t - \beta z)}\}$$

$$= \underbrace{E_x \cos(\omega t - \beta z)}$$

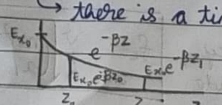
↳ represents a wave



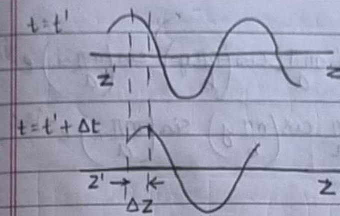
β imaginary $\Rightarrow j\beta$ becomes real

$$\tilde{E}_x = E_{x_0} e^{-\beta z}\cos(\omega t)$$

↳ there is no wave propagation
↳ there is a time variation



Just oscillates at a point with a particular value
Sinusoidal variation



Phase
$\phi = \omega t' - \beta z'$

$$\phi = \omega(t'+\Delta t) - \beta(z'+\Delta z)$$

$$\omega(t'+\Delta t) - \beta(z'+\Delta z) = \omega t' - \beta z'$$

$$\omega\Delta t - \beta\Delta z = 0$$

$$\omega\Delta t = \beta\Delta z$$

$$\boxed{\frac{\Delta z}{\Delta t} = \frac{\omega}{\beta}}$$

Small incremental distance over small incremental time
↳ phase velocity of the wave inside waveguide v_g

$$v_g = \frac{\Delta z}{\Delta t} = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$= \frac{1}{\sqrt{\mu_r\varepsilon_r}\sqrt{\mu_0\varepsilon_0}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$\boxed{v_g = \frac{c}{\sqrt{\mu_r\varepsilon_r}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}}$$

* Phase velocity of wave inside waveguide is greater than that of light

Wave impedance $\boxed{Z_g = \frac{E_x}{H_y} = \frac{-E_y}{H_x}}$

$$Z_g = \frac{E_x}{H_y} = \frac{\frac{j\omega\mu}{k_c^2} A' e^{-j\beta z} \frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)}{\frac{j\beta}{k_c^2} A' \frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}}$$

$$= \frac{\omega\mu}{\beta}$$

$$Z_g = \frac{-E_y}{H_x} = \frac{-\left\{ \frac{-j\omega\mu}{k_c^2} A' \frac{m\pi}{a} \cos\left(\frac{n\pi y}{b}\right) \sin\left(\frac{m\pi}{a}x\right) e^{-j\beta z} \right\}}{\frac{j\beta}{k_c^2} A' \frac{m\pi}{a} \cos\left(\frac{n\pi}{b}y\right) \sin\left(\frac{m\pi}{a}x\right) e^{-j\beta z}}$$

$$= \frac{\omega\mu}{\beta}$$

$$\boxed{Z_g = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\omega\mu}{\beta}}$$

$$\frac{\omega\mu}{\beta} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \sqrt{\frac{\mu}{\varepsilon}} \frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$Z_g = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

Intrinsic impedance of the medium (μ, ε) $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

In case medium is free space inside waveguide

$$\eta = \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi$$

$$Z_g = \frac{\eta_0}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \qquad Z_g > \eta_0$$

β
v_g
Z_g

Example A waveguide that is rectangular has its dimensions in the ratio $a:b = 2:1$ is supporting TE_{11} mode and it has cut-off frequency of 9 GHz. (a) Determine the dimensions of the waveguide

* TE_{mn} mode $\Rightarrow m=1,\ n=1$
* Unless otherwise specified, free space is present inside the waveguide
* Width is greater than height $(a>b)$

$$f_c = \frac{c}{2\sqrt{\mu_r\varepsilon_r}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Free space $\mu_r = 1$, $\varepsilon_r = 1$

$$9\times10^9 = \frac{3\times10^8}{2\sqrt{1\times1}} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

$$\frac{a}{b} = \frac{2}{1} \qquad \boxed{a = 2b}$$

$$60^2 = \left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2$$

$$3600 = \left(\frac{1}{2b}\right)^2 + \left(\frac{1}{b}\right)^2 = \frac{1}{4b^2} + \frac{1}{b^2} = \frac{5}{4b^2}$$

$$3600 b^2 = \frac{5}{4}$$

$$60 b = \frac{\sqrt{5}}{2}$$

$$\boxed{b = 0.0186\,m}$$

$$a = 2b = 2(0.0186)$$

$$\boxed{a = 0.037\,m}$$

(b) If the operating frequency of the wave is 18 GHz, determine (i) wave number or phase constant β (ii) phase velocity v_g (iii) wave impedance of the wave Z_g

$f > f_c \Rightarrow \beta$ is real; there is wave propagation

$$\beta = \omega\sqrt{\mu\varepsilon}\left\{\sqrt{1-\left(\frac{f_c}{f}\right)^2}\right\}$$

$$\omega = 2\pi f = 36\pi\times10^9$$

$$\sqrt{\mu\varepsilon} = \sqrt{\mu_0\varepsilon_0} = \frac{1}{c} = \frac{1}{3\times10^8}$$

$$= \frac{\omega}{c}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\beta = \frac{36\pi \times 10^9}{3\times 10^8}\sqrt{1-\left(\frac{9}{18}\right)^2} = 120\pi\sqrt{1-0.25}$$

$$= 120\pi \times \frac{\sqrt{3}}{2} = 60\sqrt{3}\pi$$

$$\boxed{\beta = 326.483 \text{ rad/m}}$$

$$\cancel{Z_g} = v_g = \frac{c}{\sqrt{\mu_r\varepsilon_r}\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{3\times 10^8}{\sqrt{1\times 1}\sqrt{1-\left(\frac{9}{18}\right)^2}}$$

$$= \frac{3\times 10^8}{1\times\frac{\sqrt{3}}{2}} = 2\sqrt{3}\times 10^8 \qquad \boxed{v_g = \frac{\omega}{\beta}}$$

$$\boxed{v_g = 3.464\times 10^8 \text{ m/s}} \qquad v_g > c$$

$$Z_g = \frac{\eta_o}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{120\pi}{\frac{\sqrt{3}}{2}} = \frac{240\pi}{\sqrt{3}} \qquad \boxed{Z_g = \frac{\omega\mu}{\beta}}$$

$$\boxed{Z_g = 435.31\ \Omega}$$

$a > b$

$\Rightarrow$ What is the lowest mode possible in TE wave propagation? (which mode will yield lowest cut-off frequency?)

$m = n = 0 \quad f_c = \frac{c}{2\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2} = 0$ Not possible $\hookrightarrow$ all fields vanish

$m = 0$
$n = 1 \quad f_c = \frac{c}{2\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{1}{b}\right)^2} = \frac{c}{2b\sqrt{\mu_r\varepsilon_r}}$

$m = 1$
$n = 0 \quad f_c = \frac{c}{2\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{1}{a}\right)^2} = \frac{c}{2a\sqrt{\mu_r\varepsilon_r}}$

$m = 1$
$n = 1 \quad f_c = \frac{c}{2\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2}$

$\because a > b \quad f_c$ is minimum for $m = 1, n = 0$

$$\Rightarrow f_c = \frac{c}{2a\sqrt{\mu_r\varepsilon_r}} \qquad H_z = A'\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$\Rightarrow \boxed{H_z = A'\cos\left(\frac{\pi}{a}x\right)e^{-j\beta z} \quad \text{exists}}$$

$$\Rightarrow E_x = \frac{j\omega\mu}{k_c^2}A'\left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{\pi}{b}ny\right)e^{-j\beta z} = 0 \quad \text{does not exist}$$

$$\Rightarrow E_y = \frac{-j\omega\mu}{k_c^2}A'\frac{m\pi}{a}\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{a}x\right)e^{-j\beta z}$$

$$\boxed{E_y = -\frac{A'j\omega\mu}{k_c^2}\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)e^{-j\beta z} \quad \text{exists}}$$

$$\Rightarrow H_x = \frac{A'j\beta}{k_c^2}\left(\frac{m\pi}{a}\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{a}x\right)e^{-j\beta z}$$

$$\boxed{H_x = A'\frac{j\beta}{k_c^2}\frac{\pi}{a}\sin\left(\frac{\pi}{a}x\right)e^{j\beta z} \quad \text{exists}}$$

$$\Rightarrow H_y = \frac{j\beta}{k_c^2}A'\frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z} = 0 \quad \text{does not exist}$$

Lowest mode TE_{10}

- $f_{c_{10}} = \frac{c}{2a\sqrt{\mu_r\varepsilon_r}}$

TE_{10} Dominant mode of operation

- E_y, H_x, H_z exist
- H_y, E_x do not exist

- $Z_g = \frac{-E_y}{H_x} = \frac{\omega\mu}{\beta}$
- $v_g = \frac{\omega}{\beta}$

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TM Waves (Transverse Magnetic) $E_z \neq 0$ $H_z = 0$

$$E_x = \frac{-j\beta}{k_c^2}\left[\frac{\partial E_z}{\partial x} + \frac{\omega\mu}{\beta}\frac{\partial H_z}{\partial y}\right]$$

$$E_y = \frac{-j\beta}{k_c^2}\left(\frac{\partial E_z}{\partial y} - \frac{\omega\mu}{\beta}\frac{\partial H_z}{\partial x}\right)$$

$$H_x = \frac{-j\beta}{k_c^2}\left[\frac{\partial H_z}{\partial x} - \frac{\omega\varepsilon}{\beta}\frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j\beta}{k_c^2}\left(\frac{\partial H_z}{\partial y} + \frac{\omega\varepsilon}{\beta}\frac{\partial E_z}{\partial x}\right)$$

$$E_x = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial x} \qquad H_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial y}$$

$$E_y = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial y} \qquad H_y = \frac{-j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial x}$$

Considering Helmholtz Wave Equation for source-free region

$$\nabla^2\bar{E} + k^2\bar{E} = 0$$
$$\nabla^2 E_z + k^2 E_z = 0$$

$$E_z = E_{z_0} e^{-j\beta z}$$
$$\frac{\partial E_z}{\partial z} = -j\beta E_z$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + k^2 E_z = 0 \qquad \frac{\partial^2 E_z}{\partial z^2} = -\beta^2 E_z$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - \beta^2 E_z + k^2 E_z = 0$$

Factoring out $E_{z_0} e^{-j\beta z}$

$$\boxed{\frac{\partial^2 E_{z_0}}{\partial x^2} + \frac{\partial^2 E_{z_0}}{\partial y^2} + (k^2-\beta^2)E_{z_0} = 0}$$

$$k^2 - \beta^2 = k_c^2 \qquad \frac{\partial^2 E_{z_0}}{\partial x^2} + \frac{\partial^2 E_{z_0}}{\partial y^2} + k_c^2 E_{z_0} = 0$$

Assume $E_{z_0} = XY$ (X → function of x, Y → function of y) $\qquad k_c^2 = k_x^2 + k_y^2$

$$\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} + (k_x^2 + k_y^2)E_{z_0} = 0$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + (k_x^2 + k_y^2)XY = 0$$

Divide by XY

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + (k_x^2 + k_y^2) = 0$$

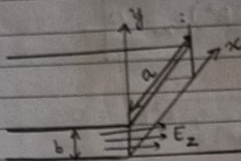
$$\left(\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + k_x^2\right) + \left(\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k_y^2\right) = 0$$

X is independent of Y
Individual components have to be zero for them to add up to zero

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + k_x^2 = 0 \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k_y^2 = 0$$

$$\frac{\partial^2 X}{\partial x^2} + Xk_x^2 = 0 \qquad \frac{\partial^2 Y}{\partial y^2} + Yk_y^2 = 0$$

$$X = A\cos(k_x x) + B\sin(k_x x)$$
$$Y = C\sin(k_y y) + D\cos(k_y y)$$



$E_{z_0} = XY$

$E_z = XYe^{-j\beta z}$

Bottom wall $(y=0)$

① $E_z\big|_{y=0} = 0$ ∵ Tangential

$XYe^{-j\beta z}\big|_{y=0} = 0$

$Xe^{-j\beta z}Y\big|_{y=0} = 0$

$Xe^{-j\beta z}\left[C\sin(k_y y) + D\cos(k_y y)\right]_{y=0} = 0$

$Xe^{-j\beta z}D = 0 \quad \Rightarrow \boxed{D=0} \quad Y = C\sin(k_y y)$

After first boundary condition

$E_z = [A\sin(k_x x) + B\cos(k_x x)][C\sin(k_y y)]\, e^{-j\beta z}$

② Top wall $(y=b)$

$E_z\big|_{y=b} = 0$ ∵ Tangential

$\underbrace{Xe^{-j\beta z}}_{\neq 0}\underbrace{C\sin(k_y b)}_{=0} = 0$

$\boxed{C \neq 0}$ Y will vanish

$\sin(k_y b) = 0$

$k_y b = n\pi$

$$\boxed{k_y = \frac{n\pi}{b}}$$

After second boundary condition

$$E_z = XC\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

③ Front wall $(x=0)$

$E_z\big|_{x=0} = 0$

$[A\sin(k_x x) + B\cos(k_x x)]\,Ye^{-j\beta z}\big|_{x=0} = 0$

$\underbrace{Ye^{-j\beta z}}_{\neq 0}B = 0 \qquad \boxed{B=0} \qquad \Rightarrow X = A\sin(k_x x)$

After third boundary condition $E_z = A\sin(k_x x)\,C\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$

④ Back wall $(x=a)$

$E_z\big|_{x=a} = 0$

$A\sin(k_x a)\underbrace{C\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}}_{\neq 0} = 0$

$A\sin(k_x a) = 0 \qquad \boxed{A \neq 0}$ X will vanish

$\sin(k_x a) = 0$

$k_x a = m\pi$

$\boxed{k_x = \frac{m\pi}{a}} \qquad X = A\sin\left(\frac{m\pi}{a}x\right)$

After applying conditions ①, ②, ③, ④

$E_z = A\sin\left(\frac{m\pi}{a}x\right)C\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$ Let $AC = A'$

$$\boxed{E_z = A'\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}}$$

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$$\frac{\partial E_z}{\partial x} = \frac{\partial}{\partial x}\left\{A' \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\right\}$$

$$= A'\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$\frac{\partial E_z}{\partial y} = \frac{\partial}{\partial y}\left\{A' \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\right\}$$

$$= A'\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$E_x = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial x} = \frac{-j\beta}{k_c^2}\left[A'\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\right]$$

$$\boxed{E_x = -\left(\frac{j\beta}{k_c^2}\right)A'\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}}$$

$$E_y = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial y} = \frac{-j\beta}{k_c^2}\left\{A'\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\right\}$$

$$\boxed{E_y = -A'\left(\frac{j\beta}{k_c^2}\right)\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}}$$

$$H_x = \frac{j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial y} = \frac{j\omega\varepsilon}{k_c^2}\left\{A'\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\right\}$$

$$\boxed{H_x = A'\left(\frac{j\omega\varepsilon}{k_c^2}\right)\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}}$$

$$H_y = \frac{-j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial x} = \frac{-j\omega\varepsilon}{k_c^2}\left\{A'\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\right\}$$

$$\boxed{H_y = -A'\left(\frac{j\omega\varepsilon}{k_c^2}\right)\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}}$$

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Phase constant (β)

$$k^2 - \beta^2 = k_c^2$$
$$\beta^2 = k^2 - k_c^2$$
$$k_c^2 = \omega_c^2\mu\varepsilon \qquad = \omega^2\mu\varepsilon - \omega_c^2\mu\varepsilon$$
$$\beta^2 = \omega^2\mu\varepsilon\left(1 - \frac{\omega_c^2}{\omega^2}\right)$$

$\omega = 2\pi f \qquad \omega_c = 2\pi f_c$

$$\boxed{\beta = \omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \qquad f > f_c$$

Wave impedance (Z_g)

$$Z_g = \frac{E_x}{H_y} = \frac{-E_y}{H_x}$$

$$\frac{E_x}{H_y} = \frac{-\left(\frac{-j\beta}{k_c^2}\right)A'\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}}{-A'\left(\frac{j\omega\varepsilon}{k_c^2}\right)\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta z}}$$

$$\frac{E_x}{H_y} = \frac{\beta}{\omega\varepsilon}$$

$$\frac{-E_y}{H_x} = \frac{-\left\{-A'\left(\frac{j\beta}{k_c^2}\right)\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}\right\}}{A'\left(\frac{j\omega\varepsilon}{k_c^2}\right)\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}}$$

$$= \frac{\beta}{\omega\varepsilon}$$

$$\boxed{Z_g = \frac{\beta}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

η → intrinsic impedance of the medium inside the waveguide

Phase velocity v_g

$$v_g = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$= \frac{1}{\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$\boxed{v_g = \frac{c}{\sqrt{\mu_r\varepsilon_r}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}}$$

Wavelength λ_g

$$\lambda_g = \frac{v_g}{f} = \frac{c/f}{\sqrt{\mu_r\varepsilon_r}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$\boxed{\lambda_g = \frac{\lambda}{\sqrt{\mu_r\varepsilon_r}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}}$$

λ → wavelength of waves in unbounded free space

Cut-off frequency f_c

$$k_c^2 = k_x^2 + k_y^2$$

$$\omega_c^2\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$(2\pi f_c)^2\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$2\pi f_c\sqrt{\mu\varepsilon} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \Rightarrow f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\boxed{f_c = \frac{c}{2\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

What is the lowest mode of operation for the TM wave propagation inside a rectangular waveguide?

* $m = n = 0$ $\quad f_c = 0$ $\quad$ Not possible → all fields vanish

* $m = 0$, $n = 1$ $\quad f_c = \dfrac{c}{2b\sqrt{\mu_r\varepsilon_r}}$

$$E_x = -A'\left(\frac{j\beta}{k_c^2}\right) \times 0 = 0$$

$$E_y = -A'\left(\frac{j\beta}{k_c^2}\right)\left(\frac{\pi}{b}\right)\underbrace{\sin(0)}_{=0}\cos\left(\frac{\pi}{b}y\right)e^{-j\beta z} = 0$$

$$H_x = A'\left(\frac{j\omega\varepsilon}{k_c^2}\right)\left(\frac{\pi}{b}\right)\underbrace{\sin(0)}_{=0}\cos\left(\frac{\pi}{b}y\right)e^{-j\beta z} = 0$$

$$H_y = -A'\left(\frac{j\omega\varepsilon}{k_c^2}\right) \times 0 = 0$$

All fields vanish ∴ Not possible

* $m = 1$, $n = 0$ $\quad f_c = \dfrac{c}{2a\sqrt{\mu_r\varepsilon_r}}$

$$E_x = -A'\left(\frac{j\beta}{k_c^2}\right) \times \frac{\pi}{a} \times \cos\left(\frac{\pi}{a}x\right) \times 0 \times e^{-j\beta z} = 0$$

$$E_y = -A'\left(\frac{j\beta}{k_c^2}\right) \times 0 = 0 \qquad H_x = +A'\left(\frac{j\omega\varepsilon}{k_c^2}\right) \times 0 = 0$$

$$H_y = -A'\left(\frac{j\omega\varepsilon}{k_c^2}\right) \times \left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right) \times 0 = 0$$

All fields vanish ∴ Not possible

* $m = 1$, $n = 1$ $\quad f_c = \dfrac{c}{2\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$

TM_{11} Dominant mode of operation for TM Waves

- $E_x = -A'\left(\frac{j\beta}{k_c^2}\right) \times \left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)e^{-j\beta z}$
- $E_y = -A'\left(\frac{j\beta}{k_c^2}\right)\left(\frac{\pi}{b}\right)\cos\left(\frac{\pi}{b}y\right)\sin\left(\frac{\pi}{a}x\right)e^{-j\beta z}$
- $H_x = A'\left(\frac{j\omega\varepsilon}{k_c^2}\right)\left(\frac{\pi}{b}\right)\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}y\right)e^{-j\beta z}$
- $H_y = -A'\left(\frac{j\omega\varepsilon}{k_c^2}\right)\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right) \times e^{-j\beta z}$

Example Determine the cut-off frequency of TM_{11} mode in a rectangular waveguide filled with dielectric having $\varepsilon_r = 4$, and having dimensions in the ratio $a:b = 4:1$ and the height (b) ~ 1 cm

TM_{11} mode $\Rightarrow m = 1,\ n = 1$

$$\frac{a}{b} = \frac{4}{1}$$

$$a = 4b$$

$$\mu_r \varepsilon_r = 1 \times 4 = 4$$

$$b \sim 1\,cm \Rightarrow a \sim 4\,cm$$

$$f_c = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{3 \times 10^8}{2 \times 2} \sqrt{\left\{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{1}\right)^2\right\} \times 10^4}$$

$$\boxed{f_c = 7.73 \times 10^9\ Hz}$$

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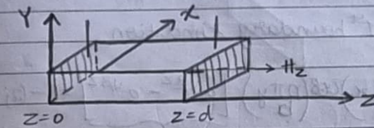
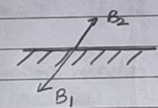
RESONATORS (used at microwave frequencies)

Rectangular Waveguide Resonator



Resonator
- Closed by metal walls
- Waveguides introduced via probes
- Waveguides taken out using slits

Perfect conductor
- electric & magnetic fields inside are zero



$$B_{N_1} = B_{N_2}$$
$$\mu_1 H_{N_1} = \mu_2 H_{N_2}$$
$$0 = 0$$

TE mode ($E_z = 0$, $H_z \neq 0$)
H_z at the boundary $z = 0$ and $z = d$ is zero

TE mode H_z in rectangular waveguide $\quad H_z = A' \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$

In resonator, the forward wave is H_z^+ and reflected wave is H_z^-.

$$H_z^+ = A' \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta z}$$

$$H_z^- = A'' \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{+j\beta z}$$

The net wave inside the resonator $\quad H_z = H_z^+ + H_z^-$

$$= \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\left[A' e^{-j\beta z} + A'' e^{j\beta z}\right]$$

Applying the boundary conditions

at $z=0$ $\quad H_z\big|_{z=0} = 0 \Rightarrow A' + A'' = 0$

$$\boxed{A'' = -A'}$$

After applying the 1st boundary condition,

$$H_z = A'\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\underbrace{\left[e^{-j\beta z} - e^{j\beta z}\right]}_{-2j\sin(\beta z)} = -(2jA')\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin(\beta z)$$

at $z=d$ $\quad H_z\big|_{z=d} = 0 \Rightarrow \underbrace{(-2jA')\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)}_{\neq 0}\sin(\beta d) = 0$

$$\sin\beta d = 0$$

$$\beta d = p\pi$$

$$\boxed{\beta = \frac{p\pi}{d}}$$

After applying both boundary conditions

$$\boxed{H_z = (-2jA')\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right)}$$

$$k^2 - \beta^2 = k_c^2$$

$$k^2 = k_c^2 + \beta^2 = k_x^2 + k_y^2 + \beta^2$$

$$= \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2$$

$\omega_r \Rightarrow$ ~~resonator~~ resonant angular frequency

$$\omega_r^2\mu\varepsilon = (2\pi f_r)^2\mu\varepsilon = 4\pi^2 f_r^2\mu\varepsilon$$

$$\boxed{k = 2\pi f_r\sqrt{\mu\varepsilon} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}}$$

Resonant frequency of the resonator

$$\boxed{f_r = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}}$$

$p \neq 0 \quad \because \sin 0 = 0 \quad H_z$ will vanish

Dominant mode in the resonator $\boxed{TE_{101}}$

Example: A rectangular waveguide resonator is operating at 9 GHz in dominant mode. If the resonator dimensions are in the ratio $a:b:d = 2:1:4$, determine the dimensions of the resonator, if it is filled with dielectric having $\varepsilon_r = 4$.

Dominant mode TE_{101} $\quad m=1$, $n=0$, $p=1$ $\quad a = 2b$, $d = 4b$

$f_r = 9$ GHz

$\mu_r\varepsilon_r = 1\times 4 = 4$

$$f_r = \frac{c}{2\sqrt{\mu_r\varepsilon_r}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$

$$9\times 10^9 = \frac{3\times 10^8}{2\times 2}\sqrt{\left(\frac{1}{2b}\right)^2 + \left(\frac{0}{b}\right)^2 + \left(\frac{1}{4b}\right)^2}$$

$$3\times 10\times 4 = \sqrt{\frac{1}{4b^2} + \frac{1}{16b^2}} = \sqrt{\frac{5}{16b^2}} = \frac{\sqrt{5}}{4b}$$

$$b = \frac{\sqrt{5}}{30\times 16} \Rightarrow \boxed{b = 4.658 \text{ mm}}$$

$$\boxed{a = 9.317 \text{ mm}}$$

$$\boxed{d = 18.63 \text{ mm}}$$

Assignment by 25/1/24, 4 p.m

$E_z = \int E_x \quad E_y, H_x, H_y$

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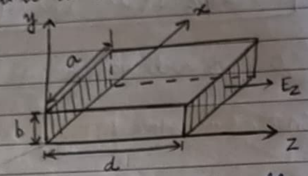
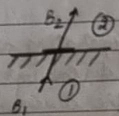
Resonator in TM mode $(H_z = 0, E_z \neq 0)$



E_z is normal component

Resonator walls $z=0$ and $z=d$

- tangential electric field to the wall is zero
- normal magnetic field to the wall is zero



$B_{N1} = B_{N2}$

$\mu_1 H_{N1} = \mu_2 H_{N2}$

$0 = 0$

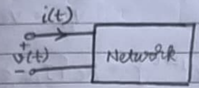
* Use E_x or E_y expression for the waveguide
* For the resonator ~~Fixed~~ Forward wave E_x^+

Reflected wave E_x^-

Net electric field

$$E_x = E_x^+ + E_x^-$$

POWER FLOW IN RECTANGULAR WAVEGUIDES



Instantaneous power flow into the network $\Rightarrow p(t) = v(t)\, i(t)$

$= VI\cos(\omega t + \theta)\cos(\omega t + \phi)$

Instantaneous voltage $v(t) = V\cos(\omega t + \theta)$

Instantaneous current $i(t) = I\cos(\omega t + \phi)$

} Not the same phase $\Rightarrow$ not purely resistive

Average power $P_{av} = \frac{1}{T}\int_0^T p(t)\,dt \qquad T = \frac{1}{f} = \frac{2\pi}{\omega}$

$$P_{av} = \frac{1}{T}\int_0^T VI\cos(\omega t + \theta)\cos(\omega t + \phi)\,dt$$

$$= \frac{VI}{2T}\int_0^T \{\cos(2\omega t + \theta + \phi) + \cos(\cancel{2\omega t}\ \theta - \phi)\}\,dt$$

$$= \frac{VI}{2T}\left[\frac{1}{2\omega}\sin(2\omega t + \theta + \phi) + t\cos(\theta - \phi)\right]_0^T$$

$$= \frac{VI}{2T}\left[\left\{\frac{1}{2\omega}\sin\left(2\omega T + \theta + \phi\right) + T\cos(\theta - \phi)\right\} - \left\{\frac{1}{2\omega}\sin(\theta + \phi) + 0\right\}\right]$$

$2\omega \times \frac{2\pi}{\omega}$

$$= \frac{VI}{2T}\left[\frac{1}{2\omega}\sin(4\pi + \theta + \phi) + \frac{2\pi}{\omega}\cos(\theta - \phi) - \frac{1}{2\omega}\sin(\theta + \phi)\right]$$

$$= \frac{VI}{2T} \times T\cos(\theta - \phi) + \frac{VI}{2T}\left[\frac{\sin(\theta + \phi) - \sin(\theta + \phi)}{2\omega}\right]$$

$$\boxed{P_{avg} = \frac{VI}{2}\cos(\theta - \phi)}$$

$$v(t) = V\cos(\omega t+\theta) = Re\{Ve^{j\omega t}e^{j\theta}\} = Re\{V_{ph}e^{j\omega t}\}$$

$$V_{ph} = Ve^{j\theta}$$

$$i(t) = I\cos(\omega t+\phi) = Re\{Ie^{j\phi}e^{j\omega t}\} = Re\{I_{ph}e^{j\omega t}\}$$

$$I_{ph} = Ie^{j\phi}$$

Evaluate $Re\{\frac{1}{2}V_{ph}I_{ph}^*\}$

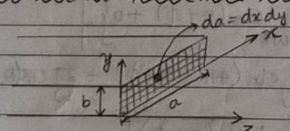
$$\frac{1}{2}V_{ph}I_{ph}^* = \frac{1}{2}(Ve^{j\theta})(Ie^{j\phi})^* = \frac{VI}{2}e^{j\theta}e^{-j\phi}$$

$$Re\left\{\frac{1}{2}V_{ph}I_{ph}^*\right\} = Re\left\{\frac{VI}{2}e^{j\theta}e^{-j\phi}\right\} = \frac{VI}{2}\cos(\theta-\phi)$$

$$= P_{av}$$

$$\boxed{P_{av} = \frac{VI}{2}\cos(\theta-\phi) = \frac{1}{2}Re\{V_{ph}I_{ph}^*\}}$$

POWER FLOW IN RECTANGULAR WAVEGUIDES



$$P_{av} = \frac{1}{2}Re\left[\int \bar{E}\times\bar{H}^*\cdot da\right]$$

$$= \frac{1}{2}Re\left[\int_{x=0}^{a}\int_{y=0}^{b}\bar{E}\times\bar{H}^*\cdot\bar{a}_z\,dx\,dy\right]$$

For the case of dominant mode propagation TE_{10}

$$\bar{E} = E_y\bar{a}_y = -A'\left(\frac{j\omega\mu}{k_c^2}\right)\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)e^{-j\beta z}\,\bar{a}_y$$

$$\bar{H} = H_x\bar{a}_x + H_z\bar{a}_z = A'\left(\frac{j\beta}{k_c^2}\right)\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)e^{-j\beta z}\,\bar{a}_x + A'\cos\left(\frac{\pi}{a}x\right)e^{-j\beta z}$$

$$\bar{H}^* = \bar{a}_x H_x^* + \bar{a}_z H_z^*$$

$$\bar{E}\times\bar{H}^*\cdot\bar{a}_z\,dx\,dy = \bar{a}_y E_y\times(\bar{a}_x H_x^* + \bar{a}_z H_z^*)\cdot\bar{a}_z\,dx\,dy$$

$$= (-\bar{a}_z E_y H_x^* + \bar{a}_x E_y H_z^*)\cdot\bar{a}_z\,dx\,dy$$

$$= -E_y H_x^*\,dx\,dy$$

$$-E_y H_x^*\,dx\,dy = -\left\{\left((-A')\left(\frac{j\omega\mu}{k_c^2}\right)\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)e^{-j\beta z}\right)\left(A'\left(\frac{-j\beta}{k_c^2}\right)\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)e^{j\beta z}\right)\right\}dx\,dy$$

$$= (A')^2\left(\frac{\omega\mu\beta}{k_c^4}\right)\left(\frac{\pi}{a}\right)^2\sin^2\left(\frac{\pi}{a}x\right)dx\,dy$$

$$= \left(\frac{A'^2\omega\mu\beta}{k_c^4}\right)\left(\frac{\pi}{a}\right)^2\left[\sin\left(\frac{\pi}{a}x\right)\right]^2 dx\,dy$$

$$k_c^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

for $TE_{10} \Rightarrow k_c^2 = \left(\frac{\pi}{a}\right)^2$

$$-E_y H_x^*\,dx\,dy = \left(\frac{A'^2\omega\mu\beta}{k_c^2}\right)\frac{1}{k_c^2}\times k_c^2\sin^2\left(\frac{\pi}{a}x\right)dx\,dy$$

$$= \frac{A'^2\omega\mu\beta}{k_c^2}\sin^2\left(\frac{\pi}{a}x\right)dx\,dy$$

$$= (A')^2\left(\frac{\omega\mu\beta a^2}{\pi^2}\right)\sin^2\left(\frac{\pi}{a}x\right)dx\,dy$$

$$\int_{x=0}^{a}\int_{y=0}^{b}\bar{E}\times\bar{H}^*\cdot\bar{a}_z\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{b}(A')^2\left(\frac{\omega\mu\beta a^2}{\pi^2}\right)\sin^2\left(\frac{\pi}{a}x\right)dx\,dy$$

$$(A')^2 \frac{\omega\mu\beta a^2}{\pi^2} \int_{x=0}^{a} \sin^2\left(\frac{\pi x}{a}\right) dx \int_{y=0}^{b} dy$$

$$(A')^2 \left(\frac{\omega\mu\beta a^2 b}{\pi^2}\right) \int_{x=0}^{a} \sin^2\left(\frac{\pi x}{a}\right) dx$$

$$\boxed{P_{av} = \frac{1}{2} \mathrm{Re}\left[(A')^2 \left(\frac{\omega\mu\beta a^2 b}{\pi^2}\right) \int_{x=0}^{a} \sin^2\left(\frac{\pi}{a}x\right) dx\right]}$$

$$\int_{x=0}^{a} \sin^2\left(\frac{\pi}{a}x\right) dx = \frac{1}{2}\int_{x=0}^{a}\left[1-\cos\left(\frac{2\pi x}{a}\right)\right] dx$$

$$= \frac{1}{2}\left[x - \frac{a}{2\pi}\sin\left(\frac{2\pi x}{a}\right)\right]_0^a$$

$$= \frac{1}{2}\left[\left(a - \frac{a}{2\pi}\underbrace{\sin(2\pi)}_{=0}\right) - \left(0 - \frac{a}{2\pi}\sin 0\right)\right]$$

$$= \frac{a}{2}$$

$$P_{av} = \frac{1}{2}\mathrm{Re}\left\{(A')^2\left(\frac{\omega\mu\beta a^2 b}{\pi^2}\right)\left(\frac{a}{2}\right)\right\}$$

$$\boxed{P_{av} = \frac{(A')^2\, \omega\mu\beta a^3 b}{4\pi^2}}$$

Average power flowing in the waveguide for TE_{10}

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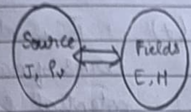
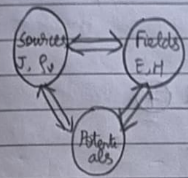


Connecting sources to get fields directly can be dedious

$\Rightarrow$ Requires a mediator



Simpler using mediator (potentials)

$$\nabla \times \bar{H} = j\omega\varepsilon\bar{E} + \bar{J} \longrightarrow (1)$$
$$\nabla \times \bar{E} = -j\omega\mu\bar{H} \longrightarrow (2)$$
$$\nabla \cdot \bar{D} = \rho_v \longrightarrow (3)$$
$$\nabla \cdot \bar{B} = 0 \longrightarrow (4)$$

Magnetic vector potential $\bar{A}$ such that

$$\bar{B} = \nabla \times \bar{A} \longrightarrow (5)$$

since div (curl vector) = 0

For electrostatic case $\quad \bar{E} = -\nabla V$

$$\bar{B} = \mu\bar{H} \Rightarrow \bar{H} = \frac{1}{\mu}\bar{B} = \frac{1}{\mu}(\nabla \times \bar{A}) \longrightarrow (6)$$

Using (6) in (2)

$$\nabla \times \bar{E} = -j\omega\mu\left(\frac{1}{\mu}\nabla \times \bar{A}\right) = -j\omega\nabla \times \bar{A}$$

$$\nabla \times (\bar{E} + j\omega\bar{A}) = 0$$

curl(Grad) = 0

Let $\bar{E} + j\omega\bar{A} = -\nabla V \longrightarrow (7)$

$$\boxed{\bar{E} = -\nabla V - j\omega\bar{A}}$$

$\bar{E}$ is defined in terms of magnetic vector potential & scalar potential

Considering

$$\nabla\times\bar{H} = j\omega\varepsilon\bar{E} + \bar{J}$$

Substituting values from 6 and 7

$$\frac{1}{\mu}\nabla\times\nabla\times\bar{A} = j\omega\varepsilon(-\nabla V - j\omega\bar{A}) + \bar{J}$$

$$\nabla\times\nabla\times\bar{A} = j\omega\varepsilon\mu(-\nabla V - j\omega\bar{A}) + \mu\bar{J}$$

$$\nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A} = -\nabla(j\omega\varepsilon\mu V) + \omega^2\mu\varepsilon\bar{A} + \mu\bar{J}$$

$$\nabla(\nabla\cdot\bar{A} + j\omega\varepsilon\mu V) = \nabla^2\bar{A} + \omega^2\mu\varepsilon\bar{A} + \mu\bar{J} \longrightarrow (8)$$

Considering

$$\varepsilon\nabla\cdot\bar{D} = \rho_v$$

$$\varepsilon\nabla\cdot\bar{E} = \rho_v$$

$$\varepsilon\nabla\cdot(-\nabla V - j\omega\bar{A}) = \rho_v$$

$$\nabla\cdot[-\nabla V - j\omega\bar{A}] = \frac{\rho_v}{\varepsilon}$$

$$-\nabla^2 V - j\omega\nabla\cdot\bar{A} = \frac{\rho_v}{\varepsilon} \longrightarrow (9)$$

Equations 8 & 9 are coupled

$\bar{B} = \nabla\times\bar{A}$ is not a unique definition for $\bar{A}$

Let $\bar{A} = \bar{a}_y x$

Only curl of $\bar{A}$ has been defined

$$\bar{B} = \nabla\times\bar{A} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix} = \bar{a}_z\frac{\partial x}{\partial x} = \bar{a}_z$$

Let $\bar{A} = -\bar{a}_x y$

$$\bar{B} = \nabla\times\bar{A} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & 0 & 0 \end{vmatrix}$$

$$= -\bar{a}_y\left(\cancelto{=0}{-\frac{\partial(-y)}{\partial z}}\right) + \bar{a}_z\left(0 - \frac{\partial(-y)}{\partial y}\right)$$

$$\nabla\times\bar{A} = \bar{a}_z$$

∴ Only defining $\nabla\times\bar{A}$ does not make $\bar{A}$ unique. To make $\bar{A}$ unique, we need to define $\nabla\cdot\bar{A}$ also.

Defining $\nabla\cdot\bar{A} = -j\omega\mu\varepsilon V$ and using it in 8 and 9;
↳ common term in both the equations;

$$\boxed{\nabla^2\bar{A} + \omega^2\mu\varepsilon\bar{A} = -\mu\bar{J}} \longrightarrow (10)$$

defining $\nabla\cdot\bar{A}$ as $-j\omega\mu\varepsilon V$ makes the equations decoupled

$$-\nabla^2 V - j\omega(-j\omega\mu\varepsilon V) = \frac{\rho_v}{\varepsilon}$$

$$\cancel{\nabla^2 V + j\omega(-j\omega\mu}$$

$$\nabla^2 V + j\omega(-j\omega\mu\varepsilon V) = -\frac{\rho_v}{\varepsilon}$$

$$\boxed{\nabla^2 V + \omega^2\mu\varepsilon V = -\frac{\rho_v}{\varepsilon}} \longrightarrow (11)$$

Equations 10 and 11 are decoupled.

$\nabla\cdot\bar{A} = -j\omega\mu\varepsilon V$ is known as LORENTZ CONDITION

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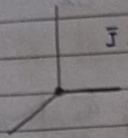
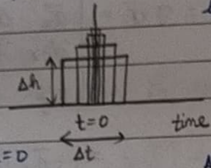
$\bar{J}$ is just a unit point source placed at origin.



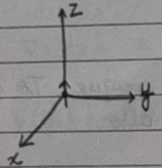
$\bar{J}$ is zero everywhere else apart from origin $\Rightarrow$ can be mapped to the concept of impulse function



area under the curve has to be 1 $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

$\delta(t) = \infty$ at $t=0$
$= 0$ elsewhere

Area
$\Delta h \Delta t = 1$ as $\Delta t \to 0$, $\Delta h \to \infty$ } total area 1



J is a vector $\rightarrow$ has the direction $\bar{a_z}$

$\delta(x)\,\delta(y)\,\delta(z) = \infty$ at $(0,0,0)$
$= 0$ elsewhere

$$\bar{J} = \bar{a_z}\,\delta(x)\,\delta(y)\,\delta(z)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\,dx\,dy\,dz = 1$$

Substituting the value of $\bar{J}$ in equation (10)

$$\Rightarrow \nabla^2 \bar{A} + \omega^2 \mu \varepsilon \bar{A} = -\mu \bar{a_z}\,\delta(x)\,\delta(y)\,\delta(z)$$

Since RHS is in the direction of $\bar{a_z}$, LHS also has to be in $\bar{a_z}$ direction

$$\bar{A} = \bar{a_z} A_z$$

$$\bar{a_z}\left[\nabla^2 A_z + \omega^2 \mu \varepsilon A_z\right] = \bar{a_z}\left[-\mu\,\delta(x)\,\delta(y)\,\delta(z)\right]$$

~~Can now be solved as a scalar equation~~

$$\nabla^2 A_z + \omega^2 \mu \varepsilon A_z = -\mu\,\delta(x)\,\delta(y)\,\delta(z)$$

At all other regions except origin,

$$\nabla^2 A_z + \omega^2 \mu \varepsilon A_z = 0$$

Let $A_z = \frac{\overset{\text{constant}}{C}}{r} e^{-jkr}$ in spherical coordinates

$\therefore \nabla^2 \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)$

For a point source, intensity is a function of r
Independent of θ and ϕ

$\nabla^2 A_z + \omega^2 \mu \varepsilon A_z = 0$
now becomes

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_z}{\partial r}\right) + \omega^2 \mu \varepsilon A_z = 0}$$

Solution : $A_z = \frac{C}{r} e^{-jkr}$ where $k = \omega\sqrt{\mu\varepsilon}$

$$\frac{\partial A_z}{\partial r} = C\left\{\frac{-e^{-jkr}}{r^2} + \frac{-jk\,e^{-jkr}}{r}\right\}$$

$$r^2 \frac{\partial A_z}{\partial r} = -Ce^{-jkr} - jrCke^{-jkr}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} A_z\right) = \cancel{jCke^{-jkr}} - rCk^2 e^{-jkr} - \cancel{jCke^{-jkr}}$$

$$= -rCk^2 e^{-jkr}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} A_z\right) = -\frac{1}{r} Ck^2 e^{-jkr} = -\frac{C}{r}\omega^2 \mu \varepsilon\, e^{-jkr} = -\omega^2 \mu \varepsilon A_z$$

$$\boxed{\begin{aligned} k &= \omega\sqrt{\mu\varepsilon} \\ k^2 &= \omega^2 \mu \varepsilon \end{aligned}}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} A_z\right) + \omega^2 \mu \varepsilon A_z = -\omega^2 \mu \varepsilon A_z + \omega^2 \mu \varepsilon A_z = 0$$

Solution verified

Consider $A_z = \frac{C}{r} e^{jkr}$

$$\frac{\partial A_z}{\partial r} = C\left\{ \frac{-e^{jkr}}{r^2} + \frac{jke^{jkr}}{r} \right\}$$

$$r^2 \frac{\partial A_z}{\partial r} = -Ce^{jkr} + jCrke^{jkr}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_z}{\partial r}\right) = -jCe^{jkr}k + jCke^{jkr} - rk^2Ce^{jkr}$$

$$= -rCk^2e^{jkr}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_z}{\partial r}\right) = -\frac{Ck^2e^{jkr}}{r} = -\omega^2\mu\varepsilon A_z$$

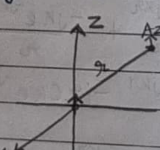
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_z}{\partial r}\right) + \omega^2\mu\varepsilon A_z = -\omega^2\mu\varepsilon A_z + \omega^2\mu\varepsilon A_z = 0$$

Both solutions satisfy

The solution $A_z = \frac{C}{r}e^{-jkr}$ is chosen

since after travelling a certain distance e^{-jkr} appears

$A_z = \frac{C}{r}e^{\pm jkr}$



e^{-jkr} phase lag

e^{jkr} phase lead

$$\boxed{A_z = \frac{C}{r}e^{-jkr}}$$

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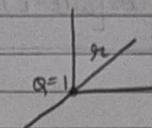
$$\nabla^2 V + k^2 V = 0 \quad \Rightarrow \quad \boxed{V = \frac{C_1}{r}e^{-jkr}}$$

Electrostatics

$$V = \frac{1}{4\pi\varepsilon r}$$



$k = \omega\sqrt{\mu\varepsilon}$

$k = 0$ in electrostatics

$V = \frac{C_1}{r}$ $\quad \boxed{C_1 = \frac{1}{4\pi\varepsilon}}$ $\quad \boxed{V = \frac{1}{4\pi\varepsilon r}e^{-jkr}}$

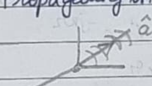
$$\left.\begin{aligned} \nabla^2 A + k^2 A &= -\mu J \\ \nabla^2 V + k^2 V &= -\frac{\rho_v}{\varepsilon} \end{aligned}\right\} \text{Solutions are similar}$$

$\Rightarrow \boxed{C = \frac{\mu}{4\pi}}$

$$\boxed{A_z = \frac{\mu}{4\pi r}e^{-jkr}}$$

$$\tilde{A}_z = Re\{A_z e^{j\omega t}\} = Re\left\{\frac{\mu}{4\pi r}e^{-jkr}e^{j\omega t}\right\}$$

$= \frac{\mu}{4\pi r}\cos(\omega t - kr)$ Propagating in +ve r-direction



$= \frac{\mu}{4\pi r}\cos\left[\omega\left(t - \frac{kr}{\omega}\right)\right]$

$$\boxed{\tilde{A}_z = \frac{\mu}{4\pi r}\cos[\omega(t - t')]}$$

$\frac{kr}{\omega} = \frac{\omega\sqrt{\mu\varepsilon}\,r}{\omega} = \sqrt{\mu\varepsilon}\,r$

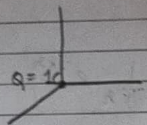
in free space $\sqrt{\mu\varepsilon} = \sqrt{\mu_0\varepsilon_0} = \frac{1}{c}$

$\Rightarrow \frac{kr}{\omega} = \frac{\overset{\text{distance}}{r}}{\underset{\text{velocity}}{c}} = t'$ Time

$J = Re\{1 e^{j\omega t}\} = \cos\omega t$

$$\boxed{\tilde{V} = \frac{1}{4\pi\epsilon r}\cos[\omega(t-t')]}$$

Retarded potentials
↳ disturbance at origin will be reflected at r after a time t'



Consider a unit point source Q at the origin

$\rho_v = \frac{Q}{V} = \frac{1}{0} = \infty$

charge

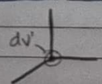
volume

point does not occupy any volume

$\rho_v = \infty$ at (0,0,0)
$= 0$ elsewhere

$\therefore \rho_v = \delta(x)\,\delta(y)\,\delta(z)$

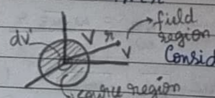
Practical Scenario: charge occupies a small volume dv'
ρ_v is no longer ∞, it exists



$\rho_v \, dV' = \Delta Q$

Electric scalar potential (in the limiting case of $dv' \to 0$)

$$\Delta V = \frac{\rho_v \, dv'}{4\pi\epsilon r} e^{-jkr}$$



Considering a bigger volume V (integrate)

If the charge density exists in a volume V, then electric scalar potential due to the charge density is

$$V = \int_{\text{Volume}} \frac{\rho_v}{4\pi\epsilon r} e^{-jkr} dv'$$

V' → to ensure integration is done w.r.t source region, not field region

The magnetic vector potential (in scalar form) due to current density J in the volume V is given by

$$A_z = \int_{\text{Volume}} \frac{\mu J}{4\pi r} e^{-jkr} dv'$$

J & ρ_v are functions of r, θ, ϕ

$$V = \int_{\text{volume}} \frac{\rho_v(r', \theta', \phi')}{4\pi\epsilon r} e^{-jkr} dv'$$

$$\bar{A} = \int_{\text{volume}} \frac{\mu \bar{J}(r', \theta', \phi')}{4\pi r} e^{-jkr} dv'$$