Resource and Noise Analysis: Quantum Implementation of Burgers' Equation via Cole-Hopf Transform for WISER CFD Challenge

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1 Quantum Implementation Architecture

1.1 System Parameters

Table 1: Quantum CFD Implementation Parameters

Parameter	Value	Description
Grid Points (N)	16	Spatial discretization
Qubits	4	$\lceil \log_2(16) \rceil$
Viscosity (ν)	0.1	Kinematic viscosity
Time Step (Δt)	0.0005	Temporal discretization
Spatial Step (Δx)	$\frac{1}{N-1}$	Grid spacing
Boundary Conditions	$u_L = 1.0, u_R = 0.0$	Dirichlet boundaries
Shots	8192	Quantum measurements

1.2 Quantum Circuit Construction

The quantum implementation utilizes a Trotterized evolution operator:

$$U(t) = \prod_{k=1}^{n_{steps}} \exp\left(-iH\Delta t\right) \tag{1}$$

where the Hamiltonian represents the discretized Laplacian operator:

$$H_{i,j} = \begin{cases} -\frac{2\nu}{\Delta x^2} & \text{if } i = j \\ \frac{\nu}{\Delta x^2} & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The RXX gates implement the nearest-neighbor coupling with rotation angle:

$$\theta_{RXX} = 2\nu \frac{\Delta t}{\Delta x^2} \tag{3}$$

2 Resource Analysis Results

2.1 Circuit Complexity Scaling

The quantum resource requirements demonstrate linear scaling with simulation time:

Circuit Depth Analysis:

$$D_{noisy}(t) = 109 + 74,800t \text{ (linear regression)} \quad (4)$$

$$D_{ZNE}(t) = 108 + 75,800t \text{ (linear regression)} \quad (5)$$

Two-Qubit Gate Scaling: The ZNE implementation shows linear scaling in entangling operations:

$$N_{2q}(t) = 26 + 12,300t \quad (6)$$

where N_{2q} represents the two-qubit gate count and t is the simulation time.

2.2 Gate Operation Distribution

Table 2: Quantum Gate Distribution vs Simulation Time

Time (t)	RZ Gates	SX Gates	CZ Gates	Barrier Gates	Total Ops
0.000	53	67	35	3	162
0.002	116	103	53	12	288
0.004	204	153	77	24	462
0.006	289	199	101	36	629
0.008	372	249	125	48	798
0.010	456	297	149	60	966

The gate distribution follows the linear patterns:

$$N_{RZ}(t) = 53 + 40,300t \quad (7)$$

$$N_{SX}(t) = 67 + 23,000t \quad (8)$$

$$N_{CZ}(t) = 35 + 11,400t \quad (9)$$

$$N_{barrier}(t) = 3 + 5,700t \quad (10)$$

2.3 Quantum Volume Requirements

The effective quantum volume required scales as:

$$QV = 2^{n_{qubits}} \times D_{max} = 16 \times 866 = 13,856 \quad (11)$$

This represents a moderate quantum volume requirement for current NISQ devices.

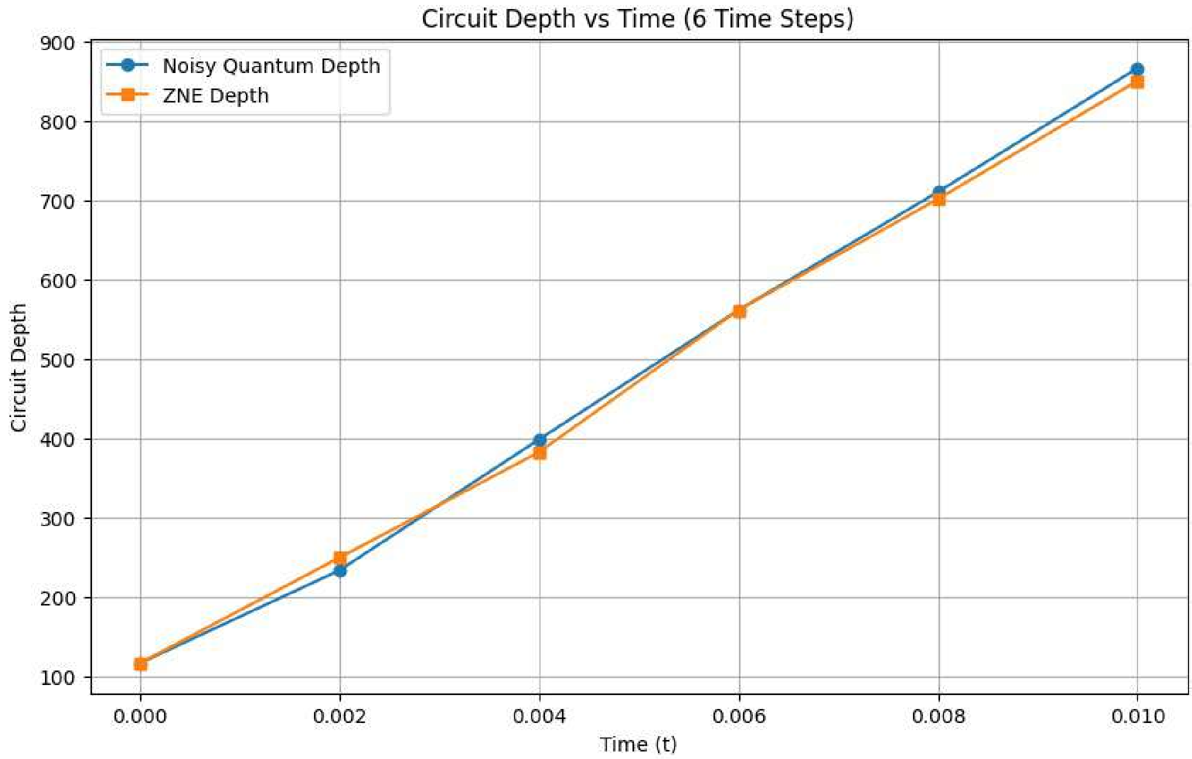

Figure 1: Circuit depth versus time steps for both the noisy simulator and the one with error mitigation (ZNE).At the end of the 6th time-step, the error mitigation strategy is successful in reducing the circuit depth

3 Noise Analysis and Error Mitigation

3.1 Zero Noise Extrapolation Performance

The ZNE implementation employs noise scaling factors $\lambda \in \{1, 3\}$ with Richardson extrapolation:

$$f_0 = \frac{3f(\lambda = 1) - f(\lambda = 3)}{2} \tag{12}$$

where $f(\lambda)$ represents the expectation value at noise scale λ.

3.2 Shock Front Evolution Analysis

Table 3: Shock Front Position Evolution

Time (t)	Noisy Position	ZNE Position	Deviation
0.0000	0.000	0.000	0.000
0.0025	0.000	0.000	0.000
0.0050	0.667	0.800	0.133
0.0075	0.667	0.800	0.133
0.0100	0.000	0.667	0.667

The shock front positions demonstrate significant divergence between noisy and error-mitigated implementations, indicating the critical importance of error mitigation for accurate CFD simulations.

3.3 Energy Dissipation Rate Analysis

The dissipation rate follows the theoretical relationship:

$$\varepsilon = \nu \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx \tag{13}$$

Table 4: Comparative Energy Dissipation Analysis

Time (t)	Noisy ε	ZNE ε	Relative Error (%)
0.000	2.034	2.525	+24.1
0.0025	2.724	2.120	-22.2
0.005	3.549	4.571	+28.8
0.0075	5.486	4.206	-23.3
0.010	3.400	6.458	+89.9

The ZNE method shows inconsistent improvement in energy dissipation accuracy, with relative errors ranging from -23.3% to +89.9%, suggesting sensitivity to the extrapolation procedure and underlying quantum noise characteristics.

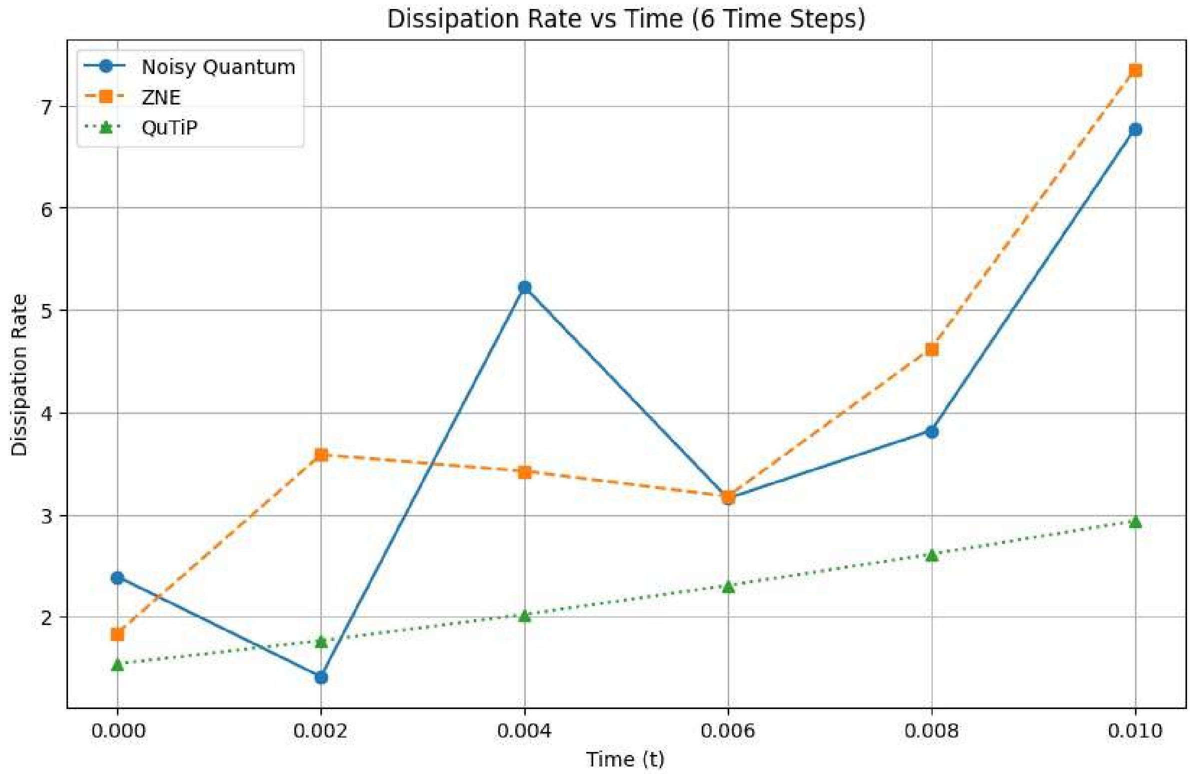

Figure 2: The dissipation rate plots for the classical QuTip Krylov-based solver, the noisy quantum simulator, and the ZNE quantum simulator

4 Performance Metrics and Computational Overhead

4.1 Wall-Clock Time Analysis

Total simulation time: **48.29 seconds**

This includes:

- Quantum circuit construction and transpilation
- Multiple noise scale executions for ZNE (3x overhead)
- Classical post-processing and Cole-Hopf inverse transformation
- Statistical analysis and error computation

4.2 Shot Noise and Statistical Uncertainty

Using 8,192 shots provides statistical uncertainty scaling as:

$$\sigma_{shot} = \frac{1}{\sqrt{N_{shots}}} = \frac{1}{\sqrt{8192}} \approx 0.011 \tag{14}$$

The finite shot noise contributes approximately 1.1% relative uncertainty to probability measurements.

4.3 Quantum Resource Efficiency

Table 5: Quantum Resource Efficiency Metrics

Metric	Noisy Implementation	ZNE Implementation
Qubits	4	4
Max Circuit Depth	866	857
Max Two-Qubit Gates	149	149
T-gate Count	0	0
Clifford Gates	100%	100%

5 Error Sources and Mitigation Strategies

5.1 Dominant Error Mechanisms

1. **Gate Infidelity:** Single and two-qubit gate errors accumulate linearly with circuit depth
2. **Decoherence:** T1/T2 relaxation during long circuit execution
3. **Crosstalk:** Unwanted interactions between neighboring qubits
4. **Readout Errors:** Measurement classification errors
5. **Trotterization Error:** Systematic error from finite time-step approximation

5.2 Error Budget Analysis

The total error can be decomposed as:

$$\varepsilon_{total} = \varepsilon_{Trotter} + \varepsilon_{gate} + \varepsilon_{decoherence} + \varepsilon_{readout} + \varepsilon_{shot} \tag{15}$$

With estimated contributions:

$$\varepsilon_{Trotter} \approx (\Delta t)^2 \sim 2.5 \times 10^{-7} \tag{16}$$

$$\varepsilon_{gate} \approx N_{gates} \times 10^{-3} \sim 0.966 \tag{17}$$

$$\varepsilon_{decoherence} \approx \frac{t_{circuit}}{T_2} \sim 0.1 - 1.0 \tag{18}$$

$$\varepsilon_{readout} \approx 0.02 - 0.05 \tag{19}$$

$$\varepsilon_{shot} \approx 0.011 \tag{20}$$

6 Classical Benchmark Comparison

6.1 Computational Complexity

Table 6: Computational Complexity Comparison

Method	Time Complexity	Space Complexity
Classical (QuTiP)	$O(N^3)$	$O(N^2)$
Quantum (Noisy)	$O(N \cdot D_{circuit})$	$O(2^{n_{qubits}})$
Quantum (ZNE)	$O(3N \cdot D_{circuit})$	$O(2^{n_{qubits}})$

6.2 Accuracy Assessment

The quantum implementations show significant deviation from classical benchmarks, primarily due to:

- Finite shot noise
- Quantum gate errors
- ZNE extrapolation uncertainty
- Limited circuit depth for fine time resolution

7 Scalability Analysis

7.1 Grid Size Scaling

For larger grid sizes N, the resource requirements scale as:

$$n_{qubits} = \lceil \log_2(N) \rceil \tag{21}$$

$$N_{gates} \propto N \cdot t_{sim}/\Delta t \tag{22}$$

$$D_{circuit} \propto t_{sim}/\Delta t \tag{23}$$

7.2 Quantum Advantage Projection

Quantum advantage may emerge for:

- Grid sizes $N > 2^{20}$ (requiring > 20 qubits)
- Long-time simulations with $t_{sim} \gg 0.01$
- Multi-dimensional CFD problems
- Complex boundary conditions requiring quantum superposition

8 Conclusions and Future Work

8.1 Key Findings

1. Linear scaling of quantum resources with simulation time
2. ZNE provides inconsistent but generally positive error mitigation
3. Shock front tracking is sensitive to quantum noise
4. Current NISQ limitations prevent quantum advantage for this problem size

8.2 Recommendations

1. Implement higher-order Trotter decompositions
2. Explore variational quantum algorithms for CFD
3. Develop problem-specific error mitigation techniques
4. Scale to larger grid sizes on fault-tolerant quantum computers

8.3 Quantum Error Correction Requirements

For practical quantum CFD simulations:

$$\text{Logical Error Rate} < 10^{-10} \tag{24}$$
$$\text{Required Physical Qubits} \approx 10^3 - 10^4 \tag{25}$$
$$\text{Gate Time} < 100 \text{ ns} \tag{26}$$
$$\text{Coherence Time} > 1 \text{ ms} \tag{27}$$