Introduction to Knock-off Filters

"Controlling False Discovery Rates via Knock-offs "
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Nandana Sengupta

October 18, 2015

Introduction

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- Consider the simple linear regession model:

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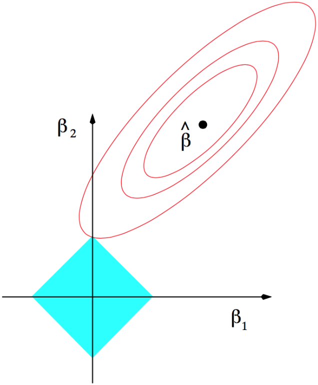

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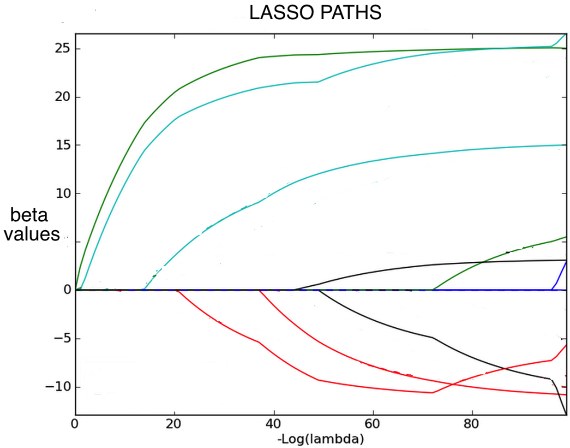

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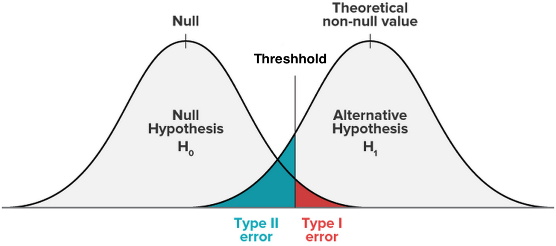

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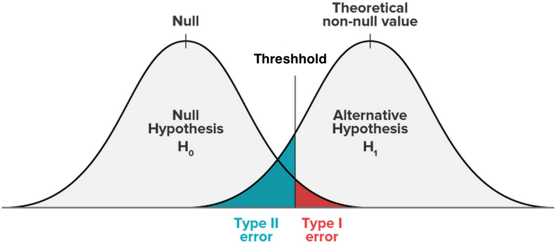

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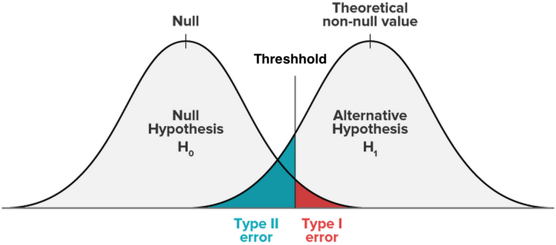
- 

- Want to control the proportion of Type I error

$$\underset{\substack{\uparrow \\ \text{False discovery rate}}}{\text{FDR}} = \mathbb{E}\left[\underbrace{\frac{\#\text{ false positives}}{\text{total \# of features selected}}}_{\text{False discovery proportion}}\right] = \mathbb{E}\left[\frac{|S \cap \mathcal{H}_0|}{|S|}\right].$$

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- Compute Lasso with Augmented Matrix

$$\beta_\lambda = \arg\min_{\beta \in \mathbb{R}^{2p}} \left\{ \frac{1}{2} \left\| y - [X \ \ \widetilde{X}] \cdot \beta \right\|_2^2 + \lambda \left\| \beta \right\|_1 \right\}$$

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$$\lambda_j = \sup \left\{ \lambda \ : \ \beta_j^\lambda \neq 0 \right\} = \text{ first time } X_j \text{ enters Lasso path}$$

$$\widetilde{\lambda}_j = \sup \left\{ \lambda \ : \ \widetilde{\beta}_j^\lambda \neq 0 \right\} = \text{ first time } \widetilde{X}_j \text{ enters Lasso path}$$

- $$W_j = \max\{\lambda_j, \widetilde{\lambda}_j\} \cdot \text{sign}(\lambda_j - \widetilde{\lambda}_j)$$

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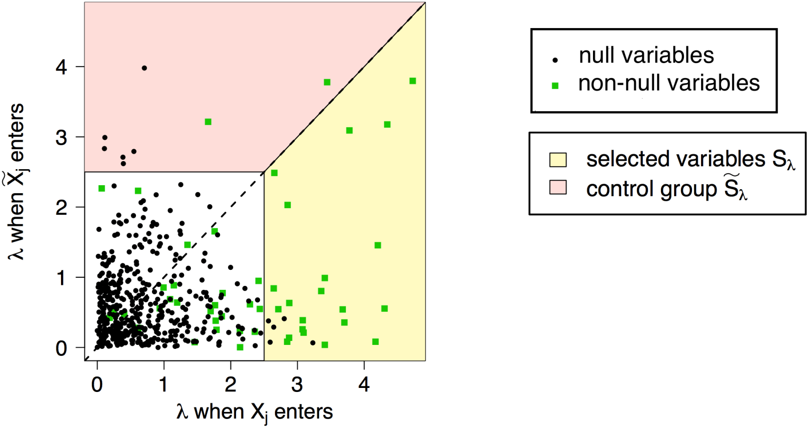
Knock-off Filters: Intuition



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Thanks!