26: Maximum Flow

26.1: Flow Network

→ $G = (V, E)$ directed

→ Each edge (u, v) has a capacity $c(u,v) \geq 0$

→ If $(u,v) \notin E$, then $c(u,v) = 0$

Exercise 26.1-3 → Source vertex 's', sink vertex 't', assume $s \leadsto v \leadsto t, \forall v \in V$.

Net flow: A function $f: V \times V \rightarrow \mathbb{R}$ satisfying

→ Capacity constraint: $\forall\, u, v \in V,\ f(u,v) \leq c(u,v)$.

→ Skew symmetry: $\forall\, u, v \in V,\ f(u,v) = -f(v,u)$.

→ Flow conservation: $\forall\, u \in V - \{s,t\},\ \sum_{v \in V} f(u,v) = 0$.

→ Another way to think of flow conservation:

$$\underbrace{\sum_{v \in V: f(v,u) > 0} f(v,u)}_{\text{total positive flow entering } u} = \underbrace{\sum_{v \in V: f(u,v) > 0} f(u,v)}_{\text{total positive flow leaving } u}.$$

"Flow in = Flow out"

→ The difference b/w positive flow p & net-flow f:

* $p(u,v) \geq 0$

* f satisfies skew symmetry.

* Equivalence of positive flow & netflow definitions

Define netflow in terms of positive flow:

→ Define $f(u,v) = p(u,v) - p(v,u)$.

→ Argue, given definition p, that this definition of f satisfies capacity constraint & flow conservation:

* Capacity constraint:

$p(u,v) \le c(u,v)$ & $p(v,u) \ge 0 \Rightarrow p(u,v) - p(v,u) \le c(u,v)$.

* Flow conservation:

$$\sum_{v\in V} f(u,v) = \sum_{v\in V} (p(u,v) - p(v,u))$$
$$= \sum_{v\in V} p(u,v) - \sum_{v\in V} p(v,u)$$
$$= 0.$$

→ Skew symmetry is trivially satisfied by this definition of $f(u,v)$:

$$f(u,v) = p(u,v) - p(v,u)$$
$$= -(p(v,u) - p(u,v))$$
$$= -f(v,u)$$

Define positive flow in terms of net flow:

→ Define

$$p(u,v) = \begin{cases} f(u,v) & ; \text{ if } f(u,v) > 0, \\ 0 & ; \text{ if } f(u,v) \le 0 \end{cases}$$

→ Argue, given definition of f, that this definition of p satisfies capacity constraint & flow conservation.

* Capacity constraint:

→ If $f(u,v) > 0$: $f(u,v) \le c(u,v) \Rightarrow 0 \le p(u,v) \le c(u,v)$

→ If $f(u,v) \le 0$: $0 = p(u,v) \le c(u,v)$

* Flow ~~constraint~~ Conservation:

$$\sum_{v\in V} p(u,v) - \sum_{v\in V} p(v,u)$$

$$= \left(\sum_{v\in V: f(u,v)>0} p(u,v) + \sum_{v\in V: f(u,v)\le 0} p(u,v) \right) - \left(\sum_{v\in V: f(v,u)>0} p(v,u) + \sum_{v\in V: f(v,u)\le 0} p(v,u) \right)$$

$$= \sum_{v\in V: f(u,v)>0} p(u,v) + \cancel{\sum_{v\in V: f(u,v)\le 0} p(u,v)} - \cancel{\sum_{v\in V: f(v,u)\le 0} p(v,u)}$$

$$- \sum_{v\in V: f(v,u)\geq 0} p(v,u)$$

$$= \sum_{v\in V: f(u,v)>0} f(u,v) - \sum_{v\in V: f(v,u)\geq 0} f(v,u)$$

$$= \sum_{v\in V: f(u,v)>0} f(u,v) - \sum_{v\in V: f(u,v)\le 0} (-f(u,v))$$

$= 0$

26-1-5 proof.

*Value of flow: $f = |f| = \underbrace{\sum_{u \in V} f(s,u)}_{\text{Total flow out of the source}} - \underbrace{\sum_{u \in V} f(u,s)}_{\text{the flow into the source}}$

* Maximum-flow ~~problem~~ problem: Given G, s, t & c. find a flow value is maximum.

* Implicit Summation notation 26.1-8 sol^n

→ We work with functions, like f, that take pairs of vertices as arguments.

→ Extends to take set of vertices, with interpretation of summing over all vertices in the set.

Ex:- If X & Y are set of vertices,

$$f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y).$$

Therefore, can express flow ~~convers~~ conservation as $f(u,V) = 0$ ∀ ∀ $u \in V - \{s,t\}$.

Notation: Omit braces in sets with implicit summations.

Ex:- $f(s, V-s) = f(s,V)$. Here, $f(s, V-s)$ really means $f(s, V-\{s\})$

Lemma: 26.1-4 sol^n

proof for any flow f in $G=(V,E)$:

① $\forall X \subseteq V$, $f(X,X)=0$

② $\forall X,Y \subseteq V$, $f(X,Y) = -f(Y,X)$

③ $\forall X,Y,Z \subseteq V \Rightarrow X \cap Y = \phi$

i.e. $f(X \cup Y, Z) = f(X,Z) + f(Y,Z)$ & $f(Z, X \cup Y) = f(Z,X) + f(Z,Y)$.

① $f(X,X) = -f(X,X) \Rightarrow f(X,X) = 0$. (part (2)) ⇓

Proof: ② $f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y)$

$= \sum_{x \in X} \sum_{y \in Y} -f(y,x)$ [Skew Symmetry]

$= \sum_{y \in Y} \sum_{x \in X} -f(y,x)$

$= -f(Y,X)$

③ $f(X \cup Y, Z) = \sum_{u \in X \cup Y} \sum_{z \in Z} f(u,z)$

$= \sum_{u \in X} \left(\sum_{z \in Z} f(u,z) \right) + \sum_{u \in Y} \left(\sum_{z \in Z} f(u,z) \right)$

$= f(X,Z) + f(Y,Z)$. [∵ $X \cap Y = \phi$]

Lemma $|f| = f(V,t)$. 26.1-1 sol^n

proof: First, show that $f(V, V-s-t) = 0$:

$f(u,V) = 0 \quad \forall u \in V - \{s,t\}$

$\Rightarrow f(V-s-t, V) = 0$ [add $f(u,V)$ ∀ $u \in V - \{s,t\}$]

$\Rightarrow f(V, V-s-t) = 0$. [by Lemma, part (2)]

Thus, $|f| = f(s,V)$ [definition of $|f|$]

$= f(V,V) - f(V-s,V)$ [lemma, part (3)]

$= -f(V-s,V)$ [Lemma, part (1)]

$= f(V, V-s)$ [Lemma, part (2)]

$= f(V,t) + f(V, V-s-t)$ [Lemma, part (3)]

$= f(V,t)$ [from above]

Cuts

A cut (S,T) of flow network $G=(V,E)$ is a partition of V into S & $T = V-S$ such that $s \in S$ & $t \in T$.

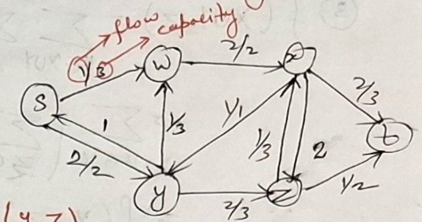
→ Similar to cut edges used in MSTs, except that here the graph is directed, & we require $s \in S$ & $t \in T$.

→ For flow f, the net flow across cut (S,T) is $f(S,T)$.

→ Capacity of cut (S,T) is $c(S,T)$.

→ A minimum cut of G is a cut whose capacity is minimum over all cuts of G.

Consider the cut



$S = \{s, w, y\}$, $T = \{x, z, t\}$

$$f(S,T) = f(w,x) + f(y,x) + f(y,z)$$
$$= 2 + (-1) + 2$$
$$= 3$$

$$c(S,T) = c(w,x) + c(y,z)$$
$$= 2+3$$
$$= 5$$

* Note: the Diff. b/w capacity & flow.

→ flow obeys skew symmetry, so $f(y,x) = -f(x,y) = -1$

→ capacity does not: $c(y,x) = 0$, but $c(x,y) = 1$.

So include flows going both ways across the cut, but capacity going only S to T.

Now, consider the cut $S = \{s, x, w, y\}$ & $T = \{z, t\}$

$$f(S,T) = f(x,z) + f(x,t) + f(y,z)$$
$$= -1 + 2 + 2$$
$$= 3.$$

$$c(S,T) = c(x,z) + c(x,t) + c(y,z)$$
$$= 2+3+3 = 8.$$

i.e. same flow as previous cut, higher capacity.

Lemma

For any cut (S,T), $f(S,T) = |f|$.

Proof: First, show that $f(S-s, V) = 0$:

$S - \{s\} \subseteq V - \{s,t\}$.

$$\therefore f(S-s, V) = \sum_{u \in S-\{s\}} f(u,V)$$
$$= \sum_{u \in S-\{s\}} 0 \quad [\text{flow conservation \& } S-\{s\} \subseteq V-\{s,t\}]$$
$$= 0.$$

So,

$$f(S,T) = f(S,V) - f(S,S) \quad [\text{Lemma, part (3), } S \cup T = V, S \cap T = \phi]$$
$$= f(S,V) \quad [\text{Lemma, part(1)}]$$
$$= f(s,V) + f(S-s, V) \quad [\text{Lemma, part (3)}]:$$
$$= f(s,V) \quad [\because f(S-s,V) = 0]$$
$$= |f|$$

Corollary

The value of any flow ≤ capacity of any cut.

Proof: Let (S,T) be any cut, f be any flow.

$$|f| = f(S,T) \quad [\because \text{Lemma}]$$
$$= \sum_{u \in S} \sum_{v \in T} f(u,v)$$
$$\leq \sum_{u \in S} \sum_{v \in T} c(u,v) \quad [\because \text{Capacity constraint}]$$
$$= c(S,T).$$

Therefore, maximum flow ≤ capacity of minimum cut.

Exercises

26.1-1: Show that splitting an edge in a flow N/w yields an equivalent N/w. More formally, suppose that flow N/w G contains edge (u,v) & we create a new flow N/w G' by creating a new vertex x & replacing (u,v) by new edges (u,x) & (x,v) with $c(u,x) = c(x,v) = c(u,v)$. Show that a maximum flow in G' has the same value as a maximum flow in G.

Using the definition of a flow, prove that if $(u,v) \notin E$ & $(v,u) \notin E$ then $f(u,v) = f(v,u) = 0$.

$f(u,v) \leq c(u,v)$

$f(u,v) + f(v,u) = 0$

$\therefore c(u,v) = 0$

Therefore, $f(u,v) = f(v,u) = 0$.

If, we can split any edges in this way, even if two vertex u&v doesn't have any connection b/w them, we can still add a vertex y & make $c(u,y) = c(y,v) = 0$

To conclude, we can transform any graph with or without antiparallel edges into an equivalent graph without antiparallel edges & have the same maximum flow value.

26.1-2 Extend the flow properties & definitions to the multiple-source, multiple-sink problem. Show that any flow in any flow in a multiple-source, multiple-sink flow N/w corresponds to a flow of identical value in the single-source, single-sink N/w obtained by adding a supersource & supersink & vice-versa.

* Capacity constraints: $\forall u,v \in V \Rightarrow 0 \leq f(u,v) \leq c(u,v)$
* Flow conservation: $\forall V-S-T \Rightarrow \sum_{v \in V} f(v,u) = \sum_{u \in V} f(u,v)$.

26.1-5 (Soln) State the maximum-flow problem as a linear-programming problem.

$$\max \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s)$$

s.t. $0 \leq f(u,v) \leq c(u,v)$

$$\sum_{v \in V} f(v,u) - \sum_{v \in V} f(u,v) = 0$$

26.1-4 (Soln) We want to prove the following lemma

Lemma: For any flow f in $G = (V,E)$:

① $\forall X \subseteq V$, $f(X,X) = 0$

② $\forall X,Y \subseteq V$, $f(X,Y) = -f(Y,X)$

③ $\forall X,Y,Z \subseteq V \Rightarrow X \cap Y = \phi$

$$f(X \cup Y, Z) = f(X,Z) + f(Y,Z) \;\&$$

$$f(Z, X \cup Y) = f(Z,X) + f(Z,Y).$$

26.1-6: (Soln) The flow sum $f_1 + f_2$ satisfies skew symmetry & flow conservation, but might violate the capacity constraint. We give proof for skew symmetry & flow conservation & an example that shows a violation of capacity constraints. Let $f(u,v) = (f_1 + f_2)(u,v)$.

For skew symmetry:

$$\begin{aligned} f(u,v) &= f_1(u,v) + f_2(u,v) \\ &= -f_1(v,u) - f_2(v,u) \quad [\text{Skew symmetry}] \\ &= -(f_1(v,u) + f_2(v,u)) \\ &= -f(v,u). \end{aligned}$$

For flow conservation,

let $u \in V - \{s,t\}$:

$$\sum_{v \in V} f(u,v) = \sum_{v \in V} (f_1(u,v) + f_2(u,v))$$

$$= \sum_{v \in V} f_1(u,v) + \sum_{v \in V} f_2(u,v)$$

$$= 0 + 0 \quad \text{[Flow Conservation]}$$

$$= 0$$

For the capacity constraint, Let $V = \{s,t\}$, $E = \{(s,t)\}$ & $c(s,t) = 1$. Let $f_1(s,t) = f_2(s,t) = 1$. Then f_1 & f_2 obey the capacity constraint, but $(f_1 + f_2)(u,v) = 2$, which violates the capacity constraint.

Solⁿ 26.1-7 & 26.1-4

To see that the flows from a convex set, we show that if f_1 & f_2 are flows, then so is $\alpha f_1 + (1-\alpha) f_2$ for all α such that $0 \le \alpha \le 1$.

→ For the capacity constraint, first observe that $\alpha \le 1$ implies that $1-\alpha \ge 0$. Thus, for $u,v \in V$, we have

$$\alpha f_1(u,v) + (1-\alpha) f_2(u,v) \ge 0 \cdot f_1(u,v) + 0 \cdot (1-\alpha) f_2(u,v) = 0.$$

Since $f_1(u,v) \le c(u,v)$ & $f_2(u,v) \le c(u,v)$, we also have

$$\alpha f_1(u,v) + (1-\alpha) f_2(u,v) \le \alpha c(u,v) + (1-\alpha) c(u,v)$$

$$= (\alpha + (1-\alpha)) c(u,v)$$

$$= c(u,v).$$

→ For skew symmetry, we have $f_1(u,v) = -f_1(v,u)$ & $f_2(u,v) = -f_2(v,u)$ for $u,v \in V$. Thus, we have

$$\alpha f_1(u,v) + (1-\alpha) f_2(u,v) = -\alpha f_1(v,u) - (1-\alpha) f_2(v,u)$$

$$= -(\alpha f_1(v,u) + (1-\alpha) f_2(v,u)).$$

→ For flow conservation, observe that since f_1 & f_2 obey flow conservation, we have $\sum_{v \in V} f_1(u,v) = 0$ & $\sum_{v \in V} f_2(u,v) = 0$ for any $u \in V - \{s,t\}$. Thus,

$$\sum_{v \in V} (\alpha f_1(u,v) + (1-\alpha) f_2(u,v)) = \alpha \sum_{v \in V} f_1(u,v) + (1-\alpha) \sum_{v \in V} f_2(u,v)$$

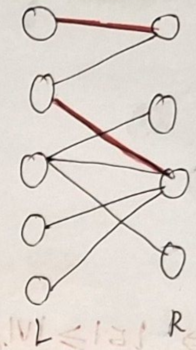
$$= \alpha \cdot 0 + (1-\alpha) \cdot 0$$

$$= 0.$$

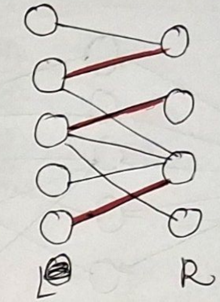
26.3: Maximum bipartite matching

Example of a problem that can be solved by turning it into flow problem.

→ $G = (V, E)$ (undirected) is bipartite if we perform partition $V = L \cup R$ such that all edges in E go b/w L and R.



matching



maximum matching

→ A matching is a subset of edges $M \subseteq E$ such that $\forall v \in V$, ≤ 1 edge of M is incident on v. (Vertex v is matched if an edge of M is incident on it; otherwise unmatched).

→ Maximum matching is a matching of maximum cardinality. (M is a maximum matching if $|M| \geq |M'|$ for all matchings M'.)

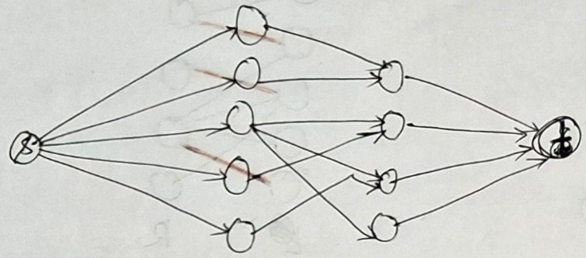
→ Problem: Given a bipartite graph (with the partition), find a maximum matching.

→ Application: Matching planes to routes

* L = Set of planes
* R = Set of routes
* $(u, v) \in E$ if plane u can fly route v.
* Want maximum # of routes to be served by planes.

Given G, define flow network $G' = (V', E')$.

* $V' = V \cup \{s, t\}$.
* $E' = \{(s,u) : u \in L\} \cup \{(u,v) : u \in L, v \in R, (u,v) \in E\} \cup \{(v,t) : v \in R\}$
* $c(u,v) = 1 \ \forall (u,v) \in E'$



Each vertex in V has ≥ 1 incident edge $\Rightarrow |E| \geq |V|/2$

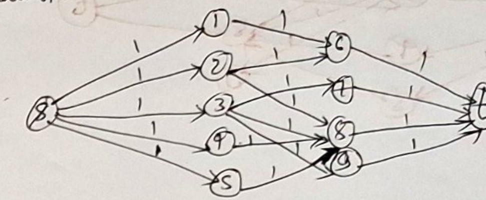
Therefore, $|E| \leq |E'| = |E| + |V| \leq 3|E|$

Therefore, $|E'| = \Theta(E)$.

Find a max flow in G'

Exercise 26.3-1

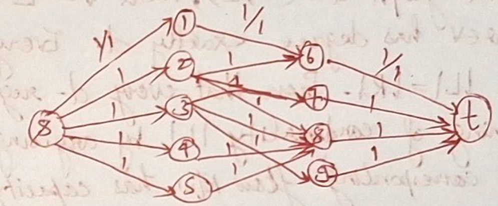
Run the Ford-Fulkerson algo. on flow N/w in figure 26-8(b) & show the residual N/w after each flow augmentation. Number the vertices in L top to bottom from 1 to 5 & in R top to bottom from 6 to 9. For each iteration, pick the augmenting path that is lexicographically smallest.



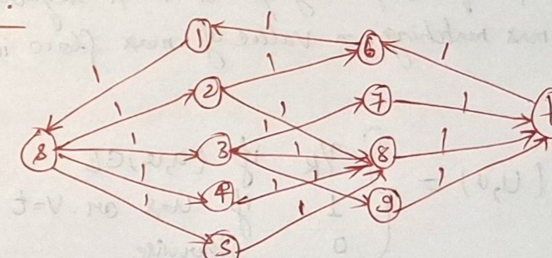
Augmenting path

S ? 1 ? 6 ? t

Flow N/w:

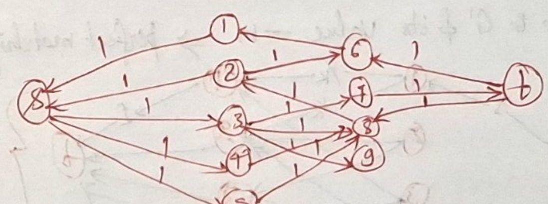


Residual N/w:



Augmenting path

S ? 1 ? 6 ? t

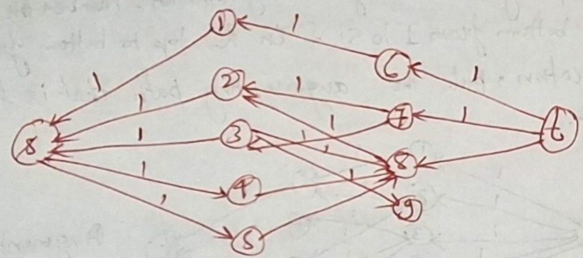


Augmenting path

S ? 2 ? 8 ? t

Final Residual graph : Augmenting path

$s ? 3 ? 7 ? t$



Exercise 26.3-5

We say that bipartite graph $G = (V, E)$, where $V = L \cup R$ is d-array if every $v \in V$ has degree exactly d. Every d-regular bipartite graph has $|L| = |R|$. Prove that every d-regular bipartite graph has a matching of cardinality $|L|$ by arguing that a minimum cut of the corresponding flow N/w has capacity $|L|$.

Consider the corresponding N/w flow graph G' to k-regular graph G :

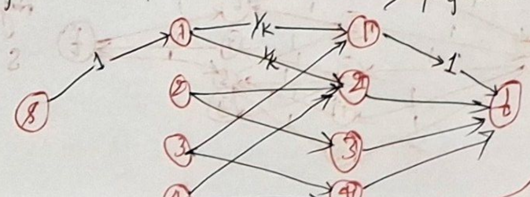
Proof:
→ size of max matching = value of max flow in G'.

→ Consider flow.

$$f(u,v) = \begin{cases} 1/k & \text{if } (u,v) \in E \\ 1, & \text{if } u=s \text{ or } v=t \\ 0 & \text{otherwise} \end{cases}$$

→ f is a flow in G' & its value = n ⇒ perfect matching.

G'



A feasible flow f of value n.