INTRODUCTION

In mathematics, a free abelian group is an abelian group with a basis. Being an abelian group means that it is a set with an addition operation that is associative, commutative and invertible. A basis, also called an integral basis, is a subset such that every element of the group can be uniquely expressed as an integer combination of finitely many basis elements.

In 1882 paper, Walter von dyck pointed out that these groups have the simplest possible presentation. The algebraic study of free groups was initiated by Jakob Nielsen in 1924, who gave them their name and established many of their basic properties.

Chapter I deals with Basic definition.

Chapter II deals with Free abelian groups.

Chapter III deals with Finitely generated abelian groups.

Chapter IV deals with Free product of groups.

Chapter V deals with Free groups

CHAPTER I

BASIC DEFINITION

DEFINITION: 1.1

A nonempty set of elements G is said to form a **group** if in G there is defined a binary operation, called the product and denoted by '·', such that

(i) For all a, b ∈ G implies that a·b ∈ G (closure law).

(ii) For all a, b, c ∈ G implies that a·(b·c) = (a·b)·c (associative law).

(iii) There exists an element e ∈ G such that a·e = e·a = a for all a ∈ G (the existence of an identity element in G).

(iv) For every a ∈ G there exists an element a^{-1} ∈ G such that a·a^{-1} = a^{-1}·a = e (the existence of inverses in G).

DEFINITION: 1.2

A group G is said to be **abelian** (or **commutative**) if for every a, b ∈ G, a·b = b·a.

DEFINITION: 1.3

Let G be a group. Then the number of elements in a group G is called the **order of the group G**.

Order of a group G is denoted by O(G).

DEFINITION: 1.4

A nonempty subset H of a group G is said to be a **subgroup** of G if, under the product in G, H itself forms a group.

DEFINITION: 1.5

A subgroup N of G is said to be a **normal subgroup** of G if for every $g \in G$ and $n \in N$, $g \cap g^{-1} \in N$.

DEFINITION: 1.6

An isomorphism of a group G to itself is called an **automorphism** of G. The set of all automorphism of G is denoted by Aut G.

DEFINITION: 1.7

Let G be a group and N_1,N_2,….,N_n **normal subgroups** of G such that

(i) $G = N_1 N_2 \ldots. N_n$

(ii) Given $g \in G$ then $g = m_1 m_2 \ldots m_n$, $m_i \in N_1$ in a unique way.

We then say that G is the **internal direct product** of N_1,N_2,…. ,N_n.

DEFINITION: 1.8

A mapping ϕ from a group G into a group $\bar{G}$ is said to be a **homomorphism** if for all a, b $\in$ G, ϕ(ab) = ϕ (a)ϕ (b).

DEFINITION: 1.9

A homomorphism ϕ from G into $\bar{G}$ is said to be an **isomorphism** if ϕ is one-to-one and onto.

DEFINITION: 1.10

Two groups G, G* are said to be **isomorphic** if there is an **isomorphism** of G onto G*. In this case we write **G ≈ G*.**

(i) $G \approx G$.

(ii) $G \approx G^*$ implies $G^* \approx G$.

(iii) $G \approx G^*, G^* \approx G^{**}$ implies $G \approx G^{**}$.

DEFINITION: 1.11

A nonempty set R is said to be an **associative ring** if in R there are defined two operations, denoted by '+' and '·' respectively, such that for all a, b, c in R:

(i) $a + b$ is in R.

(ii) $a + b = b + a$.

(iii) $(a + b) + c = a + (b + c)$.

(iv) There is an element 0 in R such that $a + 0 = a$ (for every a in R).

(v) There exists an element -a in R such that $a + (-a) = 0$.

(vi) $a \cdot b$ is in R.

(vii) $a\cdot (b \cdot c) = (a\cdot b)\cdot c$.

(viii) $a\cdot (b + c) = a\cdot b + a\cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ (the two distributive laws).

DEFINITION: 1.12

There is an element 1 in R such that $a \cdot 1 = 1 \cdot a = a$ for every a in R; if there is such we shall describe R as a **ring with unit element.**

DEFINITION: 1.13

If the multiplication of R is such that $a \cdot b = b \cdot a$ for every a, b in R, then we call R a **commutative ring.**

DEFINITION: 1.14

If R is a commutative ring, then a non-zero element a $\in$ R is called **zero-divisor** if there is non-zero element b $\in$ R, such that ab = 0.

DEFINITION: 1.15

A commutative ring is an **integral domain** if it has no zero-divisors.

DEFINITION: 1.16

A ring is said to be a **division ring** if its nonzero elements form a group under multiplication.

DEFINITION: 1.17

A **field** is a commutative division ring, if every non-zero of R has a multiplication inverse in R.

DEFINITION: 1.18

A mapping ϕ from the ring R into the ring R' is said to be a **homomorphism** if

(i) $\phi(a + b) = \phi(a) + \phi(b)$,

(ii) $\phi(ab) = \phi(a)\phi(b)$,

for all a, b $\in$ R.

DEFINITION: 1.19

If a, b $\in$ G, then b is said to be a **conjugate** of a in G if there exists an element c $\in$ G such that $b = c^{-1}ac$.

This relation a ~ b is said to be conjugacy.

DEFINITION: 1.20

An integral domain R is said to be a **Euclidean ring** if for every a $\neq 0$ in R there is defined a non-negative d(a) such that

(i) For all a, b $\in$ R, both nonzero, d(a) $\leq$ d(ab).

(ii) For any a, b $\in$ R, both nonzero, there exist t, r $\in$ R such that a = tb + r where either r = 0 or d(r) < d(b).

DEFINITION: 1.21

In the Euclidean ring R, a and b in R are said to be **relatively prime,** if their greatest common divisor is a unit of R.

DEFINITION: 1.22

A polynomial p(x) in F[x] is said to be **irreducible** over F if whenever p(x) = a(x)b(x) with a(x), b(x) $\in$ F[x], then one of a(x) or b(x) has degree 0 (i.e., is a constant).

DEFINITION: 1.23

The polynomial f(x) = $a_0 + a_1x + \cdot\cdot\cdot + a_nx^n$, where the $a_0, a_1,..., a_n$ are integers is said to be **primitive** if the greatest common divisor of $a_0, a_1, ..., a_n$ is 1.

DEFINITION: 1.24

A nonempty set V is said to be a **vector space** over a field F if V is an abelian group under an operation which we denote by '+', and if for every $\alpha \in$ F, v $\in$ V there is defined an element, written α v, in V subject to

(i) α (v + w) = α v + α w;

(ii) (α +β)v = α v + βv;

(iii) $\alpha(\beta v) = (\alpha\beta)v$;

(iv) $1\,v = v$;

for all $\alpha, \beta \in F$, $v, w \in V$ (where the 1 represents the unit element of F under multiplication).

DEFINITION: 1.25

If V is a vector space over F and if $v_1,\ldots,v_n \in V$ then any element of the form $\alpha_1 v_1 + \; + \alpha_2 v_2 + \cdots + \alpha_n v_n$ where the $\alpha_i \in F$, is a **linear combination** over F of $v_1,\ldots\ldots,v_n$.

DEFINITION: 1.26

If S is a nonempty subset of the vector space V, then L(S), the **linear span** of S, is the set of all linear combinations of finite sets of elements of S.

DEFINITION: 1.27

The vector space V is said to be **finite-dimensional** (over F) if there is a **finite subset** S in V such that V = L(S).

DEFINITION: 1.28

If V is a vector space and if $v_1,\ldots,v_n$ are in V, we say that they are **linearly dependent** over F if there exist elements $\lambda_1,\ldots,\lambda_n$ in F, not all of them 0, such that

$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$.

If the vectors $v_1,\ldots,v_n$ are not linearly dependent over F, they are said to be **linearly independent** over F.

DEFINITION: 1.29

A subset S of a vector space V is called a **basis** of V if S consists of linearly independent elements (that is, any finite number of elements in S is linearly independent) and V = L(S).

DEFINITION: 1.30

The vector space V over F is said to be an **inner product space** if there is defined for any two vectors u, v ∈ V an element (u, v) in F such that

(i) $(u, v) = (\overline{v, u})$;

(ii) $(u, u) \geq 0$ and $(u, u) = 0$ if and only if $u = 0$;

(iii) $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$;

for any u, v, w ∈ V and α, β ∈ F.

DEFINITION: 1.31

Let $\phi : G \longrightarrow \bar{G}$ be a homomorphism.

If ϕ is one-to-one, then it is called monomorphism. If ϕ is onto, then it is called an epimorphism.

DEFINITION: 1.32

Let H be a subgroup of a group G. Let a ∈ G.

Then the set $aH = \{ah \mid h \in H\}$ is called the **left coset** of H defined by a in G. Similarly $Ha = \{ha \mid h \in H\}$is called **right coset** of H defined by a.

DEFINITION: 1.33

A polynomial p(x) in F[x], the polynomial ring in x over F, we shall associate with p(x) a group, called the

Galois group of p(x).

DEFINITION: 1.34

Suppose G is finite group and O(G) = n p^m , where p is a prime number and p is not a divisor of n.

Then the subgroup H of G is said to be a **sylow subgroup** of G iff O(H) = p^m .

DEFINITION: 1.35

The linear transformations S, T∈A(V) are said to be **similar** if there exists an invertible element C ∈A (V) such that
T = CSC^{-1} .

CHAPTER II

FREE ABELIAN GROUPS

DEFINITION: 2.1

A **free abelian group** is an abelian group that has a basis in the sense that every element of the group can be written in one and only one as a finite linear combination of elements of the basis with integer coefficient

EXAMPLE:

The group Z × Z is free abelian and {(1,0), (0,1)} is a basis. Similarly, a basis for the free abelian group Z × Z × Z is {(1,0,0), (0,1,0), (0,0,1)} and so on. Thus finite direct products of the group Z with itself are free abelian groups.

DEFINITION: 2.2

If G is a free abelian group then the **rank** of G is the cardinality of a basis of G.

PROPERTIES:

(i) Every two basis of the free abelian group have the same cardinality. Also Free abelian groups of same rank are isomorphic.

(ii) A free abelian group is finitely generated if and only if its rank is a finite number n.

(iii) Direct product of two free abelian group is itself free abelian with basis the disjoint union of basis of two groups.

More generally, the direct product of any finite number of free abelian group is free abelian with basis the disjoint union of the basis of two groups. But it need not be necessarily true for infinite number of free abelian groups.

THEOREM: 2.3

Let X be the subset of non-zero abelian group G. then the following conditions are equivalent.

1. Each non-zero element a in G can be expressed uniquely in the form $a = n_1x_1 + n_2x_2 + \ldots\ldots. + n_rx_r$ for $n_i \neq 0$ in Z and distinct x_i in X

2. X generates G and $n_1x_1 + n_2x_2 + \ldots\ldots. + n_rx_r = 0$ for n_i in Z and distinct $x_i \in X$ iff $n_1 = n_2 = \ldots\ldots. n_r = 0$

PROOF:

$1 \Longrightarrow 2$

Assume that any non-zero element a in G can be expressed uniquely as

$a = n_1x_1 + n_2x_2 + \ldots\ldots. + n_rx_r.$

for $n_i \neq 0$ in Z and distinct x_i in X.

Given G is a non-zero abelian group, i.e., $G \neq 0$.

We have $X \neq 0$

It follows from 1 that $0 \notin X$.

For if $x_i = 0$ and $x_j \neq 0$ where $x_i, x_j \in X \subseteq G$ then, $x_j = x_i + x_j$, which would contradict the uniqueness of the expression for x_j.

From condition (1) it is clear that X generates G.

$n_1x_1 + n_2x_2 + ... + n_rx_r = 0$ if $n_1 = n_2 = ... n_r = 0$

Suppose that $n_1x_1 + n_2x_2 + ... + n_rx_r = 0$ with some $n_i \neq 0$ by dropping terms with zero coefficients and renumbering, we can assume all $n_i \neq 0$.

Then

$$x_1 = x_1 + (n_1x_1 + n_2x_2 + ... + n_rx_r)$$

$$x_1 = (1+n_1)\, x_1 + n_2x_2 + ... + n_rx_r$$

which gives two ways of writing $x_1 \neq 0$ contradicting the uniqueness assumption in condition 1.

If $n_1x_1 + n_2x_2 + ... + n_rx_r = 0$, then $n_1 = n_2 = ... n_r = 0$

$2 \Longrightarrow 1$

Let $a \in G$. Since X generates G.

We see a can be written in the form

$$a = n_1x_1 + n_2x_2 + ... + n_rx_r.$$

We have to show that this expression is unique.

We have if $n_1x_1 + n_2x_2 + ... + n_rx_r = 0$, then $n_1 = n_2 = ... n_r = 0$.

Suppose a has another such expression in terms of elements of X. By using some zero coefficients in the two expressions, we can assume they involve the same elements in X and are of the form,

$$a = n_1x_1 + n_2x_2 + ... + n_rx_r$$

$$a = m_1x_1 + m_2x_2 + ... + m_rx_r$$

Subtracting we get,

$$0 = (n_1 - m_1)\, x_1 + (n_2 - m_2)\, x_2 + ... + (n_r - m_r)\, x_r.$$

Then by our assumption we get

$n_1 - m_1 = n_2 - m_2 =........= n_r - m_r = 0$

$n_i - m_i = 0$

$n_i = m_i$ for i = 1, 2,..., r.

Thus the coefficients are unique.

Expression of any element a in G is unique.

PROPOSITION: 2.4

A subset X of an abelian group G is a basis of G if and only if every mapping of X into an abelian group A can be extended uniquely to a homomorphism of G → A.

PROOF:

Let X consist of the elements x_i for $i \in I$.

Suppose that X is a basis of G.

Let $x_i : \longrightarrow a_i$ be the mapping of X into any abelian group A.

The mapping $x = \sum_i \alpha_i x_i \longrightarrow \sum_i \alpha_i a_i$ is a well defined homomorphism of G into A and maps x_i into a_i.

However it is the only homomorphism of G which maps x_i to a_i for each $i \in I$.

Conversely,

Suppose that G and X have the stated property.

Let A be a free abelian group with basis a_i, $i \in I$.

There is a homomorphism $\emptyset : G \longrightarrow A$ such that $\emptyset (x_i) = a_i$; $i \in I$.

As a free abelian with basis a_i, there is a homomorphism $\psi : A \longrightarrow G$ such that $\psi(a_i) = x_i$; $i \in I$.

Then $\psi \circ \emptyset$ is an endomorphism of G which maps x_i into a_i .

Since the identity endomorphism I_G of G has the same property, by the uniqueness part of the stated condition.

It follows that $\psi \circ \emptyset = I_G$.

The endomorphism $\emptyset \circ \psi$ of A maps each of the generators a_i, $i \in I$ of A to itself and is therefore the identity endomorphism I_A of A.

Thus $\emptyset$ is an isomorphism of G with a mapping x_i to a_i.

Thus ψ is an isomorphism of G with a mapping x_i to a_i.

Since the elements a_i ; $i \in I$ forms a Z-basis of A, the elements x_i ; $i \in I$ form Z-basis of G.

Hence the proof.

PROPOSITION: 2.5

If G is any free abelian group, then any two bases of G have the same cardinality.

PROOF:

Let B and B' be any 2 bases of G.

We have to show that $| B |= | B' |$

Also we have the isomorphism.

$\oplus_{x\in B} Z \cong G \cong \oplus_{y\in B'} Z$.

Case 1:

When B and B' are finite sets.

Let $| B |= m$ and $| B' | = n$

Take $2G = \{2a|a \in G\}$ which is a subgroup of G.

Also we have G $\cong$ $\bigoplus_{x\in B}$ Z.

We obtain G/2G $\cong$ $\bigoplus_{x\in B}$Z/2Z.

Similarly since we have G $\cong$ $\bigoplus_{x\in B'}$Z.

G/2G $\cong$ $\bigoplus_{y\in B'}$Z /2Z.

This gives,

2^m=| $\bigoplus_{x\in B}$ Z /2Z| = G/2G = | $\bigoplus_{y\in B'}$Z /2Z| = 2^n

m=n

Case 2:

Consider the case if B is finite and B′ is infinite. As in previous case this would give

$\bigoplus_{x\in B}$ Z /2Z $\cong$ G/2G $\cong$ $\bigoplus_{y\in B'}$Z /2Z

This is however a contradiction.

Since $\bigoplus_{x\in B}$ Z /2Z is a finite group and $\bigoplus_{y\in B'}$Z /2Z is an infinite group.

Such a case doesn't exist.

Case 3:

Both B and B′ are infinite sets.

If B is infinite then |B| = |$\bigoplus_{x\in B}$ Z | .

It follows that

|B| = | $\bigoplus_{x\in B}$ Z | = |G| = | $\bigoplus_{y\in B'}$Z | = |B′|

|B| = |B′ |

Hence the two bases of G have the same cardinality.

THEOREM: 2.6

Let G be a free abelian group of rank n and let H be a subgroup of G. Then H is free abelian and rank (H) $\leq$ rank (G) .

PROOF:

Since G is a free abelian group of rank n.

Assume that $G = Z^n$.

We want to show that if $H \subseteq Z^n$ then, H is free abelian and rank(H) $\leq$ n.

We prove this using the method of induction on n.

If n=1; then H=kZ for $k \geq 0$.

So H ={0} or $H \cong Z$

$\Rightarrow$ H is free abelian and rank(H)$\leq$ rank(G)

Next for some n, every subgroup of Z^n is free abelian and of rank $\leq$ n.

Now let $H \subseteq Z^{n+1}$.

Take the homomorphism

$$f: Z^{n+1} \rightarrow Z$$

$$f(m_1, m_2, ..., m_{n+1}) = m_{n+1}$$

We have,

$$Ker(f) = \{(m_1, m_2, ..., 0)| m_i \in Z\} \cong Z^n.$$

Since $Im(f/H) \subseteq Z$, thus Im(f/H) is a free abelian group.

We get,

$$H \cong Im(f/H) \oplus ker(f/H)$$

We also have

$$\ker(f/H) = \ker(f) \cap H$$

It follows that ker(f/H) is a subgroup of ker(f) and since ker(f) is a free abelian group of rank n, by the inductive assumption we get that ker(f/H) is a free abelian group of rank ≤ n.

$$H \cong \mathrm{Im}(f/H) \oplus \ker(f/H)$$

Where Im(f/H) is a free abelian group of rank ≤ 1 and ker(f/H) is a free abelian group of rank ≤ n.

H is a free abelian group of rank ≤ n+1.

CHAPTER III
FINITELY GENERATED ABELIAN GROUPS

DEFINITION: 3.1

An abelian group (G,+) is called **finitely generated** if there exists finitely many elements x_1, x_2, ..., x_s in G such that every x in G can be written in the form x = n_1x_1 + n_2x_2, + ... + n_sx_s with integers n_1, ...,n_s.

In this, we say that the set {x_1, ..., x_s} is a generating set of G or that x_1, ..., x_s generate G. Every finite abelian group is finitely generated.

THEOREM: 3.2

If G is an abelian group generated by n elements. Then

$$G \cong F/H$$

where F is a free abelian group of rank n and H is some subgroup of F.

PROOF:

Let G be an abelian group generated by n elements

i.e., G = {a_1, a_2,....,a_n}.

Let F be free abelian group and {x_1,..., x_n} be a basis of F.

We have a homomorphism

$$f : F \rightarrow G \text{ defined by } f(x_i) = a_i.$$

Take H = ker(f).

Since f is an epimorphism by first isomorphism theorem.

First isomorphism theorem,

Let f : G → H be a homomorphism with kernel K. Then K is a normal subgroup of G and G/K ≅ Im(f).

we have,

$$F/Ker(f) \cong Im(f)$$

$$G \cong F/H.$$

Hence the proof.

THEOREM: 3.3

Let G be a finitely generated abelian group with generating set $\{ a_1, a_2, ..., a_n \}$. Let $\emptyset : Z \times Z \times .... \times Z \to G$ (where there are n factors of Z) be defined by

$$\emptyset\,(h_1, ..., h_n) = h_1a_1 + h_2a_2 + ... + h_na_n.$$

Then $\emptyset$ is a homomorphism onto G.

PROOF:

From the meaning of h_ia_i for $h_i \in Z$ and $a_i \in G$.

$$\begin{aligned} \emptyset\,[(h_1, ..., h_n) + (k_1, ..., k_n)] &= \emptyset\,(h_1+k_1, ..., h_n+k_n) \\ &= (h_1+k_1)\,a_1 + ... + (h_n+k_n)\,a_n \\ &= h_1a_1+k_1a_1+\ldots+h_na_n+k_na_n \\ &= (h_1a_1 + ... + h_na_n) + (k_1a_1 +...+k_na_n) \\ &= \emptyset\,(h_1, ..., h_n) + \emptyset\,(k_1, ..., k_n) \\ &\Longrightarrow \emptyset \text{ is an homomorphism.} \end{aligned}$$

Since $a_1, a_2, ..., a_n$ generates G.

Hence $\emptyset$ is a homomorphism onto G.

THEOREM: 3.4

If X = $\{x_1, x_2 ..., x_r\}$ is a basis for a free abelian group G and t ∈ Z, then for i ≠ j, the set

Y = $\{x_1, x_2, ... x_{j-1}, x_j + tx_i , x_j, x_{j+1},..., x_r\}$ is also a basis for G.

PROOF:

To show that Y is a basis of G.

We need to show that the set will span G and the elements are linearly independent.

Given $\{x_1, x_2 ..., x_i\}$ spans G.

Since $x_j = (-t) x_i + 1(x_j + tx_i)$

We see that x_j can be recovered from Y, which also this generates G.

Suppose,

$$n_1x_1 + n_2x_2 + ... + n_{j-1}x_{j-1} + n_j (x_j + tx_i) + n_{j+1}x_{j+1} + ... + n_rx_r =0$$

Then

$$n_1x_1+ n_2x_2 + ... + (n_i+ n_jt) x_i + ... n_jx_j + ... + n_rx_r = 0$$

And since X is a basis

$$n_1 = n_2=.... = n_i + n_j t = ...= n_j=...= n_r = 0$$

From $n_j = 0$ and $n_i + n_j t = 0$.

It follows that $n_i = 0$.

i.e., $n_1 = n_2=.... = n_i...= n_j=...= n_r = 0$

we get Y generates G.

By theorem Y is a basis of G.

Hence Y is also a basis of G.

THEOREM: 3.5

Let G be a non-zero free abelian group of finite rank n and let K be a non-zero subgroup of G. Then K is free abelian of rank s ≤ n. Furthermore there exists a basis $\{x_1, x_2 ..., x_n\}$ for G and positive integers d_1, ..., d_s where d_i divides d_{i+1} for
i = 1,…, S-1 such that $\{d_1x_1, d_2x_2, ..., d_sx_s\}$ is a basis of K.

PROOF:

Let G be a non-zero free abelian group of finite rank n. Let K be a non-zero subgroup of G.

Then we have K is also free abelian of rank ≤ n.

Now we have to show that K has a basis of the described form.

Suppose $Y = \{y_1, y_2,..., y_n\}$ is a basis for G.

All non-zero elements in K can be expressed in the form

$$k_1y_1+ k_2y_2+ ... + k_ny_n$$

for some $|k_i|$ is non-zero.

Among all bases Y for G, select one Y_1 that yields the minimal such non-zero value $|K_i|$ as all non-zero elements of K are written in terms of the basis elements in Y_1.

By renumbering the elements of Y_1 if necessary we can assume there is $w_1 \in K$ such that

$$w_1 = d_1y_1+ k_2y_2+ ... + k_ny_n$$

where $d_1 > 0$ and d_1 is the minimal attainable coefficient as just described.

Using the division algorithm, we write $k_j = d_1 q_j + r_j$ where $0 \leq r_j < d_1$ for $j = 2,3,...,n$.

Then,

$$w_1 = d_1 y_1 + (d_1 q_2 + r_2)\, y_2 + ... + (d_1 q_n + r_n) y_n$$

$$w_1 = d_1\, (y_1 + q_2 y_2 + ... + q_n y_n) + r_2 y_2 + ... + r_n y_n$$

Let $x_1 = y_1 + q_2 y_2 + ... + q_n y_n$

$$w_1 = d_1 x_1 + r_2 y_2 + \dots + r_n y_n$$

Then by using theorem, $\{x_1, y_2, ..., y_n\}$ is also a basis for G. Also from equation and our choice of Y_1 for minimal coefficient d_1, we see that $r_2 = ... r_n = 0$.

Thus $d_1 x_1 \in K$

We now consider bases for G of the form $\{x_1, y_2, ..., y_n\}$.

Each element of K can be expressed in the form

$$h_1 x_1 + k_2 y_2 + ... + k_n y_n.$$

Since $d_1 x_1 \in K$, we can subtract a suitable multiple of $d_1 x_1$ and then using the minimality of $d_1$to see that h_1 is multiple of d_1.

We see actually have $k_2 y_2 + ... + k_n y_n$ in K.

Among all such bases $\{x_1, y_2, ..., y_n\}$ we choose one Y_2 that leads to some $k_i \neq 0$ of minimal magnitude.

By renumbering the elements of Y_2 we can assume that there is $w_2 \in K$ such that

$$w_2 = d_2 y_2 + ... + k_n y_n$$

where $d_2 < 0$ and d_2 is minimal as just described.

Exactly as in the preceding paragraph, we can modify our basis from $Y_2 = \{x_1, y_2, ..., y_n\}$ to a basis $Y_3 = \{x_1, x_2, y_3, ..., y_n\}$ for G where $d_1x_1, d_2x_2 \in K$

Writing $d_2 = d_1q + r$ for $0 \leq r < d_1$.

We see that $\{x_1 + qx_2, x_2, y_3, ..., y_n\}$ is a basis for G and

$d_1x_1 + d_2x_2 = d_1x_1 + (d_1q + r)\, x_2$

$= d_1 (x_1 + qx_2) + rx_2 \in K$

But by our minimal choice of d_1 we see r = 0.

So d_1 divides d_2.

We now consider all bases of the form $\{x_1, x_2, y_3, ..., y_n\}$ for G and examine elements of K of the form $k_3y_3 + ... + k_ny_{n.}$

The process continues until we obtain a basis $\{x_1, x_2, ... x_s, y_{s+1}, .., y_n\}$ where the only element K of the form $k_{s+1}y_{s+1} + ... + k_ny_n$ is zero.

i.e., all k_i are zero.

Then let $x_{s+1} = y_{s+1} = ... = x_n = y_n$ and a basis for G of the described form in the proposition.

THEOREM: 3.6

Every finitely generated abelian group is isomorphic to a group of the form

$$Z_{m_1} \times Z_{m_2} \times ..... \times Z_{m_r} \times Z \times Z ...... \times Z,$$

where m_i divides m_{i+1} for i = 1,2,....r-1.

PROOF:

It will be convenient to use as notations

$Z/1Z = Z/Z \cong Z_1 = \{0\}$

Let G be a finitely generated abelian group generated by n elements.

Let F be a free abelian group of rank n.

Then we know that F = Z × Z × Z...... × Z for n factors.

Consider the homomorphism $\emptyset : F \to G$ and let K be the kernel of this homomorphism. Also K is a subgroup of F.

Then there is basis for F of the form $\{x_1, x_2, ..., x_n\}$ where $\{d_1x_1, d_2x_2, \dots, d_sx_s\}$ is a basis of K and d_i divides d_{i+1}
for i = 1,…, S-1
By using theorem 3.1,
“ If G is an abelian group generated by n elements. Then G ≅ F/H where F is a free abelian group of rank n and H is some subgroup of F ”.

We have G is isomorphic to F/K.

But,
$F/K \cong (Z \times Z \times Z.....\times Z)/(d_1Z \times d_2Z \times ..... \times d_sZ...... \times \{0\}.... \times \{0\})$
$\cong (Z_{d_1} \times Z_{d_2} \times ..... \times Z_{d_s} \times Z..... \times Z)$

It is possible that $d_1 = 1$, in which case $Zd_1 = \{0\}$ and can be dropped from this product .

similarly d_2 may be 1 and so on.

We let $m_1$be the first $d_i > 1$, m_2 be the next d_i, and so on.

CHAPTER IV

FREE PRODUCTS OF GROUPS

DEFINITION: 4.1

Let G be a group. If $\{G_\alpha\}_{\alpha \in J}$ is a family of subgroups of G, we say that these groups generate G if every element x of G can be written as a finite product of elements of the groups G_α.

This means that there is a finite sequence $(x_1,..., x_n)$ of elements of the groups G_α such that x = $x_1 ... x_n$. Such a sequence is called a **word** in the groups G_α. It is said to **represent** the element x of G.

DEFINITION: 4.2

Let G be a group, let $\{G_\alpha\}_{\alpha \in J}$ be a family of subgroups of G that generates G. Suppose that $G_\alpha \cap G_\beta$ consists of the identity element alone whenever $\alpha \neq \beta$.

We say that G is the **free product** of the groups G_α if for each x ∈ G, there is only one reduced word in the groups G_α that represents x. In this, we write

$$G = \prod_{\alpha \in J}^{*} G_\alpha$$

or in the finite case, $G = G_1 * \cdots * G_n$.

LEMMA: 4.3

Let G be a group. Let $\{G_\alpha\}$ be a family of subgroups of G. If G is the free product of the groups G_α, then G satisfies the following condition.

Given any group H and any family of homomorphisms $h_\alpha : G_\alpha \to H$, there exists a homomorphism $h : G \to H$ whose restriction to G_α equals h_α, for each α. Furthermore, h is unique.

PROOF:

Given $x \in G$ with $x \neq 1$, let $(x_1,..., x_n)$ be the reduced word that represents x.

If h exists, it must satisfy the equation

$$h(x) = h(x_1) \ldots h(x_n) = h_{\alpha_1}(x_1) \ldots h_{\alpha_n}h(x_n) \qquad \cdots(*)$$

where α_i is the index such that $x_i \in G_{\alpha_i}$.

Hence h is unique.

To show h exists, we define it by equation $(*)$ if $x \neq 1$, and we set $h(1) = 1$.

Because the representation of x by a reduced word is unique, h is well defined.

We must show it is a homomorphism.

We first prove a preliminary result.

Given a word $w = (x_1,..., x_n)$ of positive length in the elements of the groups G_α.

Let us define $\emptyset(w)$ to be the element of H given by the equation

$\emptyset$ (w) = $h_{\alpha_1}(x_1) \ldots h_{\alpha_n}(x_n)$

where α_i is any index such that $x_i \in G_{\alpha_i}$.

Now α_i is unique unless $x_i = 1$.

Hence $\emptyset$ is well-defined.

If w is the empty word, let $\emptyset$ (w) equal the identity element of H.

We show that if w' is a word obtained from w by applying one of our reduction operations, $\emptyset$ (w') = $\emptyset$ (w).

Suppose first that w' is obtained by deleting $x_i = 1$ from the word w.

Then the equation $\emptyset$ (w') = $\emptyset$ (w) follows from the fact that $h_{\alpha_i}(x_i) = 1$.

Second, suppose that $\alpha_i = \alpha_{i+1}$ and that

w' = $(x_1,..., x_i x_{i+1},\ldots, x_n)$.

The fact that

$h_\alpha (x_i)\, h_\alpha (x_{i+1}) = h_\alpha (x_i x_{i+1})$

where $\alpha = \alpha_i = \alpha_{i+1}$, implies that $\emptyset$(w) = $\emptyset$(w').

It follows at once that if w is any word in the groups G_α that represents x, then h(x) = $\emptyset$(w).

For by definition of h, this equation holds for any reduced word w, and the process of reduction does not change the value of $\emptyset$.

Now we show that h is a homomorphism.

Suppose that w = $(x_1,..., x_n)$ and w' = $(y_1,..., y_m)$ are words representing x and y respectively.

Let (w, w') denote the word (x_1,..., x_n, y_1,..., y_m), which represents xy.

It follows that

$\emptyset$ (w, w') = $\emptyset$ (w) $\emptyset$ (w').

h(xy) = h(x)h(y).

DEFINITION: 4.4

Let $\{G_\alpha\}_{\alpha\in J}$ be an indexed family of groups. Suppose that G is a group, and that $i_\alpha : G_\alpha \to G$ is a family of monomorphisms, such that G is the free product of the groups $i_\alpha(G_\alpha)$.

Then we say that G is the **external free product** of the groups G_α, relative to the monomorphisms i_α.

THEOREM: 4.5

Let a family $\{ G_\alpha\}_{\alpha\in J}$ of groups, there exists a group G and a family of monomorphisms $i_\alpha : G_\alpha \to G$ such that G is the free product of the groups $i_\alpha(G_\alpha)$.

PROOF:

We assume that the groups G_α are disjoint as sets. Then as before, we define a word in the elements of the groups G_α to be an n-tuple w = (x_1,..., x_n) of elements of $\cup G_\alpha$.

It is called a reduced word if $\alpha_i \neq \alpha_{i+1}$ for all i, where α_i is the index such that $x_i \in G_{\alpha_i}$, and if for each i, x_i is not the identity element of G_{α_i}.

We define the empty set to be the unique reduced word of length zero.

Note that we are not given a group G that contains all the G_α as subgroups, so we cannot speak of a word representing an element of G.

Let W denote the set of all reduced words in the elements of the groups G_α.

Let P(W) denote the set of all bijective functions $\pi : W \to W$.

Then P(W) is itself a group, with composition of functions as the group operation.

We shall obtain our desired group G as a subgroup of P(W).

Step 1:

For each index α and each $x \in G_\alpha$, we define a set map $\pi_x : W \to W$. It will satisfy the following conditions:

(1) If $x = 1_\alpha$, the identity element of G_α, then π_x is the identity map of W.

(2) If $x, y \in G_\alpha$ and $z = x\,y$, then $\pi_z = \pi_x \circ \pi_y$.

We proceed as follows. Let $x \in G_\alpha$. For notational purposes, let $w = (x_1,\ldots, x_n)$ denote the general nonempty element of W, and let α_1 denote the index such that $x_1 \in G_{\alpha_1}$.

If $x \neq 1_\alpha$, define π_x as follows.

(i) $\pi_x(\emptyset) = (x)$,

(ii) $\pi_x(w) = (x,x_1,\ldots, x_n)$ if $\alpha_1 \neq \alpha$,

(iii) $\pi_x(w) = (xx_1,\ldots, x_n)$ if $\alpha_1 = \alpha$ and $x_1 \neq x^{-1}$

(iv) $\pi_x(w) = (x_2,\ldots, x_n)$ if $\alpha_1 = \alpha$ and $x_1 = x^{-1}$

If $x = 1_\alpha$, define π_x to be the identity map of W.

Note that the value of π_x is in each case a reduced word, that is, an element of W.

In cases (i) and (ii), the action of π_x increases the length of the word.

In case (iii) it leaves the length unchanged, and in case (iv) it reduces the length of the word.

When case (iv) applies to a word w of length one, it maps w to the empty word.

Step 2:

We show that if $x, y \in G_\alpha$ and $z = x\, y$, then $\pi_z = \pi_x \circ \pi_y$.

The result is trivial if either x or y equals 1_α, since in that case π_x or π_y is the identity map.

So let us assume henceforth that $x \neq 1_\alpha$ and $y \neq 1_\alpha$.

We compute the values of π_z and of $\pi_x \circ \pi_y$ on the reduced word w.

There are four cases to consider.

(i) Suppose w is the empty word.

We have $\pi_y (\emptyset) = (y)$. If $z =1_\alpha$, then $y = x^{-1}$ and $\pi_x\pi_y (\emptyset) = \emptyset$ by (iv), while $\pi_z (\emptyset)$ equals the same thing because π_z is the identity map. If $z \neq 1_\alpha$, then

$$\pi_x\pi_y (\emptyset) = (x\ y) = (z) = \pi_z (\emptyset).$$

In the remaining cases, we assume $w = (x_1,..., x_n)$, with $x_1 \in G_{\alpha_1}$.

(ii) Suppose $\alpha \neq \alpha_1$.

Then $\pi_y(w) = (y, x_1,..., x_n)$. If $z = 1_\alpha$, then $y = x^{-1}$ and $\pi_x\pi_y(w) = (x_1,..., x_n)$ by (iv), while $\pi_z(w)$ equals the same because π_z is the identity map.

If $z \neq 1_\alpha$, then

$$\begin{aligned}\pi_x\pi_y(w) &= (xy,x_1,..., x_n)\\ &= (z, x_1,..., x_n)\\ &= \pi_z\,(w)\end{aligned}$$

(iii) Suppose $\alpha = \alpha_1$ and $yx_1 \neq 1_\alpha$.

Then $\pi_y(w) = (yx_1, x_2,...,x_n)$. If $xyx_1 = 1_\alpha$, then $\pi_x\pi_y(w) = (x_1,..., x_n)$, while $\pi_z\,(w)$ equals the same thing because $zx_1 = xyx_1 = 1_\alpha$.

If $xyx_1 \neq 1_\alpha$, then

$$\begin{aligned}\pi_x\pi_y(w) &= (xyx_1, x_2,..., x_n)\\ &= (zx_1, x_2,..., x_n)\\ &= \pi_z(w).\end{aligned}$$

(iv) Finally, suppose $\alpha = \alpha_1$ and $yx_1 = 1_\alpha$.

Then $\pi_y\,(w) = (x_2,..., x_n)$, which is empty if $n = 1$.

We compute

$$\begin{aligned}\pi_x\pi_y(w) &= (x, x_2,..., x_n)\\ &= (x(yx_1), x_2,..., x_n)\\ &= (zx_1, x_2,..., x_n)\\ &= \pi_z(w).\end{aligned}$$

Step 3:

The map π_x is an element of p(W), and the map

$i_\alpha : G_\alpha \rightarrow P(W)$ defined by $i_\alpha(x) = \pi_x$ is a monomorphism.

To show that π_x is bijective, we note that if $y = x^{-1}$, then conditions (1) and (2) imply that $\pi_y \circ \pi_x$ and $\pi_x \circ \pi_y$ equal the identity map of W.

Hence π_x belongs to P(W).

The fact that i_α is a homomorphism is a consequence of condition (2).

To show that i_α is a monomorphism.

We note that if $x \neq 1_\alpha$, then $\pi_x (\emptyset) = (x)$.

So that π_x is not the identity map of W.

Step 4:

Let G be the subgroup of P(W) generated by the groups $G_\alpha' = i_\alpha(G_\alpha)$.

We show that G is the free product of the groups G_α'.

First, we show that $G_\alpha' \cap G_\beta'$ consists of the identity alone if $\alpha \neq \beta$.

Let $x \in G_\alpha$ and $y \in G_\beta$. We suppose that neither π_x nor π_y is the identity map of W and show that $\pi_x \neq \pi_y$.

But this is easy, for $\pi_x (\emptyset) = (x)$ and $\pi_y (\emptyset) = (y)$, and these are different words.

Second, we show that no nonempty reduced word

$$w' = (\pi_{x_1}, \ldots, \pi_{x_n})$$

in the groups G_α' represents the identity element of G.

Let α_i be the index such that $x_i \in G_{\alpha_i}$, then $\alpha_i \neq \alpha_{i+1}$ and

$x_i \neq 1_{\alpha_i}$ for each i.

We compute

$$\pi_{x_1} (\pi_{x_2} (\cdots(\pi_{x_n} (\emptyset)))) = (x_1,..., x_n),$$

So the element of G represented by w' is not the identity element of P(W).

LEMMA: 4.6

Let $\{G_\alpha\}_{\alpha\in J}$ be a family of groups. Let G be a group. Let $i_\alpha : G_\alpha \to G$ be a family of homomorphisms. If the extension condition holds, then each i_α is a monomorphism and G is the free product of the groups $i_\alpha (G_\alpha)$.

PROOF:

We first show that each i_α is a monomorphism.

Given an index β, let us set $H = G_\beta$.

Let $h_\alpha : G_\alpha \to H$ be the identity if α = β, and the trivial homomorphism if α ≠ β.

Let $h : G \to H$ be the homomorphism given by the extension condition.

Then $h \circ i_\beta = h_\beta$, so that i_β is injective.

There exists a group G' and a family $i_\alpha' : G_\alpha \to G'$ of monomorphisms such that G' is the free product of the groups $i_\alpha' (G_\alpha)$.

Both G and G' have the extension property.

The preceding theorem then implies that there is an isomorphism $\emptyset : G \to G'$ such that $\emptyset \circ i_\alpha = i_\alpha'$.

It follows at once that G is the free product of the groups $i_\alpha(G_\alpha)$.

COROLLARY: 4.7

Let G = $G_1 * G_2$, where G_1 is the free product of the subgroups $\{H_\alpha\}_{\alpha\in J}$ and G_2 is the free product of the subgroups $\{H_\beta\}_{\beta\in K}$. If the index sets J and K are disjoint, then G is the free product of the subgroups $\{H_\gamma\}_{\gamma\in J\cup K}$.

PROOF:

If $h_\alpha : H_\alpha \rightarrow H$ and $h_\alpha : H_\beta \rightarrow H$ are families of homomorphisms, they extend to homomorphisms $h_1 : G_1 \rightarrow H$ and $h_2 : G_2 \rightarrow H$ by the preceding lemma.

Then h_1 and h_2 extend to a homomorphism.

$h : G \rightarrow H$.

Hence G is the free product of the subgroups $\{H_\gamma\}_{\gamma\in J\cup K}$.

DEFINITION: 4.8

If S is a subset of G, one can consider the intersection N of all normal subgroups of G that contain S.

It is easy to see that N is itself a normal subgroup of G. It is called the **least normal subgroup** of G that contains S.

THEOREM: 4.9

Let G = $G_1 * G_2$. Let N_i be a normal subgroup of G_i , for i = 1, 2. If N is the least normal subgroup of G that contains N_1 and N_2, then G/N $\cong (G_1/N_1) * (G_2/N_2)$.

PROOF:

The composite of the inclusion and projection homomorphisms

$$G_1 \to G_1 * G_2 \to (G_1 * G_2)/N$$

carries N_1 to the identity element, so that it induces a homomorphism

$$i_1 : G_1/N_1 \to (G_1 * G_2)/N.$$

Similarly, the composite of the inclusion and projection homomorphisms induces a homomorphism

$$i_2 : G_2/N_2 \to (G_1 * G_2)/N.$$

We show that the extension condition holds with respect to i_1 and i_2.

It follows that i_1 and i_2 are monomorphisms and that $(G_1 * G_2)/N$ is the external free product of G_1/N_1 and G_2/N_2 relative to these monomorphisms.

So let $h_1 : G_1/N_1 \to H$ and $h_2 : G_2/N_2 \to H$ be arbitrary homomorphisms.

The extension condition for $G_1 * G_2$ implies that there is a homomorphism of $G_1 * G_2$ into H that equals the composite

$$G_i \to G_i /N_i \to H$$

of the projection map and h_i on G_i , for i = 1, 2.

This homomorphism carries the elements of N_1 and N_2 to the identity element, so its kernel contains N.

Therefore it induces a homomorphism $h : (G_1 * G_2)/N \to H$ that satisfies the conditions $h_1 = h \circ i_1$ and $h_2 = h \circ i_2$.

LEMMA: 4.10

Let S be a subset of the group G. If N is the least normal subgroup of G containing S, then N is generated by all conjugates of elements of S.

PROOF:

Let N' be the subgroup of G generated by all conjugates of elements of S.

We know that N' $\subset$ N.

To verify the reverse inclusion, we need merely show that N' is normal in G.

Given x $\in$ N' and c $\in$ G.

We show that $cxc^{-1} \in$ N'.

We can write x in the form $x = x_1 x_2 \ldots x_n$, where each x_i is conjugate to an element s_i of S.

Then cx_ic^{-1} is also conjugate to s_i.

Because,

$$cxc^{-1} = (cx_1c^{-1})(cx_2c^{-1})\ldots(cx_nc^{-1})$$

cxc^{-1} is a product of conjugates of elements of S, so that $cxc^{-1} \in$ N' as desired.

CHAPTER V
FREE GROUPS

DEFINITION: 5.1

Let G be a group. Let $\{a_\alpha\}$ be a family of elements of G, for $\alpha \in J$. We say the elements $\{a_\alpha\}$ **generate** G if every element of G can be written as a product of powers of the elements a_α.

If the family $\{a_\alpha\}$ is finite, we say G is **finitely generated**.

DEFINITION: 5.2

Let $\{a_\alpha\}$ be a family of elements of a group G. Suppose each a_α generates an infinite cyclic subgroup G_α of G.

If G is the free product of the groups $\{G_\alpha\}$, then G is said to be a **free group**, and the family $\{a_\alpha\}$ is called a **system of free generators** for G.

DEFINITION: 5.3

Let $\{a_\alpha\}_{\alpha \in J}$ be an arbitrary indexed family. Let G_α denote the set of all symbols of the form a_α^n for $n \in Z$. We make G_α into a group by defining

$$a_\alpha^n \, . \, a_\alpha^m = a_\alpha^{n+m}$$

Then a_α^0 is the identity element of G_α, and a_α^{-n} is the inverse of a_α^n. We denote a_α^1 simply by a_α.

The external free product of the groups $\{G_\alpha\}$ is called the **free group on the elements** a_α.

DEFINITION: 5.4

Let G be a group. If x, y ∈ G, we denote by [x, y] the element

$$[x, y] = xyx^{-1} y^{-1}$$

of G. It is called the **commutator** of x and y.

The subgroup of G generated by the set of all commutators in G is called the **commutator subgroup** of G and denoted [G, G].

LEMMA: 5.5

Let G be a group, the subgroup [G, G] is a normal subgroup of G and the quotient group G/[G, G] is abelian. If h : G → H is any homomorphism from G to an abelian group H, then the kernel of h contains [G, G], so h induces a homomorphism k : G/[G, G] →H.

PROOF:

Step:1

First we show that any conjugate of a commutator is in [G, G].

We compute as follows.

$$\begin{aligned} g[x, y]\, g^{-1} &= g(xyx^{-1} y^{-1})g^{-1} \\ &= (gxyx^{-1})(1)(y^{-1}g^{-1}) \\ &= (gxyx^{-1})(g^{-1} y^{-1} yg)(y^{-1}g^{-1}) \\ &= ((gx)y(gx)^{-1} y^{-1})(ygy^{-1}g^{-1}) \\ &= [gx, y]\cdot[y, g] \end{aligned}$$

which is in [G, G], as desired.

Step:2

We show that [G, G] is a normal subgroup of G.

Let z be an arbitrary element of [G, G].

We show that any conjugate gzg^{-1} of z is also in [G, G].

The element z is a product of commutators and their inverses. Because ,

$$[x, y]^{-1} = (xyx^{-1} y^{-1})^{-1}$$

$$= [y, x],$$

z actually equals a product of commutators.

Let $z = z_1 \ldots z_n$, where each z_i is a commutator. Then ,

$$gzg^{-1} = (gz_1g^{-1})(gz_2g^{-1}) \cdots (gz_ng^{-1})$$

which is a product of elements of [G, G] by Step 1 and hence belongs to [G, G].

Step:3

We show that G/[G, G] is abelian.

Let G' = [G, G].

We wish to show that

$$(aG')(bG') = (bG')(aG'),$$

that is, abG' = baG' .

This is equivalent to the equation $a^{-1}b^{-1}abG' = G'$ and this equation follows from the fact that $a^{-1}b^{-1}ab = [a^{-1}, b^{-1}]$, which is an element of G' .

Step:4

To complete the proof, we note that because H is abelian, h carries each commutator to the identity element of H.

Hence the kernel of h contains [G, G].

So that h induces the desired homomorphism k .

Hence the proof.

THEOREM: 5.6

If G is a free group with free generators a_α, then G/[G, G] is a free abelian group with basis $[a_\alpha]$, where $[a_\alpha]$ denotes the coset of a_α in G/[G, G].

PROOF:

We apply Lemma

Let G be an abelian group. Let $\{a_\alpha\}$ be a family of elements of G that generates G. Then G is a free abelian group with basis $\{a_\alpha\}$ if and only if for any abelian group H and any family $\{y_\alpha\}$ of elements of H, there is a homomorphism h of G into H such that $h(a_\alpha) = y_\alpha$ for each α.

Given any family $\{y_\alpha\}$ of elements of the abelian group H.

There exists a homomorphism $h : G \to H$ such that $h(a_\alpha) = y_\alpha$ for each α.

Because H is abelian, the kernel of h contains [G, G].

Therefore h induces a homomorphism $k : G/[G, G] \to H$ that carries $[a_\alpha]$ to y_α.

CONCLUSION

The concept of basis is important in the study of real vector spaces in linear algebra, it is equally useful to consider abelian groups which possess a basis. An abelian group that possess a basis is called as free abelian group. Here we focussed on free abelian group with finite basis and discussed properties about it. Then we move onto finitely generated abelian groups and study some theorems on free abelian groups and finitely generated abelian groups. The next chapter came up with a concept called as free product of groups. In the last chapter we discussed about free group.

APPLICATIONS

Group concept can be applied to explore the symmetry of various games, which can lead to optimal strategies and decision making. Symmetries in network graphs can be studied using group theory methods, leading to better understanding and decision making regarding network structures.

This concept has application in physics, chemistry and computer science and even puzzles like Rubik’s cube can be represented using group theory. Additionally, facilitates problem solving and supplies essential tools for other branches of mathematics.

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