MODULE 2

MATRIX ARITHMETIC

OPERATIONS ON MATRICES

EIGENVALUES & EIGENVECTORS

MODULE 2

MATRIX ARITHMETIC

MATRIX OPERATIONS

- Addition and subtraction
- Scalar multiplication
- Multiplication
- Inverse

ADDITION AND SCALAR MULTIPLICATION

- The sum of two $m \times n$ matrices A and B is the $m \times n$ matrix $A + B$ in which each element is the sum of the corresponding elements of A and B.

- The product of a scalar k and a matrix A is the matrix kA, in which each element is k times the corresponding element of A.

EXAMPLE 1

Suppose $A = \begin{bmatrix} 1 & -8 & 3 \\ 9 & 11 & 2 \\ 5 & -4 & 0 \end{bmatrix}, B = \begin{bmatrix} -3 & 0 & 2 \\ 12 & 7 & -1 \\ 6 & 8 & -5 \end{bmatrix}, C = \begin{bmatrix} 41 & -6 & 11 \\ -9 & -14 & 2 \\ 0 & 10 & 4 \end{bmatrix}$. Find $2A + B - C$

$$2A + B - C = (2)\begin{bmatrix} 1 & -8 & 3 \\ 9 & 11 & 2 \\ 5 & -4 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 2 \\ 12 & 7 & -1 \\ 6 & 8 & -5 \end{bmatrix} - \begin{bmatrix} 41 & -6 & 11 \\ -9 & -14 & 2 \\ 0 & 10 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1) + (-3) - 41 & 2(-8) + 0 - (-6) & (2)3 + 2 - 11 \\ 2(9) + 12 - (-9) & 2(11) + 7 - (-14) & 2(2) + (-1) - 2 \\ 2(5) + 6 - 0 & 2(-4) + 8 - 10 & 2(0) + (-5) - 4 \end{bmatrix}$$

$$= \begin{bmatrix} -42 & -10 & -3 \\ 39 & 43 & 1 \\ 16 & -10 & -9 \end{bmatrix}$$

```python
import numpy as np

A = np.array([[1,-8,3],[9,11,2],[5,-4,0]])
B = np.array([[-3,0,2],[12,7,-1],[6,8,-5]])
C = np.array([[41,-6,11],[-9,-14,2],[0,10,4]])

result = 2*A + B - C
print(result)
```

```
[[-42 -10  -3]
 [ 39  43   1]
 [ 16 -10  -9]]
```

MATRIX MULTIPLICATION

- The product AB of two matrices A and B can be found only if the number of columns of A equals the number of rows of B.
- If A is an $m \times n$ matrix and B is an $n \times k$ matrix, the product AB will be an $m \times k$ matrix
- If A is an $m \times n$ matrix and B is an $n \times m$ matrix, the products AB and BA can both be found. *However*, $AB \neq BA$ in general.

EXAMPLE 2

Let $A = \begin{bmatrix} 1 & -3 \\ 7 & 2 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \end{bmatrix}$. Find AB and BA.

$$AB = \begin{bmatrix} 1 & -3 \\ 7 & 2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \end{bmatrix}$$

Step 1: Multiply each element of row 1 from A with each element of column 1 from B and add these together.

$$1(1) + (-3)(3) = -8$$

Step 2: Multiply each element of row 1 from A with each element of column 2 from B and add these together.

$$1(0) + (-3)(1) = -3$$

Step 3: Multiply each element of row 1 from A with each element of column 3 from B and add these together.

$$1(-1) + (-3)(4) = -13$$

Step 4: Multiply each element of row 2 from A with each element of column 1 from B and add these together.

$$7(1) + 2(3) = 13$$

Continue this algorithm for the remaining 5 quantities.

EXAMPLE 2 CONTINUED

$$AB = \begin{bmatrix} 1 & -3 \\ 7 & 2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1) + (-3)(3) & 1(0) + (-3)(1) & 1(-1) + (-3)(4) \\ 7(1) + 2(3) & 7(0) + 2(1) & 7(-1) + 2(4) \\ -2(1) + 5(3) & -2(0) + 5(1) & -2(-1) + 5(4) \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -3 & -13 \\ 13 & 2 & 1 \\ 13 & 5 & 22 \end{bmatrix}$$

EXAMPLE 2 CONTINUED

$$AB = \begin{bmatrix} -8 & -3 & -13 \\ 13 & 2 & 1 \\ 13 & 5 & 22 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 7 & 2 \\ -2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1(1) + 0(7) + (-1)(-2) & 1(-3) + 0(2) + (-1)(5) \\ 3(1) + 1(7) + 4(-2) & 3(-3) + 1(2) + 4(5) \end{bmatrix}$$

$$\begin{bmatrix} 3 & -8 \\ 2 & 13 \end{bmatrix}$$

$$AB \neq BA$$

PYTHON

```
import numpy as np

A = np.array([[1,-3],[7,2],[-2,5]])
B = np.array([[1,0,-1],[3,1,4]])

np.dot(A,B)
```

```
array([[ -8,  -3, -13],
       [ 13,   2,   1],
       [ 13,   5,  22]])
```

```
import numpy as np

A = np.array([[1,-3],[7,2],[-2,5]])
B = np.array([[1,0,-1],[3,1,4]])

np.dot(B,A)
```

```
array([[ 3, -8],
       [ 2, 13]])
```

MATRIX FORMULATION

Sam's Shoes and Frank's Footwear both have outlets in California and Arizona. Sam's sells shoes for $80, sandals for $40, and boots for $120. Frank's prices are $60, $30, and $150, respectively. Half of all sales in California stores are shoes, 1/4 are sandals, and1/4 are boots. In Arizona, the fractions are 1/5 shoes, 1/5 sandals, and 3/5 boots.

▨ Given Information:
We have:2 stores: Sam's Shoes and Frank's Footwear
2 states: California and Arizona
3 types of footwear: Shoes, Sandals, Boots

Prices:
•Sam's: Shoes = $80, Sandals = $40, Boots = $120
•Frank's: Shoes = $60, Sandals = $30, Boots = $150

Sales Distribution of Footwear (as Fractions)

State	Shoes	Sandals	Boots
California	1/2	1/4	1/4
Arizona	1/5	1/5	3/5

MATRIX FORMULATION

☑ (a) Create a 2×3 matrix P for prices
Let's order the items as:
Shoes, Sandals, Boots

Let rows represent:
Row 1 → Sam's
Row 2 → Frank's
So matrix P is:

$$P=\begin{array}{cccc} \text{Shoes,} & \text{Sandals,} & \text{Boots} & \\ \left[\begin{array}{c} 80 \\ 60 \end{array}\right. & \begin{array}{c} 40 \\ 30 \end{array} & \left.\begin{array}{c} 120 \\ 150 \end{array}\right] & \begin{array}{l} \text{Sam} \\ \text{Frank} \end{array} \end{array}$$

☑ (b) Create a 3×2 matrix F for fractions of sales
Let rows represent:
Row 1 → Shoes
Row 2 → Sandals
Row 3 → Boots
Let columns represent:
Column 1 → California
Column 2 → Arizona

$$F = \begin{array}{ccc} \text{California} & \text{Arizona} & \\ \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{array}\right. & \left.\begin{array}{c} \frac{1}{5} \\ \frac{1}{5} \\ \frac{3}{5} \end{array}\right] & \begin{array}{l} \text{Shoes} \\ \text{Sandals} \\ \text{Boots} \end{array} \end{array}$$

Sam's Shoes and Frank's Footwear both have outlets in California and Arizona. Sam's sells shoes for $80, sandals for $40, and boots for $120. Frank's prices are $60, $30, and $150, respectively. Half of all sales in California stores are shoes, 1/4 are sandals, and1/4 are boots. In Arizona, the fractions are 1/5 shoes, 1/5 sandals, and 3/5 boots.

(a) Write a 2 x 3 matrix called P representing prices for the two stores and the three types of footwear.
(b) Write a 3 x 2 matrix called F representing the fraction of each type of footwear sold in each state.
(c) Calculate the product PF and describe what the entries represent.
(d) From the answer to part (c), what is the average price for a pair of footwear at an outlet of Frank's in Arizona?

$$P = \begin{array}{c} \begin{array}{ccc} \text{Sh} & \text{Sa} & \text{B} \end{array} \\ \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Sam's} \\ \text{Frank's} \end{array}$$

$$F = \begin{array}{c} \begin{array}{cc} \text{CA} & \text{AR} \end{array} \\ \begin{bmatrix} 1/2 & 1/5 \\ 1/4 & 1/5 \\ 1/4 & 3/5 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Sh} \\ \text{Sa} \\ \text{B} \end{array}$$

$$PF = \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \begin{bmatrix} 1/2 & 1/5 \\ 1/4 & 1/5 \\ 1/4 & 3/5 \end{bmatrix}$$

$$= \begin{bmatrix} 80\left(\frac{1}{2}\right) + 40\left(\frac{1}{4}\right) + 120\left(\frac{1}{4}\right) & 80\left(\frac{1}{5}\right) + 40\left(\frac{1}{5}\right) + 120\left(\frac{3}{5}\right) \\ 60\left(\frac{1}{2}\right) + 30\left(\frac{1}{4}\right) + 150\left(\frac{1}{4}\right) & 60\left(\frac{1}{5}\right) + 30\left(\frac{1}{5}\right) + 150\left(\frac{3}{5}\right) \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cc} \text{CA} & \text{AR} \end{array} \\ \begin{bmatrix} 80 & 96 \\ 75 & 108 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Sam's} \\ \text{Frank's} \end{array}$$

The rows give the average price per pair of footwear sold by each store and the columns give the state.

The entry in the 2nd row, 2nd column represents the average price of footwear at an outlet of Frank's in Arizona: $108

PYTHON

```
import numpy as np

A = np.array([[80,40,120],[60,30,150]])
B = np.array([[1/2,1/5],[1/4,1/5],[1/4,3/5]])
np.dot(A,B)
```

```
array([[ 80.,  96.],
       [ 75., 108.]])
```

VECTOR SOLUTIONS

Consider the matrix equation $A\vec{x} = \vec{b}$, where A is a known matrix and $\vec{b}$ is a known vector. We want to find the vector $\vec{x}$ that satisfies this equation.

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 3 & -1 \\ 1 & 3 & 2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} -6 \\ 10 \\ 5 \end{bmatrix}, \quad \text{and} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In terms of the matrix equation $A\vec{x} = \vec{b}$, we have

$$\begin{bmatrix} 1 & -1 & 5 \\ 3 & 3 & -1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 10 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 - x_2 + 5x_3 \\ 3x_1 + 3x_2 - x_3 \\ x_1 + 3x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 10 \\ 5 \end{bmatrix}$$

This translates to the linear system

$$\begin{aligned} x_1 - x_2 + 5x_3 &= -6 \\ 3x_1 + 3x_2 - x_3 &= 10 \\ x_1 + 3x_2 + 2x_3 &= 5 \end{aligned}$$

VECTOR SOLUTIONS CONTINUED

$$x_1 - x_2 + 5x_3 = -6$$
$$3x_1 + 3x_2 - x_3 = 10$$
$$x_1 + 3x_2 + 2x_3 = 5$$

We can use the augmented matrix and perform row operations to find the row-reduced echelon form of this and get

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

This tells us $x_1 = 1$, $x_2 = 2$, and $x_3 = -1$ which translates to

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

As always, check the solution in the original system.

PYTHON

```
import sympy as sym

sym.init_printing()

x1,x2,x3 = sym.symbols('x1,x2,x3')

solns = sym.solve([
    x1 - x2 + 5*x3 + 6,
    3*x1 + 3*x2 - x3 - 10,
    x1 + 3*x2 + 2*x3 - 5],
    [x1, x2, x3])

x1_ans = round(solns[x1],0)
x2_ans = round(solns[x2],0)
x3_ans = round(solns[x3],0)

print(f"The solution is x1 =",x1_ans, ", x2 =",x2_ans, ", and x3 =", x3_ans)
```

```
The solution is x1 = 1 , x2 = 2 , and x3 = -1
```

MULTIPLICATIVE INVERSES

2 x 2 Identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If A^{-1} exists, then $AA^{-1} = A^{-1}A = I$

Find A^{-1} if $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$

Form the augmented matrix $[A|I]$:

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right]$$

Perform row operations to transform A to I:

$$-2R_1 + R_2 \to R_2 \Longrightarrow \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

$$R_2 + 2R_1 \to R_1 \Longrightarrow \left[\begin{array}{cc|cc} 2 & 0 & 0 & 1 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{2}R_1 \to R_1 \Longrightarrow \\ \frac{1}{2}R_2 \to R_2 \Longrightarrow \end{array} \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & \frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

Your turn: Check $AA^{-1} = A^{-1}A = I$

ALTERNATIVE METHOD WITH 2X2

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Find A^{-1} for $A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}$, if it exists

$$2(5) - 1(-3) = 13$$

$$A^{-1} = \frac{1}{13}\begin{bmatrix} 5 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix}$$

Check:

$$AA^{-1} = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix} = \begin{bmatrix} 2\left(\frac{5}{13}\right) + (-3)\left(-\frac{1}{13}\right) & 2\left(\frac{3}{13}\right) + (-3)\left(\frac{2}{13}\right) \\ 1\left(\frac{5}{13}\right) + 5\left(-\frac{1}{13}\right) & 1\left(\frac{3}{13}\right) + 5\left(\frac{2}{13}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Your turn: Verify $A^{-1}A = I$

INVERSE DOES NOT EXIST

Find A^{-1} for $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$, if it exists

$$ad - bc = 2(-2) - (-4)(1) = -4 + 4 = 0$$

Inverse does not exist

Form the augmented matrix $[A|I]$

$$\left[\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right]$$

Perform row operations to transform A to I:

$$-R_1 + 2R_2 \to R_2 \quad \Longrightarrow \quad \left[\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

No way to complete the process. A^{-1} does not exist.

EXAMPLE 3X3

Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Form the augmented matrix $[A|I]$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1 + R_2 \to R_2 \Rightarrow \\ -3R_1 + R_3 \to R_3 \Rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right]$$

Column 2 has zeros in required positions

$$\begin{array}{r} R_3 + 3R_1 \to R_1 \Rightarrow \\ -R_3 + R_2 \to R_2 \Rightarrow \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{3}R_1 \to R_1 \Rightarrow \\ -\frac{1}{2}R_2 \to R_2 \Rightarrow \\ -\frac{1}{3}R_3 \to R_3 \Rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

<u>Your turn</u>:

Verify $AA^{-1} = A^{-1}A = I$

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

PYTHON

```
import numpy as np
from scipy import linalg
a = np.array([[1,0,1],[2,-2,-1],[3,0,0]])
linalg.inv(a)
```

```
array([[-0.        ,  0.        ,  0.33333333],
       [-0.5       , -0.5       ,  0.5       ],
       [ 1.        ,  0.        , -0.33333333]])
```

USING INVERSE TO SOLVE SYSTEM

- Suppose we represent a system of linear equations by $A\vec{x} = \vec{b}$
- A is an $n \times n$ matrix of coefficients and A^{-1} exists
- $\vec{x}$ is an $n \times 1$ column vector of variables
- $\vec{b}$ is an $n \times 1$ column vector of constants
- Then $\vec{x} = A^{-1}\vec{b}$ is the solution to the system
- If A is invertible, then the equation $A\vec{x} = \vec{b}$ has exactly one solution, namely $\vec{x} = A^{-1}\vec{b}$.

Three brands of fertilizer are available to provide nitrogen, phosphoric acid, and potash. One bag of each brand provides the units of each nutrient shown in the table.

For ideal growth, the soil on a Michigan farm needs 18 units of nitrogen, 23 units of phosphoric acid, and 13 units of potash per acre. How many bags of each brand of fertilizer should be used per acre for ideal growth on the farm?

		Brand		
		Veg Health	Grow Big	Nutriplant
Nutrient	Nitrogen	1	2	3
	Phosphoric Acid	3	1	2
	Potash	2	0	1

Let x = bags fertilizer from Veg Health, y = bags of fertilizer from Grow Big, and z = bags of fertilizer from Nutriplant

$$\begin{aligned} x + 2y + 3z &= 18 \\ 3x + y + 2z &= 23 \\ 2x + z &= 13 \end{aligned} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 18 \\ 23 \\ 13 \end{bmatrix}$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1 + R_2 \rightarrow R_2 \\ -2R_1 + R_3 \rightarrow R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & -4 & -5 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} 2R_2 + 5R_1 \rightarrow R_1 \\ \\ -4R_2 + 5R_3 \rightarrow R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 5 & 0 & 1 & -1 & 2 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & 0 & 3 & 2 & -4 & 5 \end{array}\right]$$

$$\begin{array}{r} -R_3 + 3R_1 \rightarrow R_1 \\ 7R_3 + 3R_2 \rightarrow R_2 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 15 & 0 & 0 & -5 & 10 & -5 \\ 0 & -15 & 0 & 5 & -25 & 35 \\ 0 & 0 & 3 & 2 & -4 & 5 \end{array}\right]$$

$$\begin{array}{r} \frac{1}{15}R_1 \rightarrow R_1 \\ -\frac{1}{15}R_2 \rightarrow R_2 \\ \frac{1}{3}R_3 \rightarrow R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{5}{3} & -\frac{7}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \end{array}\right]$$

$$X = A^{-1}B = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{3} & -\frac{7}{3} \\ \frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 18 \\ 23 \\ 13 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$$

Use 5 bags of Veg Health, 2 bags of Grow Big, and 3 bags of Nutriplant

PYTHON

```
import numpy as np
from scipy import linalg
a = np.array([[1,2,3],[3,1,2],[2,0,1]])
b = np.array([18,23,13])

np.dot(linalg.inv(a),b)
```

```
array([5., 2., 3.])
```

QUESTIONS?

MODULE 2

OPERATIONS ON MATRICES

MATRIX TRANSPOSE

The transpose of a matrix switches rows and columns, i.e. an $m \times n$ matrix becomes an $n \times m$ matrix.

$$A = \begin{bmatrix} 2 & -1 & 7 \\ -3 & 0 & -4 \end{bmatrix} \quad \text{becomes} \quad A^T = \begin{bmatrix} 2 & -3 \\ -1 & 0 \\ 7 & -4 \end{bmatrix}$$

<u>Properties:</u>

Let A and B be matrices where the following operations are defined. Then:

- $(A+B)^{\mathrm{T}} = \mathrm{A}^{\mathrm{T}} + \mathrm{B}^{\mathrm{T}}$ and $(A-B)^{\mathrm{T}} = \mathrm{A}^{\mathrm{T}} - \mathrm{B}^{\mathrm{T}}$
- $(kA)^T = kA^T$
- $(AB)^T = B^T A^T$
- $(A^{-1})^T = (A^T)^{-1}$
- $(A^T)^T = A$

PYTHON

```
import numpy as np

arr1 = np.array([[2,-1,7],[-3,0,-4]])
print(f"Original matrix \n",arr1,"\n")

arr1_transpose = arr1.transpose()
print(f"Transposed matrix \n", arr1_transpose)
```

```
Original matrix
 [[ 2 -1  7]
 [-3  0 -4]]

Transposed matrix
 [[ 2 -3]
 [-1  0]
 [ 7 -4]]
```

MATRIX TRACE

The trace of an $n \times n$ matrix is the sum of the elements along the diagonal. This only works with square matrices.

$$A = \begin{bmatrix} 6 & 0 & -4 \\ 1 & 2 & 5 \\ 10 & -1 & 3 \end{bmatrix} \quad \text{and} \quad tr(A) = 6 + 2 + 3 = 11$$

<u>Properties:</u>

Let A and B be $n \times n$ matrices. Then:

- $tr(A+B) = tr(A) + tr(B)$ and $tr(A-B) = tr(A) - tr(B)$
- $tr(kA) = k \cdot tr(A)$
- $tr(AB) = tr(BA)$
- $tr(A^T) = tr(A)$

PYTHON

```
import numpy as np

arr2 = np.array([[6,0,-4],[1,2,5],[10,-1,3]])
arr2_trace = arr2.trace()

print(f"The trace is", arr2_trace)
```

```
The trace is 11
```

DETERMINANT: 2X2

Determinant of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$:

$$\mathrm{d}et(A) = |A| = a_{11}a_{22} - a_{12}a_{21}$$

Calculate the determinant of $\mathrm{A} = \begin{bmatrix} -5 & 3 \\ 7 & -1 \end{bmatrix}$

$$|A| = (-5)(-1) - (3)(7) = 5 - 21 = -16$$

Note: This is a square matrix with determinant ≠ 0, so the inverse exists

<u>Your turn</u>: Find A^{-1}. Use $A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

DETERMINANT: 3X3

Determinant of $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$:

Find $|A|$ for $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 7 & -3 \\ 3 & 8 & -5 \end{bmatrix}$

$$\det(A) = |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Augment A with the first two columns of A

$$\left|\begin{array}{ccc|cc} 1 & 3 & -2 & 1 & 3 \\ 2 & 7 & -3 & 2 & 7 \\ 3 & 8 & -5 & 3 & 8 \end{array}\right|$$

$$|A| = (1)(7)(-5) + (3)(-3)(3) + (-2)(2)(8) - (-2)(7)(3) - (1)(-3)(8) - (3)(2)(-5)$$

$$= -35 - 27 - 32 + 42 + 24 + 30 = 2$$

PYTHON

```python
import numpy as np

A = np.array([[1,3,-2],[2,7,-3],[3,8,-5]])
D = np.linalg.det(A)
print(D)
```

```
2.0000000000000004
```

MATRIX MINOR AND COFACTOR

Let A be an $n \times n$ matrix. The i, j minor of A, denoted $A_{i,j}$, is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the i^{th} row and the j^{th} column of A.

The i, j cofactor of A is given by

$$C_{i,j} = (-1)^{i+j} A_{i,j}$$

Find the 2,3 cofactor of A:

$$A = \begin{bmatrix} -5 & 1 & 0 & 0 \\ -3 & -5 & 2 & 5 \\ -2 & 4 & -3 & 4 \\ 5 & 4 & -3 & 3 \end{bmatrix}$$

$$A_{2,3} = \begin{vmatrix} -5 & 1 & 0 \\ -2 & 4 & 4 \\ 5 & 4 & 3 \end{vmatrix} = 46$$

```python
import numpy as np

A_23_matrix = np.array([[-5,1,0],[-2,4,4],[5,4,3]])

A_23_det = round(np.linalg.det(A_23_matrix),0)

print(A_23_det)
```

```
46.0
```

$$C_{2,3} = (-1)^{2+3} \begin{vmatrix} -5 & 1 & 0 \\ -2 & 4 & 4 \\ 5 & 4 & 3 \end{vmatrix} = -46$$

COFACTOR EXPANSION

The cofactor expansion is an alternative way to calculate the determinant of an $n \times n$ matrix.

Let A be an $n \times n$ matrix.

The cofactor expansion of A along the i^{th} row is given by

$$\det(A) = a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + \cdots + a_{i,n}C_{i,n}$$

The cofactor expansion of A along the j^{th} column is given by

$$\det(A) = a_{1,j}C_{1,j} + a_{2,j}C_{2,j} + \cdots + a_{n,j}C_{n,j}$$

COFACTOR EXPANSION

Using the cofactor expansion along the 2nd row, find the determinant of the following matrix:

$$A = \begin{bmatrix} -5 & 1 & 0 & 0 \\ -3 & -5 & 2 & 5 \\ -2 & 4 & -3 & 4 \\ 5 & 4 & -3 & 3 \end{bmatrix}$$

$$\det(A) = -3C_{2,1} + (-5)C_{2,2} + 2C_{2,3} + 5C_{2,4}$$

$$= -3(-1)^{2+1}A_{2,1} + (-5)(-1)^{2+2}A_{2,2} + 2(-1)^{2+3}A_{2,3} + 5(-1)^{2+4}A_{2,4}$$

$$= 3A_{2,1} - 5A_{2,2} - 2A_{2,3} + 5A_{2,4}$$

$$= 3\begin{vmatrix} 1 & 0 & 0 \\ 4 & -3 & 4 \\ 4 & -3 & 3 \end{vmatrix} - 5\begin{vmatrix} -5 & 0 & 0 \\ -2 & -3 & 4 \\ 5 & -3 & 3 \end{vmatrix} - 2\begin{vmatrix} -5 & 0 & 0 \\ -2 & 4 & 4 \\ 5 & 4 & 3 \end{vmatrix} + 5\begin{vmatrix} -5 & 1 & 0 \\ -2 & 4 & -3 \\ 5 & 4 & -3 \end{vmatrix}$$

$$= 3(3) - 3(-15) - 2(46) + 5(-21) = -113$$

PYTHON

```python
import numpy as np

A_21_matrix = np.array([[1,0,0],[4,-3,4],[4,-3,3]])

A_21_det = round(np.linalg.det(A_21_matrix),0)

print(A_21_det)
```

```
3.0
```

```python
import numpy as np

A_23_matrix = np.array([[-5,1,0],[-2,4,4],[5,4,3]])

A_23_det = round(np.linalg.det(A_23_matrix),0)

print(A_23_det)
```

```
46.0
```

```python
import numpy as np

A_22_matrix = np.array([[-5,0,0],[-2,-3,4],[5,-3,3]])

A_22_det = round(np.linalg.det(A_22_matrix),0)

print(A_22_det)
```

```
-15.0
```

```python
import numpy as np

A_24_matrix = np.array([[-5,1,0],[-2,4,-3],[5,4,-3]])

A_24_det = round(np.linalg.det(A_24_matrix),0)

print(A_24_det)
```

```
-21.0
```

```python
import numpy as np

A = np.array([[-5,1,0,0],[-3,-5,2,5],[-2,4,-3,4],[5,4,-3,3]])
d = round(np.linalg.det(A))
print(d)
```

```
-113
```

CRAMER'S RULE: 2X2

Consider the system of equations:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Let D be the determinant of the coefficient matrix and assume $D \neq 0$

Define

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and } D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The solution to the system is given by

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}$$

$$2x + 5y = 15$$
$$x + 4y = 9$$

$$D = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = (2)(4) - (5)(1) = 3$$

$$D_x = \begin{vmatrix} 15 & 5 \\ 9 & 4 \end{vmatrix} = (15)(4) - (5)(9) = 15$$

$$D_y = \begin{vmatrix} 2 & 15 \\ 1 & 9 \end{vmatrix} = (2)(9) - (15)(1) = 3$$

$$x = \frac{D_x}{D} = \frac{15}{3} = 5 \qquad y = \frac{D_y}{D} = \frac{3}{3} = 1$$

Your turn: Check the solution

CRAMER'S RULE: 3X3

Consider the system of linear equations

$$
\begin{aligned}
a_1x + b_1y + c_1z &= d_1 \\
a_2x + b_2y + c_2z &= d_2 \\
a_3x + b_3y + c_3z &= d_3
\end{aligned}
$$

Let D be the determinant of the coefficient matrix and assume $D \neq 0$

Define

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\ D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

The solution to the system is given by

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$$

EXAMPLE

Recall the fertilizer example

$$x + 2y + 3z = 18$$
$$3x + y + 2z = 23$$
$$2x + z = 13$$

The solution is $(5, 2, 3)$

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -3$$

$$D_x = \begin{vmatrix} 18 & 2 & 3 \\ 23 & 1 & 2 \\ 13 & 0 & 1 \end{vmatrix} = -15 \qquad x = \frac{-15}{-3} = 5$$

$$D_y = \begin{vmatrix} 1 & 18 & 3 \\ 3 & 23 & 2 \\ 2 & 13 & 1 \end{vmatrix} = -6 \qquad y = \frac{-6}{-3} = 2$$

$$D_z = \begin{vmatrix} 1 & 2 & 18 \\ 3 & 1 & 23 \\ 2 & 0 & 13 \end{vmatrix} = -9 \qquad z = \frac{-9}{-3} = 3$$

PYTHON

```python
import numpy as np

D = np.array([[1,2,3],[3,1,2],[2,0,1]])
Dx = np.array([[18,2,3],[23,1,2],[13,0,1]])
Dy = np.array([[1,18,3],[3,23,2],[2,13,1]])
Dz = np.array([[1,2,18],[3,1,23],[2,0,13]])

print('D = ', np.linalg.det(D))
print('D_x = ', np.linalg.det(Dx))
print('D_y = ', np.linalg.det(Dy))
print('D_z = ', np.linalg.det(Dz))
```

```
D =  -3.0000000000000001
D_x =  -15.0
D_y =  -5.9999999999999997
D_z =  -9.000000000000005
```

QUESTIONS?

MODULE 2

EIGENVALUES & EIGENVECTORS

MOTIVATION

Consider multiplying nonzero vectors by a given square matrix, such as

$$\begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 33 \\ 27 \end{bmatrix} \text{ or } \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}$$

In the 1st case we have a new vector with a different direction and a different length, which is uninteresting.

In the 2nd case the multiplication produces a multiple of the original vector. This means the new vector has the same direction as the original vector.

$$\begin{bmatrix} 30 \\ 40 \end{bmatrix} = 10\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

In this example we say $\lambda = 10$ is an eigenvalue and the vector $\vec{x} = [3 \quad 4]^T$ is an eigenvector.

Eigenvalues represent the scaling factor by which a vector is transformed when a linear transformation is applied.

Some applications in machine learning include feature extraction, dimensionality reduction, and clustering.

EIGENVALUES AND EIGENVECTORS

Let A be an $n \times n$ matrix, $\vec{x}$ a nonzero $n \times 1$ column vector and λ a scalar. If

$$A\vec{x} = \lambda\vec{x},$$

then $\vec{x}$ is an eigenvector of A and λ is an eigenvalue of A.

Let's take a closer look and try to solve the equation for $\vec{x}$:

$$A\vec{x} = \lambda\vec{x}$$
$$A\vec{x} - \lambda\vec{x} = \vec{0}$$
$$(A - \lambda I)\vec{x} = \vec{0}$$

<u>Recall:</u> If $A - \lambda I$ is invertible, then there is exactly one solution, namely $\vec{x} = \vec{0}$.

Thus, to have nonzero solutions we need $A - \lambda I$ to not be invertible.

Taking this further, noninvertible matrices all have a determinant of 0.

So, if we want to find eigenvalues and eigenvectors we need $\det(A - \lambda I) = 0$.

EXAMPLE

$$A = \begin{bmatrix} 3 & 12 \\ 1 & -1 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of A.

$$A - \lambda I = \begin{bmatrix} 3 & 12 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 12 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$
$$= \begin{bmatrix} 3-\lambda & 12 \\ 1 & -1-\lambda \end{bmatrix}$$

So

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 12 \\ 1 & -1-\lambda \end{vmatrix}$$
$$(3-\lambda)(-1-\lambda) - 12$$
$$\lambda^2 - 2\lambda - 15$$

We want $\det(A - \lambda I) = 0$, or $\lambda^2 - 2\lambda - 15 = 0$.

$$\lambda^2 - 2\lambda - 15 = 0$$
$$(\lambda + 3)(\lambda - 5) = 0$$

So the eigenvalues are $\lambda = -3$ and $\lambda = 5$.

EXAMPLE CONTINUED

For $\lambda = -3$ we need to solve $(A - (-3)I)\vec{x} = \vec{0}$.

$$\begin{bmatrix} 3+3 & 12 \\ 1 & -1+3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 12 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If we put this in row-reduced echelon form we have

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $x_1 = -2x_2$ and x_2 is free.

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We can pick any nonzero value for x_2 so let $x_2 = 1$.

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Alternatively, we can convert this to a linear system:

$$\begin{aligned} 6x_1 + 12x_2 &= 0 \\ x_1 + 2x_2 &= 0 \end{aligned}$$

Or equivalently:

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_1 + 2x_2 &= 0 \end{aligned}$$

Then:

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

EXAMPLE CONTINUED

For $\lambda = 5$ we need to solve $(A - 5I)\vec{x} = \vec{0}$.

$$\begin{bmatrix} 3-5 & 12 \\ 1 & -1-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 12 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If we put this in row-reduced echelon form we have

$$\begin{bmatrix} 1 & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $x_1 = 6x_2$ and x_2 is free.

$$\vec{x} = x_2 \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

We can pick any nonzero value for x_2 so let $x_2 = 1$.

$$\vec{x} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

Converting this to a linear system:

$$-2x_1 + 12x_2 = 0$$
$$x_1 - 6x_2 = 0$$

Or equivalently:

$$x_1 - 6x_2 = 0$$
$$x_1 - 6x_2 = 0$$

Then:

$$x_1 + 2x_2 = 0$$
$$0 = 0$$

PYTHON

```python
import numpy as np

A = np.array([[3,12],[1,-1]])
w = np.linalg.eigvals(A)

print('eigenvalues are:',w)
```

```
eigenvalues are: [ 5. -3.]
```

PROPERTIES OF EIGENVALUES AND EIGENVECTORS

Let A be an $n \times n$ invertible matrix. Then the following are true:

- If A is triangular, then the diagonal elements of A are the eigenvalues of A.
- If λ is an eigenvalue of A with eigenvector $\vec{x}$, then $1/\lambda$ is an eigenvalue of A^{-1} with eigenvector $\vec{x}$.
- If λ is an eigenvalue of A, then λ is an eigenvalue of A^T.
- The sum of the eigenvalues of A is equal to $tr(A)$.
- The product of the eigenvalues is equal to the determinant of $\det(A)$.

QUESTIONS?