MODULE 7

FUNCTIONS OF SEVERAL VARIABLES

PARTIAL DERIVATIVES

MAXIMA AND MINIMA

LAGRANGE MULTIPLIERS

TANGENT PLANES AND DIFFERENTIALS

MODULE 7

FUNCTIONS OF SEVERAL VARIABLES

FUNCTIONS OF TWO VARIABLES

$z = f(x, y)$ is a function of two variables if a unique value of z is associated with each ordered pair of real numbers (x, y) in some set of ordered pairs of real numbers. This set is called the domain of f.

More generally, $z = f(x_1, x_2, \dots x_n)$ is a function of n variables for any positive integer n if a unique value of z is associated with each n-tuple of real numbers $(x_1, x_2, \dots, x_n)$ in a set of such n-tuples of real numbers. Again, this set is called the domain of f.

Suppose a small company only makes two products, smartphones and tablets. The profits are given by

$$P(x, y) = 40x^2 - 10xy + 5y^2 - 80$$

where x is the number of smartphones sold and y is the number of tablets sold.

EVALUATING FUNCTIONS

Let $f(x, y) = 4x^2 + 2xy + 3/y$ and find the following:

$f(-1,3)$

Replace x with -1 and y with 3

$$f(-1,3) = 4(-1)^2 + 2(-1)(3) + 3/3$$

$$= 4 - 6 + 1 = -1$$

$f(2,0)$

Replace x with 2, however f is not defined when $y = 0$ because of $3/y$

$$\frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{f(x+h,y) - f(x,y)}{h} = \frac{4(x+h)^2 + 2(x+h)y + 3/y - [4x^2 + 2xy + 3/y]}{h}$$

$$= \frac{4x^2 + 8xh + 4h^2 + 2xy + 2hy + 3/y - 4x^2 - 2xy - 3/y}{h}$$

$$= \frac{8xh + 4h^2 + 2hy}{h}$$

$$= 8x + 4h + 2y$$

LIMIT LAWS

Let $f(x,y)$ and $g(x,y)$ be defined for all $(x,y) \neq (a,b)$ in a neighborhood around (a,b), and assume the neighborhood is contained entirely in the domain of f. Assume that L and M are real numbers such that

$$\lim_{(x,y)\to(a,b)} f(x) = L \quad \text{and} \quad \lim_{(x,y)\to(a,b)} g(x) = M$$

and let c be a constant. Then each of the following holds:

$$\lim_{(x,y)\to(a,b)} c = c$$

Constant law

$$\lim_{(x,y)\to(a,b)} x = a \quad \text{and} \quad \lim_{(x,y)\to(a,b)} y = b$$

Identity laws

$$\lim_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = L \pm M$$

Sum/difference law

$$\lim_{(x,y)\to(a,b)} cf(x,y) = cL$$

Constant multiple law

$$\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = LM$$

Product law

$$\lim_{(x,y)\to(a,b)} [f(x,y)]^n = L^n$$

for any positive integer n

Power law

$$\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$$

for all L if n is odd and positive, and for $L \geq 0$ if n is even and positive

Root law

EXAMPLES

Find the following limits, if they exist

$$\lim_{(x,y)\to(5,1)} \frac{xy}{x+y}$$

This function is not defined along the line $y = -x$, but is defined everywhere else

$$\lim_{(x,y)\to(5,1)} \frac{5(1)}{5+1} = \frac{5}{6}$$

$$\lim_{(x,y)\to(1,1)} \frac{2x^2 - xy - y^2}{x^2 - y^2}$$

The point $(1,1)$ will cause division by 0

Use factoring to simplify:

$$\frac{2x^2 - xy - y^2}{x^2 - y^2} = \frac{(2x+y)(x-y)}{(x+y)(x-y)}$$

$$= \frac{2x+y}{x+y}$$

$$\lim_{(x,y)\to(1,1)} \frac{2x^2 - xy - y^2}{x^2 - y^2} = \lim_{(x,y)\to(1,1)} \frac{2x+y}{x+1} = \frac{3}{2}$$

TECHNIQUES

Find the following limits, if they exist:

$$\lim_{(x,y)\to(0,0)} \frac{x-4y}{6y+7x}$$

$(0,0)$ is not in the domain and factoring can't be done.

Start by proceeding along the path of the x-axis, i.e. $y=0$

$$\lim_{(x,y)\to(0,0)} \frac{x-4y}{6y+7x} = \lim_{(x,0)\to(0,0)} \frac{x}{7x}$$

$$= \lim_{(x,0)\to(0,0)} \frac{1}{7} = \frac{1}{7}$$

Now proceed along the path of the y-axis, i.e. $x=0$

$$\lim_{(x,y)\to(0,0)} \frac{x-4y}{6y+7x} = \lim_{(0,y)\to(0,0)} \frac{-4y}{6y}$$

$$= \lim_{(0,y)\to(0,0)} -\frac{2}{3} = -\frac{2}{3}$$

Two different paths to $(0,0)$ produce two different limits, i.e. the limit DNE

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}$$

$(0,0)$ is not in the domain and factoring can't be done.

Proceed along the path of the x-axis, i.e. $y=0$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \lim_{(x,0)\to(0,0)} \frac{0}{x^4} = 0$$

Proceed along the path of the y-axis, i.e. $x=0$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \lim_{(0,y)\to(0,0)} \frac{0}{3y^4} = 0$$

Caution: Two paths with the same limit does not mean the limit exists.

Proceed along a 3rd path, the line $y=x$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4} = \lim_{(x,y)\to(x,x)} \frac{x^4}{4x^4} = \frac{1}{4}$$

This limit is different than the first two, so the limit DNE

MODULE 7

PARTIAL DERIVATIVES

PARTIAL DERIVATIVES

Suppose a small company only makes two products, smartphones and tablets. The profits are given by

$$P(x, y) = 40x^2 - 10xy + 5y^2 - 80$$

where x is the number of smartphones sold and y is the number of tablets sold.

How will a change in x or y affect P?

Suppose that sales of smartphones have been steady at 10 units and only the sales of tablets vary. How would we determine marginal profit with respect to y?

Create a new function $f(y) = P(10, y)$:

$$f(y) = P(10, y) = 40(10)^2 - 10(10)y + 5y^2 - 80 = 3920 - 100y + 5y^2$$

This shows the profit from the sale of y tablets, assuming x is fixed at 10

Find df/dy to get the marginal profit with respect to y

$$\frac{df}{dy} = -100 + 10y$$

PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of two independent variables. Let all indicated limits exist. Then the partial derivative of f with respect to x is

$$f_x(x, y) = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

and the partial derivative of f with respect to y is

$$f_y(x, y) = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Let $f(x, y) = -4xy + 6y^3 + 5$. Find $f_x(x, y)$ and $f_y(x, y)$

Using the formal definition of partial derivatives is not necessary

To find $f_x(x, y)$, treat y like a constant:

$$f_x(x, y) = -4y$$

To find $f_y(x, y)$, treat x like a constant:

$$f_y(x, y) = -4x + 18y^2$$

Your turn: Let $f(x, y) = 2x^2y^3 + 6x^5y^4$. Find $f_x(x, y)$ and $f_y(x, y)$.

Answer: $f_x(x, y) = 4xy^3 + 30x^4y^4$ and $f_y(x, y) = 6x^2y^2 + 24x^5y^3$

EXAMPLES

Let $f(x,y) = 4e^{3x+2y}$. Find $f_x(x,y)$ and $f_y(x,y)$

$$f_x(x,y) = 4e^{3x+2y}(3) = 12e^{3x+2y}$$

Notice the chain rule applies here

$$f_y(x,y) = 8e^{3x+2y}$$

Let $f(x,y) = (7x^2 + 18xy^2 + y^3)^{1/3}$. Find $f_x(x,y)$ and $f_y(x,y)$

Apply the chain rule and the power rule

$$f_x(x,y) = \frac{1}{3}(7x^2 + 18xy^2 + y^3)^{-2/3}(14x + 18y^2)$$

$$f_y(x,y) = \frac{1}{3}(7x^2 + 18xy^2 + y^3)^{-2/3}(36xy + 3y^2)$$

Your turn: Let $f(x,y) = \sqrt{x^4 + 3xy + y^4 + 10}$. Find $f_x(x,y)$ and $f_y(x,y)$. Then find $f_x(2,-1)$ and $f_y(-4,3)$.

Answer:

$$f_x(x,y) = \frac{4x^3 + 3y}{2(x^4 + 3xy + y^4 + 10)^{1/2}}$$

$$f_y(x,y) = \frac{3x + 4y^3}{2(x^4 + 3xy + y^4 + 10)^{1/2}}$$

$$f_x(2,-1) = \frac{29}{2\sqrt{21}}$$

$$f_y(-4,3) = \frac{48}{\sqrt{311}}$$

2ND ORDER PARTIAL DERIVATIVES

For a function $z = f(x, y)$, if the indicated partial derivative exists, then

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

Clairaut's Theorem: The mixed partials f_{xy} and f_{yx} are equal when f is defined on an open disk D containing the point (a, b) and f_{xy} and f_{yx} are continuous on D.

EXAMPLE

Find all second order partial derivatives for $f(x,y) = -4x^3 - 3x^2y^3 + 2y^2$

First find f_x and f_y

$$f_x(x,y) = -12x^2 - 6xy^3$$

$$f_y(x,y) = -9x^2y^2 + 4y$$

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = -24x - 6y^3$$

$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = -18x^2y + 4$$

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial y \partial x} = -18xy^2$$

$$f_{yx}(x,y) = \frac{\partial^2 f}{\partial x \partial y} = -18xy^2$$

<u>Your turn</u>: Let $f(x,y) = 2e^x - 8x^3y^2$. Find all second order partial derivatives.

<u>Answer</u>:

$$f_{xx}(x,y) = 2e^x - 48xy^2$$
$$f_{yy}(x,y) = -16x^3$$
$$f_{xy}(x,y) = f_{yx}(x,y) = -48x^2y$$

MODULE 7

MAXIMA AND MINIMA

RELATIVE MAXIMA AND MINIMA

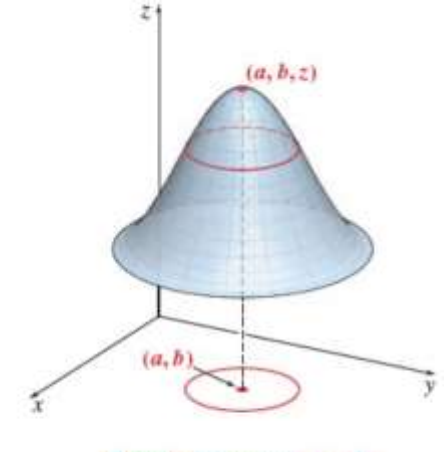
Let (a, b) be the center of a circular region contained in the xy-plane. Then for a function $z = f(x, y)$ defined for every (x, y) in the region, $f(a, b)$ is a relative maximum if

$$f(a, b) \geq f(x, y)$$

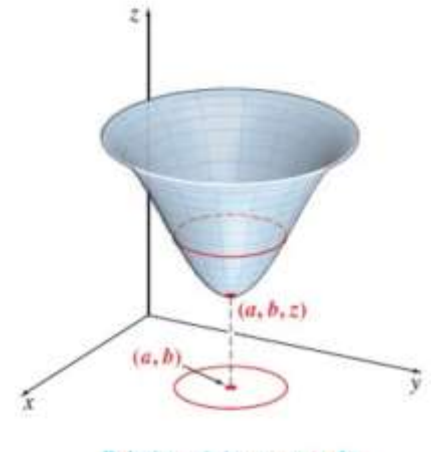
for all points (x, y) in the circular region, and $f(a, b)$ is a relative minimum if

$$f(a, b) \leq f(x, y)$$

for all points (x, y) in the circular region.



Relative maximum at (a, b)



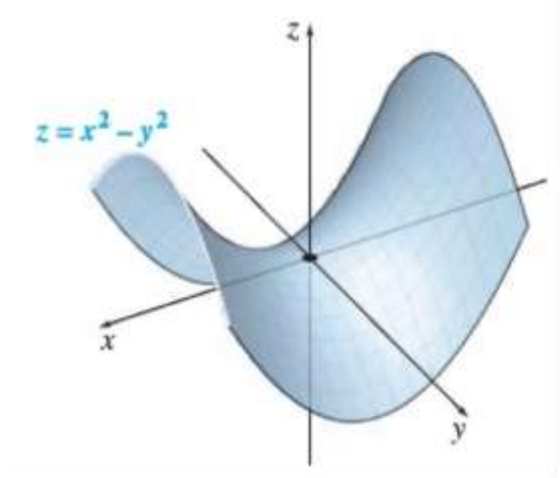
Relative minimum at (a, b)

LOCATION OF EXTREMA

Let a function $z = f(x, y)$ have a relative maximum or relative minimum at the point (a, b). Let $f_x(a, b)$ and $f_y(a, b)$ both exist. Then $f_x(a, b) = 0$ and $f_y(a, b) = 0$

If $f_x(a, b) = 0$ and $f_y(a, b) = 0$, this does not guarantee a relative minimum or relative maximum at (a, b)

Consider the graph of $z = f(x, y) = x^2 - y^2$



$f_x(0,0) = 0$ and $f_y(0,0) = 0$ but $(0,0)$ is neither a relative maximum nor a relative minimum

The point $(0,0,0)$ on the graph is called a saddle point; a minimum when approached from one direction, but a maximum when approached from another direction.

CRITICAL POINTS

Find all critical points for $f(x, y) = 6x^2 + 6y^2 + 6xy + 36x - 5$

$$f_x(x, y) = 12x + 6y + 36$$

$$f_y(x, y) = 12y + 6x$$

Set each equal to 0 and solve

$$12x + 6y + 36 = 0$$
$$12y + 6x = 0$$

$12y + 6x = 0 \rightarrow x = -2y$

$$12(-2y) + 6y + 36 = 0$$
$$-18y = -36$$
$$y = 2$$

Since $y = 2$, we have $x = -4$

$(-4,2)$ is the only critical point.

This only guarantees that ***if f has a relative extremum***, it will occur at this point.

TEST FOR RELATIVE EXTREMA

For a function $z = f(x, y)$, let f_{xx}, f_{yy}, and f_{xy} all exist in a circular region contained in the xy-plane with center (a, b). Further, let

$f_x(a, b) = 0$ and $f_y(a, b) = 0$

Define the number D, known as the discriminant by

$$D = f_{xx}(a, b) \cdot f_{yy}(a, b) - \left[f_{xy}(a, b)\right]^2$$

$f(a, b)$ is a relative maximum if $D > 0$ and $f_{xx}(a, b) < 0$

$f(a, b)$ is a relative minimum if $D > 0$ and $f_{xx}(a, b) > 0$

$f(a, b)$ is a saddle point if $D < 0$

If $D = 0$, the test gives no information

Recall in the previous example, $f(x, y) = 6x^2 + 6y^2 + 6xy + 36x - 5$ and $(-4,2)$ is the only critical point. Is this a relative maximum, a relative minimum, or neither?

$f_x(x, y) = 12x + 6y + 36$ and $f_y(x, y) = 12y + 6x$

We also know $f_x(-4,2) = 0$ and $f_y(-4,2) = 0$

$f_{xx}(x, y) = 12$, $f_{yy}(x, y) = 12$, and $f_{xy}(x, y) = 6$

$$D = 12(12) - 6^2 = 108$$

$$f_{xx}(-4,2) = 12$$

f has a relative minimum at $(-4,2)$ and $f(-4,2) = -77$

EXAMPLE

Find all points where the function $f(x, y) = 9xy - x^3 - y^3 - 6$ has any relative maxima or relative minima. Identify any saddle points.

$f_x(x, y) = 9y - 3x^2$ and $f_y(x, y) = 9x - 3y^2$

$$f_x(x, y) = 0$$
$$9y - 3x^2 = 0$$
$$9y = 3x^2$$
$$3y = x^2$$

$$f_y(x, y) = 0$$
$$9x - 3y^2 = 0$$
$$9x = 3y^2$$
$$3x = y^2$$

$$y = \frac{x^2}{3}$$

$$3x = \left(\frac{x^2}{3}\right)^2 = \frac{x^4}{9}$$

$$27x = x^4$$
$$x^4 - 27x = 0$$
$$x(x^3 - 27) = 0$$
$x = 0$ or $x = 3$

If $x = 0$, then $y = 0$ and if $x = 3$, then $y = 3$

$f_{xx}(x, y) = -6x$, $f_{yy}(x, y) = -6y$, and $f_{xy}(x, y) = 9$

$$f_{xx}(0,0) = 0$$
$$f_{yy}(0,0) = 0$$
$$f_{xy}(0,0) = 9$$

$$D = 0 \cdot 0 - 9^2 = -81$$

Since $D < 0$, there is a saddle point at $(0,0)$

$$f_{xx}(3,3) = -18$$
$$f_{yy}(3,3) = -18$$
$$f_{xy}(3,3) = 9$$

$$D = (-18)(-18) - 9^2 = 243$$

Since $D > 0$ and $f_{xx}(3,3) < 0$, there is a relative maximum at $(3,3)$

YOUR TURN

A company is developing a new energy drink. The cost in dollars to produce a batch of the drink is approximated by

$$C(x, y) = 2200 + 27x^3 - 72xy + 8y^2$$

where x is the number of kilograms of sugar per batch and y is the number of grams of flavoring per batch. Fund the amounts of sugar and flavoring that result in the minimum cost. What is the minimum cost?

Answer:

The minimum occurs by using 4 kg of sugar and 18 g of flavoring for a minimum cost of $\$1336$

Note: There is a saddle point at $(0,0)$

MODULE 7

LAGRANGE MULTIPLIERS

MOTIVATION & DEFINITION

Suppose a builder wants to know the dimensions in a new building that will maximize floor space while keeping costs fixed at $\$500{,}000$. The costs are given by

$$C(x, y) = xy + 20y + 20x + 474000$$

where x is the width and y is the length.

So, the builder would like to maximize the area $A(x, y) = xy$ subject to $xy + 20y + 20x + 474000 = 500000$

Problems like this with constraints are often solved using the method of **Lagrange multipliers**, i.e.:

Find the relative extrema for $z = f(x, y)$
subject to $g(x, y) = 0$

All relative extrema of the function $z = f(x, y)$, subject to a constraint $g(x, y) = 0$ will be found among those points (x, y) for which there exists a value of λ such that

$$F_x(x, y, \lambda) = 0, \qquad F_y(x, y, \lambda) = 0, \qquad F_\lambda(x, y, \lambda) = 0$$

where

$$F(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y)$$

and all indicated derivatives exist

EXAMPLE

Find the minimum value of $f(x, y) = 5x^2 + 6y^2 - xy$ subject to the constraint $x + 2y = 24$

Step 1

Rewrite the constraint in the form $g(x, y) = 0$

$x + 2y - 24 = 0$, so $g(x, y) = x + 2y - 24$

Step 2

Form the Lagrange function $F(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y)$

$$\begin{aligned} F(x, y, \lambda) &= 5x^2 + 6y^2 - xy - \lambda(x + 2y - 24) \\ &= 5x^2 + 6y^2 - xy - \lambda x - 2\lambda y + 24\lambda \end{aligned}$$

Step 3

Find $F_x(x, y, \lambda), F_y(x, y, \lambda)$, and $F_\lambda(x, y, \lambda)$

$$\begin{aligned} F_x(x, y, \lambda) &= 10x - y - \lambda \\ F_y(x, y, \lambda) &= 12y - x - 2\lambda \\ F_\lambda(x, y, \lambda) &= -x - 2y + 24 \end{aligned}$$

Step 4

Form the system of equations $F_x(x, y, \lambda) = 0, F_y(x, y, \lambda) = 0$, and $F_\lambda(x, y, \lambda) = 0$

$$\begin{aligned} 10x - y - \lambda &= 0 \\ 12y - x - 2\lambda &= 0 \\ -x - 2y + 24 &= 0 \end{aligned}$$

Step 5

Solve the system from Step 4

$$10x - y - \lambda = 0 \rightarrow \lambda = 10x - y$$

$$12y - x - 2\lambda = 0 \rightarrow \lambda = \frac{-x + 12y}{2}$$

Your turn: Verify intermediate steps →

$$10x - y = \frac{-x + 12y}{2}$$

$$x = \frac{2y}{3}$$

Using the 3rd equation

$$-\frac{2y}{3} - 2y + 24 = 0 \rightarrow y = 9$$

So, $x = 6$

YOUR TURN

The 2nd derivative test for relative extrema demonstrated earlier does not apply to solutions found by Lagrange multipliers

1. Convince yourself that $f(6,9) = 612$ is a minimum by trying a point very close to $(6,9)$. Your calculation should be larger than 612.

2. Solve the example used in the motivation for Lagrange multipliers, i.e. Maximize the area, $A(x,y) = xy$ subject to the cost constraint

$$xy + 20y + 20x + 474{,}000 = 500{,}000$$

then convince yourself the solution is a maximum using the method above.

You should get $x \approx 142.45$ and $y \approx 142.5$ for a maximum area of $\approx 20{,}306$ ft^2

MODULE 7

TANGENT PLANES AND DIFFERENTIALS

TANGENT PLANES

Let S be a surface defined by a differentiable function $z = f(x, y)$, and let $P = (x_0, y_0)$ be a point in the domain of f. Then the equation of the tangent plane to S at P is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Find the equation of the tangent plane to the surface defined by the function $f(x, y) = x^3 - x^2y + y^2 - 2x + 3y - 2$ at the point $(-1,3)$

$$f_x(x, y) = 3x^2 - 2xy - 2$$
$$f_y(x, y) = -x^2 + 2y + 3$$

$$f(-1,3) = (-1)^3 - (-1)^2(3) + 3^2 - 2(-1) + 3(3) - 2 = 18$$

$$f_x(-1,3) = 3(-1)^2 - 2(-1)(3) - 2 = 7$$

$$f_y(-1,3) = -(-1)^2 + 2(3) + 3 = 8$$

$$\begin{aligned} z &= 18 + 7\big(x - (-1)\big) + 8(y - 3) \\ &= 7x + 8y - 3 \end{aligned}$$

TOTAL DIFFERENTIALS

Let $z = f(x, y)$ be a function of x and y. Let dx and dy be real numbers. Then the total differential of z is

$$dz = f_x(x, y) \cdot dx + f_y(x, y) \cdot dy$$

Consider the function $z = f(x, y) = 9x^3 - 8x^2y + 4y^3$.

(a) Find dz
(b) Evaluate dz when $x = 1, y = 3, dx = 0.01$, and $dy = -0.02$

(a)

$f_x(x, y) = 27x^2 - 16xy$ and $f_y(x, y) = -8x^2 + 12y^2$

By definition

$$dz = (27x^2 - 16xy)dx + (-8x^2 + 12y^2)dy$$

(b)

$$dz = [27(1^2) - 16(1)(3)](0.01) + [-8(1^2) + 12(3^2)](-0.02) = -2.21$$

APPROXIMATIONS

Recall that with a function of one variable, $y = f(x)$, the differential dy approximates the change in y corresponding to a change in x. The change in y is given by $\Delta y = f(x + dx) - f(x)$ and the change in x is given by Δx.

The approximation of the differential dz for a function of two variables and for small values of dx and dy is given by $dz \approx \Delta z$, where $\Delta z = f(x + dx, y + dy) - f(x, y)$

Approximate $\sqrt{2.98^2 + 4.01^2}$

Notice that $2.98 \approx 3$ and $4.01 \approx 4$, and we know that $\sqrt{3^2 + 4^2} = 5$

Let $f(x, y) = \sqrt{x^2 + y^2}, x = 3, dx = -0.02, y = 4$, and $dy = 0.01$

Use dz to approximate $\Delta z = \sqrt{2.98^2 + 4.01^2} - \sqrt{3^2 + 4^2}$

$$dz = f_x(x, y)dx + f_y(x, y)dy$$

$$dz = \left(\frac{2x}{2\sqrt{x^2 + y^2}}\right)dx + \left(\frac{2y}{2\sqrt{x^2 + y^2}}\right)dy$$

$$= \left(\frac{x}{\sqrt{x^2 + y^2}}\right)dx + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)dy$$

$$= \frac{3}{5}(-0.02) + \frac{4}{5}(0.01) = -0.004$$

$$\sqrt{2.98^2 + 4.01^2} \approx 5 + (-0.004) = 4.996$$

<u>Your turn</u>: Approximate $\sqrt{5.03^2 + 11.99^2}$

<u>Answer</u>: 13.0023

APPROXIMATIONS BY DIFFERENTIALS

For a function f having all indicated partial derivatives, and for small values of dx and dy,

$$f(x + dx, y + dy) \approx f(x, y) + dz$$

or

$$f(x + dx, y + dy) \approx f(x, y) + f_x(x, y)dx + f_y(x, y)dy$$

The volume of a right circular cylinder is given by $V = \pi r^2 h$

To approximate the change in volume, find the total differential

$$dV = (2\pi rh)dr + (\pi r^2)dh$$

Using $r = 1.5$ and $h = 5$, we have

$$dV = (2\pi)(1.5)(5)dr + \pi(1.5)^2 dh = \pi(15dr + 2.25dh)$$

The factor of 15 in front of dr compared with the factor of 2.25 in front of dh indicates that a small change in radius has almost 7 times the effect on the volume as a small change in height.

The shape of a can of beer is a right circular cylinder where $r \approx 1.5$ in. and $h \approx 5$ in. How sensitive is the volume of the can to changes in the radius compared to changes in the height?

<u>Your turn</u>: A piece of bone in the shape of a right circular cylinder is 7 cm long and has a radius of 1.4 cm. It is coated with a layer of preservative 0.09 cm thick. Use total differentials to estimate the volume of the preservative used.

<u>Answer</u>: 6.65 cm^3

QUESTIONS?