MODULE 5

MAXIMA AND MINIMA

CURVE SKETCHING

L'HOPITAL'S RULE

NEWTON'S METHOD

MODULE 5

MAXIMA AND MINIMA

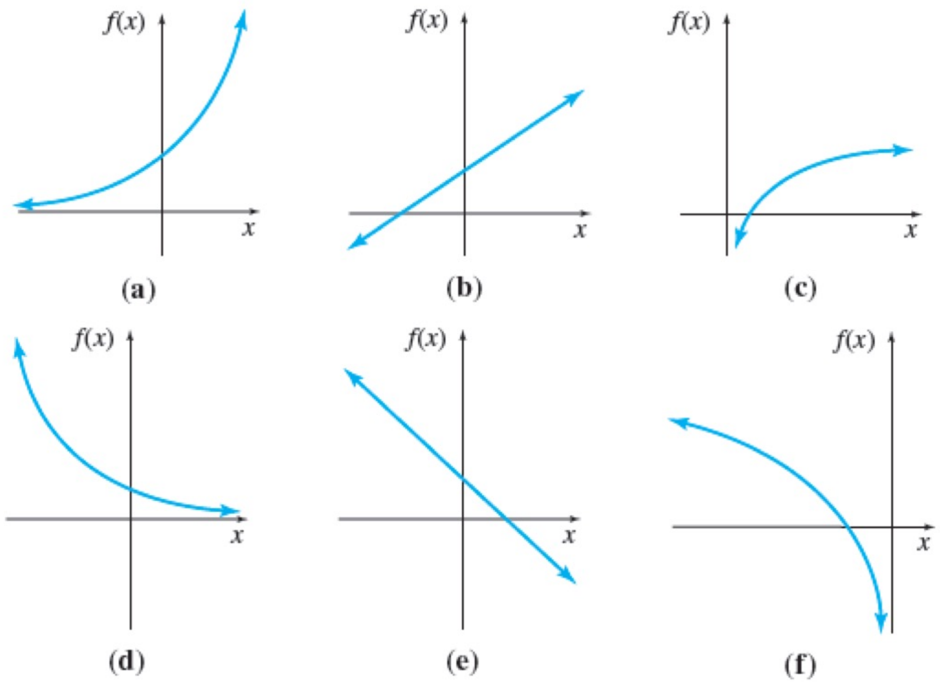
INCREASING AND DECREASING

Let f be a function defined on some interval. Then for any two numbers x_1 and x_2 in the interval, f is increasing on the interval if

$f(x_1) < f(x_2)$ whenever $x_1 < x_2$,

and f is decreasing on the interval if

$f(x_1) > f(x_2)$ whenever $x_1 < x_2$

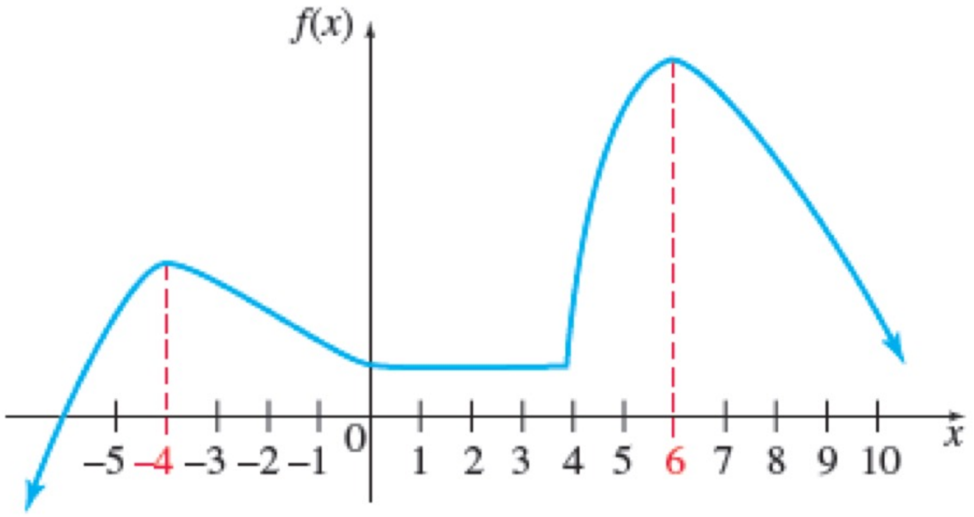


Graphs a, b, and c are increasing functions

Graphs d, e, and f are decreasing functions

INCREASING AND DECREASING

Where is this function increasing? Where is it decreasing?



f is increasing on $(-\infty, -4)$

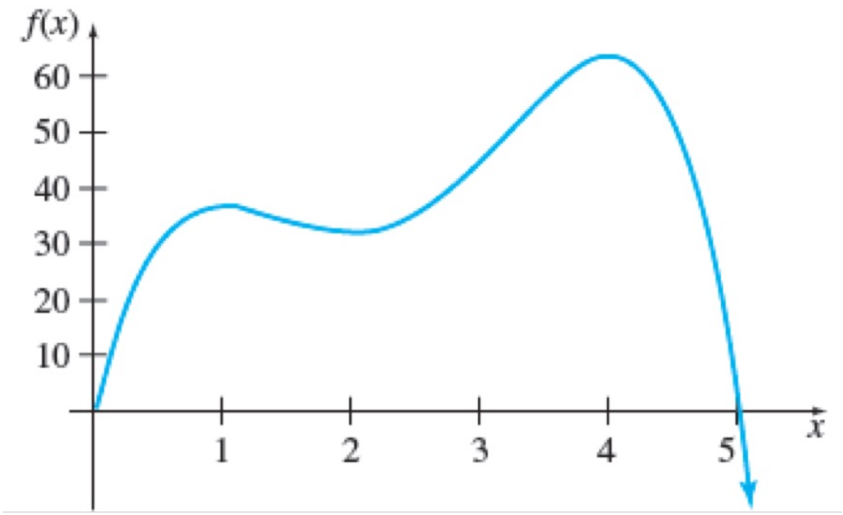
f is decreasing on $(-4,0)$

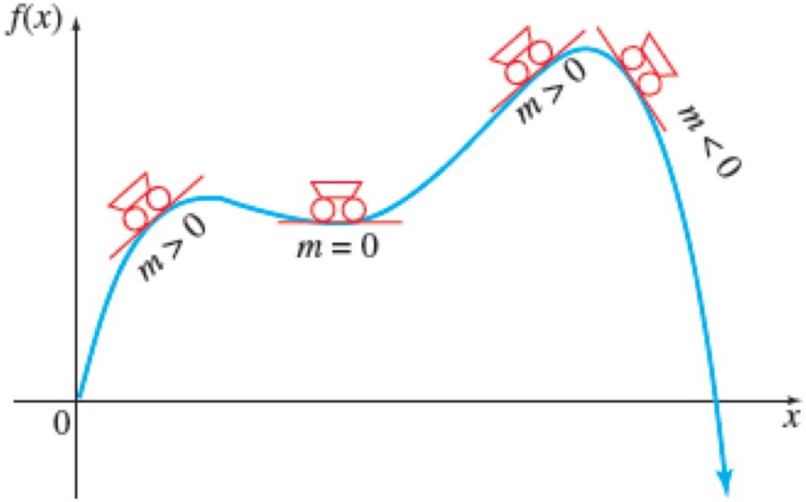
f is increasing on $(4,6)$

f is decreasing on $(6, \infty)$

f is neither increasing nor decreasing on $(0,4)$

IN TERMS OF TANGENT LINES





The slope of the tangent line will be positive when the cart moves uphill and f is increasing

The slope of the tangent line will be negative when the cart moves downhill and f is decreasing

The slope of the tangent line will be 0 at "peaks" and "valleys"

INTERVALS WHERE f IS INCREASING AND DECREASING

Suppose a function f has a derivative at each point in an open interval.

1. If $f'(x) > 0$ for each x in the interval, then f is increasing on the interval
2. If $f'(x) < 0$ for each x in the interval, then f is decreasing on the interval
3. If $f'(x) = 0$ for each x in the interval, then f is constant on the interval*

*Note: The 3rd condition must be true for each x in the interval, not a single point

The slope of the tangent line tells us whether the function is increasing, decreasing, or zero.

$m > 0 \implies$ increasing
$m < 0 \implies$ decreasing
$m = 0 \implies$ constant*

CRITICAL NUMBERS

The derivative $f'(x)$ can only change signs from positive to negative (or negative to positive) at points where $f'(x) = 0$ and at points where $f'(x)$ does not exist

The ***critical numbers*** for a function f are those numbers c in the domain of f for which $f'(c) = 0$ or $f'(c)$ does not exist.

A ***critical point*** is a point whose x-coordinate is the critical number c and whose y-coordinate is $f(c)$

Find the intervals where the following function is increasing or decreasing and locate all points where the tangent line is horizontal. Graph the function.

$$f(x) = x^3 + 3x^2 - 9x + 4$$

Step 1: Find all critical numbers

$$f'(x) = 3x^2 + 6x - 9$$

$$3x^2 + 6x - 9 = 0$$

$$3(x^2 + 2x - 3) = 0$$

$$3(x + 3)(x - 1) = 0$$

$$x = -3 \quad \text{or} \quad x = 1$$

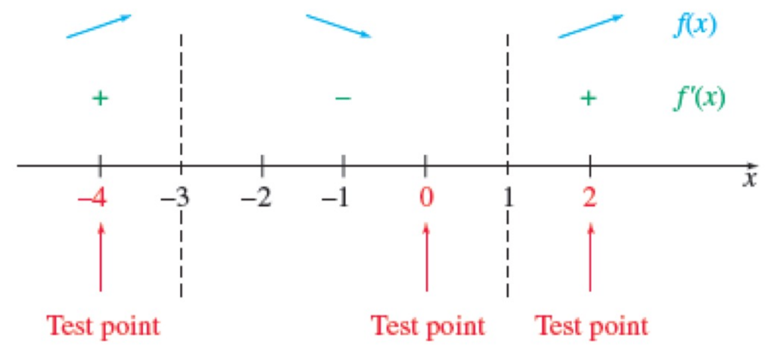
The tangent line is horizontal at $(-3, 31)$ and $(1, -1)$

These x-values determine three intervals to test the sign of the derivative:

$(-\infty, -3)$, $(-3, 1)$, and $(1, \infty)$

Step 2: Create a number line indicating critical numbers and choose test points

Note: The sign of f' can only change signs at critical numbers, so we only need to test one point on each interval



f is increasing on $(-\infty, -3)$ and $(1, \infty)$

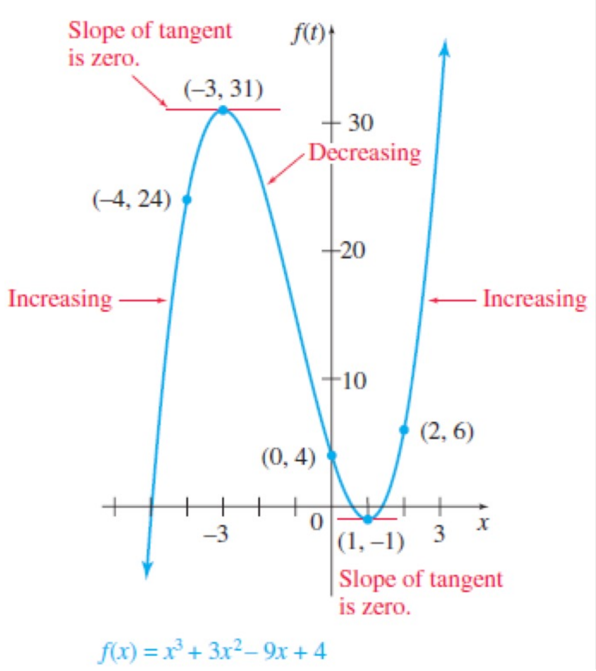
f is decreasing on $(-3, 1)$

Step 3: Graph the function

Plot critical points $(-3, 31)$ and $(1, -1)$

Also plot points for the test values: $(-4, 24)$, $(0, 4)$, and $(2, 6)$

Use the information about where the function is increasing and decreasing



EXAMPLE

The percent of concentration of a drug in the bloodstream t hours after the drug is administered is given by

$$K(t) = \frac{4t}{3t^2 + 27}$$

for $t \geq 0$. On what time intervals is the concentration of the drug increasing? On what intervals is it decreasing?

$$K'(t) = \frac{4(3t^2 + 27) - 4t(6t)}{(3t^2 + 27)^2}$$

$$= \frac{12t^2 + 108 - 24t^2}{(3t^2 + 27)^2}$$

$$= \frac{108 - 12t^2}{(3t^2 + 27)^2}$$

Denominator is never 0. Set the numerator equal to 0 and solve.

$$108 - 12t^2 = 0$$
$$12(9 - t^2) = 0$$

$$12(3 + t)(3 - t) = 0$$

$t = \pm 3$ but $t \geq 0$, so $t = 3$

Choose a test point in $(0, 3)$ and $(3, \infty)$

If $t = 2$, then $K'(t) > 0$ so K is increasing on $(0, 3)$

If $t = 4$, then $K'(t) < 0$ so K is decreasing on $(3, \infty)$

The concentration of the drug is increasing for the first 3 hours then decreases after that

MOTIVATION FOR RELATIVE EXTREMA

Suppose that the manufacturer of a product would like to find the best time within a 30-second commercial to present the name of the product. After extensive experimentation, the research group has determined the function to represent the percent of full attention that a viewer devotes to the commercial is given by

$$f(t) = -0.15t^2 + 6t + 20, \qquad 0 \le t \le 30$$

where f represents the percent of viewer's attention and t is time in seconds

The goal is to find the time at which the viewer's attention is at a maximum

Consider $f'(t) = -0.3t + 6$

$f'(t) > 0$ when

$$-0.3t + 6 > 0$$
$$-0.3t > -6$$
$$t < 20$$

Positive derivative for $t < 20$ implies f is increasing for the first 20 seconds

$f'(t) < 0$ when

$$-0.3t + 6 < 0$$
$$t > 20$$

Negative derivative for $t > 20$ implies f is decreasing after 20 seconds

This suggests the product name should be announced at 20 seconds into the commercial. At that time, the viewer will devote $f(20) = 80\%$ of their attention

RELATIVE EXTREMA

Let c be a number in the domain of a function f. Then $f(c)$ is a relative (or local) maximum for f if there exists an open interval (a, b) containing c such that

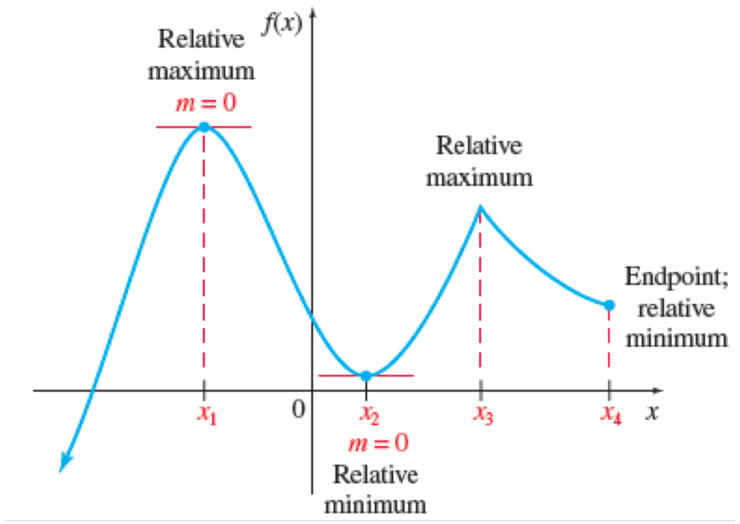
$$f(x) \leq f(c)$$

for all x in (a, b)

Likewise, $f(c)$ is a relative (or local) minimum for f if there exists an open interval (a, b) containing c such that

$$f(x) \geq f(c)$$

for all x in (a, b)



If c is an endpoint of the domain of f, we only consider the half-open interval that is in the domain

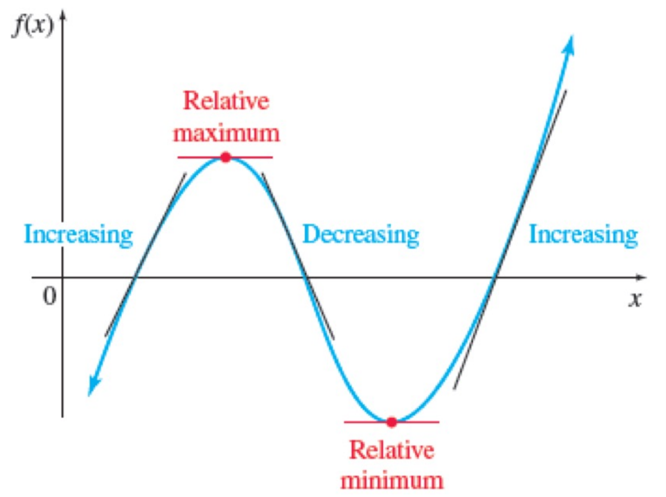
1ST DERIVATIVE TEST

Suppose all critical numbers have been found for some function f

How is it possible to tell from the equation of the function whether these critical numbers produce relative maxima, relative minima, or neither?

The graph shows tangent lines to the left of a relative maximum have positive slopes and to the right they have negative slopes

Similarly, the slopes of the tangent lines to the left of a relative minimum are negative and positive to the right



First Derivative Test

Let c be a critical number for a function f. Suppose that f is continuous on (a, b) and differentiable on (a, b) except possibly at c, and that c is the only critical number for f on (a, b).

1. $f(c)$ is a relative maximum of f if $f'(x) > 0$ on the interval (a, c) and $f'(x) < 0$ on the interval (c, b).
2. $f(c)$ is a relative minimum of f if $f'(x) < 0$ on the interval (a, c) and $f'(x) > 0$ on the interval (c, b).

EXAMPLE

Find all relative extrema for the following function as well as where the function is increasing and decreasing

$$f(x) = 6x^{2/3} - 4x$$

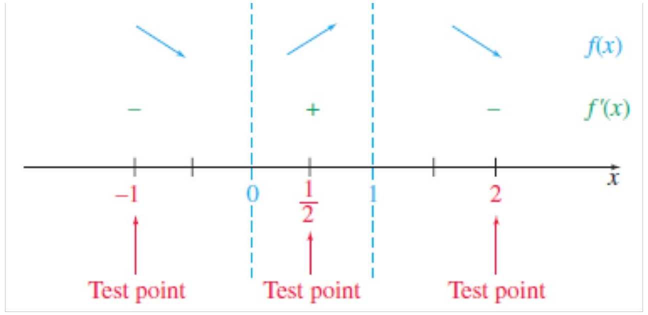
$$f'(x) = 4x^{-1/3} - 4 = \frac{4}{x^{1/3}} - 4$$

The derivative does not exist at $x = 0$, so 0 is a critical number

$$\frac{4}{x^{1/3}} - 4 = 0$$

$$\frac{4}{x^{1/3}} = 4$$

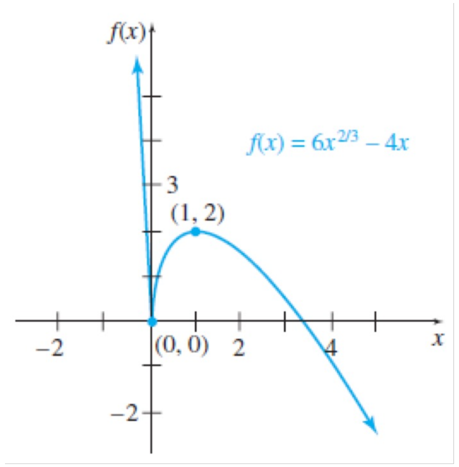
$$4 = 4x^{1/3}$$
$$1 = x^{1/3}$$
$$1 = x$$



Test points in the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$

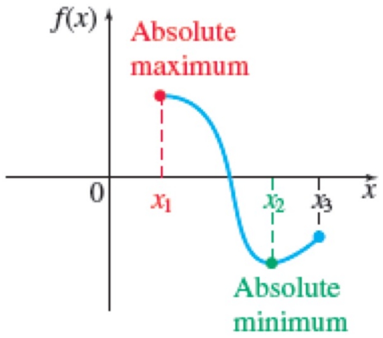
According to the 1st derivative test, f has a relative minimum at $x = 0$ and a relative maximum at $x = 1$

f is increasing on the interval $(0, 1)$ and is decreasing on the intervals $(-\infty, 0)$ and $(1, \infty)$

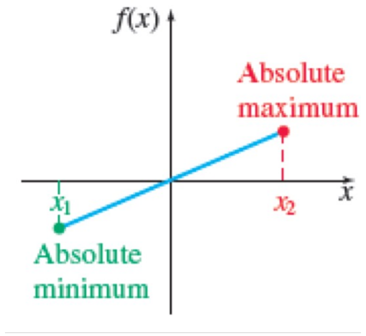


ABSOLUTE EXTREMA

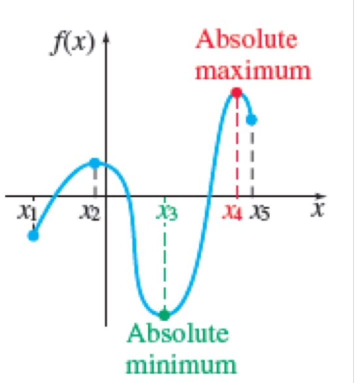
Let f be a function defined on some interval. Let c be a number in the interval. Then $f(c)$ is the absolute maximum of f on the interval if $f(x) \leq f(c)$ for all x in the interval and $f(c)$ is the absolute minimum of f on the interval if $f(x) \geq f(c)$ for all x in the interval



$f(x_1)$ = absolute maximum
$f(x_2)$ = absolute minimum
$f(x_3)$ = relative maximum



$f(x_1)$ = absolute minimum
$f(x_2)$ = absolute maximum



$f(x_1)$ = relative minimum
$f(x_2)$ = relative maximum
$f(x_3)$ = absolute minimum
$f(x_4)$ = absolute maximum
$f(x_5)$ = relative minimum

EXTREME VALUE THEOREM

A function f that is continuous on a closed interval $[a, b]$ will have both an absolute maximum and an absolute minimum on the interval

Finding Absolute Extrema

To find absolute extrema for a function f on a closed interval $[a, b]$:

1. Find all critical numbers for f in (a, b)
2. Evaluate f for all critical numbers in (a, b)
3. Evaluate f at the end points a and b
4. The largest value found in step 2 or 3 is the absolute maximum for f on $[a, b]$ and the smallest value found is the absolute minimum for f on $[a, b]$

The miles per gallon of a certain car at a speed of x mph can be modeled by

$$M(x) = -\frac{1}{45}x^2 + 2x - 20, \qquad 30 \le x \le 65$$

Find the absolute maximum and absolute minimum miles per gallon and the speeds at which they occur

$$M'(x) = -\frac{2}{45}x + 2$$

$$-\frac{2}{45}x + 2 = 0$$

$$x = 45$$

$$M(30) = 20$$
$$M(45) = 25$$
$$M(65) \approx 16.1$$

The absolute minimum is 16.1 mpg, which occurs at 65 mph
The absolute maximum is 25 mpg, which occurs at 45 mph

MEAN VALUE THEOREM

Let f be continuous on the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Given a function that is continuous on the closed interval and differentiable on the open interval, we can find a number in the open interval where the slope of the tangent at that number is equal to the slope of the secant line connecting the endpoints.

Suppose an object is dropped from a height of 400 ft. Its position at t seconds is $s(t) = -16t^2 + 400$, for $0 \leq t \leq 5$. Find the time t when the instantaneous velocity of the object equals its average velocity.

Note: f is continuous and differentiable everywhere

$$s_{avg} = \frac{f(5) - f(0)}{5 - 0} = \frac{0 - 400}{5} = -80$$

We need to find t such that $s'(t) = -32t = -80$

$$-32t = -80$$
$$t = 2.5$$

<u>Rolle's Theorem</u>

Let f be continuous on the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. Then there exists at least one number c in (a, b) such that $f'(c) = 0$.

Verify that $f(x) = 2x^2 - 8x - 6$ satisfies the conditions of Rolle's Theorem on the interval $[1,3]$ and find all points c guaranteed by Rolle's Theorem.

f is continuous and differentiable everywhere.

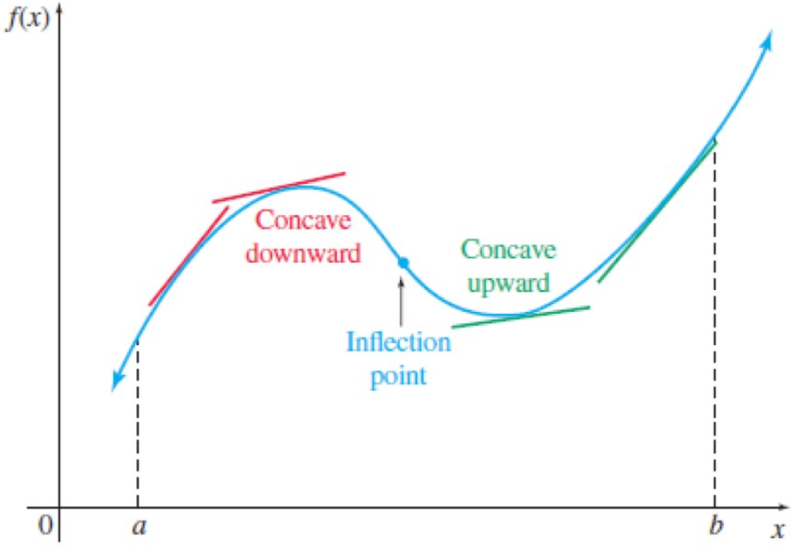
$$f(1) = -12 = f(3)$$

$$f'(x) = 4x - 8$$
$$4x - 8 = 0$$
$$x = 2$$

MODULE 5

CURVE SKETCHING

CONCAVITY OF A GRAPH



A function is concave upward on an interval (a, b) if the graph of the function lies above its tangent line at each point of (a, b)

A function is concave downward on an interval (a, b) if the graph of the function lies below its tangent line at each point of (a, b)

An inflection point is where the graph changes concavity

TEST FOR CONCAVITY

Let f be a function such that the derivatives f' and f'' exist at all points in an interval (a, b). Then f is concave upward on (a, b) if $f''(x) > 0$ for all x in (a, b) and concave downward on (a, b) if $f''(x) < 0$ for all x in (a, b).

Find all intervals where $f(x) = x^4 - 8x^3 + 18x^2$ is concave upward or downward and find all inflection points.

$f'(x) = 4x^3 - 24x^2 + 36x$

$f''(x) = 12x^2 - 48x + 36$

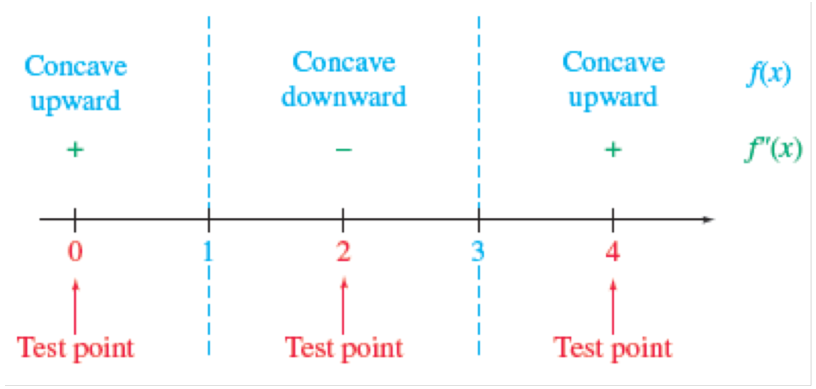
$12x^2 - 48x + 36 = 0$

$12(x^2 - 4x + 3) = 0$

$12(x - 1)(x - 3) = 0$

$x = 1$ or $x = 3$

Test points on the intervals $(-\infty, 1)$, $(1, 3)$, and $(3, \infty)$



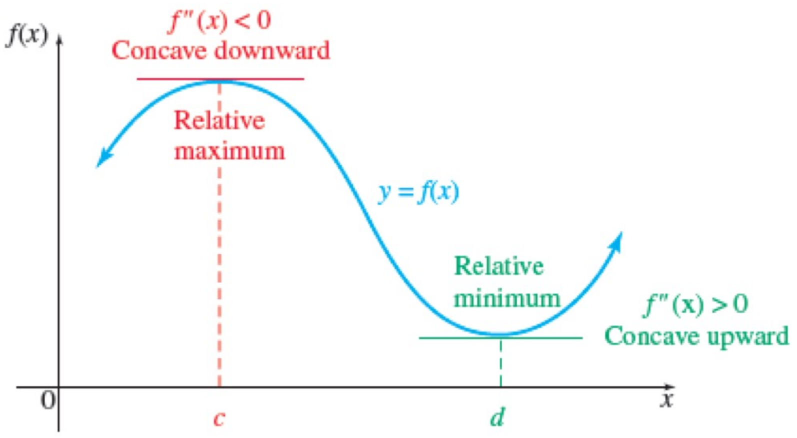
f is concave upward on $(-\infty, 1)$ and $(3, \infty)$ and concave downward on $(1, 3)$

f'' changes sign at $x = 1$ and $x = 3$

$f(1) = 11$ and $f(3) = 27$

Inflection points are $(1, 11)$ and $(3, 27)$

2^{ND} DERIVATIVE TEST



Let f'' exist on some open interval containing c, (except possibly at c itself) and let $f'(c) = 0$.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.
3. If $f''(c) = 0$ or does not exist, then the test gives no information, so the 1st derivative test should be used.

ASYMPTOTES

Horizontal asymptotes

If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, the line $y = L$ is a horizontal asymptote of f

Determine the horizontal asymptote, if any, of the function

$$f(x) = \frac{2x^2+1}{x^2}$$

$$\lim_{x\to\infty} \frac{2+\frac{1}{x^2}}{1} = \lim_{x\to\infty}\left(2+\frac{1}{x^2}\right) = 2$$

$$\lim_{x\to-\infty} \frac{2+\frac{1}{x^2}}{1} = \lim_{x\to-\infty}\left(2+\frac{1}{x^2}\right) = 2$$

$y = 2$ is a horizontal asymptote

Vertical asymptotes

These will occur in a rational function at points where the denominator (but not the numerator) is zero

Oblique asymptotes

These occur in a rational function when the degree of the numerator is larger than the degree of the denominator

CURVE SKETCHING

Graph $f(x) = \frac{x^2+1}{x}$

Step 1: Determine the domain

The only restriction is $x \neq 0$, so the domain is $(-\infty, 0) \cup (0, \infty)$

Step 2: Determine intercepts

Since $x = 0$ is not in the domain, there is no y-intercept.

If $y = 0$, then

$$\frac{x^2+1}{x} = 0$$

$$x^2 = -1$$

but $x^2 > 0$, so there is no x-intercept

Step 3: Determine any asymptotes

Horizontal asymptotes (none):

Limits at $-\infty$ and ∞ become very large in magnitude

Vertical asymptotes:

The vertical line $x = 0$ is a vertical asymptote

Oblique asymptotes:

$$f(x) = x + \frac{1}{x} \approx x$$

for very large values of x. The line $y = x$ is an oblique asymptote

Step 4: Investigate symmetry

$f(-x) \neq f(x)$ so f is not symmetric about the y-axis.

$f(-x) = -f(x)$ so f is symmetric about the origin

CURVE SKETCHING

Graph $f(x) = \frac{x^2+1}{x}$

Step 5: Find $f'(x)$ and critical points. Note where $f'(x)$ does not exist, but $f(x)$ does. Find relative extrema and determine where f is increasing and decreasing.

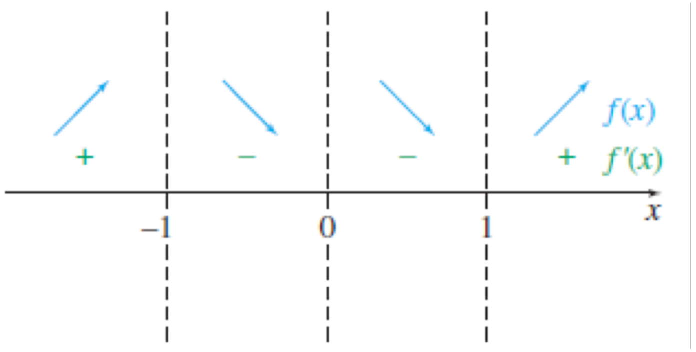
$$f'(x) = 1 - \frac{1}{x^2}$$

$$1 - \frac{1}{x^2} = 0$$

$$x^2 = 1$$
$$x^2 - 1 = 0$$
$$(x+1)(x-1) = 0$$

$$x = -1 \text{ or } x = 1$$

Critical numbers are $x = -1, 0,$ and 1



By the 1st derivative test, f has a relative maximum of $f(-1) = -2$ when $x = -1$ and a relative minimum of $f(1) = 2$ when $x = 1$

CURVE SKETCHING

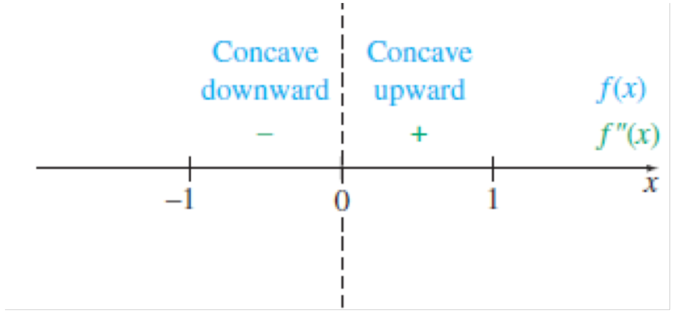
Graph $f(x) = \frac{x^2+1}{x}$

Step 6: Find $f''(x)$ and locate potential inflection points and determine where $f''(x)$ does not exist. Determine where f is concave upward or concave downward

$$f''(x) = \frac{2}{x^3}$$

which is never equal to 0 and does not exist when $x = 0$

There may be a change of concavity, but not an inflection point when $x = 0$ since $f(0)$ is not defined



$f''(x) < 0$ when $x < 0$, so f is concave downward on $(-\infty, 0)$

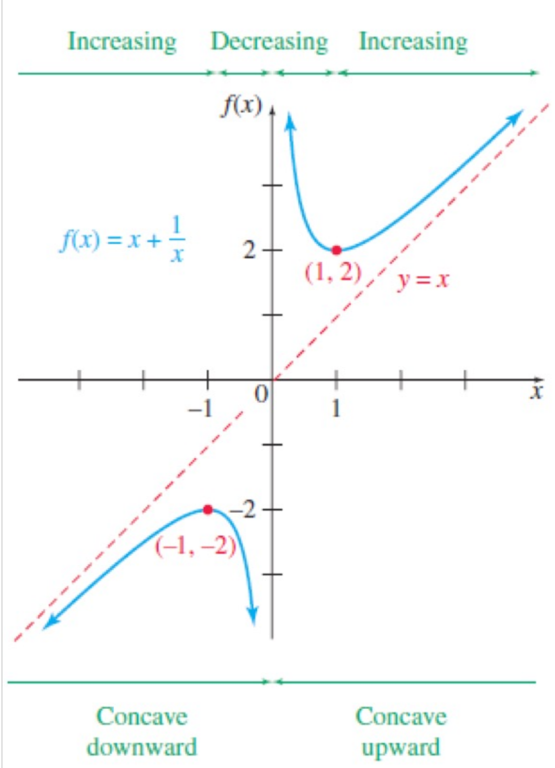
$f''(x) > 0$ when $x > 0$, so f is concave upward on $(0, \infty)$

CURVE SKETCHING

Graph $f(x) = \frac{x^2+1}{x}$

Summary:

* The domain of f is $(-\infty, 0) \cup (0, \infty)$
* No intercepts
* Asymptotes are the lines $x = 0$ and $y = x$
* Symmetric about the origin
* Critical points are $(-1, -2)$ and $(1, 2)$
* f is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1,0)$ and $(0, 1)$
* f has a relative maximum at $(-1, -2)$ and relative minimum at $(1, 2)$
* f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$



Step 7: Plot important points and asymptotes

Step 8: Connect points with a smooth curve using the correct concavity

Step 9: Check with technology

MODULE 5

L'HOPITAL'S RULE

INDETERMINATE FORM $0/0$

Suppose we wish to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

If we are interested in how F behaves near 1, we would want to calculate the value of the limit

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

We cannot apply the limit law for quotients since the limit of the denominator is 0. The limit does exist, but the value is not obvious since both the numerator and denominator approach 0 and $0/0$ is not defined.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an *indeterminate form of type* $0/0$.

INDETERMINATE FORM ∞/∞

Another possibility in which the value of a limit may not be obvious would occur when we look for a horizontal asymptote of F and need to evaluate

$$\lim_{x\to\infty} \frac{\ln x}{x-1}$$

Both the numerator and the denominator become large as $x \to \infty$.

In general, if we have a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then this limit may or may not exist and is called an *indeterminate form of type* ∞/∞.

L'HOPITAL'S RULE

Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

In particular, we have an indeterminate form of type $0/0$ or ∞/∞.

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is $\pm\infty$)

EXAMPLE

Calculate

$$\lim_{x\to 1}\frac{\ln x}{x-1}$$

Since

$$\lim_{x\to 1}\ln x = 0 \qquad \text{and} \qquad \lim_{x\to 1} x-1 = 0$$

we can apply L'Hopital's Rule:

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1}\frac{1/x}{1} = \lim_{x\to 1}\frac{1}{x} = 1$$

EXAMPLE

Calculate

$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

Since

$$\lim_{x\to\infty} e^x = \infty \qquad \text{and} \qquad \lim_{x\to\infty} x^2 = \infty$$

we can apply L'Hopital's Rule (twice):

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

MODULE 5

NEWTON'S METHOD

MOTIVATION

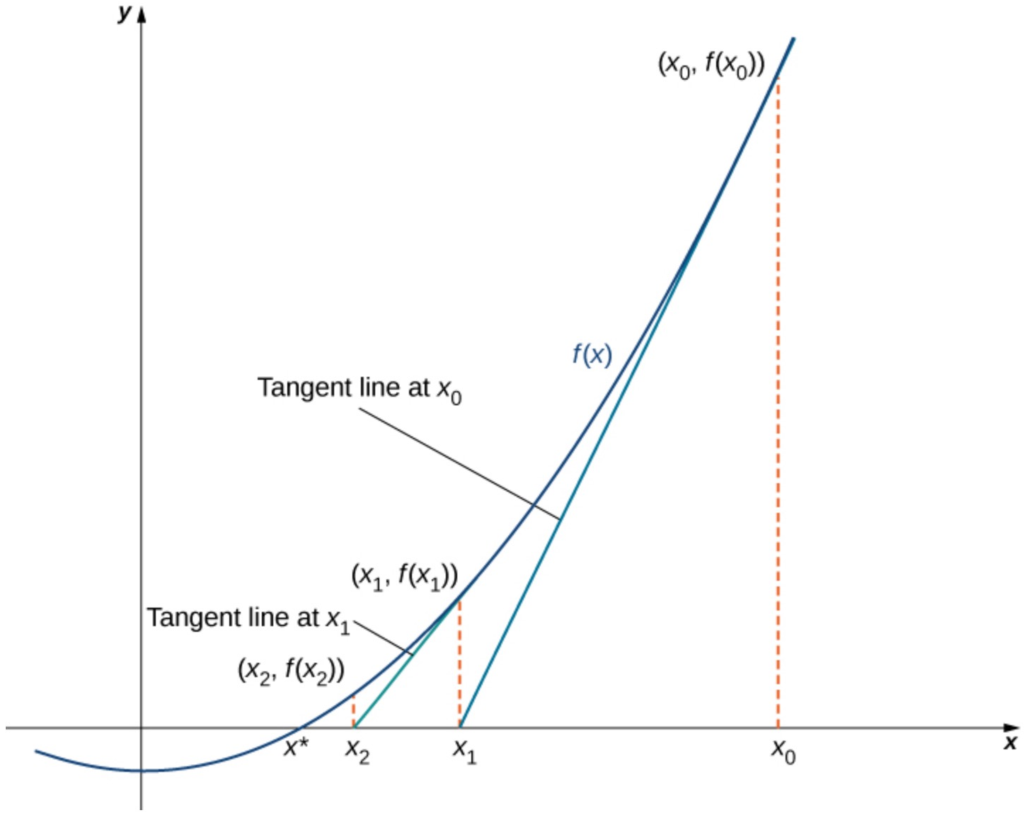
Suppose we want to solve $f(x) = 0$ where f is a polynomial.

If the degree of the polynomial is high enough, common formulas will not work.

One approach is to use Newton's Method:

1. Begin with an initial estimate, say x_0.
2. Sketch the tangent line to f at x_0.
3. If $f(x_0) \neq 0$, then this tangent line intersects the x-axis at some point, say $(x_1, 0)$.
4. Now let x_1 be the next approximation to the actual root and sketch the tangent line at $(x_1, f(x_1))$.
5. If $f(x_1) \neq 0$, then this tangent line intersects the x-axis at some point $(x_2, 0)$.
6. Now let x_2 be the next approximation and continue the process.

NEWTON'S METHOD



The equation of the tangent line at x_0 is given by:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Therefore, x_1 must satisfy

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0$$

Solving this for x_1, we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

EXAMPLE

Use Newton's Method to find $\sqrt[6]{2}$ correct to eight decimal places.

Notice finding $\sqrt[6]{2}$ is equivalent to finding the positive root of the equation $x^6 - 2 = 0$, so we let $f(x) = x^6 - 2$.

Then $f'(x) = 6x^5$ and the general formula for Newton's Method is

$$x_n = x_{n-1} - \frac{x_{n-1}^6 - 2}{6x_{n-1}^5}$$

If we choose $x_0 = 1$ as the initial approximation, we have

$$\begin{aligned} x_1 &\approx 1.16666667 \\ x_2 &\approx 1.12644368 \\ x_3 &\approx 1.12249707 \\ x_4 &\approx 1.12246205 \\ x_5 &\approx 1.12246205 \end{aligned}$$

Since x_4 and x_5 agree to eight decimal places, we conclude that $\sqrt[6]{2} \approx 1.12246205$.

QUESTIONS?