Top Trading Cycles Under Probabilistic Assignments

MTH498: Under Graduate Project - III

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Abstract

This report studies the extension of the classic Top Trading Cycle (TTC) mechanism from deterministic to probabilistic settings in house exchange markets, where agents initially possess endowments and seek to trade for preferred objects. We begin by revisiting the deterministic TTC algorithm and its desirable properties—Pareto efficiency, strategy-proofness, individual rationality, and core stability. The focus then shifts to the probabilistic domain, where outcomes are represented as lotteries over deterministic allocations. Within this framework, we formally define and examine stochastic dominance–based notions of efficiency, strategy-proofness, and individual rationality (SD-efficiency, SD-SP, and SD-IR). Building on established impossibility results for markets with four or more agents, we investigate the unresolved three-agent case. By employing the Birkhoff–von Neumann decomposition, we prove that any probabilistic assignment can be expressed as a convex combination of deterministic TTC allocations. The analysis demonstrates that, for three agents, such convex combinations satisfy SD-efficiency, SD-strategy-proofness, and SD-individual rationality simultaneously. This result provides a constructive characterization of feasible probabilistic mechanisms for small markets and highlights the boundary between possibility and impossibility in random assignment theory.

Contents

1 Introduction to Game Theory

Game theory is the name given to the methodology of using mathematical tools to model and analyze situations of interactive decision making. These are situations involving several decision makers (called players) with different goals, in which the decision of each affects the outcome for all the decision makers. This interactivity distinguishes game theory from standard decision theory, which involves a single decision maker, and it is its main focus. Game theory tries to predict the behavior of the players and sometimes also provides decision makers with suggestions regarding ways in which they can achieve their goals.

2 Matching Theory

Matching theory refers to theory of a class of models where there are two "sides" and one side is matched to the other. It is applied extensively in practice: in school admissions; in allocating houses and dorm rooms; in assigning doctors to internships; in exchanging organs (liver and kidney) among patients. There are two kinds of matching theory: one-sided matching and two-sided matching. In one-sided matching, there are agents on one side and objects on the other side, and the objective is to match agents to objects. In this model, agents have preferences over objects but objects do not have any preferences, and hence the name one-sided matching. This model is also known as the object assignment model. The other model is the two-sided matching, where agents (workers) on one side are matched to the agents (firms) on the other side. This model is often referred to as the marriage market model.

2.1 The House Exchange Problem (Object Assignment with Endowments)

The standard object assignment model assumes objects are unowned and are to be allocated by a central planner. However, in many practical scenarios, agents already possess an item and wish to trade it for a more preferred one. This is common in applications like housing markets, where individuals wish to swap houses, or in kidney exchanges, where patients have incompatible donors but could trade for a compatible one.

This leads to a specific case of the object assignment model, often called the **house exchange problem**. The model is modified as follows:

- **Agents (N):** A finite set of n agents, $\{1, \ldots, n\}$.
- **Objects (M):** A finite set of m objects, $\{o_1, \ldots, o_m\}$. Crucially, we assume the number of agents is equal to the number of objects.
- **Initial Endowments (e):** An initial allocation $e : N \to M$ is given. We can think of agent i as the "owner" of object $e(i)$. Since there are n agents and n objects, this initial allocation is a bijection. For simplicity, we can often identify each agent with their endowed object, assuming agent i initially owns object o_i.

- **Preferences ($\succ_i$):** Each agent $i \in N$ has a strict, complete, and transitive preference ordering $\succ_i$ over all objects in M.
- **Matching (μ):** The goal is to find a new matching (a permutation or bijection) $\mu : N \to M$ that reallocates the objects among the agents.

The central challenge in this exchange problem is to find a "good" final matching. We desire an allocation that is efficient (no other trade could make someone better off without making someone else worse off) and stable (no group of agents wants to break off and trade among themselves). This is precisely the problem that the **Top Trading Cycle (TTC) algorithm** is designed to solve.

3 The Top Trading Cycle (TTC) Algorithm

The Top Trading Cycles (TTC) algorithm, introduced by Shapley and Scarf (1974), is the canonical mechanism for solving the house exchange problem. It is a greedy algorithm that operates in iterative rounds to find a final allocation that is both efficient and stable.
The mechanism is defined formally as follows: Let N be the set of agents and M be the set of objects, with the initial endowment $e : N \to M$ mapping each agent to their starting object. For simplicity, let $e(i)$ be the object owned by agent i.

- **Step 1 (Initialization):** Set the initial set of active agents $N^1 = N$ and available objects $M^1 = M$. Set $k = 1$.
- **Step 2 (Graph Construction):** At step k, construct a directed graph G^k with nodes N^k. For each agent $i \in N^k$, identify their most-preferred object in the set of currently available objects, M^k. Let this object be $o_j = \text{top-choice}_i(M^k)$. Draw a directed edge from agent i to the agent $j \in N^k$ who owns o_j (i.e., $e(j) = o_j$).
- **Step 3 (Cycle Identification and Allocation):** Since every agent $i \in N^k$ points to exactly one agent in N^k, the graph G^k must contain at least one cycle. Identify all disjoint cycles in G^k.
  - For every agent i in a cycle, assign them the object they are pointing to.
  - Formally, if $(i^1, i^2, \ldots, i^p, i^1)$ is a cycle, then agent i^1 is allocated $e(i^2)$, agent i^2 is allocated $e(i^3)$, ..., and agent i^p is allocated $e(i^1)$.
- **Step 4 (Removal and Iteration):**
  - Let $\hat{N}^k$ be the set of all agents who were allocated an object in Step 3.
  - Let $\hat{M}^k$ be the set of all objects that were allocated in Step 3.
  - Set the new agent and object sets for the next round: $N^{k+1} = N^k \setminus \hat{N}^k$ and $M^{k+1} = M^k \setminus \hat{M}^k$.
  - If N^{k+1} is empty, STOP. The final allocation is the set of all assignments made.
  - Else, set $k = k + 1$ and return to Step 2.

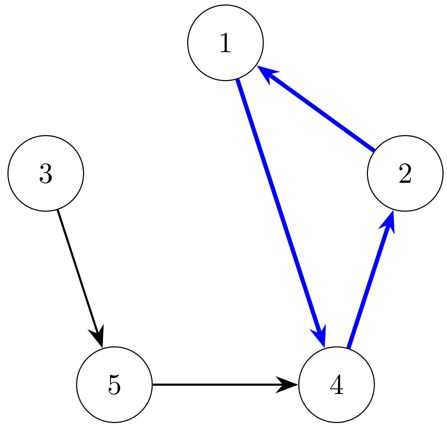


Figure 1: Graph G^1 for Iteration 1. The cycle $1 \to 4 \to 2 \to 1$ is highlighted.

3.1 An Example of TTC

Let's illustrate the algorithm with a 5-agent, 5-object market.

- **Agents & Endowments:** $N = \{1, 2, 3, 4, 5\}$, with agent i initially owning object a_i.
- **Preferences:** The preferences are given in the table below (read vertically).

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$
a_4	a_1	a_5	a_2	a_4
a_3	a_2	a_2	a_1	a_2
a_1	a_4	a_1	a_5	a_5
a_2	a_5	a_4	a_4	a_3
a_5	a_3	a_3	a_3	a_1

Iteration 1 ($k = 1$):

- Agents $N^1 = \{1, 2, 3, 4, 5\}$. Objects $M^1 = \{a_1, a_2, a_3, a_4, a_5\}$.
- **Graph G^1:** We draw a directed graph where each agent points to the owner of their most-preferred object.
  - Agent 1's top choice is a_4, owned by Agent 4. $(1 \to 4)$
  - Agent 2's top choice is a_1, owned by Agent 1. $(2 \to 1)$
  - Agent 3's top choice is a_5, owned by Agent 5. $(3 \to 5)$
  - Agent 4's top choice is a_2, owned by Agent 2. $(4 \to 2)$
  - Agent 5's top choice is a_4, owned by Agent 4. $(5 \to 4)$
- **Cycles:** We identify one cycle: $1 \to 4 \to 2 \to 1$.
- **Allocation:**

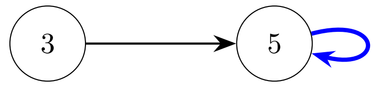


Figure 2: Graph G^2 for Iteration 2. The cycle $5 \to 5$ is highlighted.

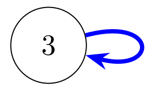


Figure 3: Graph G^3 for Iteration 3. The cycle $3 \to 3$ is highlighted.

  - Agent 1 is allocated a_4 (from Agent 4).
  - Agent 4 is allocated a_2 (from Agent 2).
  - Agent 2 is allocated a_1 (from Agent 1).

- **Removal:** $\hat{N}^1 = \{1, 2, 4\}$ and $\hat{M}^1 = \{a_1, a_2, a_4\}$.

Iteration 2 ($k = 2$):

- Agents $N^2 = \{3, 5\}$. Objects $M^2 = \{a_3, a_5\}$.
- **Graph G^2:**
  - Agent 3's top choice in M^2 is a_5, owned by Agent 5. $(3 \to 5)$
  - Agent 5's top choice in M^2 is a_5, owned by Agent 5. $(5 \to 5)$
- **Cycles:** We identify one cycle: $5 \to 5$.
- **Allocation:** Agent 5 is allocated a_5.
- **Removal:** $\hat{N}^2 = \{5\}$ and $\hat{M}^2 = \{a_5\}$.

Iteration 3 ($k = 3$):

- Agents $N^3 = \{3\}$. Objects $M^3 = \{a_3\}$.
- **Graph G^3:** Agent 3's top choice is a_3, owned by Agent 3. $(3 \to 3)$.
- **Cycles:** We identify one cycle: $3 \to 3$.
- **Allocation:** Agent 3 is allocated a_3.
- **Removal:** $\hat{N}^3 = \{3\}$ and $\hat{M}^3 = \{a_3\}$.

The set of agents is now empty, so the algorithm stops.
Final Allocation (μ): $\mu(1) = a_4$, $\mu(2) = a_1$, $\mu(3) = a_3$, $\mu(4) = a_2$, $\mu(5) = a_5$

3.2 Properties of TTC

The TTC algorithm is celebrated in matching theory because it satisfies several of the most important properties simultaneously.

- **Pareto Efficiency:** The final allocation is Pareto efficient. No other allocation exists that can make at least one agent better off without making another agent worse off.
- **Strategy-Proofness:** The mechanism is strategy-proof. It is a dominant strategy for every agent to report their true preferences. No agent can gain a more preferred object by misrepresenting their preference list.
- **Core Stability:** The allocation is in the "core." This means no subgroup of agents $S \subseteq N$ can break away and trade their own endowments among themselves to achieve an outcome that all agents in S strictly prefer.
- **Individual Rationality:** Every agent receives an object they find at least as desirable as their initial endowment.

4 The Probabilistic Exchange Problem

We now move from the deterministic setting to the **probabilistic setting**, but we remain focused on the *house exchange problem* (i.e., agents have initial endowments).

In this framework, a mechanism, or **probabilistic assignment rule (PR)**, does not output a single definite allocation. Instead, it takes the agents' preferences as input and outputs a *lottery* over all possible deterministic allocations. This outcome is represented by a **bi-stochastic matrix** $P = [p_{ij}]$, where $p_{ij} \geq 0$ is the probability that agent i receives object j.

4.1 Desirable Properties in the Probabilistic Setting

When agents compare these lotteries, we use the standard of **first-order stochastic dominance (SD)**. An agent i prefers lottery P_i to lottery Q_i if, for every subset of outcomes that i weakly prefers to its complement, P_i assigns at least as much probability to that subset as Q_i, and strictly more for at least one such subset. We seek a mechanism that satisfies the probabilistic analogues of the key properties of TTC:

Definition 1. SD-Pareto-Efficiency: We use the notation $Q_i \succeq_i^{SD} P_i$ to denote that agent i weakly prefers lottery Q_i to lottery P_i according to stochastic dominance. A random assignment P is said to be *SD-efficient* if there does not exist another random assignment Q such that

$$Q_i \succeq_i^{SD} P_i \quad \text{for all agents } i,$$

and

$$Q_j \succ_j^{SD} P_j \quad \text{for at least one agent } j.$$

Definition 2. SD-Strategy-Proofness (SD-SP): A mechanism is said to be *SD-strategy-proof* if reporting true preferences is a dominant strategy. Formally, for every agent i and every preference profile $P = (P_i, P_{-i})$, there is no misreport P'_i such that

$$\varphi_i(P'_i, P_{-i}) \succ_i^{SD} \varphi_i(P_i, P_{-i}).$$

That is, no agent can obtain a stochastically dominant lottery by misreporting their preferences.

Definition 3. SD-Individual Rationality (SD-IR): Let E be the initial probabilistic assignment matrix. Thus, the i^{th} row denoted by e_i will be a probability distribution denoting the agent $i's$ initial share over the houses. A mechanism satisfies *SD-individual rationality* if every agent weakly prefers their assigned lottery to their initial endowment. Formally, for all agents i,

$$\varphi_i(P_N) \succeq_i^{SD} e_i,$$

where e_i denotes the degenerate lottery of receiving their own initial endowment with certainty.

4.2 Impossibility Rule for ≥ 4 agents

In the deterministic setting, TTC famously satisfies all three analogous properties. A natural question is whether a "probabilistic TTC" exists that can do the same.

The answer, for most market sizes ≥ 4, is no. A major impossibility result in this field shows a fundamental conflict between these properties.

Theorem 1 (Impossibility for $n \geq 4$, (2))**.** *For markets with four or more agents ($n \geq 4$), there exists no probabilistic assignment rule that simultaneously satisfies **SD-Pareto-Efficiency**, **SD-Strategy-Proofness**, and **SD-Individual Rationality**.*

4.3 Ex-Post Properties and Ordinal Efficiency

While we have defined our core properties in terms of stochastic dominance, it is also common to analyze probabilistic mechanisms from an *ex-post* perspective, which considers the deterministic allocations that result after the lottery is resolved.

To formalize this, we assume agents have **von Neumann-Morgenstern (VNM) utility functions**, $u_i : M \to \mathbb{R}$, that are *compatible* with their preference orderings $\succ_i$. Compatibility simply means that for any two objects $o_j, o_k \in M$, we have $o_j \succ_i o_k$ if and only if $u_i(o_j) > u_i(o_k)$.

We can now define the ex-post analogues of efficiency and individual rationality.

Definition 4. Ex-Post Efficiency: A random assignment P is said to be *ex-post efficient* if it can be represented as a probability distribution (convex combination) over deterministic Pareto efficient matchings. That is,

$$P = \sum_t \lambda_t \mu_t, \quad \text{where each } \mu_t \text{ is Pareto efficient, } \lambda_t \geq 0, \text{ and } \sum_t \lambda_t = 1.$$

Definition 5. Ex-Post Individual Rationality (Ex-Post IR): A random assignment P satisfies *ex-post individual rationality* if every deterministic matching μ in its support (i.e., any μ with $\lambda_\mu > 0$ in the representation of P) is individually rational. Formally, for all agents i and all such μ,

$$\mu(i) \succeq_i e(i).$$

Our initial definition of SD-Pareto-Efficiency is often referred to in the literature as **ordinal efficiency**. As the paper by A. Bogomolnia and H. Moulin (3) notes, ordinal efficiency is a stronger requirement than ex post efficiency. The following theorem, drawn from Lemma 2 of the paper, formalizes this relationship.

Theorem 2 (Relationship between Ordinal and Ex-Post Efficiency, (3)). *Let P be a random assignment and $\succ$ be a preference profile for n agents.*

1. *If P is SD-Pareto-Efficient (ordinally efficient) at $\succ$, then it is ex-post efficient at $\succ$.*

2. *The converse does not always hold: while every ex-post efficient assignment is ordinally efficient for markets with $n \leq 3$ agents, this equivalence may fail for markets with $n \geq 4$ agents.*

Theorem 3 (Equivalence of SD-IR and Ex-Post IR). *We prove that a probabilistic assignment P is SD-IR if and only if it is Ex-Post IR.*

Proof that SD-IR $\implies$ Ex-Post IR. We prove this direction by contradiction. Assume P is **SD-IR** but **not Ex-Post IR**.

If P is not Ex-Post IR, then there exists an agent i and an object o_w such that $o_w \prec_i e(i)$ and P assigns a positive probability to i receiving o_w (i.e., $p_{io_w} > 0$).

Let $S = \{o \in M \mid o \succeq_i e(i)\}$ be the set of all objects agent i finds at least as good as their endowment. This is a "top-k" set. The SD-IR condition, $P_i \succeq_i^{SD} L_{e(i)}$, requires

$$\mathrm{Prob}(P_i \in S) \geq \mathrm{Prob}(L_{e(i)} \in S).$$

- For the endowment lottery $L_{e(i)}$, $\mathrm{Prob}(L_{e(i)} \in S) = 1$, since $e(i) \in S$.

- For the mechanism lottery P_i, the total probability of receiving an object *not* in S is $\mathrm{Prob}(P_i \notin S) = \sum_{o \prec_i e(i)} p_{io}$.

- By our "not Ex-Post IR" assumption, this sum is strictly positive, as it includes $p_{io_w} > 0$.

- Therefore, $\mathrm{Prob}(P_i \in S) = 1 - \mathrm{Prob}(P_i \notin S) < 1$.

This leads to the contradiction $\mathrm{Prob}(P_i \in S) < \mathrm{Prob}(L_{e(i)} \in S)$, or $1 < 1$. Hence, SD-IR must imply Ex-Post IR. ■

Proof that Ex-Post IR $\implies$ SD-IR. Assume P is **Ex-Post IR**. This means that for any agent i, every deterministic matching μ in the support of P satisfies $\mu(i) \succeq_i e(i)$. Equivalently, in the probabilistic representation, P assigns zero probability to any object that agent i ranks below their endowment:

$$\sum_{o \prec_i e(i)} p_{io} = 0.$$

That is, $p_{io} > 0$ only if $o \succeq_i e(i)$.

We must show that this implies $P_i \succeq_i^{SD} L_{e(i)}$. Recall that $P_i \succeq_i^{SD} L_{e(i)}$ means:

$$\text{Prob}(P_i \in S_k) \geq \text{Prob}(L_{e(i)} \in S_k)$$

for every top-k set $S_k = \{o \in M \mid o \succeq_i o_k\}$.

- **Case 1:** $e(i) \in S_k$ (i.e., S_k includes the endowment).
  For the degenerate lottery $L_{e(i)}$, we have $\text{Prob}(L_{e(i)} \in S_k) = 1$ since $e(i) \in S_k$.
  For P_i, since P assigns zero probability to objects ranked below $e(i)$, we also have $\text{Prob}(P_i \in S_k) = 1$.
  Therefore, $\text{Prob}(P_i \in S_k) = \text{Prob}(L_{e(i)} \in S_k)$.

- **Case 2:** $e(i) \notin S_k$ (i.e., S_k consists of objects strictly preferred to $e(i)$).
  Then $\text{Prob}(L_{e(i)} \in S_k) = 0$, since the degenerate lottery $L_{e(i)}$ places all probability on $e(i)$.
  Meanwhile, $\text{Prob}(P_i \in S_k) \geq 0$ always holds. Hence,

  $$\text{Prob}(P_i \in S_k) \geq \text{Prob}(L_{e(i)} \in S_k).$$

Since this inequality holds for every top-k set S_k, we conclude that $P_i \succeq_i^{SD} L_{e(i)}$. Therefore, P is SD-IR.
Hence, Ex-Post IR $\implies$ SD-IR. ■

Theorem 4 (Ex-Post Efficiency Implies Ordinal Efficiency for $n = 3$)**.** *For $n = 3$ agents, every ex post efficient assignment is ordinally efficient as well.*

Proof that Ex-Post Efficiency Implies Ordinal Efficiency for $n = 3$. We note that, there are exactly six types of deterministic preferences:

Type 1	$a \succ_1 b, c$ $b \succ_2 a, c$ $c \succ_3 a, b$	**Type 2**	$a \succ_1 b \succ_1 c$ $a \succ_2 b \succ_2 c$ $a \succ_3 b \succ_3 c$
Type 3	$a \succ_1 b \succ_1 c$ $a \succ_2 b \succ_2 c$ $a \succ_3 c \succ_3 b$	**Type 4**	$a \succ_1 c \succ_1 b$ $a \succ_2 c \succ_2 b$ $b \succ_3 a, c$
Type 5	$a \succ_1 b \succ_1 c$ $a \succ_2 b \succ_2 c$ $b \succ_3 a, c$	**Type 6**	$a \succ_1 b \succ_1 c$ $a \succ_2 c \succ_2 b$ $b \succ_3 a, c$

We analyze each type:

- **In type 1**, the only ex post efficient assignment is ordinally efficient.
- **In type 2**, any feasible assignment is ordinally efficient.
- **In type 3**, every (deterministic) priority assignment $Prio(\sigma, \succ)$ has $p_{3b} = 0$; hence every ex post efficient assignment has $p_{3b} = 0$. The latter implies ordinal efficiency: if Q stochastically dominates P, we must have $p_{ia} \leq q_{ia}$ for all i. Hence all three inequalities are equalities; next $p_{ia} + p_{ib} \leq q_{ia} + q_{ib}$ for $i = 1, 2$ implies $p_{ib} = q_{ib}$ for $i = 1, 2$, and hence $Q = P$.
- **In type 4**, every priority assignment, and hence every ex post-efficient assignment as well, has $p_{3b} = 1$. This, in turn, implies ordinal efficiency.
- **In type 5**, every priority assignment, and hence every ex post efficient assignment as well, has $p_{3a} = 0$, which implies ordinal efficiency (by an argument similar to that for type 3 above).
- **In type 6**, every priority assignment, and hence every ex post efficient assignment as well, has $p_{2b} = p_{3a} = 0$, implying ordinal efficiency by the same kind of argument again: if Q stochastically dominates P we have successively:

  $$p_{ia} \leq q_{ia} \quad \text{for } i = 1, 2 \implies p_{ia} = q_{ia} \quad \text{for } i = 1, 2$$

  $$\{p_{1a} + p_{1b} \leq q_{1a} + q_{1b}; p_{3b} \leq q_{3b}\} \implies p_{1b} = q_{1b} \quad \text{and} \quad p_{3b} = q_{3b}$$

  and $Q = P$ as desired.

■

4.4 Analyzing The Three-Player Case ($n = 3$)

Recall that the impossibility theorems apply specifically when $n \geq 4$. The case for three agents ($n = 3$) remains a significant open question in the literature.

It is not currently known whether a mechanism exists that can satisfy SD-Efficiency, SD-Strategy-Proofness, and SD-Individual Rationality for three agents, or if this specific case also admits its own impossibility result. In this report, we address this open question by investigating the $n = 3$ case. We seek to prove that, unlike the $n \geq 4$ case, a mechanism satisfying all three properties *does* exist for three agents.

5 Convex Combination of TTC Rules as Probabilistic Assignment

Theorem 5 (Birkhoff–von Neumann Theorem)**.** *Let $\mathcal{B}_n$ denote the set of all* bi-stochastic matrices *of size $n \times n$, i.e.,*

$$\mathcal{B}_n = \{A \in \mathbb{R}^{n\times n} \mid A_{ij} \geq 0,\ \sum_{j=1}^{n} A_{ij} = 1,\ \sum_{i=1}^{n} A_{ij} = 1\}.$$

Then every doubly stochastic matrix $A \in \mathcal{B}_n$ can be written as a convex combination of permutation matrices. That is, there exist permutation matrices $P_1, P_2, \ldots, P_k$ and nonnegative weights $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

$$A = \sum_{i=1}^{k} \lambda_i P_i, \quad \textit{where } \lambda_i \geq 0 \textit{ and } \sum_{i=1}^{k} \lambda_i = 1.$$

Theorem 6. *For $n = 3$ agents, convex combination of TTC rules satisy SD-IR, SD-SP and SD-efficiency.*

Proof. Following from theorem 5, notice that we can write every probabilistic assignment matrix as a convex combination of permutation matrices. Let us consider the general case of the probability assignment matrix (M) where $0 \leq \alpha, \beta, \gamma, \delta \leq 1$:

$$M = \begin{bmatrix} \alpha & \beta & 1-(\alpha+\beta) \\ \gamma & \delta & 1-(\gamma+\delta) \\ 1-(\alpha+\gamma) & 1-(\beta+\delta) & (\alpha+\beta)+(\gamma+\delta)-1 \end{bmatrix}$$

It can be written as the convex combination of the following six permutation matrices:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Thus, we have:

$$M = \sum_{i=1}^{6} c_i P_i$$

The coefficients were found to be:

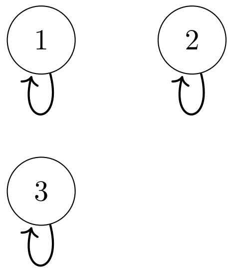
$$c_1 = \frac{\alpha}{2} + \frac{\beta}{2} + \delta - \frac{1}{2}$$
$$c_2 = \frac{\alpha}{2} - \frac{\beta}{2} - \delta + \frac{1}{2}$$
$$c_3 = \frac{\alpha}{2} + \frac{\beta}{2} + \gamma - \frac{1}{2}$$
$$c_4 = -\frac{\alpha}{2} + \frac{\beta}{2} - \gamma + \frac{1}{2}$$
$$c_5 = -\frac{\alpha}{2} - \frac{\beta}{2} + \frac{1}{2}$$
$$c_6 = -\frac{\alpha}{2} - \frac{\beta}{2} + \frac{1}{2}$$

We list all the possible type of scenarios of preference order of agents 1,2,3 over houses a,b,c.

Scenarios:

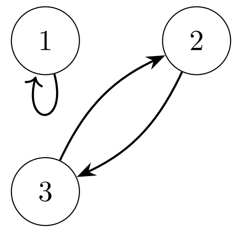
A	B	C	D	E	F	G
$1 : abc$	$1 : abc$	$1 : abc$	$1 : abc$	$1 : abc$	$1 : abc$	$1 : abc$
$2 : bac$	$2 : abc$	$2 : acb$	$2 : acb$	$2 : abc$	$2 : abc$	$2 : abc$
$3 : cba$	$3 : bac$	$3 : bac$	$3 : bca$	$3 : cba$	$3 : abc$	$3 : acb$

For each of the above scenarios we find out the assignment matrix for initial endowments P_1 to P_6 using Top Trading Cycle Mechanism. Then we take the convex combination of these assignments using our coefficients c_1 to c_6 to get our final probabilistic assignment matrix.



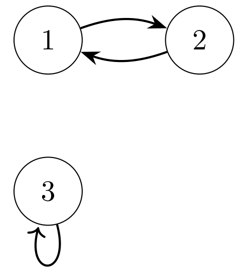
(1)

$1 \to a,\ 2 \to b,\ 3 \to c$



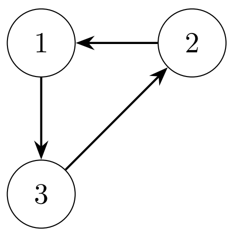
(2)

$1 \to a,\ 2 \to b,\ 3 \to c$



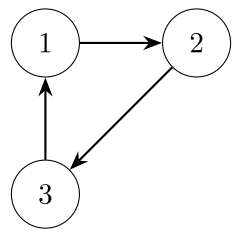
(3)

$1 \to a,\ 2 \to b,\ 3 \to c$



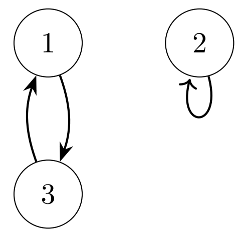
(4)

$1 \to a,\ 2 \to b,\ 3 \to c$



(5)

$1 \to a,\ 2 \to b,\ 3 \to c$



(6)

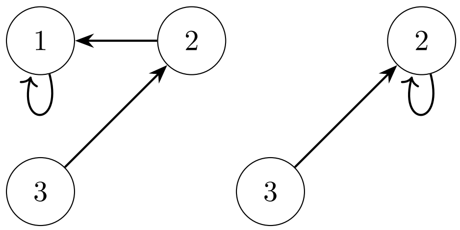
$1 \to a,\ 2 \to b,\ 3 \to c$

1 got a 6 times.
2 got b 6 times.
3 got c 6 times.

Under A:

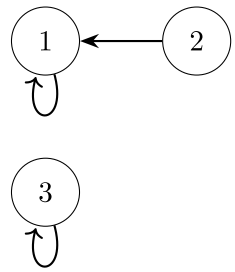
	a	b	c
1	1	0	0
2	0	1	0
3	0	0	1

Figure 4: Final allocation matrix under scenario A



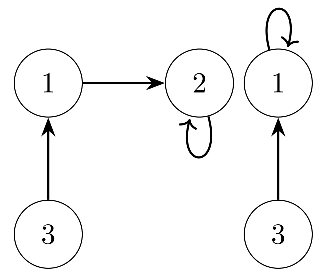
(1)

$1 \to a,\ 2 \to b,\ 3 \to c$



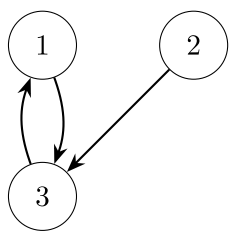
(2)

$1 \to a,\ 2 \to b,\ 3 \to c$



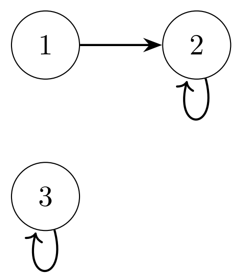
(3)

$1 \to b,\ 2 \to a,\ 3 \to c$



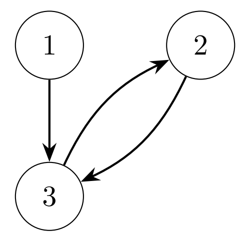
(4)

$1 \to a,\ 2 \to c,\ 3 \to b$



(5)

$1 \to c,\ 2 \to a,\ 3 \to b$



(6)

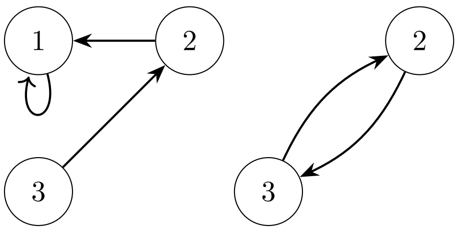
$1 \to c,\ 2 \to a,\ 3 \to b$

1 got a thrice, b once, c twice.
2 got a thrice, b once, c twice.
3 got b 4 times, c twice.

Under B:

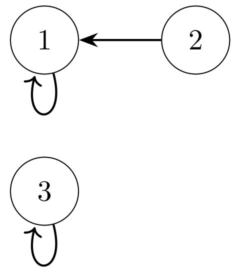
	a	b	c
1	$\frac{\alpha}{2}+\frac{\beta}{2}-\gamma+\frac{1}{2}$	$\frac{\alpha}{2}+\frac{\beta}{2}+\gamma-\frac{1}{2}$	$1-\alpha-\beta$
2	$\frac{1}{2}-\frac{\alpha}{2}-\frac{\beta}{2}+\gamma$	$\frac{\alpha}{2}+\frac{\beta}{2}+\delta-\frac{1}{2}$	$1-\gamma-\delta$
3	0	$2-(\alpha+\beta)-(\gamma+\delta)$	$(\alpha+\beta)+(\gamma+\delta)-1$

Figure 5: Final allocation matrix under scenario B



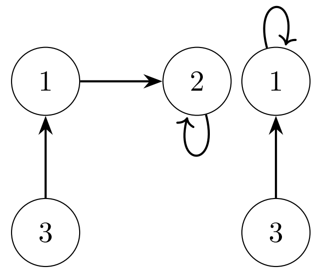
(1)

$1 \to a$, $2 \to c$, $3 \to b$



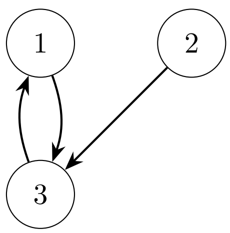
(2)

$1 \to a$, $2 \to b$, $3 \to c$



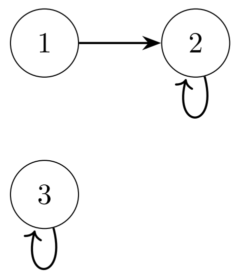
(3)

$1 \to b$, $2 \to a$, $3 \to c$



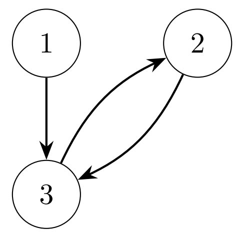
(4)

$1 \to a$, $2 \to c$, $3 \to b$



(5)

$1 \to c$, $2 \to a$, $3 \to b$



(6)

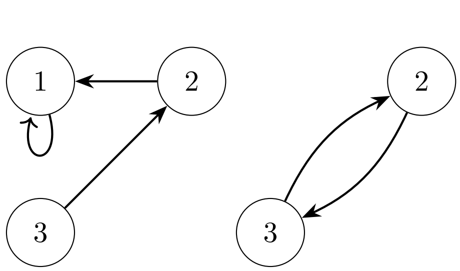
$1 \to c$, $2 \to a$, $3 \to b$

1 got a thrice, b once, c twice.
2 got a thrice, c thrice.
3 got b 5 times, c once.

Under C:

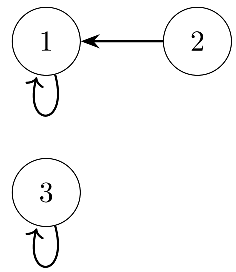
	a	b	c
1	$\frac{\alpha}{2} + \frac{\beta}{2} - \gamma + \frac{1}{2}$	$\frac{\alpha}{2} + \frac{\beta}{2} + \gamma - \frac{1}{2}$	$1 - \alpha - \beta$
2	$\frac{1}{2} - \frac{\alpha}{2} - \frac{\beta}{2} + \gamma$	0	$\frac{1}{2} + \frac{\alpha}{2} + \frac{\beta}{2} - \gamma$
3	0	$\frac{3}{2} - \frac{\alpha}{2} - \frac{\beta}{2} - \gamma$	$\frac{\alpha}{2} + \frac{\beta}{2} + \gamma - \frac{1}{2}$

Figure 6: Final allocation matrix under scenario C



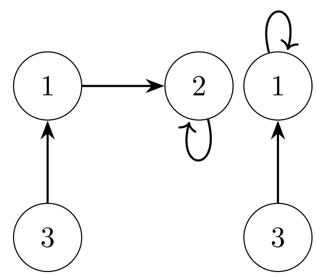
(1)

$1 \to a$, $2 \to c$, $3 \to b$



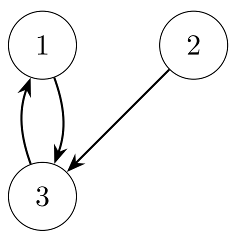
(2)

$1 \to a$, $2 \to c$, $3 \to b$



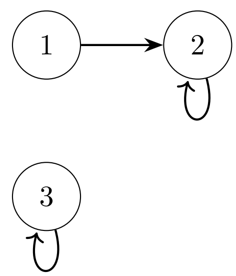
(3)

$1 \to c$, $2 \to a$, $3 \to b$



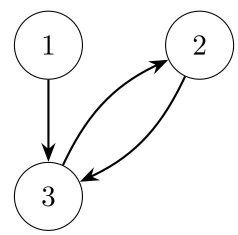
(4)

$1 \to a$, $2 \to c$, $3 \to b$



(5)

$1 \to b$, $2 \to a$, $3 \to c$



(6)

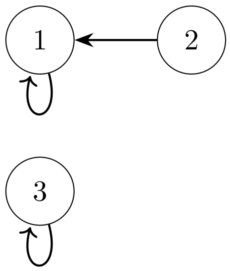
$1 \to c$, $2 \to a$, $3 \to b$

1 got a thrice, b once, c twice.
2 got a thrice, c thrice.
3 got b 5 times, c once.

Under D:

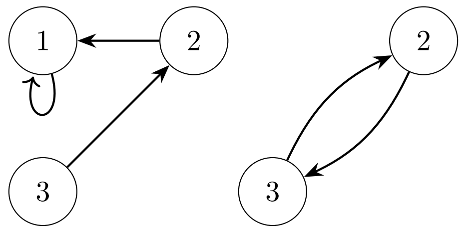
	a	b	c
1	$\frac{\alpha}{2} + \frac{\beta}{2} - \gamma + \frac{1}{2}$	$\frac{\alpha}{2} + \frac{\beta}{2} + \gamma - \frac{1}{2}$	$1 - \alpha - \beta$
2	$\frac{1}{2} - \frac{\alpha}{2} - \frac{\beta}{2} + \gamma$	0	$\frac{1}{2} + \frac{\alpha}{2} + \frac{\beta}{2} - \gamma$
3	0	$\frac{3}{2} - \frac{\alpha}{2} - \frac{\beta}{2} - \gamma$	$\frac{\alpha}{2} + \frac{\beta}{2} + \gamma - \frac{1}{2}$

Figure 7: Final allocation matrix under scenario D



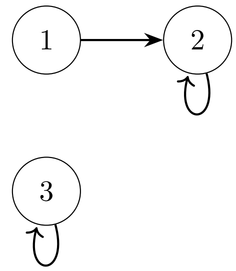
(1)

$1 \to a,\ 2 \to b,\ 3 \to c$



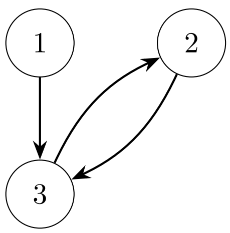
(2)

$1 \to a,\ 2 \to b,\ 3 \to c$



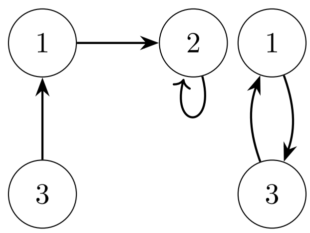
(3)

$1 \to b,\ 2 \to a,\ 3 \to c$



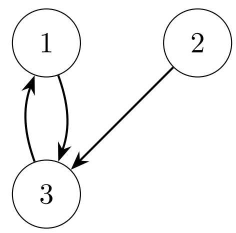
(4)

$1 \to b,\ 2 \to a,\ 3 \to c$



(5)

$1 \to b,\ 2 \to a,\ 3 \to c$



(6)

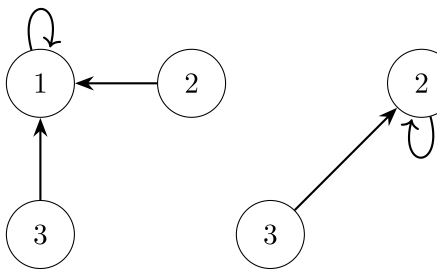
$1 \to a,\ 2 \to b,\ 3 \to c$

1 got a thrice, b thrice.
2 got a thrice, b thrice.
3 got c 6 times.

Under E:

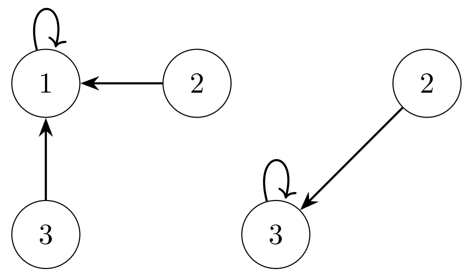
	a	b	c
1	$\frac{\alpha}{2} - \frac{\beta}{2} + \frac{1}{2}$	$-\frac{\alpha}{2} + \frac{\beta}{2} + \frac{1}{2}$	0
2	$-\frac{\alpha}{2} + \frac{\beta}{2} + \frac{1}{2}$	$\frac{\alpha}{2} - \frac{\beta}{2} + \frac{1}{2}$	0
3	0	0	1

Figure 8: Final allocation matrix under scenario E



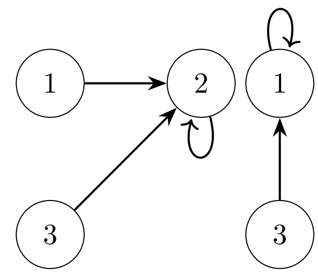
(1)

$1 \to a,\ 2 \to b,\ 3 \to c$



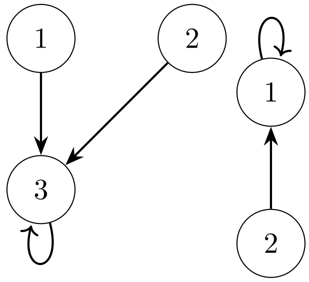
(2)

$1 \to a,\ 2 \to b,\ 3 \to c$



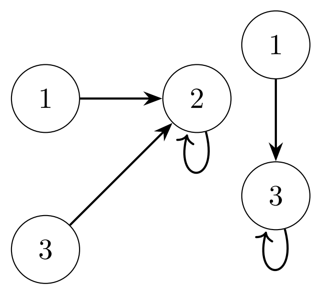
(3)

$1 \to b,\ 2 \to a,\ 3 \to c$



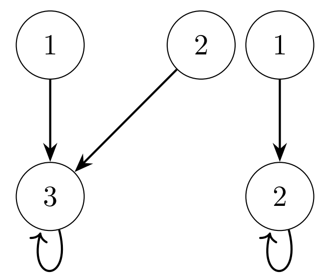
(4)

$1 \to b,\ 2 \to c,\ 3 \to a$



(5)

$1 \to c,\ 2 \to a,\ 3 \to b$



(6)

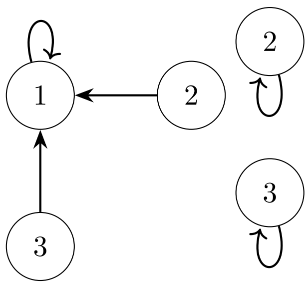
$1 \to c,\ 2 \to a,\ 3 \to b$

1 got a twice, b twice, c twice.
2 got a twice, b twice, c twice.
3 got a twice, b twice, c twice.

Under F:

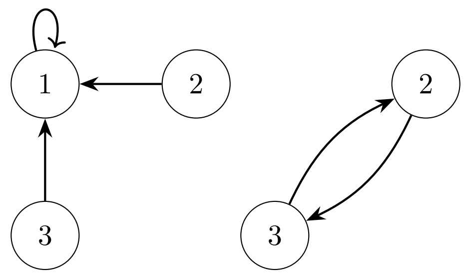
	a	b	c
1	α	β	$1-(\alpha+\beta)$
2	γ	δ	$1-(\gamma+\delta)$
3	$1-(\alpha+\gamma)$	$1-(\beta+\delta)$	$(\alpha+\beta)+(\gamma+\delta)-1$

Figure 9: Final allocation matrix under scenario F



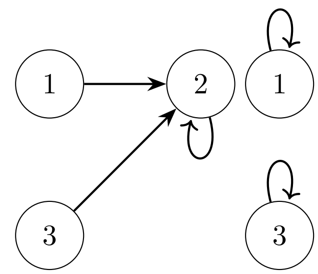
(1)

$1 \to a,\ 2 \to b,\ 3 \to c$



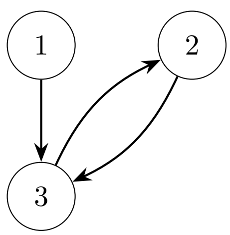
(2)

$1 \to a,\ 2 \to b,\ 3 \to c$



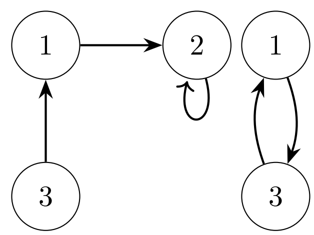
(3)

$1 \to b,\ 2 \to a,\ 3 \to c$



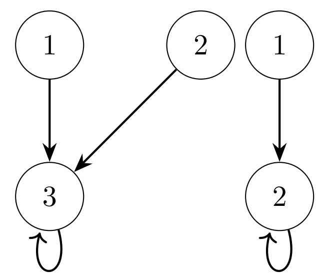
(4)

$1 \to b,\ 2 \to c,\ 3 \to a$



(5)

$1 \to b,\ 2 \to a,\ 3 \to c$



(6)

$1 \to c,\ 2 \to b,\ 3 \to a$

1 got a twice, b thrice, c once.
2 got a twice, b thrice, c once.
3 got a twice, c 4 times.

Under G:

	a	b	c
1	α	$-\frac{\alpha}{2}+\frac{\beta}{2}+\frac{1}{2}$	$-\frac{\alpha}{2}-\frac{\beta}{2}+\frac{1}{2}$
2	γ	$\frac{\alpha}{2}-\frac{\beta}{2}+\frac{1}{2}$	$\frac{1}{2}-\frac{\alpha}{2}+\frac{\beta}{2}-\gamma$
3	$1-(\alpha+\gamma)$	0	$\alpha+\gamma$

Figure 10: Final allocation matrix under scenario G

■

Now, we need to characterize the set of probabilistic assignment rules for $n = 3$ agents that simultaneously satisfy **SD-Pareto-Efficiency**, **SD-Strategy-Proofness**, and **SD-Individual Rationality**. We have thus far proved that the convex combination of TTC rules is one such set of rules. If we are able to show that these are the only type of rules that satisy the three properties, we will have a definite characterization.

6 Conclusion

The Top Trading Cycle algorithm remains one of the most powerful tools in matching theory, ensuring efficiency, individual rationality, and strategy-proofness in deterministic house exchange markets. Extending this framework to probabilistic settings introduces inherent challenges due to the trade-offs among stochastic dominance–based fairness and efficiency criteria. While impossibility results rule out the simultaneous satisfaction of all desirable properties for markets with four or more agents, this report establishes that such conflicts do not arise in the three-agent case. Using convex combinations of deterministic TTC rules, we demonstrate that SD-efficiency, SD-strategy-proofness, and SD-individual rationality can coexist in small markets. This characterization bridges deterministic and probabilistic matching models and provides a deeper understanding of how randomness can be structured to preserve the strategic and welfare properties of classical mechanisms. Future work may explore chracterisation of similar constructions that satisfy all three properties.

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[2] Stergios Athanassoglou, Jay Sethuraman. House allocation with fractional endowments.

[3] Anna Bogomolnaia, Hervé Moulin. A New Solution to the Random Assignment Problem.