Finite Difference Method for Solving the 1D Heat Equation: A Comparative Analysis of Aluminum and Silver Rods

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December 12, 2023

1 Introduction

The 1D heat equation is a fundamental partial differential equation describing the distribution of heat in a one-dimensional medium over time. This simulation utilizes the finite difference method to numerically solve the heat equation for two different materials, aluminum and silver. The finite difference method is a numerical approach to approximate the solutions of partial differential equations by discretizing both time and space.

The 1D heat equation describes how temperature $u(x,t)$ varies in one spatial dimension x and time t. The equation is given by:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Where:

- $\frac{\partial u}{\partial t}$ is the rate of change of temperature with respect to time.
- $\frac{\partial^2 u}{\partial x^2}$ is the second spatial derivative of temperature.
- α is the thermal diffusivity, a material property.

2 Thermal Diffusivity

The thermal diffusivity (α) is a material property that describes how well a material conducts heat relative to its ability to store thermal energy. It is defined as the ratio of the material's thermal conductivity (k) to its volumetric heat capacity (ρc_p), where ρ is the density and c_p is the specific heat at constant pressure.

The formula for thermal diffusivity is given by:

$$\alpha = \frac{k}{\rho c_p}$$

For Aluminum and Silver, here are approximate values for thermal conductivity, density, and specific heat:

Aluminum:
Thermal Conductivity (k): $\sim 237\, W/(mK)$
$Density(\rho) :\sim 2,700\, kg/m^3$
$SpecificHeat(c_p) :\sim 0.903\, J/(gK) or \sim 903\, J/(kgK)$

Silver:

Thermal Conductivity (k): $\sim 429\, W/(mK)$
$Density(\rho) :\sim 10,500\, kg/m^3$
$SpecificHeat(c_p) :\sim 0.235\, J/(gK) or \sim 235\, J/(kgK)$

Now, let's calculate the thermal diffusivity for both materials:

For Aluminum:

$$\alpha_{Aluminum} = \frac{237}{2700 \times 903}\, m^2/s$$

$$\alpha_{Aluminum} \approx \frac{237}{2,439,000}\, m^2/s \approx 9.71 \times 10^{-5}\, m^2/s$$

For Silver:

$$\alpha_{Silver} = \frac{429}{10,500 \times 235}\, m^2/s$$

$$\alpha_{Silver} \approx \frac{429}{2,467,500}\, m^2/s \approx 1.74 \times 10^{-4}\, m^2/s$$

So, the approximate thermal diffusivities are:
For Aluminum: $9.71 \times 10^{-5}\, m^2/s$
For Silver: $1.74 \times 10^{-4}\, m^2/s$.

3 Discretization:

To solve the heat equation numerically, the continuous spatial and temporal domains are discretized. In the code, the spatial domain is discretized into N points, and the time domain is discretized into M points.

Spatial Discretization:

The spatial domain $[0, L]$ is divided into N points with a uniform grid spacing Δx.

$\mathrm{x}_i = i \cdot \Delta x$ represents the i-th spatial point.

Temporal Discretization:

The time domain $[0, T]$ is divided into M points with a time step Δt.

$\mathrm{t}_n = n \cdot \Delta t$ represents the n-th time point.

4 Finite Difference Approximations:

The heat equation is approximated using finite differences:

Temporal Derivative:

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

Spatial Derivative:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

5 Discretized Heat Equation:

Combining the discretized temporal and spatial derivatives, the finite difference approximation of the heat equation becomes:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

This equation is solved iteratively in the code to simulate the evolution of temperature in the rod over time. The initial condition $u(x, 0)$ represents the initial temperature distribution.

6 Problem Statement

Consider a rod of length 14 units subjected to different initial conditions, and we aim to model and compare the temperature distribution over time for aluminum and silver rods. The thermal diffusivities for aluminum and silver are known, and the simulation employs the finite difference method to calculate the temperature at various spatial points over a specified time period.

7 Numerical Approach

The finite difference method discretizes the spatial and temporal domains. The rod is divided into a grid of spatial points, and the simulation advances in discrete time steps. The update of temperature at each spatial point is determined by the neighboring points using the difference equation derived from the 1D heat equation.

8 Simulation Details

8.1 Parameters

- Length of the rod (`length`) = 14 units
- Total simulation time (`time_total`) = 120 units
- Number of spatial points (`num_points_space`) = 100
- Number of time points (`num_points_time`) = 200

8.2 Material Properties

- Thermal diffusivity for Aluminum (`alpha_aluminum`) = 9.71×10^{-5}
- Thermal diffusivity for Silver (`alpha_silver`) = 1.74×10^{-4}

8.3 Initial Conditions

Different heat pulses are applied to the rods to represent distinct initial temperature distributions.

9 Results and Visualization

The temperature distribution over time is visualized in 2D plots for both aluminum and silver rods. The colormap indicates temperature variations across spatial and temporal dimensions. Additionally, temperature waveforms at a selected spatial point (`spatial_point_index` = 50) are plotted for comparison between aluminum and silver.

10 Conclusion

The presented simulation provides insights into the time-dependent temperature distribution in aluminum and silver rods using the finite difference method. Comparative analysis of the results helps understand the influence of material properties on heat conduction. This approach can be extended to study other materials or more complex scenarios, contributing to the understanding of heat transfer phenomena in various applications.

11 Application

11.1 Material Comparison

The simulation facilitates a comparative analysis of heat conduction in different materials, aiding engineers and researchers in material selection for specific applications. This is particularly relevant for designing heat sinks or thermal insulation.

11.2 Heat Transfer Studies

Understanding how heat is transferred through materials over time is crucial for designing and optimizing systems involving heat dissipation. This is important in electronics, where managing component temperature is critical for performance and reliability.

11.3 Process Optimization

The simulation can assist in optimizing manufacturing processes involving heat treatment or thermal processing. By understanding temperature distribution within a material, engineers can improve process efficiency and product quality.

11.4 Education and Research

The project serves as an educational tool for students and researchers studying numerical methods for solving partial differential equations. It provides a practical approach to understanding heat conduction principles and finite difference methods.

11.5 Material Science

In material science research, understanding thermal properties is fundamental. The simulation contributes to investigating how different materials respond to temperature changes, aiding in the development of new materials with specific thermal characteristics.

11.6 Climate Modeling

Environmental scientists and climate modelers use similar numerical methods to study heat transfer in various mediums. While this project focuses on a simplified 1D scenario, similar principles can be extended to more complex 2D or 3D models for climate simulations.

11.7 Product Design

Engineers designing products with temperature-sensitive components, such as electronic devices, can use simulations like this to predict and manage thermal behavior. This is crucial for preventing overheating and ensuring the longevity of the product.

In summary, the 1D heat equation simulation provides a versatile tool for understanding and predicting heat distribution in different materials, contributing to advancements in materials science, engineering, and various industries.

12 Class Problem

The problem for the class involves modifying parameters like the initial condition or thermal diffusivity to observe their impact on the temperature distribution over time.