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Solution of Model Question paper - I

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1 a) Evaluate: $I = \int_{-1}^{1}\int_{0}^{2}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz.$

Sol^n: I w r t y

$$I = \int_{z=-1}^{1}\int_{x=0}^{2}\left[xy+\frac{y^2}{2}+zy\Big|_{x-z}^{x+z}\right]dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{2}\Big[x\{(x+z)-(x-z)\}+\tfrac{1}{2}\{(x+z)^2-(x-z)^2\}+z\{(x+z)-(x-z)\}\Big]dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}(2xz+2xz+2z^2)\,dx\,dz.$$

$\boxed{\int_{-1}^{1}4z^3dz = 0 \quad \because 4z^3 \text{ is odd function}}$

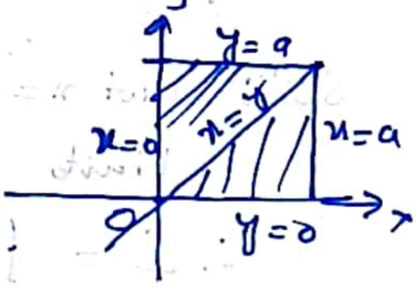
I w r t x

$$= \int_{-1}^{1}\left[2z\frac{x^2}{2}+2z\frac{x^2}{2}+2z^2x\Big|_{0}^{z}\right]dz$$

$$= \int_{-1}^{1}(z^3+z^3+2z^3)\,dz = \int_{-1}^{1}4z^3dz = \underline{0}$$

(b) Evaluate: $I = \int_{0}^{a}\int_{y}^{a}\frac{x}{x^2+y^2}\,dx\,dy$ by change of order of int^n.

Sol^n: Given $y=0$, $y=a$ & $x=y$ to $x=a$



By change of order of int^n,

new limits, $x=0$, $x=a$ & $y=0$ to $y=x$.

$$I = \int_{x=0}^{a}\int_{0}^{x}\frac{x}{x^2+y^2}\,dy\,dx = \int_{0}^{a}x\frac{1}{x}\tan^{-1}\frac{y}{x}\Big|_{0}^{x}dx$$

I w r. t y

$$= \int_0^a (\tan^{-1} 1 - \tan^{-1} 0)\, dx = \frac{\pi}{4}\int_0^a 1\, dx = \frac{\pi}{4} a.$$

(c) Derive the relation between Gamma & beta functions.

Sol^n: We have to P.T $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

UKT $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta\, d\theta \longrightarrow (1)$

$\Gamma(n) = 2\int_0^{\pi/2} e^{-x^2} x^{2n-1} dx \longrightarrow (2)$

$\Gamma(m) = 2\int_0^{\pi/2} e^{-y^2} y^{2m-1} dy \longrightarrow (3)$

$\therefore \Gamma(n)\Gamma(m) = 4\int_0^{\pi/2}\int_0^{\pi/2} e^{-(x^2+y^2)} x^{2n-1} y^{2m-1} dy\, dx$

Put $x = r\cos\theta$, $y = r\sin\theta$, limits $0 < r < \infty$, $0 < \theta < \pi/2$

$\Gamma(n)\Gamma(m) = 2\int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1} dr \times 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta\, d\theta$

$\Gamma(m)\Gamma(n) = \Gamma(m+n).\beta(m,n)$

$\therefore \beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

2(a) Evaluate; $I = \int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy$ by changing into polar coordinates.

Sol^n: Put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

Limits $0 < r < 1$ & $0 < \theta < \frac{\pi}{2}$

$\therefore I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^2 . r\, dr\, d\theta = \int_0^{\pi/2} 1\, d\theta \times \int_0^1 r^3 dr$

$= \frac{\pi}{2} \times \frac{1}{4} = \frac{\pi}{8}$.

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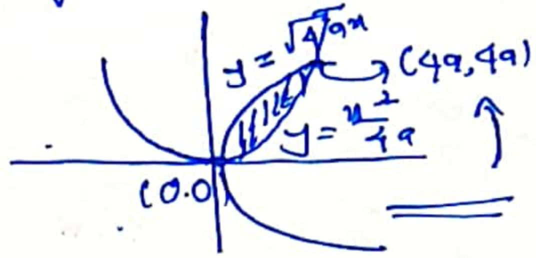
(b) Using double integration, find the area between the parabolas $y^2 = 4ax$ & $x^2 = 4ay$.

Sol^n: Area $= \iint_R dx\,dy$



$$\text{Area} = \int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{\sqrt{4ax}} 1\,dy\,dx.$$

I w r t 'y'

$$= \int_0^{4a} y\Big|_{x^2/4a}^{\sqrt{4ax}} dx = \int_0^{4a}\left[\sqrt{4a}\,x^{1/2} - \frac{1}{4a}\cdot x^2\right]dx$$

I w r t 'x'

$$\text{Area} = \sqrt{4a}\,\frac{x^{3/2}}{3/2} - \frac{1}{4a}\,\frac{x^3}{3}\Big|_0^{4a} = \frac{2}{3}(4a)^{1/2}(4a)^{3/2} - \frac{1}{12a}(4a)^3$$

$$= \frac{2}{3}(4a)^2 - \frac{\cancel{64}^{16}\,a^{\cancel{3}\,2}}{\cancel{12}_3\,\cancel{a}} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$

(c) Using beta & gamma functions, evaluate $I = \int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$

Sol^n: Let $I = \int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2}\sin^{1/2}\theta \times \cos^{-1/2}\theta\,d\theta \rightarrow ①$

WKT $\int_0^{\pi/2}\sin^p\theta$ [illegible] $= \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{v+1}{2}\right)}{2\,\Gamma\left(\frac{p+v+2}{2}\right)} \rightarrow ②$

& $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi.$

put $p = 1/2$ & $v = -1/2$

$$I = \int_0^{\pi/2}\sin^{1/2}\theta \times \cos^{-1/2}\theta\,d\theta = \frac{\Gamma\left(\frac{1/2+1}{2}\right)\Gamma\left(\frac{-1/2+1}{2}\right)}{2\,\Gamma\left(\frac{1/2-1/2+2}{2}\right)} = \frac{\Gamma(3/4)\,\Gamma(1/4)}{2\,\Gamma(1)}$$

$$= \frac{\sqrt{2}\,\pi}{2} = \frac{\pi}{\sqrt{2}}.$$

3(a) Find a directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,1)$ in the direction of the vector $2i - j - 2k$

Sol^n: $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$= (2xyz + 4z^2)\hat{i} + (x^2z + 0)\hat{j} + (x^2y + 8xz)\hat{k}$

Put $(x, y, z) = (1, -2, +1)$

$\nabla\phi = 0\hat{i} - 1\hat{j} - 6\hat{k}$

$D.D = \nabla\phi \cdot \hat{a} = (0\hat{i} - 1\hat{j} - 6\hat{k}) \cdot \frac{(2\hat{i} - 1\hat{j} - 2\hat{k})}{3}$

$= \frac{0 + 1 + 12}{3} = \frac{13}{3}$.

(b) Find div $\vec{F}$ & Curl $\vec{F}$, Where $\vec{F} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$.

Sol^n: $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz) =$

$\vec{F} = \hat{i}(\underbrace{3x^2 - 3yz}_{F_1}) + \hat{j}(\underbrace{3y^2 - 3xz}_{F_2}) + \hat{k}(\underbrace{3z^2 - 3yx}_{F_3})$

$\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 6x + 6y + 6z = 6(x + y + z)$.

$\text{Curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$

$= \hat{i}(-3x + 3x) - \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z) = \vec{0}$

(c) Define an irrotational vector. Find the constants a, b & c such that $\vec{F} = (axy - z^3)\hat{i} + (bx^2 + z)\hat{j} + (bxz^2 + cy)\hat{k}$ is irrotational.

Sol^n: Def^n: A vector point function $\vec{F}$ is said to be irrotational if $\text{curl}\vec{F} = \vec{0}$.

Let $\text{curl}\,\vec{F} = \vec{0}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy-z^3 & bx^2+z & bxz^2+cy \end{vmatrix} = \vec{0}$$

$$\hat{i}(c-1) + \hat{j}(-3z^2 - bz^2) + \hat{k}(2xb - ax) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$c-1=0 \quad -3z^2 - bz^2 = 0 \quad 2xb - ax = 0$

$c = 1 \quad \Rightarrow -3-b=0 \quad 2b-a=0$

$b = -3 \quad a = 2b = -6$

$\therefore \boxed{a=-6,\ b=-3,\ c=1}$.

4(a) Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$.

Solⁿ: Let Work done $= \int_c \vec{F}\cdot d\vec{r} = \int_c 3x^2dx + (2xz-y)dy + zdz$

Equation of the st line joining the points $(0,0,0)$ to $(2,1,3)$ is $\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$

$\Rightarrow x = 2t,\ y = t,\ z = 3t$

limits $(x,y,z) = (0,0,0) \Rightarrow t=0$ & $(x,y,z) = (2,1,3) \Rightarrow t = 1$

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} 3\cdot 4t^2\cdot 2dt + (2\cdot 2t\cdot 3t - t)dt + 3t\,3dt$$

$$= \int_0^1 (24t^2 + 12t^2 + 8t)\,dt$$

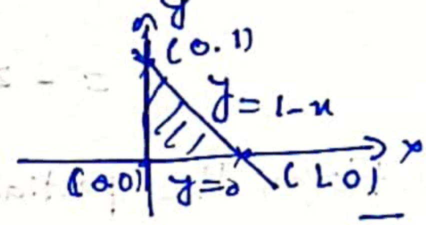
$$= \frac{24}{3} + \frac{12}{3} + \frac{8}{2}$$

$$= 8 + 4 + 4 = \underline{\underline{16}}$$

(b) Apply Green's theorem to evaluate $\int_C (3x-8y^2)\,dx + (4y-6xy)\,dy$, where C is the boundary of the region bounded by $x=0,\ y=0,\ x+y=1$.

Soln: Given $I = \int (3x-8y^2)\,dx + (4y-6xy)\,dy$

WKT $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$



$M = 3x-8y^2$, $N = 4y-6xy$

$M_y = -16y$ $\quad N_x = -6y$

$0 < x < 1$

$0 < y < 1-x$

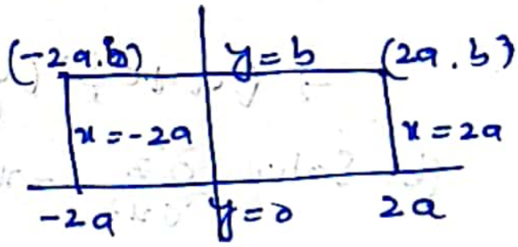
$$I = \int_0^1 \int_0^{1-x} (-6y+16y)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} 10y\,dy\,dx = \int_0^1 10\left.\frac{y^2}{2}\right|_0^{1-x} dx = \int_0^1 \frac{10}{2}(1-x)^2\,dx$$

$$= 5\int_0^1 (1+x^2-2x)\,dx = 5\left[\cancel{1}+\frac{1}{3}-\cancel{1}\right] = \frac{5}{3}.$$

(c) Apply Stoke's thm to evaluate $\iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$, taken around the rectangle bounded by the lines $x = \pm 2a$, $y=0$, $y=b$

Soln Formula; $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds \longrightarrow (1)$



$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(-2y-2y) = -4y\hat{k}$$

$\text{curl}\,\vec{F}\cdot\hat{n}\,ds = (0\hat{i}+0\hat{j}-4y\hat{k})\cdot(dy\,dz\,\hat{i}+dx\,dz\,\hat{j}+dx\,dy\,\hat{k})$

$= -4y\,dx\,dy$

Limits $x=-2a$ to $2a$ & $y=0$ to b

$$\therefore \int_s \vec{F}\cdot d\vec{r} = \int_{-2a}^{2a}\int_0^b -4y\,dy\,dx$$

$$= \int_{-2a}^{2a} -4\cdot\frac{y^2}{2}\Big|_0^b dx = -2b^2\int_{-2a}^{2a} dx$$

$$= -2b^2(2a+2a) = -2b^2(4a) = -8ab^2$$

5(a) Form the partial differential equation by eliminating the arbitrary function from the relation $ax+by+cz = f(x^2+y^2+z^2)$.

Sol^n Given $ax+by+cz = f(x^2+y^2+z^2) \longrightarrow (1)$

D. p. w. r. t. x & y of (1)

$$\boxed{\frac{\partial z}{\partial x} = p \quad \frac{\partial z}{\partial y} = q}$$

$a+0+cp = f'(x^2+y^2+z^2)(2x+2zp) \longrightarrow (2)$

$0+b+cq = f'(x^2+y^2+z^2)(2y+2zq) \longrightarrow (3)$

(2) ÷ (3) gives

$$\frac{a+cp}{b+cq} = \frac{2(x+zp)f'}{2(y+zq)f'}$$

$$\Rightarrow (a+cp)(y+zq) = (b+cq)(x+zp)$$

(b) Solve $\frac{\partial^2 z}{\partial x^2} = xy$ subject to the conditions $z_x = \log(1+y)$ when $x=1$ & $z=0$ when $x=0$.

Sol^n: Given $\frac{\partial^2 z}{\partial x^2} = xy$. I. w. r. t x

$$\frac{\partial z}{\partial x} = y\frac{x^2}{2} + f(y) \longrightarrow (1)$$

Put $z_x = \log(1+y)$ & $x=1$ in (1)

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$\log(1+y) = \frac{1}{2}y + f(y) \Rightarrow f(y) = \log(1+y) - \frac{1}{2}y$

$\therefore$ Eqn ① becomes

$$\frac{\partial z}{\partial x} = \frac{x^2}{2}y + \log(1+y) - \frac{1}{2}y$$

again I w.r.t. x

$$z = \frac{x^3}{6}\cdot\frac{y^2}{3} + \log(1+y)x - \frac{1}{2}yx + g(y) \to ②$$

put $z=0$ & $x=0$

$$0 = 0+0+0+g(y) \Rightarrow g(y) = 0$$

$\therefore$ ② becomes

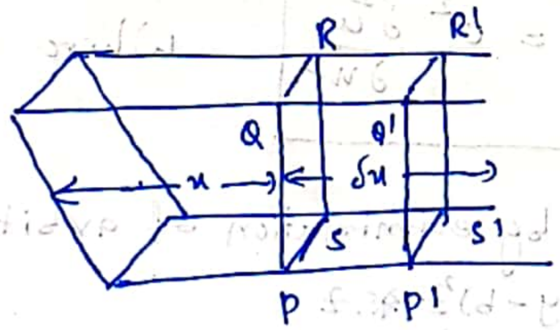
$$z = \frac{x^3 y}{6} + \log(1+y)x - \frac{1}{2}xy$$

② With usual notations derive a one dimential ~~heat eq~~ heat equation.

Soln: We shall assume the following:

① heat flows from higher tempr to lower tempr.

② the amount of heat in a body is proportional to its mass and Tempr.



Consider a homogeneous bar of constant cross sectional area A.

Let ρ = density, s = specific heat, K = thermal conductivity

u = tempr $= u(x,t)$, δu = change of tempr.

δx = thickness of the bar

A = cross sectional area.

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Let the sides are insultated, so that stream lines of heat flow are parallel and $\perp^r$ to the area A.

The mass of the element $= A \rho \delta x$

The quantity of heat stored in the slab element $\Big\} = A \rho s \delta x \delta u$

the rate of increase of heat $= R = A \rho s \delta x \dfrac{\delta u}{\delta t}$

R_1 = rate of inflow of heat; R_0 = rate of outflow of heat.

$R_1 = -KA\left(\dfrac{\delta u}{\delta x}\right)_x \qquad R_0 = -KA\left(\dfrac{\delta u}{\delta x}\right)_{x+\delta x}$

(due to emphorical formula, we take $-$ sign)

WKT $R = R_1 - R_0$

$$R = KA\left(\frac{\delta u}{\delta x}\right)_{x+\delta x} - KA\left(\frac{\delta u}{\delta x}\right)_x$$

$$\cancel{A}\rho s \delta x \frac{\delta u}{\delta t} = K\cancel{A}\left[\left(\frac{\delta u}{\delta x}\right)_{x+\delta x} - \left(\frac{\delta u}{\delta x}\right)_x\right]$$

$$\frac{\delta u}{\delta t} = \frac{K}{\rho s} \cdot \lim_{\delta x \to 0}\left[\text{---}, \text{---}\right]$$

$$\boxed{\frac{\delta u}{\delta t} = c^2 \frac{\delta^2 u}{\delta x^2}} \qquad \text{Where } c^2 = \frac{K}{\rho s}$$

6 (a) Form a p d e by elimination of arbitrary constants from $(x-a)^2 + (y-b)^2 = z$

Soln: given $z = (x-a)^2 + (y-b)^2 \longrightarrow ①$

D p w r t x & y

$\dfrac{\partial z}{\partial x} = p = 2(x-a) \longrightarrow ②$

$\dfrac{\partial z}{\partial y} = q = 2(y-b) \longrightarrow ③$

$②^2 + ③^2$ gives $\quad p^2 + q^2 = 4[(x-a)^2 + (y-b)^2]$

$\therefore p^2 + q^2 = 4z.$

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(b) Solve: $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Sol^n: AE: $\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \longrightarrow (1)$

case (i) chose $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ be the set of Multipliers

$\therefore$ Eqn (1) becomes

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{xy - xz + yz - xy + xz - yz} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$ Int

$\log x + \log y + \log z = \log c_1 \Rightarrow \log(xyz) = \log c_1$

$\boxed{xyz = c_1}$

case (ii) chose $\left(\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}\right)$ be the another set of Multipliers.

$\therefore$ Eqn (1) becomes

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\cancel{y} - \cancel{z} + \cancel{z} - \cancel{x} + \cancel{x} - \cancel{y}} \Rightarrow \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

Int: $-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = c_2 \Rightarrow \boxed{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_2}$

general sol^n is $\phi\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$

(c) Solve: $z_{yy} = z$, given that $\frac{\partial z}{\partial y} = e^{-x}$ when $y = 0$ and $z = e^x$ when $y = 0$

Sol^n: The given pde can be written as ODE is

$(D^2 - 1)z = 0 \Rightarrow$ AE: $m^2 - 1 = 0 \Rightarrow m = +1, -1$

$\therefore z = f(x)e^y + g(x)e^{-y} \longrightarrow (1)$

$\frac{\partial z}{\partial y} = f(x)e^y - g(x)e^{-y} \longrightarrow (2)$

Put $z = e^x$ & $y = 0$ in (1), we have $f(x) + g(x) = e^x \longrightarrow (3)$

$z_y = e^{-x}$ & $y = 0$ in (2), we have $f(x) - g(x) = e^{-x} \longrightarrow (4)$

① + ② gives: $f(x) = \frac{e^x + e^{-x}}{2} = \cancel{\sinh x}\ \cosh x$

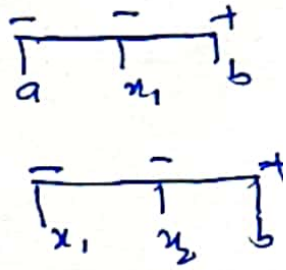
① − ② given: $g(x) = \frac{e^x - e^{-x}}{2} = \sinh x$

$\therefore$ Eqⁿ ① becomes

$$z = \sinh x\, e^{y} + \cosh x\, e^{-y}$$

7 ⓐ Find the real root of the equation $xe^x = \cos x$ which lies in $(0, 1)$ by Regula-falsi method.

Solⁿ: given $f(x) = xe^x - \cos x$



$a = 0,\ b = 1,\ f(a) = -1 \quad f(b) = 2.1780$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = 0.3147,\quad f(x_1) = -0.5198$$

$$x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)} = 0.4467,\quad f(x_2) = -0.2036\ \text{-ve}$$

$$x_3 = \frac{x_2 f(b) - b f(x_2)}{f(b) - f(x_2)} = 0.4940,\quad f(x_3) = -0.0708\ \text{-ve}$$

$$x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)} = 0.5099 \quad \therefore x \simeq 0.5099$$

(b) Using N.B.I.F, find the value of y when $x = 6$ from the given data:

x :	1	2	3	4	5
y :	1	−1	1	−1	1

Solⁿ

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1				
		−2			
2	−1		4		
		2		−8	
3	1		−4		16
		−2		8	
4	−1		4		
		2			
5	1				

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Formula; $y(x) = y_n + \frac{p}{1!}\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$

$r = \frac{x - x_n}{h} = \frac{6-5}{1} = 1$

$y(6) = 1 + (1\times 2) + \frac{1}{2}(1\times 2\times 4) + \frac{1}{6}(1\times 2\times 3\times 8) + \frac{1}{24}(1\times 2\times 3\times 4\times 16)$

$= 1+2+4+8+16 = 31.$

ⓒ Evaluate $I = \int_0^5 \frac{dx}{4x+5}$ by simpson's $(\frac{1}{3})^{rd}$ rule taking $n=10$.

Soln $a=0,\ b=5\ \ h = \frac{5}{10} = 0.5 \quad y = \frac{1}{4x+5}$

x;	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
y;	0.2	0.1429	0.1111	0.0909	0.0769	0.0667	0.0588	0.0526	0.0476	0.0434	0.04
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

Formula; $I = \frac{h}{3}\left[(y_0+y_{10}) + 4(y_1+y_3+y_5+y_7+y_9) + 2(y_2+y_4+y_6+y_8)\right]$

$= 0.4024$

8 ⓐ By N-R method, find the real root of $x\sin x + \cos x = 0$ which is near to π.

Soln given $f(x) = x\sin x + \cos x$, $f'(x) = x\cos x + \sin x - \sin x = x\cos x$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \pi - \frac{f(\pi)}{f'(\pi)} = 2.8233$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.8233 - \frac{f(2.8233)}{f'(2.8233)} = 2.7986$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7986 - \frac{f(2.7986)}{f'(2.7986)} = 2.7986$

$\therefore x = 2.7986$

(b) Using Lagrange's interpolation formula, fit a polynomial which passes through the points $(-1, 0)$, $(1, 2)$, $(2, 9)$ & $(3, 8)$ and hence find y at $x = 2.2$

Soln Let $x_0 = -1$, $x_1 = 1$ $x_2 = 2$ $x_3 = 3$

$y_0 = 0$ $y_1 = 2$ $y_2 = 9$ $y_3 = 8$, $x = 2.2$

Formula: $$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$f(x) = \frac{(x-1)(x-2)(x-3)}{(-2\times-3\times-4)} 0 + \frac{(x+1)(x-2)(x-3)}{(2\times-1\times-2)} \times 2$$
$$+ \frac{(x+1)(x-1)(x-3)}{(3\times1\times-1)} 9 + \frac{(x+1)(x-1)(x-2)}{(4\times2\times1)} \times 8$$

$$= 0 + \frac{2}{4}(x+1)(x-2)(x-3) - \frac{9}{3}(x+1)(x-1)(x-3) + \frac{8}{8}(x+1)(x-1)(x-2)$$
$$= \frac{1}{2}(x^3 - 4x^2 + x + 6) - 3(x^3 - 3x^2 - x + 3) + 1(x^3 - 2x^2 - x + 2)$$
$$= \left(\frac{1}{2} - 3 + 1\right)x^3 + \left(-\frac{4}{2} + 9 - 2\right)x^2 + \left(\frac{1}{2} - 1 + 3\right)x + \left(\frac{6}{2} - 9 + 2\right)$$
$$= -\frac{3}{2}x^3 + 5x^2 + \frac{5}{2}x - 4 = \frac{1}{2}(-3x^3 + 10x^2 + 5x - 8)$$

$f(2.2) = 9.728.$

(c) Evaluate $I = \int_4^{5.2} \log x \, dx$ using Simpson's $\left(\frac{3}{8}\right)^{th}$ rule taking $n = 6$

Soln Let $y = \log_e x$ $a = 4$, $b = 5.2$, $n = 6$ $h = 0.2$

x:	4	4.2	4.4	4.6	4.8	5	5.2
y:	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6470

$I = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 3y_3\right] = 1.8279$

Substitute & Simplify

9 (a) Using Taylor's Series method, find the value of $y(0.1)$ given $y' = 3x + y^2$, with $y(0) = 1$.

Soln: given $y' = 3x + y^2$, $x_0 = 0$, $y_0 = 1$

$y' = 3x + y^2$ . . . . $y'(0) = 1$

$y'' = 3 + 2yy'$ . . . . $y''(0) = 5$

$y''' = 2yy'' + 2(y')^2$ . . . . $y'''(0) = 12$

$y^{iv} = 2yy''' + 2y''y' + 4y'y''$ . . . . $y^{iv}(0) = 54$

$= 2yy''' + 6y'y''$

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y'(0) + \frac{(x-x_0)^2}{2!}y''(0) + \cdots$$

$$y(x) = 1 + \frac{x}{1}(1) + \frac{x^2}{2}(5) + \frac{x^3}{6}(12) + \frac{x^4}{24}(54)$$

$$= 1 + x + \frac{5x^2}{2} + 2x^3 + \frac{9}{4}x^4$$

$y(0.1) = 1.1272$ //

(b) Apply Runge-Kutta method, find $y(0.1)$, given $y' = \frac{y^2 - x^2}{y^2 + x^2}$, with $y(0) = 1$

Soln: given $f(x,y) = \frac{y^2 - x^2}{y^2 + x^2}$, $x_0 = 0$, $y_0 = 1$ $h = 0.1$

$K_1 = hf(x_0, y_0) = 0.1 f(0,1) = 0.1$

$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) = 0.1 f(0.05, 1.05) = 0.0995$

$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}) = 0.1 f(0.05, 1.0498) = 0.0995$

$K_4 = hf(x_0 + h, y_0 + K_3) = 0.1 f(0.1, 1.0995) = 0.0984$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.0994$

$\therefore y_1 = y_0 + K = 1.0994$ //

(c) Using Milne's P & C method, find $y(4.5)$ given $y' = \frac{2-y^2}{5x}$ and

x:	4.1	4.2	4.3	4.4
y:	1.0049	1.0097	1.0143	1.0187

Sol^n

x	y	$y' = \frac{2-y^2}{5x}$
$x_0 = 4.1$	$y_0 = 1.0049$	$y_0' = 0.0483$
$x_1 = 4.2$	$y_1 = 1.0097$	$y_1' = 0.0466$
$x_2 = 4.3$	$y_2 = 1.0143$	$y_2' = 0.0452$
$x_3 = 4.4$	$y_3 = 1.0183$	$y_3' = 0.0438$
$x_4 = 4.5$	$y_4 = ?$	

$$y_4^p = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') = 1.0230$$

$$y_4' = f(x_4, y_4^p) = \cancel{0.04238}\ 0.0424$$

$$y_4^c = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') = 1.0230$$

$$y_4' = f(x_4, y_4^c) = 0.0424$$

$$y_4^c = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') = 1.0230 \qquad \therefore y_4 = 1.0230$$

19 (a) Using Modified Euler's method, find $y(0.1)$, given $y' = x^2 + y$, with $y(0) = 1$, taking $h = 0.05$

Sol^n given $f(x, y) = x^2 + y$, $x_0 = 0$, $y_0 = 1$, $h = 0.05$, $x_1 = 0.05$, $x_2 = 0.1$

Step (1) $y_0' = y_0 + h f(x_0, y_0) = 1 + 0.05 f(0, 1) = 1.05$

$$y_1' = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_0')]$$

$$= 1 + 0.025[f(0, 1) + f(0.05, 1.05)] = 1.0513$$

$$y_1^2 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x, y_1')]$$

$$= 1 + 0.025[f(0, 1) + f(0.05, 1.0513)] = 1.0513$$

M.H.G

$\therefore \boxed{y_1 = 1.0513}$

Step ② $y_2^0 = y_1 + hf(x_1, y_1) = 1.0513 + 0.05f(0.05, 1.0513) = 1.1040$

$y_2^1 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^0)]$

$= 1.0513 + 0.025[f(0.05, 1.0513) + f(0.1, 1.1040)] = 1.1055$

$y_2^2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^1)]$

$= 1.0513 + 0.025[f(0.05, 1.0513) + f(0.1, 1.1055)] = 1.1055.$

$\therefore \boxed{y_2 = 1.1055}$

10(b) Using Runge-Kutta method, find $y(0.2)$, given that $y' = 3x + \frac{y}{2}$, with $y(0) = 1$

Solⁿ: Given $f(x, y) = 3x + \frac{1}{2}y$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$

$K_1 = hf(x_0, y_0) = 0.2f(0, 1) = 0.1$

$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) = 0.2f(0.1, 1.05) = 0.1650$

$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}) = 0.2f(0.1, 1.0825) = 0.1683$

$K_4 = hf(x_0 + h, y_0 + K_3) = 0.2f(0.2, 1.1683) = 0.2368$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.1672$

$\therefore y_1 = y_0 + K = 1.1672$ //

10 c. given $y' = \frac{(1+x^2)y^2}{2}$, $y(0)=1$, $y(0.1)=1.06$, $y(0.2)=1.12$, $y(0.3)=1.21$; evaluate $y(0.4)$ by using M P & C method.

Sol^n:

x	y	$y' = \frac{1}{2}(1+x^2)y^2$
$x_0 = 0$	$y_0 = 1$	$y_0' = 0.5$
$x_1 = 0.1$	$y_1 = 1.06$	$y_1' = 0.5353$
$x_2 = 0.2$	$y_2 = 1.12$	$y_2' = 0.5824$
$x_3 = 0.3$	$y_3 = 1.21$	$y_3' = 0.6545$
$x_4 = 0.4$	$y_4 = ?$	

$$y_4^p = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') = 1.2396$$

$$y_4' = f(x_4, y_4^p) = 0.7189$$

$$y_4^c = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') = 1.2506$$

$$y_4' = f(x_4, y_4^c) = 0.7253$$

$$y_4^c = 1.2508$$

$$\therefore y_4 \simeq 1.2508.$$

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