Math Olympiad

Algebra

1. For how many pairs of positive integers (x, y) is x + 3y = 1007

2. Rama was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3. She got 43 as the answer. What would have been her answer if she had solved the problem correctly?

3. The letters R, M, and O represent whole numbers. If $R \times M \times O = 240$, $R \times O + M = 46$, and $R + M \times O = 64$, what is the value of $R + M + O$?

4. Let $P(n) = (n+1)(n+3)(n+5)(n+7)(n+9)$ What is the largest integer that is a divisor of P(n) for all positive even integers n?

5. How many integer pairs (x, y) satisfy $x^2 + 4y^2 - 2xy - 2x - 4y - 8 = 0$?

6. Let $S_n = n^2 + 20n + 12$, n a positive integer. What is the sum of all possible values of n for which S_n is a perfect square?

7. Suppose that $4^{x_1} = 5$, $5^{x_2} = 6$, $6^{x_3} = 7, \ldots, 126^{x_{123}} = 127$, $127^{x_{124}} = 128$. What is the value of the product $x_1 x_2 \ldots x_{124}$?

8. If $\frac{1}{\sqrt{2011}+\sqrt{2012}} = \frac{\sqrt{m}-\sqrt{n}}{\sqrt{m+n}}$, where m and n are positive integers, what is the value of $m + n$?

9. If $a = b - c$, $b = c - d$, $c = d - a$, and $abcd \neq 0$, then what is the value of $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$?

10. How many non-negative integral values of x satisfy the equation

$$\left[\frac{x}{5}\right] = \left[\frac{x}{7}\right]?$$

(Here $[x]$ denotes the greatest integer less than or equal to x. For example, $[3.4] = 3$ and $[-2.3] = -3$.)

11. Let x_1, x_2, x_3 be the roots of the equation $x^3 + 3x + 5 = 0$. What is the value of the expression

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\left(x_3 + \frac{1}{x_3}\right)?$$

12. What is the sum of the squares of the roots of the equation

$$x^2 - 7[x] + 5 = 0?$$

(Here $[x]$ denotes the greatest integer less than or equal to x. For example, $[3.4] = 3$ and $[-2.3] = -3$.

Probability

1. A postman has to deliver ive letters to five different houses. Mischievously, he posts one letter through each door without looking to see if it is the correct address. In how rnany different way could he do this so that exactly two of the five houses receive the correct letters?

Geometry

1. PS is a line segment of length 4 and O is the midpoint of PS. A semicircular arc is drawn with PS as diameter. Let X be the midpoint of this arc. Q and R are points on the arc PXS such that QR is parallel to PS and the semicircular arc drawn with QR as diameter is tangent to PS. What is the area of the region $QXROQ$ bounded by the two semicircular arcs?

2. O and I are the circumcentre and incentre of $\triangle ABC$ respectively. Suppose O lies in the interior of $\triangle ABC$ and I lies on the circle passing through B, O, and C. What is the magnitude of $\angle BAC$ in degrees?

3. In $\triangle ABC$, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is the length of the segment BD?

4. In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on line segment CD so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. What is the area of $\triangle ABE$?

5. A triangle with perimeter 7 has integer side lengths. What is the maximum possible area of such a triangle?

6. $ABCD$ is a square and $AB = 1$ Equilateral triangles AYB and CXD are drawn such that X and Y are inside the square.What is the length of XY?

Functions

1. Let N be the set of natural numbers. Suppose $f : N \to N$ is a function satisfying the following conditions:

   (a) $f(mn) = f(m) f(n)$,

   (b) $f(m) < f(n)$ if $m < n$,

   (c) $f(2) = 2$.

   What is the value of $\sum_{k=1}^{20} f(k)$?