Operations on sets :-

i) Union ii) Intersection iii) Complement

i) Union :- A, B ; then $A \cup B$ contains $\{x \mid x \in A \text{ or } x \in B\}$.
ii) Intersection :- A, B ; then $A \cap B$ contains $\{x \mid x \in A \text{ and } x \in B\}$.
iii) Complement : A, U ; then $A' = \{x \mid x \in U, x \notin A\}$

Difference Of a Set :-

If A, B ; then the set of all elements which belongs to A but does not belong to B is called difference of A.
Represented by A - B or $A \setminus B$.

$A \setminus B = \{x \mid x \in A, x \notin B\}$
$B \setminus A = \{x \mid x \in B, x \notin A\}$

Symmetric Difference :-

A, B ; then symmetric difference is given as $A \setminus B \cup B \setminus A$.

De-Morgan's law :- A, B ;
then i) $(A \cup B)^c = A^c \cap B^c$
ii) $(A \cap B)^c = A^c \cup B^c$

Cartesian Product :-

A cartesian product of A & B is denoted by :
$A \times B = \{(a,b) \mid a \in A, b \in B\}$ has ordered pair (a, b) such that $a \in A$, $b \in B$.

Relation :- $R : A \to B$ is a subset of cartesian Product and also the subset of Ordered Pair.

Fuzzy Set Theory was developed by Zadeh in 1965. Fuzzy means uncertain or which is not clear (ambiguous). Fuzzy logic is a logic which is not precise but it can be defined by looking at the surroundings.

Fuzzy set deal with data that we collected in a precise manner. We (human beings) are more powerful than the computers in handling imprecise data. Fuzzy logic helps in conveying our reasoning capacity to a computer system so that the computer system will be able to solve such type of problems which involves parameters like height, weight, temperature etc.

Membership Function :-

Denoted by $\mu_A(x)$,
where $x \in X$ or U (Universal set)

Defined as
$\mu_A(x) = 1$, if $x \in A$
$= 0$, if $x \notin A$.

where $A(x) \in \{0, 1\}$, $A \subseteq U$.

Eg Let X = set of Natural Numbers
$A \subset X$ ; set of even numbers.

$\therefore 2 \in A$
$\therefore$ Degree of belongingness for 2 = 1
or $A(2) = 1$

$3 \notin A$
$\therefore$ Degree for 3 = 0
or $A(3) = 0$

$\therefore$ set is represented as :- $\{(1,0), (2,1), (3,0), (4,1) \ldots\ldots\}$

It should be noted that every member of $X$ is assigned to 0 or 1.

If $X$ is a universal set,

$x_i$ be element of $X$

then Fuzzy set $\tilde{A}$ is defined as

$\tilde{A} = \{ (x_i, A(x_i)) ; x_i \in A \}$ where $x_i \in \{0,1\}$

Eg. 14.9°C is a cold temperature while 15.1°C is a normal temperature. In fuzzy system boundaries are soft.

For $H = \{x \mid 25 < x\}$

$C = \{x \mid x \le 15\}$

$N = \{x \mid 15 < x \le 25\}$

Eg.

In Classic Set Theory, a person whose Height is 5.9 is not tall. But a person whose height is 6.1 is tall. This difficulty is resolved in a fuzzy set. In a fuzzy set, person whose height is 6.1 as well as 5.9 are considered to be tall.

Cardinality of set :- cardinality is number of elements in a set while in fuzzy set if it is finite then cardinality = $\Sigma$ of all members of set $A$.

Fuzzy cardinality = $|\tilde{A}|$

$= \Sigma A(x_i), \quad x_i \in X$

Relative Cardinality $= \dfrac{|\tilde{A}|}{|X|} = \dfrac{\Sigma A(x_i)}{|X|}$ ← No of elements in $X$

Eg. Let $X = \{1,2,3,4\}$

$A = \{(1,0.1),(2,0.9),(3,0.7),(4,0.2)\}$

Then scalar cardinality $= \Sigma A(x_i) = 1.9$

Relative Cardinality $= \dfrac{1.9}{4} = 0.475$.

Power set :- set of all Fuzzy sets on $X$ is called a Fuzzy Power set, denoted by $F(X)$, $x \in X$.

If $X$ contains $n$ elements, then

$F(X)$ contains $= 2^n$ elements. (For crisp set)

For fuzzy set, there ~~can be~~ are infinite elements.

$\alpha$ cut :-

Let $\tilde{A}$ be a fuzzy set defined on $X$. Then $\alpha$-cut denoted by $\alpha\tilde{A}$, given by

$\alpha\tilde{A} = \{ x \mid x \in A ; A(x) \ge \alpha , \alpha \in [0,1] \}$

Strong $\alpha$-cut :-

Let $\tilde{A}$ be a fuzzy set defined on $X$. Then strong $\alpha$-cut

$^{+}\alpha A = \{ x \mid x \in A ; A(x) > \alpha , \alpha \in [0,1] \}$

Height :-

Height of a fuzzy set is maximum value of membership function attended in a fuzzy set.

Normal Fuzzy set :-

If height of fuzzy set is 1 then fuzzy set is said to be normal.

Level set :-

let $\tilde{A}$ be fuzzy set defined on $X$. Then level of fuzzy set is

$${}^{\alpha}\tilde{A} = \{x \mid A(x) = \alpha, \alpha \in (0,1]\}$$

= collection of membership function which are not 0.

Support of a Fuzzy set :-

let $\tilde{A}$ be fuzzy set on $X$. Then support of $\tilde{A}$ is

$$\text{support of } \tilde{A} = \{x \mid A(x) > 0, x \in X\}$$
$$= {}^{0+}\tilde{A}$$

$$\text{Core } \tilde{A} = \{x \mid A(x) = 1\}$$

1) Find ${}^{\alpha}\tilde{A}$, ${}^{+\alpha}\tilde{A}$ of fuzzy set whose membership function are given as $A(x) = \frac{x}{x+1}$, $B(x) = 1 - \frac{x}{10}$, $x \in X = \{0, \ldots 10\}$ & $\alpha = 0.2, 0.4, 0.7, 0.9$

$\tilde{A} = \{(0,0), (1,0.5), (2,0.66), (3,0.75), (4,0.8), (5,0.83), (6,0.85), (7,0.87), (8,0.88), (9,0.9), (10,0.91)\}$

${}^{.2}\tilde{A} = \{1,2,3,4,5,6,7,8,9,10\}$ ${}^{+.2}\tilde{A} = \{1,2 \ldots 10\}$

${}^{.4}\tilde{A} = \{1,2, \ldots 10\}$ ${}^{+.4}\tilde{A} = \{1,2, \ldots 10\}$

${}^{.7}\tilde{A} = \{3,4 \ldots 10\}$ ${}^{+.7}\tilde{A} = \{3,4 \ldots 10\}$

${}^{.9}\tilde{A} = \{9,10\}$ ${}^{+.9}\tilde{A} = \{10\}$

$\tilde{B} = \{(0,1), (1,0.9), (2,0.8), (3,0.7), (4,0.6), (5,0.5), (6,0.4), (7,0.3), (8,0.2), (9,0.1), (10,0)\}$

${}^{.2}\tilde{B} = \{0,1 \ldots 8\}$ ${}^{+.2}\tilde{B} = \{0,1 \ldots 7\}$

${}^{.4}\tilde{B} = \{0,1 \ldots 6\}$ ${}^{+.4}\tilde{B} = \{0,1 \ldots 5\}$

${}^{.7}\tilde{B} = \{0,1,2,3\}$ ${}^{+.7}\tilde{B} = \{0,1,2\}$

${}^{.9}\tilde{B} = \{0,1\}$ ${}^{+.9}\tilde{B} = \{0\}$

2) Find $\alpha$-cut & strong $\alpha$-cut of fuzzy sets A & B whose membership fns are given as follows

$A(x) = \frac{x}{x+2}$ $B(x) = 2^{-x}$ $x \in X = \{0,1,2,3,4,5,6,7,8,9,10\}$

& $\alpha = 0.3, 0.7, 0.5$

$\tilde{A} = \{(0,0), (1,0.33), (2,0.5), (3,0.6), (4,0.66), (5,0.71), (6,0.75), (7,0.77), (8,0.8), (9,0.81), (10,0.83)\}$

${}^{.3}\tilde{A} = \{1,2 \ldots 10\}$ ${}^{+.3}\tilde{A} = \{1,2 \ldots 10\}$

${}^{.5}\tilde{A} = \{2,3 \ldots 10\}$ ${}^{+.5}\tilde{A} = \{3 \ldots 10\}$

${}^{.7}\tilde{A} = \{5, \ldots 10\}$ ${}^{+.7}\tilde{A} = \{6, \ldots 10\}$

$\tilde{B} = \{(0,1), (1,0.5), (2,0.25), (3,0.12), (4,0.06), (5,0.03), (6,0.0156), (7,0.0078), (8,0.0039), (9,0.00195), (10,0.00098)\}$

${}^{.3}\tilde{A} = \{0,1\}$ ${}^{+.3}\tilde{A} = \{0,1\}$

${}^{.5}\tilde{A} = \{0,1\}$ ${}^{+.5}\tilde{A} = \{0\}$

${}^{.7}\tilde{A} = \{0\}$ ${}^{+.7}\tilde{A} = \{0\}$

3) $A(x) = \frac{100}{1+10x}$ $x = \{10,20,30,40,50\}$

$\alpha = 0.1, 0.99$

$B(x) = \frac{20x}{20x+5}$ $x = \{10,20,30,40,50\}$ $B(x) = \frac{4x}{4x+1}$

$\tilde{A} = \{(10,0.99), (20,0.49), (30,0.33), (40,0.249), (50,0.199)\}$

${}^{.1}\tilde{A} = \{10,20,30,40,50\}$ ${}^{+.1}\tilde{A} = \{10,20,30,40,50\}$

${}^{.99}\tilde{A} = \{10\}$ ${}^{+.99}\tilde{A} = \{10\}$

$\tilde{B} = \{(10,0.975), (20,0.987), (30,0.992), (40,0.9937), (50,0.9950)\}$

${}^{.1}\tilde{A} = \{10,20,30,40,50\}$ ${}^{+.1}\tilde{A} = \{10,20,30,40,50\}$

${}^{.99}\tilde{A} = \{30,40,50\}$ ${}^{+.99}\tilde{A} = \{30,40,50\}$

Level set :-

Let $\tilde{A}$ be fuzzy set defined on $X$. Then level of fuzzy set is

$$\Lambda\tilde{A} = \{x \mid A(x) = \alpha, \alpha \in (0,1]\}$$

= collection of membership function which are not 0.

Support of a Fuzzy set :-

Let $\tilde{A}$ be fuzzy set on $X$. Then support of $\tilde{A}$ is

$$\text{support of } \tilde{A} = \{x \mid A(x) > 0, x \in X\}$$
$$= 0^{+}\tilde{A}$$

$$\text{Core } \tilde{A} = \{x \mid A(x) = 1\}$$

1) Find $\alpha\tilde{A}$, $+\alpha\tilde{A}$ of fuzzy set whose membership function are given as $A(x) = \dfrac{x}{\boxed{x+1}}$, $B(x) = 1 - \dfrac{x}{10}$, $x \in X = \{0, \ldots, 10\}$ & $\alpha = 0.2, 0.4, 0.7, 0.9$

→ $\tilde{A} = \{(0,0), (1,0.5), (2,0.66), (3,0.75), (4,0.8), (5,0.83), (6,0.85), (7,0.87), (8,0.88), (9,0.9), (10,0.91)\}$

$.2\tilde{A} = \{1,2,3,4,5,6,7,8,9,10\}$ $\quad +.2\tilde{A} = \{1,2 \ldots 10\}$

$.4\tilde{A} = \{1,2, \ldots 10\}$ $\quad +.4\tilde{A} = \{1,2, \ldots 10\}$

$.7\tilde{A} = \{3,4 \ldots 10\}$ $\quad +.7\tilde{A} = \{3,4 \ldots 10\}$

$.9\tilde{A} = \{9,10\}$ $\quad +.9\tilde{A} = \{10\}$

$\tilde{B} = \{(0,1), (1,0.9), (2,0.8), (3,0.7), (4,0.6), (5,0.5), (6,0.4), (7,0.3), (8,0.2), (9,0.1), (10,0)\}$

$.2\tilde{B} = \{0,1 \ldots 8\}$ $\quad +.2\tilde{B} = \{0,1 \ldots 7\}$

$.4\tilde{B} = \{0,1 \ldots 6\}$ $\quad +.4\tilde{B} = \{0,1 \ldots 5\}$

$.7\tilde{B} = \{0,1,2,3\}$ $\quad +.7\tilde{B} = \{0,1,2\}$

$.9\tilde{B} = \{0,1\}$ $\quad +.9\tilde{B} = \{0\}$

2) Find $\alpha$-cut & strong $\alpha$-cut of fuzzy sets A & B whose membership fns are given as follows

$A(x) = \dfrac{x}{x+2}$ $\quad B(x) = 2^{-x}$ $\quad x \in X = \{0,1,2,3,4,5,6,7,8,9,10\}$

& $\alpha = 0.3, 0.7, 0.5$

→ $\tilde{A} = \{(0,0), (1,0.33), (2,0.5), (3,0.6), (4,0.66), (5,0.71), (6,0.75), (7,0.77), (8,0.80), (9,0.81), (10,0.83)\}$

$.3\tilde{A} = \{1,2 \ldots 10\}$ $\quad +.3\tilde{A} = \{1,2 \ldots 10\}$

$.5\tilde{A} = \{2,3 \ldots 10\}$ $\quad +.5\tilde{A} = \{3 \ldots 10\}$

$.7\tilde{A} = \{5, \ldots 10\}$ $\quad +.7\tilde{A} = \{6, \ldots 10\}$

$\tilde{B} = \{(0,1), (1,0.5), (2,0.25), (3,0.12), (4,0.06), (5,0.03), (6,0.0156), (7,0.0078), (8,0.0034), (9,0.00195), (10,0.00098)\}$

$.3\tilde{A} = \{0,1\}$ $\quad +.3\tilde{A} = \{0,1\}$

$.5\tilde{A} = \{0,1\}$ $\quad +.5\tilde{A} = \{0\}$

$.7\tilde{A} = \{0\}$ $\quad +.7\tilde{A} = \{0\}$

3) $A(x) = \dfrac{100}{1+10x}$ $\quad x = \{10,20,30,40,50\}$

$\alpha = 0.1, 0.99$

$B(x) = \dfrac{20x}{20x+5}$ $\quad x = \{10,20,30,40,50\}$ $\quad B(x) = \dfrac{4x}{4x+1}$

→ $\tilde{A} = \{(10,0.99), (20,0.49), (30,0.33), (40,0.249), (50,0.199)\}$

$.1\tilde{A} = \{10,20,30,40,50\}$ $\quad +.1\tilde{A} = \{10,20,30,40,50\}$

$.99\tilde{A} = \{10\}$ $\quad +.99\tilde{A} = \{10\}$

$\tilde{B} = \{(10,0.975), (20,0.987), (30,0.992), (40,0.9937), (50,0.9950)\}$

$.1\tilde{A} = \{10,20,30,40,50\}$ $\quad +.1\tilde{A} = \{10,20,30,40,50\}$

$.99\tilde{A} = \{30,40,50\}$ $\quad +.99\tilde{A} = \{30,40,50\}$

4) Find $\alpha$-cut & $+\alpha$-cut for fuzzy set whose membership function given by

$$B(x) = [1+(x-10)^2]^{-1}, \quad \alpha = 0.3,\ 0.5,\ 0.8$$

→ As by definition of $\alpha$-cut & $+\alpha$ cut, we have,

$$B(x) \geq \alpha, \quad +B(x) > \alpha$$

Now, $[1+(x-10)^2]^{-1} \geq \alpha$

i.e. $\dfrac{1}{[1+(x-10)^2]^{-1}} \geq \alpha \Rightarrow (x-10)^2 \leq \dfrac{1}{\alpha} - 1$

or $\pm(x-10) \leq \sqrt{\frac{1}{\alpha}-1}$

$\Rightarrow x-10 \leq \sqrt{\frac{1}{\alpha}-1}$ or $-x+10 \leq \sqrt{\frac{1}{\alpha}-1}$

i.e. $x \leq 10+\sqrt{\frac{1}{\alpha}-1}$ or $x \geq 10-\sqrt{\frac{1}{\alpha}-1}$

Hence $10-\sqrt{\frac{1}{\alpha}-1} \leq x \leq 10+\sqrt{\frac{1}{\alpha}-1}$

or $x \in \left[10-\sqrt{\frac{1}{\alpha}-1},\ 10+\sqrt{\frac{1}{\alpha}-1}\right]$

Hence, $\alpha B = \left[10-\sqrt{\frac{1}{\alpha}-1},\ 10+\sqrt{\frac{1}{\alpha}-1}\right]$

$+\alpha B = \left(10-\sqrt{\frac{1}{\alpha}-1},\ 10+\sqrt{\frac{1}{\alpha}-1}\right)$

Now,

$\cdot 3\alpha = [8.472, 11.528]$ $\quad +\cdot 3\alpha = (8.472, 11.528)$

$\cdot 8\alpha = [9.5, 10.5]$ $\quad +\cdot 8\alpha = (9.5, 10.5)$

$\cdot 5\alpha = [9, 11]$ $\quad +\cdot 5\alpha = (9, 11)$

5) Find $\alpha$ & $+\alpha$ for a fuzzy A whose membership fn given as

$$A(x) = (x+3)/3, \quad -3 < x \leq 0$$
$$= (3-x)/3, \quad 0 \leq x \leq 3$$
$$= 0 \quad \text{otherwise.}$$

$\alpha = 0.5, 0.9$

→ $A(x) \geq \alpha$, $\quad +A(x) > \alpha$.

Now, $\dfrac{(x+3)}{3} \geq \alpha$, $\quad \dfrac{3-x}{3} \geq \alpha$

$x+3 \geq 3\alpha$ $\quad 3-x \geq 3\alpha$

$x \geq 3\alpha-3$ $\quad \Rightarrow x \leq 3-3\alpha$

$\therefore x \in [3\alpha-3, 0]$ $\quad \therefore x \in [0, 3-3\alpha]$

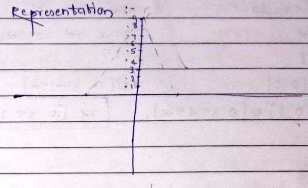
for $-3 < x \leq 0$ $\quad$ for $0 \leq x < 3$

$\therefore x \in [3\alpha-3, 3-3\alpha]$

$\therefore \cdot 5\tilde{A} = [-1.5, 1.5]$ $\quad +\cdot 5\tilde{A} = (-1.5, 1.5)$

$\cdot 9\tilde{A} = [-0.3, 0.3]$ $\quad +\cdot 9\tilde{A} = (-0.3, 0.3)$

Graphical Representation :-



6) Find $\alpha$ & $+\alpha$ for $A(x) = \dfrac{x+1}{2}$ $\quad -1 < x \leq 1$ $\quad = 0$ otherwise

$= \dfrac{3-x}{2}$, $\quad 1 < x < 3$. $\quad \alpha = 0.2, 0.5, 0.8$

→ $A(x) \geq \alpha$.

$\therefore \dfrac{x+1}{2} \geq \alpha$ $\quad \dfrac{3-x}{2} \geq \alpha$

$\therefore x \geq 2\alpha - 1$ $\quad x \leq 3-2\alpha$

$\therefore x \in [2\alpha-1, 0]$ $\quad x \in [0, 3-2\alpha]$

for $-1 < x \leq 1$ $\quad$ for $1 < x < 3$

$\therefore x \in [2\alpha-1, 3-2\alpha]$

$\therefore \cdot 2\alpha = [-0.6, 2.6]$ $\quad +\cdot 2\alpha = (-0.6, 2.6)$

$\cdot 5\alpha = [0, 2]$ $\quad +\cdot 5\alpha = (0, 2)$

$\cdot 8\alpha = [0.6, 1.4]$ $\quad +\cdot 8\alpha = (0.6, 1.4)$

8) $X = \{1,2,3,4\}$, $\tilde{A} = \{(1,2),(2,5),(3,8),(4,1)\}$

$\tilde{B} = \{(1,3),(2,6),(3,9),(4,1)\}$ Find ~~U~~, $\cap$

→ $(\tilde{A}\cap\tilde{B}) : X \rightarrow [0,1]$ is given by

$(\tilde{A}\cap\tilde{B}) = \{(1,2),(2,5),(3,8),(4,1)\}$

9) $\tilde{A} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\}$ $\tilde{B} = \{(x_1, 0.8), (x_2, 0.2), (x_3, 1)\}$

→ $(\tilde{A}\cap\tilde{B}) = \{(x_1, 0.5), (x_2, 0.2), (x_3, 0)\}$

10) $X = \{1,2,3,4\}$, $\tilde{A} = \{(1,2),(2,5),(3,8),(4,1)\}$

$\tilde{B} = \{(1,3),(2,6),(3,9),(4,1)\}$ Find $\cup$

→ $(\tilde{A}\cup\tilde{B}) = \{(1,3),(2,6),(3,9),(4,1)\}$

→ $(\tilde{A}\cup\tilde{B}) = \{(x_1, 0.8), (x_2, 0.7), (x_3, 1)\}$

Degree of subset :-

Let A & B be two fuzzy sets defined on X.
Then degree of subsethood for A is defined as :-

$$S(\tilde{A},\tilde{B}) = \frac{|\tilde{A}\cap\tilde{B}|}{|\tilde{A}|}$$

$$= \sum_{x\in X} (\tilde{A}\cap\tilde{B})(x)$$

& $|\tilde{A}| = \sum_{x \in X} A(x)$

Similarly degree of subsethood for B is defined as

$$S(\tilde{A},\tilde{B}) = \frac{|\tilde{B}\cap\tilde{A}|}{|\tilde{B}|} = \sum_{x\in X} (\tilde{B}\cap\tilde{A})(x)$$

1) If $\tilde{A} = \{\frac{1}{2}, \frac{.5}{3}, \frac{.3}{4}, \frac{.2}{5}\}$

$\tilde{B} = \{\frac{.5}{2}, \frac{.7}{3}, \frac{.2}{4}, \frac{.4}{5}\}$

Find $\tilde{A}\cup\overline{\tilde{B}}$ and $\overline{\tilde{A}\cap\tilde{B}}$

→ As, $\tilde{B} = \{\frac{.5}{2}, \frac{.7}{3}, \frac{.2}{4}, \frac{.4}{5}\}$

$\overline{\tilde{B}} = 1 - \tilde{B} = \{\frac{.5}{2}, \frac{.3}{3}, \frac{.8}{4}, \frac{.6}{5}\}$

$\therefore \tilde{A}\cup\overline{\tilde{B}} = \{1/2, .5/3, .8/4, .6/5\}$

Now, $\tilde{A}\cap\tilde{B} = \{0.5/2, .5/3, .2/4, .2/5\}$

$\therefore \overline{\tilde{A}\cap\tilde{B}} = \{0.5/2, 0.5/3, 0.8/4, 0.8/5\}$

2) $\tilde{A}, \tilde{B}$, $A(x) = \frac{2x}{2x+5}$, $B(x) = \frac{x}{x+1}$ for $x \in X$, $X = \{6,7,8,9,10\}$

Find i) $\tilde{A}\cup(\overline{\tilde{A}\cap\tilde{B}})$ ii) $\alpha[\tilde{A}\cup(\overline{\tilde{A}\cap\tilde{B}})]$ if $\alpha = 0.9$

iii) Scalar cardinality of $\tilde{A}\cup(\overline{\tilde{A}\cap\tilde{B}})$

→ $\tilde{A} = \left\{\frac{6}{0.70588}, \frac{7}{0.7368}, \frac{8}{0.76190}, \frac{9}{0.7826}, \frac{10}{0.8}\right\}$

$\tilde{B} = \left\{\frac{6}{0.8571}, \frac{7}{0.875}, \frac{8}{0.888}, \frac{9}{0.9}, \frac{10}{0.909}\right\}$

i) $\tilde{A}\cap\tilde{B} = \left\{\frac{6}{0.70588}, \frac{7}{0.7368}, \frac{8}{0.7619}, \frac{9}{0.7826}, \frac{10}{0.8}\right\}$

$\therefore \overline{\tilde{A}\cap\tilde{B}} = 1 - \tilde{A}\cap\tilde{B}$

$= \left\{\frac{6}{0.29412}, \frac{7}{0.2632}, \frac{8}{0.2381}, \frac{9}{0.2174}, \frac{10}{0.2}\right\}$

$\therefore \tilde{A}\cup(\overline{\tilde{A}\cap\tilde{B}}) = \left\{\frac{6}{0.70588}, \frac{7}{0.7368}, \frac{8}{0.7619}, \frac{9}{0.7826}, \frac{10}{0.8}\right\} = \tilde{A}$

ii) Now, $\alpha(\tilde{A}\cup(\overline{\tilde{A}\cap\tilde{B}})) = \alpha(\tilde{A}) = \{\ \}$ for $\alpha = 0.9$

iii) Scalar cardinality of $\tilde{A}\cup(\overline{\tilde{A}\cap\tilde{B}})$

$\overline{\tilde{A}\cup(\tilde{A}\cap\tilde{B})} = \left\{\frac{6}{0.29412}, \frac{7}{0.2632}, \frac{8}{0.2381}, \frac{9}{0.2174}, \frac{10}{0.2}\right\}$

29412
2632
2381
2174
2
1.21282

$\therefore$ Scalar cardinality = 1.2129.

CONVEX FUZZY SET:-

A fuzzy set is said to be convex defined on $X$ if all $\alpha$-cuts are convex where $\alpha \in [0,1]$.

Theorem :-

Let $A$ be a set defined on $X$. Then $\tilde{A}$ is convex, {when defined on $R$}. if & only if

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min(A(x_1), A(x_2));$$

for $\forall\, x_1, x_2 \in R$ and $\lambda \in [0,1]$.

Case I :- Let $\tilde{A}$ be convex on $R$.

To show that,

$A(\lambda x_1 + (1-\lambda)x_2) \geq \min(A(x_1), A(x_2))$ for $x_2, x_1 \in R$

Let $\alpha = A(x_1) \leq A(x_2)$

$\therefore \alpha = A(x_1)$

$\Rightarrow x_1 \in {}^{\alpha}A$.

Similarly, $x_2 \in {}^{\alpha}A$

Hence, $x_1, x_2 \in {}^{\alpha}A$.

$\therefore A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha$, or $\min(A(x_1), A(x_2))$

for $x_1, x_2 \in R$

Case II :- Consider

$A(\lambda x_1 + (1-\lambda)x_2) \geq \min(A(x_1), A(x_2))$, $x_1, x_2 \in R$

To show $\tilde{A}$ is convex, or

to show $\alpha$-cuts are convex, $\alpha \in [0,1]$.

$\lambda(x_1) + (1-\lambda)x_2 \in {}^{\alpha}A$. , $x_1, x_2 \in {}^{\alpha}A$.

$A(x_1) \geq \alpha$

$A(x_2) \geq \alpha$

$\Rightarrow A(\lambda x_1 + (1-\lambda)x_2) \geq \min(A(x_1), A(x_2))$

$\geq \min(\alpha, \alpha) = \alpha$.

Hence,

$A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha$ , $\alpha \in [0,1]$

$\Rightarrow {}^{\alpha}\tilde{A}$ is convex for all $x_1, x_2 \in R$

$\therefore \tilde{A}$ is convex.

Special Fuzzy set

Let $A$ be fuzzy set defined on $X$ then special fuzzy set given as ${}_{\alpha}\tilde{A}$ & is defined as:

$${}_{\alpha}\tilde{A} \begin{cases} = \alpha, & x \in {}^{\alpha}A \\ = 0, & x \notin {}^{\alpha}A. \end{cases}$$

First Decomposition Theorem:-

For every fuzzy set $\tilde{A} \in F(X)$; where $F(X)$ is a family of fuzzy sets.

Then $\bigcup_{\alpha} {}_{\alpha}\tilde{A} = \tilde{A}$

where ${}_{\alpha}\tilde{A}$ is a special fuzzy set.

Proof:- Let $A(x) = a$ $\quad a \in [0,1]$

That is

$\bigcup_{\alpha} A = \sup_{\alpha} A$ $\qquad [0,1] = [0,a] \cup (a,1]$

$\alpha \in [0,1] \quad \alpha \in [0,1]$

$\sup_{\alpha} A \because \alpha \in [0,a] \cup (a,1]$

$$= \operatorname{Max}\left\{ \sup_{\alpha \in [0,a]} A, \sup_{\alpha \in (a,1]} A \right\}$$

When $\alpha \in [0,a]$ $\quad 0 \leq \alpha \leq a$.

$A(x) \geq \alpha$ ---- (1)

$\Rightarrow \sup_{\alpha} A(x) = a$ $\quad \alpha \in [0,a]$

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ii) When $\alpha \in (a,1]$

$A(x) < \alpha$

$\Rightarrow \sup_\alpha A = 0, \quad$ as $x \notin A$. ...... (2)
$\alpha \in (a,1]$

Hence from (1) & (2)

$\sup_\alpha A = \text{Max}\{a, 0\} = a.$
$\alpha \in [0,a] \cup (a,1]$

Fuzzy Cardinality

Fuzzy Cardinality is the Probablity of number of elements in $\alpha$cut is $\alpha$.

$$\text{Fuzzy Cardinality} = \frac{\alpha}{|{}^\alpha A|}, \quad \forall \alpha \in \tilde{A}$$

$$\wedge\tilde{A} = \{\alpha \mid \alpha \in A(x), \alpha \in (0,1]\}.$$

1.) $\tilde{A} = \left\{ \frac{0}{x_1}, \frac{.33}{x_2}, \frac{.5}{x_3}, \frac{.71}{x_4}, \frac{.21}{x_5}, \frac{.41}{x_6}, \frac{.11}{x_7} \right\}$

Find fuzzy cardinality of $\tilde{A}$.

$\wedge\tilde{A} = \{.33, .5, .71, .21, .41, .11\}$ (Exclude 0)

$.33A = \{x_2, x_3, x_4, x_6\} = 4$

$.5A = \{x_4, x_3\} = 2$

$.71A = 1$

$.21A = 5$

$.41A = 3$

$.11A = 6$

$\therefore \frac{\alpha}{|{}^\alpha A|} = \left\{ \frac{.33}{4}, \frac{.5}{2}, \frac{.71}{1}, \frac{.21}{5}, \frac{.41}{3}, \frac{.11}{6} \right\}$

2.) Let $\tilde{A}$ & $\tilde{B}$ defined on $X$.

$\tilde{A} = \left\{ \frac{0}{1}, \frac{.2}{1.5}, \frac{.35}{2}, \frac{.15}{2.5}, \frac{.5}{3}, \frac{.25}{3.5}, \frac{.4}{4} \right\}$

$\tilde{B} = \left\{ \frac{1}{1}, \frac{.15}{1.5}, \frac{.2}{2}, \frac{.35}{2.5}, \frac{.4}{3}, \frac{.125}{3.5}, \frac{.4}{4} \right\}$

Find i) $(\tilde{A} \cap \bar{\tilde{B}}) \cup \tilde{A}$

ii) $\overline{\tilde{A \cup B}}$

iii) Find Fuzzy cardinality of $\overline{\tilde{A \cup B}}$

$\bar{\tilde{B}} = \left\{ \frac{0}{1}, \frac{.85}{1.5}, \frac{.8}{2}, \frac{.65}{2.5}, \frac{.6}{3}, \frac{.875}{3.5}, \frac{.6}{4} \right\}$

i) $(\tilde{A} \cap \bar{\tilde{B}}) = \left\{ \frac{0}{1}, \frac{.2}{1.5}, \frac{.35}{2}, \frac{.15}{2.5}, \frac{.5}{3}, \frac{.25}{3.5}, \frac{.4}{4} \right\}$

$\therefore (\tilde{A} \cap \bar{\tilde{B}}) \cup \tilde{A} = \left\{ \frac{0}{1}, \frac{.2}{1.5}, \frac{.35}{2}, \frac{.15}{2.5}, \frac{.5}{3}, \frac{.25}{3.5}, \frac{.4}{4} \right\}$

ii) $\tilde{A \cup B} = \left\{ \frac{1}{1}, \frac{.2}{1.5}, \frac{.35}{2}, \frac{.35}{2.5}, \frac{.5}{3}, \frac{.25}{3.5}, \frac{.4}{4} \right\}$

$\therefore \overline{\tilde{A \cup B}} = \left\{ \frac{0}{1}, \frac{.8}{1.5}, \frac{.65}{2}, \frac{.65}{2.5}, \frac{.5}{3}, \frac{.75}{3.5}, \frac{.6}{4} \right\}$

iii) Now, $\wedge\tilde{A} = \{.8, .65, .5, .75, .6\}$

$.8A = \{1.5\} = 1$

$.65A = \{.8, .65, .75\} = 3$

$.5A = \{.8, .65, .75, .6\} = 5$

$.75A = \{.8, .75\} = 2$

$.6A = \{.8, .65, .75, .6\} = 4$

Special Fuzzy set on strong $\alpha$-cut :-

let $\tilde{A}$ be a fuzzy set defined on X.

The strong fuzzy special set is defined as :-

$$ {}_{+\alpha}\tilde{A} = \alpha \quad , \; x \in {}^{+\alpha}\tilde{A} $$
$$ = 0 \quad , \; x \notin {}^{+\alpha}\tilde{A} $$

Second Decomposition Theorem :-

For every $\tilde{A} \in F(x)$, where $F(x)$ is the family of fuzzy sets, then for every $\tilde{A}$

$$ \tilde{A} = \bigcup_{+\alpha} \tilde{A} \quad , \alpha \in [0,1] $$

Proof :-

Consider

$$ \bigcup_{\substack{+\alpha \\ \alpha \in [0,1]}} \tilde{A} = \sup_{\substack{+\alpha \\ \alpha \in [0,1]}} \tilde{A} $$

Hence

$$ \sup_{\substack{+\alpha \\ \alpha \in [0,1] \\ = [0,a) \cup [a,1]}} \tilde{A} = \operatorname{Max} \left\{ \sup_{\substack{+\alpha \\ \alpha \in [0,a)}} A(x) , \sup_{\substack{+\alpha \\ \alpha \in [a,1]}} A(x) \right\} $$

Now, consider $\alpha \in [0,a)$

i.e. $0 \leq \alpha < a$

or $A(x) > \alpha$

$\Rightarrow x \in {}^{+\alpha}\tilde{A}$.

$\Rightarrow \sup\limits_{\alpha \in [0,a)} A(x) = a$

And when $\alpha \in [a,1]$

$A(x) < \alpha$

$\Rightarrow x \notin {}^{+\alpha}\tilde{A}$. $\Rightarrow \sup\limits_{\alpha \in [a,1]} A(x) = 0$.

$$ \therefore \bigcup_{\substack{+\alpha \\ \alpha \in [0,1]}} A(x) = \operatorname{Max}\{a, 0\} $$
$$ = a $$
$$ = \tilde{A}(x) $$
$$ = \tilde{A}. $$

1.) Verify First Decomposition Theorem ~~for~~

$$ \tilde{A} = \left\{ \frac{0.2}{x1}, \frac{0.4}{x2}, \frac{0.6}{x3}, \frac{0.8}{x4}, \frac{1}{x5} \right\} = \bigcup_{\alpha \in [0,1]} {}_{\alpha}\tilde{A} $$

$$ \bigcup_{\alpha \in [0,1]} {}_{\alpha}\tilde{A} = \tilde{A} $$

$[0,1] = [0,0.2] \cup (.2,.4] \cup (.4,.6] \cup (.6,.8] \cup (.8,1]$

Now, ${}^{.2}\tilde{A} = \{x1, x2, x3, x4, x5\}$

${}^{.4}\tilde{A} = \{x2, x3, x4, x5\}$

${}^{.6}\tilde{A} = \{x3, x4, x5\}$

${}^{.8}\tilde{A} = \{x4, x5\}$

${}^{1}\tilde{A} = \{x_5\}$

Also,
$$ \bigcup_{\alpha} \tilde{A} = \bigcup_{\alpha \in [0,.2]} \tilde{A} \cup \bigcup_{\alpha \in [0.2,.4]} \tilde{A} \cup \bigcup_{\alpha \in [.4,.6]} \tilde{A} \cup \bigcup_{\alpha \in [.6,.8]} \tilde{A} \cup \bigcup_{\alpha \in [.8,1]} $$
$$ = \tilde{A} $$
$$ = \bigcup_{\alpha \in [0,1]} {}_{\alpha}\tilde{A} = \tilde{A} = \left\{ \frac{0.2}{x1}, \frac{0.4}{x2}, \frac{0.6}{x3}, \frac{0.8}{x4}, \frac{1}{x5} \right\} $$

2.) Calculate degree of subsethood of

$A = \{ (u,.2), (v,.4), (w,.8), (x,.7), (y,.1), (z,.3) \}$

& $B(x) = \frac{x}{1+x}$, ~~$\forall x \in [0,1]$~~. $\forall x \in (0,10)$

$$ B(x) = \left\{ \frac{0}{0}, \frac{1}{0.5}, \frac{2}{0.67}, \frac{3}{0.75}, \frac{4}{0.8}, \frac{5}{0.83}, \frac{6}{0.85}, \frac{7}{0.875}, \frac{8}{0.888}, \frac{9}{0.9}, \frac{10}{0.909} \right\} $$

0 0.1 0 0.05 0.7 0.53

~~A(x)~~ =

32
909
90
50
67
75
80
83
85
875
888

1
05
70
53

$A(x) = \frac{x}{1+x}$, ~~$x \in X$~~ $B(x) = 2^{-x}$

$x \in X = \{0, \ldots 5\}$

Find $\overline{\tilde{A}}$, $\overline{\tilde{B}}$, $\tilde{A} \cup \tilde{B}$, $\tilde{A} \cap \tilde{B}$, $\overline{\tilde{A} \cup \tilde{B}}$

$\tilde{A} = \left\{ \frac{0}{0}, \frac{1}{0.5}, \frac{2}{0.66}, \frac{3}{0.75}, \frac{4}{0.8}, \frac{5}{0.83} \right\}$

$\tilde{B} = \left\{ \frac{0}{1}, \frac{1}{0.5}, \frac{2}{0.25}, \frac{3}{0.125}, \frac{4}{0.0625}, \frac{5}{0.03125} \right\}$

$\therefore \overline{\tilde{A}} = \left\{ \frac{0}{1}, \frac{1}{0.5}, \frac{2}{0.34}, \frac{3}{0.25}, \frac{4}{0.2}, \frac{5}{0.17} \right\}$

& $\overline{\tilde{B}} = \left\{ \frac{0}{1}, \frac{1}{0.5}, \frac{2}{0.75}, \frac{3}{0.875}, \frac{4}{0.9375}, \frac{5}{0.96875} \right\}$

$\tilde{A} \cup \tilde{B} = \left\{ \frac{0}{1}, \frac{1}{0.5}, \frac{2}{0.66}, \frac{3}{0.75}, \frac{4}{0.8}, \frac{5}{0.83} \right\}$

$\tilde{A} \cap \tilde{B} = \left\{ \frac{0}{0}, \frac{1}{0.5}, \frac{2}{0.25}, \frac{3}{0.125}, \frac{4}{0.0625}, \frac{5}{0.03125} \right\}$

$\overline{\tilde{A} \cup \tilde{B}} = \left\{ \frac{0}{0}, \frac{1}{0.5}, \frac{2}{0.34}, \frac{3}{0.25}, \frac{4}{0.2}, \frac{5}{0.17} \right\}$

## EXTENSION FUNCTION:-

Consider a function defined from $f : x \to y$ then the extension $f^n$ is defined as

$F : F(x) \to F(y)$ where $F(x)$ & $F(y)$ are family of fuzzy sets defined on $x$ & $y$ respectively.

i.e., if $A \in F(x)$

or $A = \left\{ \frac{x_1}{A(x_1)}, \frac{x_2}{A(x_2)}, \frac{x_3}{A(x_3)} \ldots \right\}$

then

$F(A) = B = \{ y \mid y = f(x), x \in A \}$

with the membership function given as

$B(y) = \mathrm{Max}(A(x))$; $B(y) \in [0,1]$

i.e.

$F(A) = B = \left\{ \frac{y_1}{B(y_1)}, \frac{y_2}{B(y_2)}, \frac{y_3}{B(y_3)} \ldots \right\}$

then

$F^{-1} : F(y) \to F(x)$

$D \in F$

$F^{-1}(D) = \{ x \mid f(x) \in D \}$

& its membership function will be :-

$\{ F^{-1}(D) \}(x) = D\{ f(x) \}$

Here $F(A)$ & $F^{-1}(D)$ are also fuzzy sets.

1.) let $y = f(x) = 0.6x + 4$ & $X = \{3, 5, 7\}$, $\tilde{A}$ defined on $X$.

$\tilde{A} = \left\{ \frac{0.2}{3}, \frac{0.1}{5}, \frac{0.3}{7} \right\}$. Find $F(A)$.

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~~No~~ Here :

| $x$ | $f(x) = y$ |
|---|---|
| 3 | 5.8 |
| 3 | 7 |
| 7 | 8.2 |

$\therefore f(x) = \{5.8, 7, 8.2\}$

Now, $B(y) = Max(A(x))$

$\therefore B(5.8) = Max(A(3)) = 0.3$

$B(7) = Max(A(5)) = 0.1$

$B(8.2) = Max(A(7)) = 0.3$

Therefore, $F(A) = \left\{ \frac{0.3}{5.8}, \frac{0.1}{7}, \frac{0.3}{8.2} \right\}$

FUZZY RELATION :-

Let $R$ be a relation defined on $\tilde{A}$ & $\tilde{B}$. for any $x \in A$, $y \in B$ we have

$$R = \left\{ (x,y) ; \mu_R(x,y) / (x,y) \in A \times B \text{ \& } \mu_R(x,y) \in [0,1] \right\}$$

where $\mu$ is any membership function.

PROOF :- Suppose $R$ is relation defined for $A \times B$ & $S$ is another relation defined on $A \times B$. Then,

$$U_{R,S}(x,y) = \{ max(\mu_R(x,y), \mu_S(x,y)) \}$$

&

$$\cap_{R,S}(x,y) = \{ \cancel{max}\ min(\mu_R(x,y), \mu_S(x,y)) \}$$

&

$$\mu_{\bar{R},\bar{S}}(x,y) = 1 - \mu_{R,S}(x,y)$$

$\alpha$-cut for fuzzy relations :-

$$\alpha_R = \{ (x,y) \mid \mu_R(x,y) \geq \alpha \}$$

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Note :- If $\tilde{A}$ is a fuzzy set defined on $X$ & $\tilde{B}$ is a fuzzy set defined on $Y$ then from $A \times B$ we get a relation $R$ whose membership grades are

$$\mu_R(x,y) = \mu_{A\times B}(x,y) = min(A(x), B(y))$$

COMPOSITION OPERATION :-

let $P$ be defined on $X \times Y$

$Q$ be defined on $Y \times Z$

where

$$P = \{ (x,y), \mu_P(x,y), (x,y) \in X \times Y \}$$

$$Q = \{ (y,z), \mu_Q(y,z), (y,z) \in Y \times Z \}$$

then

$$P(x,y) \circ Q(y,z) = R(x,z)$$

and membership value/function defined as :-

$$R = P \circ Q = \{ (x,z) \mid max( min(\mu_P(x,y), \mu_Q(y,z)) ) \}$$

If $f : x_1 \times x_2 \times x_3 \cdots \times x_n$ then extension function for this is

$f(x_1) \times f(x_2) \times \cdots \times f(x_n) \longrightarrow f(y)$ where $A_1 \in f(x_1)$ & $A_2 \in f(x_2)$ which are fuzzy sets defined on $X_n$

Further $F^{-1}(B)\{x_1, \cdots x_n\} = B\{F(x_1, x_2 \cdots x_n)\}$

If $P, Q$ are relation then

$$\overline{P \circ Q} = \bar{P} \circ \bar{Q}$$

Theorem :- If $A, B \in F(x)$; $x \in X$ and $\alpha \in [0,1]$ then

i) $A \subseteq B$ if & only if ${}^{\alpha}A \subseteq {}^{\alpha}B$ and
& ${}^{+\alpha}A \subseteq {}^{+\alpha}B$

ii) $A = B$ if & only if ${}^{\alpha}A = {}^{\alpha}B$
& ${}^{+\alpha}A = {}^{+\alpha}B$

Proof :- i) Let $A \subseteq B$; $\alpha \in [0,1]$

$\Rightarrow x \in A$, $x \notin B$. $\Rightarrow$

$\Rightarrow {}^{\alpha}A \neq {}^{\alpha}B$

Also $y \in B \Rightarrow y \in A$ $(\because A \subseteq B)$

$B(y) \geq \alpha$, $A(y) \geq \alpha$

$\therefore {}^{\alpha}A = {}^{\alpha}B$

There fore ${}^{\alpha}A = {}^{\alpha}B$ & ${}^{\alpha}A \neq {}^{\alpha}B$

$\Rightarrow {}^{\alpha}A \subseteq {}^{\alpha}B$

Conversely, ${}^{\alpha}A \subseteq {}^{\alpha}B$

There fore $x \in {}^{\alpha}A \subseteq {}^{\alpha}B$

$\Rightarrow A(x) \leq B(x)$ $\forall x \in X$ and $A \subseteq B$

There fore given statement is true.

b) For ${}^{+\alpha}A \subseteq {}^{\alpha}B$

$B(y) > \alpha$, $A(y) > \alpha$

$\therefore {}^{+\alpha}A = {}^{+\alpha}B$

There fore ${}^{+\alpha}A = {}^{+\alpha}B$ and ${}^{+\alpha}A \neq {}^{+\alpha}B$

$\therefore$ $\boxed{{}^{+\alpha}A \subseteq {}^{+\alpha}B}$

ii) Let $A = \{(1, .2), (2, .4), (3, .6), (4, .1)\}$

$B = \{(1, .2), (2, .4), (3, .6), (4, .1)\}$ $(\because A = B)$

Now for $\alpha = 0.4$

${}^{\alpha}A = \{2, 3\}$, ${}^{\alpha}B = \{2, 3\}$

${}^{+\alpha}A = \{3\}$ ${}^{+\alpha}B = \{3\}$

$\therefore {}^{\alpha}A = {}^{\alpha}B$

${}^{+\alpha}A = {}^{+\alpha}B$.

Now, $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$, $\forall x \in X$, $\alpha \in [0,1]$

$\mu_A(x) \geq \alpha$, $\mu_B(x) \geq \alpha$.

$\therefore x \in {}^{\alpha}A$, $x \in {}^{\alpha}B$

$\Leftrightarrow {}^{\alpha}A = {}^{\alpha}B$ & ${}^{+\alpha}A = {}^{+\alpha}B$.

Theorem :- Let $A, B \in F(x)$; and $\alpha, \beta \in [0,1]$

i) ${}^{+\alpha}A \subseteq {}^{\alpha}A$;

ii) $\alpha \leq \beta \Rightarrow {}^{\alpha}A \supseteq {}^{\beta}A$ and ${}^{+\alpha}A \supseteq {}^{+\beta}A$;

iii) ${}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B$ ;- and

iv) ${}^{\alpha}\bar{A} = {}^{+(1-\alpha)}A$ $\boxed{+(1-\alpha)} \Rightarrow$ strong $(1-\alpha)$ cut.

Proof :- By definition,

${}^{\alpha}A = \{x / A(x) \geq \alpha, \forall x \in X, \alpha \in [0,1]\}$.

${}^{+\alpha}A = \{x / A(x) > \alpha, \forall x \in X, \alpha \in [0,1]\}$

From these definitions, we get

${}^{+\alpha}A \subseteq {}^{\alpha}A$ or ${}^{\alpha}A \supseteq {}^{+\alpha}A$

Let $X = \{1, 2, 3, 4\}$ $A = \{(1, .2), (2, .4), (3, .6), (4, .1)\}$

$\therefore {}^{.4}A = \{2, 3, 4\}$, ${}^{+.4}A = \{3, 4\}$

$\therefore$ (1) is true.

Now at $\alpha = 0.5$

${}^{.5}A = \{3, 4\}$ ${}^{+.5}A = \{3, 4\}$

(2) $\alpha \le \beta$. ~~A~~gain by definition we have

$\alpha A = \{x / A(x) \ge \alpha, x \in X, \alpha \in [0,1]\}$

$\beta A = \{x / A(x) \ge \beta, x \in X, \alpha \in [0,1]\}$

since, $\alpha \le \beta$

we have $\cancel{\beta A \le \alpha A}$. ${}^{\beta}A \le {}^{\alpha}A$

Take $X = \{1,2,3,4\}$

$A = \{(1,.2),(2,.4),(3,.6),(4,.8)\}$

Now for $\alpha = 0.3$ & $\beta = 0.4$

$\alpha A = \{2,3,4\}$ $\quad \beta A = \{2,3,4\}$

Also for $\alpha = 0.4$ & $\beta = 0.4$

$\alpha A = \{2,3,4\}$ $\quad \beta A = \{2,3,4\}$

And for $\alpha = 0.4$ & $\beta = 0.5$

$\alpha A = \{2,3,4\}$ $\quad \beta A = \{3,4\}$

Hence

$\beta A \subseteq {}^{\alpha}A$ or $\alpha A \ge {}^{\beta}A$.

For $\alpha = 0.3$ & $\beta = 0.4$

${}^{+\alpha}A = \{2,3,4\}$ $\quad {}^{+\beta}A = \{3,4\}$

$\alpha = 0.4$ & $\beta = 0.4$

${}^{+\alpha}A = \{3,4\}$ $\quad {}^{+\beta}A = \{3,4\}$

$\alpha = 0.4$ & $\beta = 0.6$

${}^{+\alpha}A = \{3,4\}$ $\quad {}^{+\beta}A = \{4\}$

Hence

${}^{+\beta}A \le {}^{+\alpha}A$ or ${}^{+\alpha}A \ge {}^{+\beta}A$.

(3) $\alpha(A \cap B) = \alpha A \cap \alpha B$, $\quad \alpha(A \cup B) = \alpha A \cup \alpha B$

Consider;

$x \in {}^{\alpha}(A \cap B)$

$\therefore (A \cap B)(x) \ge \alpha$, $\forall x \in X$

$A(x) \cap B(x) \ge \alpha$, $\forall x \in X$

$= \text{Min}(A(x), B(x)) \ge \alpha$, $\forall x \in X$.

But $A(x) \ge \alpha$ & $B(x) \ge \alpha$

$\therefore$ ~~Min~~ $x \in \min({}^{\alpha}A, {}^{\alpha}B)$

$\Rightarrow x \in {}^{\alpha}A \cap {}^{\alpha}B$.

$\therefore \alpha(A \cap B) \subseteq {}^{\alpha}A \cap {}^{\alpha}B$.

Conversely, let $x \in {}^{\alpha}A \cap {}^{\alpha}B$

$\Rightarrow x \in {}^{\alpha}A$ and $x \in {}^{\alpha}B$.

$\therefore A(x) \ge \alpha$ and $B(x) \ge \alpha$.

$\therefore [A(x) \cap B(x)] \ge \alpha$.

$\therefore (A \cap B)(x) \ge \alpha$.

$\Rightarrow x \in {}^{\alpha}(A \cap B)$.

Hence $\alpha(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B$.

Similarly, consider $x \in {}^{\alpha}(A \cup B)$

$\therefore (A \cup B)(x) \ge \alpha$, $\forall x \in X$

$\therefore A(x) \cup B(x) \ge \alpha$, $\forall x \in X$

$= \text{Max}(A(x), B(x)) \ge \alpha$, $\forall x \in X$.

But $A(x) \ge \alpha$ or $B(x) \ge \alpha$, $\forall x \in X$.

$\therefore x \in \text{Max}({}^{\alpha}A, {}^{\alpha}B)$.

$\Rightarrow x \in {}^{\alpha}A \cup {}^{\alpha}B$.

$\therefore \alpha(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B$.

Conversely, let $x \in \alpha A \cup {}^{\alpha}B$

$\Rightarrow x \in {}^{\alpha}A$ or $x \in {}^{\alpha}B$.

$\therefore$ ~~$\alpha(A \cup B)$~~ $A(x) \ge \alpha$ or $B(x) \ge \alpha$

$\therefore [A(x) \cup B(x)] \ge \alpha$ $\quad \therefore (A \cup B)(x) \ge \alpha$

$\therefore x \in {}^{\alpha}(A \cup B)$.

Hence $\alpha(A \cup B) = \alpha A \cup {}^{\alpha}B$.

(4) $^{\alpha}\bar{A} = \overline{^{+(1-\alpha)}A}$

When $x \in {}^{\alpha}\bar{A}$

$\bar{A}(x) \geq \alpha$

$\therefore [1 - A(x)] \geq \alpha$

$\therefore A(x) \leq 1 - \alpha$

$\Rightarrow x \notin {}^{+(1-\alpha)}A$

Hence $x \in \overline{^{+(1-\alpha)}A}$

$\therefore {}^{\alpha}\bar{A} = \overline{^{+(1-\alpha)}A}$

Conversely,

$x \in \overline{^{+(1-\alpha)}A}$

$\Rightarrow \bar{A}(x) \leq (1-\alpha)$

$\bar{A}(x) \geq \alpha$

EQUILLIBRIUM OF FUZZY SET :-

THEOREM :-

The equillibrium of a fuzzy compliment is degree of membership in a fuzzy set A which equals the degree of membership in compliment of A. i.e. it is the value of A for which $c(a) = a$.

Every fuzzy compliment has atmost 1 equillibrium (if there are more than one then they are equal.)

PROOF :- Let $c$ be the fuzzy compliment then we have

$c(a) - a = 0; \quad a \in [0,1]$

If $b$ is another constant such that

$c(a) - a = b \quad \text{----} (1)$

Suppose $a_1$ & $a_2$ are two solutions of (1);

such that ~~c(a) - a = b~~ $a_1 < a_2$.

We have, $c(a_1) - a_1 = b$

$c(a_2) - a_2 = b$

$\therefore c(a_1) - a_1 = c(a_2) - a_2$

$\therefore c(a_1) - c(a_2) = a_1 - a_2$

as, $a_1 < a_2$ we have

$c(a_1) > c(a_2)$, as $c$ is decreasing function.

$\therefore c(a_1) - a_1 > c(a_2) - a_2$,

which is not true

Hence we must have $c(a) - a = b$

$\therefore$ This equation has atmost one solution.

THEOREM :- If $A \in f(x)$, then $^{\alpha}A = \bigcap_{\beta<\alpha} {}^{\beta}A = \bigcap_{\beta<\alpha} {}^{+\beta}A$

given

$^{\alpha}A = \{x \mid A(x) \geq \alpha, \alpha \in [0,1], \beta < \alpha, \alpha \in [0,1]\}$

PROOF :-

given, $\beta < \alpha$

$\Rightarrow {}^{\alpha}A \subseteq {}^{\beta}A$

$\Rightarrow {}^{\alpha}A \subseteq \bigcap_{\beta<\alpha} {}^{\beta}A$

if $x \in X$, $\exists\ \varepsilon > 0$; such that $\alpha - \varepsilon < \alpha$.

and $x \in {}^{\beta}A \Rightarrow x \in {}^{\alpha-\varepsilon}A$

(Assumption) as $\alpha$ is arbitrarily small, we take $\varepsilon \to 0$.

$\Rightarrow A(x) \geq \alpha$.

$\Rightarrow x \in {}^{\alpha}A$.

$\therefore x \in \bigcap {}^{\beta}A \Rightarrow x \in {}^{\alpha}A$

$\Rightarrow \bigcap_{\beta>\alpha} {}^{\beta}A \subseteq {}^{\alpha}A \quad \text{----} (2)$

from ① & ②,

$$^{\alpha}A = \bigcap_{\beta < \alpha}{}^{\beta}A$$

Similarly, $^{\alpha}A = \bigcap_{\beta < \alpha}{}^{+\beta}A$

Hence the statement is true