# Proof of Q(n,k) = R(n,k)

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## Theorem

Let $Q(n,k)$ denote the number of partitions in which the number $n$ is partitioned into numbers where the largest number is atmost k. Let $R(n,k)$ denote the number of partitions where the number $n$ is partitioned into $j$ positive numbers and any number among $j$ numbers is atmost $k$. Prove that $Q(n,k) = R(n,k)$ for all values of n and k.

## Proof

Let $P(n,k)$ denote the number of partitions in which the number $n$ is partitioned into exactly k positive numbers. Let $S(n,k)$ denote the number of partitions where the number $n$ is partitioned into $j$ positive numbers and maximum of $j$ numbers is exactly $k$.
Claim: $P(n,k) = P(n-1,k-1) + P(n-k,k)$ and the same is true for $S(n,k)$
For any partition P(n,k), either it has 1 in the partition or it doesn't. If 1 is present in the partition, then $P(n,k) = P(n-1,k-1)$. In the other case, every number in the partition is greater than 1. We can subtract 1 from each number in the partition $P(n,k)$ to obtain $P(n-k,k)$ since 1 is subtracted k times from n.
Hence, $P(n,k) = P(n-1,k-1) + P(n-k,k)$.
The base case is $P(n,1) = 1$ and $P(n,k) = 0$ if $n < k$.

Now, we prove the same for $S(n,k)$. Clearly the base case holds, as $S(n,1)$ a number partitioned into 1's. So $S(n,1) = 1$. A partition can not have a number greater than n, so $S(n,k) = 0$ if $n < k$.
$S(n,k)$ has either one number equal to k or more then one number equal to k in the partition. In the first case, $S(n,k) = S(n-1,k-1)$ where we have just subtracted 1 from both sides of equality, from n as well as from the number k. In the second case, even if we subtract k from both sides the maximum is still k. So $S(n,k) = S(n-k,k)$.
Hence, $S(n,k) = S(n-1,k-1) + S(n-k,k)$.
Since the recursion and base case for both functions is same, $P(n,k) = S(n,k)$. This implies that $\sum_{i=1}^{k} P(n,k) = \sum_{i=1}^{k} S(n,k)$. We know that $Q(n,k) = \sum_{i=1}^{k} S(n,k)$ and $R(n,k) = \sum_{i=1}^{k} P(n,k)$. Therefore, we can say that $Q(n,k) = R(n,k)$ for all values of n and k.