

Midterm

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Problem 1

1.1

Proof.

$$\begin{aligned}
 (\forall n) \quad & A^{n+2}\mathbf{v} - A^{n+1}\mathbf{v} = A^{n+1}\mathbf{v} - A^n\mathbf{v} \\
 \iff & A^2\mathbf{v} - A\mathbf{v} = A\mathbf{v} - \mathbf{v} \\
 \iff & (A - I)^2\mathbf{v} = 0
 \end{aligned}$$

Since $(A - I)^2\mathbf{v} = 0$ for some non-zero \mathbf{v} , we know that $\det((A - I)^2) = 0$, which means $\det(A - I) = 0$. Thus A has eigenvalue 1 and \mathbf{v} is in its generalized eigenspace. So \mathbf{v} is a generalized eigenvector for A . \square

1.2

CounterExample.

$$\begin{aligned}
 (\forall n) \quad & A^{n+2}\mathbf{v} = A^{n+1}\mathbf{v} + A^n\mathbf{v} \\
 \iff & A^2\mathbf{v} = A\mathbf{v} + \mathbf{v} \\
 \iff & (A^2 - A - I)\mathbf{v} = 0
 \end{aligned}$$

Suppose

$$x_1 = \frac{1 + \sqrt{5}}{2}, \quad x_2 = \frac{1 - \sqrt{5}}{2}$$

Then x_1, x_2 are distinct roots for equation $x^2 - x - 1 = 0$. Let

$$A = \begin{bmatrix} x_1 & \\ & x_2 \end{bmatrix}$$

The only two generalized eigenspaces are $\text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $\text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, so pick $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Clearly \mathbf{v} is in neither of the two generalized eigenspaces. Also, we know that $A^2 - A - I = O$, which means the sequence $\{A^k\mathbf{v}\}$ here satisfies the required condition $A^{n+2}\mathbf{v} = A^{n+1}\mathbf{v} + A^n\mathbf{v}$, but \mathbf{v} is NOT a generalized eigenvector for A . \square

1.3

Proof. Suppose M has a generalized eigenspace for eigenvalue λ . Let I be the identical mapping. We know for some non-zero $p(x) \in V$, $\exists t \in \mathbb{N}$ s.t.

$$(M - \lambda I)^t p(x) = 0$$

Which means

$$(x - \lambda)^t p(x) = 0$$

Since $(x - \lambda)^t$ is not a zero polynomial, then we must have $p(x) = \mathbf{0}$. Contradiction. So M has no generalized eigenvector. \square

1.4

Proof. Let $A = XJX^{-1}$, where J is the Jordan normal form of A . We know that all the Jordan blocks in J are nilpotent.

Suppose J has a Jordan block N which is larger than 5×5 . We immediately know that $N^5 \neq O$. I.e. $J^5 \neq O$. This is contradictory to $A^5 = O$. So every Jordan block of J is not larger than 5×5 .

Let $d = \dim \text{Ker}(A) = \dim \text{Ker}(J)$. So J have d Jordan blocks. Since the sum of the size of all Jordan blocks is the size of J , which is n , we know that $5d \geq n$, i.e. $d \geq \frac{n}{5}$. \square

1.5

Proof. For this subproblem, restrict the matrices to real matrices. From singular value decomposition we know

$$A = U_r D V_r^T$$

where D is a diagonal matrix with the singular value of A (all positive real number), U_r and V_r are respectively a orthonormal basis for $\text{Ran}(A)$ and $\text{Ran}(A^T)$. From $AA^T AA^T = O$, we have

$$U_r D V_r^T V_r D U_r^T U_r D V_r^T U_r D V_r^T V_r D U_r^T = O$$

Simplify, we get

$$U_r D^3 V_r^T U_r D^2 U_r^T = O$$

Multiply $U_r (D^{-1})^2 U_r^T$ from the left and $U_r D^{-1} V_r^T$ from the right, we get

$$A^2 = U_r D V_r^T U_r D V_r^T = O$$

Thus A and all its Jordan blocks are nilpotent. For all $l \times l$ nilpotent Jordan block N , $N^k = O \iff k \geq l$. So the size of any the Jordan block of A is at most 2, so $2 \times \dim \text{Ker}(A) \geq n$, we get $\dim \text{Ker}(A) \geq \frac{n}{2}$. \square

Problem2

2.1

Let

$$P = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Note that

$$PA \otimes BP^{-1} = P \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix} P^{-1} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{11} & a_{11}b_{12} & a_{12}b_{12} \\ a_{21}b_{11} & a_{22}b_{11} & a_{21}b_{12} & a_{22}b_{12} \\ a_{11}b_{21} & a_{12}b_{21} & a_{11}b_{22} & a_{12}b_{22} \\ a_{21}b_{21} & a_{22}b_{21} & a_{21}b_{22} & a_{22}b_{22} \end{bmatrix} = B \otimes A$$

\square

2.2

Proof. First,

$$I \otimes e^B = \begin{bmatrix} e^B & \\ & e^B \end{bmatrix} = e \begin{bmatrix} B & \\ & B \end{bmatrix} = e^{I \otimes B}$$

Then we know

$$\begin{aligned}
 e^{A \otimes I} &= e^{PI \otimes AP^{-1}} \\
 &= P e^{I \otimes A} P^{-1} \\
 &= PI \otimes e^A P^{-1} \\
 &= e^A \otimes I
 \end{aligned}$$

□

2.3

Proof. Note that $(A \otimes I)(I \otimes B) = (AI) \otimes (IB) = (IA) \otimes (BI) = (I \otimes B)(A \otimes I)$. So

$$e^{A \otimes I + I \otimes B} = e^{A \otimes I} e^{I \otimes B} \quad (*)$$

We have

$$\begin{aligned}
 e^A \otimes e^B &= (e^A I)(I e^B) \\
 &= (e^A \otimes I)(I \otimes e^B) \\
 &= e^{A \otimes I} e^{I \otimes B} && \text{(From the previous subproblem)} \\
 &= e^{A \otimes I + I \otimes B} && \text{(From (*))} \\
 &= e^{A \oplus B}
 \end{aligned}$$

□

2.4

Calculate directly, we have

$$\begin{aligned}
 \text{trace}(A \oplus B) &= \text{trace}(A \otimes I) + \text{trace}(I \otimes B) \\
 &= \text{trace}\left(\begin{bmatrix} a_{11}I & a_{12}I \\ a_{21}I & a_{22}I \end{bmatrix}\right) + \text{trace}\left(\begin{bmatrix} B & \\ & B \end{bmatrix}\right) \\
 &= 2\text{trace}(A) + 2\text{trace}(B)
 \end{aligned}$$

□

2.5

Proof. Case 1. If A is diagonalizable, then let

$$A = P \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} P^{-1}$$

Note that A is invertible, so λ_1, λ_2 are non-zero. Let $\lambda_1 = e^{\mu_1}, \lambda_2 = e^{\mu_2}$, where $\mu_1, \mu_2 \in \mathbb{C}$. Let $X = P \begin{bmatrix} \mu_1 & \\ & \mu_2 \end{bmatrix} P^{-1}$.

We have

$$\begin{aligned} e^X &= e^{P \begin{bmatrix} \mu_1 & \\ & \mu_2 \end{bmatrix} P^{-1}} \\ &= P e^{\begin{bmatrix} \mu_1 & \\ & \mu_2 \end{bmatrix}} P^{-1} \\ &= P \begin{bmatrix} e^{\mu_1} & \\ & e^{\mu_2} \end{bmatrix} P^{-1} \\ &= P \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} P^{-1} \\ &= A \end{aligned}$$

Case 2. If A is not diagonalizable, then let

$$A = P \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix} P^{-1}$$

Note that A is invertible, so $\lambda \neq 0$. Let $\lambda = e^\mu$, where $\mu \in \mathbb{C}$. Let $X = P \begin{bmatrix} \mu & \lambda^{-1} \\ & \mu \end{bmatrix} P^{-1}$. We have

$$X = P \begin{bmatrix} \lambda^{-1} & \\ & 1 \end{bmatrix} \begin{bmatrix} \mu & 1 \\ & \mu \end{bmatrix} \begin{bmatrix} \lambda & \\ & 1 \end{bmatrix} P^{-1}$$

So

$$\begin{aligned} e^X &= P \begin{bmatrix} \lambda^{-1} & \\ & 1 \end{bmatrix} e^{\begin{bmatrix} \mu & 1 \\ & \mu \end{bmatrix}} \begin{bmatrix} \lambda & \\ & 1 \end{bmatrix} P^{-1} \\ &= P \begin{bmatrix} \lambda^{-1} & \\ & 1 \end{bmatrix} \begin{bmatrix} e^\mu & e^\mu \\ & e^\mu \end{bmatrix} \begin{bmatrix} \lambda & \\ & 1 \end{bmatrix} P^{-1} \\ &= P \begin{bmatrix} \lambda^{-1} & \\ & 1 \end{bmatrix} \begin{bmatrix} \lambda & \lambda \\ & \lambda \end{bmatrix} \begin{bmatrix} \lambda & \\ & 1 \end{bmatrix} P^{-1} \\ &= P \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix} P^{-1} \\ &= A \end{aligned}$$

□

2.6

Consider $A \otimes B$, we have

$$A \otimes B = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix} := \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}$$

where $\mathbf{v}_i \in \mathbb{C}^4$. When A is not invertible, we have λ_1, λ_2 s.t. $\lambda_1^2 + \lambda_2^2 \neq 0$ and

$$\lambda_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + \lambda_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = 0$$

So

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_3 = \begin{bmatrix} b_{11}(\lambda_1 a_{11} + \lambda_2 a_{12}) \\ b_{21}(\lambda_1 a_{11} + \lambda_2 a_{12}) \\ b_{11}(\lambda_1 a_{21} + \lambda_2 a_{22}) \\ b_{21}(\lambda_1 a_{21} + \lambda_2 a_{22}) \end{bmatrix} = 0$$

which means $\mathbf{v}_1, \mathbf{v}_3$ are linearly dependent, so $\det(A \otimes B) = \det^2(A)\det^2(B) = 0$.

When B is not invertible, we have λ_1, λ_2 s.t. $\lambda_1^2 + \lambda_2^2 \neq 0$ and

$$\lambda_1 \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} + \lambda_2 \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = 0$$

So

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 = \begin{bmatrix} a_{11}(\lambda_1 b_{11} + \lambda_2 b_{12}) \\ a_{11}(\lambda_1 b_{21} + \lambda_2 b_{22}) \\ a_{21}(\lambda_1 b_{11} + \lambda_2 b_{12}) \\ a_{21}(\lambda_1 b_{21} + \lambda_2 b_{22}) \end{bmatrix} = 0$$

which means $\mathbf{v}_1, \mathbf{v}_2$ are linearly dependent, so $\det(A \otimes B) = \det^2(A)\det^2(B) = 0$.

When both A, B are invertible, let $A = e^X, B = e^Y$. We get

$$\begin{aligned} \det(A \otimes B) &= \det(e^X \otimes e^Y) && \text{(subproblem2.5)} \\ &= \det(e^{X \oplus Y}) && \text{(subproblem2.3)} \\ &= e^{\text{trace}(X \oplus Y)} \\ &= e^{2\text{trace}(X) + 2\text{trace}(Y)} && \text{(subproblem2.4)} \\ &= (e^{\text{trace}(X)} e^{\text{trace}(Y)})^2 \\ &= (\det(e^X) \det(e^Y))^2 \\ &= \det^2(A) \det^2(B) \end{aligned}$$

□

Problem3

3.1

Example. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}$$

Then we have the Jordan decomposition

$$A = \begin{bmatrix} -i & 0 & i & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -i & 0 & i \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} i & 1 & & \\ & i & & \\ & & -i & 1 \\ & & & -i \end{bmatrix} \begin{bmatrix} -i & 0 & i & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -i & 0 & i \\ 0 & 1 & 0 & 1 \end{bmatrix}^{-1}$$

□

3.2

Example. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ & 0 & 1 \\ & & 0 \end{bmatrix}$$

Calculate directly, we get

$$AB = \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}, BA = O$$

Clearly, the rank of AB and BA are different. So AB and BA are not similar. □

3.3

Example. Let

$$A = \begin{bmatrix} 1 & i \\ & 1 \end{bmatrix}$$

Clearly A has non-real entry but only has real eigenvalue 1. □

3.4

Example. Let

$$A = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

$$\text{By definition, } e^A = \begin{bmatrix} e^0 & \frac{d}{dx}e^x|_{x=0} & & \\ & e^0 & & \\ & & e^0 & \frac{d}{dx}e^x|_{x=0} \\ & & & e^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix} = A + I.$$

□

Let $A = XJX^{-1}$, where J is the Jordan normal form of A . We have $e^A = A + I \iff e^J = J + I$. Consider the single Jordan blocks of J .

When $J(\lambda)$ is 1×1 , then we have $e^\lambda = \lambda + 1$.

When $J(\lambda)$ is 2×2 , then we have $\begin{bmatrix} e^\lambda & e^\lambda \\ & e^\lambda \end{bmatrix} = \begin{bmatrix} \lambda + 1 & 1 \\ & \lambda + 1 \end{bmatrix}$, which yields

$$\begin{cases} e^\lambda = 1 \\ e^\lambda = \lambda + 1 \end{cases}$$

We know $\lambda = 0$. All 2×2 Jordan block of J is nilpotent.

When $J(\lambda)$ is 3×3 , then we have $\begin{bmatrix} e^\lambda & e^\lambda & \frac{e^\lambda}{2!} \\ & e^\lambda & e^\lambda \\ & & e^\lambda \end{bmatrix} = \begin{bmatrix} \lambda + 1 & 1 & \\ & \lambda + 1 & 1 \\ & & \lambda + 1 \end{bmatrix}$, which yields

$$\begin{cases} e^\lambda = 1 \\ e^\lambda = \lambda + 1 \\ \frac{e^\lambda}{2!} = 0 \end{cases}$$

We know it has no solution in \mathbb{C} , so J does not have a 3×3 single Jordan block. Similarly there is no 4×4 single Jordan block.

So all the possible Jordan canonical forms of A are

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}, \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}, \begin{bmatrix} \mu_1 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & \mu_2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & \mu_1 & \\ & & & \mu_2 \end{bmatrix}, \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \mu_3 & \\ & & & \mu_4 \end{bmatrix}$$

Where μ_i are solutions to $e^\mu - \mu - 1 = 0$ in \mathbb{C} . (They can be distinct or some of them are equal.) □

Problem 4

4.1

Note that

$$M(1) = x, \quad M(x) = x^2, \quad M(x^2) = x^3, \quad M(x^3) = -d - cx - bx^2 - ax^3$$

Let

$$A = \begin{bmatrix} 0 & 0 & 0 & -d \\ 1 & 0 & 0 & -c \\ 0 & 1 & 0 & -b \\ 0 & 0 & 1 & -a \end{bmatrix}$$

We know that

$$M([1 \ x \ x^2 \ x^3]) = [1 \ x \ x^2 \ x^3]A$$

4.2

Proof. Let's consider the case of $n \times n$ matrix A . Let V be the space of polynomials degree strictly less than n and $q(x)$ be a degree n polynomial.

For all polynomials $r(x)$, $s(x)$, $t(x)$. We have $r(x) = m(x)t(x) + (r(x) \bmod t(x))$, $s(x) = n(x)t(x) + (s(x) \bmod t(x))$ for some polynomial $m(x)$, $n(x)$. So we have

$$(r(x) + s(x)) \bmod t(x) = ((r(x) \bmod t(x)) + (s(x) \bmod t(x)))$$

$$(r(x) \cdot s(x)) \bmod t(x) = ((r(x) \bmod t(x)) \cdot (s(x) \bmod t(x)))$$

We know $Mp(x) = xp(x) \bmod q(x)$, so for $\forall k$

$$\begin{aligned} M^k p(x) &= M(M^{k-1}p)(x) \\ &= xM^{k-1}p(x) \bmod q(x) \\ &= (x \bmod q(x))(M^{k-1}p(x) \bmod q(x)) \\ &= (x \bmod q(x))((xM^{k-2}p(x) \bmod q(x)) \bmod q(x)) \\ &= (x \bmod q(x))(xM^{k-2}p(x) \bmod q(x)) \\ &= x^2 M^{k-2}p(x) \bmod q(x) \end{aligned}$$

Follow this, by induction, we know

$$M^k p(x) = x^k p(x) \bmod q(x)$$

Note that the polynomial of the operator M is the linear combination of M^k for some k . For \forall polynomial $P(x)$

$$(P(M)p)(x) = P(x)p(x) \bmod q(x)$$

So we know $q(M)p(x) = q(x)p(x) \bmod q(x) = \mathbf{0}$. For \forall polynomial $Q(x)$ s.t. $Q(M) = O$, we have

$$Q(x)p(x) \bmod q(x) = \mathbf{0}$$

So $Q(x)$ is a multiple of $q(x)$. Thus $q(x)$ is the minimal polynomial of operator M and matrix A . So the characteristic polynomial of A is a monic multiple of $q(x)$ with degree n . Note that $\deg(q(x)) = n$. Thus, there can be no more factors for the characteristic polynomial, which means $q(x)$ is the characteristic polynomial. \square

Collaborator for **4.2**: Chen Siyuan.

4.3

We claim that A has a single Jordan block for each eigenvalue λ_i . If not, then the degree of factor $(x - \lambda_i)$ in the minimal polynomial will be strictly less than the degree of factor $(x - \lambda_i)$ in the characteristic polynomial, which is contradictory to the conclusion in the previous subproblem, that the characteristic polynomial and the minimal polynomial are the same.

Then we know that $AB=BA$ implies $B = f(A)$ for some polynomial f . (Proven in the lecture note) \square

4.4

From **subproblem 4.2** we know that $p(x)$ is the minimal polynomial, which means the Jordan canonical form of A can be

$$\begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & 3 \end{bmatrix}$$

Or dependent on the choice of basis, there are other 23 possible forms, which correspond to other permutations of $\{1, 2, 3, 4\}$. \square

4.5

From **subproblem 4.2** we know that $p(x)$ is the minimal polynomial, which means the Jordan canonical form of A can be

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 1 & \\ & & & 2 \end{bmatrix}$$

Or dependent on the choice of basis, there are other 5 possible forms, which correspond to other permutations of $\{0, 1, 2\}$. \square

Problem 5

5.1

Proof. Note that $AA^T = I$.

$$\begin{aligned} & A + A^T - 2I \\ &= -(AA^T - A - A^T + I) \\ &= -(A - I)(A^T - I) \end{aligned}$$

\square

5.2

Proof.

$$\begin{aligned}
\lim_{t \rightarrow 0} \frac{-(f(t) - I)(f(t)^T - I)}{t} &= \lim_{t \rightarrow 0} \frac{(f(t) - f(0)) + (f(t)^T - f(0)^T)}{t} \\
&= \lim_{t \rightarrow 0} \frac{(f(t) - f(0))}{t} + \lim_{t \rightarrow 0} \frac{(f(t)^T - f(0)^T)}{t} \\
&= f'(0) + f'(0)^T
\end{aligned}$$

□

5.3

Proof.

$$\begin{aligned}
\lim_{t \rightarrow 0} \frac{-(f(t) - I)(f(t)^T - I)}{t} &= \lim_{t \rightarrow 0} -(f(t) - f(0)) \frac{f(t)^T - f(0)^T}{t} \\
&= \lim_{t \rightarrow 0} -(f(t) - f(0)) \cdot \lim_{t \rightarrow 0} \frac{f(t)^T - f(0)^T}{t} \\
&= O \cdot f'(0)^T \\
&= O
\end{aligned}$$

So $f'(0) + f'(0)^T = \lim_{t \rightarrow 0} \frac{-(f(t) - I)(f(t)^T - I)}{t} = O$, $f'(0)$ is skew-symmetric.

□

5.4

Proof. Note that $(f(t+h)f(t)^T)^T = f(t)f(t+h)^T$, we have

$$\begin{aligned}
f'(t)f(t)^T + f(t)f'(t)^T &= \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} f(t)^T + f(t) \lim_{h \rightarrow 0} \frac{f(t+h)^T - f(t)^T}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(t+h)f(t)^T + f(t)f(t+h)^T - 2I}{h} \\
&= - \lim_{h \rightarrow 0} \frac{f(t+h)f(t)^T - f(t)f(t)^T}{h} (f(t)f(t+h)^T - I) \\
&= -f'(t)f(t)^T \lim_{h \rightarrow 0} (f(t)f(t+h)^T - I) \\
&= O
\end{aligned}$$

Note that $f(t)^{-1} = f(t)^T$, we have $f'(t)f(t)^{-1} + f(t)f'(t)^{-1} = O$. So $f'(t)f(t)^{-1}$ is skew-symmetric.

□

5.5

Proof. From $f'(t)f(t)^{-1} = A$, we know $f'(t) = Af(t)$. Let $f(t) = [v_1(t) \ \cdots \ v_n(t)]$. For $\forall i$, $\frac{dv_i(t)}{dt} = Av_i(t)$. So $v_i(t) = e^{At}w_i$ for some $w_i \in \mathbb{C}^n$. Let $B = [w_1 \ \cdots \ w_n]$, we have $f(t) = e^{At}B$.

Since A is skew-symmetric, we have $(e^{At})^{-1} = (e^{At})^T$. Thus $B = (e^{At})^T f(t)$, we know $\det(B) = \det((e^{At})^T) \det(f(t)) = 1$ and $BB^T = (e^{At})^T f(t)f(t)^T (e^{At}) = I$. I.e. $B \in SO_n$.

□