Homework 4

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1 Exercise 1.4.1

1. Let

$$B = \begin{bmatrix} I & X \\ & I \end{bmatrix}, \quad A = \begin{bmatrix} M & C \\ & N \end{bmatrix}$$

Then

$$BAB^{-1} = \begin{bmatrix} M & XN + C - MX \\ & N \end{bmatrix} = \begin{bmatrix} M & \\ & N \end{bmatrix}$$

Thus

$$XN - MX = -C$$

Let $f : M_2(\mathbb{R}) \to M_2(\mathbb{R}), f(X) = XN - MX$. Then f is a linear map.

Pick basis for $M_2(\mathbb{R})$

$$T = \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \}$$

Then the corresponding matrix of f under basis T is

$$F = \begin{bmatrix} 2 & 0 & -2 & 0 \\ 5 & 3 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 5 & 3 \end{bmatrix}$$

Let the coordinate of C under basis T be $\hat{c}$, the coordinate of X under basis T be $\hat{x}$. Then Solve the equation $F\hat{x} = -\hat{c}$.

Then we get

$$\hat{x} = \begin{bmatrix} -2 \\ \frac{31}{9} \\ -\frac{3}{2} \\ \frac{7}{6} \end{bmatrix}$$

So, let

$$
B = \begin{bmatrix} 1 & 0 & -2 & \frac{31}{9} \\ & 1 & -\frac{3}{2} & \frac{7}{6} \\ & & 1 & 0 \\ & & & 1 \end{bmatrix}
$$

Then $BAB^{-1} = \begin{bmatrix} 1 & 2 & & \\ & 1 & & \\ & & 3 & 5 \\ & & & 4 \end{bmatrix}$. $\square$

2.   Since $V_3 \cap V_4 = \{0\}$, only need to find basis for V_3 and V_4 respectively.

$$
V_3 = Ker(A - 3I) = span(\begin{bmatrix} 4 \\ 3 \\ 2 \\ 0 \end{bmatrix})
$$

$$
V_4 = Ker(A - 4I) = span(\begin{bmatrix} \frac{59}{9} \\ \frac{19}{3} \\ 5 \\ 1 \end{bmatrix})
$$

Then, a basis for $V_3 + V_4$ is $\{ \begin{bmatrix} 4 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{59}{5} \\ \frac{19}{3} \\ 5 \\ 1 \end{bmatrix} \}$.

Also, we can pick a basis directly from the previous calculation. I.e. the last 2 row of B^{-1} forms a basis of $V_3 + V_4$. So,

$$
\{ \begin{bmatrix} 2 \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{31}{9} \\ -\frac{7}{6} \\ 0 \\ 1 \end{bmatrix} \}
$$

is a basis of $V_3 + V_4$. $\square$

2   Exercise 1.4.2

The minimal polynomial of A is not the square of the minimal polynomial of B.

Counter example   Let

$$B = \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$$

Note that all the eigenvalue of B is 0, we know that the minimal polynomial of B is $p(x) = x^2$.
But

$$A^2 = \begin{bmatrix} B & I \\ & B \end{bmatrix} \begin{bmatrix} B & I \\ & B \end{bmatrix} = \begin{bmatrix} O & 2B \\ & O \end{bmatrix} , \quad A^3 = O$$

So the minimal polynomial of A is $q(x) = x^3 \neq p^2(x)$.   □

3   Exercise 1.4.3

1. Proof.   Let

$$A = XUX^{-1}$$

where U is an upper triangluar matrix.
From $AB = BA$, we know that

$$UX^{-1}BX = X^{-1}BXU$$

Let $T = X^{-1}BX$.

We try to prove that T is upper triangular by mathematical induction. Let T and U be nxn matrix. Let's state the proposition to be prove: **For any U, $T \in M_n(\mathbb{C})$, if U is upper triangular and $UT = TU$, then T is upper triangular.**
First, when $n = 1$, T is clearly upper triangular.

Induction Hypothesis   When $n = k - 1$, if U is upper triangular and $UT = TU$, then T is upper triangular.

When $n = k$, let

$$UT = TU$$

The last row of UT is

$$\begin{bmatrix} u_{nn}t_{n1} & u_{nn}t_{n2} & u_{nn}t_{n3} & \cdots & u_{nn}t_{nn} \end{bmatrix}$$

Compare the entries of UT and TU.

The $(n, 1)$ entry of TU is

$$u_{11}t_{n1}$$

So $u_{11}t_{n1} - u_{nn}t_{n1} = 0$ is always true, but u_{11} is not always equal to u_{nn}.
So we conclude that

$$t_{n1} = 0 \tag{1}$$

The $(n, 2)$ entry of TU is

$$u_{12}t_{n1} + u_{22}t_{n2}$$

From (1), we know that $t_{n1} = 0$, so the $(n, 2)$ entry of TU is $u_{22}t_{n2}$.

So $u_{22}t_{n2} - u_{nn}t_{n2} = 0$ is always true, but u_{22} is not always equal to u_{nn}.

So we conclude that

$$t_{n2} = 0 \qquad\qquad (2)$$

. . .

The (n, k) entry of TU is

$$u_{1k}t_{n1} + u_{2k}t_{n2} + \cdots + u_{kk}t_{nk}$$

From (1) to (k-1), we know that $t_{ni} = 0, \quad \forall i \in \{1, 2, \cdots, k-1\}$, so the (n, k) entry of TU is $u_{kk}t_{nk}$.

So $u_{kk}t_{nk} - u_{nn}t_{nk} = 0$ is always true, but u_{kk} is not always equal to u_{nn}.

So we conclude that

$$t_{nk} = 0 \qquad\qquad (k)$$

. . .

Follow this iterating process, we conclude that for $\forall i \in \{1, 2, \cdots, n-1\}$, $t_{ni} = 0$. As for the (n, n) entry, we can finally get

$$u_{nn}t_{nn} = u_{nn}t_{nn}$$

which is always true.

So, we get

$$T = \begin{bmatrix} T_{n-1} & \beta_1 \\ & t_{nn} \end{bmatrix}$$

Thus,

$$\begin{bmatrix} U_{n-1} & \alpha_1 \\ & u_{nn} \end{bmatrix} \begin{bmatrix} T_{n-1} & \beta_1 \\ & t_{nn} \end{bmatrix} = \begin{bmatrix} T_{n-1} & \beta_1 \\ & t_{nn} \end{bmatrix} \begin{bmatrix} U_{n-1} & \alpha_1 \\ & u_{nn} \end{bmatrix}$$

where U_{n-1} and T_{n-1} is respectively the (n-1) the sequential principal matrix of U and T.

According to the calculation of partitioned matrices, we get

$$U_{n-1}T_{n-1} = T_{n-1}U_{n-1}$$

From the **induction hypothesis**, we know that T_{n-1} is upper triangular, so T is upper triangular.

Hence, we have prove that **when $n = k$, if U is upper triangular and $UT = TU$, then T is upper triangular**. According to mathematical induction, for $\forall n \in \mathbb{N}_+$

$$
T = \begin{bmatrix}
t_{11} & t_{12} & \cdots & t_{1n} \\
 & t_{22} & \cdots & t_{2n} \\
 & & \ddots & \vdots \\
 & & & t_{nn}
\end{bmatrix}
$$

T is upper triangluar, which means

$$B = XTX^{-1}$$

So A,B are simultaneously triangularized by X. $\square$

2. Counter example

Let

$$
A = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ & 0 \end{bmatrix}
$$

We know that A and B commutes and A is already in Jordan Normal Form.

We try to find all the matrix X s.t. $B = X \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} X^{-1}$.

Note that

$$
Ker(B) = span(\begin{bmatrix} 1 \\ 0 \end{bmatrix})
$$

$$
Ker(B^2) - Ker(B) = span(\begin{bmatrix} 0 \\ 1 \end{bmatrix})
$$

So $\forall m \in (Ker(B^2) - Ker(B))$

$$
m = \begin{bmatrix} 0 \\ c \end{bmatrix}
$$

where $c \in \mathbb{C} - \{0\}$. So we get all the matrix X

$$
X = c \begin{bmatrix} 2 & \\ & 1 \end{bmatrix}
$$

But

$$
AX = c \begin{bmatrix} 2 & 1 \\ & 1 \end{bmatrix} \quad XA = c \begin{bmatrix} 2 & 2 \\ & 1 \end{bmatrix}
$$

So $AX \neq XA$, which means $A \neq XAX^{-1} = X \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} X^{-1}$.

So A and B can not be simultaneously put into Jordan normal form. $\square$

4   Exercise 1.4.4

There is no such function f.

Lemma   If $f \circ g$ is bijective, then f is surjective and g is injective.

Proof. Let $f : B \rightarrow C$, $g : A \rightarrow B$.

If f is not surjective, then $\exists c \in C$ s.t. $\forall b \in B$, $f(b) \neq c$. But $f \circ g$ is bijective, which means that

$$\forall c \in C, \exists a \in A \ s.t. \ c = f(g(a))$$

We get a contridiction.

If g is not injective, which means $\exists a_1 \neq a_2$ s.t. $g(a_1) = g(a_2)$, so $f(g(a_1)) = f(g(a_2))$. But $f \circ g$ is bijective, which means

$$\forall a_1 \neq a_2, \ f(g(a_1)) \neq f(g(a_2))$$

We get a contridiction.   □

From $f \circ f = -x$ and **Lemma**, we know that f is bijective. So $\forall y \in \mathbb{R}$, $\exists ! x \in \mathbb{R}$ s.t. $y = f(x)$. Note that $f \in C(\mathbb{R})$. So f either strictly increases or decreases.

Without loss of generalization, let f strictly increases.

Consider $f \circ f \circ f$, from $f \circ f(x) = -x$, we know that

$$f(-x) = f \circ f \circ f(x) = -f(x)$$

So f is an odd function. $f(0) = 0$, $\forall x > 0$, $f(x) > 0$.

Pick $\forall c > 0$, we have $f(c) > 0$ and thus $f(f(c)) > 0$. Note that $f(f(c)) = -c < 0$. We get a contradiction.

As a result, there is no function that satisfies the all the following 3 properties:

1. $f : \mathbb{R} \rightarrow \mathbb{R}$

2. $f \in C(\mathbb{R})$

3. $f \circ f(x) + x = 0$

□