HW 8 Multilinear Maps

Liu Mingdao 2020011156

Exercise 1.8.1

1. We first express α, β, γ in the standard dual basis. Let

$$\alpha = \alpha_1 \boldsymbol{e}_1^T + \alpha_2 \boldsymbol{e}_2^T$$

$$\beta = \beta_1 \boldsymbol{e}_1^T + \beta_2 \boldsymbol{e}_2^T$$

$$\gamma = \gamma_1 \boldsymbol{e}_1^T + \gamma_2 \boldsymbol{e}_2^T$$

Then the (i, j, k) entry of $\alpha \otimes \beta \otimes \gamma$ is

$$\alpha_1\beta_1\gamma_1 \boldsymbol{e}_1^T \otimes \boldsymbol{e}_1^T \otimes \boldsymbol{e}_1^T + \alpha_2\beta_1\gamma_1 \boldsymbol{e}_2^T \otimes \boldsymbol{e}_1^T \otimes \boldsymbol{e}_1^T + \alpha_1\beta_2\gamma_1 \boldsymbol{e}_1^T \otimes \boldsymbol{e}_2^T \otimes \boldsymbol{e}_1^T + \alpha_2\beta_2\gamma_1 \boldsymbol{e}_2^T \otimes \boldsymbol{e}_2^T \otimes \boldsymbol{e}_1^T$$
$$+\alpha_1\beta_1\gamma_2 \boldsymbol{e}_1^T \otimes \boldsymbol{e}_1^T \otimes \boldsymbol{e}_2^T + \alpha_2\beta_1\gamma_2 \boldsymbol{e}_2^T \otimes \boldsymbol{e}_1^T \otimes \boldsymbol{e}_2^T + \alpha_1\beta_2\gamma_2 \boldsymbol{e}_1^T \otimes \boldsymbol{e}_2^T \otimes \boldsymbol{e}_2^T + \alpha_2\beta_2\gamma_2 \boldsymbol{e}_2^T \otimes \boldsymbol{e}_2^T \otimes \boldsymbol{e}_2^T$$

2. We construct M_E. For $\forall M \in (\mathbb{R}^2)^* \otimes (\mathbb{R}^2)^* \otimes (\mathbb{R}^2)$, decompose it into rank one tensors $M = \sum_{i=1}^{r} \alpha_i \otimes \beta_i \otimes \gamma_i$. Then define $M_E : M \to \sum_{i=1}^{r} \alpha_i E \otimes \beta_i \otimes \gamma_i$. Then M_E is well-defined transformation on $(\mathbb{R}^2)^* \otimes (\mathbb{R}^2)^* \otimes (\mathbb{R}^2)$.

3. Note that for arbitrary rank-one tensor in $(\mathbb{R}^l)^* \otimes (\mathbb{R}^m)^* \otimes (\mathbb{R}^n)^*$ like $\alpha_i \otimes \beta_i \otimes \gamma_i$, its image under M_E is still rank-one.

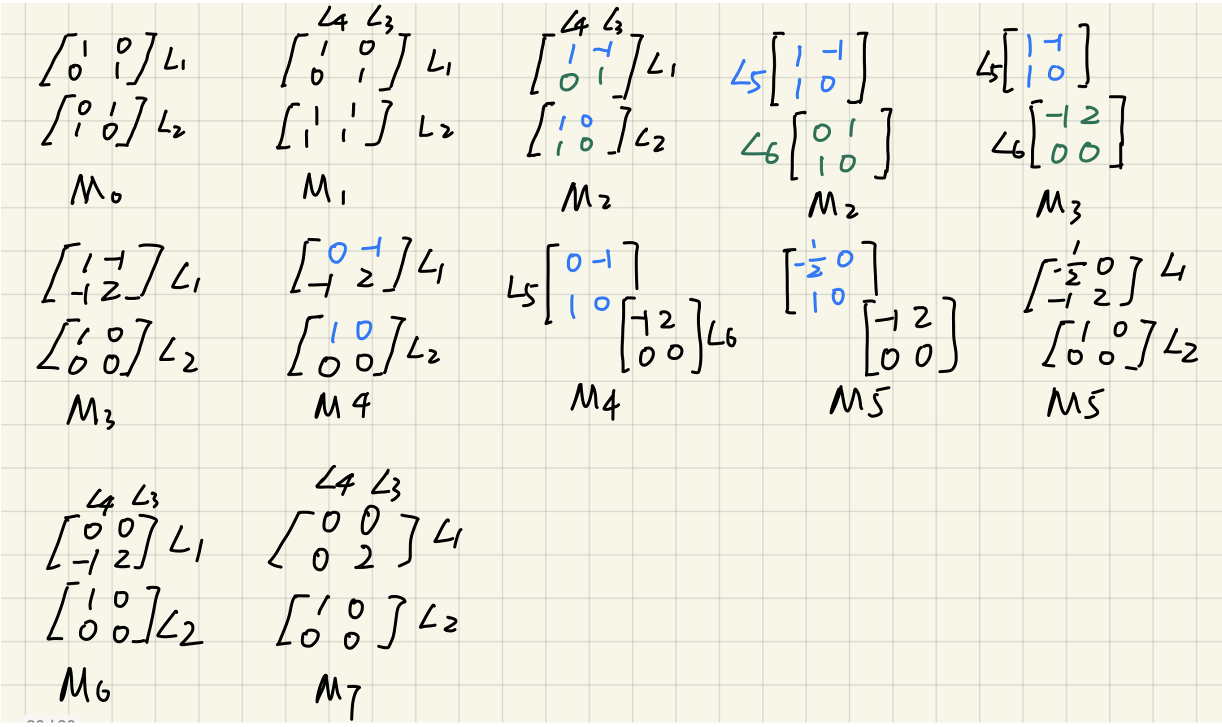
For $\forall M \in (\mathbb{R}^l)^* \otimes (\mathbb{R}^m)^* \otimes (\mathbb{R}^n)^*$ with rank r. Let $M' = M_E(M)$. Suppose the rank of M' is r'. From the definition of rank, we decompose $M = \sum_{i=1}^{r} \alpha_i \otimes \beta_i \otimes \gamma_i$, where $\{\alpha_i \otimes \beta_i \otimes \gamma_i\}$ are distinct rank one tensors. Then $M' = M_E(M) = \sum_{i=1}^{r} \alpha_i E \otimes \beta_i \otimes \gamma_i$. Since we have expressed M' in r rank one tensors, we know $r' \leq r$.

We decompose $M' = \sum_{i=1}^{r'} \delta_i \otimes \epsilon_i \otimes \zeta_i$, where $\{\delta_i \otimes \epsilon_i \otimes \zeta_i\}$ are distinct rank one tensors. Then $M_{E^{-1}}(M') = \sum_{i=1}^{r'} \delta_i E^{-1} \otimes \epsilon_i \otimes \zeta_i$. Note that $M_{E^{-1}}(M') = M$, we have $M = \sum_{i=1}^{r'} \delta_i E^{-1} \otimes \epsilon_i \otimes \zeta_i$. Since we have expressed M in r' rank one tensors, we know $r \leq r'$.

Combine the two conclusions above, we know $r = r'$, i.e. $rank(M) = rank(M_E(M))$.

4. Suppose the i-th "2D" layer of $M \in (\mathbb{R}^l)^* \otimes (\mathbb{R}^m)^* \otimes (\mathbb{R}^n)^*$ has rank r. For $M(-,-,\boldsymbol{e}_i) \in (\mathbb{R}^l)^* \otimes (\mathbb{R}^m)^*$, we have $rank(M(-,-,\boldsymbol{e}_i)) = r$. Suppose $rank(M) = r_M$, i.e. $M = \sum_{i=1}^{r_M} \alpha_i \otimes \beta_i \otimes \gamma_i$. Then feed $\boldsymbol{e}_i$, we know $M(-,-,\boldsymbol{e}_i) = \sum_{i=1}^{r_M} \gamma_i(\boldsymbol{e}_i)\alpha_i \otimes \beta_i$. We have expressed $M(-,-,\boldsymbol{e}_i)$ in r_M rank one tensors, we get $r \leq r_M$, i.e. $rank(M)$ is at least r.

5. $rank(M) = 2$. Note that the first layer of M has rank 2. From the previous subproblem we know that $rank(M) \geq 2$. Now we conduct a series of elementry "layer" action. To make it clearer, I express it in the following diagram. Let $M_0 = M$.



1. Add L_1 to L_2, get M_1.
2. Add $-L_4$ to L_3, get M_2.
3. In M_2, let L_5 be the layer “behind”, and L_6 be the layer “forward”.
4. Add $-L_5$ to L_6, get M_3.
5. In M_3, let L_1 be the layer “upward”, and L_2 be the layer “downward”.
6. Add $-L_2$ to L_1, get M_4.
7. In M_4, let L_5 be the layer “behind”, and L_6 be the layer “forward”.
8. Add $\frac{1}{2}L_6$ to L_5, get M_5.
9. In M_5, let L_1 be the layer “upward”, and L_2 be the layer “downward”.
10. Add $\frac{1}{2}L_2$ to L_1, get M_6.
11. Add $\frac{1}{2}L_3$ to L_4, get M_7.

Note that $M_7 = \boldsymbol{e}_1^T \otimes \boldsymbol{e}_1^T \otimes \boldsymbol{e}_1^T + 2\boldsymbol{e}_2^T \otimes \boldsymbol{e}_2^T \otimes \boldsymbol{e}_2^T$. So $rank(M) = rank(M_7) \leq 2$. Conbine the two conclusions above, we have $rank(M) = 2$. $\square$

Exercise 1.8.2 We suppose $M \in (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*$, then $M = \sum\limits_{i,j,k\in\{1,2,3\}} (i+j+k)\boldsymbol{e}_i^T \boldsymbol{e}_j^T \boldsymbol{e}_k^T$.

1. By direct calculation, we have $M(\boldsymbol{v},\boldsymbol{v},\boldsymbol{v}) = 3x^3 + 6y^3 + 9z^3 + 12x^2y + 15x^2z + 15y^2x + 21y^2z + 21z^2x + 24z^2y + 36xyz$.

2. We try to find the (i,j,k) entry of M^σ. For $\forall i,k,j \in \{1,2,3\}$, construct $p : \{1,2,3\} \to \{1,2,3\}$ s.t. $p(1) = i, p(2) = j, p(3) = k$. Then

$$\begin{aligned} M^\sigma(\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k) &= M^\sigma(\boldsymbol{e}_{p(1)}, \boldsymbol{e}_{p(2)}, \boldsymbol{e}_{p(3)}) \\ &= M(\boldsymbol{e}_{p(\sigma(1))}, \boldsymbol{e}_{p(\sigma(2))}, \boldsymbol{e}_{p(\sigma(3))}) \\ &= p\circ\sigma(1) + p\circ\sigma(2) + p\circ\sigma(3) \\ &= p(1) + p(2) + p(3) \\ &= i + j + k \end{aligned}$$

Then M and M^σ have the same entries, i.e. $M = M^\sigma$.

3. Consider $M(-,-,\boldsymbol{e}_1)$. For $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$, $M(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{e}_1) = \boldsymbol{x}^T A \boldsymbol{y}$, where $A = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}$. Note that the row reduced echelon form of A is $\begin{bmatrix} 1 & 0 & -1 \\ & 1 & 2 \\ & & 0 \end{bmatrix}$, so $rank(A) = 2$. From **1.8.1.4**, $rank(M) \geq 2$. Let $A = \alpha_1\beta_1^T + \alpha_2\beta_2^T$, $\gamma = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \delta = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, then we know that

$$\begin{aligned} M &= (\alpha_1^T \otimes \beta_1^T + \alpha_2^T \otimes \beta_2^T) \otimes \gamma^T + \gamma^T \otimes \gamma^T \otimes \delta^T \\ &= \alpha_1^T \otimes \beta_1^T \otimes \gamma^T + \alpha_2^T \otimes \beta_2^T \otimes \gamma^T + \gamma^T \otimes \gamma^T \otimes \delta^T \end{aligned}$$

M can be expressed as the sum of three “rank-one” things, so $rank(M) \leq 3$.

$\square$

Exercise 1.8.3

1. For $\forall \lambda, \mu \in \mathbb{R}$, $\forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$, $\forall \boldsymbol{w}_1, \boldsymbol{w}_2 \in W$, we have

$$\begin{aligned} X \otimes Y(\lambda\boldsymbol{v}_1 + \mu\boldsymbol{v}_2, \boldsymbol{w}_1) &= X(\lambda\boldsymbol{v}_1 + \mu\boldsymbol{v}_2) \otimes Y(\boldsymbol{w}_1) \\ &= X(\lambda\boldsymbol{v}_1) \otimes Y(\boldsymbol{w}_1) + X(\mu\boldsymbol{v}_2) \otimes Y(\boldsymbol{w}_1) \\ &= \lambda X \otimes Y(\boldsymbol{v}_1, \boldsymbol{w}_1) + \mu X \otimes Y(\boldsymbol{v}_2, \boldsymbol{w}_1) \\ X \otimes Y(\boldsymbol{v}_1, \lambda\boldsymbol{w}_1 + \mu\boldsymbol{w}_2) &= X(\boldsymbol{v}_1) \otimes Y(\lambda\boldsymbol{w}_1 + \mu\boldsymbol{w}_2) \\ &= X(\boldsymbol{v}_1) \otimes Y(\lambda\boldsymbol{w}_1) + X(\boldsymbol{v}_1) \otimes Y(\mu\boldsymbol{w}_2) \\ &= \lambda X \otimes Y(\boldsymbol{v}_1, \boldsymbol{w}_1) + \mu X \otimes Y(\boldsymbol{v}_1, \boldsymbol{w}_2) \end{aligned}$$

So $X \otimes Y$ is bilinear.

2. Note that trace is independent of basis, we try to solve this problem under a certain basis. Suppose $dimV = n$, $dimW = m$, $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$ forms a basis of V, $\{\boldsymbol{w}_1, \cdots, \boldsymbol{w}_m\}$ forms a basis of W. Then

$$\boldsymbol{T} : \{\underbrace{\underbrace{\boldsymbol{v}_1 \otimes \boldsymbol{w}_1, \boldsymbol{v}_1 \otimes \boldsymbol{w}_2, \cdots, \boldsymbol{v}_1 \otimes \boldsymbol{w}_n}_{n \text{ tensors as a group}}, \cdots, \underbrace{\boldsymbol{v}_m \otimes \boldsymbol{w}_1, \boldsymbol{v}_m \otimes \boldsymbol{w}_2, \cdots, \boldsymbol{v}_m \otimes \boldsymbol{w}_n}_{n \text{ tensors as a group}}}_{m \text{ groups in total}}\}$$

form a basis of $V \otimes W$, and in this problem we always arrange the basis in the same order as presented above. Suppose the matrix of X under $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$ is $A = [a_1, \cdots, a_n]_{n\times n}$, and the matrix of Y under $\{\boldsymbol{w}_1, \cdots, \boldsymbol{w}_m\}$ is $B = [b_1, \cdots, b_m]_{m\times m}$.

From the previous subproblem, we treat $X \otimes Y$ as a linear trasformation on $V \otimes W$. We know

$$X \otimes Y(\boldsymbol{v}_i \otimes \boldsymbol{w}_j) = \boldsymbol{T} a_i \otimes b_j$$

where $\boldsymbol{T} a_i \otimes b_j$ denotes using $a_i \otimes b_j$ as coefficients to form a linear compsition of basis $\boldsymbol{T}$. Thus, the matrix of $X \otimes Y$ under basis $\boldsymbol{T}$ is

$$M = [\underbrace{a_1 \otimes b_1, a_1 \otimes b_2, \cdots, a_1 \otimes b_n}_{n \text{ vectors as a group}}, \cdots, \underbrace{a_m \otimes b_1, a_m \otimes b_2, \cdots, a_m \otimes b_n}_{n \text{ vectors as a group}}]_{mn\times mn}$$

Then $trace(M) = \sum_{i=1}^{m}(a_{ii} \cdot \sum_{j=1}^{n} b_{jj}) = (\sum_{i=1}^{m} a_{ii})(\sum_{j=1}^{n} b_{jj}) = trace(X)trace(Y)$. □

Exercise 1.8.4 Map *trace* is in the space $(\mathbb{R}^n)^* \otimes \mathbb{R}^n$.

For $\forall M \in \mathbb{R}^n \otimes (\mathbb{R}^n)^*$, decompose it into the tensors of the standard basis, we have

$$M = \sum_{i,j\in\{1,\cdots,n\}} m_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j^T$$

From the definition of *trace* map, we know $trace(M) = \sum_{k=1}^{n} m_{kk}$. Thus, the entries of *trace* in $(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ under the standard basis is $trace = \sum_{k=1}^{n} \boldsymbol{e}_k^T \otimes \boldsymbol{e}_k$. □