HW 7 Tangent Vectors and Cotangent Vectors

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Exercise 1.7.1

1. Consider $g(x) = c$, $c \in \mathbb{R}$. If $c = 0$, note that $\boldsymbol{v}$ is a linear map, $v(g) = v(0) = 0$. If $c \neq 0$, then for $\forall \boldsymbol{p} \in \mathbb{R}^3$, $c \cdot v(g) = v(c \cdot g) = v(g^2) = g(\boldsymbol{p})v(g) + g(\boldsymbol{p})v(g) = 2c \cdot v(g)$, i.e. $v(g) = 0$.

2. From Leibniz rule we have $v((x - p_1)f) = (x - p_1)\big|_{\boldsymbol{p}} v(f) + f(\boldsymbol{p})v(x - p_1)$. Note that $(x - p_1)\big|_{\boldsymbol{p}} = 0$ and $v(x - p_1) = v(x) - v(p_1) = v(x)$, we have $v((x - p_1)f) = f(\boldsymbol{p})v(x)$. Similarly, we have

$$v((y - p_2)f) = (y - p_2)\big|_{\boldsymbol{p}} v(f) + f(\boldsymbol{p})v(y - p_2) = f(\boldsymbol{p})v(y) - f(\boldsymbol{p})v(p_2) = f(\boldsymbol{p})v(y)$$

$$v((z - p_3)f) = (z - p_3)\big|_{\boldsymbol{p}} v(f) + f(\boldsymbol{p})v(z - p_3) = f(\boldsymbol{p})v(z) - f(\boldsymbol{p})v(p_3) = f(\boldsymbol{p})v(z)$$

3. Suppose $a < 1,\ b < 1,\ c < 1$, then $a = b = c = 0$, contradiction. So at least one number in $\{a, b, c\}$ is larger or equal to 1. Without loss of generalization, assume $a \geq 1$. Let $f = (x - p_1)^{a-1}(y - p_2)^b(z - p_3)^c$, from $a + b + c > 1$, we know $(a - 1) + b + c \geq 1$, so at least one factor of f has a positive exponent and other factors have exponent 1. Note that $(x - p_1)\big|_{\boldsymbol{p}} = (y - p_2)\big|_{\boldsymbol{p}} = (z - p_3)\big|_{\boldsymbol{p}} = 0$. So $f(\boldsymbol{p}) = 0$. From the previous subproblem we have $v((x - p_1)^a(y - p_2)^b(z - p_3)^c) = v((x - p_1)f) = f(\boldsymbol{p})v(x) = 0$. (When $b \geq 1$ subsititue $\{a, b, c\}$ in this subproblem with $\{b, a, c\}$, in the case of $c \geq 1$, subsititue $\{a, b, c\}$ with $\{c, b, a\}$.)

4. Using Taylor Expansion at $\boldsymbol{p}$ for f, we have $f = f(p_1, p_2, p_3) + \frac{\partial f}{\partial x}(\boldsymbol{p})(x - p_1) + \frac{\partial f}{\partial y}(\boldsymbol{p})(y - p_2) + \frac{\partial f}{\partial z}(\boldsymbol{p})(z - p_3) + R(x, y, z)$, where $R(x, y, z)$ is the remainder term. Note that $R(x, y, z)$ can be write as linear composition of terms $(x - p_1)^a(y - p_2)^b(z - p_3)^c$, where $a + b + c \geq 2$ (since we have listed the terms when $a + b + c \leq 1$). Then, from the previous subproblem, we have $v(R(x, y, z)) = 0$. So $v(f) = v(f(p_1, p_2, p_3)) + v(\frac{\partial f}{\partial x}(\boldsymbol{p})(x - p_1)) + v(\frac{\partial f}{\partial y}(\boldsymbol{p})(y - p_2)) + v(\frac{\partial f}{\partial z}(\boldsymbol{p})(z - p_3)) = \frac{\partial f}{\partial x}(\boldsymbol{p})v(x) + \frac{\partial f}{\partial y}(\boldsymbol{p})v(y) + \frac{\partial f}{\partial z}(\boldsymbol{p})v(z)$.

5. From the definition of "direction derivative", we know for $\forall f$ and given point $\boldsymbol{p} \in \mathbb{R}^3$,

$$\begin{aligned}
\nabla_v f &= \lim_{t \to 0^+} \frac{f(\boldsymbol{p} + t\boldsymbol{v}) - f(\boldsymbol{p})}{t} \\
&= \lim_{t \to 0^+} \frac{\frac{\partial f}{\partial x}(\boldsymbol{p})v(x)t + \frac{\partial f}{\partial y}(\boldsymbol{p})v(y)t + \frac{\partial f}{\partial z}(\boldsymbol{p})v(z)t + o(||\boldsymbol{v}||t)}{t} \\
&= \frac{\partial f}{\partial x}(\boldsymbol{p})v(x) + \frac{\partial f}{\partial y}(\boldsymbol{p})v(y) + \frac{\partial f}{\partial z}(\boldsymbol{p})v(z) \\
&= v(f)
\end{aligned}$$

□

Exercise 1.7.2 In this problem, assume that X is linear.

1. $X_{\boldsymbol{p}}(fg) = X(fg)\big|_{\boldsymbol{p}} = (fX(g) + gX(f))\big|_{\boldsymbol{p}} = f(\boldsymbol{p})X(g)\big|_{\boldsymbol{p}} + g(\boldsymbol{p})X(f)\big|_{\boldsymbol{p}} = f(\boldsymbol{p})X_p(g) + g(\boldsymbol{p})X_p(f)$. For any given $\boldsymbol{p} \in \mathbb{R}^3$, X_p satisify the Leibniz rule, so X_p is a derivation at $\boldsymbol{p}$.

2. For any given $\boldsymbol{p} \in \mathbb{R}^3$, let $\boldsymbol{w}$ denote the corresponding vector of X at $\boldsymbol{p}$, then $df(X)\big|_{\boldsymbol{p}} = df(\boldsymbol{w})$. Note that $\boldsymbol{w} = \begin{bmatrix} X_{\boldsymbol{p}}(x) \\ X_{\boldsymbol{p}}(y) \\ X_{\boldsymbol{p}}(z) \end{bmatrix}$, so $df(X)\big|_{\boldsymbol{p}} = \begin{bmatrix} \frac{\partial f}{\partial x}(\boldsymbol{p}) & \frac{\partial f}{\partial y}(\boldsymbol{p}) & \frac{\partial f}{\partial z}(\boldsymbol{p}) \end{bmatrix} \begin{bmatrix} X_{\boldsymbol{p}}(x) \\ X_{\boldsymbol{p}}(y) \\ X_{\boldsymbol{p}}(z) \end{bmatrix}$. From the previous subproblem $X_{\boldsymbol{p}}(f)$ is a derivation at $\boldsymbol{p}$, and from **1.7.1.4** we know a derivation $X_{\boldsymbol{p}}(f) = X(f)(\boldsymbol{p}) = \frac{\partial f}{\partial x}(\boldsymbol{p})X_{\boldsymbol{p}}(x) + \frac{\partial f}{\partial y}(\boldsymbol{p})X_{\boldsymbol{p}}(y) + \frac{\partial f}{\partial z}(\boldsymbol{p})X_{\boldsymbol{p}}(z) = df(X)\big|_{\boldsymbol{p}}$.

3. For $\forall f, g \in V$, we have

$$\begin{aligned}
(X \circ Y - Y \circ X)(fg) &= X \circ Y(fg) - Y \circ X(fg) \\
&= X(Y(fg)) - Y(X(fg)) \\
&= X(f \cdot Y(g) + g \cdot Y(f)) - Y(f \cdot X(g) + g \cdot X(f)) \\
&= X(f \cdot Y(g)) + X(g \cdot Y(f)) - Y((f \cdot X(g)) - Y(g \cdot X(f)) \\
&= (Y(g) \cdot X(f) + f \cdot X \circ Y(g)) + (X(g) \cdot Y(f) + gX \circ Y(f))) \\
&\quad - (Y(f) \cdot X(g) + fY \circ X(g)) - (Y(g) \circ X(f) + gY \circ X(f)) \\
&= f(X \circ Y - Y \circ X)(g) + g(X \circ Y - Y \circ X)(f)
\end{aligned}$$

$(X \circ Y - Y \circ X)$ satisfies the Leibniz rule, so $(X \circ Y - Y \circ X)$ is a vector field.

4. For skew-symetric matrix A, B, we have

$$\begin{aligned}
(AB - BA)^T &= (AB)^T - (BA)^T \\
&= (B^T A^T) - (A^T B^T) \\
&= (-B)(-A) - (-A)(-B) \\
&= -(AB - BA)
\end{aligned}$$

So $(AB - BA)$ is skew-symetric.