Project Option One

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Problem 2.1

Note that

$$\begin{aligned}\frac{d}{dt}(e^{-At}y) &= -e^{-At}Ay + e^{-At}\frac{dy}{dt} \\ &= e^{-At}(\frac{dy}{dt} - Ay) \\ &= e^{-At}f(t,y)\end{aligned}$$

Integral from 0 to t on both sides, we have

$$\begin{aligned}\int_0^t d(e^{-At}y) &= \int_0^t e^{-As}f(s,y)ds \\ e^{-At}y(t) - e^{-A\cdot 0}y(0) &= \int_0^t e^{-As}f(s,y)ds \\ y(t) - e^{At}c &= \int_0^t e^{A(t-s)}f(s,y)ds\end{aligned}$$

Finally

$$y(t) = e^{At}c + \int_0^t e^{A(t-s)}f(s,y)ds$$

□

Problem 2.2

We know that both $f(x) = cos(xt)$ and $g(x) = x^{-1}sin(xt)$ are even functions of x. Note that f and g are analytic functions, so $f(x)$ and $g(x)$ can be expanded at $x = 0$ as power series with only even powers of x. Then $f(\sqrt{A})$ and $g(\sqrt{A})$ are just the power series of A, which is totally independent from the choice of $\sqrt{A}$.

Therefore the value of $y(t) = f(\sqrt{A})y_0 + g(\sqrt{A})y_0'$ doesn't depend on the choice of $\sqrt{A}$, so it can be unique. □

Problem 2.3

First we decompose A into Jordan normal form

$$A = Z\begin{bmatrix} J_1 & \\ & J_2\end{bmatrix}Z^{-1}$$

where the eigenvalues of J_1 is in the left half plane and the eigenvalues of J_2 in the right half plane. Partition Z as $[Z_1\ Z_2]$. We have

$$W = \frac{1}{2}(sign(A) + I) = [O\ Z_2]Z^{-1}$$

Let V_+ be the sum of all A-invariant subspaces whose eigenvalues have a positive real part, V_- be the sum of all A-invariant subspaces whose eigenvalues have a negative real part. From Jordan canonical form, we konw

Z_1 form a basis of V_-, Z_2 form a basis of V_+. Since $W = [O\ Z_2]Z^{-1}$, the linearly independent columns of W from a basis of V_+, i.e. $Ran(W) = V_+$. Partition Q as $[Q_1\ Q_2]$. Then we get

$$W = Q_1[R_{11}\ R_{12}]\Pi^T$$

So the columns of Q_1 form a basis of $Ran(W)$, which is V_+. Then $AQ_1 = Q_1B$ for some $B \in M_q(\mathbb{C})$.

Then we know by direct calculation

$$Q^TAQ = \begin{bmatrix} Q_1^TAQ_1 & Q_1^TAQ_2 \\ Q_2^TAQ_1 & Q_2^TAQ_2 \end{bmatrix} = \begin{bmatrix} B & Q_1^TAQ_2 \\ Q_2^TQ_1B & Q_2^TAQ_2 \end{bmatrix} = \begin{bmatrix} B & Q_1^TAQ_2 \\ O & Q_2^TAQ_2 \end{bmatrix} := \begin{bmatrix} A_{11} & A_{12} \\ & A_{22} \end{bmatrix}$$

For any eigenpair (λ, x) of B (which is A_{11}), we have $Bx = \lambda x$, thus $AQ_1x = Q_1Bx = \lambda Q_1x$, so (λ, Q_1x) is an eigenpair of A. Note that $Q_1x \in V_+$, we have $Re(\lambda) > 0$.

Note that A have no eigenvalue on the imaginary axis, so $\mathbb{C}^n = V_- \oplus V_+$. Thus, the columns of Q_2 form a basis of V_-. Let $AQ_2 = Q_2C$ for some $C \in M_{(n-q)}(\mathbb{C})$. Note that $C = Q_2^TAQ_2 = A_{22}$. For any eigenpair (μ, y) of C, we have $Cy = \mu y$, thus $AQ_2y = Q_2Cy = \mu Q_2y$, so (μ, Q_2y) is an eigenpair of A. Note that $Q_2y \in V_-$, we have $Re(\mu) < 0$. □

Problem 2.4

First show that $A\#B = A^{\frac{1}{2}}B^{\frac{1}{2}}$. Notice that

$$\begin{aligned}(A^{\frac{1}{2}}B^{-\frac{1}{2}})^2 &= A^{\frac{1}{2}}B^{-\frac{1}{2}}A^{\frac{1}{2}}B^{-\frac{1}{2}} \\ &= A^{\frac{1}{2}}A^{\frac{1}{2}}B^{-\frac{1}{2}}B^{-\frac{1}{2}} \\ &= AB^{-1}\end{aligned}$$

$A^{\frac{1}{2}}$ and $B^{-\frac{1}{2}}$ commute, so they can be simultaneously diagonoalized. Let $A^{\frac{1}{2}} = X\Lambda_AX^{-1}$, $B^{-\frac{1}{2}} = X\Lambda_BX^{-1}$, we know $A^{\frac{1}{2}}B^{-\frac{1}{2}} = X\Lambda_A\Lambda_BX^{-1}$, $A^{\frac{1}{2}}B^{-\frac{1}{2}}$ is positive definite. Note that the principal square root of a matrix is unique. So $A^{\frac{1}{2}}B^{-\frac{1}{2}} = (AB^{-1})^{\frac{1}{2}}$. Thus, $A\#B = (AB^{-1})^{\frac{1}{2}}B = A^{\frac{1}{2}}B^{-\frac{1}{2}}B = A^{\frac{1}{2}}B^{\frac{1}{2}}$.

Then show that for this hermitian positive definite A, $e^{\frac{1}{2}log(A)} = A^{\frac{1}{2}}$. We know that $(e^{\frac{1}{2}log(A)})^2 = e^{\frac{1}{2}log(A)}e^{\frac{1}{2}log(A)} = e^{log(A)} = A$. So $e^{\frac{1}{2}log(A)} = \sqrt{A}$. Note that all eigenvalues of A are positive so all the eigenvalues of $log(A)$ are real, which means the eigenvalues of $e^{\frac{1}{2}log(A)}$ are all positive. So $e^{\frac{1}{2}log(A)} = A^{\frac{1}{2}}$.

Note that $\frac{1}{2}log(A)$ and $\frac{1}{2}log(B)$ commute. We have

$$\begin{aligned}E(A,B) &= e^{\frac{1}{2}log(A)+\frac{1}{2}\log(B)} \\ &= e^{\frac{1}{2}log(A)}e^{\frac{1}{2}\log(B)} \\ &= A^{\frac{1}{2}}B^{\frac{1}{2}}\end{aligned}$$

□

Problem 2.5

(2.27a) Note that $AA^{-1}A = A$. From definition, $A\#A = A$.

(2.27b)

$$\begin{aligned}(A\#B)^{-1} &= (B(B^{-1}A)^{\frac{1}{2}})^{-1}\\ &= (B^{-1}A)^{-\frac{1}{2}}B^{-1}\\ &= ((B^{-1}A)^{-1})^{\frac{1}{2}}B^{-1}\\ &= (A^{-1}B)^{\frac{1}{2}}B^{-1}\\ &= A^{-1}\#B^{-1}\end{aligned}$$

(2.27c)

$$\begin{aligned}A\#B &= B(B^{-1}A)^{\frac{1}{2}}\\ &= AA^{-1}B(B^{-1}A)^{\frac{1}{2}}\\ &= A(B^{-1}A)^{-1}(B^{-1}A)^{\frac{1}{2}}\\ &= A(B^{-1}A)^{-\frac{1}{2}}\\ &= A(A^{-1}B)^{\frac{1}{2}}\\ &= B\#A\end{aligned}$$

(2.27d) Use "≥ 0" to denote hermitian positive semidefinite. For any complex square matrix $C \geq 0$ and $D \geq 0$, for $\forall x \in \mathbb{C}^n$, $x^*CDCx = x^*C^*DCx = (Cx)^*D(Cx) \geq 0$. So $CDC \geq 0$.
Let $T = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$. Then $T \geq 0$. Let$T = P\Lambda P^*$. we have $(T^{\frac{1}{2}} - I)^2 = P(\Lambda^{\frac{1}{2}} - I)^2P^*$, which means $(T^{\frac{1}{2}} - I)^2 \geq 0$. Then we know $(T + I - 2T^{\frac{1}{2}}) \geq 0$. Simultaneously multiply hermitian positive definite $B^{\frac{1}{2}}$ from both sides, we get $(B^{\frac{1}{2}}TB^{\frac{1}{2}} + B - 2B^{\frac{1}{2}}T^{\frac{1}{2}}B^{\frac{1}{2}}) \geq 0$. I.e. $(A + B - 2A\#B) \geq 0$. □

Problem 2.6

Let $X = (A^{-1}B)^{\frac{1}{2}}$ and the two eigenvalues of X be $\lambda_1,\ \lambda_2$. We know $tr(X) = \lambda_1 + \lambda_2$, $det(X) = \lambda_1\lambda_2 = \frac{\beta}{\alpha}$. Then

$$\begin{aligned}det(\alpha^{-1}A + \beta^{-1}B) &= det(A \cdot (\alpha^{-1}I + \beta^{-1}X^2))\\ &= det(A) \cdot det(\alpha^{-1}I + \beta^{-1}X^2)\\ &= \alpha^2(\frac{1}{\alpha} + \frac{\lambda_1^2}{\beta})(\frac{1}{\alpha} + \frac{\lambda_2^2}{\beta})\\ &= \frac{\alpha}{\beta}(\lambda_1 + \lambda_2)^2\\ &= \frac{\alpha}{\beta}tr(X)^2\end{aligned}$$

We know that X is the principal square root of $A^{-1}B$, which means $tr(X) \geq 0$. So

$$tr(X) = \sqrt{\frac{\beta}{\alpha}det(\alpha^{-1}A + \beta^{-1}B)}$$

From Cayley-Hamilton Theorem, we have $X^2 - tr(X)X + det(X)I = O$, i.e.

$$tr(X)(A^{-1}B)^{\frac{1}{2}} = A^{-1}B + det(X)I$$

Multiply A from the left, we get

$$tr(X)(A\#B) = det(X)A + B$$

Simplify this equation, we have

$$
\begin{aligned}
A\#B &= \sqrt{\frac{\alpha}{\beta det(\alpha^{-1}A+\beta^{-1}B)}}B + \frac{\beta}{\alpha}\sqrt{\frac{\alpha}{\beta det(\alpha^{-1}A+\beta^{-1}B)}}A \\
&= \frac{\sqrt{\alpha\beta}}{\sqrt{det(\alpha^{-1}A+\beta^{-1}B))}}(\alpha^{-1}A+\beta^{-1}B)
\end{aligned}
$$

□

Problem 2.7

Use "> 0" to denote hermitian positive definite. For $\forall x \in \mathbb{C}^n$, $x^*RBR^*x = (R^*x)^*B(R^*x) > 0$, so $RBR^* > 0$, thus $(RBR^*)^{\frac{1}{2}} > 0$. $x^*R^{-1}(RBR^*)^{\frac{1}{2}}R^{-*}x = (R^{-*}x)^*(RBR^*)^{\frac{1}{2}}(R^{-*}x) > 0$, then $R^{-1}(RBR^*)^{\frac{1}{2}}R^{-*} > 0$.

We know that the hermitian positive definite solution to $XAX = B$ is unique (proven in Page 45). So we verify that $X = R^{-1}(RBR^*)^{\frac{1}{2}}R^{-*}$ is a solution. We have

$$XAX = R^{-1}(RBR^*)^{\frac{1}{2}}R^{-*}R^*RR^{-1}(RBR^*)^{\frac{1}{2}}R^{-*} = B$$

So $X = R^{-1}(RBR^*)^{\frac{1}{2}}R^{-*}$ is the hermitian positive definite solution to $XAX = B$. We can compute the solution using the formula of X. □