In [1]:
```
#Ref to the paper Calibrating and completing the volatility cube in the s
#J. de Kock
import math
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

The SABR model assumes volatility of the forward price is stochastic. We will look at a swaption with maturity T_m and tenor = $T_n - T_m$. The equations are

$$dS_{m,n}(t) = V_{m,n}(t)S_{m,n}(t)^{\beta_{m,n}}dW_{m,n}(t)$$

$$dV_{m,n}(t) = \sigma_{m,n}V_{m,n}(t)dZ_{m,n}(t)$$

Let $\rho_{m,n}$ be the correlation between the wiener processes $W_{m,n}(t)$ and $Z_{m,n}(t)$. $S_{m,n}(t)$ is the forward swap rate with maturity T_m and tenor $T_n - T_m$. $V_{m,n}(t)$ is the volatility of the forward swap rate. $\sigma_{m,n}$ is the volatility of $V_{m,n}(t)$ which is assumed to be constant. The initial conditions are $V_{m,n}(0) = \alpha$ and $S_{m,n}(0) = S_0$ are known.

The steps to calibrate the market data with the model are as follows. Derive an expression for swaption volatility $\hat{\sigma}_{m,n}(K)$ in the SABR model

1. Derive an expression for option price in the SABR model

   A. Derive pde for call price
   B. solve pde
   C. Set option price to the option price in black model and solve for swaption volatility.

   The function below $SABR(T, K, S_0, \sigma_{m,n}, \alpha, \beta_{m,n}, \rho_{m,n})$ gives us this.

   We should have $\hat{\sigma}_{m,n}(K_{ATM})$ for all m and n. This plane should be complete.

2. Calibrate the model to attain implied volatility for all set of m, n and K. We can use method of least squares to fit. We shall minimize the following function where the sum is over K and m,n fixed.

$$\sqrt{(}(\sum(V_{SABR} - V_{market})^2)$$

Along each m,n set we will callibrate for $\sigma, \alpha, \beta, \rho$. We use Hagan 2012 for the formula

https://www.itwm.fraunhofer.de/fileadmin/ITWM-Media/Zentral/Pdf/Berichte_ITWM/2011/bericht_202.pdf

In [45]:
```
def SABR(T,K,S0,sigma,alpha,beta,rho):  #T = T_m, time to maturity, All 
```

```
    S0K = S0*K
    lS0K = np.log(S0/K)

    z = (sigma/alpha)*((S0K)**((1-beta)/2))*(lS0K)
    x = np.log((np.sqrt(1-2*rho*z+z**2)+z-rho)/(1-rho))

    denom = 1+(((1-beta)*lS0K)**2)/24 + (((1-beta)*lS0K)**4)/1920

    numer = 1 + T*(((((1-beta)*alpha)**2)/(24*(S0K**(1-beta))) + \
    (rho*beta*sigma*alpha)/(4*(S0K**((1-beta)/2))) + ((sigma**2)*(2-3*(rh

    imp_vol = (alpha*numer*(z/x))/(denom*(S0K**((1-beta)/2)))

    return (alpha*numer)/(denom*(S0K**((1-beta)/2))) if np.any(S0==K) els
```

This table gives the at the money implied volatility(Black76 log normal formula) of the strike prices. The first row gives basis points to add to the ATM strike, to get the out of money strikes. For example, 1M10Y ATM, -200bps strike implied volatility is 24.47% + 16.71bps.

We need to bootstrap the swap curve at ATM to calculate the forward swap rate. $S_{m,n}(t)$. So the strike = forward swap rate +/- bps as mentioned on the broker page.

For each maturity/tenor pair (m,n), we have the implied volatility across different strikes, we will calibrate one set of $V_{m,n}(t), \beta, \rho, \sigma$ parameters. The β parameter in SABR is typically selected to match a particular backbone (say β = 0.25 or 0.5). We can either fix this or calibrate it. β si typically around 0.5. We will calibrate $V_{m,n}(t)$ (this controls the overall level of the implied vol), ρ (this controls the skew), and σ (this controls the smile/kurtosis). So for instance, once we have calibrated our SABR model to the swaption market data, we should have as many set of (α, ρ, σ) parameters as the row of the matrix. SABR is convenient because the function returns an implied vol, which can be directly compared to the one quoted by brokers.

In [3]:

```
import pandas as pd
import xlrd

#file_input = xlrd.open_workbook('Swaptions Market Data.xlsx')
#Market_data = file_input.sheet_by_name('Swaptions Prof Tee')

m_data=pd.read_excel("Swaptions Market Data.xlsx",  sheetname= "Market Vo
m_data
#xls_file = pd.ExcelFile('Swaptions Market Data.xlsx')
#mkt_data = xls_file.parse('Swaptions Prof Tee')
```

Out[3]:

	-200	-150	-100	-50	-25	ATM	25	50	100	150	
1M1Y	69.07	39.530	21.630	8.724	3.682	22.50	-1.54	-1.10	1.84	4.988	7
1M2Y	54.55	32.520	17.850	7.087	2.992	28.72	-1.60	-1.88	-0.21	2.305	4
1M3Y	44.14	27.090	14.990	5.965	2.537	29.78	-1.49	-1.98	-1.01	0.945	3
1M5Y	29.12	18.570	10.440	4.172	1.781	26.07	-1.09	-1.51	-0.95	0.466	2
1M10Y	16.71	10.570	5.737	2.149	0.881	24.47	-0.49	-0.65	-0.22	0.734	1
3M1Y	40.54	21.830	11.140	4.225	1.800	27.26	-1.22	-1.94	-2.32	-1.940	-1
3M2Y	28.05	16.580	9.203	3.823	1.701	29.83	-1.27	-2.18	-3.12	-3.290	-3
3M3Y	23.45	14.460	8.200	3.457	1.556	29.98	-1.22	-2.16	-3.31	-3.780	-3
3M5Y	15.39	9.924	5.726	2.405	1.077	26.60	-0.87	-1.58	-2.54	-3.030	-3
3M10Y	9.907	6.438	3.638	1.444	0.626	24.51	-0.52	-0.95	-1.60	-2.020	-2
6M1Y	26.62	15.780	8.828	3.719	1.670	28.54	-1.23	-2.09	-2.93	-3.020	-2
6M2Y	21.89	13.620	7.798	3.342	1.520	29.28	-1.19	-2.08	-3.16	-3.560	-3
6M3Y	18.82	12.030	7.000	3.039	1.396	29.40	-1.13	-2.02	-3.22	-3.820	-4
6M5Y	13.81	9.151	5.441	2.404	1.117	26.74	-0.94	-1.72	-2.87	-3.570	-3
6M10Y	9.231	6.139	3.608	1.556	0.716	24.37	-0.61	-1.13	-1.93	-2.470	-2
9M2Y	17.14	11.320	6.793	3.032	1.406	29.53	-1.14	-2.05	-3.28	-3.930	-4
9M5Y	11.46	7.824	4.773	2.162	1.016	26.48	-0.86	-1.60	-2.71	-3.430	-3
9M10Y	8.164	5.554	3.348	1.490	0.695	24.12	-0.59	-1.10	-1.88	-2.410	-2

In [4]:

```
#The first column is parsed as unicode. Change it to float for ease of ca
for ind in range(0,18):
    m_data[-200][ind] = np.float64(m_data[-200][ind])
```

```
C:\Users\User\Anaconda2\lib\site-packages\ipykernel\__main__.py:3: Setting
WithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame

See the caveats in the documentation: http://pandas.pydata.org/pandas-doc
s/stable/indexing.html#indexing-view-versus-copy
  app.launch_new_instance()
```

In [5]:

```
import scipy.stats as ss
#black76 formula to calculate price call price, T is time to maturity T_i
#This formula returns the undiscounted call price

def d1(T,K,F,sigma):
    return (np.log(F/K) + (sigma**2 / 2) * T)/(sigma * np.sqrt(T))
```

```
def d2(T,K,F,sigma):
    return (np.log(F/K) - (sigma**2 / 2) * T) / (sigma * np.sqrt(T))

def Black76(T,K,F,sigma):
    return (F * ss.norm.cdf(d1(T,K,F,sigma)) - K * ss.norm.cdf(d2(T,K,F,s
```

Bootstrap the ATM strike to attain the forward rate

Method to bootstrap the required forward rates

ref: https://www.math.nyu.edu/~alberts/spring07/Lecture1.pdf

We will use linear interpolation of discount factors that are between 2 nearest libor rates that is given by the data.

We are using libor rates from a different time period. The rates are supposed to be a little higher for the quoted implied volatility. To prevent forward rates being negative later, we shall add 2.5% to all of the rates. In the case, if the forward rates are negative later, we would simply not take the implied volatility for the calibration.

In [6]:
```
#USDLibor = xls_file.parse('USDLibor')
USDLibor= pd.read_excel("Swaptions Market Data.xlsx",  sheetname= "USDLib
USDLibor['Mid'] = 0.5*(USDLibor['Ask']+USDLibor['Bid']) + 2.5 #added to p
USDLibor['Disc_Factor(0,yr)'] = 1/(1+USDLibor['Year']*USDLibor['Mid']/100
USDLibor
```

Out[6]:

	Year	Ask	Bid	Mid	Disc_Factor(0,yr)
0	1	0.309	0.269	2.789	0.972867
1	2	0.363	0.323	2.843	0.946199
2	3	0.463	0.423	2.943	0.918873
3	4	0.614	0.574	3.094	0.889870
4	5	0.794	0.754	3.274	0.859328
5	6	0.975	0.935	3.455	0.828295
6	7	1.152	1.112	3.632	0.797296
7	8	1.319	1.279	3.799	0.766918
8	9	1.471	1.431	3.951	0.737686
9	10	1.607	1.567	4.087	0.709874
10	11	1.728	1.688	4.208	0.683583
11	12	1.834	1.794	4.314	0.658900
12	13	1.927	1.887	4.407	0.635764
13	14	2.006	1.966	4.486	0.614236
14	15	2.070	2.030	4.550	0.594354
15	16	2.123	2.083	4.603	0.575878
16	17	2.165	2.125	4.645	0.558768
17	18	2.198	2.158	4.678	0.542876
18	19	2.225	2.185	4.705	0.527997
19	20	2.246	2.206	4.726	0.514086

In [7]:

```
D = []
D.append(1.0)
for ind in range(0,20):
    D.append(USDLibor['Disc_Factor(0,yr)'][ind])

#Linear interpolation of discount factors, where T1 - 1 < T < T1
def Interpolate(T,T1):
    return float(T1-T)*D[T1-1] + float(T-(T1-1))*D[T1]
```

In [8]:

```
Zero_Disc_1M = np.zeros((4,11)) #To create a matrix of discount factors
for columns in range(0,11):
    for rows in range(0,4):
        Zero_Disc_1M[rows,columns] = Interpolate(columns+(1.0+(3.0*rows))

pd.DataFrame(Zero_Disc_1M)
```

```
#The rows give the 1m,4m,7m,10m discount factors of each year to today
#for example the 1st row, last column gives us D(0,10yr1m)
```

Out[8]:

	0	1	2	3	4	5	6	
0	0.997739	0.970644	0.943922	0.916456	0.887325	0.856742	0.825711	0.7947
1	0.990956	0.963978	0.937090	0.909205	0.879689	0.848984	0.817962	0.7871
2	0.984172	0.957311	0.930259	0.901954	0.872054	0.841225	0.810212	0.7795
3	0.977389	0.950644	0.923427	0.894704	0.864418	0.833467	0.802462	0.7719

In [9]:

```
def PVBasisPoint(year,DiscountFac):
    denom = 0.0
    for n in range(year+1):
        for m in range(0,4):
            denom =denom + DiscountFac[m][n]
    denom =denom - DiscountFac[0][0] + DiscountFac[0][year+1]
    return denom
def ForwardRateCalc(year,DiscountFac):
    forward = (DiscountFac[0][0] - DiscountFac[0][year+1])/(0.25*PVBasisI
    return forward
```

In [10]:

```
LiborForw = []
LiborForw.append(ForwardRateCalc(0,Zero_Disc_1M))  #The 1m1y forward rate
LiborForw.append(ForwardRateCalc(1,Zero_Disc_1M))  #The 1m2y forward rate
LiborForw.append(ForwardRateCalc(2,Zero_Disc_1M))
LiborForw.append(ForwardRateCalc(4,Zero_Disc_1M))
LiborForw.append(ForwardRateCalc(9,Zero_Disc_1M))

LiborForw
```

Out[10]:

```
[0.027625121400132166,
 0.027815930709971987,
 0.028405148101906648,
 0.030465523081602643,
 0.034076696885275745]
```

In [11]:

```
#For the 3m discount factors

Q_Zero_Disc_3M = np.zeros((4,11)) #To create a matrix of discount factors
for columns in range(0,11):
    for rows in range(0,4):
        Q_Zero_Disc_3M[rows,columns] = Interpolate(columns+(3.0+(3.0*rows

pd.DataFrame(Q_Zero_Disc_3M)
#The rows give the 3m,6m,9m,12m discount factors of each year to today
#for example the 1st row, last column gives us D(0,10yr3m)
```

Out[11]:

	0	1	2	3	4	5	6	
0	0.993217	0.966200	0.939368	0.911622	0.882234	0.851570	0.820545	0.7897
1	0.986433	0.959533	0.932536	0.904371	0.874599	0.843811	0.812795	0.7821
2	0.979650	0.952866	0.925704	0.897120	0.866963	0.836053	0.805045	0.7745
3	0.972867	0.946199	0.918873	0.889870	0.859328	0.828295	0.797296	0.7669

In [12]:

```
LiborForw.append(ForwardRateCalc(0,Q_Zero_Disc_3M))    #The 3m1y forward
LiborForw.append(ForwardRateCalc(1,Q_Zero_Disc_3M))
LiborForw.append(ForwardRateCalc(2,Q_Zero_Disc_3M))
LiborForw.append(ForwardRateCalc(4,Q_Zero_Disc_3M))
LiborForw.append(ForwardRateCalc(9,Q_Zero_Disc_3M))

LiborForw  #rates are attached to the initial list
```

Out[12]:

```
[0.027625121400132166,
 0.027815930709971987,
 0.028405148101906648,
 0.030465523081602643,
 0.034076696885275745,
 0.027673044172992461,
 0.027962278621461837,
 0.028650203993624765,
 0.030762898968053731,
 0.034254664145327594]
```

In [13]:

```
Q_Zero_Disc_6M = np.zeros((4,11)) #To create a matrix of discount factors

for columns in range(0,11):
    for rows in range(0,3):
        Q_Zero_Disc_6M[rows,columns] =  Q_Zero_Disc_3M[rows+1,columns]
for columns in range(0,10):
    Q_Zero_Disc_6M[3,columns] =  Q_Zero_Disc_3M[0,columns+1]

pd.DataFrame(Q_Zero_Disc_6M)
#The rows give the 6m,9m,12m,1y3m discount factors of each year to today
#for example the 1st row, last column gives us D(0,10yr6m)
```

Out[13]:

	0	1	2	3	4	5	6	
0	0.986433	0.959533	0.932536	0.904371	0.874599	0.843811	0.812795	0.7821
1	0.979650	0.952866	0.925704	0.897120	0.866963	0.836053	0.805045	0.7745
2	0.972867	0.946199	0.918873	0.889870	0.859328	0.828295	0.797296	0.7669
3	0.966200	0.939368	0.911622	0.882234	0.851570	0.820545	0.789701	0.7596

In [14]:
```
LiborForw.append(ForwardRateCalc(0,Q_Zero_Disc_6M))   #The 6m1y forward i
LiborForw.append(ForwardRateCalc(1,Q_Zero_Disc_6M))
LiborForw.append(ForwardRateCalc(2,Q_Zero_Disc_6M))
LiborForw.append(ForwardRateCalc(4,Q_Zero_Disc_6M))
LiborForw.append(ForwardRateCalc(9,Q_Zero_Disc_6M))

LiborForw
```

Out[14]:
```
[0.027625121400132166,
 0.027815930709971987,
 0.028405148101906648,
 0.030465523081602643,
 0.034076696885275745,
 0.027673044172992461,
 0.027962278621461837,
 0.028650203993624765,
 0.030762898968053731,
 0.034254664145327594,
 0.027744930168349651,
 0.028184554041248405,
 0.029023407049133077,
 0.031216390119784259,
 0.034525241556635944]
```

In [15]:
```
Q_Zero_Disc_9M = np.zeros((4,11)) #To create a matrix of discount factors

for columns in range(0,11):
    for rows in range(0,3):
        Q_Zero_Disc_9M[rows,columns] =  Q_Zero_Disc_6M[rows+1,columns]
for columns in range(0,10):
    Q_Zero_Disc_9M[3,columns] =  Q_Zero_Disc_6M[0,columns+1]

pd.DataFrame(Q_Zero_Disc_9M)
#The rows give the 9m,12m,1y3m,1y6m discount factors of each year to toda
#for example the 1st row, last column gives us D(0,10yr9m)
```

Out[15]:

	0	1	2	3	4	5	6	
0	0.979650	0.952866	0.925704	0.897120	0.866963	0.836053	0.805045	0.7745
1	0.972867	0.946199	0.918873	0.889870	0.859328	0.828295	0.797296	0.7669
2	0.966200	0.939368	0.911622	0.882234	0.851570	0.820545	0.789701	0.7596
3	0.959533	0.932536	0.904371	0.874599	0.843811	0.812795	0.782107	0.7523

In [16]:
```
LiborForw.append(ForwardRateCalc(1,Q_Zero_Disc_9M))   #The 9m2y forward i
LiborForw.append(ForwardRateCalc(4,Q_Zero_Disc_9M))
LiborForw.append(ForwardRateCalc(9,Q_Zero_Disc_9M))

LiborForw
```

Out[16]:
```
[0.027625121400132166,
 0.027815930709971987,
 0.028405148101906648,
 0.030465523081602643,
 0.034076696885275745,
 0.027673044172992461,
 0.027962278621461837,
 0.028650203993624765,
 0.030762898968053731,
 0.034254664145327594,
 0.027744930168349651,
 0.028184554041248405,
 0.029023407049133077,
 0.031216390119784259,
 0.034525241556635944,
 0.028410164662534036,
 0.03167871859155269,
 0.034800310366418091]
```

Break down the values

Let's change the values in the table into the same units for easier calculation and coding. We will work with a copy of mkt_data

In [17]:
```
# change all values in the data to a common base(%).
# mkt_data.ix[:,1]    to return 1st column
# mkt_data[mkt_data.columns[0]] to return 1st column
# mkt_data.iloc[[2]] to return 3rd row

DataTable = m_data.copy()
for col in DataTable.columns:
        DataTable[col] = m_data['ATM']/100 + m_data[col]/10000
DataTable['ATM'] = m_data['ATM']/100
DataTable
```

Out[17]:

	-200	-150	-100	-50	-25	ATM	25	
1M1Y	0.231907	0.228953	0.227163	0.225872	0.225368	0.2250	0.224846	0.2
1M2Y	0.292655	0.290452	0.288985	0.287909	0.287499	0.2872	0.287040	0.2
1M3Y	0.302214	0.300509	0.299299	0.298397	0.298054	0.2978	0.297651	0.2
1M5Y	0.263612	0.262557	0.261744	0.261117	0.260878	0.2607	0.260591	0.2
1M10Y	0.246371	0.245757	0.245274	0.244915	0.244788	0.2447	0.244651	0.2
3M1Y	0.276654	0.274783	0.273714	0.273023	0.272780	0.2726	0.272478	0.2
3M2Y	0.301105	0.299958	0.299220	0.298682	0.298470	0.2983	0.298173	0.2
3M3Y	0.302145	0.301246	0.300620	0.300146	0.299956	0.2998	0.299678	0.2
3M5Y	0.267539	0.266992	0.266573	0.266240	0.266108	0.2660	0.265913	0.2
3M10Y	0.246091	0.245744	0.245464	0.245244	0.245163	0.2451	0.245048	0.2
6M1Y	0.288062	0.286978	0.286283	0.285772	0.285567	0.2854	0.285277	0.2
6M2Y	0.294989	0.294162	0.293580	0.293134	0.292952	0.2928	0.292681	0.2
6M3Y	0.295882	0.295203	0.294700	0.294304	0.294140	0.2940	0.293887	0.2
6M5Y	0.268781	0.268315	0.267944	0.267640	0.267512	0.2674	0.267306	0.2
6M10Y	0.244623	0.244314	0.244061	0.243856	0.243772	0.2437	0.243639	0.2
9M2Y	0.297014	0.296432	0.295979	0.295603	0.295441	0.2953	0.295186	0.2
9M5Y	0.265946	0.265582	0.265277	0.265016	0.264902	0.2648	0.264714	0.2
9M10Y	0.242016	0.241755	0.241535	0.241349	0.241269	0.2412	0.241141	0.2

The objective function

Ok, we want to find sigma alpha and rho for each set of (T_m,T_m+T_n). Let abrs be a vector such that abrs[0] = sigma, abrs[1] = alpha, abrs[2] = rho. And mrkt be the market volatility. We will fix beta to be 0.5. We can estimate beta using linear regression. For more details, you can refer to http://www.frouah.com/finance%20notes/The%20SABR%20Model.pdf

In [18]:

```
DataTable=DataTable.rename(columns = {'ATM': 0})
#Create strike grid
Str = np.zeros((18,11))
for i in range(18):
    for j in range(11):
        Str[i][j] = LiborForw[i] + 0.0001*(np.float64(DataTable.columns[j
```

In [19]:
```
df = pd.DataFrame(Str)
df
```

Out[19]:

	0	1	2	3	4	5	6	
0	0.007625	0.012625	0.017625	0.022625	0.025125	0.027625	0.030125	0.032
1	0.007816	0.012816	0.017816	0.022816	0.025316	0.027816	0.030316	0.032
2	0.008405	0.013405	0.018405	0.023405	0.025905	0.028405	0.030905	0.033
3	0.010466	0.015466	0.020466	0.025466	0.027966	0.030466	0.032966	0.035
4	0.014077	0.019077	0.024077	0.029077	0.031577	0.034077	0.036577	0.039
5	0.007673	0.012673	0.017673	0.022673	0.025173	0.027673	0.030173	0.032
6	0.007962	0.012962	0.017962	0.022962	0.025462	0.027962	0.030462	0.032
7	0.008650	0.013650	0.018650	0.023650	0.026150	0.028650	0.031150	0.033
8	0.010763	0.015763	0.020763	0.025763	0.028263	0.030763	0.033263	0.035
9	0.014255	0.019255	0.024255	0.029255	0.031755	0.034255	0.036755	0.039
10	0.007745	0.012745	0.017745	0.022745	0.025245	0.027745	0.030245	0.032
11	0.008185	0.013185	0.018185	0.023185	0.025685	0.028185	0.030685	0.033
12	0.009023	0.014023	0.019023	0.024023	0.026523	0.029023	0.031523	0.034
13	0.011216	0.016216	0.021216	0.026216	0.028716	0.031216	0.033716	0.036
14	0.014525	0.019525	0.024525	0.029525	0.032025	0.034525	0.037025	0.039
15	0.008410	0.013410	0.018410	0.023410	0.025910	0.028410	0.030910	0.033
16	0.011679	0.016679	0.021679	0.026679	0.029179	0.031679	0.034179	0.036
17	0.014800	0.019800	0.024800	0.029800	0.032300	0.034800	0.037300	0.039

In [20]:
```
#create market grid of volatilities
mkt_vol = np.zeros((18,11))
for i in range(18):
    for j in range(11):
        mkt_vol[i][j] = DataTable[DataTable.columns[j]][i]
mkt_vol[0][6]
```

Out[20]: 0.22484600000000002

In [21]:
```
def obj_fun(abrs):        #This function solves for one tenor, maturity pa
    beta = 0.5
    for j in range(11):  #for each m,n we have 11 sets of strikes
        S0K = LiborForw[0]*Str[0][j]
        lS0K = np.log(LiborForw[0]/Str[0][j])
```

```
        z = (abrs[0]/abrs[1])*((S0K)**((1-beta)/2))*(lS0K)
        x = np.log((np.sqrt(1-2*abrs[2]*z+z**2)+z-abrs[2])/(1-abrs[2]))

        denom = 1+(((1-beta)*lS0K)**2)/24 + (((1-beta)*lS0K)**4)/1920

        numer = 1 + (1.0/12)*((((1-beta)*abrs[1])**2)/(24*(S0K**(1-beta))
        (abrs[2]*beta*abrs[0]*abrs[1])/(4*(S0K**((1-beta)/2))) + \
        ((abrs[0]**2)*(2-3*(abrs[2]**2)))/24)

        imp_vol = (abrs[1]*numer*(z/x))/(denom*(S0K**((1-beta)/2)))

        diff = imp_vol - mkt_vol[0][j]
        sum_sq_diff=0
        sum_sq_diff = sum_sq_diff+diff**2
    obj = math.sqrt(sum_sq_diff)

    return obj
```

In [22]:

```
#set starting guess for sigma,alpha,beta,rho
starting_guess = np.array([0.001,0.001,0])
```

In [23]:

```
from scipy.optimize import minimize
bnds = ( (0.001,None), (0.001,None), (-0.999,0.999) )

res = minimize(obj_fun, starting_guess, bounds = bnds, method='SLSQP')
res
```

Out[23]:

```
     fun: 9.882778566339123e-07
     jac: array([  4.67558391e-03,   5.23463775e+00,   2.83680484e-03,
         0.00000000e+00])
 message: 'Optimization terminated successfully.'
    nfev: 59
     nit: 10
    njev: 10
  status: 0
 success: True
       x: array([ 0.01043176,  0.04312627,  0.00038034])
```

In [24]:

```
res.x   #The values for sigma, alpha, beta, rho for the 1m1y pair.
```

Out[24]:

```
array([  2.84373158e-03,   2.26200923e-01,   2.05476277e-05])
```

In [24]:

```
#Let's do this for each row. Input time to maturity, Tm
TimePer=[]
for i in range(0,5):
    TimePer.append(1.0/12)
for i in range(0,5):
    TimePer.append(3.0/12)
for i in range(0,5):
    TimePer.append(6.0/12)
for i in range(0,3):
```

```
    TimePer.append(9.0/12)
```

In [25]:

```
def obj_fun_array(abrs,T,K,S0,mrkt,beta):
    for j in range(11):  #for each m,n we have 11 sets of strikes, 11 rov

        S0K = S0*K[j]
        lS0K = np.log(S0/K[j])

        z = (abrs[0]/abrs[1])*((S0K)**((1-beta)/2))*(lS0K)
        x = np.log((np.sqrt(1-2*abrs[2]*z+z**2)+z-abrs[2])/(1-abrs[2]))

        denom = 1+(((1-beta)*lS0K)**2)/24 + (((1-beta)*lS0K)**4)/1920

        numer = 1 + T*((((1-beta)*abrs[1])**2)/(24*(S0K**(1-beta))) +  \
        (abrs[2]*beta*abrs[0]*abrs[1])/(4*(S0K**((1-beta)/2))) + \
        ((abrs[0]**2)*(2-3*(abrs[2]**2))))/24)

        imp_vol = (abrs[1]*numer*(z/x))/(denom*(S0K**((1-beta)/2)))

        diff = imp_vol - mrkt[j]
        sum_sq_diff=0
        sum_sq_diff = sum_sq_diff+diff**2
    obj_ar = math.sqrt(sum_sq_diff)

    return obj_ar
```

In [26]:

```
sigma = []
alpha = []
rho = []

def Calibrate(guess,T,K,S0,mrkt,beta):
    for i in range(18):
        x0 = guess
        bnds = ( (0.001,None), (0.001,None), (-0.999,0.999) )

        result = minimize(obj_fun_array, x0, (T[i],K[i],S0[i],mrkt[i],bet
                          bounds = bnds, method='SLSQP')
        sigma.append(result.x[0])
        alpha.append(result.x[1])
        rho.append(result.x[2])
```

In [27]:

```
Calibrate(starting_guess,TimePer,Str,LiborForw,mkt_vol,0.5)
```

In [28]:

```
print sigma
```

```
[0.010431757970530179, 0.009926644761636948, 0.008791842514464999 5, 0.0081
418759120351122, 0.0072155205306021104, 0.0045751854656676284, 0.003887389
5914603713, 0.0037687441249995048, 0.004236830506348305 8, 0.00427294664435
99592, 0.0022698881632834721, 0.0020790387065854937, 0.001919512888532290
7, 0.0020068445409937032, 0.0022004426637856726, 0.0019281675023870165, 0.
0019480930115593938, 0.0019244965388465764]
```

In [29]:

```python
Parameters = np.zeros((18,3))
for i in range(18):
    Parameters[i][0] = sigma[i]
    Parameters[i][1] = alpha[i]
    Parameters[i][2] = rho[i]

df1=pd.DataFrame(Parameters)
```

In [30]:

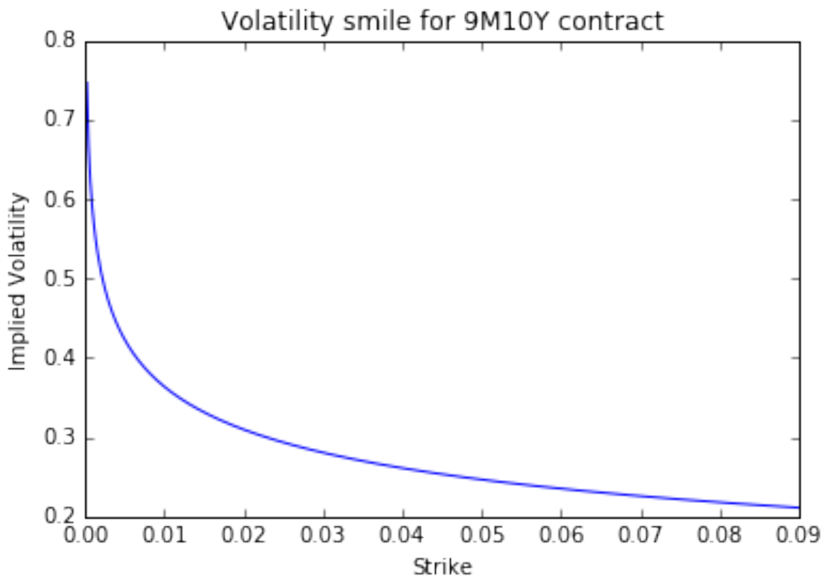
```python
df1.columns = ['Sigma', 'Alpha', 'Rho']
df1.index = m_data.index
df1
```

Out[30]:

	Sigma	Alpha	Rho
1M1Y	0.010432	0.043126	3.803417e-04
1M2Y	0.009927	0.055099	1.232439e-03
1M3Y	0.008792	0.057567	9.997264e-04
1M5Y	0.008142	0.051796	5.338583e-04
1M10Y	0.007216	0.050847	4.033205e-04
3M1Y	0.004575	0.052078	1.242433e-04
3M2Y	0.003887	0.057185	1.746840e-04
3M3Y	0.003769	0.058009	1.691422e-04
3M5Y	0.004237	0.052942	4.644313e-05
3M10Y	0.004273	0.050947	4.308092e-06
6M1Y	0.002270	0.054539	1.439053e-05
6M2Y	0.002079	0.056283	4.402620e-05
6M3Y	0.001920	0.057159	4.183079e-05
6M5Y	0.002007	0.053507	1.471557e-05
6M10Y	0.002200	0.050798	-8.127135e-07
9M2Y	0.001928	0.056916	4.550726e-05
9M5Y	0.001948	0.053290	3.288724e-05
9M10Y	0.001924	0.050432	9.741254e-06

In [47]:
```
Strike_Range = np.linspace(0, 0.09, 400, endpoint=True)
ImpVol_1m1y = SABR(TimePer[0],Strike_Range,LiborForw[0],sigma[0],alpha[0]
ImpVol_9m10y = SABR(TimePer[17],Strike_Range,LiborForw[17],sigma[17],alph

plt.plot(Strike_Range,ImpVol_9m10y)
plt.xlabel('Strike')
plt.ylabel('Implied Volatility')
plt.title('Volatility smile for 9M10Y contract')
plt.show()
```



In [32]:
```
PVBP = []
PVBP.append(PVBasisPoint(0,Zero_Disc_1M))  #The 1m1y discount factors
PVBP.append(PVBasisPoint(1,Zero_Disc_1M))  #The 1m2y discount factors
PVBP.append(PVBasisPoint(2,Zero_Disc_1M))
PVBP.append(PVBasisPoint(4,Zero_Disc_1M))
PVBP.append(PVBasisPoint(9,Zero_Disc_1M))
PVBP.append(PVBasisPoint(0,Q_Zero_Disc_3M))  #The 3m1y discount factors
PVBP.append(PVBasisPoint(1,Q_Zero_Disc_3M))  #The 3m2y discount factors
PVBP.append(PVBasisPoint(2,Q_Zero_Disc_3M))
PVBP.append(PVBasisPoint(4,Q_Zero_Disc_3M))
PVBP.append(PVBasisPoint(9,Q_Zero_Disc_3M))
PVBP.append(PVBasisPoint(0,Q_Zero_Disc_6M))  #The 6m1y discount factors
PVBP.append(PVBasisPoint(1,Q_Zero_Disc_6M))  #The 6m2y discount factors
PVBP.append(PVBasisPoint(2,Q_Zero_Disc_6M))
PVBP.append(PVBasisPoint(4,Q_Zero_Disc_6M))
PVBP.append(PVBasisPoint(9,Q_Zero_Disc_6M))
PVBP.append(PVBasisPoint(1,Q_Zero_Disc_9M))  #The 9m2y discount factors
PVBP.append(PVBasisPoint(4,Q_Zero_Disc_9M))
PVBP.append(PVBasisPoint(9,Q_Zero_Disc_9M))
```

In [41]:
```
def SABR_VOL(T,K,S0,sig,alph,bet,ro):
    Imp_Vol = np.zeros((18,11))
    for i in range(18):
        for j in range(11):
            Imp_Vol[i][j] = SABR(T[i],K[i][j],S0[i],sig[i],alph[i],bet,ro
    return Imp_Vol
```

```
def Black76_Price(T,K,F,Vol,BP):
    B76price = np.zeros((18,11))
    for i in range(18):
        for j in range(11):
            B76price[i][j] = BP[i]*Black76(T[i],K[i][j],F[i],Vol[i][j])
    return B76price
```

In [48]:

```
Implied_Volatilities = SABR_VOL(TimePer,Str,LiborForw,sigma,alpha,0.5,rh

pd.DataFrame(Implied_Volatilities)
```

Out[48]:

	0	1	2	3	4	5	6	
0	0.351990	0.313632	0.289747	0.272659	0.265690	0.259487	0.253907	0.248
1	0.446358	0.398566	0.368576	0.347039	0.338238	0.330397	0.323339	0.31
2	0.456155	0.409766	0.380011	0.358405	0.349529	0.341599	0.334446	0.32
3	0.383127	0.349936	0.327280	0.310276	0.303173	0.296774	0.290961	0.28
4	0.340848	0.317359	0.300083	0.286536	0.280744	0.275462	0.270615	0.26
5	0.424328	0.378296	0.349576	0.329006	0.320612	0.313137	0.306412	0.30
6	0.460812	0.412097	0.381358	0.359219	0.350161	0.342083	0.334808	0.32
7	0.455756	0.410333	0.380953	0.359527	0.350705	0.342816	0.335692	0.32
8	0.388168	0.355234	0.332585	0.315514	0.308366	0.301920	0.296059	0.29
9	0.340107	0.316898	0.299779	0.286330	0.280574	0.275322	0.270500	0.26
10	0.443382	0.395551	0.365631	0.344175	0.335415	0.327611	0.320588	0.31
11	0.449990	0.403293	0.373590	0.352112	0.343307	0.335447	0.328363	0.32
12	0.443462	0.400554	0.372464	0.351850	0.343337	0.335711	0.328816	0.32
13	0.387234	0.355369	0.333217	0.316419	0.309364	0.302991	0.297187	0.29
14	0.337022	0.314355	0.297565	0.284339	0.278669	0.273494	0.268738	0.26
15	0.451538	0.405532	0.376038	0.354627	0.345832	0.337975	0.330887	0.32
16	0.380726	0.350327	0.328975	0.312688	0.305826	0.299617	0.293956	0.28
17	0.332526	0.310483	0.294086	0.281137	0.275578	0.270499	0.265830	0.26

In [49]:

```
Arbitrage_free_price = Black76_Price(TimePer,Str,LiborForw,Implied_Volat

pd.DataFrame(Arbitrage_free_price)
```

Out[49]:

	0	1	2	3	4	5	6	
0	0.078463	0.058847	0.039232	0.019630	0.010220	0.003238	0.000483	0.000
1	0.154780	0.116085	0.077390	0.038871	0.021133	0.008188	0.002022	0.000
2	0.228925	0.171694	0.114463	0.057594	0.031685	0.012786	0.003447	0.00
3	0.370247	0.277685	0.185123	0.092952	0.050384	0.019270	0.004603	0.000
4	0.680947	0.510711	0.340474	0.171170	0.093555	0.036797	0.009319	0.00
5	0.078103	0.058577	0.039076	0.020406	0.012588	0.006743	0.003083	0.00
6	0.154062	0.115547	0.077144	0.040930	0.025982	0.014680	0.007298	0.00
7	0.227837	0.170879	0.114128	0.060853	0.038929	0.022291	0.011306	0.005
8	0.368358	0.276270	0.184413	0.097426	0.061334	0.034090	0.016490	0.006
9	0.677082	0.507816	0.339070	0.179737	0.113681	0.063637	0.031067	0.01
10	0.077565	0.058190	0.039159	0.022147	0.015302	0.009922	0.006022	0.00
11	0.152985	0.114788	0.077411	0.044234	0.030897	0.020353	0.012619	0.00
12	0.226197	0.169751	0.114699	0.066085	0.046539	0.031013	0.019520	0.01
13	0.365508	0.274277	0.185089	0.105882	0.073957	0.048668	0.030092	0.01
14	0.671294	0.503782	0.340124	0.194657	0.135889	0.089265	0.054999	0.03
15	0.151917	0.114229	0.078335	0.047570	0.035193	0.025107	0.017267	0.01
16	0.362673	0.272754	0.187040	0.113304	0.083550	0.059292	0.040465	0.026
17	0.665590	0.500643	0.343317	0.207620	0.152738	0.107973	0.073265	0.04

In [ ]:

In [ ]:

In [ ]:

Comparison to the Displaced Diffusion model.

The SABR model gives us the Black76 implied volatilities. And we input that into the Black76 model to get the prices of the Swaption. Let us now compare this price with the price given by the displaced diffusion model. We shall calibrate volatility, Vol and the displaced diffusion parameter b using the prices found above.

In [36]:

```
def DisplcedDiffusion(T,K,F,Vol,BP,b):  #input b as a decimal
    DDprice = np.zeros((18,11))
    coeff = (1-b)/b
    for i in range(18):
        for j in range(11):
            DDprice[i][j] = BP[i]*Black76(T[i],K[i][j]+coeff*F[i],F[i]/b,
    return DDprice
```

In [66]:

```
from scipy.optimize import root
#DDVol be the volatility to solve for
def root_fun_array(Param,T,K,F,BP,Price):
    coeff = (1-Param[1])/Param[1]
    fun_val = []
    fun_val.append(BP*Black76(T,K+coeff*F,F/Param[1],Param[0]*Param[1])-
    fun_val.append(BP*Black76(T,K+coeff*F,F/Param[1],Param[0]*Param[1])-
    return fun_val

DD_Vol = np.zeros((18, 11))
colms = range(11)
colms.pop(5)  #Exclude ATM price
DD_b=np.zeros((18,11))

start_value = np.array([0.001,0.5])

def Solve_Root(guess,T,K,F,BP,Price):
    for i in range(18):
        for j in colms:
            result = root(root_fun_array, guess, (T[i],K[i][j],F[i],BP[i]
            DD_Vol[i][j] = (result.x[0])
            DD_b[i][j] = (result.x[1])
```

In [67]:

```
Solve_Root(start_value,TimePer,Str,LiborForw,PVBP,Arbitrage_free_price)

dfDD_Vol=pd.DataFrame(DD_Vol)
```

```
print dfDD_Vol

dfDD_b=pd.DataFrame(DD_b)
print dfDD_b
```

```
        0      1         2         3         4    5          6          7  \
0   0.001  0.001  0.001000  0.001000  0.001038  0.0   0.001000   0.001000   
1   0.001  0.001  0.001000  0.001064  0.001001  0.0   0.298664   0.001000   
2   0.001  0.001  0.001000  0.001000  0.001000  0.0   0.001000   0.001000   
3   0.001  0.001  0.001000  0.001000  0.001000  0.0   0.266437   0.001000   
4   0.001  0.001  0.001000  0.001000  0.001000  0.0   0.249113   0.001000   
5   0.001  0.001  0.001000  0.001000  0.001000  0.0   0.277248   0.001000   
6   0.001  0.001  0.001000  0.001000  0.001000  0.0   7.711076   0.001000   
7   0.001  0.001  0.001000  0.001000  0.001000  0.0   5.327473   0.311511   
8   0.001  0.001  0.001000  0.001245  0.001000  0.0   0.268791   0.273077   
9   0.001  0.001  0.001000  0.001000  0.001000  0.0   0.249870   0.256547   
10  0.001  0.001  0.001000  0.001000  0.001000  0.0  18.458346   0.308688   
11  0.001  0.001  0.001000  0.001000  0.001000  0.0  13.117544  18.560164   
12  0.001  0.001  0.001000  0.001000  0.001000  0.0  13.490984  22.810117   
13  0.001  0.001  0.001009  0.001000  0.001000  0.0  17.409424   0.279227   
14  0.001  0.001  0.001037  0.001000  0.001000  0.0   5.701153   0.252563   
15  0.001  0.001  0.001000  0.001000  0.001000  0.0  28.443589  12.229631   
16  0.001  0.001  0.001000  0.001000  0.001000  0.0  18.106886  22.469217   
17  0.001  0.001  0.001000  0.001000  0.001000  0.0  17.856829  39.274514   

           8         9        10  
0   0.001000  0.001000  0.001000  
1   0.001000  0.001000  0.001000  
2   0.001000  0.001000  0.001000  
3   0.001000  0.001000  0.001000  
4   0.001000  0.001000  0.001000  
5   0.001000  0.001000  0.001000  
6   0.001000  0.001000  0.001000  
7   0.287363  0.001000  0.001000  
8   0.001000  0.294302  0.001000  
9   0.001000  0.282876  0.001000  
10  0.001000  0.001000  0.001000  
11  0.001000  0.001000  0.001000  
12  0.001000  0.001000  0.001000  
13  0.001000  0.001000  0.001000  
14  0.001000  0.001000  0.001000  
15  0.311850  0.001000  0.001000  
16  0.290445  0.001000  0.243705  
17  0.001000  0.001000  0.001000  
      0         1          2             3             4    5          6  \
0   0.5  0.500000   0.500000      0.500000 -17806.148549  0.0   0.500000   
1   0.5  0.500000   0.500000 -19088.889807 -22275.662341  0.0   0.203502   
2   0.5  0.500000   0.454342      0.500000      0.500000  0.0   0.500000   
3   0.5  0.500000   0.500000      0.500000      0.500000  0.0   0.533673   
4   0.5  0.500000   0.500000      0.500000      0.500000  0.0   0.576813   
5   0.5  0.500000   0.500000      0.500000      0.500000  0.0   0.684057   
6   0.5  0.500000   0.500000      0.500000      0.500000  0.0  36.331766   
7   0.5  0.501744   0.500000      0.500000      0.500000  0.0  35.298229   
8   0.5  0.500000   0.500000  -7200.887348 -10167.722874  0.0   0.783782   
9   0.5  0.500000   0.500000      0.500000      0.500000  0.0   0.491815   
```

```
10  0.5  0.500000     0.500000      0.500000      0.500000  0.0  21.345994
11  0.5  0.500000     0.500000      0.500000      0.500000  0.0  20.305425
12  0.5  0.500000     0.500000      0.500000      0.500000  0.0  19.870087
13  0.5  0.500000 -7729.414042      0.500000      0.500000  0.0  22.267386
14  0.5  0.500000 -7521.150745 -11347.673039      0.500000  0.0  24.472746
15  0.5  0.500000     0.500000      0.500000      0.500000  0.0  14.598370
16  0.5  0.500000     0.500000      0.500000      0.500000  0.0  16.415662
17  0.5  0.500000     0.500000      0.500000      0.500000  0.0  18.118320

            7         8         9        10
0    0.500000  0.500000  0.500000  0.500000
1    0.500000  0.500000  0.500000  0.500000
2    0.500000  0.500000  0.500000  0.500000
3    0.500000  0.500000  0.500000  0.500000
4    0.500000  0.500000  0.500000  0.500000
5    0.500000  0.500000  0.500000  0.500000
6    0.500000  0.500000  0.500000  0.500000
7    0.537430  1.339032  0.500000  0.500000
8    0.663494  0.500000  0.462991  0.500000
9    0.341527  0.500000  0.204706  0.500000
10   0.089581  0.500000  0.500000  0.500000
11  36.085133  0.500000  0.500000  0.500000
12  34.561406  0.500000  0.500000  0.500000
13   0.427396  0.500000  0.500000  0.500000
14   0.479834  0.500000  0.500000  0.500000
15  22.478009  0.626071  0.500000  0.500000
16  25.537032  0.314223  0.500000  1.424165
17  28.399347  0.500000  0.500000  0.500000
```

In [59]:

```
root_fun_array(start_value,TimePer[0],Str[0][0],LiborForw[0],PVBP[0],Arb
```

Out[59]: 1.3170890159654386e-09