Fixed Income Derivatives*
Lecture Notes

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Contents

Preface

The motivation behind this course in Fixed Income Derivatives is to introduce students to a range of common derivatives on both a theoretical and practical level. A great number of text books and academic articles have been published on the pricing and risk management of derivatives. While many of these go through the theoretical aspects in detail, few actually address the practical implementation leaving students with a gap to bridge in order apply their theoretical knowledge. The aim of these notes is to bridge this gap between turning relatively simple theoretical models into applications that can be used to produce fairly accurate numbers on real world data.

The theoretical level is intended to be somewhere in between Hull (2006) and Björk (2004) but with a focus on implementation. The mathematical treatment is therefore not very formal. While the treatment here of some topics (especially risk neutral valuation and martingale pricing) is rather heuristic in nature, students are encouraged to explore these topics greater detail in other courses or as self-study. A good starting point to improve your mathematical and theoretical foundation for some of the results used in these notes is Björk (2004). Finally for students interested in cutting edge modeling the three volume book Andersen & Piterbarg (2010a), Andersen & Piterbarg (2010b) and Andersen & Piterbarg (2010c) is a definitive source.

A key tool in the course is a VBA library with a set of functions that can be used in Excel to price and risk manage derivatives. These notes go through the theory behind the functions and describe how to implement them. These notes should therefore be read together with the VBA code. When references to the VBA code are made, the relevant function and argument names are written in a special `format`. To facilitate easy understanding of the functions, all the code has — as best coding practice would suggest — been commented. For students who are unfamiliar with VBA, I recommend reading McDonald (2000). Together these resources will provide students with the necessary knowledge to modify and expand the functions as a part of the exercises and ultimately in the exam.

While VBA has a number of limitations that make it unsuitable for large scale pricing applications, it is an easy programming language to use in an introductory course and does not require much prerequisite knowledge to learn. Furthermore, using VBA has the benefit of the full flexibility of Excel as the interface for inputting data and solving numerical problems. Most large financial institutions have their own in-house developed analytics library that in terms of their basic functionality are much like the library we will develop in the course. These libraries are typically written in C++ or some other advanced language that allows for much greater flexibility and computing speed than VBA. As the code examples are a victim of my limited programming skills any true code aficionados are recommended to check out Quantlib.org which is an open source library that includes many advanced functionalities. It is important to stress that the VBA code has been written to facilitate clarity rather than performance. The purpose is thus to demonstrate just one, easy way of implementing basic pricing models rather than *the* best way.

1 Introduction

Derivatives play a tremendously important role in the modern financial markets. Understanding how these instruments work and knowing how to price should therefore be of concern to anyone with interest in the financial markets. Why is focus of these notes solely on fixed income- rather than derivatives in general? First of all, the interest rate, foreign exchange and credit derivatives that we will work with here are by far the most widely traded contracts as can be seen in figure 1. Secondly, these derivatives have the broadest user base — they are extensively used by both financial institutions, asset managers, corporations and even sovereigns. Thirdly, from a theoretical point of view especially interest rates derivatives provide some additional theoretical challenges compared to, say, equity derivatives since there is an inherent interplay between the interest rates used for discounting and the computation of future cash flows. If you are able to understand interest rate derivatives, you are thus well equipped to work with equity or commodity derivatives at a later stage. Finally, since the specific markets and products treated in these notes are among the most liquid derivatives markets they pose extremely high requirements to any person trying to model the them. The margin of error is simply very small in these markets as they often trade in huge volume at very limited bid-offer spreads.

Figure 1: Notional outstandings according to the ISDA Market Survey
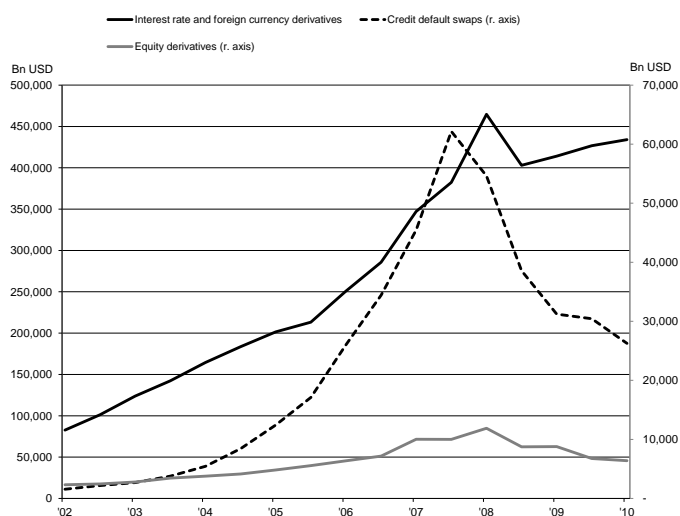
2 Basic building blocks

2.1 Motivation

Before we define the most basic concepts, let us motivate why are interested in them. An instrument that we will work extensively with in the course is the fixed-for-floating Interest Rate Swap (IRS). This a bilateral contract between two counterparties to exchange a series of fixed interest rate payments for a series of floating interest rate payments over specified period of time e.g. 10 years. We denote the two cash flow streams as *legs*.

For each of the world's major currencies such a contract exist on a set of pre-specified standard terms. Swap contracts that fulfill these standard terms are called *plain vanilla* in the industry jargon.[1]

In order to complete the financial characterization of such a contract we need — at least — the following information:

- Start date: When should the accrual of interest commence?

- End date: When should it end?

- Payment frequencies: How often is interest paid on each leg?

- Day count conventions: How are periods converted into year fractions when calculating interest rate payments?

- Floating rate index: Which index is the floating rate fixed against?

- Fixing frequency: How often is the floating rate reset?

- Rolling conventions: If an event is scheduled to take place on a non-business day how is this date adjusted to a business day?

As a motivating example, let us look at the 10 year IRS in EUR. The standard for this contract is to start the interest accrual on the so called *spot date* which is two business days after the *trading date*. The EUR plain vanilla IRS is indexed against the 6M EURIBOR® rate, which is then fixed at the start of each interest period and paid out at the end of the period, implying that the fixing and payment frequencies for the floating leg are semi-annual. Furthermore, these floating interest rate payments are calculated using the Actual/360 day count convention. On the fixed leg, interest is paid annually at the end of each period according to the 30/360 convention. In general the fraction of a year used to calculate interest rate payments are often called the *coverage* or simply year fraction. Finally, all non-business days (except for the start date) are adjusted to business days according to the *Modified Following* rolling convention. Before we can think about trying to value the 10Y EUR IRS, we thus need to work out a schedule containing 20 semi-annual periods for the floating leg and another schedule containing 10 annual periods for the fixed leg as well as their associated coverages. While this derivative — as we will see — is indeed fairly easy price and risk manage it does require a few basic tools just to identify what is to happen when over the life of the contract.

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[1]The term "plain vanilla" is often used to describe slightly different things. Sometimes the term is used to convey information about the details of a contract and sometimes it is used to characterize the risk factors in a contract. Some market participants will thus characterize a contract done on non-standard terms as plain vanilla as long as it can by hedged by trading standard instruments.

Basically, pricing derivatives is a question of working out the value of one or more future financial events. Examples of such events are fixings of floating rates, exercise of options and the payments of their associated cash flows. Such events typically follow some schedule, fixings could for example take place every 6 months, or payments from an option could be made 1 week after exercise. A basic building block for derivative pricing will therefore be to construct schedules for events and assign dates to these. Finally, as we will work extensively with derivative contracts with interest and other periodic payments we will need functionalities that can determine the number of interest days in a given period.

2.2 Dates

The first task we need to master is to add and subtract a given period to a specific date referred to as the *Start date*. These periods can be Days (D), Business Days (B), Weeks (W), Months (M) or Years (Y). In short hand notation, we will thus write 2B or 4M to denote periods of 2 business days or 4 months respectively. Often we will refer to these periods as *tenors*. The addition and subtraction of periods is done by utilizing the fact that any date can be expressed as a serial relative to given reference date, that is then day 1. In Excel this is done using January 1st 1900 as reference. As an example, January 12th 2010 has the serial representation 40190 — that specific date is 40190 calendar days after January 1st 1900. In the following we want to construct a new date $DD - MM - YYYY$ with the serial representation S from an anchor date denoted $dd - mm - yyyy$ with the serial representation s. Once the serial representation is in place, we can easily add periods according to the following "cook book":

Add n days, nD: Set $S = s + n$ and convert to S into $DD - MM - YYYY$

Add n weeks, nW: Set $S = s + n \cdot 7$ and convert to S into $DD - MM - YYYY$

Add n months, nM: Set $YYYY = yyyy + \text{INT}((mm + n - 1)/12)$ and $MM = \text{mod}(mm + n - 1, 12) + 1$ with the exception rule that if $MM < 1$ the new date should be constructed using $YYYY'$ and MM' where $YYYY' = YYYY - 1$, $MM' = MM + 12$.[2] Finally, set $DD = \min(dd, \text{DaysInMonth}(MM))$ or $DD = \min(dd, \text{DaysInMonth}(MM'))$ if we are in the $MM < 1$ case.

Add n years, nY: Set $YYYY = yyyy + n$, $MM = mm$ and $DD = dd$ except if $yyyy + n$ is a not leap year and $dd = 29$ and $mm = 2$, then $DD = 28$

As an example, adding 1M to the 29 January 2010, will result in 28 February 2010 since February only has 28 days in non-leap years. Luckily, we de not need to implement the above rules explicitly, as Excel already has the VBA `DateAdd` function that does this. The only drawback is that the arguments for this function is a bit annoying (the notation for week is e.g. WW rather than W) and it does not support adding business days. For this reason, we will build our own function on top of the `DateAdd` function. This function is called `fidAddTenor` and takes an argument called `DayRule` in addition to the `StartDate` and `Tenor` arguments.

The day rule specifies how to roll a non-business day into a good business day. In the industry jargon, "good" business days are days where a certain event can take place.

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[2]Here $\text{mod}(x, 12)$ means x *modulu* 12, which is simply the remainder of x after division by 12. As an example $\text{mod}(28, 12) = 4$ since $28 - 2 \cdot 12 = 4$.

As an example payments in EUR between Eurozone banks can only take place when the TARGET is open.[3] Payments can e.g. not be made on weekends or on the 25$^{\text{st}}$ of December. Different holiday calendars govern which days are good days and which days are holidays in e.g. London or New York. In professional trading systems these calendars are typically implemented either as databases or as algorithms. In this course we will however disregard all holidays except for weekends.

A non-business day can be rolled using the following conventions:

- None (`None` or `[Blank]`), do not roll the date.

- Following (`F`), roll the date to the next good business day.

- Preceding (`P`), roll the date to the previous good business.

- Modified Following (`MF`), roll the date to the following good business except if this falls in the following month, in that case roll the date back to the previous good business day.

The tenors and date rules described above have been implemented as a function called `fidAddTenor` in VBA. Note that as a special case we can use `fidAddTenor` to adjust potential non-business by adding 0D to a given start date and subsequently apply a rolling convention. This special case has been implemented as `fidAdjustDate`.

2.3 Coverages

As mentioned in the motivation we will need the concept of coverages or year fractions as we will work extensively with interest bearing instruments. It is customary to communicate interest as a rate per annum. What does this however mean, if we are to pay interest on period that is not a full year? In that case we will count the days in relevant period defined by a *Start Date*, T_S and an *End Date*, T_E and relate these to the number of days in a standardized year according to a *day count convention*. Although this seems rather trivial there are actually quite a few ways of doing this — there are many different conventions. The most common ones (which are the ones we will use) are:

- Actual/360 (`Act/360`), subtract the serial representation of the start date from the end date and divide by 360 ($\text{Cvg} = \frac{T_E - T_S}{360}$).

- Actual/365 Fixed (`Act/365`), same as Act/360 but divide by 365 instead ($\text{Cvg} = \frac{T_E - T_S}{365}$).

- Actual/365.25 (`Act/365.25`), same as Act/360 but divide by 365.25 instead to account for leap years ($\text{Cvg} = \frac{T_E - T_S}{365.25}$).

- 30/360, (`30/360`), assume that each month has 30 days and that each year thus have 360 days and calcuate

$$\text{Cvg} = \frac{1}{360}\Big[(\text{Year}(T_E) - \text{Year}(T_S)) * 360 + (\text{Month}(T_E) - \text{Month}(T_S)) * 30$$

$$+ \min(30, \text{Day}(T_E)) - \min(30, \text{Day}(T_S))\Big]$$

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[3]TARGET is short for Trans-European Automated Real-time Gross Settlement Express Transfer System and is the payment system used to transfer EUR funds between banks in the Eurozone.

2.4 Schedules

For many financial contracts we will be interested in laying out a whole series of events rather than just a single one. As mentioned earlier, the plain vanilla interest rate swap entails a series of fixed rate payments that — in the EUR market — will typically be annual as well as a series of floating rate fixings and payments that occur semi-annually. To facilitate the pricing of the IRSs and many other contracts we will therefore need to be able to create a full schedule for these events with some frequency over a given period of time. A minimum of three pieces of information are required to construct a schedule of dates:

- Start date, on what date should the schedule begin?

- Maturity, for how long will the schedule run? This can either be a specific date e.g. the 27th of November 2052 or be implied from a tenor e.g. 20Y.

- Frequency, how often should events occur in the schedule? This will typically be some even period such as 1Y, 6M, 3M or every business day (1B).

- Day rule, how should non-business days in the schedule be rolled?

Note that the start date in itself could actually be specified by a tenor if we provide an *Anchor date*. Now, how is such a schedule then constructed? There are some obvious problems that need to be addressed. First of all, if the length of the schedule is not divisible by the tenor setting the periodicity of the schedule (e.g. a schedule with a maturity of $9\frac{1}{2}$ years and a frequency of 1Y) we will end up with a number of regular periods and an "odd" period of a different length. Such an odd period is called a *stub*. Dealing with stubs is however not a problem, as long as we can decide on whether put the stub in the beginning or end of the schedule as well as decide if the stub should be shorter or longer than the regular periods.[4] In line with market practice, we place a stub in the beginning and shorten it relative to the regular periods. We then say that we roll out schedules using *short first stubs*.

Having decided on how to deal with potential stubs, it is now obvious that we will need to construct our schedules backwards. To do this, we first need to identify the start and maturity dates. This is done by first identifying the *unadjusted start date* (either by adding a tenor to the anchor date or by using a specific given date) and roll this according to the day rule to find the *Adjusted start date*.

Next, we find the maturity date by finding the *unadjusted maturity date* (either directly or by adding a tenor to the adjusted start date) and roll this into the *adjusted maturity date*. The date in the month for the unadjusted maturity date is called the *anniversary date*. Now, we can subtract multiples of regular periods from the unadjusted maturity date until we reach the adjusted start date or are left with a short stub. Suppose, we are looking for a 3M schedule of dates maturing on Monday 3 August 2020 (which is both the unadjusted and adjusted maturity date) that are all adjusted according to the modified following rolling convention. Subtracting 3M from this date, we arrive on 3 May 2020 but since this is a Sunday, we roll modified following without further problems to Monday 4 May. But what should we do from here, should we now subtract another 3M from this date? Doing that would cause our roll dates to gradually increase over time for longer dated schedules (until we reach the "modified" part of our MF convention).

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[4]Disregarding the possibility of placing the stub in between the regular periods.

Table 1: The fidGenerateSchedule output

UnAdj Start	UnAdj End	Adj Start	Adj End	Coverage
26-Jul-10	31-Oct-10	26-Jul-10	29-Oct-10	0.2639
31-Oct-10	30-Apr-11	29-Oct-10	29-Apr-11	0.5056
30-Apr-11	31-Oct-11	29-Apr-11	31-Oct-11	0.5139
31-Oct-11	30-Apr-12	31-Oct-11	30-Apr-12	0.5056
30-Apr-12	31-Oct-12	30-Apr-12	31-Oct-12	0.5111

To prevent this, we will construct our schedule in two steps. First, will construct a set of *unadjusted* days, which in this case would be the 3rd of February, May, August and November every year and then roll these dates into a set of *adjusted* dates. The two step approach ensures that the roll dates are always close the anniversary date. The schedule construction can be seen in figure 2.

Figure 2: Constructing a schedule from multiples of the relevant tenor.
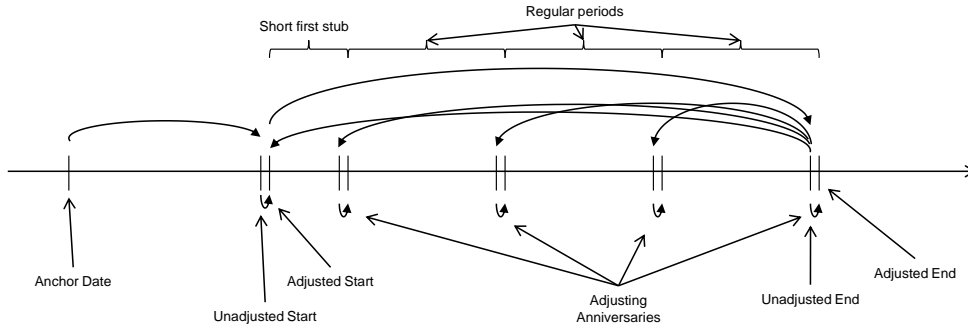
As we will often want not only to find a schedule of dates but also, their associated coverages it is worth pointing out that these should be calculated between the adjusted dates. The above logic have been implemented in the VBA code as `fidGenerateSchedule`. This function takes the arguments listed above as well as the day count basis as the calculation of coverages has been build into the function. The output of the function rather than just a single value — as you know from most functions in Excel — is a matrix, or *array* in the Excel terminology. If you are not already familiar with array functions, you can read about them in McDonald (2000).[5]

As an example let us study the output of `fidGenerateSchedule` using the following inputs `Anchordate` = 26-Mar-10, `Start` = 4M, `Maturity` = 31-Oct-2012, `Frequency` =6M, `DayCountBasis` = Act/360 and `DayRule` = MF. The output is shown in table 1.

This example reveals a number of interesting observations. The adjusted start date is rolled since 26 Sep 2010 is a Sunday. Secondly, since the subtraction of 6M periods are done on a multiplicative basis (i.e. subtracting 6M, 12M, 18M and so on) rather than on a sequential basis, unadjusted end dates correctly alternates between the 30th and 31st in each month. Thirdly, the combination of a 6M frequency and the fixed maturity date causes a stub that is correctly placed in front as a shorter than regular period.

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[5]Importantly, you should know that array functions are entered into a spreadsheet by selecting a range of cells, typing your array formula in the upper left corner cell and then hitting Ctrl+Shift+Enter. The last part puts curly brackets around the formula and accesses the full array.

2.5 Interest rate concepts

From micro- and macroeconomics the concept of interest rates is well known. However, there are many different ways of calculating interest rate payments and we need to be explicit when using these different methodologies. We therefore need to define a few different interest rate concepts and understand how they are related.

2.5.1 Discount factors and zero coupon rates

The (credit) risk free *zero coupon bond* is the financial contract that pays its (unit) face value at some maturity date with certainty. Such an obligation can be traded today and we can thus assign a market value to it. We define $P(t, T)$ to be the price observed at time t for a zero coupon bond maturing at time T. The time t present value of receiving \$1 at T is thus simply $P(t, T)$. If the time-value-of-money is positive (money now is worth more than money tomorrow), it must hold that $P(t, T) \leq 1 \quad \forall \quad t \leq T$. Note also, that since the zero coupon bond does not entail credit risk it must hold that $P(T, T) = 1$. Throughout the course we will show a special interest in the set of zero coupon bond prices observed at a specific date, $t = 0$. The reason for this is that knowing $P(0, T)$ for all possible values of T enables us to price — i.e. find the present value of — all certain cash flows simply by multiplying a cash flow with the relevant $P(0, T)$. We call the set of $P(0, T)$'s *discount factors* as they exactly tell us how to discount future cash flows.

Note that even though $P(t, T)$ for $t \geq 0$ is a price on a future date — a *forward price* — we can actually calculate it at $t = 0$ using a replication argument. Suppose we at $t = 0$ buy 1 unit of the zero coupon bond maturing at T at a cost of $P(0, T)$ and finance this purchase by selling $P(0, T)/P(0, t)$ units of the zero coupon bond maturing at t. The latter sale generates an initial cash flow of $P(0, T)$ thus making the two transactions self-financing. At time t we are now required to pay out $P(0, T)/P(0, t)$ while we are guaranteed to receive 1 unit at time T. These two cash flows are in fact the forward zero coupon bond in itself. Assuming absence of arbitrage, the forward price must therefore fulfill

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \tag{2.1}$$

Effectively, we can therefore lock in a rate of return today on a future investment.

Since the zero coupon bond is a tradeable asset with a unique price, we can use it to define and relate various interest rate accrual methods. As we will see later on, it can be very convenient to work with continuous accrual. We can illustrate this concept by thinking of a bank account

Letting $r_{Cont}(t, T)$ and $r_{Disc}(t, T)$ denote the zero coupon rates using continuous, respectively, discrete compounding these are related defined by

$$P(t, T) = \exp\big(-r_{Cont}(t, T) \cdot (T - t)\big) \tag{2.2}$$

$$P(t, T) = \frac{1}{(1 + r_{Disc}(t, T))^{T-t}} \tag{2.3}$$

If we can observe a set of zero coupon bond prices (or infer them from other instruments), we use the two definitions above to uniquely represent these as there is a 1:1 relation between zero coupon bond prices and their *zero coupon rates*. For a given observation time t, we call any mapping from $T \rightarrow r(t, T)$ a zero coupon *yield curve*. Importantly, we will thus speak of interest rates in plural — there a no longer a single rate of interest.

2.5.2 xIBOR rates

Many interest rate derivatives are contracts written either directly or indirectly on a set of official interest rates called LIBOR fixings. LIBOR® is short for the London InterBank Offered Rate and is an official fixing set for maturities ranging from 1B to 12M each day at 11:00 GMT by the British Bankers Association for all major currencies.[6] The rates are supposed to reflect the rate, i.e. the price, at which prime banks can borrow money on an *unsecured* basis in each currency. The fixing is calculated as a truncated average of rates submitted by a number of panel banks. The set of panel banks varies in size and composition for each currency. In addition to the BBA LIBOR fixings there are a number of similar fixings set by other fixing entities. Among these are the European Banking Federation, who sets the EURIBOR® fixing and NASDAQ OMX who sets the CIBOR fixing, which are the most widely used reference rates for interest rate derivatives in EUR and DKK. Collectively, we will refer all the rates as *xIBOR rates*. Although the fixing methodology differs, all of these xIBOR are used in in the same way in derivatives contracts for each currency.

xIBOR fixings are reported using the *Money Market convention*, which means that the interest paid at maturity on a notional of N is simply $\delta \cdot N \cdot L$ where δ denotes the coverage and L the xIBOR rate. This is sometimes called *simple interest*. Suppose we want to borrow 1 unit at time $t = 0$ maturing at time T. This can be done either by selling $1/P(0, T)$ units of the zero coupon bond maturing at T or by borrowing 1 unit in the xIBOR market. Assuming no arbitrage and that we can actually fund ourselves at the xIBOR rate, these two funding strategies must be equivalent. We therefore define the *spot xIBOR rate* between $t = 0$ and T, $L(0, T)$ as

$$1 + \delta L(0, T) = \frac{1}{P(0, T)} \Leftrightarrow$$
$$L(0, T) = \frac{1}{\delta}\left(\frac{1}{P(0, T)} - 1\right) \tag{2.4}$$

By $F(t, T, T + \delta)$ we denote the future simple interest rate contracted at time t at which we can borrow funds between time T and $T + \delta$. We will refer to $F(t, T, T + \delta)$ as the *Forward xIBOR rate*. The formula for the spot xIBOR rate can easily be generalized using the same arguments and recalling (2.1), it must therefore hold that

$$1 + \delta F(t, T, T + \delta) = \frac{1}{P(T, T + \delta)} \Leftrightarrow$$
$$1 + \delta F(t, T, T + \delta) = \frac{P(t, T)}{P(t, T + \delta)} \Leftrightarrow \tag{2.5}$$
$$F(t, T, T + \delta) = \frac{1}{\delta}\left(\frac{P(t, T)}{P(t, T + \delta)} - 1\right)$$

Note that the spot xIBOR rate is a special case of the forward xIBOR rate.

Before we can start to price specific xIBOR contracts, we need one final building block. Specifically, we need to be able to calculate the time t value of being being paid the δ-tenor xIBOR rate at time $T + \delta$ for the fixing done at time T. It turns out (as we will discuss in more detail in section 5.1) that there is a link between absence of arbitrage and risk neutral expectations. This link means that we can write

$$\text{PV of xIBOR payment}_t = P(t, T + \delta) \cdot E_t^{Q_{T+\delta}}[L(T, T + \delta)] \tag{2.6}$$

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[6]BBA has LIBOR fixings in USD, GBP, EUR, JPY, CHF, CAD, AUD, NZD, DKK and SEK.

where $E_t^{Q_{T+\delta}}[X_T]$ is the time t expectation of the stochastic variable X_T under the probability measure $Q^{T+\delta}$ (the meaning of which we will elaborate further on in section 5.1). By using this specific choice of probability measure, it turns out that calculating the expectation in (2.6) is particularly easy since

$$E_t^{Q_{T+\delta}}[L(T, T + \delta)] = F(t, T, T + \delta) \tag{2.7}$$

Taken together, (2.6) and (2.7) means, that we can actually assign an arbitrage free price to the claim of receiving a (stochastic) xIBOR payment by simply calculating $P(t, T + \delta)$ and $F(t, T, T + \delta)$. We will not pursue the deeper mathematics of these results here, but focus on applying the result to a number of specific contracts in the following sections.

3 Linear interest rate derivatives

Armed with our knowledge of the basic building blocks we are now ready to start pricing derivative contracts written on xIBOR rates. We will begin with a class of instruments that are usually referred to as the "linear" instruments in the market place. Later on we will work with non-linear instruments which are options written on interest rates. Both classes of instruments are extremely widely used and are traded in great volume on a daily basis in many currencies.

Derivative contracts can be traded in two different ways; either on an exchange with central clearing between counter parties or on a bilateral basis as so-called *over-the-counter* or *OTC* contracts. While the exchange traded market consist of standardized contracts the OTC market offers fully flexible instruments since counter parties can negotiate every detail of the individual contract.

3.1 Forward Rate Agreements

A *Forward Rate Agreement* or simply *FRA* is an OTC interest rate derivative in which the two parties agree to pay, respectively, receive the difference between a pre-specified fixed interest rate called the *FRA rate* or *strike* and a xIBOR rate over a given period of time on a given notional. Assuming that we can borrow funds at some xIBOR rate, entering into a FRA contract thus enables us to lock in a future funding rate by buying a FRA. By buying a FRA, we would effectively be paying a fixed rate against receiving a floating interest rate (the xIBOR that we assumed we could fund us at). On a net basis we would in this case be left with a fixed future interest rate payment. As such FRAs are extensively used to hedge interest rate exposure among both corporations and financial institutions.

Although the definition of xIBOR rates stipulate that interest is paid at the end of the accrual period, FRAs are designed to pay off the difference between the xIBOR fixing and the strike rate at the begging of the accrual period. This timing mismatch is however adjusted by discounting the interest rate payment back to the start date of the FRA using the xIBOR rate itself.

Formally, the party buying the FRA starting accrual at time T and maturing at time $T + \delta$ on a notional of N with a FRA rate of K receives the following cash payment at time T

$$\text{FRA payoff at time } T = \frac{N\delta(L(T, T + \delta) - K)}{1 + \delta L(T, T + \delta)} \tag{3.1}$$

It is market standard to let the day count convention of the coverage δ follow the day count convention of the xIBOR rate. For EUR FRAs written on EURIBOR®, δ is therefore calculated using Act/360. When we value a FRA contract, we want to calculate the present value of this cash flow. Note that we can plug in (2.5) and at time t lock in a future cash flow of on the FRA contract which can then be discounted back. Since it is market standard to trade FRAs "at market", that is trade them at an NPV of zero, we define the forward rate $F^{\text{FRA}}(t, T, T + \delta)$ to be the rate at which the FRA has zero NPV. Mathematically, you will typically see the FRA PV expressed as

$$\text{PV}_t^{\text{FRA}} = P(t, T)E_t^{Q_T}\left[\frac{N\delta(L(T, T + \delta) - K)}{1 + \delta L(T, T + \delta)}\right] \tag{3.2}$$

Again, we will be rather cavalier about the underlying mathematics and simply postulate that it is in fact when $F^{\mathrm{FRA}}(t, T, T+\delta) = F(t, T, T+\delta)$ that the FRA contract is "fair".[7]

FRAs are usually liquidly quoted on 1M, 3M and 6M xIBOR rates with a maturity of up to two years. Market activity is typically concentrated around "even" start dates such as 1M, 2M, 3M etc. from the trading date. In the industry jargon, FRAs will be denoted by short hand notation like 1X4, which refers to the FRA starting 1M from now maturing 4-1=3 months later. The 1x4 FRA is thus a contract on what the fixing of the 3M xIBOR rate will be in one month's time. A typical broker screen shown in figure 3.[8] Note that the screen shows both *bid* and *ask* prices. In the market terminology is is common to talk about three prices: The *mid* which is the price used as reference and for evaluating profit and loss and the bid and ask (also called *offer* price), which are the prices you sell, respectively, buy at. The so called bid-ask (or bid-offer) spread is an indication of how liquid a market is.

Figure 3: EUR FRA quotes from the broker ICAP, 4 February 2010.

3.2 Money market futures

Another important set of xIBOR based instruments are *Money Market futures*. These are exchange traded contracts where the most important ones are written on 3M USD LIBOR or 3M EURIBOR® rates. These are called Eurodollar (ED) and EURIBOR (ER) futures and trade on the Chicago Mercantile Exchagne (CME) and the London

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[7]This is actually a bit imprecise once we introduce the dual curve setup in section 4.6.

[8]A *broker* is a party that intermediates between parties in a given market. In the fixed income market broker firms such as ICAP, Tullet-Prebon and BGC Partners matches interests to buy and sell either via electronic platforms or via traditional voice broking. Broker screens often serve as reference in OTC markets.

International Financial Futures and options Exchange (LIFFE), respectively.[9] The payoff of these contracts on their settlement date per unit of notional is simply

$$\text{Money Market future payoff at time } T = \delta \cdot 100 \cdot (1 - L(T, T + \delta)) \tag{3.3}$$

Each contract has pre-specified notional of 1,000,000 USD or EUR. These contracts trade on a set of standardized settlement dates known as *International Monetary Market dates* or simply *IMM dates.* These are the third Wednesday in March, June, September and December. In addition to the IMM dated futures, a limited set of short dated non-IMM contracts trade. At any given time, ED and ER contracts are very liquid for maturities of up to 2 years.[10]

Money Market futures are like FRAs traded as NPV zero contracts. Today's futures price is thus the price that ensures zero NPV. Looking at the money market future settling at time T written on the xIBOR fixing set at time T with a tenor of δ, we will denote this futures time t price by $p^{\text{FUT}}(t, T, \delta)$. From this price we will define a corresponding futures implied rate $F^{\text{FUT}}(t, T, T + \delta)$ by

$$F^{\text{FUT}}(t, T, T + \delta) = \frac{1}{100}(100 - p^{\text{FUT}}(t, T, \delta)) \tag{3.4}$$

As is customary for exchange traded futures, these contracts are marked-to-market every day and any gain or loss on the position is settled. In this regard they differ from FRAs written on 3M xIBOR rates, otherwise the contracts are very similar.

Figure 4: The strip of LIFFE 3M EURIBOR Futures on 4 February 2010.

The fact that FRA can accrue value over time while futures are settled on a daily basis turns out to be important. Suppose you have bought an Eurodollar future (leaving you

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[9]For historical reasons a market for USD denominated deposits placed outside the United States arose after WWII. The market for these deposits is called the Eurodollar market.

[10]Although contracts are listed further out the curve, the liquidity in these are much smaller.

Table 2: Futures-FRA convexity adjustments in basis points, per 4 February 2010 using $\sigma = 1.2\%$.

Futures contract	Adjustment
Mar-10	0.0
Jun-10	0.2
Sep-10	0.4
Dec-10	0.7
Mar-11	1.1
Jun-11	1.6
Sep-11	2.2
Dec-11	2.8
Mar-12	3.6
Jun-12	4.5
Sep-12	5.4
Dec-12	6.4

positioned for lower rates) and that you at the same time have bought the corresponding FRA (leaving you positioned for higher rates). If rates drop, you receive a cash flow from the mark-to-market on future but you do not have the opposite cash flow on the FRA position, since this is not settled daily. You will therefore receive cash if rates drop. If rates on the other hand were to go up, you would have to pay cash to settle the future position while the FRA position will still not generate any cash flow. You will therefore pay out cash when rates go up. The combined position thus leaves you paying out cash exactly when it is expensive to borrow (rates have gone up) and receiving cash when it only earns a low interest (rates have gone down). This phenomenon creates a bias that prevents the rates implicit in Money Market futures prices from being identical to FRA rates. The is called the forward-futures *convexity adjustment* or *financing bias.*

The financing bias means that a market participant positioning herself for lower rates in a future will require a slightly higher break even rate relative to the corresponding FRA. We can conclude that is must be the case that $F^{\mathrm{FRA}}(t, T, T+\delta) < F^{\mathrm{FUT}}(t, T, T+\delta)$. This means that when observing futures implied rates, we must subtract something from these to correct for the bias. In real life, the adjustment between the two set of rates is done in a term structure model and will be increasing in the time to settlement as well as the (implied) volatility of interest rates. A simple version of such an adjustment is

$$F^{\mathrm{FUT}}(t, T, T+\delta) - \mathrm{FRA}(t, T, T+\delta) = \frac{1}{2}\sigma^2 T(T+\delta) \tag{3.5}$$

where σ denotes the per-annum implied volatility of the short rate.[11] A typical value for σ could be 1.2%, thus yielding the set of convexity adjustments found in table 2 expressed as basis points.[12] We note that the adjustment is increasing roughly to the square of time, so this phenomenon matters most for longer dated contracts.

3.3 Interest rate swaps

As mentioned in section 2.1 a plain vanilla interest rate swap is a bilateral contract between two counter parties to exchange a series of fixed interest rate payments for a

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[11]See Hull (2006) for more information on this.

[12]One basis point is 1/100 of a percent.

series of floating interest rate payments over specified period of time. The plain vanilla IRS thus have a fixed leg and a floating leg. For the counter party paying the fixed rate, the fixed leg is thus a liability while the floating leg is an asset. For each counter party the position in the swap is denoted relative to the fixed leg (i.e. the counter party paying the fixed rate has entered into a *payer swap*, while the counter party paying the floating rate has entered into a *receiver swap*).

We value an IRS by finding the value of each leg separately. Let us consider a swap starting at time T_S and maturing at time T_E. Note that most swaps are traded with spot start (meaning that $t \approx T_S$), but in the below we will cover the more general case of *forward starting swaps*. Although most IRSs are written on a so-called bullet notional profile (a constant notional), we will for the sake of generality consider a swap where the notional corresponding to the accrual period starting at time T_{i-1} and ending at time T_i, N_i, can vary over time.

The floating leg in this swap is linked to some xIBOR rate fixed-in-advance (i.e. the floating rate is reset at the begging of each accrual period) and paid-in-arrears (i.e. the interest rate is paid at the end of the accrual period). The xIBOR rate has a tenor of δ (typically 3M or 6M) which is calculated according some day count convention and paid on an coverage of δ^{Float}. For the plain vanilla IRS these will be identical, that is, 6M EURIBOR® fixings are paid on 6M coverages implying that fixings and payments occur with the same frequency.[13] The standard market conventions for the IRSs in the major currencies are listed in table 3.

Table 3: The plain vanilla IRS conventions.

Currency	Index name	Spot start	Roll	Floating Leg Term	Floating Leg Freq.	Floating Leg Day count	Fixed Leg Freq.	Fixed Leg Day count
EUR	Euribor	2B	MF	6M	S	Act/360	A	30/360
USD	USD Libor	2B	MF	3M	Q	Act/360	S	30/360
GBP	GBP Libor	0B	MF	6M	S	Act/365	S	Act/365
JPY	JPY Libor	2B	MF	6M	S	Act/360	S	Act/365
SEK	Stibor	2B	MF	3M	Q	Act/360	A	30/360
NOK	Nibor	2B	MF	6M	S	Act/360	A	30/360
DKK	Cibor	2B	MF	6M	S	Act/360	A	30/360

Knowing the start-, end dates, frequency and day count convention of the floating leg, we define a set of coverages $\delta_{S+1}^{\text{Float}}, \ldots, \delta_E^{\text{Float}}$ and dates $T_S, \ldots, T_E$ spaced apart by δ_i^{Float}. We can now write the present value of this floating leg at time t as

$$PV_t^{\text{Float}} = \sum_{i=S+1}^{E} \delta_i^{\text{Float}} F(t, T_{i-1}, T_i) N_i P(t, T_i) \tag{3.6}$$

Note that we because we can replicate all future xIBOR payments using zero coupon bonds (remembering (2.5)), we can easily value a future floating rate cash flow stream without any advanced modelling. In fact looking at the case of bullet (i.e. a constant) notional, since we are calculating our forward xIBOR rates on the same zero coupon curve as we are discounting on, we can simplify equation (3.6) even further by using the definition in (2.5):

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[13]In practice — because of the way schedules are rolled out — the accrual periods on the floating leg in an IRS does not necessarily match the accrual period of each xIBOR fixing perfectly, we will however disregard this phenomenon in this course.

$$PV_t^{\text{Float}} = N \cdot (P(t, T_S) - P(t, T_E)) \qquad (3.7)$$

This formula looks suspiciously simple, so we need to understand how it can arise. Note first, that we can easily add an exchange of notional to the floating rate stream by adding $N \cdot P(T_E)$ to (3.7) thus giving us a so-called *Floating Rate Note* (FRN).[14] If we furthermore focus on $T_S = 0$, the special case of a spot starting FRN, we realize that it must have a PV of simply N. Put differently, this means that the spot starting bullet FRN must trade at par. The key insight here is, that we accrue interest at the exact same rate as we discount with. We are thus exactly compensated enough to be indifferent between receiving N dollars today or receiving them in future. If we discounted at a different rate or if we earned a floating rate plus or minus a spread, this par result would not hold. In section 4.4 we will get back to this point and break this close relationship between forward rates and discount factors.

The fixed leg will have the same start and end dates as the floating leg, but the set of coverages and dates we need to create will typically be different because of differences in payment frequency and day count conventions. For the fixed leg we therefore define a set of coverages $\delta_{S+1}^{\text{Fixed}}, \ldots, \delta_E^{\text{Fixed}}$ and dates $T_S, \ldots, T_E$ spaced apart by δ_i^{Fixed}. Denoting by K the fixed rate paid in the swap, we easily value the fixed leg simply by discounting back the coupon payments

$$PV_t^{\text{Fixed}} = \sum_{i=S+1}^{E} \delta_i^{\text{Fixed}} K N_i P(t, T_i) \qquad (3.8)$$

For the payer swap starting at T_S and maturing at T_E we can now summarize its time t PV as

$$PV_t^{\text{Payer}} = \sum_{i=S+1}^{E} \delta_i^{\text{Float}} F(t, T_{i-1}, T_i) N_i P(t, T_i) - \sum_{i=S+1}^{E} \delta_i^{\text{Fixed}} K N_i P(t, T_i) \qquad (3.9)$$

since the party paying fixed, will see the floating leg as an asset and the fixed leg as a liability. Note that it follows that $PV^{\text{Receiver}} = -PV^{\text{Payer}}$.

As with FRAs it is customary to trade swaps as zero NPV contracts, we thus define $R(t, T_S, T_E)$, the *par swap rate*, to be the fixed rate that ensures $PV^{\text{Float}} = PV^{\text{Fixed}}$, yielding the following definition:

$$R(t, T_S, T_E) = \frac{\sum_{i=S+1}^{E} \delta_i^{\text{Float}} F(t, T_{i-1}, T_i) N_i P(t, T_i)}{\sum_{i=S+1}^{E} \delta_i^{\text{Fixed}} N_i P(t, T_i)} \qquad (3.10)$$

When trading IRSs, market participants will communicate this par swap rate, a notional and the pay or receive position. Although IRSs are OTC contracts they trade extremely frequently, often via electronic trading screens as seen in figure 5. Screens like this can thus serve as a market reference.

Again turning our focus to the bullet notional case, we see that can write the par swap rate as

$$R(t, T_S, T_E) = \sum_{i=S+1}^{E} w_i F(t, T_{i-1}, T_i), \text{where} \quad w_i = \frac{\delta_i^{\text{Float}} P(t, T_i)}{\sum_{j=S+1}^{E} \delta_j^{\text{Fixed}} P(t, T_j)} \qquad (3.11)$$

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[14]A FRN is simply a bond paying a floating, rather than fixed coupon.

Figure 5: EUR swaps against 6M EURIBOR®.

Since $\sum w_i \approx 1$, we can think of the par swap rate as being a weighted average of the forward rates. This average is "front loaded" in the sense that the weights applied to first forwards are larger than the weights applied to the last forwards because of discounting. Intuitively, this makes good sense since — if the contract is fair — we must on average expect to pay and receive the same in interest.

Another important fact about IRSs is that we can think of the value of a swap as being proportional to difference between the fixed rate and the par swap rate. Say, we are currently paying a fixed rate K (against receiving xIBOR) on a swap starting at T_S and maturing at T_E. Were we to enter into a matching receiver swap done at the par rate, the two floating legs will cancel each other out. The two fixed legs will however leave us with a series of netted fixed rates of $R(t, T_S, T_E) - K$ paid on the fixed rate coverages. We can thus write the value of the payer swap as

$$PV_t^{\mathrm{Payer}} = A(t, T_S, T_E)(R(t, T_S, T_E) - K) \tag{3.12}$$

where $A(t, T_S, T_E) = \sum \delta_i^{\mathrm{Fixed}} P(t, T_i)$ denotes the swap's *annuity factor* or *level* of the swap. Properly scaled the annuity factor is the value of receiving 1 bps for $T_E - T_S$ years. Since $\partial PV_t^{\mathrm{Payer}}/\partial R = A$ we note that the sensitivity — the risk — of the payer with respect to the par swap rate is exactly the swap annuity (and similarly $-A$ for the receiver swap).

When trading IRSs it is customary to close out trades by entering into an offsetting position rather than terminating the existing trade. For this reason (3.12) is a very common way determining the value of swap positions. Looking at (3.12) it is clear that a position in a payer swap, leaves you positioned for par swap rates to increase (and vice versa for receiver swaps). We can think of a forward starting IRS as a forward contract on the underlying par swap rate. The payer is long this forward contract while the receiver

Table 4: Swap annuities on a 100m notional for a 1 bps payment. The annuities are based on 30/360 annual payments on flat zero coupon yield curves while varying the level between 0% and 5%.

Maturity	0%	1%	2%	3%	4%	5%
1Y	10,000	9,900	9,802	9,704	9,608	9,512
2Y	20,000	19,702	19,410	19,122	18,839	18,561
5Y	50,000	48,526	47,106	45,736	44,414	43,140
10Y	100,000	94,685	89,726	85,097	80,773	76,731
20Y	200,000	180,352	163,176	148,124	134,901	123,255
30Y	300,000	257,863	223,305	194,807	171,174	151,463
50Y	500,000	391,444	312,821	254,992	211,772	178,937
100Y	1,000,000	628,786	427,823	311,831	240,393	193,600

is short the contract.[15]

Note that the swap annuity is increasing in the length of the swap but that this effect diminishes over time if discounting rates are positive as seen in table 4. As can be seen, at a flat yield curve of 4% the value of receiving 1 bps on a EUR 100m notional for an additional 20 years after the 30th year is just around EUR 40,000 compared to the approx. 135,000 that the first 20 years are worth.

3.4   Calibrating a swap market

While the previous sections have shown how to price FRAs and IRSs given a set of discount factors (or equivalently a zero coupon yield curve), we have still yet to see how we can infer these from market observables. In fact, rather than pricing than swaps based on discount factors, we are typically interested in using our pricing formulae "the other way around", that is, we want to find a set of discount factors consistent with market rates. Typically, market participants model the underlying zero curve and fit this to match a set of market quotes. This fitting is often referred to as curve stripping, bootstrapping or simply curve calibration.[16]

Suppose we are interested in constructing a zero curve with a maximum maturity of 10 years. Suppose further we can observe a quote for the 10Y EUR IRS. From section 3.3 we know that this quote depends on 20 forward xIBOR rates (since the floating leg is semi-annual and there is coinciding payments on the two legs), which in turn are spanned by 20 discount factors. Just knowing this one quote is obviously not enough to uniquely determine the zero curve. Had we instead been able to observe all 20 FRAs that implicitly make up the swap, we could have easily determined a curve that could uniquely price all swaps that fall within this 10 year semi-annual grid.[17] In reality we can typically observe many quotes at a time (and thus put many restrictions on our zero curve) but we cannot hope to observe enough market quotes to uniquely determine the entire yield curve, as we would need non-overlapping instruments for every possible maturity date. For this reason, market practice is to specify a set of knot points and an interpolation method through

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[15]In the market lingo, you are *long* an asset if you want the underlying to increase in price and *short* if you want it to go down.

[16]While bootstrapping is actually a well defined mathematical method that only works for certain interpolation methods, the term is often used interchangeably with curve calibration.

[17]More precisely, we could say that the IRS consist of 19 forward rates as the first 6M floating rate is fixed already on the trading day for spot starting EUR swaps.

which we can calculate any discount factor that we may need. An addition to this, we typically also need an extrapolation rule. As discussed in Hagan & West (2006), it is customary to do this inter- and extrapolation in continuously compounded zero coupon rates, although we could have chosen to do our interpolation in other domains such as instantaneous forwards or even the discount factors themselves. This means that the only parameters for our "model" of the yield curve is a number of zero rates (and their associated maturity date) as well as inter- and extrapolation rules.

Formally, we will formulate our curve calibration problem as a least squares optimization problem. For a set of market quotes $\mathbf{A} = \{a_1, \ldots, a_N\}^\top$, a set of parameters $\mathbf{P} = \{p_1, \ldots, p_M\}$, we use the formulae from the previous sections to compute a set of model quotes $\mathbf{B}(\mathbf{P}) = \{b_1, \ldots, b_N\}^\top$ based on these inputs. Calibrating our zero coupon yield curve is now a question of solving

$$\min_{\mathbf{P}} \sum_{i=1}^{N} (a_i - b_i)^2 \tag{3.13}$$

More conveniently (as well will see), we could have stated this in matrix format as:

$$\min_{\mathbf{P}} ||\mathbf{B}(\mathbf{P}) - \mathbf{A}||^2 \tag{3.14}$$

Depending on our choice of interpolation, this can become a complex non-linear problem to solve, why we will resort to numerical routines. It is however worth noting that in any circumstance we know that the optimization problem will have the following first set of order conditions (using the matrix formulation):

$$(\mathbf{B}(\mathbf{P}) - \mathbf{A})^\top \frac{\partial \mathbf{B}(\mathbf{P})}{\partial \mathbf{P}} = \mathbf{0} \tag{3.15}$$

where $\frac{\partial \mathbf{B}(\mathbf{P})}{\partial \mathbf{P}}$ denotes the jacobian matrix defined by:

$$\frac{\partial \mathbf{B}(\mathbf{P})}{\partial \mathbf{P}}_{N \times M} = \begin{pmatrix} \frac{\partial b_1}{\partial p_1} & \cdots & \frac{\partial b_1}{\partial p_M} \\ \vdots & \ddots & \vdots \\ \frac{\partial b_N}{\partial p_1} & \cdots & \frac{\partial b_N}{\partial p_M} \end{pmatrix} \tag{3.16}$$

This means that at the heart of any calibration algorithm, the jacobian matrix $\frac{\partial \mathbf{B}(\mathbf{P})}{\partial \mathbf{P}}$ must be calculated. This will turn out to have a very useful application not only in the context of hedging of swap risks in section 3.7 but also in more general cases as we will see chapters 5 and 6.

To ensure nice properties of the problem and fast calibration, it is customary to specify a number of knot points equal to the number of market quotes (i.e. set $N = M$) and ensure that no instruments are perfectly overlapping. In practice this is typically done by selecting a number of FRAs as well as the relevant xIBOR fixing for the first section of the curve and then use IRSs for the rest of the curve.

3.5   Interpolation

Having understood the basic problem of curve calibration, we will in the following apply some of methodologies laid out in Hagan & West (2006) while we examine some practical implications. In particular, we will work with *linear-* and *hermite spline* interpolation.

Some of the interpolation methods from Hagan & West (2006) has been implemented in the VBA function `fidInterpolate`. This function takes the arguments `KnownX`, `KnownY`, `OutputX` and `Method`. The first two arguments are assumed to be two vectors of equal length that are sorted in increasing order for the known X's. `OutputX` is the X value for which we want to interpolate and `Method` is finally a string specifying which interpolation method to use. The choices here are `Constant`, `Linear`, `LogLinear` or `Hermite`. Before we proceed, it is perhaps worth remembering that we often not only need an interpolation rule — how should we price a 31 year swap if we can only observe market quotes out to 30 years? If we apply some extrapolation rule, we can at least get some qualified guidance on how to quote this. While we will only use the flat extrapolation scheme here, we note that there are just as many extrapolation methods as there are interpolation methods.

3.6 Curve construction in practice

A typical set of market quotes for constructing a 6M based EURIBOR® curve could be something like the inputs found in table 5. Note that none of the quotes are perfectly overlapping (the 2Y swap consists e.g. of the 6M fixing, the 6X12, 12X18 and 18X24 FRAs, but the two latter are not among the quotes). The input corresponds to the mid-market quotes found in figure 3 and figure 5.

Table 5: EUR swap curve input quotes, as of 4 February 2010

EURIBOR®	Fixing	FRA	Quote, mid	IRS	Quote, mid
6M	0.967	1X7	0.98	2Y	1.652
		2X8	1.043	3Y	2.019
		3X9	1.13	4Y	2.319
		4X10	1.217	5Y	2.577
		5X11	1.317	7Y	2.995
		6X12	1.399	10Y	3.395
		7X13	1.476	15Y	3.753
		8X14	1.56	20Y	3.873
		9X15	1.637	30Y	3.975

Letting our anchor date be 4-Feb-2010, we can now construct the set of knot points that we need to calibrate our FRA/IRS market by setting these equal to the maturity date for each instrument. The exact dates can in turn be found using `fidAddTenor` with the MF day rule. In order to solve the problem in (3.13), we need some functions that can actually calculate market rates from a set knot points and zero rates, these are `fidForwardRate` and `fidSwapRate`. These in turn build on top of `fidZeroRate` and `fidDiscFactor` which both simply interpolate in a set of continuously compounded Act/365 rates denoted `CurveRates` and their corresponding knot points `CurveMaturities` (which are to be specified as dates).

For both `fidForwardRate` and `fidSwapRate` to work, each function needs some additional instrument specification such as `Start`, `Maturity`, `DayCountBasis` (both fixed and floating for `fidSwapRate`) and `DayRule`. In addition both `FloatTenor` and `FixedTenor` should be specified for `fidSwapRate`. Both start- and maturity dates can either be specified as dates or as periods (e.g. 5Y). Finally, both functions need `CurveRates`, `CurveMaturities` as well as `Method` as they link back to the interpolation in zero rates. Actually, the function `fidSwapRate` takes the arguments `FwdCurveMat`, `FwdCurveRates`,

`DiscCurveMat` and `DiscCurveRates` rather than just one set of dates and rates. The two latter are optional arguments - later on in the course we will see, why we need to distinguish between forward and discounting curves. For now, it suffices to know that when the optional inputs are not provided, `fidSwapRate` resorts to using the same curve for calculated forward rates and discount factors.

Let us see how all this works in practice. Having calculated the maturity dates as shown in table 5 we can setup the optimization problem (3.13) and find the associated zero coupon rates using the `SOLVER` function in Excel. The errors between the observed market rates and the computed rates are negligible for all methods, within a a single basis point.

Table 6: EUR swap curve calibrated per 4 February 2010 using different interpolation methods, zero rates are continuously compounded Act/365 rates.

End point	Knot date	Constant	Linear	LogLinear	Hermite
6M	04/08/2010	0.978%	0.976%	0.976%	0.98%
7M	06/09/2010	0.989%	0.989%	0.989%	0.99%
8M	04/10/2010	1.036%	1.033%	1.034%	1.034%
9M	04/11/2010	1.089%	1.088%	1.088%	1.089%
10M	06/12/2010	1.131%	1.13%	1.13%	1.131%
11M	04/01/2011	1.172%	1.17%	1.17%	1.171%
12M	04/02/2011	1.197%	1.196%	1.196%	1.198%
13M	04/03/2011	1.22%	1.215%	1.215%	1.216%
14M	04/04/2011	1.267%	1.266%	1.266%	1.267%
15M	04/05/2011	1.314%	1.313%	1.313%	1.314%
2Y	06/02/2012	1.638%	1.638%	1.638%	1.638%
3Y	04/02/2013	2.004%	2.002%	2.002%	2.002%
4Y	04/02/2014	2.307%	2.305%	2.305%	2.305%
5Y	04/02/2015	2.57%	2.569%	2.569%	2.569%
7Y	06/02/2017	3.01%	3.005%	3.005%	3.004%
10Y	04/02/2020	3.453%	3.435%	3.435%	3.433%
15Y	04/02/2025	3.88%	3.838%	3.839%	3.832%
20Y	04/02/2030	4.007%	3.965%	3.965%	3.958%
30Y	06/02/2040	4.119%	4.072%	4.073%	4.062%

Importantly however, it is not only on their ability to fit market rates that we should evaluate the different interpolation schemes. Typically, market participants would require smoothness on the estimated curves - in particular the forward xIBOR curve since these projects futures payments. If the forward xIBOR curve is not smooth, instruments with near identical fixing and cash flow profiles can be valued very differently — something that can be intuitively hard to justify. Looking at figure 6 and figure 7 we can see while the zero coupon rate curves behave much the same (with the exception of constant interpolation) but that the different methods yield highly different results in terms of 6M forward xIBOR rates. Since we can intuitively think of forward xIBOR rates as being linked to the slope of the zero curve, it is no surprise that the non-differentiable methods `constant`, `linear` and `loglinear` does not perform very well. We see however that `hermite` interpolation does a good job using this eyeball statistic. Another test that we could reasonably subject our interpolation methods to, is to check how well market rates calculated on our interpolated curves match up with the observable market that were not used in the curve construction.

Figure 6: Calibrated zero coupon curves (Act/365)
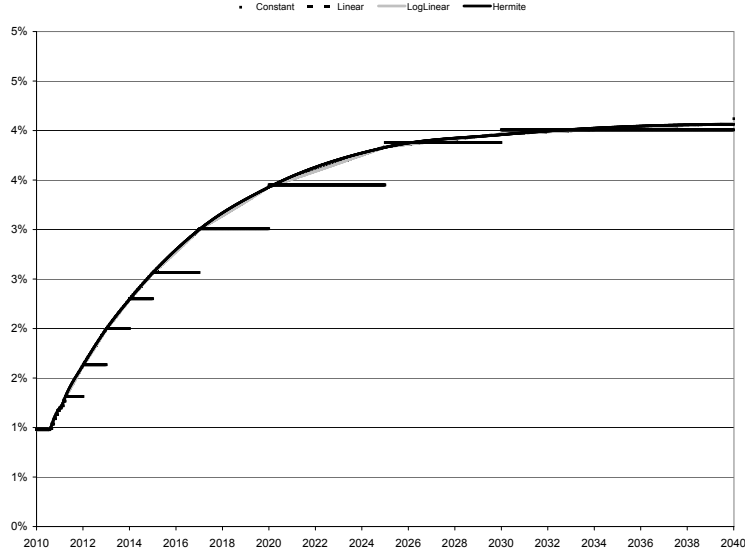
3.7 Hedging swap risks

In the previous sections we have seen how the value of FRAs and IRSs depend on zero rates and how these can be calibrated to market rates. As a consequence, we must expect that the value of positions in these products change in value when market rates change. In order to assess these risks we need to quantify our exposure. This is typically done in a simple, yet very intuitively manner by bumping the yield curve used to value swaps and FRAs. In line with traditional derivatives "lingo", first order interest rate sensitivities for interest rate derivatives are referred to as *delta risks*.

The crudest measure is perform a parallel shift of 1 bps on the zero curve and calculate the resulting change in value to our positions. This is called calculating the *Dollar value of a basis point* or simply *DV01*. Given that the yield moves in many other ways than just parallel, such a risk measure is typically not enough. Instead, we will compute a sensitivity with regard to several rates. As our delta risk in this case will be expressed as several numbers, we will then describe them as a *delta vector*. In this course, we will be working extensively with two different delta vectors — on that this calculated based on our vector model parameters $\mathbf{P}$ (the zero coupon rates) and one that is calculated on our market rates $\mathbf{Q}$. While both will obviously describe the same underlying economics, it will often be much easier for market participants to relate their risk profile on market observables (i.e. market rates) rather than more or less intuitive model parameters (something that become even clearer when working with the more advanced option pricing models in section 5.7).

Letting $V(\mathbf{P})$ denote the value of a derivative or portfolio of derivatives, we define DV01 (with a slight abuse of notation) as

$$
\begin{aligned}
\text{DV01} &= \frac{1}{10,000} \frac{\partial V(\mathbf{P})}{\partial \mathbf{P}} \\
&\approx \frac{1}{10,000} \frac{V(\mathbf{P}+\epsilon) - V(\mathbf{P})}{\epsilon}
\end{aligned}
\tag{3.17}
$$

Before we turn our attention to some numerical results for specific instruments, it is worthwhile to explicitly calculate the derivative of the time T maturity zero coupon bond

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Figure 7: Calibrated 6M Forward xIBOR curves (Act/360)
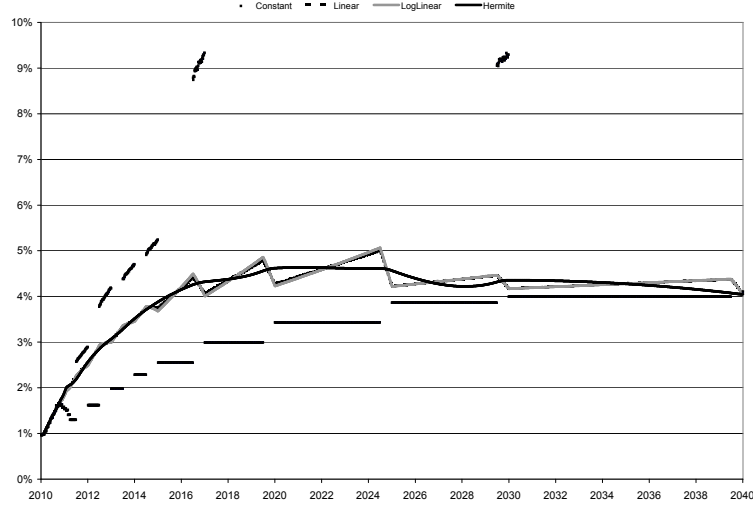
$P(t,T)$ with respect to its corresponding continuously compounded zero coupon rate $r(t,T)$

$$\frac{\partial P(t,T)}{\partial r(t,T)} = \frac{\partial \exp(-r(t,T)(T-t))}{\partial r(t,T)}$$
$$= -(T-t)P(t,T) \tag{3.18}$$

This shows us (unsurprisingly) that the holder of a zero coupon bond is positioned for lower rates and that this sensitivity is proportional to the product of time-to-maturity and the zero coupon bond price itself.

Since we are typically interested in calculating risk figures for trades we have already priced, we will typically use a single sided finite difference computation like the one above to avoid the overhang of calculating PVs in both an up and a down shift. If we are looking to let our DV01 capture local risks, we will set $\epsilon = 1$ bp. As an example, let us first try to calculate the DV01 for a EUR 100m 10Y EUR payer swap with a fixed equal to the par rate on the curve calibrated in section 3.6. On the initial curve, this trade has (per definition) a PV of EUR 0 while its value changes to EUR 85,973 in the +1 bp scenario. Noting that 1 bp $= 1/10,000$, the DV01 of this trade is thus

$$\text{DV01} = \frac{1}{10,000} \frac{85,973 - 0}{\frac{1}{10,000}} = 85,973$$

This means that if the zero coupon curve increases in parallel fashion by 1 bp, the above swap will gain EUR 85,973. What about the delta vectors then? Instead of bumping the entire zero coupon curve in one go, we now shift to changing only one point at a time. Formally, we are looking to calculate:

$$\underset{1 \times M}{\frac{\partial V(\mathbf{P})}{\partial \mathbf{P}}} = \left(\frac{\partial V(\mathbf{P})}{\partial p_1}, \ldots, \frac{\partial V(\mathbf{P})}{\partial p_M} \right) \tag{3.19}$$

Again, we use bump-and-revalue approach and use a bump of 1 bps to approximate $\partial V(\mathbf{P})/\partial p_i$. The results can be seen figure 8 and are summarized in table 7.

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Table 7: Zero rate deltavectors using hermite interpolation, all trades are on a EUR 100m notional.

Bucket	10Y at-market Payer	5Y10Y 3% Receiver	8x14 Long at-market FRA	9Y 5% Payer
6M	-110	0	0	-110
7M	0	0	0	0
8M	0	0	-6,585	0
9M	0	0	0	0
10M	0	0	0	0
11M	-12	0	0	-18
12M	325	0	0	479
13M	26	0	0	38
14M	-1	0	11,537	-2
15M	0	0	0	0
2Y	654	0	0	964
3Y	958	0	0	1,411
4Y	949	256	0	1,397
5Y	2,280	43,290	0	-1,039
7Y	5,399	-3,871	0	33,915
10Y	75,586	-9,753	0	50,816
15Y	-80	-92,001	0	-2,381
20Y	0	481	0	0
30Y	0	0	0	0
Total	85,973	-61,598	4,953	85,469

We see that the spot starting 10Y payer swap is positioned for higher rates, with most of its risk placed in the 10Y bucket. Recalling equation (3.9) and the fact that — under some assumptions — the floating leg could be written as the difference between two discount factors, the picture is no surprise. In fact looking at (3.20), we can see that since $P(t, T_S) \approx 1$ for a spot starting swap (and since it has very limited sensitivity towards changes in rates), it must be the second term that is providing most of sensitivity — at least for fixed coupons, K, in the 0%-5% range. Since we can write the value of the payer swap as

$$PV_t^{\text{Payer}} = P(t, T_S) - P(t, T_E) - \sum_{i=S+1}^{E} \delta_i^{\text{Fixed}} K P(t, T_i) \tag{3.20}$$

we can also intuitively explain the delta vector for the forward starting swap. In the case of forward starting swaps, both of the first two terms in (3.20) appear as providing the bulk of sensitivity. Remembering (3.18) we can also understand why the 15Y sensitivity is numerically larger than the 5Y sensitivity. Finally, since we can actually think of a FRA as a one period forward starting swap, we appeal to the same intuition when explaining the FRA's deltavector. Finally, we note that the sum of the delta vector risks is identical to the DV01 calculated above. For linear products such as IRSs and FRAs this will generally be the case, since bumping all the points by 1 bp one at a time will be roughly identical to bumping the entire curve by 1 bp.

Looking towards calculating the delta vector with respect to market — rather than zero coupon — rates we quickly face a problem in terms of computing performance. Since the curve calibration is a computationally expensive task we would rather not like to

Figure 8: Zero rate delta vectors using hermite interpolation
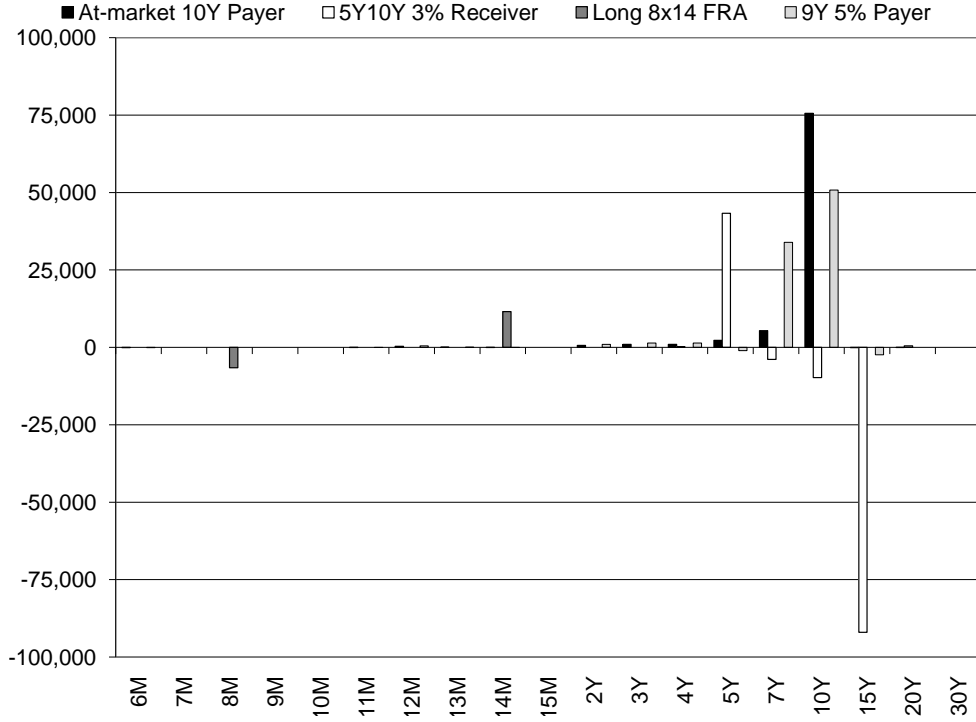
calculate the market rate delta vector by bumping each market rate and then subsequently recalibrate our curve. Instead, we can use a trick relying on multivariate calculus.[18] While this trick is widely used in industry, there are only few references to it in the literature with the notable exception of Andersen & Piterbarg (2010a).[19]. First, note that while we in the above calculated $\frac{\partial V(\mathbf{P})}{\partial \mathbf{P}}$ our calibration problem involved calculating $\frac{\partial \mathbf{B}(\mathbf{P})}{\partial \mathbf{P}}$. Now the trick is that we can use the chain rule of multivariate calculus to write

$$\underset{1 \times M}{\frac{\partial V(\mathbf{P})}{\partial \mathbf{P}}} = \underset{1 \times N}{\frac{\partial V(\mathbf{P})}{\partial \mathbf{B}}^{\top}} \underset{N \times M}{\frac{\partial \mathbf{B}(\mathbf{P})}{\partial \mathbf{P}}} \tag{3.21}$$

implying that we can simply produce the sensitivities with respect to market rates rather than model parameters (i.e. zero coupon rates) if we can invert $(\partial \mathbf{B}(\mathbf{P})/\partial \mathbf{P})^{\top}$ to find

$$\underset{N \times 1}{\frac{\partial V(\mathbf{P})}{\partial \mathbf{B}}} = \left(\underset{N \times M}{\frac{\partial \mathbf{B}(\mathbf{P})^{\top}}{\partial \mathbf{P}}} \right)^{-1} \underset{M \times 1}{\frac{\partial V(\mathbf{P})^{\top}}{\partial \mathbf{P}}} \tag{3.22}$$

This is an important result since it explicitly gives us the link between our calibration routine and our hedging problem.[20] It is extremely useful from a computational point since we basically just need to store information from the calibration problem and retrieve this in order to calculate risk figures. Unfortunately, we cannot retrieve the jacobian directly

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[18]Thanks to Jesper Andreasen for pointing this out.

[19]Albeit in the context of interest rate options the jacobian methodology is also presented in Chibane, Miao & Xu (2009).

[20]It is worth noting that the above implicitly assumes that $N = M$ otherwise we cannot invert $(\partial \mathbf{B}(\mathbf{P})/\partial \mathbf{P})^{\top}$. We will get back to this, and potential solutions in section 5.7.3.

from Excels Solver routine so we do need the extra overhang of estimating the jacobian $\frac{\partial \mathbf{B}(\mathbf{P})}{\partial \mathbf{P}}$.[21] Excel can however help us to calculate the market based delta vector via the functions for matrix inversion and multiplication (`MMULT()` and `MINVERSE()`). Intuitively, we are simply solving a set of linear equations by some matrix algebra in order to work out what shifts to the model parameters (the zero rates) correspond to in terms of market rates. Using the above results, a set of market rate delta vectors are shown in table 8.

Table 8: Market rate delta vectors using hermite interpolation, all trades are on a EUR 100m notional.

Instrument	10Y at-market Payer	5Y10Y 3% Receiver	8x14 Long at-market FRA	9Y 5% Payer
Fixing 2B 6M	0	66	0	99
FRA 1M 6M	0	-14	0	-22
FRA 2M 6M	0	4	0	6
FRA 3M 6M	0	0	0	0
FRA 4M 6M	0	0	0	0
FRA 5M 6M	0	-2	0	-3
FRA 6M 6M	0	43	0	68
FRA 7M 6M	0	3	0	5
FRA 8M 6M	0	0	4,982	-1
FRA 9M 6M	0	0	0	0
IRS 2B 2Y	0	172	0	271
IRS 2B 3Y	0	261	0	410
IRS 2B 4Y	0	469	0	423
IRS 2B 5Y	0	47,218	0	-3,873
IRS 2B 7Y	0	720	0	32,504
IRS 2B 10Y	85,632	2,449	0	57,943
IRS 2B 15Y	0	-114,119	0	-2,884
IRS 2B 20Y	0	-86	0	-19
IRS 2B 30Y	0	0	0	0
Total	85,632	-62,817	4,981	84,928

From table 8 we can observe the intuitive natural result that a 10Y payer swap at the par rate is only exposed to the 10Y swap rate (recal (3.12)). The same result goes for the 8x14 FRA since both of these instruments were part of the original curve estimation. Interestingly, we also see that the 9Y swap has — somewhat counter intuitively — exposure towards the 15Y swap. Why is this? Because of the interpolation, the bump-and-revalue of individual zero rate points has a potential effect for a larger section of the curve. In the case of non-local interpolation schemes (i.e. rules that rely on more that just neighboring points) such as hermite spline, bumping one point has the potential to cause changes beyond the neighboring points. While the calibration problem per construction removes this effect in optimum ($\mathbf{B}(\mathbf{P}) = \mathbf{A}$) for instruments that are included in the calibration problem, the effect can very well arise for instruments that were not part of calibration problem. To further illustrate this point, table 9 shows the market rate delta vector for the same instruments calculated on a curve with linear interpolation. By using this local

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[21]Had we instead used more specialized software such as Matlab or alternatively built our own optimizing algorithm, we could have retrieved the Jacobian directly from the calibration problem.

interpolation scheme, we see that the sensitivities of the 9Y swap towards the 5Y and 15Y swaps are significantly reduced.[22]

Table 9: Market rate delta vectors using linear interpolation, all trades are on a EUR 100m notional.

Instrument	10Y at-market Payer	5Y10Y 3% Receiver	8x14 Long at-market FRA	9Y 5% Payer
Fixing 2B 6M	0	59	0	90
FRA 1M 6M	0	-3	0	-4
FRA 2M 6M	0	0	0	-1
FRA 3M 6M	0	0	0	0
FRA 4M 6M	0	0	0	0
FRA 5M 6M	0	0	0	0
FRA 6M 6M	0	40	0	62
FRA 7M 6M	0	5	0	7
FRA 8M 6M	0	0	4,982	-1
FRA 9M 6M	0	0	0	0
IRS 2B 2Y	0	171	0	266
IRS 2B 3Y	0	259	0	402
IRS 2B 4Y	0	348	0	539
IRS 2B 5Y	0	47,491	0	1,088
IRS 2B 7Y	0	780	0	28,239
IRS 2B 10Y	85,669	2,232	0	54,470
IRS 2B 15Y	0	-114,496	0	-64
IRS 2B 20Y	0	132	0	0
IRS 2B 30Y	0	0	0	0
Total	85,670	-62,984	4,981	85,093

It is worth mentioning that while we emphasized the link between the original calibration problem and the trick of (3.21), we could just as well have applied the same methodology to produce delta vectors with respect to arbitrary instruments. As long as our potential hedge instruments span the yield curve, we should expect to be able to produce sensible risk figures. In particular, this means that while we may want to calibrate our yield curve with respect to a combination of different instruments (e.g. FRAs and interest rate swaps) we could choose to view our risk solely e.g. in spot starting swaps or in 3M forward buckets.

When trying to hedge swap risks, a trader will run a reports looking much like the one in table 8 for her entire portfolio and try to work out which trades to take on in order immunize the current risks. As an example, most of the risk in a 5Y10Y forward starting swap like the one above, could then be hedged with a position in a 5Y and a 10Y spot starting swap. Finally, it is important to note that this risk management technique of bump-and-revalue could be used for many other products than just FRAs and swaps. In fact, the exact same methodology could be applied to bonds or even interest rate sensitive pension liabilities.

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[22]It is worth noticing that the 9Y swap still carries risk towards the shorter dated instruments. Can you explain why?

4   Currency contracts

The previous sections have focused on interest rate derivatives where fixings and payments took place in the same, single currency. In addition to these, there exist a very large market for so called cross currency swaps as well as other currency derivatives. In the market terminology currency contracts are often referred to *Foreign Exchange-* or simply *FX derivatives*. The contracts covered in the below all belong to the class of linear FX derivatives. In addition to these, there exist a large and rather well developed market for more or less exotic *FX options*. Finally, there is the special class of derivatives called *quanto* products that pay off in a currency different from the one the underlying is denominated in (e.g. an interest rate swap that fixes against USD LIBOR but pays in EUR). In the below we will however only focus on linear FX products.

4.1   The spot exchange rate

Before we define the basic FX contracts, we need to define the concept of exchange rates. As the exchange rate between any two currencies can be defined in two ways, this is classic source of confusion. We define the *spot exchange rate* S_t to be the price in the domestic currency of buying one unit of foreign currency:[23]

$$S_t = \frac{\text{Units of domestic currency}}{\text{Units of foreign currency}}$$

As an example the official fixing from the Danish Centralbank on 12 February 2010 for EUR/DKK was 7.4445, implying that it will cost 7.4445 Danish Kroner to buy a single Euro. Note when quoting currency pairs — FX *crosses*— the order of the pair matters. The convention is to treat the the first currency (EUR in the above example) as the foreign and the latter (DKK above) as the domestic. Note finally, that it follows that DKK/EUR must then be approximately 0.1343.

4.2   FX Forwards

As many foreign exchange rates exhibit substantial volatility, and many market participants know their cash flow profiles in advance, a well developed market for trading currency forwards has existed for many years. An FX forward is a bilateral OTC contract to exchange cash flows in two different currencies at a future date at a pre-specified exchange rate. To describe this market, we will define $X(t, T)$ to be the time T forward exchange rate as seen from time t.

Note that, since the FX forward is simply two known, opposing cash flows in two different currencies at a future date, we can easily value the contract by simply discounting them back using their respective discount factors and collect the NPV in a single currency via the spot exchange rate. Suppose, we have agreed to buy $1/X(t, T)$ units of foreign currency at time T against selling 1 unit of domestic currency, our NPV in domestic

---

[23]The term spot exchange rate implies that the transaction of delivering and receiving the two cash flows happens with a *spot lag*. This is typically 2 good business days after the trade date in both currencies (i.e. both markets have to be open). We will however disregard this spot lag and use the spot exchange rate to compare $t = 0$ NPVs in different currencies.

currency at time t must be:

$$\text{FX Fwd NPV}_t = \underbrace{1 \cdot P^D(t,T)}_{\text{NPV in domestic currency}} - S_t \cdot \underbrace{\frac{1}{X(t,T)} \cdot P^F(t,T)}_{\text{NPV in foreign currency}} \qquad (4.1)$$

where $P^D(t,T)$ and $P^F(t,T)$ denote domestic, respectively, foreign zero coupon bond prices.

The standard in the FX forward market is — like in any other forward market — to initiate contracts with an NPV of zero. That is, we set $X(t,T)$ exactly such that the contract is fair. By setting the NPV to zero and rearranging (4.1), we easily obtain

$$\begin{aligned} X(t,T) &= S_t \frac{P^F(t,T)}{P^D(t,T)} \\ &= S_t e^{(r_D(t,T) - r_F(t,T))(T-t)} \end{aligned} \qquad (4.2)$$

where r_D and r_F denotes the continuously compounded risk-free zero coupon rates in the domestic, respectively, foreign currency. Equation (4.2) is continuous time version of the well known *covered interest rate parity* from international economics. The formula imposes a no arbitrage restriction between the interest rates earned on — risk free — local currency deposit accounts and the spot and forward exchange rates. To understand the parity result, let us consider the following example: Suppose we want to convert one unit of foreign currency at time t into domestic currency at time T. We now have two strategies:

- Exchange the unit of foreign currency to domestic, which yields $1 \cdot S_t$ units of domestic currency. Place these funds in a risk free deposit account (that yields the continuously compounded rate of r_D) until time T. In total this strategy provides $S_t e^{r_D(T-t)}$ units of domestic currency at time T.

- Place the unit of foreign currency in the risk free foreign deposit account (that yields r_F), which leaves us with $1 \cdot e^{r_F T}$ units of foreign currency at time T. Finally, we can exchange this amount on a forward basis providing a total of $X(t,T)e^{r_F(T-t)}$ units of domestic currency at time T.

Assuming that borrowing and lending rates are identical for the respective risk free accounts, these two strategies must be equivalent in order to avoid arbitrage. Were they not identical, we could borrow money using one strategy and place money using the other while ensuring a guaranteed profit. Note that although (4.2) is formulated to calculate the FX forward rates, we could also apply the result to find e.g. the foreign interest rate assuming that we know the domestic rate as well as spot and forward exchange rates.

The quoting convention for FX forwards is to communicate the difference between the forward and spot exchange rates — the *forward points* — measured in *pips* or 1/10.000's. If the EUR/USD spot rate is 1.3628 and the 1Y forward exchange rate is 1.3615, a FX forward trader would say that the 1Y forward is trading at -13 pips. An example of a series of FX forward quotes can be seen in figure 9.

4.3 FX swaps

An *FX swap* is a widely traded OTC combination of a spot exchange trade and the reverse FX foward. That is, a FX swap is simply the agreement to e.g. buy euros and

Figure 9: EUR/USD FX foward quotes from the broker Tullet, 26 February 2010.


```
Backpage                                              ComdtyTTSF
200<Go> to view in Launchpad
19:20 Tullett Prebon Information            PAGE  1 / 2
   EUR
   Spot           Bid      Ask     Time
 1)  EUR        1.3626   1.3629   2/26
   Forward Rates
 2)  O/N        -0.0200  -0.0100  2/26
 3)  T/N        -0.0140  -0.0100  2/26
 4)  S/N        -0.0180  -0.0150  2/26
 5)  1 Week     -0.1100  -0.0900  2/26
 6)  2 Week     -0.3700  -0.3200  2/26
 7)  3 Week     -0.4500  -0.4000  2/26
 8)  1 Month    -0.6800  -0.6300  2/26
 9)  2 Month    -1.2300  -1.0800  2/26
10)  3 Month    -1.8800  -1.7300  2/26
11)  4 Month    -2.8000  -2.5000  2/26
12)  5 Month    -4.6000  -4.2000  2/26
13)  6 Month    -6.7500  -6.2500  2/26
14)  7 Month    -8.9500  -8.2000  2/26
15)  8 Month   -10.1000  -9.1000  2/26
16)  9 Month   -11.8500 -10.3500  2/26
17) 10 Month   -13.0000 -11.5000  2/26
18) 11 Month   -13.8000 -12.3000  2/26
19)  1 Year    -14.0500 -12.0500  2/26
Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000
Japan 81 3 3201 8900    Singapore 65 6212 1000    U.S. 1 212 318 2000    Copyright 2010 Bloomberg Finance L.P.
                                                   SN 619685 H197-874-2 27-Feb-2010 19:20:56
```

sell dollars today against selling euros and buying dollars in the future. While the FX forward contract in itself carries outright exposure towards exchange rates, the FX swap has a more specialized risk profile since the simultaneous spot trade neutralizes (most of) the outright exposure. The FX swap exchanges cash flows on the spot date against exchange the reversed flow on the maturity date. The FX swap is also quoted using pips, with one leg having the maturity date cash flow adjusted by the number of forward points. An example of this could be the trade in table 10 (1Y EUR/USD swap traded at -13.05 pips on 26 February). FX swaps are used extensively to manage short term FX positions

Table 10: 1Y EUR 10m EUR/USD FX swap.

Date	Buy	Sell
26-Feb-10	EUR 10,000,000	USD 13,627,500
28-Feb-11	USD 13,614,450	EUR 10,000,000

in financial institutions and hedge FX exposure for asset managers and corporations.

4.4 Cross Currency Swaps

Both FX forwards and -swaps are traded liquidly as relatively short dated instruments (up to about 1-2Y). For longer dated FX related contracts, the market activity — and thus liquidity — is focused around the so called *Cross Currency Swap* or simply *CCS*. The CCS is an OTC agreement to exchange a series of floating rate payments in one currency against a series of floating rate payments in another currency. Unlike the IRSs, the market standard for CCSs is to have both initial and final exchange of notional. Furthermore, standard in the CCS market is to exchange the 3M xIBOR rates in the two currencies with a spread applied to one of the legs. For swaps quoted against USD, this *basis swap*

spread is applied to the non-USD leg. The majority of interbank CCSs are in fact quoted against USD, so in the below we should think of "domestic" as being USD. The position in a CCS is denoted relative to the spread (are we paying or receiving the spread).

Why is the market activity split between the FX Forward and CCS market across maturities? Intuitively, FX Forwards carry interest rate risk against the fixed cash flow in each currency. This interest rate risk is roughly proportional to the time-to-maturity. For longer dated FX Forward, this interest rate risk can become quite significant. The interest rate risk can in turn be offset by exchanging floating rate payments — exactly as is done in the CCS. Intuitively, we think of the CCS as being an FX Forward contract, where you simultaneously trade an IRS in each currency to remove the interest rate risk. It is because of the significant interest rate risk above the 1Y maturity, that the market activity is split as it is.

As with the other swaps considered in this course, the market standard is to trade the swap at NPV 0. The *par basis swap spread* is thus chosen to ensure that the CCS is at par. Finally, the notionals on the legs are set according to the spot exchange rate. As with other OTC products, price indications are available on request from market making banks or from brokers. An example of a broker screen with CCS quotes can be seen in figure 10.

Figure 10: CCS quotes against 3M USD LIBOR flat from the broker ICAP, 14 February 2010.

Denoting by C the basis swap spread and by N the EUR notional, table 11 shows the cash flows taking place in a 1Y EUR/USD CCS. We see that we can think of the CCS as being a EUR loan collateralized by a USD deposit. We receive the EUR notional at inception, pay EUR xIBOR rates plus the basis swap spread in interest and repay the EUR notional at maturity (our loan). Oppositely, we pay out a USD notional at inception against receiving USD xIBOR rates in interest as well as the notional at maturity (our deposit). Note, that once the initial exchange of notional has been made, we can think

of the remainder of the CCS as a long position in a USD FRN and a short position in a EUR FRN.

Table 11: 1Y EUR/USD Cross Currency Swap cash flows.

Date	Pay (EUR)	Receive (USD)
2B	$+N$	$-S_0 N$
3M	$-\delta(L^{\mathrm{EUR}}(T_{2B}, T_{3M}) + C)N$	$+\delta L^{\mathrm{USD}}(T_{2B}, T_{3M})S_0 N$
6M	$-\delta(L^{\mathrm{EUR}}(T_{3M}, T_{6M}) + C)N$	$+\delta L^{\mathrm{USD}}(T_{3M}, T_{6M})S_0 N$
9M	$-\delta(L^{\mathrm{EUR}}(T_{6M}, T_{9M}) + C)N$	$+\delta L^{\mathrm{USD}}(T_{6M}, T_{9M})S_0 N$
1Y	$-(\delta(L^{\mathrm{EUR}}(T_{9M}, T_{1Y}) + C)N + N)$	$+\delta L^{\mathrm{USD}}(T_{9M}, T_{1Y})S_0 N + S_0 N$

Having seen an example of the cash flows in a CCS, we are ready to focus on some general pricing formulae where we will reuse the machinery from the IRS valuation. First of all, let us consider the shortest possible plain vanilla CCS — the 3M contract that entails only a single exchange of xIBOR payments. Noting that we could have stated (4.2) in terms of simple (xIBOR) rates, we realize that knowing the domestic currency xIBOR fixing with a tenor of δ^D, $L^D(t, T_\delta)$, and the corresponding maturity FX forward rate, $X(t, T_\delta)$, allows us to calculate an implied foreign simple interest rate $\overline{L}^F(t, T_\delta)$ as:

$$X(t, T_\delta) = S_t \frac{P^F(t, T_\delta)}{P^D(t, T_\delta)} \Rightarrow$$
$$X(t, T_\delta) = S_t \frac{1 + \delta^D L^D(t, T_\delta)}{1 + \delta^F \overline{L}^F(t, T_\delta)} \Rightarrow \tag{4.3}$$
$$\overline{L}^F(t, T_\delta) = \frac{1}{\delta^F} \left[\frac{S_t}{X(t, T_\delta)} (1 + \delta^D L^D(t, T_\delta)) - 1 \right]$$

where superscripts D and F denote domestic, respectively, foreign coverages, interest rates and discount factors. For any tenor δ, the difference between the implied rate calculated above and the actual foreign xIBOR rate, $L^F(0, T_\delta)$, is called the *CCS (spot) break* or *-roll*:

$$\delta \text{ tenor CCS break}_t = \overline{L}^F(t, T_\delta) - L^F(t, T_\delta) \tag{4.4}$$

Supposing for example that the 3M EUR/USD FX forward on February 26th, 2010 was -2, the EUR/USD spot was 1.3630, the 3M EURIBOR fixing was 0.656% while the 3M USD LIBOR fixing was 0.25169%, we can calculate the 3M EUR CCS break as:

$$\frac{1}{92/360} \left[\frac{1.3630}{1.3628} (1 + 92/360 \cdot 0.25169\%) - 1 \right] - 0.656\% \approx -0.347\%$$

That is, the FX forward implied difference between the actual 3M EURIBOR fixing and FX forward implied EUR rate is -35 bps. Below, we will discuss in detail what exactly causes the break to deviate from zero.

CCS contracts can be valued in two ways: Either we use FX forwards to exchange all future foreign cash flows into the domestic currency and discount these back using the domestic discount factors, or we use foreign discount factors and exchange the PV to the domestic currency via the spot exchange rate. These two approaches must be equivalent according to (4.2).

Using the local discount factor pricing approach (and converting the currency values to a single currency using the spot exchange rate), we can write the value (in foreign

currency) of a CCS starting at time T_S and ending at time T_E on a foreign currency notional of N as:

$$\text{CCS NPV}_t = \underbrace{\underbrace{NP^F(t,T_S) - NP^F(t,T_E)}_{\text{Notional exchanges}} - \underbrace{\sum_{i=S+1}^{E} \delta_i^F (F^F(t,T_{i-1},T_i) + C) NP^F(t,T_i)}_{\text{xIBOR payments (incl. spread)}}}_{\text{Foreign Leg (in foreign currency)}}$$

$$- \frac{1}{S_0} \cdot \underbrace{\left[S_0 NP^D(t,T_S) - S_0 NP^D(t,T_E) - \sum_{i=S+1}^{E} \delta_i^D F^D(t,T_{i-1},T_i) S_0 NP^D(t,T_i) \right]}_{\text{Domestic Leg (in domestic currency)}}$$

(4.5)

where superscripts D and F denote domestic, respectively, foreign coverages, forward xIBOR rates and discount factors.[24] Note that the domestic leg notional is set according to the spot exchange rate — even if the CCS is forward starting. This has the implication that S_0 simply cancels out in (4.5). We can think of the notional on the domestic leg as being constantly rescaled with the spot exchange rate up until the trade is agreed and the exchange rate is fixed. Once the exchange rate has been fixed, the CCS becomes FX sensitive with the value of each leg. Note however, that it is typically only once the initial exchange has been made that the value of each leg becomes significant.

Focusing on the domestic leg in (4.5) and remembering equation (3.7), we recall that we can simply write the value of the floating interest rate cash flow stream as $S_0 N[P^D(t,T_S) - P^D(t,T_E)]$ provided that we discount these cash flows on the same xIBOR curve. Adding the value of the initial and final exchanges of notional, we see that the *domestic leg* must a total NPV of zero (in both foreign and domestic terms). Note the generality of this argument — any domestic leg regardless of maturity — must have a value of zero before the initial exchange and par on any future fixing date.[25] This means that when looking at CSSs where initial exchange is yet to be made, we can simply disregard the domestic leg since it has no value.

Corresponding to the par swap rate for IRSs, we can now define the *par basis spread* B as the spread that provides a NPV of zero (and explicitly leave out the domestic leg):

$$B = \frac{P^F(t,T_S) - P^F(t,T_E) - \sum_{i=S+1}^{E} \delta_i^F F^F(t,T_{i-1},T_i) P^F(t,T_i)}{\sum_{i=S+1}^{E} \delta P^F(t,T_i)}$$

(4.6)

Note that when trading CCSs we can think of par CCS spreads as being related to forward CCS breaks in the same way as par swap rates are related to forward xIBOR rates. That is, we can think of the par CCS spread as being a weighted average of forward CCS breaks.

Remembering the discussion in section 3.3 and equation (3.7) in particular, it can seem a little surprising that B can be anything but zero. In fact, were it true that all spot FRNs in all currencies trade at par, the basis spread must be zero. So why is this not the case? Why do most CCS spreads (as seen in figure 10) deviate substantially from zero?

First of all, xIBOR rates are a reflection of unsecured credit risk. But since we are implicitly collateralizing our EUR loan with a USD deposit in table 11, we must purge out

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[24]Note that the potential for differences in xIBOR day count conventions require us to allow for differences in coverages.

[25]The value of a FRN can deviate from par once the first floating rate has been fixed — then we can no longer guarantee that the accrued interest offsets the discounting.

any differences in credit premia embedded in EUR xIBOR against USD xIBOR rates. In fact, if it is the case that the banks in the USD BBA LIBOR panel are on average a better credit than the banks in the EURIBOR panel, then we must purge the relatively high EURIBOR fixings of the extra credit premium relative to USD BBA LIBOR. A negative CCS spread would do exactly this. Stated generally, differences in the credit quality of xIBOR panel banks can have an impact on the CCS market. Note that if B is negative (positive), then the discounting rates must be below (above) the foreign xIBOR rates. Intuitively, we can think of the CCS spread as being a spread between forward xIBOR and discounting rates.

The last part is key, as it implies that we cannot use the same zero coupon curve to project foreign xIBOR rates and discount the associated cash flows. For this reason, we must introduce the concept of two distinct curves: A *forward* curve and a *discounting* curve for all other currencies than our domestic. For the domestic currency we will *define* the forward and discount curve to be the same curve. Note that formulae like (2.4) and (2.5) still hold — we can still compute xIBOR rates from a zero coupon curve, but we have to accept that this pseudo zero coupon curve has no economic interpretation in itself - only the forward rates calculated on it have a financial meaning. When we stick to representing the forward curve via zero coupon rates it is strictly because of convenience. We thus have to be specific and distinguish between our zero curve used for forwarding and our zero curve used for discounting.

To help us compute par CCS spreads, we need the optional arguments in the `fidFloatingPv` function to allow for an additional set of curves so that the curve inputs are now given as `FwdCurveMat` (the maturities for the forward curve), `FwdCurveRates` (the forward curve zero rates), `DiscCurveMat` (the maturities for the discounting curve) and finally `DiscCurveRates` (the discounting curve zero rates).

Finally, the CCS spreads themselves can be calculated with the function `fidCcsSpread`. This function is an implementation of (4.6), which does not admit any arguments relating to the domestic currency leg as this — as mentioned — has zero value under the assumption that the domestic forward and discounting curve is identical.

4.5 Case study: The financial crisis, xIBOR rates and CCS spreads

In the years leading up to the financial crisis that began in August 2007, CCS markets were perhaps considered a bit dull by many market participants. The basis spreads in most currencies relative to USD were all close to zero for almost all maturities. The spreads were furthermore extremely stable. As can be seen in figure 11 this has changed substantially since then. What has caused this regime shift?

First of all, the financial crsis — especially around the time of the collapse of Lehman Brothers on Monday September 15, 2008 — brought massive market attention to the (short term) credit worthiness of banks. This caused a repricing of xIBOR rates which had two effects:

- The market began differentiating between xIBOR rates of different tenors. Since these rates reflect unsecured credit among prime banks, markets began to realize that 6M rates (which should correspond to a 6M loan) should probably be relatively higher than 1W rates because of the added credit risk caused by the longer term. This caused a massive repricing of IRS trades done against different xIBOR tenors. If we pay higher fixings on an IRS against 6M floating rates compared to 3M ditto,

Figure 11: 1Y par CCS spreads
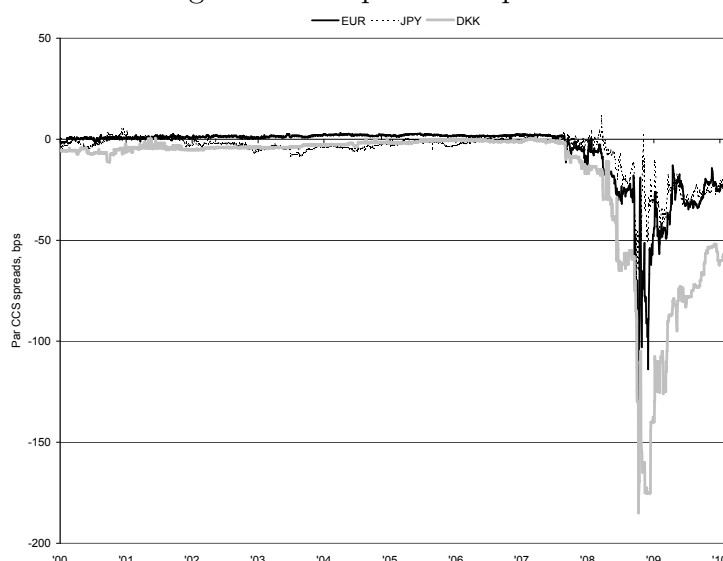
we should require a higher fixed rate on the former swap. This phenomenon is known as the *single currency basis*, *tenor basis* or simply *money market basis*. We will not look into single currency basis in this text, but it is worth mentioning that there are a host important of issues relating to this phenomenon which could be addressed in master's theses or student seminars. A nice survey of the modelling of money market basis can be found in Fujii, Shimada & Takahashi (2009).

- Just as importantly, markets began differentiating much more between xIBOR rates from different panels. Were the 3M EURIBOR rate set relatively high or low compared to the 3M USD BBA LIBOR rate? The market also began speculating if some panel banks were deliberately trying to push the xIBOR fixings up or down relative to what was considered fair. This obviously has implications for CCS spreads.

The primary driver for the massive widening of CCS spreads however came from another source. Throughout much of the financial crisis, there was a structural mismatch between supply and demand of USD cash among non-US banks. Many banks outside the US had bought USD denominated assets for borrowed funds that they began loosing money on. At the same time, markets for short term USD funding completely seized up.[26] Not only did banks have to cover their losses — they also found it difficult to refinance their short term debt, causing an increased demand for USD liquidity. However, these banks could not just simply buy USD spot for funds raised in their domestic currencies because this would give them FX exposure, they had to borrow USD on a currency hedged term basis (by swapping their domestic currencies into USD).

After some time, the Federal Reserve began flooding the US market with liquidity. This liquidity was however "stuck" inside the US financial institutions who was hoarding USD liquidity to finance their own positions. This increased demand and reduced supply for USD cash caused a massive dislocation of the CCS market. Since CCSs provide liquidity from the initial exchange, European banks were willing receive substantially less than 3M

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[26]One such key market was the asset backed commercial paper (ABCP) market, which cover short term (up to about 3M) secured loans.

EURIBOR on their EUR liquidity if they could obtain USD liquidity in exchange. This caused the 3M EUR/USD break to become significantly negative (exceeding -100 bps at some points). As the first break went very negative, longer maturity par basis spreads followed because they — as noted above — can be thought of as weighted averages of the forward breaks. This phenomenon can be seen in figure 12, as the most violent changes has been in short end of the curve. In order to address this shortage of USD liquidity among non-US financial institutions, the Federal Reserve initiated a series of very substantial FX swap facilities with a number of other central banks such as ECB, Bank of England and Danmarks Nationalbank. Via these FX swaps, financial institutions outside the United States could obtain USD funding from their local Central Bank, see Goldberg, Kennedy & Miu (2011) for more details on disruptions in USD funding market and the swap lines.

The dates shown in figure 12 are chosen to represent different regimes: Before the financial crisis (02/01/2007), the financial crisis just before and after the collapse of Lehman Brothers (01/09/08 and 01/10/08) and finally recent conditions (04/01/10)

Figure 12: Term structures of par CCS spreads, source ICAP
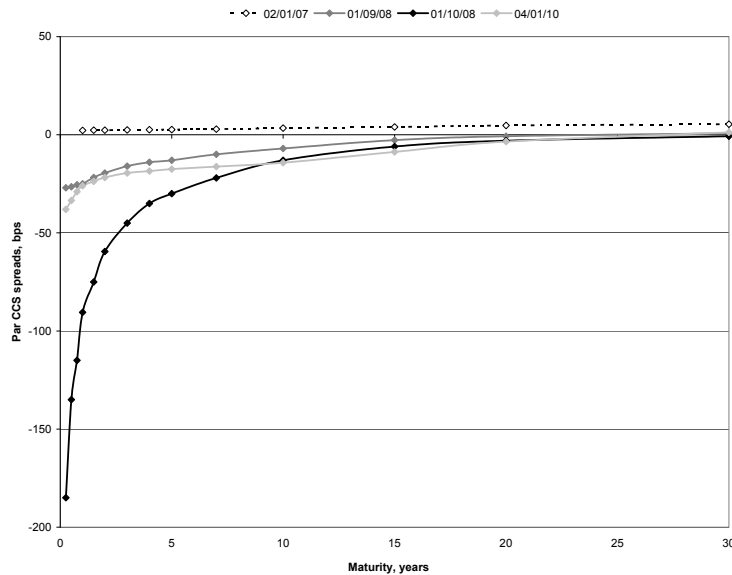
4.6   Swap market calibration revisited

The fact that discounting rates for foreign currency cash flows depends on CCS spreads has implications beyond the CCS markets themselves. In fact, in order to ensure an arbitrage free pricing setup we want to make sure that we treat the floating rate legs in IRSs and FRAs identical to the floating rate legs in CCSs. This means that we must extend the dual curve setup with separate forward and discounting curves to cover IRSs and FRAs as well. Furthermore, if CCSs depend on discounting rates and we know from (4.2) that FX forwards also does, we must ensure that these two products are priced consistently. In practice these considerations are highly relevant as desks are often allowed to trade all four products (FRA, IRS, CCS and FX forwards).[27]

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[27]In the industry jargon, a desk is a trading unit with its own (daily) profit and loss statement. Typically, the trading operations in banks are divided into different desks according to the risks they take and the primary products they are responsible for quoting prices in.

It has been known for years, that using the simple formula (3.7) did not hold for foreign currency floating legs, see e.g. Fruchard, Zammouri & Willems (1995). Nevertheless, many banks did not recognize this before the break out of the financial crisis as the level (close to zero) and low volatility made the effect negligible. Today all advanced trading operations recognize the relationship between IRSs, FRAs, CCSs and FX forwards — but the financial crisis made them to revisit the way they calibrate swap curves.

We recall from section 3.3 that we can think of par swap rates as being just a weighted average of forward xIBOR rates. Also, we can recall that we can think of CCS spreads as being the difference between xIBOR rates and discounting rates. Intuitively, we can thus calibrate a swap market using both inputs as they determine their own curve. IRSs tie down the forward curve (and the absolute level of interest rates) while CCSs tie down the discounting curve (and the relative level of discounting rates). Between the CCSs and FX forwards, the latter is typically used for the short end of the curve while the former is used for maturities above approximately 1Y. For the simplified set of market quotes shown in table 12, the EUR forward and discounting curve can be calibrated under the assumption that USD leg in the EUR/USD CCS has zero value. The forward curves resulting from the calibration are shown in figure 13.[28]

Table 12: Reduced EUR swap curve input quotes

IRS (vs 3M EURIBOR)	Quote	EUR/USD CCS	Quote
1Y	0.967%	1Y	-0.310%
2Y	1.652%	2Y	-0.283%
3Y	2.019%	3Y	-0.245%
4Y	2.319%	4Y	-0.223%
5Y	2.577%	5Y	-0.205%
7Y	2.995%	7Y	-0.188%
10Y	3.395%	10Y	-0.125%
15Y	3.753%	15Y	-0.065%
20Y	3.873%	20Y	-0.015%
30Y	3.975%	30Y	0.035%

A note of caution on the curve construction is warranted as hinted by the curves in figure 13. As CCS quotes depend on the spread between the forward and discounting curves, even minor differences in the interpolation between knot points can cause very unsmooth curves. Even if we use the same interpolation method for both curves (as is consequently done in the fidAnalytics library) differences in the maturity grid of knot points can cause large interpolation differences in the spread. In practice, these problems can be addressed by e.g. letting the discounting curve "inherit" the shape of the forward curve by interpolating directly in a spread rather than in the absolute rates. For the reminder of this text, we will keep the dual curve setup. Any references to swap curves thus entail both forward- and discounting curves.

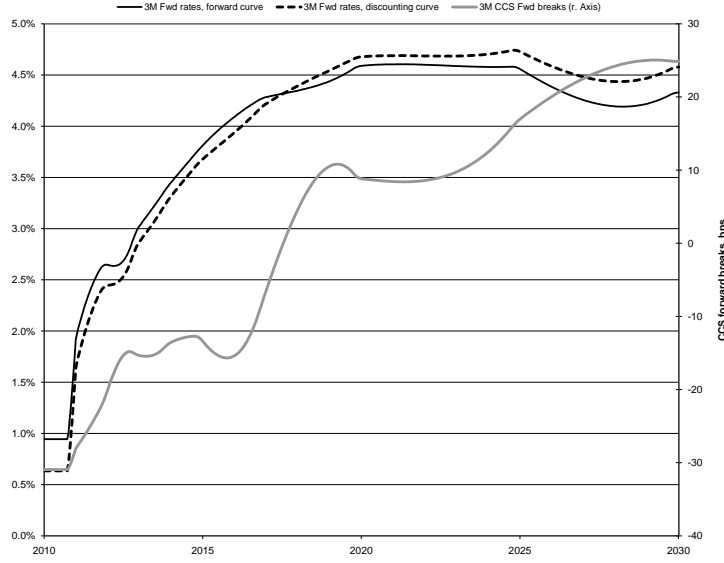## 4.7   IR risks in the dual curve setup

The introduction of a dual curve setup for pricing CCSs and IRSs naturally impacts the way we measure interest rate risks. The bump and re-run approach introduced in

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[28]Note here that while we can still use formulae like (2.5) to calculate forward rates on the discounting curve, these rates are not actual xIBOR fixings.

Figure 13: Calibrated dual curves shown as 3M forward rates.

section 3.7 is still the same — we just have to bump more inputs to correctly measure interest rate risk in the dual curve setup. We can thus easily calculate both forward- and discounting curve Dv01s as well as delta vectors. Formally, we are simply augmenting our parameter vector such that we we now include the discounting curve rates in addition to the forward curve rates i.e. $\mathbf{P} = \{p_1^{\text{Fwd}}, p_2^{\text{Fwd}}, \ldots, p_{M-1}^{\text{Disc}}, p_M^{\text{Disc}}\}$. In table 13 the zero (discounting and forward) delta vectors for three different trades are calculated on the curves shown in figure 13.

Firstly we note that both IRSs have most of their exposure linked to the forward curve. This is natural as a 1 bp move in the forward curve will imply a higher cash flow at all future fixings — a first order effect. Importantly, this effect will be identical for IRSs of a given length regardless of their fixed rate why the forward curve risk is identical between the two 10Y IRSs. Recalling table 7, we see that we approximately recover the result from that table as the net delta vector. When splitting risk between the forward and discounting curve we are simply allocating the same underlying risk to two different curves.[29]

Secondly, we note that while the at market trade has very little exposure towards the discounting curve, the off-market trade with a coupon of 5% has much more discounting risk. This comes from the fact that when paying fixed at 5% (vs. a market rate of 3.395%) we have a substantial negative mark-to-market on the swap. To reduce the PV of this negative cash flow stream we want to see discounting rates increase. For the par IRS this effect is typically small since we are — per construction of the par swap rate — on average paying and receiving an equal amount.[30] This dependency between the fixed rate and the discounting risk profile is what causes convexity in IRSs. Interestingly, this also tells us that the choice of discounting curve among different market participants can be difficult to deduce if we can only observe par rates. Even if two counter parts agree on

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[29]The risk figures cannot be completely reconciled between the two tables as the underlying curve calibration is slightly different.

[30]Note however, that on a steep forward curve, the par swap actually cover significant net cash flows that are discounting sensitive.

which swaps are priced to par, they can disagree massively on the valuation of off-market swaps because of the larger (in absolute terms) discounting Dv01 of these trades. As the financial crisis caused many market participants to modify their valuation models, this has become a major problem in swap terminations and novations as well as in the daily margining of collateralized trades.[31]

Finally, we see that the at market 10Y CCS has very little net interest rate risk (which was exactly the motivation for trading this product rather than the FX Forward). Rather this instrument is exposed to movements of the spread between the forward and discounting curve. This observation confirms the intuition from section 4.6 that the CCSs tie down discounting curve relative to an outright level of (forward) interest rates given by the IRSs.

Table 13: Dual curve zero rate delta vectors, EUR 100m notional paying fixed rate (respectively paying the CCS spread)

Maturity	10Y IRS at market				10Y IRS 5%				10Y CCS at market		
	Fwd	Disc	Net		Fwd	Disc	Net		Fwd	Disc	Net
1Y	-14	212	198		-14	370	212		14	2	16
2Y	493	128	621		493	442	935		-493	508	15
3Y	776	148	924		776	604	1,380		-777	781	4
4Y	863	61	924		863	513	1,376		-863	860	-3
5Y	2,476	-291	2,185		2,476	798	3,274		-2,476	2,509	33
7Y	7,038	-1,585	5,453		7,038	1,039	8,077		-6,872	6,680	-193
10Y	76,870	-268	76,602		76,870	1,924	78,794		-77,054	76,787	-267
15Y	-138	54	-84		-138	2	-137		78	-70	8
20Y	0	0	0		0	0	0		0	0	0
30Y	0	0	0		0	0	0		0	0	0
Total	88,365	-1,542	86,823		88,365	5,691	94,055		-88,444	88,056	-387

Having computed delta vectors with respect to our model parameters (forward and discounting zero coupon rates), we are faced with the same problem as in section 3.7 — we would rather want our exposure to be expressed in terms of market rates. We can however easily apply the jacobian methodology by augmenting our model quote vector $\mathbf{B}(\mathbf{P}) = \{b_1^{\text{IRS}}, \ldots, b_{N-1}^{\text{CCS}}, b_N^{\text{CCS}}\}^\top$. Formally, we will now have an enlarged — relative to the single curve case — jacobian matrix given by

$$\frac{\partial \mathbf{B}(\mathbf{P})}{\partial \mathbf{P}}_{N \times M} = \begin{pmatrix} \frac{\partial b_1^{\text{IRS}}}{\partial p_1^{\text{Fwd}}} & \cdots & \frac{\partial b_1^{\text{IRS}}}{\partial p_M^{\text{Disc}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial b_N^{\text{CCS}}}{\partial p_1^{\text{Fwd}}} & \cdots & \frac{\partial b_N^{\text{CCS}}}{\partial p_M^{\text{Disc}}} \end{pmatrix} \tag{4.7}$$

Again, we calculate the jacobian by bump and re-run for the three instruments shown above and do appropriate matrix manipulations as given by (3.22). The result can be seen in table 14. Again, we recognize the intuitive result (that we saw in table 8) that a 10Y IRS at market is only exposed towards its own rate. We also see however, that the 10Y

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[31]Counter parts often agree to terminate a given trade against a cash payment when they want to take profit or loss on a given position. Another often used possibility is simply to transfer — to *novate* — the obligations of a given transaction to a third party. This is also done against a cash exchange. The collateralization of derivative trades is covered in section 7.1.

Table 14: Dual curve market rate delta vectors, EUR 100m notional paying the fixed rate (paying the CCS spread).

Instrument	10Y IRS at market	10Y IRS 5%	10Y CCS at market
IRS 1Y	0	141	0
IRS 2Y	0	252	0
IRS 3Y	0	379	0
IRS 4Y	0	384	0
IRS 5Y	0	962	0
IRS 7Y	0	2,641	1
IRS 10Y	86,484	88,976	11
IRS 15Y	0	-62	0
IRS 20Y	0	0	0
IRS 30Y	0	0	0
CCS 1Y	0	145	0
CCS 2Y	0	256	0
CCS 3Y	0	388	0
CCS 4Y	0	396	0
CCS 5Y	0	982	0
CCS 7Y	0	2,709	1
CCS 10Y	0	2,552	-88,813
CCS 15Y	0	-66	0
CCS 20Y	0	0	0
CCS 30Y	0	0	0

5% IRS is exposed to shorter dates IRSs as well as CSSs. The intuition behind is that in our curve setup where 3M USD LIBOR is the reference we are implicitly working out present values relative to this benchmark. This means that any profit or loss should be FX term hedged into the relevant benchmark currency. In other words, in order to hedge the off-market IRS we would need to trade some EUR/USD cross currency basis as well as some EUR IRSs. By doing this we effectively create a fixed-for-float cross currency swap, where the interest rate risk on the negative future cash flows is removed and the cash flows are forward exchanged into USD. Finally, table 14 also yields the intuitive result that a 10Y at market CCS is only exposed towards itself.

5 Interest rate options

5.1 The Black formula

Before we introduce the specific interest rate options that we will work with, a brief recap of some basic option pricing theory is in place. Rather than formally introducing the deeper mathematics, the arguments here will be heuristic of nature. The section borrows heavily from chapters in Hull (2006), Björk (2004) as well as the reprise section of Hagan, Kumar, Lesniewski & Woodward (2002).

First of all, a *European call option* is the right — but not the obligation to buy an asset V_T, at a future date T at a pre-specified price of K called the *strike* price.[32] The payoff of such a contract can thus be written as $(V_T - K)^+$ which is short-hand for $\max(V_T - K, 0)$. Likewise, a *European put option* is the right but not the obligation to sell the asset. The payoff of the put can be written as $(K - V_T)^+$.

Although V_T and hence also the option pay-offs at time T are stochastic, it turns out that we can price the option contracts simply by computing the expected value of their payoffs. Before we proceed we therefore need a few insights into why this is case and learn a few things about stochastic processes and probability measures.

Loosely stated, a *probability measure* is a map that tells how likely different values of a random variable are. As a example, think of a dice with its six sides. A probability measure tells us how likely it is for each of the sides to come up. We know that for a fair dice each side has the same *physical* probability of coming up and that the combined probability of any of the sides coming up must sum to 1. Knowing for instance that a dice is fair allows us to calculate the expected value of a roll of the dice — the expected value will obviously depend on the probability measure.[33]

In mathematical finance, we are often not interested physical probability measures but rather in theoretical constructions called *risk neutral probability measures*. Think of these alternative measures as having some reallocation of probability between different outcomes relative to the physical measure. Changing the probability measure can alter the expected value of a random variable - think of what would happen to the expected value of the dice roll if we used a probability measure under which the side 6 would come up with certainty. This is a clever alternative to specifically modelling risk premia, since we can intuitively allocate more probability to the states of the world to which market participants are risk averse.

Next, we need to introduce the concept of a *martingale* process. A stochastic process f_t is called a martingale under the probability measure Q if $E_t^Q[f_T] = f_t$ for $t \leq T$.[34] In words this means that the expected future value of the process is today's value — the process is thus driftless. We note that if f_T/g_T is a martingale, it must hold that

$$\frac{f_t}{g_t} = E_t\left[\frac{f_T}{g_T}\right] \qquad \Leftrightarrow \qquad f_t = g_t E_t\left[\frac{f_T}{g_T}\right] \tag{5.1}$$

This will turn out to be quite useful in the following, as we will see that using some specific choice of g as our numeraire will simplify our pricing formulas. Mathematical

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[32]The term *European* option is market terminology for indicating that the option can only be exercised on a single date. Options that can be exercised on any date before expiration are called *American*, while options that only can be exercised on specific dates are called *Bermudan*.

[33]As an example the expected value of a dice roll under the physical measure is $1/6 \cdot 1 + 1/6 \cdot 2 + 1/6 \cdot 3 + 1/6 \cdot 4 + 1/6 \cdot 5 + 1/6 \cdot 6 = 3.5$.

[34]The t subscript on the expectations operator means that the expectation is calculated at time t, while the Q superscript refers to the measure under which the expectation is calculated.

finance is closely linked to this martingale concept since the *first fundamental theorem of asset pricing* states that a market is arbitrage free if and only if there exists a martingale measure.[35] In words, this means that absence of arbitrage means that we can price derivative contracts simply by calculating expected values.

Another important result is the so-called *martingale representation theorem*. This results shows that any stochastic variable f_t that is a martingale can be written as

$$df_t = c(\cdot)dW_t \tag{5.2}$$

where $c(\cdot)$ is a — possibly stochastic — function and W_t is a brownian motion.[36] This is as far as the theory of arbitrage free pricing brings us. In order to operationalize the theory we need to postulate some model for $c(\cdot)$.

One particularly simple and popular model for $c(t, \cdot)$ is the so-called Black (1976) model:

$$df_t = \sigma f_t dW_t \tag{5.3}$$

which thus postulate $c(t, f_t) = \sigma f_t$. This model implies that $\log(f_T)$ is normally distributed with a standard deviation of $\sigma\sqrt{T}$. A final, central result in mathematical finance is the so-called Black-Scholes-Merton (BSM) formula. In the general case presented below, it is simply the expected value of the call option pay-off assuming that V_T is lognormally distributed and that $\log(V_T)$ has a standard deviation of ω:

$$E_t[(V_T - K)^+] = E_t[V_T]\Phi(d_1) - K\Phi(d_2) \qquad \text{where}$$
$$d_1 = \frac{\log(E_t[V_T]/K) + \frac{1}{2}\omega^2}{\omega}$$
$$d_2 = \frac{\log(E_t[V_T]/K) - \frac{1}{2}\omega^2}{\omega} = d_1 - \omega \tag{5.4}$$

where Φ denotes the cumulative standard normal distribution. The proof of BSM formula can be found in e.g. Hull (2006). We note that can easily apply the BSM formula to the case of the Black (1976) model.

5.2 Caps and floors

Having been introduced to some basic concepts from option pricing theory, we are now ready to focus on specific interest rate option contracts. A *caplet* is a call option on a forward xIBOR rate. The option has European style exercise meaning that the caplet written on $L(T, T + \delta)$ is exercised at time T. While the option pay-off is fixed at the beginning of the period (fix-in-advance), it is not paid until the end of the period (pay-in-arrears). Remembering that $L(T, T + \delta) = F(T, T, T + \delta)$ the caplet payoff can thus be valued at time T as:

$$\text{Caplet PV}_T = P(T, T + \delta)\delta(F(T, T, T + \delta) - K)^+ \tag{5.5}$$

Notice that at first glance this contract seems difficult to price since the pay-off depends not only on the stochastic $F(T, T, T + \delta)$ but also on $P(T, T + \delta)$ which is also stochastic seen from time t. Rather than trying to work out expectations over the simultaneous

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[35]Moreover, the *second fundamental theorem of asset pricing* states that if and only if the market is complete this martingale measure is unique.

[36]The function $c(\cdot)$ can be dependent on time, f_t itself or other stochastic variables.

distribution of the two variables, we will employ a little trick from section 5.1. By using the zero coupon bond maturing at time $T + \delta$ as numeraire, we can price the caplet as

$$
\begin{aligned}
\text{PV Caplet}_t &= P(t, T + \delta)\delta E_t^{Q_{T+\delta}}\left[\frac{P(T, T + \delta)(F(T, T, T + \delta) - K)^+}{P(T, T + \delta)}\right] \\
&= P(t, T + \delta)\delta E_t^{Q_{T+\delta}}[(F(T, T, T + \delta) - K)^+]
\end{aligned}
\tag{5.6}
$$

where $Q_{T+\delta}$ denotes the probability measure that is forward risk neutral with respect to $P(t, T+\delta)$. Using the insights gained in section 5.1, we can — assuming that $F(T, T, T+\delta)$ is log-normally distributed — formulate the Black'76 formula for caplets as

$$
\begin{aligned}
\text{PV Caplet}_t &= P(t, T + \delta)\delta[F(t, T, T + \delta)\Phi(d_1) - K\Phi(d_2)] \qquad \text{where} \\
d_1 &= \frac{\log(F(t, T, T + \delta)/K) + \frac{1}{2}\sigma^2(T - t)}{\sigma\sqrt{T - t}} \\
d_2 &= \frac{\log(F(t, T, T + \delta)/K) - \frac{1}{2}\sigma^2(T - t)}{\sigma\sqrt{T - t}} = d_1 - \sigma\sqrt{T - t}
\end{aligned}
\tag{5.7}
$$

where we have used the fact that $E_t^{Q_{T+\delta}}[F(T, T, T + \delta)] = F(t, T, T + \delta)$.

Pricing the floorlet can now be done by invoking the put-call parity:

$$
\text{Forward}(K) = \text{Call}(K) - \text{Put}(K)
\tag{5.8}
$$

This is a model free result that links the forward, call- and put option struck at K together. Noting that the present value of the forward contract is $P(t, T + \delta)\delta(F(t, T, T + \delta) - K)$ we easily obtain

$$
\text{PV Floorlet}_t = P(t, T + \delta)\delta[K\Phi(-d_2) - F(t, T, T + \delta)\Phi(-d_1)]
\tag{5.9}
$$

which is the Black (1976) formula for floorlets.[37]

In order to price cap- or floorlets we simply need to calculate the relevant forward xIBOR rate and discount factor and plug that into (5.7) or (5.9) together with the strike K, time-to-fixing T and the volatility σ. Assuming that we have a calibrated swap curve at our disposal all these inputs are readily observable except for σ. Since there the option PVs are increasing in σ there is a unique value $\overline{\sigma}$ that matches any observable market price. That is, we can imply $\overline{\sigma}$ from market prices. We therefore define the *implied volatility* to be the annualized volatility that equates the Black (1976) formulas with market prices. Unfortunately, it is not possible invert (5.7) or (5.9) algebraically to find $\overline{\sigma}$. Instead, we need some numerical method to find the $\overline{\sigma}$ that solves Black Formula$(\overline{\sigma}, \cdot)$ − Market Price = 0. We will return to this in section 5.4.

Note that since $\sigma\sqrt{T - t}$ enters the pricing formulas as the generalized standard deviation ω from (5.4), we need to specify σ and T relative to some common day count convention. Throughout the text we will use Act/365 when calculating our (annualized) volatilities.

In the market terminology, a *cap* is a portfolio of caplets while a *floor* is a portfolio of floorlets. Valuing these is easy — we simply value each individual cap- or floorlet and

---

[37]We note that the value of the forward contract here is not necessarily equal to the value of the corresponding FRA contract. The potential pricing difference arises from the discounting. The FRA contract contract pays out in-advance based on discounting using the xIBOR rate itself whereas the forward contract valued here is discounting on the discounting curve.

sum up their PVs. For the cap and floor starting at time T_S and maturing at time T_E we can thus write:

$$\text{PV Cap}_t = \sum_{i=S+1}^{E} P(t,T_i)\delta_i[F(t,T_{i-1},T_i)\Phi(d_1) - K\Phi(d_2)]$$

$$\text{PV Floor}_t = \sum_{i=S+1}^{E} P(t,T_i)\delta_i[K\Phi(-d_2) - F(t,T_{i-1},T_i)\Phi(-d_1))] \qquad \text{where}$$

$$d_1 = \frac{\log(F(t,T_{i-1},T_i)/K) + \frac{1}{2}\sigma_{i-1}^2(T_{i-1}-t)}{\sigma_{i-1}\sqrt{T_{i-1}-t}}$$

$$d_2 = \frac{\log(F(t,T_{i-1},T_i)/K) - \frac{1}{2}\sigma_{i-1}^2(T_{i-1}-t)}{\sigma_{i-1}\sqrt{T_{i-1}-t}} = d_1 - \sigma_i\sqrt{T_{i-1}-t}$$

(5.10)

Note that in this formulation, the last fixing is observed at time $T_E - \delta$ while the last payment is made at T_E. Note furthermore, that the volatilities σ_i have been given an i subscript. This has been done, as the market typically price cap- and floorlets with different time-to-expiration at different levels of implied volatility. This has an implication when quoting implied volatilities for caps and floors. Either we can communicate the *spot* implied volatility or the *flat* implied volatility. The former method uses individual volatilities for each cap- or floorlet while the latter uses just a single implied volatility for all the cap- or floorlets in the cap or floor in question. Intuitively, we can thus think of the flat volatility as being some weighted average volatility.

Caps and floors are actively traded OTC instruments in all major currencies. Like the other OTC products we have seen, price indications can be obtained from broker screens as seen in table 15. Although caps and floors can be traded on any δ-tenor xIBOR rate, the market activity is concentrated on the main xIBOR index from the IRS market. The conventions for the caps and floors are taken from the relevant xIBOR rate. For EUR this means that caps and floors are quoted against the 6M EURIBOR® using the Act/360 day count convention and that days are adjusted to the modified following convention.[38]

Table 15: Mid-market EUR Cap floor quotes 12 March 2010, premia shown as basis points of notional.

			2.00%		2.25%		2.50%		3.00%		3.50%		4.00%		4.50%		5.00%	
Mat.	K_{ATM}	ATM	C	F	C	F	C	F	C	F	C	F	C	F	C	F	C	F
1Y	0.99%	11	1		1		1											
18M	1.23%	31	10		7		5		3		1		1		1			
2Y	1.44%	59	31		24		18		11		7		4		3		2	
3Y	2.08%	113		102			78		49		31		20		13		9	
4Y	2.37%	188		122		165			114		75		50		34		24	
5Y	2.62%	267		137		185		239	203		139		94		65		46	
6Y	2.84%	349		150		201		259			221		152		105		74	
7Y	3.03%	430		161		214		275		420	314		220		153		108	
8Y	3.19%	508		172		227		291		442	415		294		206		146	
9Y	3.32%	582		183		241		306		464	519		372		262		186	
10Y	3.43%	654		194		254		322		485	624		452		321		228	
12Y	3.61%	794		218		281		354		527		741	624		449		323	
15Y	3.79%	980		256		325		403		591		823	863		632		459	
20Y	3.90%	1234		328		407		497		712		979	1162		864		640	
25Y	3.86%	1455		424		517		622		873		1186	1346		1013		764	
30Y	3.78%	1663		538		646		769		1061		1425	1478		1126		864	

Although the contracts listed in table 15 are spot starting, their schedules are different from the IRS market. In particular, it is customary to disregard the first cap- of floorlet

---

[38]Again, it is important to stress that when working with interest rate options, we should — as always — be careful to use the correct conventions and distinguish between their use. When pricing caps and floors we will thus typically use the Act/365 convention when specifying the time-to-expiration, while we will use the Act/360 convention to compute coverages and forward xIBOR rates.

since its underlying is already known at the trade date. The IRS that is comparable to the 10Y EUR cap is thus a 6M forward starting swap that matures 9.5 years later. In fact it is the fixed par rate of such a swap (albeit quoted with a semi-annual Act/360 coupon) that sets the ATM rate in table 15. The *At-The-Money* (ATM) strike is the rate that is equal to the underlying forward rate. If an interest rate option has a strike equal to the forward rate, the option is said to be struck At-the-Money. If the forward is is below the strike for a call (put) option it is said to be Out-of-the-Money (In-the-Money) and vice versa. It is customary to abbreviate these concepts as OTM and ITM.

For individual cap- or floorlets the concept of ATM is clear but for portfolios of these, we need the IRS analogy to compute a single rate. Note, that put-call parity tells us that using the ATM level as strike, must entail that the cap and floor have identical values (as the forward swap has zero value).

We will not develop a `fidCapletPV` function as it turns out that we can treat cap- and floorlets as special cases of the swaption instrument that are covered in the following section.

5.3 Swaptions

The right but not the obligation to enter into an IRS on a future date with a prespecified fixed rate is called a European swap option or simply *swaption*. The option at time T_S to enter into a swap maturing at time T_E paying (receiving) a fixed rate of K is called a T_S into $(T_E - T_S)$ payer (receiver) swaption. Swaptions can be settled using either *physical* or *cash* settlement. When there is physical settlement the underlying swap is initiated which means that — recalling (3.12) — at T_S the holder of the payer swaption receives

$$
\begin{aligned}
\text{Payer swaption PV}_{T_S}^{\text{Physical}} &= [A(T_S, T_S, T_E)(R(T_S, T_S, T_E) - K)]^+ \\
&= A(T_S, T_S, T_E)(R(T_S, T_S, T_E) - K)^+ \qquad \text{where} \\
A(T_S, T_S, T_E) &= \sum_{i=S+1}^{E} \delta_i P(T_S, T_i)
\end{aligned} \tag{5.11}
$$

Note that we can actually think of a caplet on a δ-tenor forward xIBOR rate as being the special case of a single period payer swaption with physical settlement.

In the case of cash settlement, the par swap rate itself is used to discount back the difference between the par swap- and the strike rate. This can be done in several ways, but one simple way is

$$
\text{Payer swaption PV}_{T_S}^{\text{Cash}} = \tilde{A}(T_S, T_S, T_E)(R(T_S, T_S, T_E) - K)^+
$$
$$
\tilde{A}(T_S, T_S, T_E) = \sum_{i=S+1}^{E} \frac{\delta_i}{(1 + \delta_i \cdot R(T_S, T_S, T_E))^{T_i - T_S}} \tag{5.12}
$$

The price difference between cash- and physically settled swaptions will depend on the curve shape and — given our split between forward and discounting curves — also depend on CCS spreads. Because of this, a cash settled swaption cannot be used to perfectly hedge a physically settled ditto and vice versa. An example of swaption payoffs can be seen in figure 14. It is important to note that swaptions payoff as a function of the swap rate is non-linear. This non-linearity is caused by the convexity of the underlying swap and is thus dependent on the length of the swap. Also, note the difference in value between the cash and physical settlement. This difference is caused by having a discounting that is

100 bps below the forward curve — discounting using the swap rate will thus intuitively correspond to discounting 100 bps above the discounting curve, causing the value of the cash settled swaption to be lower.

Figure 14: Payoff for strike 5% swaptions written on the 30Y swap rate using either cash- or physical settlement. Calculated using a 100 bp spread between forward and discounting curves.
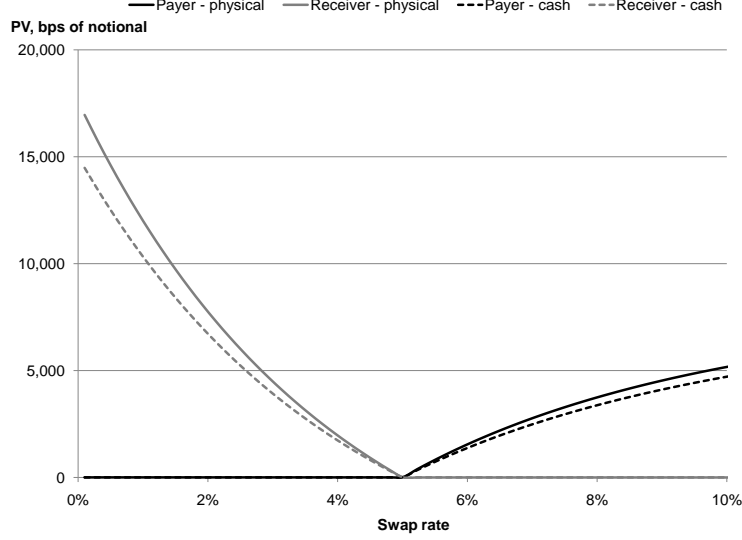
Valuing swaptions again allows us to employ the numeraire trick from section 5.1, this time by using $A(T_S, T_S, T_E)$ (or its cash settling equivalent) as the numeraire.[39]

$$\text{Payer swaption PV}_t = A(t, T_S, T_E) E_t^A \left[\frac{A(T_S, T_S, T_E)(R(T_S, T_S, T_E) - K)^+}{A(T_S, T_S, T_E)} \right]$$
$$= A(t, T_S, T_E) E_t^A [(R(T_S, T_S, T_E) - K)^+] \qquad (5.13)$$

Doing so lead us to the Black (1976) formula for payer swaptions:

$$\text{Payer Swaption PV}_t = A(t, T_S, T_E)[R(t, T_S, T_E)\Phi(d_1) - K\Phi(d_2)] \qquad \text{where}$$
$$d_1 = \frac{\log(R(t, T_S, T_E)/K) + \frac{1}{2}\sigma^2(T_s - t)}{\sigma\sqrt{T_s - t}} \qquad (5.14)$$
$$d_2 = \frac{\log(R(t, T_S, T_E)/K) - \frac{1}{2}\sigma^2(T_s - t)}{\sigma\sqrt{T_s - t}} = d_1 - \sigma\sqrt{T_s - t}$$

To value the payer swaption, we simply need to work out the forward swap rate as well as the annuity factor (physical or cash) as seen from time t and plug these into (5.14) together with the time-to-expiration and the volatility σ.

Valuing the receiver swaption can easily be done by applying the put-call parity for swaptions:

$$\text{Forward Starting Payer Swap}(K) = \text{Payer Swaption}(K) - \text{Receiver Swaption}(K) \qquad (5.15)$$

---

[39]In the case of cash settlement, this is actually a bit harder than presented. First of all, (5.12) only tells us how to discount back to time T_S. We need to discount all the way back to time t, why we write $\tilde{A}(t, T_S, T_E) = P(t, T_S) \sum_{i=S+1}^{E} \frac{\delta_i}{(1+\delta_i \cdot R(t, T_S, T_E))^{T_i - T_S}}$. As noted in Mercurio (2008), doing so does actually not guarantee arbitrage free pricing — a phenomenon that we will disregard in this course.

49

Plugging in (5.14) and rearranging gives us the Black (1976) formula for receiver swaptions:

$$\text{PV Receiver}_t = A(t, T_S, T_E)[K\Phi(-d_2) - R(t, T_S, T_E)\Phi(-d_1)] \qquad (5.16)$$

As with caps and floors, swaptions are actively traded OTC products and are quoted in all major currencies. Swaptions are typically quoted either as *straddle* options premia or as Black (1976) implied volatilities as seen in figure 15. A straddle is the combination of a bought call option (i.e. a payer swaption) and a bought put option (i.e. a receiver swaption). In section 5.5 we will see why it makes sense to trade these combinations.

Figure 15: Implied volatilites for EUR Swaptions, February 1st 2011. Source: ICAP

5.4   IR options in fidAnalytics

As argued in the previous section, we can think of cap- and floorlets as being swaptions on very short dated swaps. For this reason, we will only develop a set of pricing functions in `fidAnalytics` that cover swaptions. Basically, we just need two functions — `fidSwaptionPv` and `fidImpSwaptionVol` — to complete our pricing setup.

As the forward (swap) rate enter the Black (1976) formulas, the `fidSwaptionPv` function will obviously require at least the same input as `fidSwapRate`. In addition, the function requires:

- `TypeFlag`: `Payer` or `Receiver` (use `Payer` for caplets and `Receiver` for floorlets).

- `K`: The strike rate of the option.

- `Settlement`: `Cash` or `Physical` (use `Physical` for cap- and floorlets).

- `ImpVol`: The implied Black volatility.

Basically,the function is simply an implementation of (5.14) and (5.16) with the `TypeFlag` used to toggle between them (`cash` vs. `physical`). However, we also need to be able to compute the annuity values consistent with both cash and physical settlement. For this purpose there is a switch between using `fidAnnuityPv` (for physical settlement) and an expression for the cash settlement annuity. The output of `fidSwaptionPv` is the premium on a unit notional, that is, simply the fraction of the notional.

Finally, we want to be able to imply Black (1976) volatilities from market prices. This is done in `fidImpSwaptionVol`. The function takes the exact same inputs as `fidSwaptionPv` except for `ImpVol` which has been replaced with `Premium`. The `premium` should be provided as the option premium on a unit notional (the same convention as the output of `fidSwaptionPv`). The function then uses Newton-Raphson's method to find the implied volatility $\overline{\sigma}$ that solves Black Formula$(\overline{\sigma}, \cdot)$ − Market Price = 0.[40]

Note, that when we want to apply `fidSwaptionPv` or `fidImpSwaptionVol` to cap- or floorlets, we need a little care when specifying the conventions on the fixed leg as well as the length of the underlying swap. In fact, we need to set the conventions of the fixed leg equal to the conventions of the floating leg (e.g. a 6M tenor and Act/360 day count basis in EUR). If we also set the length of the underlying swap equal to the xBIOR tenor (e.g. 6M), we end up with a underlying swap with only a single fixing and payment of a xIBOR rate. By doing so, we have created a cap- or floorlet from the full set of swaption conventions.

5.5 Plain vanilla greeks

Traditional text-book risk management of options revolve around computing the so-called greek numbers or simply *greeks* — sensitivities with respect to the underlying, implied volatility etc. As our option pricing models for interest rate options are specified for individual forward rates (e.g. the 6M forward xIBOR rate that fixes in 8.5 year's time or the 10Y10Y forward swap rate), reconciling these risks can be difficult. The challenge is that we cannot directly aggregate the risks toward the two above forward rates. Rather than focusing on this traditional way of calculating greeks with respect to the underlying forward rate, we will instead focus on calculating the Dv01s and delta vectors we were introduced to in section 4.7.

5.5.1 Dv01 - Delta

One way of calculating *Delta* — i.e. the sensitivity towards the underlying — is simply the Dv01. We simply apply the same methodology that we saw in (3.17). Simultaneously bumping both the forward- and the discounting curve up and down has two opposing effects in the case of payer swaptions. An increase in forward rates increases the expected pay-off of the swaption (positive forward curve Dv01) while an increase in the discounting curve reduces the present value of the expected payments (negative discounting curve Dv01).[41] The net Dv01 for payer swaptions of varying strikes and varying time-to-expiry can be seen in figure 16. The figure clearly shows that the net Dv01 actually starts to decrease as the strike is lowered beyond some point (thus moving the payer swaption

---

[40]Recall, that from a given starting value x_0, we can iteratively solve $f(x) = 0$ by using the updating rule $x_{n+1} = x_n - f(x_n)/\frac{df(x)}{dx}\big|_{x=x_n}$. This is known as Newton-Raphsons method.

[41]Strictly speaking, this segregation of effects only holds true for physically settled swaptions. Cash settled payer swaptions will see some of this present value effect be attributed to the forward curve as this is implicitly used for discounting via the forward swap rate.

deeper in-the-money (ITM)). This is known as *negative convexity* and will be covered below.

For receiver swaptions the Dv01s (in plural due to the dual curve setup) will be negative for both the forward and the discounting curve. Because of this, we note that the absolute Dv01 in figure 16 are lower compared to figure 17, which shows the net Dv01 for the corresponding receiver swaptions. Comparing the two figures to each other, we can now also understand why swaptions are traded as straddle positions. For options that are at-the-money-forward (3% in the present case), the net delta risk of the straddle is limited as the payer and receiver mostly offset each other's sensitivities.

Figure 16: Net Black Dv01 (forward and discounting curve) on EUR 100m physically settled payer swaptions with 10Y underlying swap for varying time-to-expiry. The sensitivities are calculated on flat yield curves at 3% and $\sigma = 14\%$.
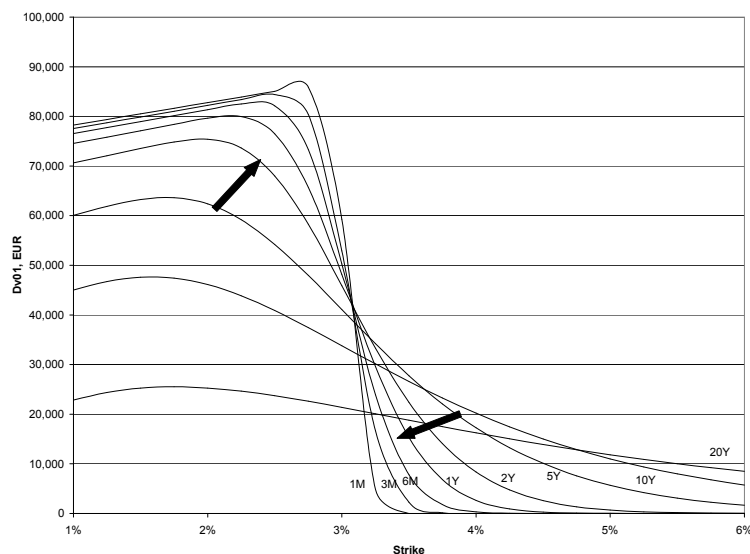
From both figures we see that the Dv01 graphs compresses around the at-the-money-forward level as expiration comes closer. As indicated by the arrows the Dv01 profile becomes less and less smooth. In fact, the Dv01 profile turns into a digital profile in the limit $(T_S - t_0 \to 0)$. The intuition is that if the payer (receiver) swaption is in-the-money just before expiry, then a 1 bps increase (drop) in rates will almost certainly imply a 1 bps higher cash flow over the life of the underlying swap. The value of this will be equal to the swap's annuity level. Obviously, the same does not hold if the swaption is expiring right at-the-money. In that case, a 1 bps decrease (increase) in rates will bring the payer (receiver) just out-of-the-money.

5.5.2 Gamma - Dv01^2

Just as we saw in (3.17) how to approximate first order derivatives numerically, we can proceed to higher orders using the same basic methodology. In particular, we want to calculate the second order sensitivity with respect to changes in yield curves. This second order effect is called *gamma* in the option slang whereas it is often referred to as *convexity* in the interest rate world. As with the delta risks, we could either calculate gamma with respect to the underlying forward rate or with respect to the entire curves (i.e. a Dv01^2 sensitivity).

Figure 17: Net Black Dv01 (forward and discounting curve) on EUR 100m receiver swaptions with 10Y underlying swap for varying time-to-expiry. The sensitivities are calculated on flat yield curves at 3% and $\sigma = 14\%$.
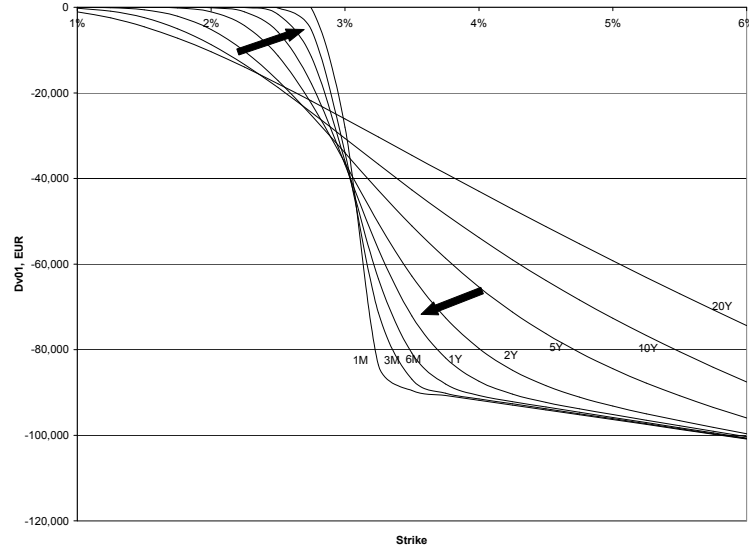

Letting this time $\mathcal{R}$ denote the joint set of both the forward- and discounting curve zero coupon rates, we will define the gamma Dv01 to be

$$
\begin{aligned}
\text{Gamma Dv01} &= \frac{1}{10,000^2} \frac{\partial^2 \text{PV}(\mathcal{R})}{\partial \mathcal{R}^2} = \frac{\partial \text{DV01}}{\partial \mathcal{R}} \\
&\approx \frac{1}{10,000^2} \frac{\text{PV}(\mathcal{R} + \epsilon) + \text{PV}(\mathcal{R} - \epsilon) - 2 \cdot \text{PV}(\mathcal{R})}{\epsilon^2}
\end{aligned}
\tag{5.17}
$$

where $\text{PV}(\mathcal{R})$ denotes the present value of a derivatives portfolio dependent on $\mathcal{R}$. Note that whereas the Dv01 can be approximated by just a single-sided shift (requiring two calculations of the portfolio value), the Gamma Dv01 calculation needs a two-sided shift (requiring three calculations of the portfolio value).

The gamma Dv01 for payer swaptions with varying strikes and time-to-expiration can be seen in figure 18. Note that the figure show the negative convexity region for payer swaptions. The intuition behind this negative convexity region is that when a deep ITM payer sees the forward and discounting curve increase by 1 bps, the expected annuity payment will be increased by approx. 1 bps — however the *entire* expected payment will be discounted harder. If the expected annuity payment is large enough the negative discounting risk on this will offset the increase in the (positive) forward risk.

From figure 18 we also see that gamma is largest around the ATM strike and that the gamma increases towards expiration. Options for which the underlying is traded close to ATM just before expiry will experience a so-called *gamma spike*. This spike means that the exposure towards parallel changes in the yield curve changes very rapidly towards expiration. Finally, we note that being long convexity is valuable. If we are long convexity, we will become more exposed towards higher rates as rates climb and become less exposed as the drop — a win-win situation. If an option trader is long convexity, she can thus use delta hedges to lock in profit from movements in the underlying. If you locally remove your delta sensitivity using a (close to) linear instrument to hedge your

Figure 18: Net Black Gamma Dv01 (forward and discounting curve) on EUR 100m payer swaptions with 10Y underlying swap for varying time-to-expiry. The sensitivities are calculated on flat yield curves at 3% and $\sigma = 14\%$.
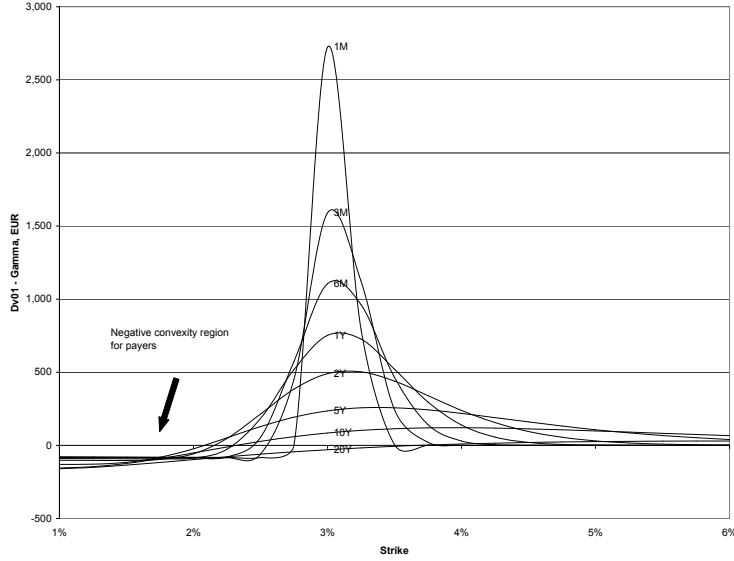
delta risk, you will gain from movements either up or down in the underlying.

5.5.3 Vega

The sensitivity towards changes in implied volatility is known as *vega*. Again, we apply our bump-and-re-run approach to approximate the first-order derivative of our option pricing formulas with respect to σ. That is we define and calculate vega as

$$\text{Vega} = \frac{1}{100} \frac{\partial \text{Swaption PV}(\sigma_{\text{Black}}, \cdot)}{\partial \sigma} \approx \text{Swaption PV}(\sigma_{\text{Black}} + 1\%, \cdot) - \text{Swaption PV}(\sigma_{\text{Black}}, \cdot) \tag{5.18}$$

As indicated, vega is typically reported scaled to a shift of 1%. Such sensitivities can be seen in figure 19. In the Black (1976) model for swaptions vega can alternatively be calculated in closed form:

$$\text{Vega}_t^{\text{Black}} = \frac{1}{100} A(t, T_S, T_E) R(t, T_S, T_E) \phi(d_1) \sqrt{T_S - t} \tag{5.19}$$

where ϕ denotes the standard normal density and d_1 is defined as in (5.14). Comparing the closed form expression to the approximation in (5.18) will typically yield minor differences since vega is not linear for large shifts in σ.

Intuitively, the shape of the vega profile is caused by the convexity of the option pay-offs. The convexity means that the increase in pay-off for payer swaptions for a large upward move in the underlying forward rate will be larger than the decrease in pay-off for a similarly large downward move in the underlying forward rate.[42] This is a result of *Jensen's inequality*. As σ increases, these large moves become more likely causing vega to be positive. We note that vega tends to be largest around at the ATM level. This

---

[42]The same holds true for receiver swaptions. Here a large downward move in rates will be more valuable compared to a large upward move in rates.

is not surprise as the convexity is largest in this region. Finally, vega is increasing in time-to-expiration. This is also intuitive, as a longer time-to-expiration makes it possible for the larger moves in the underlying to be realized. In fact, vega (in the Black (1976) model) is proportional to the square root of time-time-to-expiration as seen in (5.19).

Figure 19: Vega on EUR 100m payer swaptions with 10Y underlying swap for varying time-to-expiry. The sensitivities are calculated on flat yield curves at 3% and $\sigma = 14\%$.
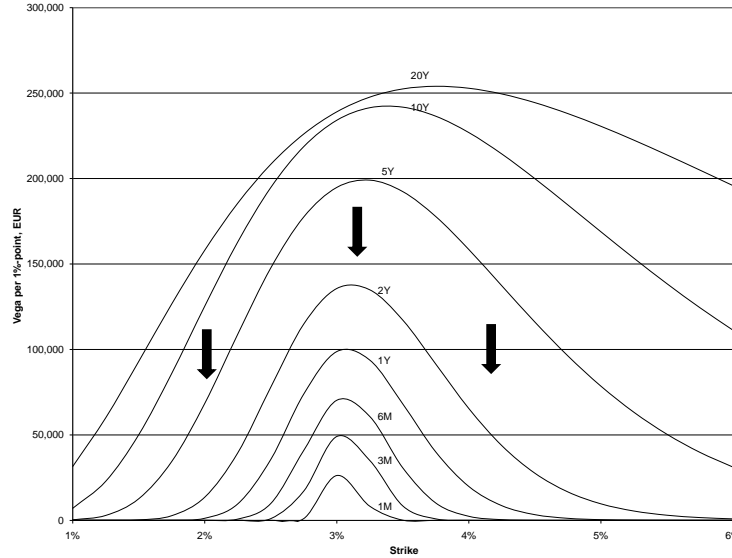
5.5.4 Theta

As discussed above, holders of options benefit from convexity. To gain this benefit they pay a premium to buy the options. However to be able monetize on the convexity, option holders must see a large move in the underlying before expiration. This means that for each day that passes without these movements materializing, the option holder will loose money — she has paid for convexity that did not provide a pay-off. To measure this effect, it is customary to calculate the change in value of an option as the time-to-expiration shortens. This sensitivity is referred to as *Theta*. For an option expiring at T_S (which is measured in years using Act/365), we define and calculate theta as

$$\text{Theta} = -\frac{1}{365}\frac{\partial \text{Swaption}PV(T_S - t, \cdot)}{\partial T_S - t} \approx \text{Swaption PV}(T_S - (t + 1/365), \cdot) - \text{Swaption PV}(T_S - t, \cdot)$$

(5.20)

Note that although theta is often calculated with respect to a change of 1 business day in many real life trading systems, we report it with a change of 1 calendar day for simplicity. Such sensitivities are shown in figure 20. As seen in figure 20, theta is typically negative and increasing towards expiration around the ATM level with a profile that resembles gamma. In fact, it can be shown that for a delta neutral portfolio of options theta is a proxy for gamma (gamma long and theta short or vice versa). For deep ITM payer swaptions we can however see that theta turn positive. The intuition here is that the swap annuity itself is also sensitive towards time changes as the present value of the annuity rises as time-to-expiration shortens. For deep ITM options this effect dominates the time-value loss on the optionality.

55

Figure 20: Theta (1 day) on EUR 100m payer swaptions with 10Y underlying swap for varying time-to-expiry. The sensitivities are calculated on flat yield curves at 3% and $\sigma = 14\%$.
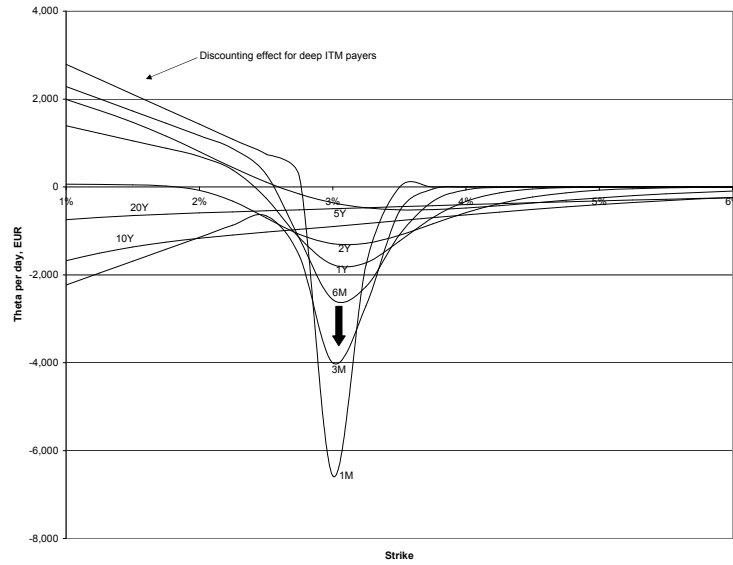

5.5.5 Hedging swaptions

As the options covered above are all OTC products, an option market maker will rarely be able to hedge a sold option by buying back the exact same instrument (and if she could, it would likely not be profitable to do so). Instead, the trader will using different combinations of options and swaps to hedge her risks. As an example, the trader will use plain vanilla IRSs (and CCSs) to manage the delta risks and use options to manage the gamma, vega and theta risks. This is done as the bid-offer spreads in the swap market is typically much tighter compared to the options market — the hedge should obviously be undertaken where it is cheapest.

Table 16: Risk figures for ATMF swaptions on EUR 100m notional. Calculated on flat yield curves at 3% and $\sigma = 14\%$.

Expiration	ATMF	Payer Dv01	Receiver Dv01	Net Dv01	Net Gamma	Net Vega
1M	3.048%	45,728	-41,786	3,942	6,090	57,174
3M	3.048%	45,189	-41,886	3,303	3,398	101,379
6M	3.048%	44,915	-41,502	3,413	2,362	143,390
1Y	3.048%	44,460	-40,656	3,804	1,634	200,301
2Y	3.048%	43,411	-39,129	4,282	1,113	274,368
5Y	3.048%	39,733	-35,722	4,011	631	393,156
10Y	3.048%	33,104	-31,804	1,300	369	472,262
20Y	3.047%	21,137	-26,909	-5,772	184	482,024

As can be seen in table 16 the combination of a long position in payer and receiver swaption has a limited net Dv01 but a large vega or gamma (depending on the time-to-expiration) exposure. This is reason why swaptions primarily trade as straddles in the interbank market — it lets the option trader focus on the risk characteristics that are special to the options, the delta risk can always be created via the swap market. A

final observation from table 16 (and the figures above) is that options with short time-to-expiration (below 1Y) are called *gamma-vol* positions while options with longer time-to-expiration are called *vega-vol* positions.

Finally, we note that we can move simple parallel shift in yields curves and calculate deltavectors for swaptions. This can both be done using the zero rate approach and market rate approach using the tricks explored in section 3.7.

5.6 The volatility smile

Although the Black (1976) model assumes a constant volatility for options of all strikes, it has been a stylized fact in the market place for many years that options trade at different implied volatilities depending on their strike. Typically, these differences are referenced to the ATM implied volatility level: Are out-of-the-money (OTM) and in-the-money (ITM) options expensive or cheap — in implied volatility terms — to their ATM cousins? Either way, non-constant implied vol-strike relationships are referred to as *volatility smiles* or *-skews*.

We will not cover the empirical regularities in detail here, since they are covered in Hagan et al. (2002) which is also on the reading list. Instead, we will develop a bit of intuition on what causes volatility smiles. Letting $A(\cdot)$ denote the swap annuity we can write (5.13) as

$$
\begin{aligned}
\text{PV Payer}_t &= A(t, T_S, T_E) E_t^A[(R(T_S, T_S, T_E) - K)^+] \\
&= A(t, T_S, T_E) \int_{-\infty}^{\infty} (R(T_S, T_S, T_E) - K)^+ g^A(R(T_S, T_S, T_E)) dR(T_S, T_S, T_E) \\
&= A(t, T_S, T_E) \int_{K}^{\infty} (R(T_S, T_S, T_E) - K) g^A(R(T_S, T_S, T_E)) dR(T_S, T_S, T_E)
\end{aligned}
$$
(5.21)

where $g^A(R(T_S, T_S, T_E))$ is simply the density of the relevant swap rate on the expiration date under the probability measure Q^A. This notation is model-free, our choice of model e.g. Black (1976) simply sets the distributional properties of $g^A(R(T_S, T_S, T_E))$. Now, a trick due to Breeden & Litzenberger (1978) allows us express relationships between between observable payer swaption premia scaled by $A(t, \cdot)$ and the underlying density $g^A(R(\cdot))$ of the swap rate at time T_S and its distribution function $G^A(R(\cdot)) = \int_{-\infty}^{R} g^A(S) dS(\cdot)$. These relationships are obtained by differentiating the scaled call option prices with respect to K:

$$
\begin{aligned}
\frac{\partial \text{PV Payer}_t / A(t, \cdot)}{\partial K} &= \frac{\partial}{\partial K} \left[\int_{K}^{\infty} (R - K) g^A(R) dR \right] \\
&= - \int_{K}^{\infty} g^A(R) dR \\
&= G^A(K) - 1
\end{aligned}
$$
(5.22)

$$
\begin{aligned}
\frac{\partial^2 \text{PV Payer}_t / A(t, \cdot)}{\partial K^2} &= \frac{\partial}{\partial K} \left[- \int_{K}^{\infty} g^A(R) dR \right] \\
&= g^A(K)
\end{aligned}
$$
(5.23)

where we have used *Leibniz' rule* for integration.[43] This — still model-free — result tells us how to back out risk neutral probability distributions from option prices. That is, if we can observe prices for options of neighboring strikes, we can infer information about the underlying probability distribution. This means that the presence of volatility smiles is simply a rejection of the log-normal distribution. If the market is pricing e.g. low strike options relatively expensive in volatility terms, it is equivalent to saying that the market assigns more probability to lower rates.

As a final note, it worth oting that for the Black (1976) model (5.23) has an easy closed form expression. In fact, for the Black (1976) model we have that

$$\frac{\partial \text{PV Payer}_t / A(t, \cdot)}{\partial K} = -\Phi(d_2) \tag{5.24}$$

5.7 The SABR model

5.7.1 Specifying the model

Before we introduce a model capable of handling volatility smiles, it is in order to remind ourselves of why they present a challenge to our Black (1976) framework laid out above. Volatility smiles are problematic for two reasons.

- Pricing: Typically liquidity in OTC option markets prevents us from observing option prices for all strikes. Instead we typically only see relatively few prices away from ATM. We want to be able to interpolate well between the few points that we actually can observe. Also, we want to be able to extrapolate well beyond the OTM prices that actually are observable. Without a proper model for this inter-/extrapolation quoting options away from ATM can be extremely difficult.

- Hedging: As pointed out in Hagan et al. (2002) the volatility smile itself seems to move together with the underlying. This introduces some dynamic behavior that we want to incorporate into a good option pricing model. In particular, we should be concerned that we do not try to hedge delta risks as vega risks and vice versa.

While fairly accurate, simple corrections to ATM implied volatilities can remedy the first challenge, we need a more advanced model to remedy the second. Recalling (5.3), we simply want to come up with a another model and thus another function $c(\cdot)$ to use in (5.2).

One such model is the so-called *Stochastic Alpha Beta Rho* — or simply *SABR* — model. The model was proposed in Hagan et al. (2002) and had successfully been implemented in several large banks some years prior to that. The model is a two-factor model with one set of dynamics driving the stochastic volatility and another set driving the forward rate. Rewritten a bit, the model is formulated as

$$
\begin{aligned}
df_t &= \sigma_t f_t^\alpha dW_t^1 \\
d\sigma_t &= \epsilon \sigma_t dW_t^2 \\
dW_t^1 dW_t^2 &= \rho dt
\end{aligned}
\tag{5.25}
$$

where $0 \leq \alpha \leq 1$, $\epsilon > 0$ and $-1 \leq \rho \leq 1$. We note that as special cases of the SABR model, we can recover the Black (1976) model (for $\alpha = 1$ and $\epsilon = 0$) as well as the Constant Elasticity of Variance (CEV) formulation of Cox (1975) (for $\epsilon = 0$).

---

[43]Leibniz rule tells us how to differentiate definite integrals over variables that enter the integral limits: $\frac{\partial}{\partial z} \int_{a(z)}^{b(z)} f(x, z) dx = f(x, b(z)) \frac{\partial b(z)}{\partial z} - f(x, a(z)) \frac{\partial a(z)}{\partial z} + \int_{a(z)}^{b(z)} \frac{\partial f(x, z)}{\partial z} dx$

The main contribution in Hagan et al. (2002) is that this advanced model actually results in option *prices* that can be *approximated* by a closed form expression for a Black (1976) implied volatility $\sigma_{\text{Black}}(K, f)$. This approximation is known as the *SABR formula*.

$$\sigma_{Black}(K, f) = \frac{\sigma_0}{(fK)^{(1-\alpha)/2}\left(1 + \frac{(1-\alpha)^2}{24}\log^2(f/K) + \frac{(1-\alpha)^4}{1920}\log^4(f/K) + ...\right)}\left(\frac{z}{x(z)}\right)$$

$$\cdot \left\{1 + \left[\frac{(1-\alpha)^2}{24}\frac{\sigma_0^2}{(fK)^{1-\alpha}} + \frac{1}{4}\frac{\rho\alpha\epsilon\sigma_0}{(fK)^{(1-\alpha)/2}} + \frac{2-3\rho^2}{24}\epsilon^2\right] \cdot t_{exp} + ...\right\}$$

$$z = \frac{\epsilon}{\sigma_0}(fK)^{(1-\alpha)/2}\log(f/K)$$

$$x(z) = \log\left\{\frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1 - \rho}\right\}$$

(5.26)

For options that are struck AMTF ($K = f$) the formula collapses into the simpler expression

$$\sigma_{Black}^{ATM}(f, f) = \frac{\sigma_0}{f^{(1-\alpha)}}\left\{1 + \left[\frac{(1-\alpha)^2}{24}\frac{\sigma_0^2}{f^{2-2\alpha}} + \frac{1}{4}\frac{\rho\alpha\epsilon\sigma_0}{f^{(1-\alpha)}} + \frac{2-3\rho^2}{24}\epsilon^2\right] * t_{exp} + ...\right\} \quad (5.27)$$

We can use this formula to find σ_0 as function of σ_{Black}^{ATM}. Doing however requires us to solve a third order polynomial. In practice this is done numerically as only one of the three possible roots have the right order of magnitude. In the `fidAnalytics` library this has been done in `fidBlackToSabr` using Newton-Raphson.

Although the formulas (5.26) and (5.27) look a bit cumbersome they are actually simple to implement as can be seen in `fidSabr`. The equations are simply a lot of algebra on the four model parameters $\alpha, \epsilon, \sigma_0, \rho$ in addition to the time-to-expiration t_{Exp}, strike K and the relevant forward swap (or xIBOR) rate f. Subsequently, these are the inputs that `fidSabr` admits. The ease of implementation means that `fidSabr` is actually all we need to implement the SABR model. We can then re-use our black pricing formula `fidSwaptionPv` with an implied volatility that comes out of `fidSabr`. This ease of implementation is certainly one of the reasons why the model has become so popular.

5.7.2 Calibrating a SABR model

Assuming that we have a calibrated swap curve at our disposal, we want to calibrate the four parameters of the SABR model $\Omega = \{\alpha, \epsilon, \sigma_0, \rho\}$ such that we fit market data. This market data will be a set of strikes $K = \{K_1, \ldots, K_n\}$ and corresponding Black (1976) implied volatilities $\Sigma = \{\overline{\sigma}_1, \ldots, \overline{\sigma}_n\}$. Mathematically speaking, we set up a least squares minimization problem to find our parameters:

$$\min_{\Omega} = \sum_{i=1}^{n}(\overline{\sigma}_i - \sigma_{Black}(K_i, f, \Omega))^2 \qquad \text{s.t.}$$

$$0 \leq \alpha \leq 1$$
$$-1 \leq \rho \leq 1$$
$$0 \leq \epsilon$$
$$0 \leq \sigma_0$$

(5.28)

While these restrictions ensure that the model is properly specified, they are not enough to ensure that the approximation in (5.26) is valid. In particular, the approximation can

degenerate for various combinations of $\epsilon = 0$, $\sigma_0 = 0$ and for $\rho = \pm 1$. Because of this, we will typically specify some boundaries in `SOLVER` that prevent the restrictions in (5.28) from holding with equality. Recalling the discussion in section 3.4, we can actually think of our SABR model is being just another layer in our model setup. That is, we could simply now specific a single large calibration problem where we would be interested in finding a parameter vectors that contains both zero coupon rates (for the yield curve part) and SABR paramaters (for the option part). While we will take a calibrated yield curve as input (via the relevant forward swap rate) when calibrating our SABR model, this nesting of models will be useful when looking a full risk reports.

Typically, we will be able to observe between 4 and, say, 10 different strikes for each relevant forward swap rate. As noted in Hagan et al. (2002) the model is actually somewhat overspecified since α and ρ affect the volatility smile in a qualitatively similar way. For this reason, it is common to fix the α parameter and simply use the remaining three free parameters to fit market data. Doing so — as we will see below — allows us to fine-tune the delta hedging properties of the model. As a calibration example, consider the quotes in table 17.

The quoting of volatility smile uses double relative references why an explanation of table 17 is in place. First of all, the smile quotes are typically updated rather infrequently (say, once a week). We are thus interested in quoting them in a way that is fairly stable across different levels of interest rates and ATM volatility. The strike quote is therefore given as a relative reference to the current ATMF level e.g. 200 bps below the ATMF (that is, a strike of 3.238%). The volatility quote is then calculated as an offset to the ATM level e.g. $\sigma(3.238\%) = 11.0\% + 4.7\% = 15.7\%$. This can be confirmed by looking at figure 21. If ATMF rates the following day would drop by, say, 20 bps to 5.038% while the ATM volatility level would rise to, say, 11.4% we could still use the relative quotes and now compute $\sigma(3.038\%) = 11.4\% + 4.7\% = 16.1\%$, this is why the smile is quoted this way.

Table 17: Market quotes for 10Y10Y swaption smile.

| Swaption | ATMF | ATM Vol | Offset to ATMF (bps) | | | | | | | |
			-300	-200	-100	-50	+50	+100	+200	+300
10Y10Y	5.238%	11.0%	+9.0%	4.7%	1.8%	+0.8%	-0.50%	-0.70%	-0.60%	-0.20%

Using a fixed $\alpha = 0.54$ these quotes have been used to calibrate the model shown in table 18. The model is shown in figure 21. As can be seen, the calibration errors are small compared to the typical bid-offer spread in the swaption market of approximately 1 vega.[44] Besides the reference model that uses $\alpha = 0.54$, another model using $\alpha = 0.30$ is also shown both the in figure and in the table. As can be seen, the two models give rise to almost identical volatility smiles, thus convincing us that we do in fact have the required degree of freedom to set the α parameter freely.

5.7.3 Hedging in the SABR model

As previously mentioned , an important argument for moving from the Black (1976) model to the SABR model (or some other more advanced model) is the improved hedging

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[44]As the option trader will see a 10Y10Y swaption as primarily vega risk, bid-offer spreads will typically be set using the vega risk metric as a rule-of-thumb. A plausible bid-to-mid (i.e. the difference between the mid-market- and the bid price) would be 0.5 vega.

Table 18: Calibrated parameters for 10Y10Y swaption SABR model.

| Parameter	Reference model ($\alpha = 54\%$)	Alternative model ($\alpha = 30\%$)
σ_0	2.697%	1.316%
ϵ	28.837%	27.529%
ρ	-27.922%	-15.537%

Figure 21: 10Y10Y Swaption SABR model.
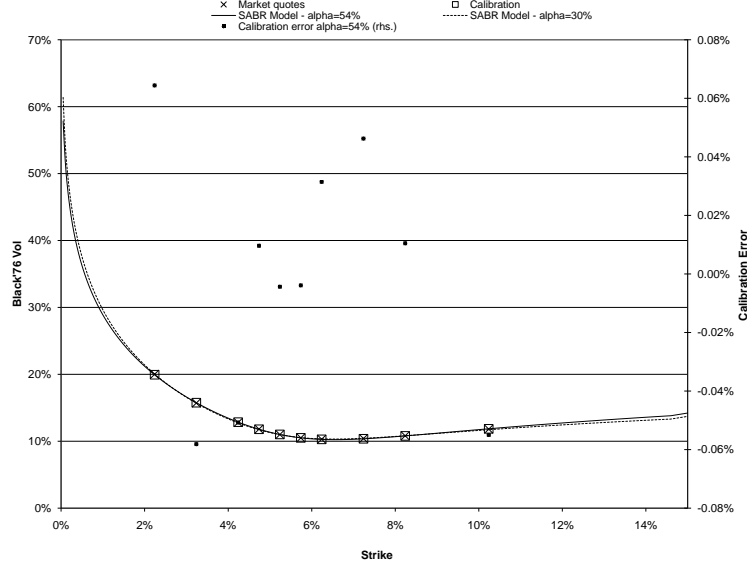
properties of the model. This shows up when we want to calculate Dv01 or delta vectors in the SABR model. Since $\sigma_{Black}(K, f)$ is a function of the underlying forward (swap) rate f, any bump to the underlying forward- or discounting zero coupon curves will result in a change in implied volatility. It is important to remember this effect when we build spreadsheets to calculate greeks in the SABR model.

As shown above, different choices of α can give rise to (almost) identical volatility smiles. However, when we turn to hedging the different choices will provide different risk figures. This can be seen in figure 22, where the models listed in table 18 have been used to calculate Dv01. Note that the models have been re-calibrated to an ATM Black (1976) implied volatility of $\sigma_{\text{Black}}^{\text{ATM}} = 14\%$ as to make the risk figures comparable to the above sections. We note from the figure, that the SABR model is capable of generating significantly different Dv01 sensitivities and that these can be altered using different values of α.

Besides the traditional Dv01 and gamma Dv01 risk measures, there are a number of other risks that we can calculate in the SABR model. In particular we can calculate derivatives with respect to α, σ_0, ϵ and ρ. As these parameters determine the volatility smile, these risks can help us manage the risks associated with having options positions across a wide range of strikes. The problem with these model parameter greeks is that that can be difficult to translate into market instruments. Rather we will once again invoke our jacobian risk methodology to produce a full risk report. Now, our parameter vector can be written as $\mathbf{P} = \{p_1^{\text{Fwd}}, p_2^{\text{Fwd}}, \ldots, p_{M-4}^{\text{Disc}}, p_{M-3}^{\sigma_0}, p_{M-2}^{\alpha}, p_{M-1}^{\epsilon}, p_M^{\rho}\}$. We simply append the SABR parameters to our two curve parameter vector. In terms of model quotes we will be looking at a vector that will likewise be appended with implied volatilities of different

Figure 22: Dv01's in the SABR model vs. the Black'76 model. The sensitivities are calculated for 10Y10Y physically settled payer swaptions on flat yield curves at 3% and the volatility models are re-calibrated to $\sigma_{\text{Black}}^{\text{ATM}} = 14\%$.
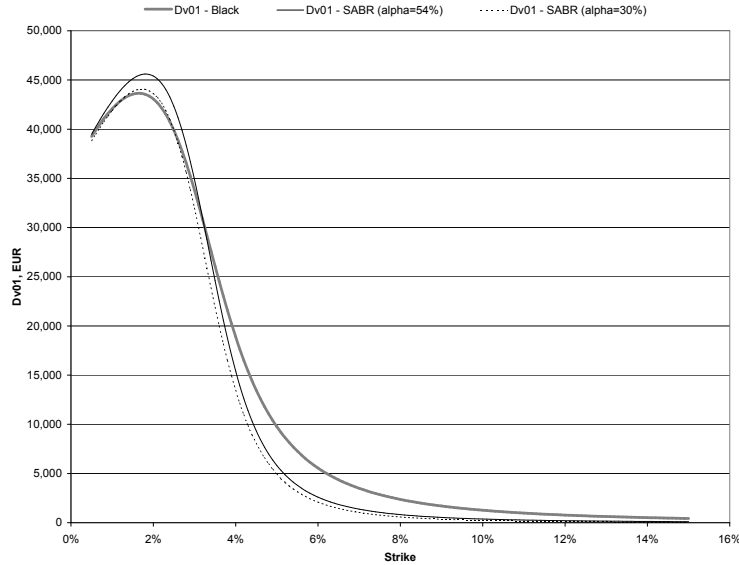
strikes $j = 1, \ldots, J$. $\mathbf{B}(\mathbf{P}) = \{b_1^{\text{IRS}}, \ldots, b_{N-J}^{\text{CCS}}, b_{N-J+1}^{\sigma(K_1)}, b_N^{\sigma(K_J)}\}^\top$. While we can easily calculate the corresponding jacobian matrix (remembering in particular that the implied volatilities will depend on the forward rate), our simple risk methodology of inverting the jacobian is likely to fail. The reason for this is one of dimensionality. In previous sections we focused only the case where the number of parameters matched the number of quotes ($N = M$), which means that the jacobian is a square matrix. However, if we use more than four strikes to calibrate the four parameters of the SABR model (as indicated by table 17), we cannot invert the jacobian. If the jacobian matrix fulfills some assumptions (which we will not go into here), it is possible to calculate a *pseudo inverse matrix*. This matrix can be computed using a technique called Single Value Decomposition (SVD) or other advanced methods see e.g. Andersen & Piterbarg (2010c).[45] While SVD methods are beyond the scope of this course, it is important to note the link between linear algebra and risk calculations. In particular, it is possible to specify hedging problems under weighting schemes that penalizes hedges in less liquid instrument.

Limiting our focus back to the case of a square jacobian, table shows a SABR risk report for IRS, CCS and the ATM implied volatility quote for four different instruments. Note the difference between the risk profile of the two 6% payer swaptions with cash, respectively, physical settlement. Since the physically settled contract will depend have an explicit dependence on discount factors all the way out to 20 years while the cash settling variant only has discounting (and thus CCS) risk out to 10 years.

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[45]For a matrix M, the matrix expression $M = U\Sigma V^\top$ is called the single value decomposition. For M the matrix $M^+ = V\Sigma^+ U^\top$ is called the pseudo inverse of M. The matrix Σ is a diagonal matrix with the so called singular values, while Σ^+ is constructed from replacing each of the diagonal elements with its reciprocal value.

Table 19: Dual curve market rate Black delta vectors, 10Y10Y swaptions EUR 100m notional with different settlement types. Flat forward curve at 4% and flat discounting curve at 3.5%.

Market quote		6% Payer, Cash	6% Payer, Physical	3.5% Receiver, Cash	3% Payer, Cash
IRS	1Y	2	2	5	18
IRS	2Y	3	3	10	35
IRS	3Y	1	1	4	16
IRS	5Y	58	77	212	707
IRS	10Y	-14,335	-14,514	13,804	-75,841
IRS	15Y	6	-671	-74	77
IRS	20Y	23,242	24,313	-27,226	115,474
IRS	30Y	-4	7	6	-23
CCS	1Y	2	2	5	19
CCS	2Y	4	4	11	36
CCS	3Y	2	1	5	17
CCS	5Y	74	97	222	752
CCS	10Y	-821	-92	-2,440	-8,306
CCS	15Y	-3	-696	-11	-32
CCS	20Y	-1	-488	-3	2
CCS	30Y	0	12	0	0
Vol 1%	ATM	184,450	189,247	256,502	214,502

5.8 Digital options

An important building block for structured interest rate products is *digital options*. A digital call option is an option that pays off one unit currency if the underlying rate f is at or above above a strike rate K. In this course we will focus on the *digital caplet* that fixes-in-advance at time T and pays-in-arrears at time $T+\delta$ on an accrual fraction δ (just like the plain vanilla caplets). When written on the xIBOR rate $F(T,T,T+\delta)$ can be written as

$$\text{Digital Caplet Pay-off}_T = P(T,T+\delta)\delta \cdot \mathbf{1}_{\{K \leq F(T,T,T+\delta)\}} \tag{5.29}$$

where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function. Similarly, the *digital floorlet* can be written as

$$\text{Digital Floorlet Pay-off}_T = P(T,T+\delta)\delta \cdot \mathbf{1}_{\{K \geq F(T,T,T+\delta)\}} \tag{5.30}$$

Valuing such contracts can easily be done using the methodologies previously introduced. In particular, we can value the digital caplet at time t as

$$
\begin{aligned}
\text{Digital Caplet PV}_t &= P(t,T+\delta)\delta E_t^{Q_{T+\delta}}\left[\frac{P(T,T+\delta)\cdot\mathbf{1}_{\{K\leq F(T,T,T+\delta)\}}}{P(T,T+\delta)}\right] \\
&= P(t,T+\delta)\delta E_t^{Q_{T+\delta}}[\mathbf{1}_{\{K\leq F(T,T,T+\delta)\}}] \\
&= P(t,T+\delta)\delta(1-G^{Q_{T+\delta}}(K))
\end{aligned}
\tag{5.31}
$$

where $G^{Q_{T+\delta}}(K)$ denotes the distribution function of $F(T,T,T+\delta)$ under the measure that uses the zero coupon bond maturing at time $T+\delta$ as numeraire. Note that we can think of the value of the digital caplet as simply the discounted probability (multiplied with the accrual fraction δ) of the forward xIBOR rate F fixing above the strike K. Likewise, the value of the digital floorlet is the probability of fixing below the strike.

Remembering (5.22), we can calculate the prices of digital cap- and floorlets from the price differences of plain vanilla cap- and floorlets that are struck infinitely close to each other. In practice, digital options are often priced as so-called call- or put-spreads. That is, we calculate the digital caplet as

$$
\begin{aligned}
\text{Digital Caplet PV}_t &= -P(t, T + \delta)\delta\frac{\partial\text{Caplet PV}(t, K)/[P(t, T + \delta)\delta]}{\partial K} \\
&= -\frac{\partial\text{Caplet PV}(t, K)}{\partial K} \\
&\approx \frac{\text{Caplet PV}(t, K - \epsilon) - \text{Caplet PV}(t, K + \epsilon)}{2\epsilon}
\end{aligned}
\tag{5.32}
$$

This basically means that we can price — and hedge — a digital caplet by taking a leveraged long position in the plain vanilla caplet struck at $K - \epsilon$ and a leveraged short position in the ditto struck at $K + \epsilon$. The positions are leveraged relative to the notional on the digital caplet, since the plain vanilla caplets will have a notional that is scaled by $\frac{1}{2}\epsilon^{-1}$. Likewise, digital floorlets can be replicated with a long position in a plain vanilla floorlet struck at $K + \epsilon$ and a short position in a ditto struck at $K - \epsilon$.

5.9 Static replication of arbitrary European payoffs

Having first introduced the plain vanilla European call option, we saw in the previous section how to replicate digital options via a replication argument. It turns out that we can actually generalize this even further to price arbitrary European payoffs using static replication using only plain vanilla European call- and put options. Formally, let X_T denote the time T value of some underlying and let f denote some (twice differentiable) pay-off function for a given contract V such that we can write $V_T = f(X_T)$. It turns out that we can write

$$
V_T = f(\kappa) + f'(\kappa)(X_T - \kappa) + \int_{-\infty}^{\kappa} f''(K)(K - X_T)^+dK + \int_{\kappa}^{\infty} f''(K)(X_T - K)^+dK \tag{5.33}
$$

as an expansion around some scalar κ, you can find a nice little proof of this in appendix A.

We can now take expectations (under the relevant forward measure) while setting $\kappa = X(t, T)$ and arrive at

$$
V_t = P(t, T)f(X(t, T)) + \int_{-\infty}^{X(t,T)} f''(K)PV_t^{\text{Put}}(K)dK + \int_{X(t,T)}^{\infty} f''(K)PV_t^{\text{Call}}(K)dK
\tag{5.34}
$$

this result is often sometimes to as the *Carr formula* as it has been popularized by Peter Carr. Importantly, the result shows us that we can price arbitrary European payoffs in a model free setting if we can price plain vanilla options at all possible strikes. Although the formula might look a little daunting at first glance, its implementation is actually fairly straight forward. The first term is easy to price if we can just work out the forward value $X(t, T)$. The two integrals over the complete range of strikes K represent the "tricky" part but can be computed using numerical integration. This simply means that we will divide the strike domain into a finite set of points ΔK (setting some fixed lower and upper bound for our integration) and compute a sum of option prices with a notional of $f''(K)\Delta K$. In the following section, we will see an explicit example of how to use — and implement — the Carr formula.

Although the result in (5.34) is model independent, it is important to stress that the prices we will ultimately compute are not. First of all, we will only be able to approximate the value of the integrals via numerical integration (potentially, we also need to approximate f'' if we cannot compute this in closed form). More importantly, we will introduce some model dependency in our computation of option prices with arbitrary strikes. Intuitively, we will need some model to inter- and extrapolate in the volatility smile. If we choose a poor model for this, we will obviously also produce poor prices for complex payoffs using the Carr formula. If we believe that e.g. the SABR model possesses good properties for this inter- and extrapolation of option prices we can however easily start using it to compute arbitrarily exotic payoffs as long as they are European in nature (i.e. only depend on a single observation of the underlying).

5.10 Constant Maturity Swaps

5.10.1 Introducing CMS products

So far we have looked at floating rate instruments that are written on xIBOR rates. There is however several other reference rates that can be used as underlying for floating rate instruments. An important class among these are the so called *Constant Maturity Swap-* or simply *CMS* rates. Just as the British Bankers Association and the European Banking Federation publish daily xIBOR fixings, the *International Swaps and Derivatives Association* (ISDA) publishes daily swap rate fixings. These fixings are simply the par swap rates reported by a number of swap dealers for a number of maturities. The conventions for these swaps rates are the plain vanilla conventions listed in table 3. Each day, ISDA thus collects the fixings for swaps with e.g. 2Y, 10Y and 30Y maturities. As the maturities are the same each day, the fixings are called constant maturity swap fixings. In the below we will denote by $\overline{R}(T, T, T + S)$ the CMS fixing for the S-year par swap rate observed at time T.

What makes CMS products interesting from a theoretical pricing perspective is that the fixing rates are typically applied to accrual periods that do not match the length of the underlying swap. In fact, CMS rates are typically applied to 3M periods using money market interest accrual. This mismatch between, say, a 10Y rate and a 3M accrual period introduces the need for a so-called *convexity correction.* The more formal introduction to this topic is found in Hagan (2003) which is also on the reading list. What is covered below is thus some more detailed intuition and the relationship to the Carr formula as well as practicalities of a numerical procedure that accounts for the correction.

The most common CMS product is the so called CMS swap. Such a swap is the exchange of floating xIBOR rates against floating CMS rates in the same currency. The standard EUR version of this product is to pay 3M EURIBOR plus a spread against receiving some CMS (e.g. 10Y) rate every three months with both floating rates being fixed-in-advance and paid-in-arrears over some period of time (e.g. 5Y).[46] Throughout the life of the CMS swap, the value at each fixing date T_i of the following payment can be written as

$$\text{CMS payment } \text{PV}_{T_i} = P(T_i, T_i + \delta_i) \cdot \delta_i \cdot \left[\overline{R}(T_i, T_i, T_i + S) - (L(T_i, T_{i+\delta_i}) + C) \right] \quad (5.35)$$

where δ_i is the coverage (typically 3M), $L(T_i, T_{i+\delta_i})$ is the δ-tenor xIBOR fixing at time T_i and C is the CMS spread. As such, the CMS swap is simply the exchange of a series of

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[46]Note that is important to distinguish between the tenor of the underlying CMS rate and the maturity of the CMS swap.

rate differential between short- and long dated interest rates. On an upward sloping yield curve, the spread applied to the xIBOR leg tends to be positive to compensate for the positive rate differentials. As CMS swaps are OTC traded instruments they are quoted on broker screens and on request with dealers. An example of a CMS broker screen can be seen in figure 23.

Figure 23: EUR CMS quotes from the broker ICAP, 23 April 2010

```
 20                                              N207 n Corp   ICAE
200<Go> to view in Launchpad
15:40 EUR  CMS                                             PAGE  1 / 2

   Object     Ask     Bid     Time      Object     Ask     Bid     Time
   2 Year Index                         10 Year Index
1)  5Y Swap  75.2000  69.2000  15:39   9)  5Y Swap 189.1000 179.1000  15:39
2) 10Y Swap  61.6000  55.6000  15:36   10) 10Y Swap 141.6000 131.6000  15:38
3) 15Y Swap  53.8000  47.8000  15:36   11) 15Y Swap 114.1000 104.1000  15:39
4) 20Y Swap  48.1000  42.1000  15:33   12) 20Y Swap 102.0000  82.0000  15:38


   5 Year Index                         20 Year Index
5)  5Y Swap 136.9000 127.9000  15:39   13)  5Y Swap 213.4000 193.4000  15:39
6) 10Y Swap 105.9000  96.9000  15:38   14) 10Y Swap 148.0000 128.0000  15:38
7) 15Y Swap  87.8000  78.8000  15:36   15) 15Y Swap 113.0000  93.0000  15:36
8) 20Y Swap  76.2000  67.2000  15:36   16) 20Y Swap  97.4000  77.4000  15:36



   EUR CMS quoted A/360 vs 3M Euribor


Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000
Japan 81 3 3201 8900      Singapore 65 6212 1000      U.S. 1 212 318 2000      Copyright 2010 Bloomberg Finance L.P.
                                                                   SN 619685 H197-874-1 23-Apr-2010 15:40:32
```

5.10.2   Pricing the CMS swaplet

As we have previously seen how to price a series of floating rate xIBOR payments (and the annuity that pays the constant spread C) we will focus exclusively on pricing the CMS rate payment. In fact, we will refer to the value of such a single payment as the CMS *swaplet*. Mathematically speaking, we price the CMS swaplet by computing

$$\text{CMS Swaplet PV}_t = \widetilde{A}(t, T_i, T_i + S) \cdot \delta_i E_t^{\widetilde{A}}\left[\frac{P(T_i, T_i + \delta_i)\overline{R}(T_i, T_i, T_i + S)}{\widetilde{A}(T_i, T_i, T_i + S)}\right] \quad (5.36)$$

This pricing problem is not nearly as well-behaved as the ones we saw for cap- and floorlets or swaptions. The problem is that there is no "clever" choice of numeraire this time — we need to model the future behavior of the ratio $G(t) = P(t, T_i + \delta_i)/\widetilde{A}(t, T_i, T_i + S)$ (for $t = T_i$ in particular). If we can however rewrite $G(t)$ to become a function of $R(\cdot)$, we can apply the results from section 5.9.

Intuitively, the problem is that the swap rate per construction pays off relative to $\widetilde{A}(\cdot)$ while the CMS swaplet pays relative to $P(\cdot)$.[47] Suppose as an example that we tried to hedge a CMS swaplet payment on a unit notional (that is made against a payment of $K \cdot \delta \cdot P(t, T_i + \delta)$) with a forward starting swap struck at K with a scaled notional of $\delta \cdot G(t)$. If we subsequently bump the swap curve up and down, we will see that the CMS swaplet payment is almost linear in the rate changes (there is not much interest rate risk on the single discount factor 3 months into the future) while the forward starting swap can exhibit substantial convexity as shown in figure 24. Because of this convexity gap,

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[47]Strictly speaking, the plain vanilla IRS rather pays off relative to $A(\cdot)$ rather than $\widetilde{A}(\cdot)$.

the CMS swaplet payment is more valuable than the forward swap hedge. The results in Hagan (2003) are different ways of closing this gap. For even more details on the subject, see Andersen & Piterbarg (2010c)

Figure 24: The convexity gap for a EUR 100m payment in 10Y time on a 30Y CMS fixing for a 3M coverage (Act/360). Values are calculated for parallel shifts to flat yield curves at 3%.
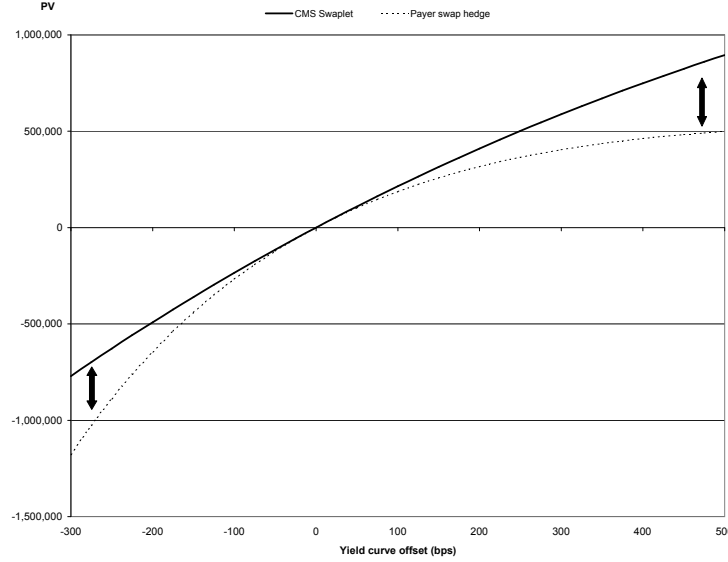
As is shown in Hagan (2003), the gap can be closed by taking long positions payer- and receiver swaptions of varying strikes if we use the Carr formula machinery. This approach to the pricing (and also the risk management) of CMS products is thus often referred to as *static CMS replication*. In order to actually price a CMS swaplet, we however need to postulate a model for the ratio $G(t)$. This is done in two steps. First, we assume that we can write both $P(T, T_i + \delta_i)$ and $\widetilde{A}(t, T_i, T_i + S)$ as functions of $R(T_i, T_t, T_i + S)$. One particular simple model, that is covered in Appendix A of Hagan (2003) is that we assume that the yield curve is flat at time T_i such we can discount at a flat rate of $R(T_i, T_t, T_i + S)$. Using a bit of bond math, this model yields (focusing only on the case of quarterly payments and an underlying swap tenor of S years:

$$G(R) = \frac{R}{(1+R)^{1/4}} \frac{1}{1 - \frac{1}{(1+R)^S}} \tag{5.37}$$

What Hagan (2003) shows is how to relate this function to the convexity correction. We will not go into specific details here, but only note that we can further ease our computational problem is we use an expansion trick to approximate the swaption replication weights as $f''(x) \approx 2G'(R(t, \cdot))/G(R(t, \cdot))$. This allows us to restate one of the results

from Hagan (2003) (equation (2.19a), p. 41):

$$\text{CMS Swaplet PV}(t, T_i, S) = \underbrace{\delta_i P(t, T_i + \delta_i) R(t, T_i, T_i + S)}_{\text{PV of cash flow projected by forward swap rate}}$$

$$+ \underbrace{\delta_i \frac{P(t, T_i + \delta_i)}{\widetilde{A}(t, T_i, T_i + S)} \left[\int_{R(t, \cdot)}^{\infty} \text{Payer}(x) f''(x) dx + \int_{-\infty}^{R(t, \cdot)} \text{Receiver}(x) f''(x) dx \right]}_{\text{PV of cash settling swaptions}}$$

$$(5.38)$$

Note that in theory, this replication recipe instructs us to hedge a CMS swaplet with swaptions across all strikes from $-\infty$ (that is, 0% assuming non-negative rates) to $+\infty$ with infinitely small notionals. Rather than attempting that, we will bracket the strike range into discrete buckets, where the notional on each swaption will then be $\Delta x \cdot \delta_i \frac{P(t, T_i + \delta_i)}{\widetilde{A}(t, T_i, T_i + S)} \cdot 2 \frac{G'(R(t, \cdot))}{G(R(t, \cdot))}$.

This is the final step in our pricing since we need to formulate how to do the numerical integration of the two integrals in (5.38). We will do this by breaking the strike interval up into n discrete steps. Since it is computationally intensive, we should be careful when choosing the strike range over which to use our n steps (typically between 50 and 100). This can be done in an easy and pragmatic manner by setting:

$$\text{Lower bound} = \max(0\%, R \exp(-m\sigma\sqrt{T_i})$$
$$\text{Upper bound} = \min(20\%, R \exp(m\sigma\sqrt{T_i}) \qquad (5.39)$$
$$\text{Integration step} = \frac{\text{Upper bound} - \text{Lower bound}}{n}$$

where m is the number of standard deviations to integrate over, σ is the ATM Black (1976) volatility and T_i is the time-to-fixing.[48] The hard-coded upper bound of 20% is somewhat arbitrary, but is a typical value for this purpose.

To sum up, we can state the following algorithm for computing the CMS swaplet value:

1. Calculate relevant forward swap rate, R.

2. Calculate $G(R)$ and estimate $G'(R) \approx \frac{G(R + 1/10{,}000) - G(R)}{10{,}000}$.

3. Calculate the ATM volatility and update lower- and upper bounds for integration as well as the integration step size using (5.39)

4. Integrate numerically over the relevant strike the payer- and receiver swaption premia while looking up a new Black (1976) volatility for each new strike using the SABR parameters.

5. Calculate the discounting factor corresponding to the CMS swaplet payment date, the forward annuity factor (for cash settling swaptions) as well as the coverage.

6. Calculate the value of the projected forward swap rate payment and add the integrated swaption premia.

This algorithm has been implemented in the fidAnalytics library as `fidCmsSwapletPv`. As inputs the function takes the following inputs:

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[48]Thanks to Jesper Andreasen for pointing this out

- `AnchorDate`, the anchor date.

- `Start`, the start of the accrual period for the CMS rate. This is also the fixing date for the CMS rate. Can be specified as a date or period.

- `Maturity`, the end accrual date for the CMS rate. This is also the payment date as the CMS swaplet pays-in-arrears. Can be specified as a date or period.

- `CmsDayCountBasis`, the day count basis used for the CMS accrual period.

- `CmsTenor`, the length of the underlying swap. Should be specified as a tenor in years (e.g. 1Y-30Y).

- `DayRule`, the day rule used to roll the `Start` and `Maturity` dates for the CMS accrual period if they are specified as periods.

- `Sigma`, the σ_0 parameter for a SABR model calibrated to the fixing time corresponding to the `Start` date and the `CmsTenor`.

- `Alpha`, the SABR α parameter.

- `Epsilon`, the SABR ϵ parameter.

- `Rho`, the SABR ρ parameter.

In addition to these, the functions takes the by now well-known inputs that specifies the forward and discounting curves as well as the interpolation method. Note the absence of conventions that specifies the fixed and floating legs of the underlying swap. These have instead been hard-coded into the function for the sake of brevity. The conventions have been fixed to 6M floating leg (Act/360) and 1Y fixed leg (30/360). Taking a step back, why is a SABR model required as input to the `fidCmsSwapletPv` function? Since the convexity gap is closed by taking positions in swaptions across the entire smile, the pricing of CMS products is quite sensitive to the volatility smile. To correctly price and risk manage CMS products, we thus need a good model of the volatility smile. In fact, the popularity of CMS-linked structured products have created an actively traded market for payer swaptions with strikes in the 10-20% range — even though it is extremely hard to justify any real expectations of rates rising that high for the foreseeable future.

5.10.3 Applying the CMS pricing results

To price a full CMS swap rather that just a single CMS swaplet, we simply need to set up a schedule of accrual dates and calculate the sum of the corresponding CMS swaplet PVs. If we compare these to the value of simply projecting the forward swap rates, we can calculate a *convexity adjusted forward swap rate*. This enables us to conveniently define the convexity correction as a difference in rates (when comparing this adjusted rate to the forward swap rate) rather than difference in present values.[49] Mathematically, we thus have

$$\text{Convexity Correction}_T = \frac{\text{CMS Swaplet PV}(t, T, S)}{\delta P(t, T)} - R(t, T, T + S) \qquad (5.40)$$

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[49]In Hagan (2003), the convexity correction is defined as a present value.

This correction is increasing in both T and S as a longer time-to-fixing (T) means that the replicating swaptions are more expensive and a longer CMS tenor (S) introduces more convexity relative to the length of the accrual period (implying larger notional weights in the replication scheme). An example of the convexity correction can be seen figure 25. The figure shows the correction resulting from two different volatility smile specifications. Model 1 is thus equivalent to a Black (1976) model with a flat smile of $\sigma_{\text{Black'76}} = 15\%$ while Model 2 is SABR with a pronounced smile ($\alpha = 50\%$, $\epsilon = 30\%$ and $\rho = 0\%$) recalibrated to yield the same ATM volatility for all the underlying swaptions. We see that the convexity correction is significant in size and that the smile handling matters quite a lot as well.

Figure 25: The convexity correction 30Y CMS rates using either a flat smile with a vol of 15% or a full smile adjustment with σ_0 re-calibrated to yield a ATM vol of 15%. Yield curves are flat at 3%.
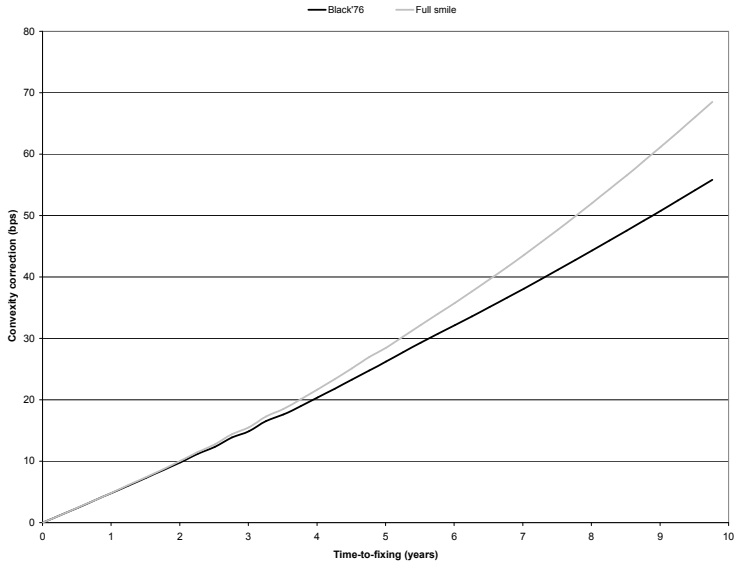
Returning to the pricing of the CMS swap, we want to find the spread that — when applied to the xIBOR leg — causes the CMS swap to have zero NPV. As can be seen in table 20, the can easily be done together with `fidGenerateSchedule`, `fidFloatingPv` as well as `fidAnnuityPv`.

Table 20: CMS spread for 10Y swap on 30Y CMS fixing using 3M accrual on Act/360. Calculated for flat yield curves at 3% and a flat smile of 15%.

PV xIBOR	-25,932,179
PV CMS	28,781,851
Net	2,849,671
Pv of 1 bps	87,313
CMS Spread	32.6

5.10.4 The risk profile in a CMS swap

As previously mentioned, the CMS swap is fundamentally an exchange of short-dated xIBOR rates against long-dated swap rates. Therefore, the primary risk factor to a CMS swap is the curve slope. If the curve steepens, the (convexity adjusted) forward swap rates will increase relative to the forward xIBOR rates, thus increasing the rate differentials exchanged in the CMS swap. Importantly, the CMS swap will be sensitive to the curve far beyond its own maturity (the last fixing in a standard 5Y EUR CMS swap on the 30Y tenor, depends on the 30Y swap rate in 5Y-3M=57M time). CMS swaps are thus important building blocks for structured products with payoffs that are linked to the slope of the yield curve.

While the CMS swap is very sensitive towards changes to the curve slope, it is relatively immune to parallel shifts to the forward and discounting curves. Again, the intuition is that the CMS swap exchanges rate differentials that are not influenced by parallel shifts.

Finally — as previously discussed — the CMS swap will be sensitive towards swaption volatilities across the entire smile for all the underlying swaptions. The counterparty who receives (pays) the CMS fixing will thus benefit from increases (decreases) to implied volatility and is thus long (short) vega. Because of the influence on CMS spreads via deep out-of-money swaptions, some dealers actually use CMS spreads to calibrate their SABR models as the CMS spreads puts restrictions on the extrapolation of the volatility smile.

6   Credit derivatives

Since the mid 1990s a derivatives market for credit risk has arisen. Although the market has shrunk substantially in size since the breakout of the financial crisis, it remains a tremendously important market. Furthermore, the market can — and should — be seen together with the market for traditional credit sensitive cash bonds.

Formally, we say that *credit risk* is the risk that the value of a financial contract changes due to an unexpected change in credit quality.[50] That is, the risk that some entity *defaults* — ie. that it ceases to meet its contractual obligations. While the notion of default is often used somewhat interchangeably with concept of *bankruptcy*, the two are not the same. Default is wider concept that includes breaches of covenants, delays in payments and other more or less technical failures. Bankruptcy, on the other hand is a specific legal construction that allows debtors to seek protection from their creditors.

In the credit derivatives market contracts are traded whose pay-offs are linked to the credit worthiness of corporations or governments. As with other markets there exists a class of linear credit derivatives as well as more advanced products such a credit options and basket derivatives. In this course we will focus on the linear derivatives and in the process also address the pricing of cash bonds. A good starting point for students wanting learn more about the pricing of credit risk is Giesecke (2004), which the below section on intensity models borrows heavily from.

6.1   Asset swaps

6.1.1   Introducing asset swaps

While the credit derivatives market is a fairly new market, it is important to note that credit risk has been actively traded for many years. It has for example been possible to buy and sell otherwise similar bonds that traded at markedly different yields because of differences in credit quality among different bond issuers. If two otherwise identical bonds issued by two different entities are trading at different prices, it is typically a reflection of the fact that the market is factoring in different likelihoods of default.

It turns out that we can use our existing tools developed for pricing and hedging interest rate swaps in combination with traditional cash bonds to address differences in credit risk.

An *asset swap* is a customized interest rate swap where the coupons paid on the one leg is structured to match an existing asset. This leg is called the *asset leg*. The other leg is typically just a plain vanilla xIBOR with a spread called the *asset swap spread* or simply *ASW spread*. This leg is called the *funding leg*. As an example, we could structure a fixed-for-float asset swap such that the fixed asset leg matches an Italian government bond and calculate some spread above or below xIBOR that matches the value of the asset leg coupons. The basic construction is sketched in figure 26.

Importantly, asset swaps are legally isolated from the underlying asset — you still have to make and receive payments in the asset swap even if the underlying asset defaults. This means that the party paying the asset leg passes on all other risk factors (primarily interest rate risk) while retaining the credit risk.[51]

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[50]This definition is given in Duffie & Singleton (2003) which is an excellent reference on the topic.

[51]Importantly, the coupons can be "exotic", e.g. variable coupons that have embedded optionality (as is the case for capped floating rate mortgage bonds) or are linked to a consumer price index (such as inflation indexed bonds). The former is fairly widely used in the Danish mortgage bond market.

The basic idea when pricing asset swaps (ie. finding the spread that makes the swap fair) is a simple yet powerful one. Since the swap curve is readily available for a wide range of maturities it is a good reference curve. Also, compared to bond based yield curves the swap curve has the benefit of being generic. This is unlike bonds where specific bonds from the same issuer may have differences in their pricing because of differences in, say, issue size, tax treatments and other technical factors.

When we price asset swaps, we simply evaluate the credit quality of some given asset relative to the credit premium in the swap curve. For this reason ASW spreads can be negative, if the underlying bond represents a better credit quality than the unsecured interbank credit quality embedded in the xIBOR fixings and thus the swap curve. Intuitively, the more expensive a given bond is, the lower will the ASW spread be and vice versa. While ASW spreads are often interpreted as a credit measure, it is more correct to view the spreads as the product of the price residual on a given bond once differences in coupon and maturity has been accounted for.

As many bond issuers have issued multiple bonds of varying maturity it is often customary to plot their ASW spreads as a function of time. Such a collection of ASW spreads and maturities is often referred to as an *asset swap curve*.

Asset swaps are used both as a trading strategy where an asset swap and the underlying is traded as a package and as a relative value measure. That is, even market participants who do not actively trade asset swaps can use ASW spreads to make judgements on which bonds to buy or sell.

6.1.2 Par-par asset swaps

The cleanest type of asset swap is called a *par-par asset swap* and is sometimes also referred to as the *true asset swap*. Since bonds of varying credit quality will trade at different prices it can be difficult to compare the spread on asset swap packages that require different upfront cash outlays. Because of this, the par-par asset swap package is constructed such that the investor pays par for the entire package by offsetting any discount or premium to par via the swap. As an example, consider the package outlined in figure 26. The par-par asset swap package has the benefit that the investor can now

Figure 26: Par-par asset swap of a bond.
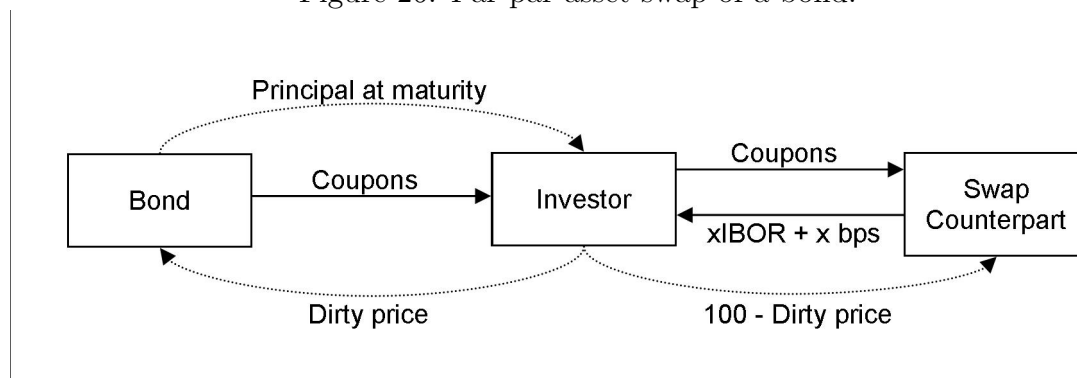
directly compare how much she is paid above (or below) some reference xIBOR rate by making an investment of USD 100. The ASW spread is thus a measure of the excess return generated by taking on the credit risk - provided that the credit quality remains unchanged. If the spread widens — perhaps as a result of a deterioration of credit — the

investor will experience a mark-to-market loss on the position and vice versa if the spread tightens. An investor who has bought the bond in an asset swap package is receiving the ASW spread and thus positioned for this spread to fall (just like a receiver in a plain vanilla IRS is positioned for rates drop). The sensitivity towards such changes in credit quality can be measured by the annuity value of the floating leg (i.e. what is the value of an extra basis point paid on the funding leg?). Note, that we can easily extend the par-par asset swap package to cover different currencies — ie. the investor can compare which return a USD 100 investment yields by asset swapping Swedish mortgage bonds, Italian government bonds or German corporate bonds. Since our swap curve setup sets the correct relative price of liquidity in different currencies via the discounting curve's dependence on CCS spreads, we automatically obtain the correct alternative cost of liquidity.

If the bond is trading above par, investor is borrowing money from the swap counter-party to buy the bond. This loan is paid back over time as the coupons on the asset leg is worth more than the expected cash flows on the funding leg. Oppositely, if the bond is trading below par, the investor is placing money with the swap counter part. This implicit deposit is also paid back over time as the funding leg is worth more than the asset leg. If this loan or placement becomes large — the bond is trading far above or below par — then the asset swap itself can pose a credit risk. To avoid taking on too much counterparty credit risk — and because of issues relating to the discounting of large net cash flows, many market participants prefer trading a another type of asset swap — the *yield-to-maturity asset swap*.

6.1.3 Yield-to-maturity asset swaps

The yield-to-maturity (YTM) asset swap is simply an odd-dated plain vanilla IRS. That is, the maturity date of the IRS is matched to the maturity date of the bond (why the construction is sometimes also referred to as the match-maturity swap). The start date is set according to the standard convention for the particular swap market e.g. 2B for EUR swaps. The YTM asset swap spread is defined as the difference between the bond's yield-to-maturity and the fixed rate in the swap.

Whereas the par-par asset swap is theoretically clean in the sense, that the par-par ASW spread exactly tells the investor how much pick-up she will earn over some xIBOR rate per fixing, the YTM ASW spread is less clean because of the mismatches in cash flow. Effectively, the YTM asset swap is more of a hedging strategy than a theoretically sound relative value measure. The lack of upfront payments is however preferred in the market place and it turns out that YTM and par-par ASW spreads are fairly close to each for standard fixed rate bullet bonds that are trading relatively close to par. A final note on YTM asset swaps is that the mismatch between the fixed coupon on the bond and the fixed rate in the YTM asset swap means that the investor obtains some interest rate risk that the investor potentially needs to address.

6.1.4 Other uses of asset swaps

Very importantly, the methodologies used for asset swaps can be used to evaluate funding strategies. For many corporations and financial institutions it is important to obtain a diversified funding base.[52] In order to appeal to different investor preferences, companies

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[52]One of the compounding effects of the financial crisis was the overreliance on certain types of funding. The previously mentioned ABCP market was a funding source that many structured investment vehicles relied on. When the ABCP market seized up, these vehicles found themselves unable to attract funding

often choose to issue both fixed and floating rate debt and furthermore do so in different currencies. But why would a domestically focused Danish industrial company ever want to issue, say, a fixed rate JPY bond? Since many investors — asset managers in particular — for various technical reasons are not active users of derivatives, bond issuers can use derivatives to design more or less specialized bonds to attract certain investors. One such example could be the JPY bond above. If a specific group of Japanese investors are interested in buying bonds issued by the particular company in question (they believe in the soundness of the credit), the company can offer them a JPY denominated bond and use a CCS to convert the bond into a DKK loan. In that case, the funding manager of the company would use the asset swap methodology to evaluate whether to issue fixed or floating bonds in e.g. EUR or JPY. It is therefore customary also to talk about *liability swaps*. Corporations with large funding programmes and all larger financial institutions thus constantly compare what they — through the use of these liability swaps — will effectively pay to borrow money over different time horizons. They will then have a *funding curve* against a single benchmark currency and xIBOR rate and use the CCS markets to calculate funding targets in other currencies.

6.2 Modeling credit risk

Fundamentally speaking, there are two approaches to modeling credit risk: The *structural* and the *reduced form* approach. The structural models build on the Merton (1974) model and extensions of this. They attempt to model how the asset value of a firm evolves over time and compare this to the debt structure of the company to identify whether a default occurs or not. Since it turns out that the credit risk on a company's debt can be priced via a put option on the firm's assets, the structural model approach is sometimes referred to as *option-based modeling*.

The structural models presents an economic model of what causes defaults. Although these models can provide guidance on possible drivers of credit risk, they are not tractable when it comes to pricing credit sensitive securities as they are typically dependent on several unobservable inputs.

The market preference is thus to use reduced form models to price and hedge credit risk. This class of models have arisen from the work of Jarrow & Turnbull (1995), Jarrow, Lando & Turnbull (1997), Duffie & Singleton (1999) and Lando (1998). The basic idea is to give up the economic model default and instead focus on a probabilistic model of default. That is, we no longer concern ourselves with what exactly causes default but only try to model the probability of such an event occurring. As we will see below, reduced form model rely probabilistic tools from point processes which are governed by intensities — the modeling approach is thus also known as *intensity-based modeling*.

6.3 Intensity models

When working with intensity models to price credit sensitive securities, we are interested in calculating probabilities of default over different time horizons. Just as we have seen before, these probabilities will be calculated under some risk neutral probability measure. Letting τ denote the random default time of some *name*, we define the point process N_t

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form other sources.

by[53]

$$N_t = 1_{\{\tau \le t\}} \tag{6.1}$$

N_t thus simply indicates whether the name has defaulted up until time t (by counting the number of default events that have arrived). By postulating a model for the evolution of N_t we create different distributions of τ which in turn will result in different prices for the credit sensitive security we are looking at.

We will model N_t using the well-known concept of *intensities*. Letting $Q(\tau \le t)$ denote the *probability of default*, the intensity λ_t is given by

$$\lambda_t = \lim_{h \downarrow 0} \frac{1}{h} Q(\tau \le t + h | t < \tau) \tag{6.2}$$

That is, conditional on no default having taken place up until t, $\lambda_t \cdot \Delta t$ is the probability that a default will happen over the interval $(t, t + \Delta t]$. As such, any non-negative process for λ_t will do. In practice, three different specifications are used:

- **Constant λ.** In this case, N_t is then a *homogenous Poisson process* with intensity λ and τ is exponentially distributed. We can then write the (risk neutral) time T default probability as

$$Q(\tau \le T) = 1 - \exp(-\lambda T) \tag{6.3}$$

- **Deterministic, time-varying $\lambda(t)$.** This specification implies that N_t is an *inhomogeneous Poisson process* and the default probability is given by

$$Q(\tau \le T) = 1 - \exp\left(- \int_0^T \lambda(u) du \right) \tag{6.4}$$

- **Stochastic λ_t.** This is the most general specification under which N_t is known as a *Cox-* or *doubly stochastic process*. The default probability can be written as

$$Q(\tau \le T) = 1 - E_0^Q \left[\exp\left(- \int_0^T \lambda_u du \right) \right] \tag{6.5}$$

The deterministic but time-varying specification is sufficient for our purposes since we are only looking to price linear credit derivatives, but for more advanced products we would need the Cox processes. Note that when $Q(\tau \le T)$ denotes the probability of default then $1 - Q(\tau \le T) = Q(T < \tau)$ must necessarily define the *probability of survival*.

Now, we are ready to introduce the pricing of an important building block when modeling credit risk — the credit risky zero coupon without recovery. By $B(t, T)$ we denote the time t price of the risky zero coupon bond that pays 1 if $T < \tau$ and 0 otherwise (if the name defaults before T). It turns out that pricing $B(t, T)$ is surprisingly simple since it is shown in Lando (1998) (in the most general Cox process case) that

$$
\begin{aligned}
B(t, T) &= E_t^Q \left[\exp\left(- \int_t^T r_s ds \right) \mathbf{1}_{\{T < \tau\}} \right] \\
&= E_t^Q \left[\exp\left(- \int_t^T r_s ds \right) \exp\left(- \int_t^T \lambda_s ds \right) \right] \\
&= E_t^Q \left[\exp\left(- \int_t^T (r_s + \lambda_s) ds \right) \right]
\end{aligned} \tag{6.6}
$$

---

[53]In the industry jargon, a *name* is an entity on which one or more credit securities have been written.

Since $P(t,T) = E_t^Q[\exp(-\int_t^T r_s ds)]$, we can see that we can actually think of the intensity as being a spread that is added to the risk-free rate r. As such, it turns out that intensity-based modeling is very similar to traditional yield curve modeling. Under the simplifying assumption — which we will use in subsequent sections — that the evolution of interest rates and default intensities are independent, we furthermore have that:

$$
\begin{aligned}
B(t,T) &= E_t^Q\left[\exp\left(-\int_t^T (r_s + \lambda_s)ds\right)\right] \\
&= E_t^Q\left[\exp\left(-\int_t^T r_s ds\right)\right]E_t^Q\left[\exp\left(-\int_t^T \lambda_s)ds\right)\right] \\
&= P(t,T)Q(T < \tau)
\end{aligned}
\tag{6.7}
$$

Note that under the assumption of zero recovery, $B(t,T)$ is the price of the security that pays out one unit of currency if no default have occurred. On the other hand, this must mean that the price of the complement contingent claim $\widetilde{B}(t,T)$ that pays out one unit of currency only if a default has occurred must be $\widetilde{B}(t,T) = P(t,T)Q(\tau \leq T)$. Armed with pricing results for these two securities (under the simplifying assumption of independence between discounting rates and default intensities), we are ready to price the most important credit derivative — the credit default swap.

6.4 Credit Default Swaps

The *Credit Default Swap* or simply *CDS* is an insurance like contract in which the credit risk on given reference security is transferred in exchange of fixed, periodic payments.[54] The CDS consists of a *fee leg*, where the *protection buyer* pays a fixed periodic premium to the *protection seller* until the contract matures or a *credit event* takes place. The other leg in the CDS is the *protection leg* on which the protection seller pays the *Loss Given Default* or simply *LGD* to the protection buyer if a credit event takes place. Note that per definition the *Recovery Rate* (RR) on some security is defined as $RR = 1 - LGD$. To conclude the specification of the CDS contract, counter parts needs to agree on what constitutes a credit event and how to determine the LGD payment.

Writing broadly accepted documentation that helps market participants agree on which corporate events counts as a credit event has actually been a hard legal task. While it is easy to agree that bankruptcy (i.e. legal protection from the creditors) and failure to pay constitute so-called *hard credit events*, it is more difficult to agree on what constitutes a *soft credit event*. Currently, CDS contracts traded in Europe treat various types of debt restructurings such as reduction/postponement of interest/principal, increased subordination and change to non-permitted currency as soft credit events.[55]

Should a credit event take place, CDS contracts can be settled in two ways: *Physical-* or *Cash settlement*. In the case of physical settlement, the protection buyer is obligated

---

[54]Although the CDS contract is economically equivalent to an insurance contract, it is legally not considered an insurance contract. The primary reason for this, is that insurance can typically only be bought on assets that you already own (where you can demonstrate an insurable interest). Since you enter into CDS contracts on either the long or short side without owning the underlying reference securities, they are not considered insurance.

[55]CDS contracts traded in the US does not need to consider restructurings as such events would typically take place in under Chapter 11 (which already constitutes a credit event).

to deliver the reference security (or any bond that ranks pari-passu with this).[56] Against this, the protection seller is obligated to pay par value for the delivered bonds. On a net basis, the protection buyer thus receives the difference between par and the bond value — which is the LGD. While physical settlement was standard in the early days of the CDS market, the fact that the notional outstanding on CDS contracts can be many times the outstanding notional of bond issued by a given name has proven problematic. In fact, there has been several cases where the price of a defaulted entity's bonds sharply increased upon default as the result of a *short squeeze* among protection buyers who did not own the underlying bonds. In the case of cash settlement, an auction process is set up to determine the post-default price of the reference securities.

6.4.1   The CDS big-bang of 2009

In early 2009 the standard conventions for trading CDS contracts in both Europe and North America were changed. Since the nature of the changes were rather significant, they are referred to as the *ISDA CDS big-bang*. Because of the dramatic growth in credit derivatives market participants wanted to strengthen the underlying infrastructure of the market. The three most important changes were

- The formation of determination committees. The purpose of these committees is to decide when a credit event has taken place.

- Hardwiring of the cash settlement. Following the big-bang, all CDS contracts were converted into cash settlement.

- Fixed coupons. Instead of trading CDSs with different CDS spreads, contracts now trade with standardized coupons against upfront cash payments. The market still communicate prices as spreads, but settle actual trades by paying an upfront fee to enter into a contract with a fixed (off-market) coupon.

6.4.2   Pricing and risk managing CDS contracts

If we make the simplifying assumption that default can only take place on coupon dates, the valuation of the CDS contract is relatively simple. We make this simplification to avoid having to deal with fractional coupons (if a default happens half way through a coupon period, the coupon is paid in half and the protection leg is settled). Finally, we will limit ourselves to the case of stepwise constant $\lambda(t)$ (a special case of the deterministic, time-varying $\lambda(t)$ that is particularly easy integrate) as this is sufficient to price CDSs. Note that, we could in principle have chosen many different interpolation methods between a set of intensity knot points to construct an intensity curve. The problem is however that we both need ensure that the intensity curve remains positive (which can be a problem for downward sloping CDS curves) and that we want to avoid the computationally intensive problem of integrating arbitrary interpolation methods.

Starting with fee leg, we can now write the value of the CDS starting at time T_S and

---

[56]The fact that multiple bonds can be delivered into the CDS contract, introduces a so-called *delivery option*. Obviously, the protection buyer will deliver the cheapest possible bond. This option is typically not modeled explicitly when working with credit derivatives.

maturing at time T_E paying the spread C as

$$\text{Fee leg PV}_t = C \cdot \underbrace{\sum_{S+1}^{E} \delta_i P(t, T_i) Q(T_i < \tau)}_{\text{Risky annuity PV}} \tag{6.8}$$

The protection leg is at first glance a bit more tricky, since it involves both the stochastic LGD payment and the stochastic default time. The market standard approach to pricing this is however extremely simple. We simply postulate a fixed recovery rate of 40% (implying a LGD of 60%) and adjust the intensities to match the market prices. This can be done as the intensities represent risk neutral and not physical probabilities, what matters is basically the *expected loss* which is a product of probability of default and LGD.

$$\text{Protection leg PV}_t = (1 - RR) \cdot \sum_{i=S+1}^{E} P(t, T_i) Q(T_{i-1} \leq \tau < T_i) \tag{6.9}$$

Finally, we define the *par CDS spread G* as the spread that causes the fee- and protection leg to have the same value.

$$G = \frac{LGD \cdot \sum_{S+1}^{E} P(t, T_i)[\exp(-\int_t^{T_{i-1}} \lambda(u)du) - \exp(-\int_t^{T_i} \lambda(u)du)]}{\sum_{S+1}^{E} \delta_i P(t, T_i) \exp(-\int_t^{T_i} \lambda(u)du)} \tag{6.10}$$

As can be be seen, pricing CDS contracts it simply a question of being able to calculate survival (or default-) probabilities by integrating an intensity curve over time.

As it is typically possible to trade CDS with several maturities, we will typically be able to observe several market quotes for the par CDS spreads on a given name. Just like have seen for IRSs, we will then want to calibrate intensity curve in order to be able to price a full *CDS curve* with contracts of any maturity.

When marking CDS contracts to market, we can use the same methodology as we saw for IRSs, that is, we can price the annuity of the difference between the par spread and the coupon on the specific contract. We thus close out a position by entering into the offsetting contract where the two protection legs net out, leaving just an annuity (positive or negative) on the fee leg. Since this annuity payment stream will stop if the underlying name defaults, we will need to discount the coupon difference by the risky annuity PV (see (6.8)). Notably, a trader who has closed out her position by entering into the opposite contract rather than terminating the trade will thus still have some sensitivity towards the underlying name's intensity curve. Note also, that the valuation of the risky annuity is what enables us to switch between quoting spreads and settling new trades with upfront payments against fixed coupons (cf. the CDS Big-Bang).

6.4.3 Credit derivatives in fidAnalytics

As we saw in the section above, the basic building block when pricing CDS contracts is the survival probability $Q(T < \tau) = \exp(-\int_0^T \lambda(u)du)$ which requires us to integrate intensity curve (remember that we are only looking deterministic, time-varying λ specification). Such a function has been implemented in fidAnalytics as `fidSurvivalProb`. This function takes an `AnchorDate`, a `MaturityDate` (it only allows for dates), a set of `IntensityMaturities` knot points (a vector of dates) and finally a set of `Intensities`

knot points (a vector of intensity values). Because we explicitly limit ourselves to constant interpolation, we will not need an interpolation method input to the function.

Building on top on of `fidSurvivalProb` are the functions

- `fidRiskyAnnuityPv`

- `fidProtectionLegPv`

- `fidParCdsSpread`

- `fidCdsPv`

which all take the "usual" schedule related inputs as well as inputs for the discounting curve and intensity curve. The only new inputs are the `Recovery` (the recovery rate in percent) for the last three functions and the `TypeFlag` input (`Buyer` or `Seller`) for the `fidCdsPv` function.

6.4.4 Comparing CDS- and ASW spreads

As previously discussed, the ASW spread can be interpreted as the spread over xIBOR that an investor can obtain by assuming the credit risk of a bond. Now, since the CDS spread is a direct contract on assuming the credit risk on a given bond issuer, it seems natural that the two spreads are related. In fact, for a while it was wrongly argued that any divergence of the two spreads constituted close to true arbitrage. The argument went that if an investor could place funds — without assuming additional credit risk — at xIBOR, then she could synthetically replicate an ASW package by selling protection (earning the CDS spread).

Today, it is widely recognized that the two spreads should not be identical. In fact, for many bonds the *Cash-CDS basis* (the difference between the par-par ASW spread and the CDS spread of a matched maturity CDS contract) was traded at several hundred basis points during the peaks of the financial crisis. Although a detailed overview of the possible drivers of the cash-CDS basis is beyond the scope of this course, important reasons are:

- **Funding liquidity:** In order to buy a bond, investors need to pay cash upfront. In a climate where liquidity is scarce (and it is difficult to repo finance the purchase of the bond), investors will require a higher expected return by buying the cash bond relative to selling protection via the CDS. Also, a bond purchase will come onto the investor's balance sheet while the CDS position is kept off-balance sheet.

- **Counter party risk:** While the counter party credit risk in a collateralised asset swap is limited (rates change relatively slowly over time) the counter party risk in CDS contracts can be quite significant as the value can change rapidly when a default occurs (credit risk is more "jumpy" in nature). This phenomenon will dampen the expected return for investors selling protection (as protection buyers are only willing to pay a smaller spread because of the counter party credit risk).

- **Technical default and loss of coupons:** The CDS contract pays off under a set of legally specified terms. This means that even if the value of the cash bond is unchanged by some event, this event could potentially trigger a so called technical default. Such event could potentially broaden the scenarios where the protection pays off. Imagine for instance that a default event has occurred because of a missed

payment that is settled just a single day late. Let us assume that the bond is trading at par and its post default value is unchanged (since it was just a technical default). Had you bought bond and asset swapped it, you will will continue to to receive coupons and you have had no loss of principal. Had you instead written protection on the underlying name, you are now required to pay out the LGD (which is zero since the bond is unchaged at par) — but you will no longer receive the CDS spread. Receiving the ASW spread would in this case be worth more, why the CDS spread should be above the ASW spread.

7 Risk management

The last section is a brief introduction to some important topics in modern risk management. A easy to read introduction to many of the topics covered below can be found in Hull (2007).

7.1 Documentation

So far we have looked at derivative contracts from a relatively theoretical point of view. In practice, a foundation for well functioning derivatives markets is a solid legal documentation base. Much of the legal work address questions such as which parties should do what and when in a particular transaction and which fall-backs are provided if e.g. fixings are no longer available or payments cannot be processed. However, two important standard documents have ramifications for pricing and risk management and they therefore warrant a brief introduction here.

Typically, OTC derivatives between two counterparties are traded under an *ISDA Master Agreement*. Such a master agreement (or a similar contract done under local law) is a legal umbrella under which all the derivative transactions between the counterparties are governed. Importantly, the ISDA master agreement ensures that if one party defaults the entire portfolio of derivatives can be netted against each other. This *netting principle* actually goes against standard bankruptcy law under which an estate administrator is free distribute funds to pay off individual claims. As such, it is extremely important since it guarantees that a counterparty cannot choose to default on just the contracts where she is out-of-the-money and still require payment in full on the contracts where she is in-the-money (what is referred to as *cherry picking*). The existence of an enforceable netting agreement means that the counterparty credit risk present in derivatives transactions should be addressed on the *portfolio level* against each counterparty and not on the *trade level*.

To further reduce counterparty credit risk it is common to supplement the ISDA master agreement with a *Credit Support Annex (CSA)*.[57] This is a standardized legal document developed by ISDA that facilitates the pledging of collateral against the net present value of the all the derivatives transactions traded under the ISDA master. The idea is that if a default should occur, the collateral can be netted against the value of the derivatives portfolio. Although the broader legal framework for collateralization is standardized in the CSA, important economic features are bilaterally negotiated making the documentation very flexible. Counterparties will typically negotiate factors like:

- **Threshold amount.** Some CSAs provide either one or both counter parties with a blanco credit in the form of a threshold — collateral is only posted for a negative market value in excess of the threshold. Some extremely credit worthy counterparties like central banks and sovereigns have historically required infinite threshold on their part. This implies single sided CSAs where only their counterparty will post collateral. The standard between most counter parties is however double sided CSAs with zero thresholds.

- **Eligible collateral.** Which collateral will be accepted in the CSA? This will typically be cash and government bonds in one or more specified currencies.

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[57]There also exist a similar document known as a *Credit Support Deed*, while the two are legally different they address the same underlying economic principles, and it is thus common to collectively refer to either type of document as a CSA.

- **Collateral interest.** If cash is posted as collateral it will earn interest (typically an Over/Night rate).

- **Frequency.** How often is the market value of the collateral and derivatives portfolio calculated. In the interbank market daily collateral calls are standard.

- **Independent amount.** For some counter parties of limited credit worthiness it can common to require the pledging of an amount that is independent of the value of the derivatives portfolio. A bank would typically require this from, say, a leveraged hedge fund that the bank considers risky. This can be seen as an equivalent to the initial margin on an exchange.

- **Minimum transfer amount** To avoid the operational annoyance of transferring minuscule amounts, counter parts often agree on a MTA.

The exact specifications of the CSA thus have implications for the counterparty credit risk embedded in derivatives transactions. Note that while CSA agreements are quite common among institutional market participants (banks, asset managers, insurance companies and pension funds) it is fairly rare that banks have CSA agreements with the corporate clients. Aside from the fact that corporations do typically not have an operational setup to facilitate margin calls, they are also lacking the liquidity to continuously post collateral.[58]

7.2 Counterparty credit risk

The pricing results shown in previous sections implicitly assume that all cash flows in the derivative transactions will be paid in full. However, as has been proven repeatedly this will not always be the case. When trading derivatives we should thus be concerned with the choice of counterparty — two otherwise identical IRSs traded with, say, JP Morgan and Lehman Brothers did ex post turn out to have quite different values. To correctly assess and manage the risk of default among counterparties, financial institutions try to model, price and even hedge the credit risk embedded in their derivatives transactions. This is done via the so called Credit Value Adjustments (CVA).

While modeling of counterparty credit risk is an inherently different task and is beyond the scope of this text, we will develop a bit of intuition on the topic. First, let us consider a derivative trade with a time T value of V_T. We can think of the credit exposure of this contract as being $\max(V_T, 0)$, that is, we should only be concerned with the credit risk if we are owed money on the trade.[59] We will however not so much be interested in what our exposure is today (where the derivative's PV is known) but rather be concerned about its possible future values and the corresponding future exposure. Typically, risk managers thus operate with the concept of *Potential Future Exposure* (PFE) and *Expected Positive Exposure* (EPE). This is typically estimated by working out some percentile from the distribution of future values. A typical PFE profile for a 10Y IRS and CCS is sketched in figure 27 as a function of time. What is the intuition behind these PFE profiles? We can think of the future value of the IRS as being the difference between the fixed rate and the prevailing future par swap rate multiplied by the future annuity. For an IRS traded at market, the initial PFE must necessarily be zero. Likewise, the swap must also

---

[58]Many corporations have fairly "lumpy" cash flow profiles and rely on the revolving credit facilities offered by banks to smooth this. It will therefore not make sense for a bank to require collateral postings from a corporation if the implication is that it will just lead to an increased draw on its credit facility.

[59]Note that intuitively, the credit risk thus seems to be related an option on the derivative contract.

have zero value once it matures. In between these two extremes the possible difference in coupons will increase over time, while the annuity value will decrease as maturity comes closer. This explains the hump shape. As the floating- and fixed leg payment frequencies are typically not identical, the hump will have a jagged behavior. For the CCS it is the exchange of notional at maturity that drives the PFE profile. As the contract has FX exposure on the full notional, calculating the PFE is simply a question of how far FX rates can move between the start date and the maturity date of the CCS.

Figure 27: The credit exposure for a 10Y IRS and a 10Y CCS.
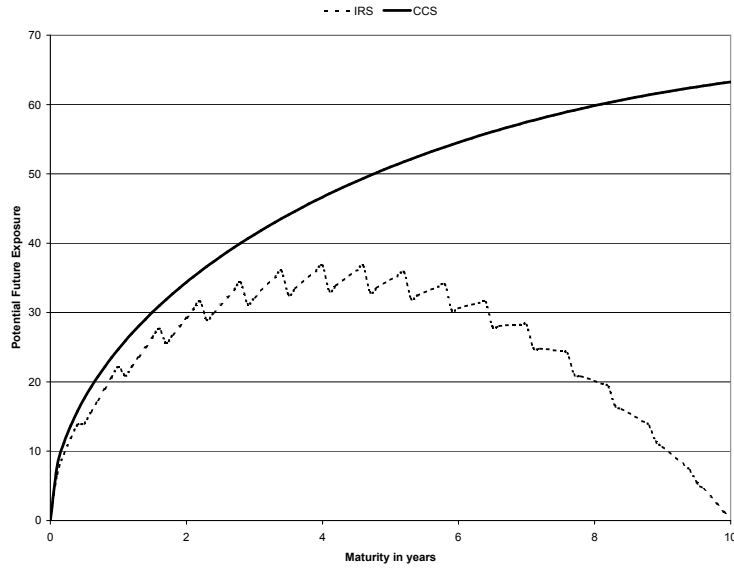
Typically, we will have done multiple trades against a given counter party. In that case, our credit exposure will be $\max(\sum V_T^i, 0)$ assuming that a netting contract is in place. This makes the PFE estimation tricky, since we basically need to model the joint distribution of the future value of all the derivative contracts. If these contracts are moreover written on a number of different asset classes (e.g. interest rates, FX, equities, commodities and inflation) it becomes clear that we are looking at a problem that is sensitive towards correlations. Finally, if a CSA agreement is in place the credit exposure would become $\max([\sum V_T^i] - C_T, 0)$ where C_T is the collateral value at time T. This adds another layer of difficulty as there is a potential need to model the value process of the collateral (remember that this can be securities that can be credit sensitive on their own).

7.3 Value at Risk

As the number of risk factors on a given trading desk increases, it can be difficult to consolidate risk reports via the bump and re-run approach. In particular, it can difficult to assess the overall riskiness of a position by looking just partial risk figures.[60] A commonly used aggregate risk measure is the so-called *Value-at-Risk* or simply *VaR*. Denoting the — stochastic — time T profit and loss by X_T, the VaR at the safety level k is implicitly defined by

$$P(X_T < \text{VaR}_k) = 1 - k \tag{7.1}$$

---

[60]Remember how the interplay between implied volatilities and interest rates complicated the delta risk figures in the Black (1976) model.

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Intuitively, the VaR number provides a lower boundary for our profit and loss that we can expect to keep above 95% of the time. Equivalently, we must then also expect our profit and loss to breach this level 5% of the time. It is important to note that the VaR number does not tell us what happens once this threshold is breached. In order to provide a risk measure for rare events, it is therefore customary to supplement the VaR figure with an *Expected Shortfall* (ES) calculation. ES is defined as

$$\text{Expected Shortfall}(k) = E[X_T | X_T < \text{VaR}_k) \tag{7.2}$$

and is thus used to capture tail risks. It is thus the expected loss once the VaR_k safety level has been breached. The VaR and ES measures are graphically illustrated in figure 28. Note that both VaR and ES are typically calculated under physical rather than risk neutral probabilities. As such VaR and ES is typically estimated by looking at historical data. Estimates can be constructed either by revaluing positions on full sets of market data and calculating the change in value from today's value. Alternatively, the measures can be estimated by multiplying a set of, say, historically observed market rate changes to a delta vector risk report and use this estimate to construct a profit and loss distribution.

Figure 28: Value-at-Risk and Expected shortfall at a 95% safety level.
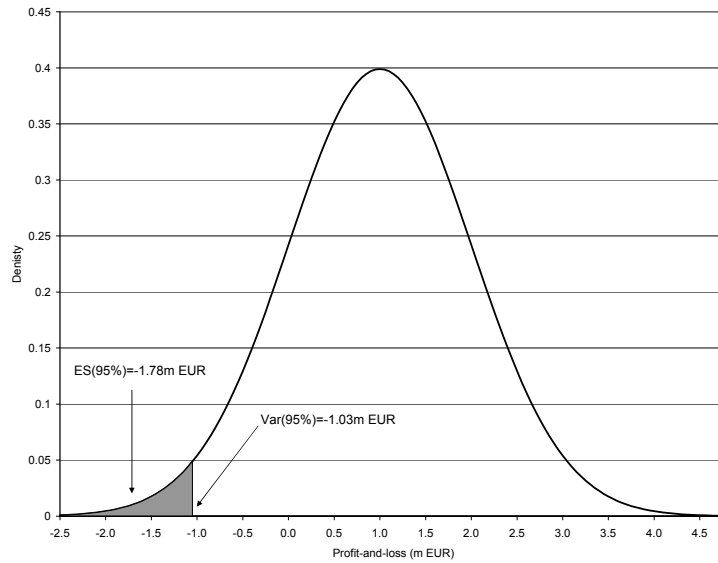
When calculating VaR, it is typical to look at safety levels ranging from 95% to 99.5% and consider time horizons ranging from one day to one year. Such measures are often at the heart of the internal models that banks and pension funds use for their risk budgeting and their regulatory solvency calculations.

8 Further reading

These notes are meant to serve as an introduction to the pricing and risk management of some of the most widely used fixed income derivatives. In the above sections we have in general terms looked at derivatives with linear- and European payoffs written on single underlyings. The natural extension for students wanting to learn more about fixed income derivatives would be to expand into path-dependent payoffs such as Bermudan swaptions or options with ratchet features or into multi underlying products such CMS spread options. The commonality among such products is that we would need more advanced modeling — in particular full term structure models — to deal with them. Students are again encouraged to read in particular Björk (2004) in order to prepare themselves with the additional mathematical tools that are required for more advanced modeling. Again, the books of Andersen & Piterbarg (2010b) and Andersen & Piterbarg (2010c) are excellent references for not only advanced modeling of interest rate derivatives but also for an extensive catalogue of exotic products. Finally, a growing area of both theoretical and applied research is CVA. This field combines some of the pricing methodologies for in particular interest rate- and credit derivatives to asses how to price and risk manage the credit risk inherent on any OTC transaction. For an introduction to CVA, a good source is Gregroy (2010)

A   Deriving the Carr formula

Restating the derivation given in Carr (2005), we can for any continuously twice differentiable function f write

$$
\begin{aligned}
f(X) =& f(\kappa) + \mathbf{1}_{\{X>\kappa\}} \int_\kappa^X f'(u)du - \mathbf{1}_{\{X<\kappa\}} \int_X^\kappa f'(u)du \\
=& f(\kappa) + \mathbf{1}_{\{X>\kappa\}} \int_\kappa^X \left[f'(\kappa) + \int_\kappa^u f''(v)dv \right] du \\
& - \mathbf{1}_{\{X<\kappa\}} \int_X^\kappa \left[f'(\kappa) - \int_u^\kappa f''(v)dv \right] du
\end{aligned}
$$

Since $f'(\kappa)$ is independent of u, we can easily integrate it out. Subsequently, we can use Foubini's theorem (change of the integration order for double integrals) to rewrite the above as

$$
\begin{aligned}
f(X) =& f(\kappa) + f'(\kappa)(X - \kappa) + \mathbf{1}_{\{X>\kappa\}} \int_\kappa^X \int_v^X f''(v)dudv \\
& + \mathbf{1}_{\{X<\kappa\}} \int_X^\kappa \int_X^v f''(v)dudv
\end{aligned}
$$

Now, integrating over u (since $f''(v)$ is a constant in the inner integral) we get

$$
\begin{aligned}
f(X) =& f(\kappa) + f'(\kappa)(X - \kappa) + \mathbf{1}_{\{X>\kappa\}} \int_\kappa^X f''(v)(X - v)dv \\
& + \mathbf{1}_{\{X<\kappa\}} \int_X^\kappa f''(v)(v - X)dv \\
=& f(\kappa) + f'(\kappa)(X - \kappa) + \int_\kappa^X f''(v)(X - v)^+dv \\
& + \int_X^\kappa f''(v)(v - X)^+dv
\end{aligned}
$$

which is exactly equation (5.33) - Q.E.D.

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