UNIVERSITY OF COPENHAGEN
FACULTY OF SCIENCE

Master thesis in Mathematics-Economics

Nanna Ingemann Ohrt

Swaptions pricing

Advisor: Rolf Poulsen

Submitted: May 31th 2024

Abstract

Contents

1 Introduction

In this thesis we will investigate swaptions pricing.

2 Swaptions as a missing link in asset allocation

Look at pdf form Noamura

3 Mathematics of pricing swaptions

To determine swaptions prices, it is important to understand which things there affects the price of the swaption. This chapter simplifies these concepts by explaining interest rates, bonds, swaps, and options, and then shows how they come together to determine the price of a swaption.

3.1 Time value of money

Understanding the concept of interest rates begins with the fundamental idea that a dollar today holds more value than the same dollar in the future. To understand these concept, a discount factor is introduce

$$B(t,T) = \text{value at time t of a dollar received at time T}$$

$B(t,T)$ refer to a contract that pays one dollar maturity, T, which can be illustrated as below

$$t < T \rightarrow B(t,T) < 1$$
$$t = T \rightarrow B(t,T) = 1$$

The concept "time value of money" it asserts that the value of money today is worth more than the same amount in the future due to its potential earning capacity, inflation, and risk. This principle underpins various financial decisions, including investing, borrowing, and pricing financial instruments. Essentially, it recognizes that a dollar received today can be invested and earn interest over time, thereby increasing its value. Conversely, a dollar received in the future is subject to uncertainty and may not retain its purchasing power due to inflation or other factors. The discount factor represents the present value of future cash flows, taking into account the time value of money. It reflects the idea that receiving a certain amount of money in the future is less valuable than receiving the same amount today.

3.2 Zero coupon bonds

One of the most common applications of the concept "time value of money" is zero coupon bonds. By there construction the mechanism of "time value of money" is present. This instrument have the common property that they provide the owner with a deterministic cash flow.

Definition 1. *A zero coupon bond with maturity data T, also called a T-bons, is a contract which guarantees the holder one dollar to be paid on the date T. The price at time t of a bond with maturity data T is denoted by $p(t,T)$ [1]*

3.3 The yield curve

Where the concept "time value of money" and the discount factor are fundamental concepts used to assess the present value of future cash flows, while the yield curve provides insights into market expectations regarding future interest rates. Understanding the interplay between these concepts is crucial for making informed investment decisions and pricing financial instruments. The yield curve is a graphical representation illustrating the interest rates (bond yields) for various maturities. Yield curve can provide a intuition about future interest rates and give insight in the bond market today. The general intuition is that longer-term rates is higher then short-term rates, which in other words means that a lager premium is expect for lending money over a longer period of time. This case sketches a yield cure with a positive slope.

3.4 Interest rates

3.4.1 Spot rates

The spot rate represents the yield-to-maturity of a zero coupon bond, while the forward rate refers to the anticipated interest rate in the future. The definition for determined spot rates is listed below

Definition 2. *The simple spot rate for* $[S, T]$*, henceforth referred to as the LIBOR spot rate, is defined as [1]*

$$L(t; S, T) = -\frac{p(t, T) - p(t, S)}{(T - S)p(t, T)}$$

3.4.2 Forward rates

Forward rates play a crucial role in financial markets, particularly in the realm of interest rate analysis and derivative pricing. They represent the interest rate applicable to a future period, agreed upon today. Understanding forward rates requires grasping the concept of forward contracts and the expectations theory of interest rates. Forward rates can be derived from the yield curve. The yield curve plots the yields of bonds with different maturities. By analyzing the yield curve, one can infer the implied forward rates for future periods. For example, the forward rate between year 1 and year 2 is the rate at which an investor can borrow or lend money for the period between year 1 and year 2, starting at year 1.

Lets consider three time points on the yield curve $t = 0, 1, 2$, where it is assumed that $t_0 < t_1 < t_2$. At time t_0 we have the spot rates $p(t_0, t_1)$ and $p(t_1, t_2)$, which represent the yields for bonds maturing at time t_1 and t_2 respectively. Hence the forward rate, $R(t_1, t_2)$, can med determined using the equation below

$$R(t_1, t_2) = \frac{(1 + p(t_0, t_2))^2}{(1 + p(t_0, t_1))} - 1$$

Imagine investing one dollar in a one-year zero-coupon bond, $B(t_0, t_1)$, and instantly reinvesting the money received at time t_1 in a new one-year zero-coupon bond, $B(t_1, t_2)$, at rate $R(t_1, t_2)$. This strategy should yield the same return as investing one dollar in a two-year zero coupon bond $B(t_0, t_2)$ and holding it for two years. This strategy illustrated the idea of forward rates. Let us then look a the general formula for forward rates.

Definition 3. *The continuously compounded forward rate for* $[S, T]$ *contracted at* t *is defined as [1]*

$$R(t; S, T) = -\frac{\log p(t, T) - \log p(t, S)}{(T - S)}$$

3.5 Financial derivatives

3.5.1 Bonds

A bond is a debt security, like a loan. Borrowers issue bonds to raise money from investors willing to lend them money for a certain amount of time. When you purchase a bond you are lending money to the issuer, which in some cases is a government or company. In return, from the construction of the bond, the issuer guarantees to pay a predetermined rate during the term of the bond and repay the principal at maturity.

Earlier a zero coupon bond has ben introduced and when talk about bonds, a zero coupon bond is the simplest representation of a bond. The zero coupon bond contract is only given by two cash flows. One for the buyer, there pays the issuer at time t $= t_0$, and another where the buyer receives the principal at time t $=$ T. Unlike other types of bonds, a zero coupon bond does not offer periodic interest payments (coupons) throughout its term. [1]

The price of a zero coupon bond is represented as p(t,T), where an individual lends an amount, K, with the intention of earning a return in the future. Therefore, the price of a zero coupon bond, with its principal (also known as face value) K, at time t and with maturity T, is denoted as.

$$p(t, T) = B(t, T) \cdot K$$

3.5.2 Fixed Coupon Bonds

As describe a zero coupon bond does not involve coupons throughout the term of the bond. But moving forward we will introduce various bond with coupon there are either fixed or floating. First we will consider the simplest form of a coupon bond, which is a fixed coupon bond. Fixed coupon bonds are a type of debt security that offers investors a predictable return in the form of regular interest payments, known as coupons, until the bond's maturies. These coupons are set at a fixed rate at the time of issuance, based on the bond's face value, and are typically paid annually or semi-annually. Upon reaching maturity, the issuer repays the principal amount (face value) to the issuer, concluding the bond contract. The purpose of a fixed coupon bond is there ability to provide a steady stream of income, making them an attractive option for conservative investors seeking to minimize risk and secure predictable returns.

Continuing we will compute the price of a fixed coupon bond. First we note that the fixed coupon bond, can be replicated by holding a portfolio consisting of zero coupon bond with maturities T_i, for $i = 1, ..., n$.

So we will hold c_i zero coupon bonds of maturities T_i for $i = 1, ..., n-1$, and $K + c_n$ bonds with maturity T_n. Hence we have that the price, p(t), at time t, where $t < T$, of the fixed coupon bonds becomes. [1]

$$p(t) = K \cdot p(t, T_n) + \sum_{i=1}^{n} c_i \cdot p(t, T_i)$$

When taking about coupons, there are typically determined in terms of return than in monetary terms. So the return of the i'th coupon is denoted as a simple rate acting in the face value K, over the time period $[t_{i-1}, T_i]$. So for the i'th coupon the return is equal to r_i, and the face value is K, hence we have that

$$c_i = r_i(T_i - T_{i-1})K$$

Where for standardized coupon, the time intervals will be equally spaced, which means that

$$T_i = T_0 + i\delta$$

This also means the the coupon rates $r_1, ..., r_n$ will be equal to a common coupon rate r. Hence the price p(t,T) of a fixed coupon bond where $t \leq T_1$ will be determined as below [1]

$$p(t) = K\Big(p(t, T_n) + r\delta \sum_{i=1}^{n} \cdot p(t, T_i)\Big)$$

3.5.3 Floating Rate Bonds

Now a short introduction to fixed coupon bonds has be given, as mentioned there are also many type of bonds there have floating coupon. When it is listed that there are bonds there have floating coupon, what there is really said is that the rate is floating. So with the fixed coupon bond, the coupon was at predetermined when the agreement was made. But there are also bond, where the coupon is reset for every coupon period. These types of bond is referred to as floating rate bonds. The most simple floating rate bond, is where the coupon rate r_i is set to the spot LIBOR rate $L(T_{i-1}, T_i)$. Thus we have that

$$c_i = (T_i - T_{i-1})L(T_{i-1}, T_i)K$$

Here we have that $L(T_{i-1}, T_i)$ is determined at time T_{i-1}, but the coupon is first delivered at time T_i. [1]

The LIBOR rate stands for London InterBank Offered Rate, which is a rate the the British Bankers Association sets every business day. Like the LIBOR rate, there is many types of xIBOR rates, one that we we encounter later is EURIBOR rate. Which is a rate the European Banking Federation sets every business day.

These different type of xIBOR rates are sets differently, but they all use the money market convention. So when taking about business day, the money market convention is important. This is a day-count convention is a standardized methodology for calculating the number of days between two dates. This means that when $t < T_0$ the coupon dates are equally spaced with

$$\delta = T_i - T_{i-1}$$

To determined the value of a the simplest floating rate bond, the LIBOR spot rate we can without loss is generality assume that K=1 and insert the Definition 2 of the LIBOR spot rate to obtain

$$\begin{aligned} c_i &= (T_i - T_{i-1})L(T_{i-1}, T_i)K \\ &= \delta L(T_{i-1}, T_i) \\ &= \frac{1 - p(T_{i-1}, T_i)}{\delta p(T_{i-1}, T_i)} = \frac{1}{p(T_{i-1}, T_i)} - 1 \end{aligned}$$

This lead to a formula for the floating rate bond, which is listed below [1]

$$p(t) = p(t, T_n) + \sum_{i=1}^{n} \Big[p(t, T_{i-1}) - p(t, T_i) \Big] = p(t, T_0)$$

where we note that if $t = T_0$ we get that $p(T_0) = 1$

This leads to some general assumption there guarantee the existence of a sufficiently rich and regular bond market

Assumption 1. *We assume the following*

- *There exists a (frictionless) market for T-bond for every T>0*
- *The relation p(t,t)= 1 holds for all t*
- *For each fixed t, the bond price p(t,T) is differentiable w.r.t time of maturity T [1]*

3.6 Interest rate swaps

Now some simple cases of different type of bonds has be introduce. Then we will combined the knowledge we have gained to move on to take interest rate derivatives in consideration. Again we will consider the simplest type of a interest rate derivative, which is a interest rate swap. The construction of a interest rate swap is that there is exchange a payment stream of a fixed rate of interest, which is know as the swap rate. This fixed rate is exchange for some floating rate, such as the LIBOR rate or the EURIBOR rate. As mentioned the fixed rate is know as the swap rate, this swap rate is determined from forward rate extracted from the yield curve, so it makes the present value of the swap equal to zero. This we will formulate this later.

As stated in the interest rate swap, two cash flow are exchanged, where one of is a fixed cash flow and the other is a floating cash flow. These components of the interest rate swap are known ad the "fixed leg" and the "floating leg". The role of each participant in the swap is determined in relation to the fixed leg: the party making fixed payments is engaged in a "payer swap," while the party making floating payments (and receiving fixed payments) is involved in a "receiver swap."

Again we have that K is the principal also know as the face value and we will denote the swap rate, R.

Further we have that the payments arises at the dates $T_1, ..., T_n$, this means that at time T_i buyer of the interest rate swap will pay

$$K\delta L(T_{i-1}, T_i) \tag{1}$$

where we have that $L(T_{i-1}, T_i)$ is the spot rate, which could be the LIBOR spot rate or the EURIBOR spot rate. It is also assumed that the days $T_0, ..., T_n$ is equally spaced this $\delta = T_i - T_{i-1}$ as mentioned above in the section for floating rate bonds. The it is noticed the expression in Equation 1 is the same as Kc_i, where again c_i is the i'th coupon for the floating rate. So at time T_i the buyer will pay $K\delta R$, where the cash flow at time T_i is given by below

$$K\delta\Big[L(T_{i-1}, T_i) - R\Big]$$

Then by applying the results from the section for floating rate bonds again, we are able to compute the value of the cash flow at time $t < T_0$. The value of the cash flow is listed below

$$Kp(t, T_{i-1}) - K(1 + \delta R)p(t, T_i)$$

Hence we have that the total value denote by $\Pi(t)$, so the total value at time t of the swap is given as below

$$\pi(t) = K\sum_{i=1}^{n}\Big[p(t, T_{i-1}) - (1 + \delta R)p(t, T_i)\Big] \tag{2}$$

Moving forward we simplify Equation 2 in the below Proposition 1 [1]

Proposition 1. *The price, for $t < T_0$, of the swap in Equation 2 above is given by*

$$\Pi(t) = Kp(t, T_0) - K\sum_{i=1}^{n} d_i p(t, T_i)$$

where

$$d_i = R\delta, \quad i = 1, ..., n-1$$
$$d_n = 1 + R\delta$$

3.7 Options

3.8 Swaptions

4 SABR Implied Volatility and Option Prices

Look at The SABR model

4.1 Process for the forward rate

4.2 The SABR model

4.3 Estimating Parameters

5 Data and the Volatility Risk Premium

Look at Broekmans

5.1 Data

5.2 The volatility Risk Premium

References

[1] Björk, Arbitrage Theory in Continuous Time, Oxford, fourth edition, 2020