# Master Thesis

## Swaption Pricing Using the SABR Model

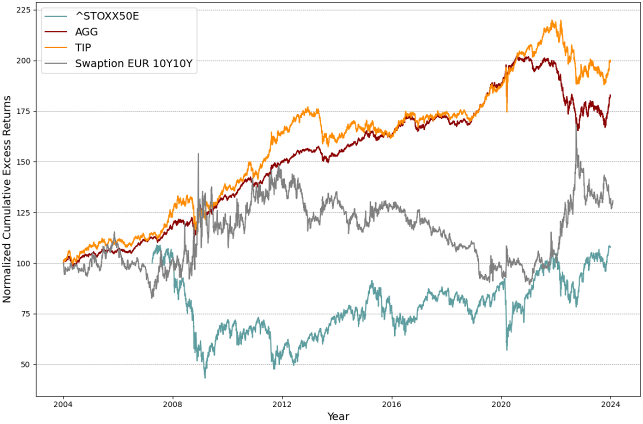
Nanna Ingemann Ohrt

UNIVERSITY OF COPENHAGEN

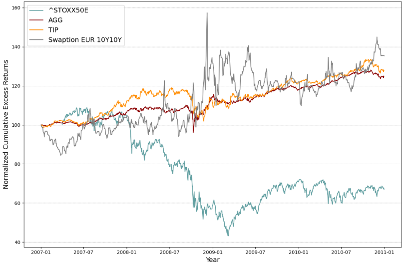
# Outline

- Swaption As a Missing Link in Asset Allocation
- Mathematics of Pricing Swaptions
- One-Factor Short-Rate Model
- Constant Volatility
- The SABR Model
- Risk Related to The SABR Model
- Conclusion

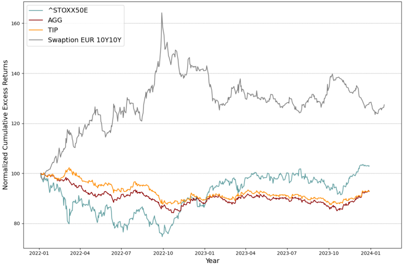
# Swaption As a Missing Link in Asset Allocation



# Swaption As a Missing Link in Asset Allocation




Global Financial Crisis




High inflation period

# Mathematics of Pricing Swaptions

A swaption is a financial derivative that can be described as an option to exchange a fixed rate bond for floating rate bonds for a predetermined principal. There are two types of swaptions, payer swaptions and receiver swaptions. A payer swaption gives the holder the right to pay a fixed interest rate and receive a floating rate, similar to a call option in the stock market. On the other hand, a receiver swaption allows the holder to pay a floating interest rate and receive a fixed rate, resembling a put option.

# Mathematics of Pricing Swaptions

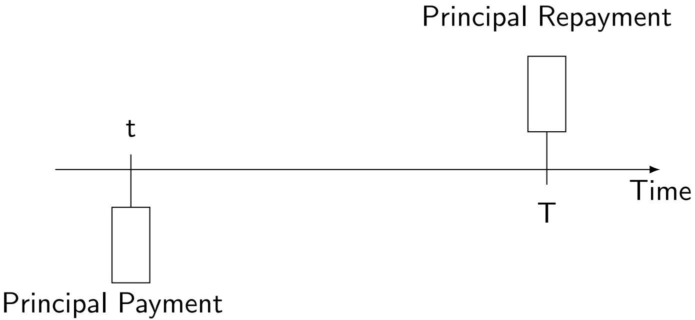



Illustration 3.1: Cashflow for a zero coupon bond

# Mathematics of Pricing Swaptions

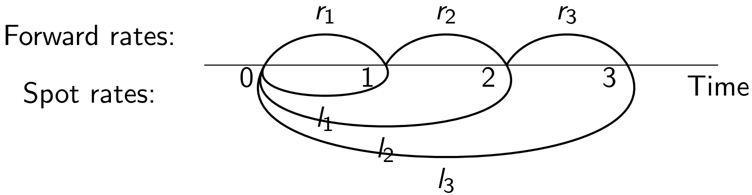



Illustration 3.3: Forward and spot rates

# Mathematics of Pricing Swaptions

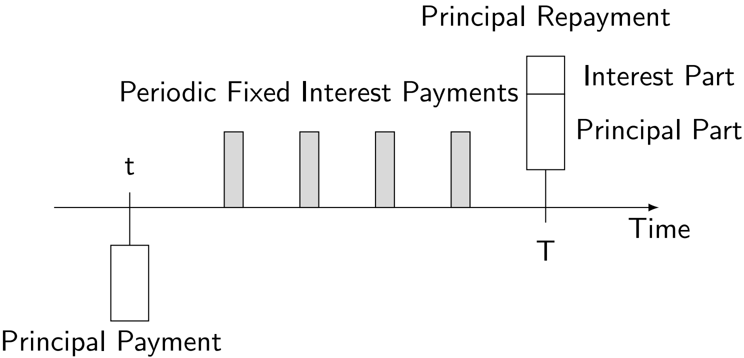



Illustration 3.4: Cashflow for a fixed coupon bond

# Mathematics of Pricing Swaptions

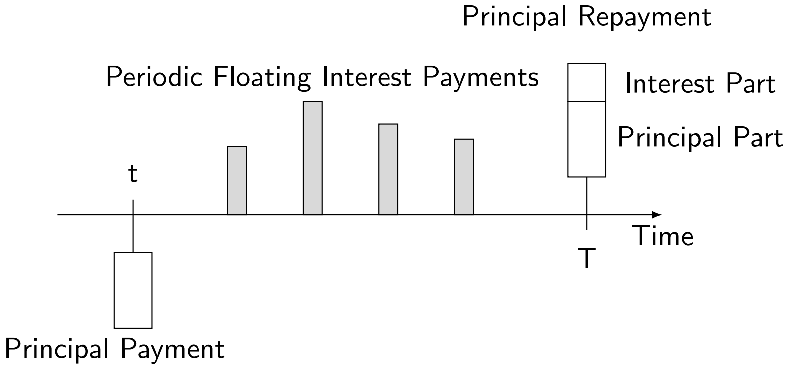



Illustration 3.5: Cashflow for a floating rate bond

# Mathematics of Pricing Swaptions

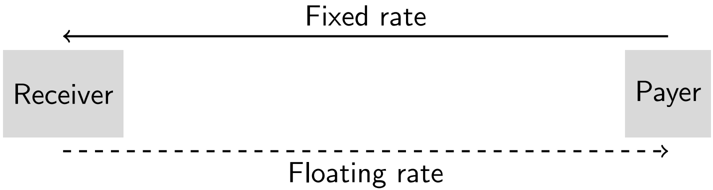



Illustration 3.6: Cashflow for fixed and floating rate exchanges

# Mathematics of Pricing Swaptions

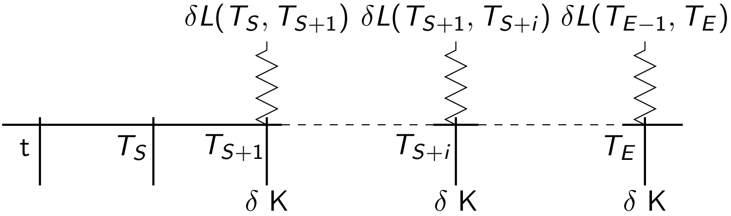



Illustration 3.7: Cashflow for a payer swap

# One-Factor Short-Rate Model

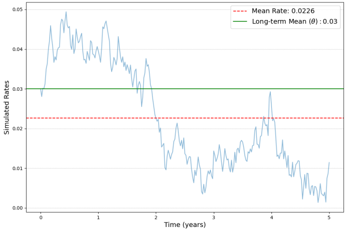
The Vasicek model

$$dr_t = \kappa\Big[\theta - r(t)\Big]dt + \sigma dW(t) \tag{1}$$

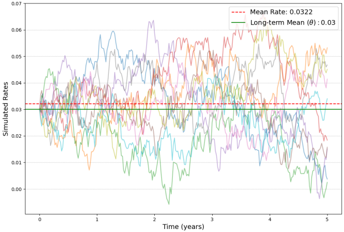
$$r(0) = r_0 \tag{2}$$

The parameters in the short rate dynamic $\kappa$, $\theta$ and $\sigma$ are positive constants. Where $\kappa$ represent the mean reversion speed, $\theta$ is the long-term average rate, $\sigma$ is the volatility and $W(t)$ is a Wiener process.

# One-Factor Short-Rate Model

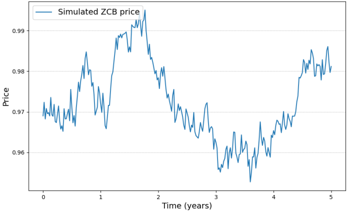



Plot of one simulated rate path using the Vasicek model.

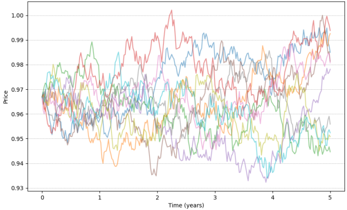



Plot of 10 simulated rates paths using the Vasicek model.

# One-Factor Short-Rate Model

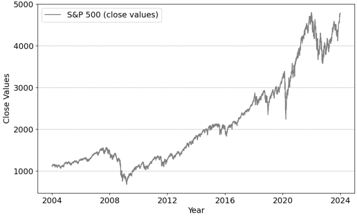


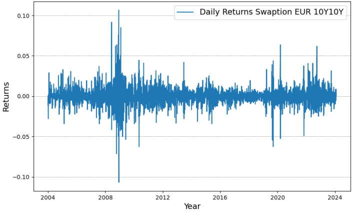

Plot of one simulated ZCB price path using the Vasicek model.

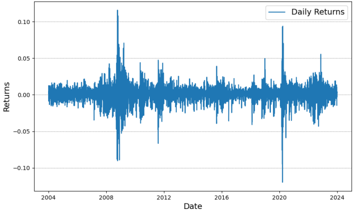



Plot of 10 simulated ZCB prices paths using the Vasicek model.

# Constant Volatility

# The SABR Model

The SABR model consists of a dynamic for the forward price and one for the volatility, since the SABR model is a two-factor model. The SABR model also formulates how the two processes are correlated.

$$d\hat{F}_t = \hat{\alpha}_t \hat{F}_t^{\beta} dW_t^1, \qquad \hat{F}(0) = f \tag{3}$$

$$d\hat{\alpha}_t = \nu \hat{\alpha}_t dW_t^2, \qquad \hat{\alpha}(0) = \alpha \tag{4}$$

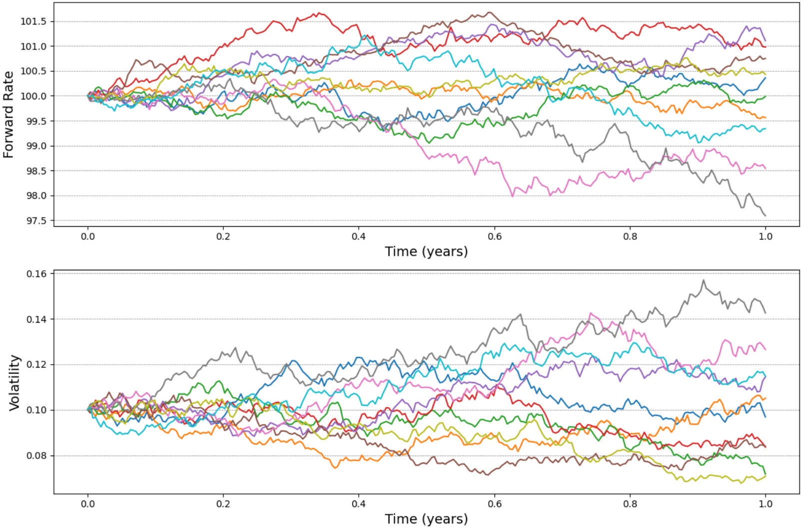
where $W_t^1$ and $W_t^2$ are correlated Wiener process and it is assumed that.

$$dW_t^1 dW_t^2 = \rho dt \tag{5}$$

# The SABR Model

- $F_0$ — Initial forward rate or asset price.
- $\alpha_0$ — Initial volatility.
- $\beta$ — Elasticity parameter.
- $\nu$ — Volatility of the volatility parameter.
- $\rho$ — Correlation between the asset price and its volatility.

# The SABR Model




Ten simulated paths for the forward rate and the volatility in the SABR model.

# The SABR Model

The paper Managing Smile Risk of Hagen (2002) states that under the SABR model, the prices of European options is given by Black formula

$$V_{\text{call}} = D(t_{\text{set}})fN(d_1) - KN(d_2) \tag{6}$$

$$V_{\text{put}} = V_{\text{call}} + D(t_{\text{set}})[K - f] \tag{7}$$

with

$$d_{1,2} = \frac{\log \frac{f}{K} \pm \frac{1}{2}\sigma_B^2 t_{\text{ex}}}{\sigma_B \sqrt{t_{\text{ex}}}} \tag{8}$$

here $t_{\text{set}}$ is the settlement date and $t_{\text{ex}}$ os the exercise date

# The SABR Model

The implied volatility $\sigma_B(f, K)$ is given by

$$\sigma_B(K, f) = \frac{\alpha}{(fK)^{(1-\beta)/2}} \left\{ 1 + \frac{(1-\beta)^2}{24} \log^2 \frac{f}{K} + \frac{(1-\beta)^4}{1920} \log^4 \frac{f}{K} + \dots \right\} \left( \frac{z}{x(z)} \right). \tag{9}$$

where

$$z = \frac{\nu}{\alpha} (fK)^{(1-\beta)/2} \log \frac{f}{K}, \tag{10}$$

$$\tag{11}$$

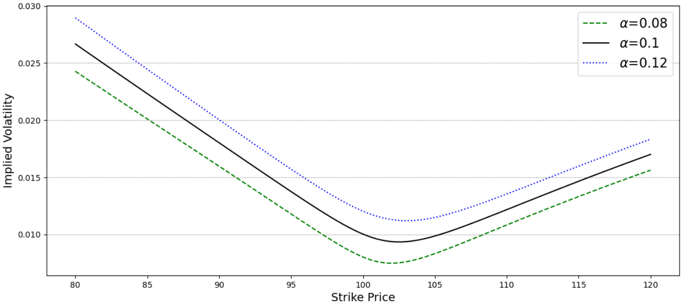
and $x(z)$ is defined by

$$x(z) = \log \left\{ \frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1 - \rho} \right\}. \tag{12}$$

For the special case of at-the-money (ATM) options, options strike at $K = f$, this formula reduces to

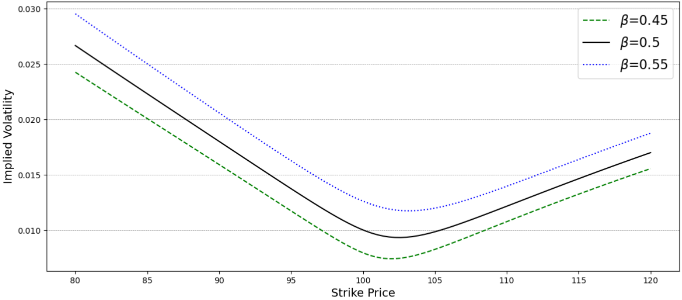
$$\sigma_{ATM} = \sigma_B(f, f) = \frac{\alpha}{f^{1-\beta}} \left\{ 1 + \left( \frac{(1-\beta)^2}{24} \frac{\alpha^2}{f^{2-2\beta}} + \frac{\rho\beta\nu}{4} \frac{\alpha}{f^{1-\beta}} + \frac{2 - 3\rho^2}{24} \nu^2 \right) t_{\text{ex}} + \dots \right\}. \tag{13}$$

# The SABR Model



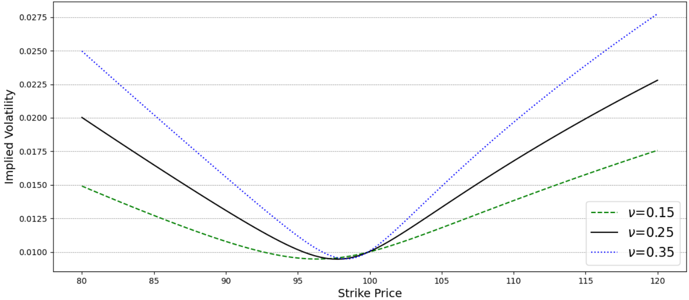

SABR model volatility smiles at various $\alpha_0$ levels

# The SABR Model



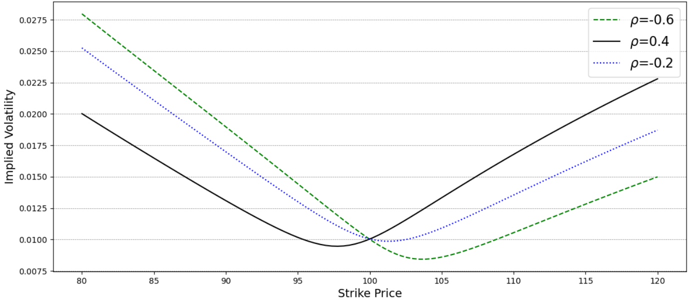

SABR model volatility smiles at various $\beta$ levels

# The SABR Model



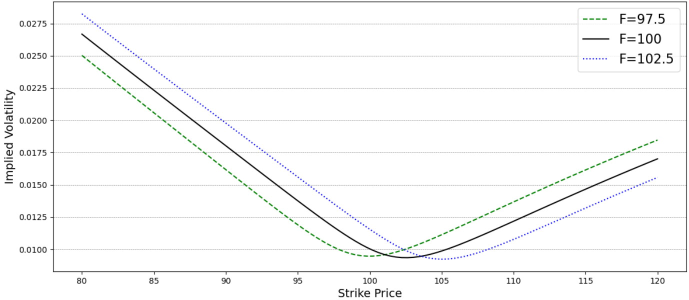

SABR model volatility smiles at various $\nu$ levels

# The SABR Model




SABR model volatility smiles at various $\rho$ levels

# The SABR Model




SABR model volatility smiles at various F levels

# Risk Related to The SABR Model

**Changes in $\alpha$ parameter:**

- Affects the level of the implied volatility smile.
- Higher $\alpha$ values increase the overall level of implied volatility.
- Lower $\alpha$ values decrease the overall level of implied volatility.

**Fixing $\beta$ parameter:**

- Common practice since similar movements occur when changing both $\beta$ and $\nu$ parameters.

**Effect of $\beta$ on the implied volatility smile:**

- Lower $\beta$ values make the smile steeper (more skewed).
- Higher $\beta$ values flatten the smile.

# Risk Related to The SABR Model

**Effect of $\nu$ parameter:**

- Higher $\nu$ values increase the curvature of the smile, making it more pronounced.
- Lower $\nu$ values decrease the curvature of the smile.

**Effect of $\rho$ parameter:**

- Positive $\rho$ values skew the smile to the right (towards higher strikes).
- Negative $\rho$ values skew the smile to the left (towards lower strikes).

**Effect of the forward price, $f$:**

- Adjusts the center of the implied volatility smile.

**Importance:**

- Awareness of these movements in implied volatility is crucial if swaptions are included in asset allocation.

# Conclusion

- Volatility is not constant.
- Model selection impacts swaption prices.
- Chose a two-factor model to handle stochastic volatility.
- Understanding risks of the SABR model is important.
- Swaption is a missing link i asset allocation.