UNIVERSITY OF COPENHAGEN
FACULTY OF SCIENCE

# Master thesis in Mathematics-Economics

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# Swaptions pricing

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**Submitted: May 31th 2024**

# Abstract

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# 1 Introduction

# 2 Swaptions as a missing link in asset allocation

# 3 Mathematics of pricing swaptions

To determine a swaption price it is important to understand what affects the price of the swaption. This chapter simplifies this concepts by explaining interest rates, bonds, swaps-and options, and then shows how they come together to determine the price of a swaption.

## 3.1 Time Value Of Money

Understanding the concept of interest rates begins with the fundamental idea that a dollar today holds more value than the same dollar in the future. To understand this concept, a discount factor is introduced as

$$B(t, T) = \text{value at time t of a dollar received at time T}$$

$B(t, T)$ refer to a contract that pays one dollar at maturity, T, which can be illustrated as below

$$t < T \rightarrow B(t, T) < 1$$
$$t = T \rightarrow B(t, T) = 1$$

The concept "Time Value of Money" asserts that the value of a dollar today is worth more than the same amount in the future due to its potential earning capacity and inflation. The "Time Value Of Money" concept underpins various financial decisions, such as investing, borrowing, and pricing financial instruments. Essentially, it recognizes that a dollar received today can be invested and earn interest over time, thereby increasing its value. Conversely, a dollar received in the future is subject to uncertainty and may not retain its purchasing power due to inflation or other factors. The discount factor represents the present value of future cash flows, taking into account the time value of money. It reflects the idea that receiving a certain amount of money in the future is less valuable than receiving the same amount today.

## 3.2 Zero Coupon Bonds

One of the most common applications of the concept "Time Value Of Money" is zero coupon bonds. By construction, the mechanism of "Time Value Of Money" is present. This instrument have the common property of providing the owner with a deterministic (future) cash flow.

**Definition 1.** *A zero coupon bond with maturity date T, also called a T-bond, is a contract which guarantees the holder one dollar to be paid at date T. The price at date t of a bond with maturity date T is denoted by $p(t, T)$. [1]*

Before moving forward we will look at the cashflow for a zero coupon bond. The illustration below shows that a time t, the principal payment is made at the price $P(t, T)$ and at maturity T the principal is repaid.

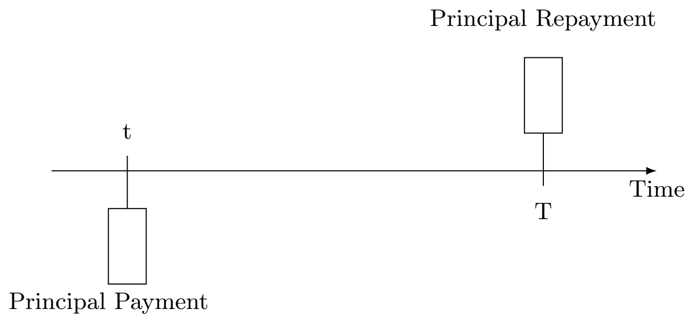



Illustration 1: Cashflow for a zero coupon bond

## 3.3 The Yield Curve

Where the concept "Time Value Of Money" and the discount factor are fundamental concepts used to assess the present value of future cash flows, the yield curve provides insights into market expectations regarding future interest rates. Understanding the interplay between these concepts is crucial for making informed investment decisions and pricing financial instruments. The yield curve is a graphical representation illustrating the interest rates (bond yields) for various maturities. Yield curves provides information about future interest rates and gives insight in the bond market today. The general intuition is that longer-term rates is higher than short-term rates, which in other words means that a larger premium is expected for lending money over a longer period of time. This case sketches a yield curve with a positive slope, which is illustrated below.

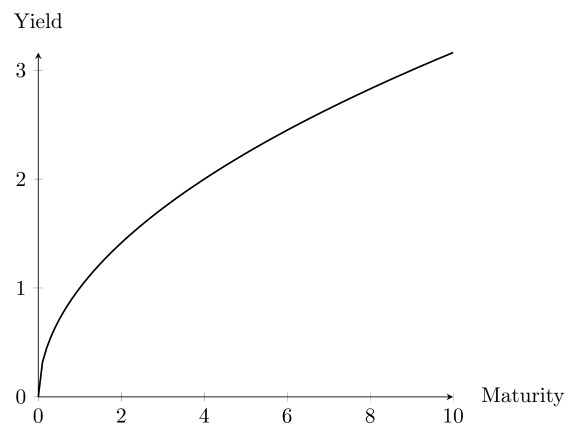



Illustration 2: Yield curve with a positive slope

## 3.4 Interest Rates

### 3.4.1 Spot Rates

The spot rate represents the yield-to-maturity of a zero coupon bond, while the forward rate refers to the anticipated interest rate in the future. The definition for determined spot rates as follows below

**Definition 2.** *The simple spot rate for $t < S < T$, henceforth referred to as the LIBOR spot rate, is defined as [1]*

$$L(t; S, T) = -\frac{p(t,T) - p(t,S)}{(T-S)p(t,T)}$$

where $p(t,T)$ and $p(t,S)$ represent the price at time t of zero coupon bonds, that pay 1 dollar at time T and S, respectively. Intuitively, the spot rate $L(t;S,T)$ is the average rate of interest for borrowing or lending money over the time period from S to T, where such borrowing or lending is made at time t.

### 3.4.2 Forward rates

Forward rates play a crucial role in financial markets, particularly in the realm of interest rate analysis and derivative pricing. They represent the interest rate applicable to a future period, agreed upon today. Understanding forward rates requires grasping the concept of forward contracts and the expectations theory of interest rates. Forward rates can be derived from the yield curve. The yield curve plots the yields of bonds with different maturities. By analyzing the yield curve, one can infer the implied forward rates for future periods. For example, the forward rate between year 1 and year 2 is the rate at which an investor can borrow or lend money for the period between year 1 and year 2, starting at year 1.

Lets consider three time points on the yield curve $t = 0, 1, 2$, where it is assumed that $t_0 < t_1 < t_2$. At time $t_0$ we have the spot rates $p(t_0, t_1)$ and $p(t_1, t_2)$, which represent the yields for bonds maturing at time $t_1$ and $t_2$ respectively. Hence the forward rate, $R(t_1, t_2)$, can med determined using the equation below [1]

$$R(t_1, t_2) = \frac{(1 + p(t_0, t_2))^2}{(1 + p(t_0, t_1))} - 1$$

Imagine investing one dollar in a one-year zero-coupon bond, $B(t_0, t_1)$, and instantly reinvesting the money received at time $t_1$ in a new one-year zero-coupon bond, $B(t_1, t_2)$, at rate $R(t_1, t_2)$. This strategy should yield the same return as investing one dollar in a two-year zero coupon bond $B(t_0, t_2)$ and holding it for two years. This strategy illustrated the idea of forward rates. Let us then look a the general formula for forward rates.

**Definition 3.** *The continuously compounded forward rate for $[S, T]$ contracted at t is defined as [1]*

$$R(t; S, T) = -\frac{\log p(t,T) - \log p(t,S)}{(T-S)}$$

So now formulas for spot rate and forward rates has been determined. To illustrate the differences between the two types of rates, a simple illustration below shows at which times the rates are determined. From the illustration we see that all the spot rates are determine a time $t = 0$, to each time point to maturity. Where the forward rates starts a different time points.

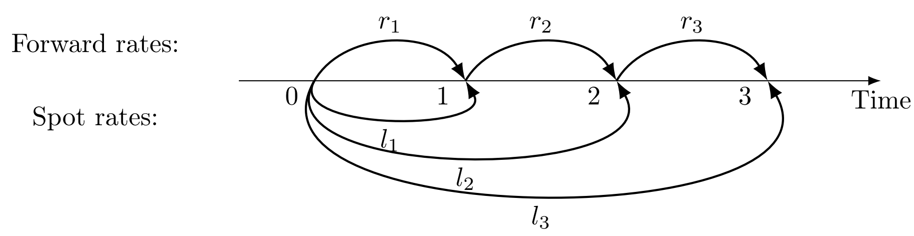



Illustration 3: Forward and spot rates

## 3.5 Financial Derivatives

### 3.5.1 Bonds

A bond is a debt security, like a loan. Borrowers issue bonds to raise money from investors willing to lend them money for a certain amount of time. When you purchase a bond you are lending money to the issuer, which in some cases is a government or company. In return, from the construction of the bond, the issuer guarantees to pay a predetermined rate during the term of the bond and repay the principal at maturity.

Earlier a zero coupon bond was introduced, and when talking about bonds, a zero coupon bond is the simplest representation of a bond. The zero coupon bond contract is only given by two cash flows. One for the buyer, that pays the issuer at time t = $t_0$, and another where the buyer receives the principal at time t = T. Unlike other types of bonds, a zero coupon bond does not offer periodic interest payments (coupons) throughout its term. [1]

The price of a zero coupon bond is represented as p(t,T), where an individual lends an amount, K, with the intention of earning a return in the future. Therefore, the price of a zero coupon bond, with its principal (also known as face value) K, at time t and with maturity T, is denoted as.

$$p(t, T) = B(t, T) \cdot K$$

### 3.5.2 Fixed Coupon Bonds

As describe, a zero coupon bond does not involve coupons throughout the term of the bond. But moving forward we will introduce various bond with coupons that are either fixed or floating. First we will consider the simplest form of a coupon bond, which is a fixed coupon bond. Fixed coupon bonds are a type of debt security that offers investors a predictable return in the form of regular interest payments, known as coupons, until the bond's maturies. These coupons are set at a fixed rate at the time of issuance, based on the bond's face value, and are typically paid annually or semi-annually. Upon reaching maturity, the issuer repays the principal amount (face value) to the issuer, concluding the bond contract. The purpose of a fixed coupon bond is the ability to provide a steady stream of income, making them an attractive option for conservative investors seeking to minimize risk and secure predictable returns.

Continuing, we will compute the price of a fixed coupon bond. First we note that the fixed coupon bond, can be replicated by holding a portfolio consisting of zero coupon bond with maturities $T_i$, for $i = 1, ..., n$. So we will hold $c_i$ zero coupon bonds of maturities $T_i$ for $i = 1, ..., n-1$, and $K + c_n$ bonds with maturity $T_n$. Hence we have that the price, p(t), at time t, where $t < T$, of the fixed coupon bonds becomes. [1]

$$p(t) = K \cdot p(t, T_n) + \sum_{i=1}^{n} c_i \cdot p(t, T_i)$$

When talking about coupons, they are typically determined in terms of return rather than in monetary terms. So the return of the i'th coupon is denoted as a simple rate, acting on the face value K, over the time period $[t_{i-1}, T_i]$. So for the i'th coupon the return is equal to $r_i$, and the face value is K, hence we have that

$$c_i = r_i(T_i - T_{i-1})K$$

Where for standardized coupons, the time intervals will be equally spaced, which means that

$$T_i = T_0 + i\delta$$

This also means the the coupon rates $r_1, ..., r_n$ will be equal to a common coupon rate r. Hence the price p(t,T) of a fixed coupon bond where $t \leq T_1$ will be determined as below [1]

$$p(t) = K\Big(p(t, T_n) + r\delta \sum_{i=1}^{n} \cdot p(t, T_i)\Big)$$

To end this section a illustration of the cashflow for a fixed coupon bond is illustrated below.

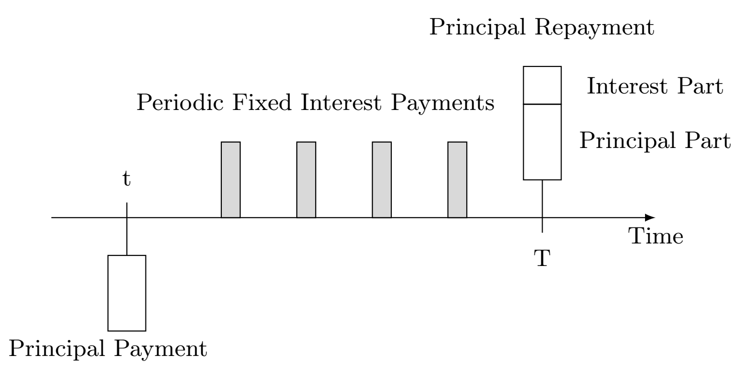



Illustration 4: Cashflow for a fixed coupon bond

### 3.5.3 Floating Rate Bonds

Now a short introduction to fixed coupon bonds has been given, as mentioned there are also many other types of bonds that have floating coupons. When it is listed that there are bonds that have floating coupons, what there is really said is that the rate is floating. So with the fixed coupon bond, the coupon was predetermined when the agreement was made. But there are also bonds, where the coupon is reset for every coupon period. These types of bonds is referred to as floating rate bonds. The most simple floating rate bond, is where the coupon rate $r_i$ is set to the spot LIBOR rate $L(T_{i-1}, T_i)$. Thus we have that

$$c_i = (T_i - T_{i-1})L(T_{i-1}, T_i)K \quad \text{for} \quad i = 1, ..., n$$

Here we have that $L(T_{i-1}, T_i)$ is determined at time $T_{i-1}$, but the coupon is first delivered at time $T_i$. [1]

The LIBOR rate stands for London InterBank Offered Rate, which is a rate the the British Bankers Association sets every business day. Like the LIBOR rate, there is many types of xIBOR rates, one is EURIBOR rate which is a rate the European Banking Federation sets every business day.

These different type of xIBOR rates are sets differently, but they all use the money market convention. So when taking about business day, the money market convention is important. This is a day-count convention is a standardized methodology for calculating the number of days between two dates. This means that when $t < T_0$ the coupon dates are equally spaced with

$$\delta = T_i - T_{i-1}$$

To determined the value of a the simplest floating rate bond, the LIBOR spot rate we can without loss of generality assume that K=1 and insert Definition 2 of the LIBOR spot rate to obtain

$$\begin{aligned} c_i &= (T_i - T_{i-1})L(T_{i-1}, T_i)K \\ &= \delta L(T_{i-1}, T_i) \\ &= \frac{1 - p(T_{i-1}, T_i)}{\delta p(T_{i-1}, T_i)} = \frac{1}{p(T_{i-1}, T_i)} - 1 \end{aligned}$$

Then we have found the price of the LIBOR spot rate, where $\frac{1}{p(T_{i-1}, T_i)}$ is zero coupon bonds prices. So the next step is to determine the price for the floating rate bond. If we look at the $T_i$-payment of the floating rate bond, it has the value of

$$P(t, T_{i-1}) - P(t, T_i)$$

If we then sum over all the i payments of the floating rate we obtain that the floating rate bond value is

$$p(t) = p(t, T_n) + \sum_{i=1}^{n} \Big[ p(t, T_{i-1}) - p(t, T_i) \Big] = p(t, T_0)$$

where we note that if $t = T_0$ we get that $p(T_0) = 1$ [1].

Like for the section for the fixed coupon bond, a illustration of the cashflow for a floating rate bond is illustrated below. To the to illustrations we clearly see the different in the periodic interest payments.

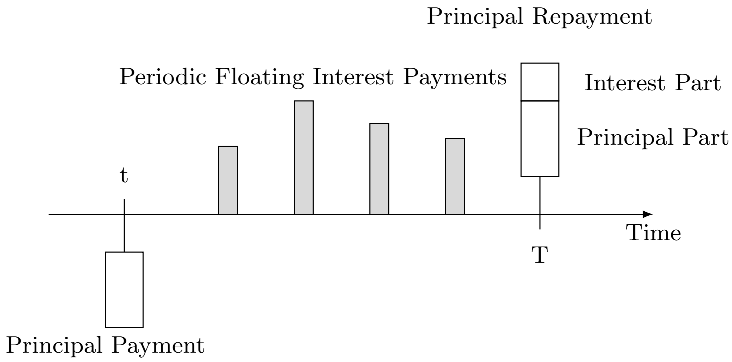



Illustration 5: Cashflow for a floating rate bond

## 3.6 Interest Rate Swaps

Now some simple cases of different types of bonds has be introduce. Then we will combine the knowledge we have gained to move on to take interest rate derivatives into consideration. Again we will consider the simplest type of a interest rate derivative, which is a interest rate swap. The construction of a interest rate swap is that there is an exchange of a payment stream of a fixed rate of interest, which is know as the swap rate. This fixed rate is exchanged for some floating rate, such as the LIBOR rate. As mentioned the fixed rate is know as the swap rate, this swap rate is determined from forward rate extracted from the yield curve, so it makes the present value of the swap equal to zero. This we will formulate formally later.

As stated in the interest rate swap, two cash flow are exchanged, where one of is a fixed cash flow and the other is a floating cash flow. These components of the interest rate swap are known ad the "fixed leg" and the "floating leg". The role of each participant in the swap is determined in relation to the fixed leg: the party making fixed payments is engaged in a "payer swap," while the party making floating payments (and receiving fixed payments) is involved in a "receiver swap.". The two involved cashflow there are exchanged form the "receiver" to the "payer" is illustrated below.

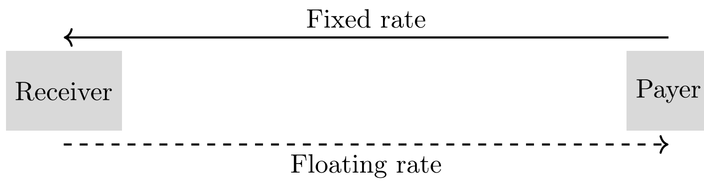



Illustration 6: Cashflow for fixed and floating rate exchanges

Again we have that K is the principal also know as the face value and we will denote the swap rate, R. Further we have that payments arises at the dates $T_1, ..., T_n$, this means that at time $T_i$ the buyer of the interest rate swap will pay

$$K\delta L(T_{i-1}, T_i) \tag{3.1}$$

where we have that $L(T_{i-1}, T_i)$ is the spot rate, which could be the LIBOR spot rate. It is also assumed that the days $T_0, ..., T_n$ is equally spaced with $\delta = T_i - T_{i-1}$ as mentioned above in the section for floating rate bonds. Then it is noticed that the expression in Equation 3.1 is the same as $Kc_i$, where again $c_i$ is the i'th coupon for the floating rate. So at time $T_i$ the buyer will pay $K\delta R$, where the cash flow at time $T_i$ is given by below

$$K\delta\Big[L(T_{i-1}, T_i) - R\Big]$$

Then by applying the results from the section for floating rate bonds again, we are able to compute the value of the cash flow at time $t < T_0$. The value of the cash flow is listed below

$$Kp(t, T_{i-1}) - K(1 + \delta R)p(t, T_i)$$

Hence we have that the total value denote by $\Pi(t)$, so the total value at time t of the swap is given as below

$$\pi(t) = K\sum_{i=1}^{n}\Big[p(t, T_{i-1}) - (1 + \delta R)p(t, T_i)\Big] \tag{3.2}$$

Moving forward we simplify Equation 3.2 in the below Proposition 1 [1].

**Proposition 1.** *The price, for $t < T_0$, of the swap in Equation 3.2 above is given by*

$$\Pi(t) = Kp(t, T_0) - K\sum_{i=1}^{n} d_i p(t, T_i)$$

*where*

$$d_i = R\delta, \quad i = 1, ..., n-1$$
$$d_n = 1 + R\delta$$

To sum up on interest rate swaps, let consider a timeline for a payer swap contract. Where the issuer is paying the fixed leg and receiving the floating leg. The timeline of the contract is illustrated below, where a time t the contract is made. Then the swap start a time $T_S$ and maturities at time $T_E$. The squiggly lines denote the floating interest payments that the payer will make based on the interest rate observed at the beginning of the period and the end of the period. The vertical lines at the beginning of each period represent the fixed payment dates, and the horizontal dotted line indicates the continuation of the swap contract over time.

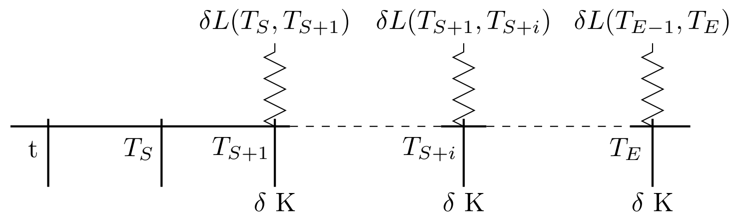



Illustration 7: Cashflow for a payer swap

Earlier we left behind a discussion of how the swap rate, R, is determined. It was noted that the swap is determined such that the present value of the swap is equal to zero. Now we will give a more accurate definition of how swap rates is determined in Proposition 2.

**Proposition 2.** *If, by convention, we assume that the the contract is written at* $t = 0$*, the swap rate is given by [1]*

$$R = \frac{p(0, T_0) - p(0, T_n)}{\delta \sum_{i=1}^{n} p(t, T_i)}$$

If we have that $T_0 = 0$ the formula for the swap rate, R, becomes

$$R = \frac{1 - p(0, T_n)}{\delta \sum_{1}^{n} p(0, T_i)}$$

## 3.7 Options

In this section will introduce the framework of options in the over-the-counter-market. The purpose of this section is to establish a pricing formula for European call options. The meaning of introducing pricing of options before introducing swaptions pricing, is that a swaption is a more complex derivative. So the idea is to get a fundamental understanding of pricing derivatives in a more simple case.

Firstly, let's clarify what the over-the-counter market (OTC) is. It is a marketplace where numerous trades occur. In the OTC market private companies exchange trades, these companies are firms as banks, other large financial institutions and funds managers [2]. Then we have established the market where options is traded, so moving forward we will look in to options contracts.

A call options gives the holder the right to buy the underlying asset at a fixed strike price, K, at a predetermined time, T. Where a put option gives the holder the right to sell the underlying asset at a fixed strike price, K, and a predetermined time, T. Options contracts come in various types, with the most common being the European and American options, followed by Bermudan options. European options can only be exercised at the maturity date, while American options can be exercised at any time point upon to the maturity date. Bermudan options allow exercise at specific predetermined time points. For the purpose of understanding the basics of options pricing, we will focus on the European option. The contract functions, $\Phi$, for European call and put options are as follows.

$$\Phi(x)_{\text{call}} = \max[S - K, 0] \tag{3.3}$$

$$\Phi(x)_{\text{put}} = \max[K - S, 0] \tag{3.4}$$

where K is the strike price, S denotes the market price of the underlying asset [1]. From Equation 3.3 and Equation 3.4 we see that the value of the contract function can not be negative, since in both cases the contract function is a function there takes the maximum of the payoff and zero. So the holder maximum lost is the paid premium. Below the described contract function for a European call option is illustrated.

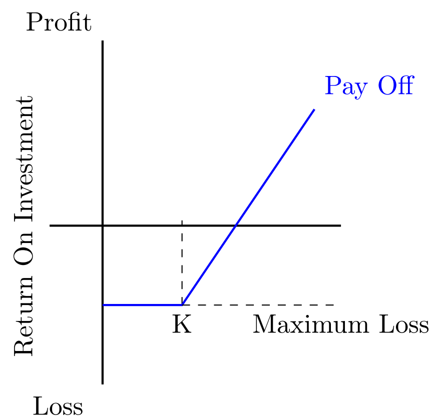



Illustration 8: Contract function for a European call option

### 3.7.1 Risk Neutral Measure

Before options pricing a brief introduction to the risk neutral measure will be covered. When options are priced the value of the options is calculated by discounting the options expected payoff at time T under the risk neutral measure $\mathbb{Q}$.

The value of the options is calculated under the risk neutral measure $\mathbb{Q}$ also know as the pricing measure, and not under the actual measure $\mathbb{P}$. The $\mathbb{P}$- measure reflects th real world probabilities. If prices was determined under $\mathbb{P}$ is could lead to arbitrage opportunities, because it would reflect the actual risk preferences of investors, who demand different rates of return for different risks. Under the risk neutral measure

$\mathbb{Q}$ probabilities are shifted or adjusted in such way that the expected rate of return on assets becomes the risk-free rate. This adjustment removes the risk premiums that are present in the actual probability measure $\mathbb{P}$. This lead to the First Fundamental Theorem of Asset Pricing, to develop this theorem we consider the concept of martingales. A stochastic process $X_t$ is a $\mathbb{Q}$-martingale, if the process has no drift term (dt-term). Which is satisfied if it holds that $\mathbb{E}_t^{\mathbb{Q}}\Big[X_T\Big] = X_t$ for all $t < T$. Next we will consider a price process $X_t$ with the following dynamic.

$$dX_t = r_t X_t dt + \sigma X_t dW_t^{\mathbb{Q}} \tag{3.5}$$

where $W_t^{\mathbb{Q}}$ is $\mathbb{Q}$-Wiener process and $r_t$ is the process for the risk free interest rate. $r_t$ can be looked at the locally risk-free rate return from a continuously compounded bank account $B(t) = \exp\Big[\int_0^t r(s)ds\Big]$. Where the bank account has the following dynamic

$$dB(t) = r(t)B(t)dt \tag{3.6}$$

$$B(0) = 1 \tag{3.7}$$

If we look at Equation 3.5 we see there is a dt-term present, so it is not a martingale. But if we discount the price process, this will be a martingale do to the martingale property below

**Proposition 3.** ***(The Martingale Property)*** *In the Black-Scholes model, the price process $\Pi_t$ for every traded asset, be it the underlying or derivative asset, has the property that the normalized price process*

$$Z_t = \frac{\Pi_t}{B_T}$$

*is a martingale under the measure $\mathbb{Q}$ [1]*

This lead os to the First Fundamental Theorem of Asset Pricing in Theorem 1.

**Theorem 1.** ***(First Fundamental Theorem of Asset Pricing)*** *Given a time horizon, a risky asset with price process $X_t$ and a risk-free asset with price process $B_t$, the market is arbitrage free (under the probability measure $\mathbb{P}$) if and only if there exists an equivalent probability measure $\mathbb{Q}$ such that the discounted price process $\Big[\frac{X_t}{B_t}\Big]$ is a $\mathbb{Q}$- martingale [1]*

Hence we have establish the First Fundamental Theorem of Asset Pricing. So to sum up in order to be able to calculate option prices, the "fair" or arbitrage-free price, there must exist a risk neutral measure $\mathbb{Q}$, such the the discount prices is a $\mathbb{Q}$-martingale.

#### 3.7.2 Options Pricing

The next question to be answered is what is the "fair" price of these options, we will denote the price of the option by $\Pi(t)$. Again to simplify we will consider the European call option moving forward. To determine the price of a European call potion, we wil use the Black-Scholes formula. This requires a review of Risk

Neutral Valuation and the Black-Scholes model.

Risk Neutral Valuation determine the value of an asset by discounting the expected values of the assets future pay-offs at the risk-free rate of return, this formalized in Theorem 2 below

**Theorem 2.** ***(Risk Neutral Valuation)*** *The arbitrage free price of the claim $\Phi(S_t)$ is given by $\Pi(t)[\Phi]=F(t,S_t)$, where $F$ is given by the formula*

$$F(t,s) = -e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}_{t,s}\Big[\Phi(S_T)\Big]$$

*where the Q-dynamics os S is*

$$dS_t = rS_t dt + S_t\sigma(t,S_t)dW^{\mathbb{Q}}$$
$$S_0 = s$$

*and $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Wiener process [1]*

The Risk Neutral Valuation has been introduced, hence the only thing left before we are able to price a European call option, is to establish the model the price in found under. In this case it is the Black-Scholes model. It consists of two asses, a risk free asset with price process, B, and a stock price with price process, S. The dynamics of the two assets is listed below

$$dBt = rB_t dt$$
$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where the short rate, r, is a deterministic constant, $\mu$ and $\sigma$ is two constants. It is also assumed that the stock price process is lognormal distributed. From Theorem 2 (Risk Neutral Valuation) the formulas for determine the arbitrage free price is available. Finally the requirements for being able to price a European option is satisfied, hence we have the Black-Scholes Formula below.

**Proposition 4.** ***(Black-Scholes Formula)*** *The price of a European call option with strike K and time of maturity T is given by the formula $\Pi = F(t,S_t)$*

$$F(t,S_t) = sN[d_1(t,s)] - e^{-r(T-t)}KN[d_2(t,s)]$$

*Here N is the cumulative distribution function for the $N[0,1]$ distribution and*

$$d_1(t,s) = \frac{1}{\sigma\sqrt{T-t}}\Big[\ln\Big(\frac{s}{K}\Big) + \Big(r + \frac{1}{2}\sigma^2)(T-t)\Big]$$
$$d_2(t,s) = d_1(t,s) - \sigma\sqrt{T-t}$$

*[1]*

## 3.8 Swaptions

Now that we have establish a foundational understanding of interest rates, bonds, swaps and options, we can now go deeper into swaptions. First we will explain what constitutes a swaption and the we will continuing to develop the framework of pricing a swaption, buibuiltld on the knowledge we have established.

A swaption is a financial derivative that can be describeed as an option to exchange a fixed rate bond for floating rate bonds for a predetermined principal. There are two types of swaptions, payer swaptions and receiver swaptions. A payer swaption gives the holder the right to pay a fixed interest rate and receive a floating rate, similar to a call option in the stock market. On the other hand, a receiver swaption allows the holder to pay a floating interest rate and receive a fixed rate, resembling a put option [3] .

### 3.8.1 Swaption Pricing

Swaptions pricing purpose is to calculating the present value of expected payments from the swap contract, should the option be exercised. The pricing model must take various factor into account, such as the volatility of interest rates, the term structure of interest rates, and the time value of money. To price swaptions, the Black model will be used, which is an extension of the Black Scholes model for equity options. The choice of using the Black model for pricing swaption is commonly used, especially when is purpose is the price European swaption. Likewise for swaps, there is also different type of swaptions. Again European options can only be exercised at the maturity date, while American options can be exercised at any point in time up to the maturity date. Moving forward we will only consider European swaptions. The Black model assumes that the underlying swap rate follows a lognormal distribution and uses a risk neutral valuation approach. These concept has been reviewed earlier, hence we can the move on to formulating pricing swaptions.

First we consider a swaption that is settled such that the holder has the right to pay a fixed rate, $S_K$, and receive a floating rate on the swap that will expire in n year starting in T years. Further we will assume that there are m payments per year under the swaption and we will let the notional principal be denoted by L. These m payments has assumed that each fixed payment on the swap is the fixed rate times L/m. Next we suppose that the given swap rate for an n-year swap staring a time T, is denoted by $S_T$. From the knowledge on swaps we formulate the payoff function of the swaption, which is listed in Equation 3.8 below

$$\frac{L}{M} \max\Big(S_T - S_K, 0\Big) \tag{3.8}$$

We note that the cashflow generated from the payoff function of the swaption, is reviewed m times a year. The most commonly frequency payments is semi-annually and annually. These payments at m times of a year, is paid throughout the life of the swap. The payment of the swap, have the following payments dates

$$T_1, T_2, ..., T_{mn}$$

Let us be reminded that a swaption is a option on the swap rate, which is the one that generated the payoffs. Then we formulated the price of the payer swaption as [2]

$$\Pi(t)_{\text{Payer swaption}} = \sum_{i=1}^{mn} \frac{L}{m} p(0,T_i)\Big[S_F N(d_1) - S_K N(d_2)\Big] \quad \text{for} \quad T_i = T + i/m$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\Big[\ln\Big(\frac{S_F}{S_k}\Big) + \sigma^2\Big(\frac{T}{2}\Big)\Big]$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

Where N is the cumulative distribution for the $N[0,1]$ distribution, $S_F$ is the forward swap rate a time zero, $\sigma$ is the volatility of the forward swap rate. The term $\sum_{i=1}^{mn} \frac{L}{m} p(0,T_i)$ is the discount factor for the mn payoffs. To simplify we define A as the values of the contract that pays $1/m$ at times $T_i(1 \leq i \leq mn)$ a in Equation 3.9

$$A = \frac{1}{m}\sum_{i=1}^{mn} p(0,T_i) \tag{3.9}$$

Hence we have that the value of the swaption can be expressed as

$$LN\Big[S_F N(d_1) - S_K N(d_2)\Big]$$

Which leads to the case we looked at, where the contract of the swaption was made such that the holder has the right to receive a fixed rate of $S_K$ instead of paying it. It also leads to the payoff of the swaption as listed in Equation 3.10 below. We note that the payoff is a payoff function of a put option on $S_T$.

$$\frac{L}{M}\max\Big(S_K - S_T, 0\Big) \tag{3.10}$$

Finally we can end this section with the value of a swaption in a standard market model [2].

$$\Pi(t)_{\text{swaption}} = LA\Big[S_K N(-d_2) - S_F N(-d_1)\Big]$$

To summarize the mathematics of pricing a swaption has been reviewed, which include introducing interest rates, bonds, swaps and option. It is important to remember which choices was made along the way, because the price of the swaption depends on the choice of the model. But now the simplest case has been introduced, so we can move forward with the analysis.

# 4 One-Factor Short-Rate Model

As we have established, interest rate is crucial for pricing swaptions. Therefore, this chapter will begin by examining the general one-factor short rate model. It will then continue with a closer look at the Vasicek model, where we will determine its properties. To support and investigate these properties, a small simulation analysis of the Vasicek model will be performed.

The risk-free short rate, r, is sometimes referred to as the instantaneous short rate. The concept is used in finance modeling to represent the continuously compounded interest rate for short time intervals. The short rate, r, is often modeled using stochastic differential equations in mathematical finance. The typical models for modeling the short rate is the Vasicek model and the Cox–Ingersoll–Ross model, later the Vasicek model will be covered. When pricing derivatives as bonds and options, the price depends on the process followed by r in the risk-neutral world [2].

As discussed in the Section 3.7.1 - Risk Neutral Measure, $r_t$, can be looked at the locally risk-free rate from a continuously compounded bank account $B(t) = \exp\left[\int_0^t r(s)ds\right]$. Where the bank account has the dynamic listed in Equation 3.6 and Equation 3.7. Postulation: the considered market is arbitrage-free, wich due to the First Fundamental Theorem of Asset Pricing, if stating that there exist a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, all asset prices discounted by $B(t)$ are $\mathbb{Q}$-martingales. In other words under the considered market for any T we have that

$$\frac{P(0,T)}{B(0)} = P(0,T) = \mathbb{E}^{\mathbb{Q}}\Big(\frac{P(T,T)}{B(T)}\Big) = \mathbb{E}^{\mathbb{Q}}\Big(\frac{1}{B(T)}\Big) = \mathbb{E}^{\mathbb{Q}}\Big(\exp\Big[-\int_0^t r(s)ds\Big]\Big) \tag{4.1}$$

where $P(0,T)$ is the price at time zero of the asset and note that $P(T,T) = 1$. So Equation 4.1 says that the time zero price of the asset are $\mathbb{Q}$-expectations of the payoff [5]. In other words in a market free of arbitrage, bond prices are determined by the risk-neutral expectations of how the short-term interest rate will behave. Because all types of interest rate instruments are based on bond prices, the entire term structure or zero coupon curve can be described by the distributional properties of just one state variable - the short rate [5].

## 4.1 The Vasicek model

So all interest rate instruments are fundamentally dependent on bond prices. Understanding the movements of these prices is essential for accurately describing the term structure or zero coupon curve. The behavior of the short rate, a key variable, underlies this understanding due to its distributional properties.

The Vasicek model, introduced by Oldrich Vasicek in 1977, serves as a robust framework to analyze these dynamics. The Vasicek model is renowned for its simplicity and the ease with which it facilitates bond price calculations, the model assumes that the short-term interest rate adheres to a mean-reverting stochastic process. This process is characterized by parameters that dictate the rate's mean reversion speed, its long-term average level, and its volatility. The model is used for forecasting how interest rates in the market will develop in the future. The model is a mathematical result of interest rates and it is a one-factor short rate model and the model is constructed in the term of that the evolution of interest rates only depends on one stochastic variable.

So now a short introducing to the Vasicek model has been covered and the next step is to look closer at the mathematical framework of the Vasicek model. The Vasicek model consists of the dynamic of the short rate under the $\mathbb{P}$-measure (the real world measure). Where the dynamic of the short rate is governed by a stochastic differential equation. The dynamic for the short rate in the Vasicek model is presented below in Equation 4.2 and Equation 4.3 [1].

$$dr_t = \kappa\Big[\theta - r(t)\Big]dt + \sigma dW(t) \tag{4.2}$$

$$r(0) = r_0 \tag{4.3}$$

The dynamic for the short rate in Equation 4.2 is a Ornstein-Uhlenbeck process, which is a type of stochastic differential equation that describes the evolution of a mean-reverting behavior. So the process is consisting of a tendency to revert towards the mean of the process. This tendency is illustrated in Figure 1 below, where rates is simulated using the Vasicek model for some chosen parameters listed in Table 1 below. Note in Figure 1 one simulated path of the short rate is illustrated. Where in Figure 2 ten simulated paths are illustrated, but the same tendency appears. The parameters in the short rate dynamic $\kappa$, $\theta$ and $\sigma$ are positive constants. Where $\kappa$ represent the mean reversion speed, $\theta$ is the long-term average rate, $\sigma$ is the volatility and $W(t)$ is a Wiener process [5].

| **Parameter** | **Parameter explanation** | **Value** |
|---|---|---|
| $T$ | Time to maturity | 10 |
| $r_0$ | Initial short rate | 0.05 |
| $\kappa$ | Mean reversion speed | 0.2 |
| $\theta$ | Long-term average rate | 0.03 |
| $\sigma$ | Volatility | 0.02 |

Table 1: Summary of parameters used for simulation the short rate in the Vasicek model

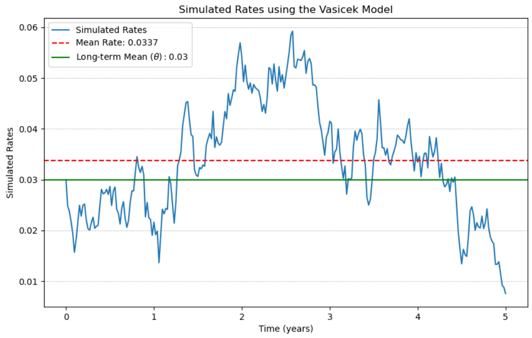



Figure 1: Plot of one simulated rate path using the Vasicek model.

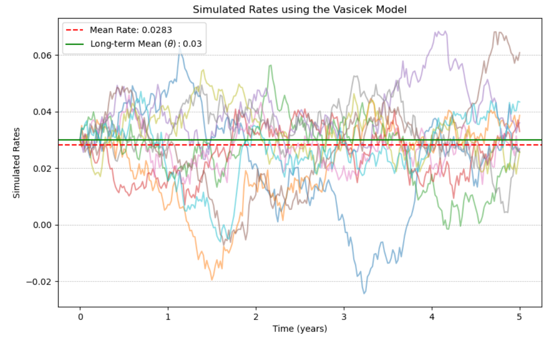



Figure 2: Plot of 10 simulated rates paths using the Vasicek model.

Then we look closer at the dynamic of the short rate in Equation 4.2. This equation can be rearranged and integrated to express the short rate, $r(t)$, as a function of its value at any prior time point s, so we have to have that for $s < t$ [5].

$$dr_t = \kappa\left[\theta - r(t)\right]dt + \sigma dW(t) \tag{4.4}$$

$$dr(t) = k\theta dt - kr(t)dt + \sigma dW(t), \tag{4.5}$$

$$dr(t) + kr(t)dt = k\theta dt + \sigma dW(t), \tag{4.6}$$

$$e^{kt}dr(t) + ke^{kt}r(t)dt = e^{kt}k\theta dt + e^{kt}\sigma dW(t), \tag{4.7}$$

$$\frac{d}{dt}\left(e^{kt}r(t)\right) = e^{kt}\frac{d}{dt}r(t) + ke^{kt}r_t dt, \tag{4.8}$$

$$d\left(e^{kt}r(t)\right) = e^{kt}dr(t) + ke^{kt}r_t dt, \tag{4.9}$$

$$\int_s^t d\left(e^{ku}r(u)\right) = k\theta\int_s^t e^{ku}du + \sigma\int_s^t e^{ku}dW(u), \tag{4.10}$$

$$e^{kt}r(t) - e^{ks}r(s) = \frac{k\theta}{k}\left(e^{kt} - e^{ks}\right) + \sigma\int_s^t e^{ku}dW(u), \tag{4.11}$$

$$r(t) = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right) + \sigma\int_s^t e^{-k(t-u)}dW(u). \tag{4.12}$$

From Equation 4.4 to Equation 4.12 the expression for the short rate in the Vasicek model is integrated, but first the terms in the stochastic differential equation is rearranged. Then both sides of the equation are multiplied by the integrating factor $e^{\kappa t}$ to facilitate the integration. Next, both sides is integrated from s to t. The result of the integration shows the change in the short rate, r, over time, adjusted by the integrating factor. The final expression in Equation 4.12 for the short rate, $r(t)$, at time t, showing how it depends on the initial rate, $r(s)$, the mean-reverting term and the stochastic term [5]. We note that final expression in Equation 4.12 for the short rate is Gaussian.

Further we find that $r(t)$ is normally distributed with mean and variance determined as follows [1].

$$\begin{aligned}
\mathbb{E}\Big[r(t)\Big] &= \mathbb{E}\Big[r(s)e^{-k(t-s)}\Big] + \mathbb{E}\Big[\theta\left(1-e^{-k(t-s)}\right)\Big] + \underbrace{\mathbb{E}\Big[\sigma\int_s^t e^{-k(t-u)}dW(u)\Big]}_{:=0} \\
&= \mathbb{E}\Big[r(s)e^{-k(t-s)}\Big] + \mathbb{E}\Big[\theta\left(1-e^{-k(t-s)}\right)\Big] \\
&= r(s)e^{-k(t-s)} + \theta\left(1-e^{-k(t-s)}\right)
\end{aligned}$$

Using stochastic calculus, it can be demonstrated that the stochastic integral of a deterministic function $f(s)$ with respect to a Wiener process is distributed according to a Gaussian distribution, having a mean of zero and a variance given by $\int_0^t f(s)ds$. We use this to find the variance of the short rate, $r(t)$ [1].

$$\begin{aligned}
Var\Big[r(t)\Big] &= Var\Big[\sigma\int_s^t e^{-k(t-u)}dW(u)\Big] \\
&= \sigma^2\int_s^t e^{-2k(t-u)}dW(u) \\
&= \frac{\sigma^2}{2\kappa}\Big(e^{-2k(t-s)}\Big)
\end{aligned}$$

This lead to the theoretical distribution of the the short rate in the Vasicek model, which is represented below

$$R(t) \sim \mathcal{N}\Big[r(s)e^{-k(t-s)} + \theta\left(1-e^{-k(t-s)}\right), \frac{\sigma^2}{2\kappa}\Big(e^{-2k(t-s)}\Big)\Big] \tag{4.13}$$

Moving towards simulating the distribution, we can use the above attributes of the short rate being Gaussian. Given that the stochastic integral of a deterministic function with respect to a Wiener process is Gaussian distributed with a mean of zero, we can simulate the short rate distribution by generating a large number of possible paths for the Wiener process. Each path would then correspond to a realization of the short rate over time. In Figure 3 below the described method is used and the simulated paths for the short rate in the Vasicek model and the distribution of one the simulated paths is plotted in a histogram. In Figure 3 the probability density function (PDF) of the fitted normal distribution is also plotted. This verifies that the short rate process in the Vasicek model is Gaussian distributed.

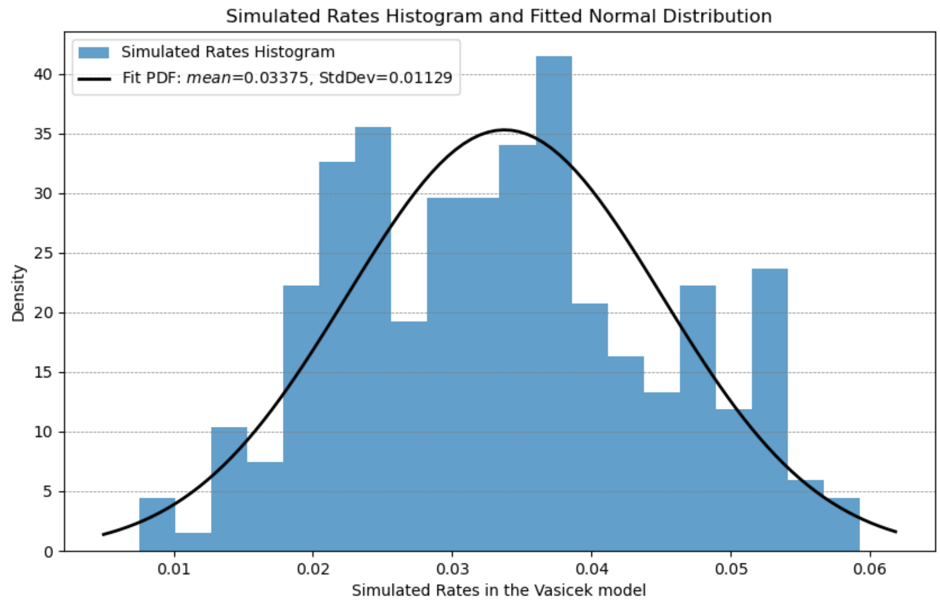



Figure 3: Histogram of one simulated rate path using the Vasicek model.

Now some common properties of the Vasicek model has been reviewed and supported by simulations. Moving forward we will focus on bond prices using the Vasicek model. Earlier when we discussed bond prices, the Black Scholes model was introduced.

Now we will examine another aspect of the Vasicek model, namely the ability to derive explicit formulas for bond pricing based on model expectations. But first some words on how these bond prices are determined in the Vasicek model. When using the Vasicek model, it is possible to derive closed-form solutions for zero coupon bond prices. These prices are determined by an equation that factors in the current short rate, the speed of mean reversion, the long-term mean level, the volatility of the short rate, and the bond's time to maturity. The bond prices formula from the Vasicek model considers the expected path of future short rates under the risk neutral measure, discounted back to the present value.

As mentioned we will now look a the formula for pricing bonds using the Vasicek model. We consider a zero coupon bond with maturity T and at time t the price is given as in Equation 4.14 below [1].

$$P(t,T) = A(t,T)e^{-rB(t,T)} \tag{4.14}$$

Then we find the partial derivatives with respect to $r$ and $t$ of the zero coupon bond listed in Equation 4.14.

$$\begin{aligned}\frac{\partial P}{\partial t} &= \frac{\partial}{\partial t}(Ae^{-rB}) = e^{-rB}\frac{\partial A}{\partial t} + Ae^{-rB}\frac{\partial}{\partial t}(-rB) \\ &= e^{-rB}\frac{\partial A}{\partial t} - rAe^{-rB}\frac{\partial B}{\partial t} = -\frac{P}{A}\frac{\partial A}{\partial t} - rP\frac{\partial B}{\partial t}\end{aligned}$$

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}(Ae^{-rB}) = Ae^{-rB}\frac{\partial}{\partial r}(-rB) = -PB$$

$$\frac{\partial^2 P}{\partial r^2} = \frac{\partial}{\partial r}(-PB) = -B\frac{\partial}{\partial r}(P) = PB^2$$

Then by applying Ito's lemma [1] and inserting the derivatives with respect to $r$ and $t$ we found above as well as the formula for the short rate in the Vasicek model present in Equation 4.2 we obtain the following

$$\begin{aligned}dP(t,T) &= \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial r}dr + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}dr^2 \\ dP(t,T) &= \left(\frac{P}{A}\frac{\partial A}{\partial t} - rP\frac{\partial B}{\partial t}\right)dt + (-PB)dr + \frac{1}{2}(PB^2)dr^2 \\ \frac{dP}{P} &= \left(\frac{1}{A}\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)dt - Bdr + \frac{1}{2}B^2dr^2 \\ \frac{dP}{P} &= \left(\frac{1}{A}\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)dt - B(\kappa\theta dt - \kappa r dt + \sigma dW_t) + \frac{1}{2}B^2\sigma^2 dt \\ \frac{dP}{P} &= \left(\frac{1}{A}\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} - \kappa\theta B + \kappa r B + \frac{1}{2}B^2\sigma^2\right)dt - \sigma B dW_t\end{aligned}$$

Under the risk neutral measure, the expected return of the bond must be equal to the risk free rate. Thus we have that

$$\begin{aligned}r &= \frac{1}{A}\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} - \kappa\theta B + \kappa r B + \frac{1}{2}B^2\sigma^2 \\ r\left(1 + \frac{\partial B}{\partial t} - \kappa B\right) &= \frac{1}{A}\frac{\partial A}{\partial t} - \kappa\theta B + \frac{1}{2}B^2\sigma^2\end{aligned}$$

Since this holds for all values of $r$, which does not feature in the left hand side, we deduce that

$$0 = \frac{1}{A}\frac{\partial A}{\partial t} - \kappa\theta B + \frac{1}{2}B^2\sigma^2 \tag{4.15}$$

$$0 = 1 + \frac{\partial B}{\partial t} - \kappa B \tag{4.16}$$

Considering that the price of zero coupon bond at maturity $P(T,T) = 1$, the function $P = Ae^{-rB}$ suggests $B(T,T) = 0$ and $A(T,T) = 0$. By then applying the integrating factor to the Equation 4.16, reorganizing and integrating form t to T, we obtain

$$\begin{aligned}
0 &= 1 + \frac{\partial B}{\partial t} - \kappa B \\
0 &= e^{-\kappa t} + e^{-\kappa t}\frac{\partial B}{\partial t} - e^{-\kappa t}\kappa B \\
-e^{-\kappa t} &= e^{-\kappa t}\frac{\partial B}{\partial t} - e^{-\kappa t}\kappa B \\
-e^{-\kappa t} dt &= d\left(e^{-\kappa t} B(t,T)\right) \\
-\int_t^T e^{-\kappa u} du &= \int_t^T d\left(e^{-\kappa t} B(t,T)\right) \\
\frac{1}{\kappa}\left(e^{-\kappa T} - e^{-\kappa t}\right) &= e^{-\kappa T} B(T,T) - e^{-\kappa t} B(t,T) \\
B(t,T) &= \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)
\end{aligned} \tag{4.17}$$

Finally by substituting into Equation 4.15 and integrating we get that

$$\begin{aligned}
0 &= \frac{1}{A}\frac{\partial A}{\partial t} - \kappa\theta B + \frac{1}{2}B^2\sigma^2 \\
0 &= \frac{1}{A}\frac{\partial A}{\partial t} - \kappa\theta\left(\frac{1-e^{-\kappa(T-t)}}{\kappa}\right) + \frac{\sigma^2}{2\kappa^2}\left(1-e^{-\kappa(T-t)}\right)^2 \\
0 &= \frac{1}{A}\frac{\partial A}{\partial t} - \theta\left(1-e^{-\kappa(T-t)}\right) + \frac{\sigma^2}{2\kappa^2}\left(1+e^{-2\kappa(T-t)} - 2e^{-\kappa(T-t)}\right) \\
\frac{1}{A}\frac{\partial A}{\partial t} &= \theta\left(1-e^{-\kappa(T-t)}\right) - \frac{\sigma^2}{2\kappa^2}\left(1+e^{-2\kappa(T-t)} - 2e^{-\kappa(T-t)}\right) \\
\int_t^T \frac{dA(u,T)}{A(u,T)} &= \theta\int_t^T \left(1-e^{-\kappa(T-u)}\right) du - \frac{\sigma^2}{2\kappa^2}\int_t^T \left(1+e^{-2\kappa(T-u)} - 2e^{-\kappa(T-u)}\right) du \\
\ln A(T,T) - \ln A(t,T) &= \theta(T-t) - \theta\left(\frac{1-e^{-\kappa(T-t)}}{\kappa}\right) - \frac{\sigma^2}{2\kappa^2}\left((T-t) + \left(\frac{1-e^{-2\kappa(T-t)}}{2\kappa}\right) - 2\frac{1-e^{-\kappa(T-t)}}{\kappa}\right) \\
-\ln A(t,T) &= \theta(T-t) - \theta\left(\frac{1-e^{-\kappa(T-u)}}{\kappa}\right) - \frac{\sigma^2}{4\kappa^3}\left(2\kappa(T-t) + 1 - e^{-2\kappa(T-t)} - 4 + 4e^{-\kappa(T-t)}\right) \\
-\ln A(t,T) &= \theta(T-t) - \theta\left(\frac{1-e^{-\kappa(T-u)}}{\kappa}\right) - \frac{\sigma^2}{4\kappa^3}\left(2\kappa(T-t) - \left(1+e^{-2\kappa(T-t)} - 2e^{-\kappa(T-t)}\right) - 2 + 2e^{-\kappa(T-t)}\right) \\
-\ln A(t,T) &= \theta(T-t) - \theta\left(\frac{1-e^{-\kappa(T-u)}}{\kappa}\right) - \frac{\sigma^2}{2\kappa^2}\left((T-t) - \frac{1-e^{-\kappa(T-t)}}{\kappa}\right) + \frac{\sigma^2}{4\kappa^3}\left(1-e^{-\kappa(T-t)}\right)^2 \\
\ln A(t,T) &= \left(\theta - \frac{\sigma^2}{2\kappa^2}\right)\left(\frac{1-e^{-\kappa(T-u)}}{\kappa} - (T-t)\right) - \frac{\sigma^2}{4\kappa}\left(\frac{1-e^{-\kappa(T-t)}}{\kappa}\right)^2 \\
A(t,T) &= \exp\left\{\left(\theta - \frac{\sigma^2}{2\kappa^2}\right)\left(\frac{1-e^{-\kappa(T-u)}}{\kappa} - (T-t)\right) - \frac{\sigma^2}{4\kappa}\left(\frac{1-e^{-\kappa(T-t)}}{\kappa}\right)^2\right\}
\end{aligned} \tag{4.18}$$

Combining this we get the formula for pricing bonds using the Vasicek model, which present in Proposition 5 below.

**Proposition 5.** ***(The Vasicek term structure)*** *In the Vasicek model, bond prices are given by*

$$P(t,T) = A(t,T)e^{-rB(t,T)} \tag{4.19}$$

*where*

$$A(t,T) = \exp\left\{\left(\theta - \frac{\sigma^2}{2\kappa^2}\right)\left(\frac{1-e^{-\kappa(T-u)}}{\kappa} - (T-t)\right) - \frac{\sigma^2}{4\kappa}\left(\frac{1-e^{-\kappa(T-t)}}{\kappa}\right)^2\right\}$$

$$B(t,T) = \frac{1}{\kappa}\left(1-e^{-\kappa(T-t)}\right)$$

*[1]*

To illustrate the behavior of bond price in the Vasicek model, a small simulation study is performed. For the study some parameters are chosen, these are listed in Table 2. In Figure 4 one simulate path for the development of the ZCB price is illustrated and in Figure 5 ten simulated paths are illustrated. We see that the price fluctuate over the price period, as expected.

| **Parameter** | **Parameter explanation** | **Value** |
|---|---|---|
| $T$ | Time to maturity | 10 |
| $r_0$ | Initial short rate | Simulated |
| $\kappa$ | Mean reversion speed | 0.2 |
| $\theta$ | Long-term average rate | 0.03 |
| $\sigma$ | Volatility | 0.02 |

Table 2: Summary of parameters used simulation ZCB prices in the Vasicek model

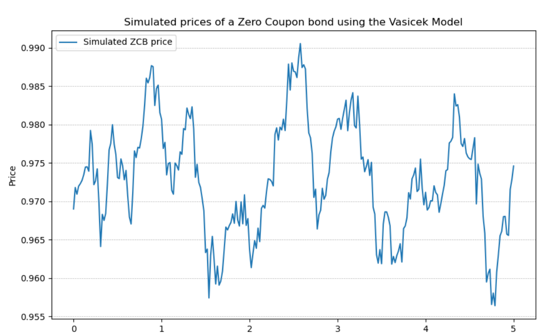



Figure 4: Plot of one simulated ZCB price path using the Vasicek model.

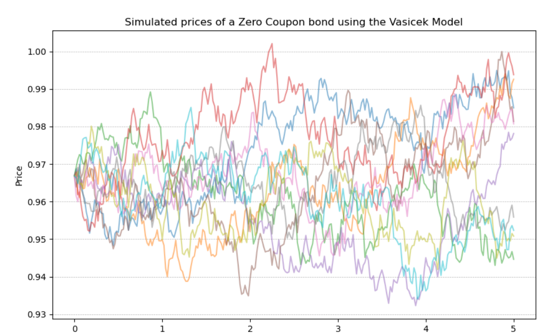



Figure 5: Plot of 10 simulated ZCB prices paths using the Vasicek model.

The Vasicek model has now been introduced and we have taken a closer look at the distribution in the short rate model. This examination leads us to the Vasicek term structure model, which outlines the method for pricing bonds using the Vasicek framework. Similarly, in Section 3, specific decisions are made regarding the calculation of bond prices with the Vasicek model. A critical decision is the assumption that the volatility,

$\sigma$, is a positive constant. However, we should consider whether volatility remains constant over a given period.

Subsequently, we will introduce another model, the SABR (Stochastic Alpha, Beta, Rho) model. The SABR model is a stochastic volatility model utilized to estimate the implied volatility smile. Although it does not provide a direct formula for option pricing, the estimated implied volatility curve can be employed in the Black Scholes model, discussed earlier, to price swaptions. Before we dig into the SABR model, we will examine the assumption of constant volatility inherent in the Black Scholes model. The purpose of this small analysis, it to make an attachment to the market and see some examples of not constant volatility. This can be seen as a motivation for looking into two-factor models, instead of a one-factor model, like the Vasicek model.

# 5 Constant Volatility

This chapter will look into the limitations of the Black Scholes model, particularly its assumption of constant volatility, and introduce the SABR model as a more dynamic alternative for estimating volatility. The chapter will illustrate the inconsistency of this assumption with real-world market data, using the S&P 500 index and the 10Y10Y EUR swaption as examples.

The Black Scholes model, introduced earlier, outlines the pricing formula for an European call option in Proposition 4. It is essential to recall that in the Black Scholes formula, volatility is considered constant. This implies that the volatility of the asset's returns does not vary over time, establishing a direct correlation between the option's price and its volatility. Consequently, understanding implied volatility becomes crucial. Although the Black Scholes model does not provide a closed-form solution for implied volatility, it can be determined numerically, a topic not covered in this analysis. Instead, we introduce the SABR model to estimate volatility, which can then be applied to the Black Scholes model for option pricing. This will be covered in the Section 7 about the SABR model.

We will briefly demonstrate why the assumption of constant volatility is inconsistent with market data. Our analysis includes an examination of the S&P 500 index and a 10Y10Y EUR swaption, which are frequently used financial indicators.

As depicted in Figure 6 and Figure 7, the development of the 10Y10Y EUR swaption and S&P 500 index levels illustrates fluctuations over time. This variability is further emphasized by the return patterns shown in Figure 8 and Figure 9, where the returns of the 10Y10Y EUR swaption and S&P 500 index are plotted. These fluctuations suggest that market volatility is not constant. Our analysis underlines that volatility varies significantly from day to day, reflecting the market's response to new information and events. This observation challenges the applicability of the Black Scholes model, which assumes constant volatility and highlights the need for models like the SABR model that more accurately capture market dynamics and provide nuanced volatility estimates.

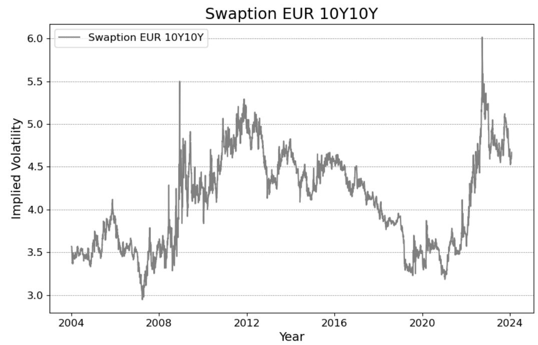



Figure 6: Swaption EUR 10Y10Y from 01.01.2004 to 01.01.2024. Data source Citi Velocity

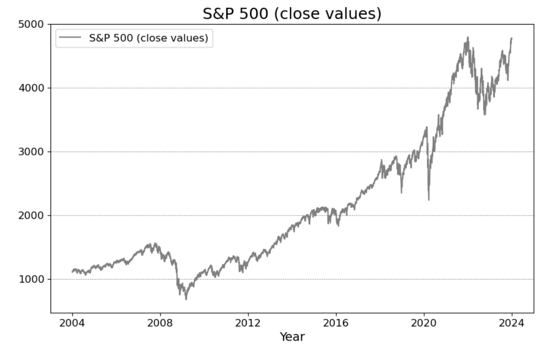



Figure 7: S&P500 index (close values) from 01.01.2004 to 01.01.2024. Data source Yahoo Finance.

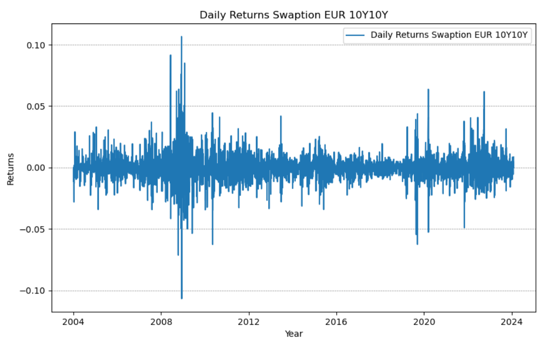



Figure 8: Swaption EUR 10Y10Y return from 01.01.2004 to 01.01.2024.

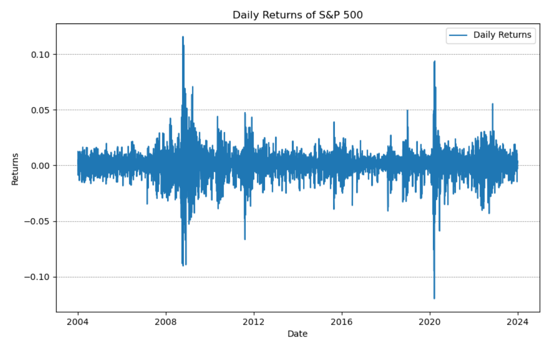



Figure 9: S&P500 index (close values) return from 01.01.2004 to 01.01.2024.

The market data reveals a tendency in volatility, indicating that it is not constant. Given this fluctuation, how should we proceed when considering to be able to price a swaption? We will continue our analysis by exploring a two-factor model rather than the previously described one-factor Vasicek model. This approach allows us to model volatility stochastically and use the resulting volatility estimates to price a swaption in the Black Scholes model.

# 6 Data

In Section 5 wee looked into the assumption of constant volatility in the Black Scholes model. Where during the study, a 10Y10Y EUR swaption was introduced. This chapter will provide a introduction to market data on swaption. In this thesis the data source is Citi Velocity, which is a market data plantformed owned by Citi Bank.

Before delving into the market data on swaptions, let's revisit the definition of a swaption. A swaption gives the holder the right, but not the obligation, to enter into an interest rate swap in the future. There are two types of swaptions: payer swaptions and receiver swaptions, as previously described.

The four main parameters defining a swaption are:

- **Expiry** when in the future the holder can exercise their right.
- **Tenor** the maturity of the underlying interest rate swap.
- **Strike** the pre-specified interest rate of the swap.
- **Pay/Receive** whether the investor pays or receives the fixed rate.

As mentioned a 10Y10Y EUR swaption was analyzed, but let's clarify what a 10Y10Y EUR swaption entails. Assuming the swaption is a receiver swaption, it represents a contract that, in ten years, gives the holder the right to enter into a ten-year interest rate swap in EUR, receiving a fixed rate. Thus, the swaption has an expiry of 10 years, a tenor of 10 years, and a strike equivalent to the value of the fixed rate. It's important to note that the underlying interest rate swap in the swaption is a forward swap, commencing 10 years from now.

Then before we will look at some displayed data, lets first cover some lingo of swaption. A swaption with a strike set at the current market rate for the underlying forward swap is referred to as at-the-money-forward (ATMF). If the strike differs from the ATMF rate, the swaption is categorized as out-of-the-money (OTM).

When discussing swaptions, it is essential to recognize how the volatility surface varies across different expiries and swap forward tenors. Displayed below in Table 3 is the ATMF swaption volatility surface. The presentation of the volatility surface in swaptions differs notably from that in other asset classes like equities. In the case of equities, different tenors are not a consideration, allowing both at-the-money (ATM) and out-of-the-money (OTM) levels to be represented on a single chart. Conversely, the out-of-the-money-forward (OTMF) swaption volatility surface, as shown in Table 4, adopts a different layout. The horizontal axis of Table 4 spans a range of strikes, from -200 basis points below to +200 basis points above the ATM strike.

Now a short introduction to swaption data has been covered, then the next step is to illustrated some volatility surface. The purpose of during this is first of all, to underline that the volatility surface for a given swaption change over time. Secondly it is to see that the volatility smile for swaption with a fixed expiry and various tenor can also be slightly different. From Figure 10 we see the volatility surface for a 10Y10Y EUR swaption at five days over the time period from 20.02.2020 to 20.02.2024. From the plot we clearly see that the volatility surface is change over the period. We see a pattern of the volatility surface moving up and down, without large change in the curvature of the volatility surface. This implies changes in the alpha parameter i the SABR, which will be covered later in Section 7.4. Then we look at the OTM volatility surface where the swaption expiry is fixed at ten years, but for various tenors. We look at this case over to days, where the first on the dates 20.02.2023 and 20.02.2024. This leads to at plot for each combination of the fixed expiry (10Y) and the various tenor. These volatility surface plots is illustrated in Figure 20a to Figure 20j below. From these plots we clearly see that the volatility surface change for the various tenors.

| | Tenor | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Expiry | 1Y | 2Y | 3Y | 5Y | 7Y | 10Y | 15Y | 20Y | 30Y |
| 1M | 5.2896 | 6.0865 | 5.9731 | 5.6770 | 5.4074 | 5.0402 | 4.8121 | 4.6578 | 4.5098 |
| 2M | 5.4361 | 6.1242 | 5.9983 | 5.7206 | 5.4838 | 5.1569 | 4.9592 | 4.7942 | 4.6228 |
| 3M | 5.5244 | 6.1032 | 5.9953 | 5.7530 | 5.5240 | 5.1959 | 5.0318 | 4.8662 | 4.6948 |
| 6M | 6.0007 | 6.3187 | 6.1810 | 5.9009 | 5.7098 | 5.4304 | 5.2822 | 5.1272 | 4.9696 |
| 9M | 6.2264 | 6.3990 | 6.2515 | 6.0103 | 5.8553 | 5.6185 | 5.4686 | 5.3155 | 5.1284 |
| 12M | 6.2698 | 6.3692 | 6.2157 | 6.0080 | 5.8713 | 5.6575 | 5.4984 | 3.5614 | 5.1749 |
| 18M | 6.3333 | 6.3395 | 6.2046 | 6.0073 | 5.8874 | 5.7166 | 5.5284 | 5.3998 | 5.2215 |
| 2Y | 6.2967 | 6.2853 | 6.1613 | 5.9773 | 5.8477 | 5.7289 | 5.5014 | 5.3854 | 5.2146 |
| 3Y | 6.1824 | 6.1546 | 6.0463 | 5.8606 | 5.7536 | 5.6654 | 5.4234 | 5.2916 | 5.1171 |
| 4Y | 6.0380 | 6.0008 | 5.8975 | 5.7407 | 5.6613 | 5.5681 | 5.3010 | 5.1479 | 4.9677 |
| 5Y | 5.8966 | 5.8683 | 5.7524 | 5.6119 | 5.5376 | 5.4563 | 5.1646 | 5.0015 | 4.8213 |
| 7Y | 5.6398 | 5.6152 | 5.5170 | 5.3796 | 5.2946 | 5.2228 | 4.9248 | 4.7421 | 4.5575 |
| 10Y | 5.3399 | 5.3311 | 5.2101 | 5.0614 | 4.9531 | 4.8315 | 4.5247 | 4.3559 | 4.1757 |
| 12Y | 5.1687 | 5.1598 | 5.0635 | 4.8726 | 4.7460 | 4.5879 | 4.2716 | 4.1312 | 3.9422 |
| 15Y | 4.9532 | 4.9412 | 4.8215 | 4.5966 | 4.4391 | 4.2880 | 3.9736 | 3.8382 | 3.6523 |
| 20Y | 4.5909 | 4.5840 | 4.4649 | 4.2116 | 4.0605 | 3.8790 | 3.5666 | 3.4626 | 3.2743 |
| 30Y | 4.1109 | 4.1065 | 3.9893 | 3.6882 | 3.5119 | 3.2952 | 3.0168 | 2.9317 | 2.7602 |

Table 3: At-the-money-forward (ATMF) swaption volatility surface. Normal absolute values. Data source Citi Velocity 20.02.2024.

| | Strike | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Expiry x Tenor | -200 | -100 | -75 | -50 | -25 | **ATM** | 25 | 50 | 75 | 100 | 200 |
| 10Y x 1Y | 5.1416 | 5.1819 | 5.2105 | 5.2465 | 5.2898 | **5.3399** | 5.3966 | 5.4596 | 5.5283 | 5.6025 | 5.9448 |
| 10Y x 2Y | 5.1588 | 5.1864 | 5.2116 | 5.2442 | 5.2841 | **5.3311** | 5.3845 | 5.4444 | 5.5102 | 5.5815 | 5.9138 |
| 10Y x 3Y | 5.0699 | 5.0806 | 5.1017 | 5.1304 | 5.1665 | **5.2101** | 5.2599 | 5.3166 | 5.3795 | 5.4480 | 5.7709 |
| 10Y x 5Y | 4.9781 | 4.9597 | 4.9735 | 4.9950 | 5.0241 | **5.0614** | 5.1043 | 5.1549 | 5.2119 | 5.2750 | 5.5792 |
| 10Y x 7Y | 4.9234 | 4.8694 | 4.8762 | 4.8916 | 4.9157 | **4.9531** | 4.9896 | 5.0388 | 5.0955 | 5.1594 | 5.4749 |
| 10Y x 10Y | 4.8882 | 4.7796 | 4.7750 | 4.7806 | 4.7966 | **4.8315** | 4.8598 | 4.9065 | 4.9626 | 5.0275 | 5.3610 |
| 10Y x 12Y | 4.7487 | 4.6379 | 4.6305 | 4.6334 | 4.6170 | **4.6781** | 7.7064 | 4.7517 | 4.8068 | 4.8709 | 5.2033 |
| 10Y x 15Y | 4.6292 | 4.4963 | 4.4859 | 4.4862 | 4.4974 | **4.5247** | 4.5530 | 4.5970 | 4.6509 | 4.7143 | 5.0457 |
| 10Y x 20Y | 4.4983 | 4.3478 | 4.3329 | 4.3288 | 4.3360 | **4.3559** | 4.3846 | 4.4256 | 4.4772 | 4.5385 | 4.8650 |
| 10Y x 30Y | 4.3647 | 4.1915 | 4.1706 | 4.1605 | 4.1619 | **4.1757** | 4.2003 | 4.2370 | 4.2847 | 4.3428 | 4.6603 |

Table 4: Out-of-the-money-forward (OTMF) swaption volatility surface. Normal absolute values. Data source Citi Velocity 21.02.2024.

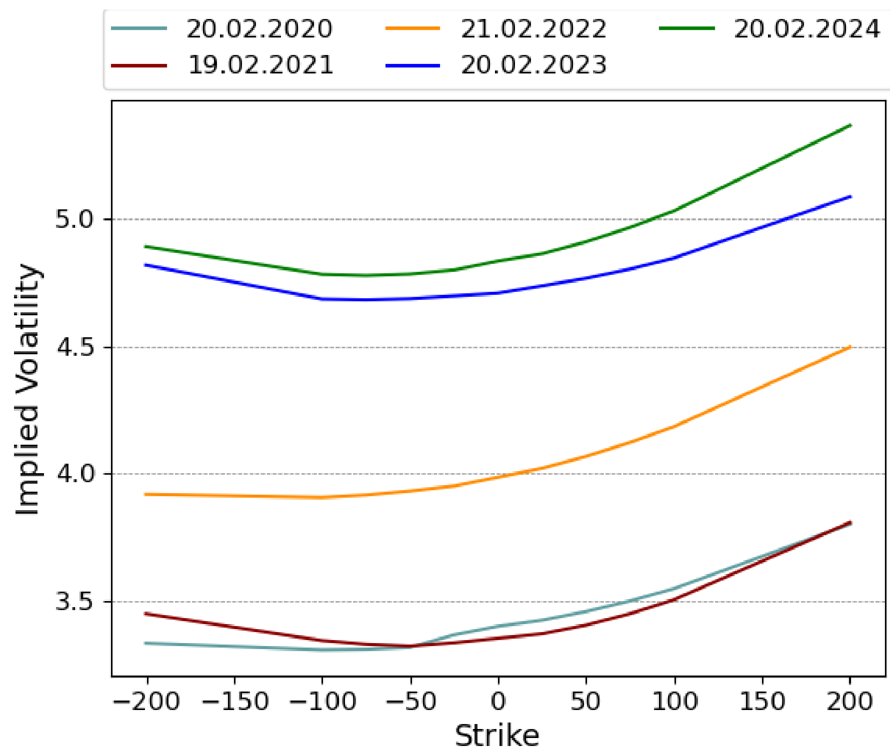
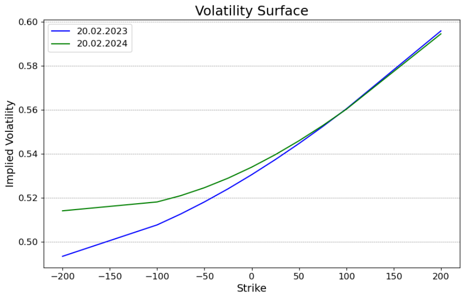
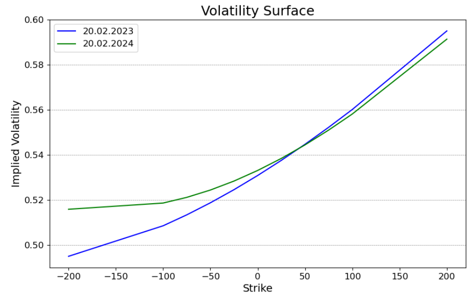


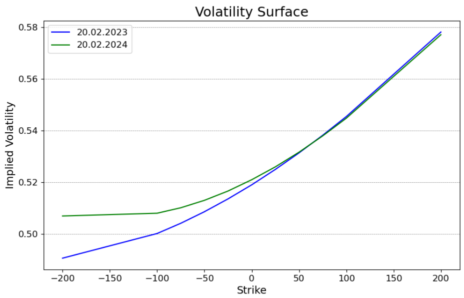

Figure 10: EUR swaption 10Y10Y skew over five years. Data source Citi Velocity.



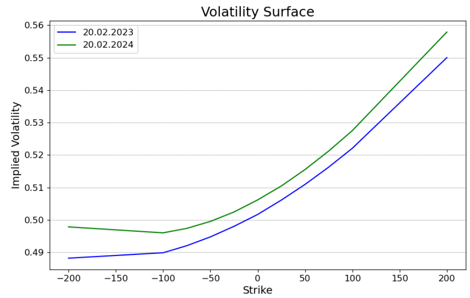

(a) EUR swaption 10Y1Y



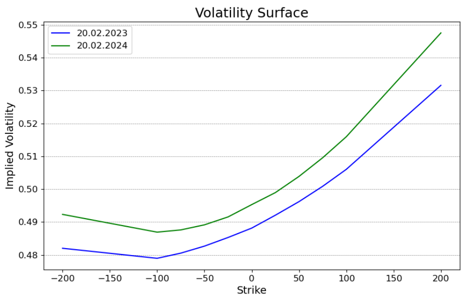

(b) EUR swaption 10Y2Y



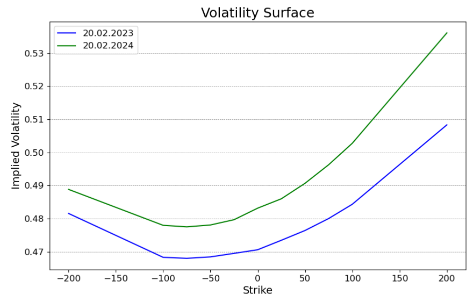

(c) EUR swaption 10Y3Y



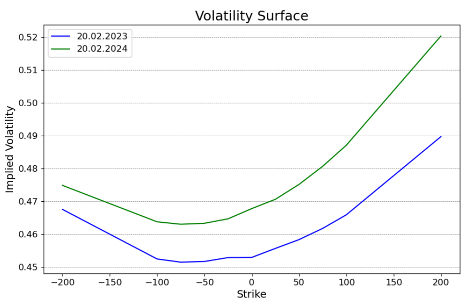

(d) EUR swaption 10Y5Y



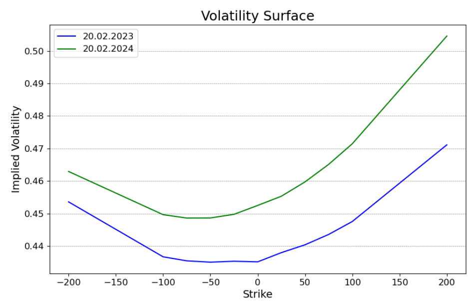

(e) EUR swaption 10Y7Y



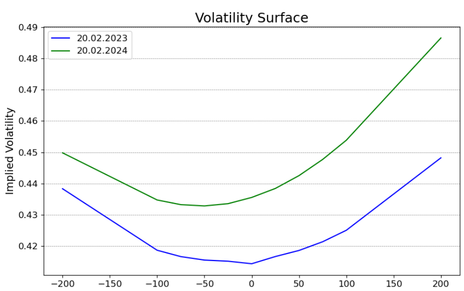

(f) EUR swaption 10Y10Y



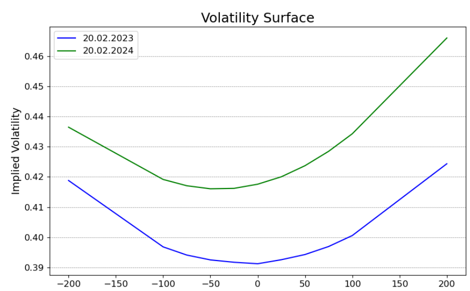

(g) EUR swaption 10Y12Y




(h) EUR swaption 10Y15Y




(i) EUR swaption 10Y20Y




(j) EUR swaption 10Y30Y

# 7 The SABR model

In this chapter, we will first introduce the SABR model and explain how to determine the forward rate and volatility using this model. We will then focus on calculating implied volatility and pricing European options with the SABR model. Finally, the chapter will discuss methods for estimating the parameters of the SABR model.

The SABR (Stochastic Alpha, Beta, Rho) model marks a pivotal advancement in financial modeling, effectively addressing the significant limitations found in traditional methods like the Black Scholes model, which assumes constant volatility. Developed in 2002 by Patrick Hagan, Deep Kumar, Andrew Lesniewski, and Diana Woodward, the SABR model is highly respected for its ability to manage the dynamic and unpredictable nature of market volatility.

As a two-factor model, the SABR framework models both the forward rate (or asset price) and its volatility as a stochastic processes. This approach is vital as it incorporates a stochastic behavior in volatility, significantly improving the model's ability to capture the true, skewed, and heavy-tailed nature of financial market data. By allowing for volatility fluctuations, the SABR model provides a flexible and realistic framework for pricing derivatives, proving especially useful for options with long maturities where the assumption of constant volatility falls short [6].

## 7.1 Specification for the SABR model

The main different between the SABR model and the Black Scholes model is the assumptions regrading the volatility, as mentioned earlier. In the Black Scholes model the volatility is a assumed to be constant and in the SABR model the volatility evolves as a function of time, t, the strike price, K, and the current forward price, $f_t$. Futhermore the volatility itself is random. So we chose the unknown coefficient $C(t,*)$ to be $\hat{\alpha}\hat{F}^{\beta}$, where the "volatility" $\hat{\alpha}$ is a stochastic process itself. The extra randomness is scaled thought the inclusion of a "volatility of volatility" parameter $\nu$.

Now we will formulate the SABR model mathematically. The SABR model consists of a dynamic for the forward price and one for the volatility, since the SABR model is a two-factor model. The SABR model also formulate the how the two process is correlated.

$$dF_t = \alpha_t f_t^{\beta} dW_t^1, \qquad F(0) = f \tag{7.1}$$

$$d\alpha_t = \nu\alpha_t dW_t^2, \qquad \alpha(0) = \alpha \tag{7.2}$$

where $W_t^1$ and $W_t^2$ are two correlated Wiener process and it is assumed that [6].

$$dW_t^1 dW_t^2 = \rho dt \tag{7.3}$$

So we have that parameters in the SABR model is as follows, $\alpha$ represents the initial volatility level , $\nu$ represents the volatility of volatility, or the rate at which volatility itself changes , $\beta$ represents the elasticity of the volatility; a common practice is to fix beta based on the underlying asset and as mentioned $\rho$ is the correlations between the two Wiener process, the asset price and its volatility.

So the SABR model is characterized by the stochastic process $\alpha_t$, the parameter $\beta$, and the correlation coefficient $\rho$, which is also reflected in its name - Stochastic Alpha Beta Rho. In a specific variant of the SABR model, by setting $\beta = 1$ and $\nu = 0$, the model reverts to the classic Black Scholes framework. This configuration leads to a constant volatility, denoted $\alpha_0$, and a forward process where returns follow a normal distribution with a mean of zero and a standard deviation of $\alpha_0\sqrt{t}$. So now the SABR model has been introduced and the analysis will continuing forward on how to price a swaption using the SABR model to determine the implied volatility.

## 7.2 Simulation the SABR model

In this section a short view on how the process for the forward price and the volatility develops over time will be covered. To provide the intuition on how the dynamic listed in Equation 7.1 and Equation 7.2 behave, the dynamics is simulated ten times for some chosen parameters is listed in Table 5. The simulated paths is illustrated in Figure 12 below. From Figure 12 we see that the dynamics are driven by the randomness in the Wiener process and it develops from the initial value of the forward price and volatility.

| **Parameter** | **Parameter explanation** | **Value** |
|---|---|---|
| $F_0$ | Initial forward rate or asset price | 100 |
| $\alpha_0$ | Initial volatility | 0.2 |
| $\beta$ | Elasticity parameter | 0.5 |
| $\nu$ | Volatility of the volatility parameter | 0.25 |
| $\rho$ | Correlation between the asset price and its volatility | -0.4 |

Table 5: Summary of parameters used for simulating the SABR model

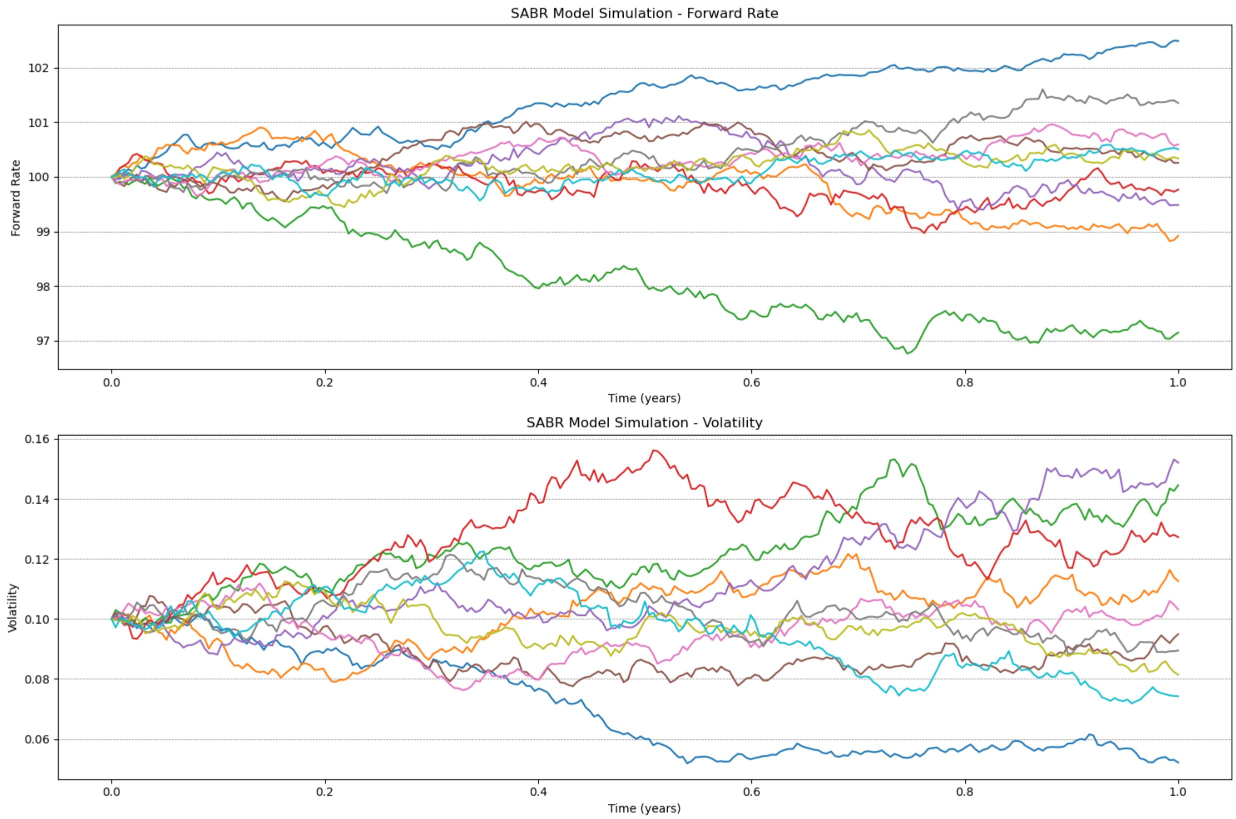



Figure 12: Ten simulated paths for the forward rate and the volatility in the SABR model.

## 7.3 SABR Implied Volatility and Option Prices

Before we are able to move forward with the analysis, we need to formulate how to determine implied volatility in the SABR model. But these calculations are out of the scope for this analysis, so we will use the formula in the paper Managing Smile Risk of Hagen (2002) [6]. The paper states that under the SABR model, the prices of European options is given by Black formula in Equation 7.4 to Equation 7.6 below

$$V_{\text{call}} = D(t_{\text{set}})fN(d_1) - KN(d_2), \tag{7.4}$$

$$V_{\text{put}} = V_{\text{call}} + D(t_{\text{set}})[K - f], \tag{7.5}$$

with

$$d_{1,2} = \frac{\log \frac{f}{K} \pm \frac{1}{2}\sigma_B^2 t_{\text{ex}}}{\sigma_B \sqrt{t_{\text{ex}}}}, \tag{7.6}$$

where the implied volatility $\sigma_B(f, K)$ is given by

$$\sigma_B(K, f) = \frac{\alpha}{(fK)^{(1-\beta)/2}} \left\{ 1 + \frac{(1-\beta)^2}{24} \log^2 \frac{f}{K} + \frac{(1-\beta)^4}{1920} \log^4 \frac{f}{K} + \dots \right\} \left( \frac{z}{x(z)} \right). \tag{7.7}$$

where

$$z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2} \log \frac{f}{K}, \tag{7.8}$$

$$\tag{7.9}$$

and x(z) is defined by

$$x(z) = \log \left\{ \frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1 - \rho} \right\}. \tag{7.10}$$

For the special case of at-the-money (ATM) options, options strike at $K = f$, this formula reduces to

$$\sigma_{ATM} = \sigma_B(f, f) = \frac{\alpha}{f^{1-\beta}} \left\{ 1 + \left( \frac{(1-\beta)^2}{24} \frac{\alpha^2}{f^{2-2\beta}} + \frac{\rho\beta\nu}{4} \frac{\alpha}{f^{1-\beta}} + \frac{2 - 3\rho^2}{24} \nu^2 \right) t_{\text{ex}} + \dots \right\}. \tag{7.11}$$

So we have that the parameters $\alpha, \beta, \nu$ and $\rho$ in the SABR model is estimated and the implied volatility $\sigma_B$ is a function of the forward price and the strike. Now that we have a simplified formula for the implied volatility from the SABR model, we can start analyzing how the model works. We will do this by continuing our analysis with investigating how the different parameters affects implied volatility in the SABR model.

## 7.4 Investigating the parameters of the SABR model

In this section, we will explore the parameters of the SABR model. This analysis will offer insights into how different parameters influence the behavior of implied volatility. We will conduct this investigation using the closed-form solution established in the previous section. Our study will be build on Equation 7.7 and Equation 7.11, where we will test various parameter values. To see how changes in the different parameters affects on the implied volatility smile so realistic as possible we fix the values of the parameters. Below in Table 6 the chosen values of the parameters are listed, together with a short explain of the parameters. Then the study is performed by changing the parameters one by one, to illustrate the isolated effect of the parameters.

| **Parameter** | **Parameter explanation** | **Value** |
|---|---|---|
| $f$ | Initial forward rate or asset price | 100 |
| $T$ | Time to maturity | 1 |
| $K$ | Strike | $K \in (80, 120)$ |
| $\alpha_0$ | Initial level of volatility | 0.1 |
| $\beta$ | Elasticity of the volatility | 0.5 |
| $\rho$ | Correlation between the asset price and its volatility | -0.4 |
| $\nu$ | Volatility of the volatility parameter | 0.25 |

Table 6: Summary of parameters used for investigating the SABR model

We will start the study with investigating how $\alpha_0$ affects the volatility smile in the SABR model. First we note that the fixed value of $\alpha_0$ is 0.1 Hereafter we adjust the value of $\alpha_0$ to 0.08 and 0.12, where the other parameters are kept fixed as listed in Table 6 above. So having changed the parameter up and down from the initial value of $\alpha_0$. We note that the parameter $\alpha_0$ represents "the initial level of volatility" as it is the starting point for the stochastic volatility process. Below in Figure 13 the volatility smile is illustrated for three different value of $\alpha_0$. From Figure 13 we see that the up and down movements don't change the shape of the volatility smile, it only shifts the volatility smile respectively up and down given the movement in $\alpha_0$.

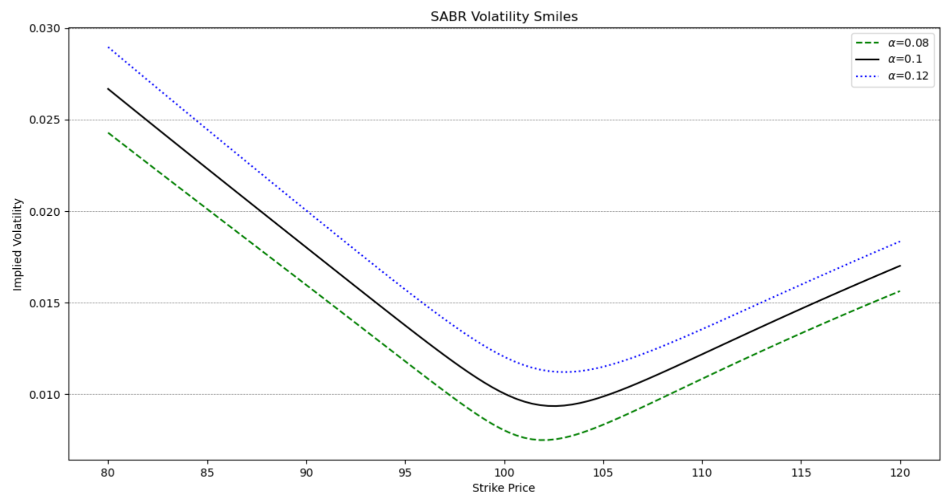



Figure 13: SABR model volatility smiles at various $\alpha_0$ levels

Then we will continue the study of the parameters affects by looking into shift in the beta value in the SABR model. Again we shift the value of $\beta$ up and down from the fixed beta in Table 6. So we will look at $\beta$ equal to 0.45, 0.5 (the fix beta) and 0.55. Although not explicitly constrained in the original article by Hagan et al. (2002), we restrict $\beta$ to the range $[0, 1]$. Choosing a $\beta$ less than 0 would lead to an illogical scenario where higher forward prices F result in a smaller relative change in the process $f_t$, thus we set the lower limit at $\beta \leq 0$. Similarly, a $\beta$ greater than 1 would suggest that the expected deviation from the current state of $F_t$ exceeds the product of volatility and the current forward level (times $\alpha_0\sqrt{t}$), which is also impractical. Therefore, we set the upper limit at $\beta \geq 1$.

Below in Figure 14 the volatility smile in the SABR model for the different beta values are illustrated. From Figure 14 we see a small effect of the curvature of the volatility smile. We also note that for higher beta the shift is larger, than for lower beta. We also note that the change in the beta parameter is more present in the left side from of strike value at-the-money (ATM). Other than that we see the same patterns from the change in the alpha parameter, namely the up and down shifts in the volatility smile, respectively to the movement in the beta parameter.

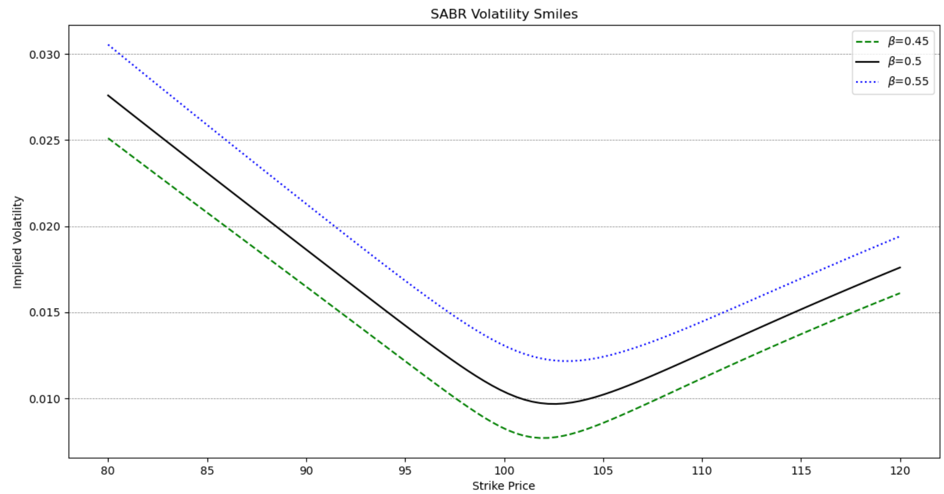



Figure 14: SABR model volatility smiles at various $\beta$ levels

Moving forward we will shift the $\nu$ parameter, which can be describe as the volatility of the volatility parameter. Again we shift the $\nu$ parameter up and down from the fixed $\nu$ parameter listed in Table 6. The volatility smile calculated from the SABR for the different $\nu$ parameters are illustrated in Figure 15 below. A very clear pattern emerges, namely that the parameter $\nu$ controls the curvature of the volatility smile in the SABR model. When we looked at the shifts in the beta parameter we made a comment that it made small changes in the curvature. But after this part of the analysis, we clearly see that the curvature is determine from the value of $\nu$. Which makes sense, when we think of the interpretation of the parameter describe before. We also note that increasing $\nu$ increased the level of the implied volatility for the out-of-the-money (OTM) strikes also increase and vise vera.

Then we look at the parameter for the correlation between the asset price and its volatility, namely $\rho$. So we have that $\rho$ is the correlation between to two Wiener process in the SABR model, note that $\rho$ is bounded and takes values between $-1$ and 1. In Figure 16 the shifted valus of $\rho$ from the fixed value is illustrated. We see that higher positive correlations generally show a decreasing trend in implied volatility with an increase in strike price, while strong negative correlations can cause the implied volatility to increase with strikes that are in-the-money (ITM). We alo note that when the correlation is close to zero the smile is more symmetric around the ATM strike. So in total we see that the correlation parameter $\rho$ has a significantly affects on the shape and slope of the volatility smile in the SABR model, and hence on the implied volatilities values.

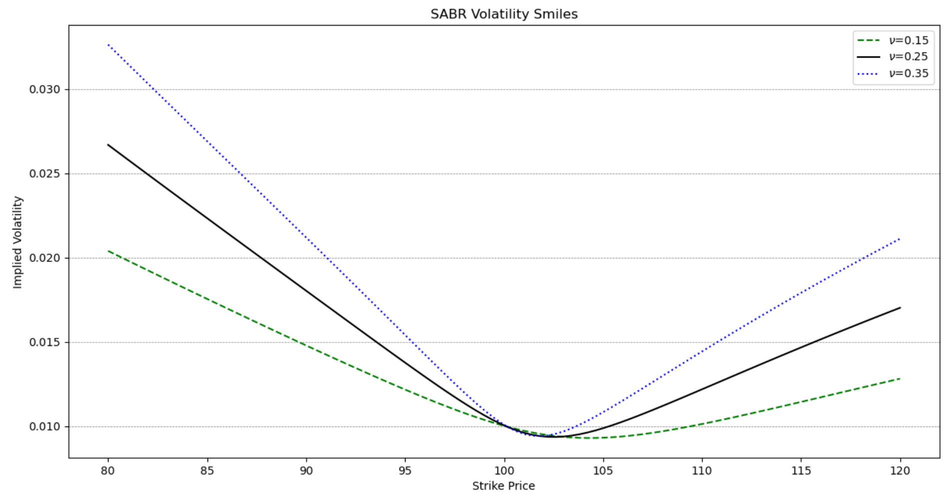



Figure 15: SABR model volatility smiles at various $\nu$ levels

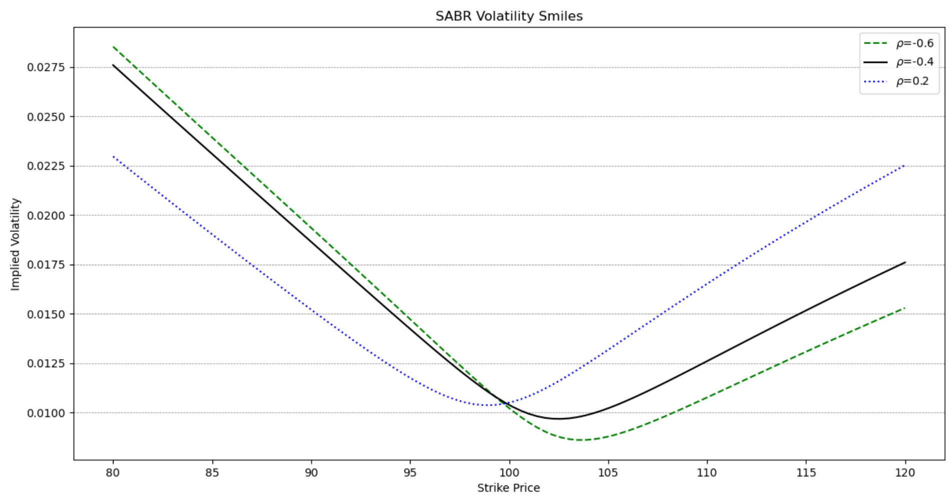



Figure 16: SABR model volatility smiles at various $\rho$ levels

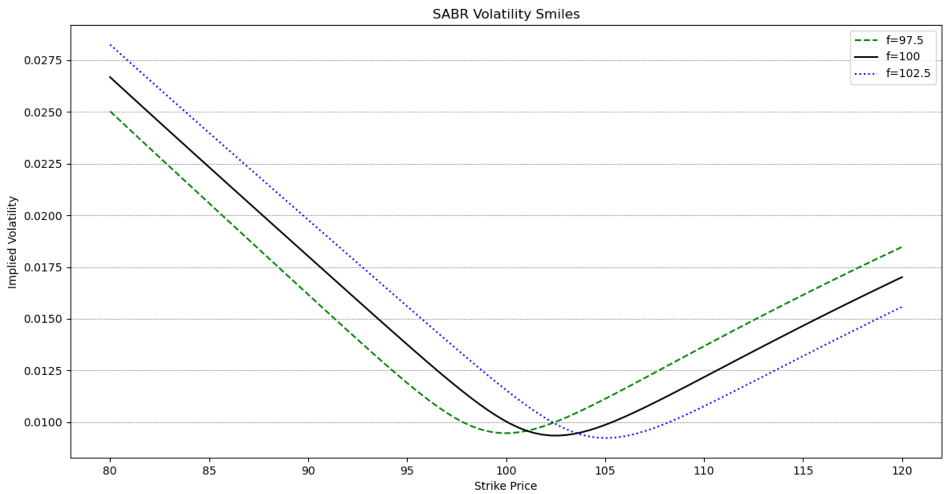



Figure 17: SABR model volatility smiles at various f levels

Finally we will investigate how the forward price, f, affect the volatility smile in the SABR model. The same procedure as for the other parameters will be used. So in Figure 17 the volatility smile for the forward prices are illustrated. We see that the slope, shape and curvature is retained, when there is only change in the forward price. Which is as we expected given how the forward price, f, enters in the closed solution for the implied volatility in Equation 7.7 and Equation 7.11.

To summarize this section provides knowledge of the impact of the various parameters in the closed-form solution for the implied volatility in the SABR model. This knowledge gives a good idea of how the model works, and positions us better to understand estimations of the model. So now we have looked at the closed-form solution and are ready for the next step of our analysis which is to look at estimating the parameters in the SABR model.

## 7.5 Estimating Parameters in the SABR model

In this section we will look into how we can estimate the parameters in the SABR model. So we want to estimate $\alpha$, $\beta$, $\rho$ and $\nu$. There are different methods for estimating the parameters in the SABR model. The various approaches distinguish themselves based on their methods for estimating or selecting $\beta$ and $\alpha$. $\beta$ may be either set to a predetermined best guess or fitted together with other parameters. Similarly, $\alpha$ may be estimated using Equation 7.14 or fitted concurrently with the other parameters.

We will proceed with the approach, where we chose some fixed $\beta$'s and the estimate $\alpha$ using Equation 7.14 and the setup a minimization problem there minimize $\rho$ and $\nu$. We chose this approach since earlier in Section 7.4 we noticed that changes in $\beta$ and $\rho$ gave some of the same effect on the volatility smile. Therefor we chose to fixe $\beta$, for different values of $\beta$. Lets then remind yourself of what the change in $\beta$ did to the volatility smile. We saw that when moving the $\beta$ value op and down, the corresponding change in the volatility smile appeared. Then remind yourself of the dynamic of the forward rate in the SABR model, listed in Equation 7.1. Where we consider the term $\alpha_t F_t^{\beta}$ as the total volatility of the forward rate process, it becomes apparent that decreasing $\beta$ generally amplifies the volatility and vise vera.

Then we consider the minimization problem to estimate the SABR model, where the objective is to align the market implied volatilities with the SABR model's volatility, which is computed for a set of strikes and the current forwards rate for combination of expiry and tenor [3]. Below the minimization problem is present with the parameters condition we argued for in Section 7.4.

$$\min_{\rho,\nu} \sum_{i=0}^{N_k} \Big( \sigma_{\text{BS}} - \sigma_{\text{SABR}}(f, \sigma_{\text{ATM}}, k_i, T, \rho, \nu) \Big)^2 \tag{7.12}$$

$$0 \leq \beta \leq 1, \quad -1 \leq \rho \leq 1, \quad 0 \leq \nu, \quad 0 \leq \alpha \tag{7.13}$$

There are also different approaches to estimating the $\alpha$ parameter. To estimate $\alpha$ we will use the method proposed in (Hagan et al., 2002), where the ATM SABR-formula is inverted numerically, where the ATM SABR-formula is listed in Equation 7.14 [6]. This is performed in Equation 7.14 to Equation 7.16 below. Another way to estimate $\alpha$ is to solve the minimization problem, but this will not be covered in the thesis.

$$\sigma_{\text{ATM}} = \frac{\alpha}{f^{1-\beta}} \left\{ 1 + \left[ \frac{(1-\beta)^2 \alpha^2}{24 f^{2-2\beta}} + \frac{\rho\beta\nu\alpha}{4 f^{1-\beta}} + \frac{(2-3\rho^2)\nu^2}{24} \right] t_{\text{ex}} \right\} \tag{7.14}$$

$$0 = A \cdot \alpha^3 + B \cdot \alpha^2 + C \cdot \alpha - \sigma_{\text{ATM}} f^{1-\beta} \tag{7.15}$$

where

$$A = \frac{(1-\beta)^2 T}{24 f^{2-2\beta}}, \quad B = \frac{\rho\beta\nu T}{4 f^{1-\beta}}, \quad C = \left[ 1 + \frac{2-3\rho^2}{24} \nu^2 \right] t_{\text{ex}} \tag{7.16}$$

Before we are ready to estimate the parameters numerically, lets first take a short look at the required data to performed the estimation study. As mentioned we need the current forward rates for any combination of expiry and tenor. In this analysis we will consider data where the expiry is 10 years and has various tenors namely [1Y,2Y,3Y,5Y,7Y,10Y,12Y,15Y,20Y,30Y]. Below in Table 7 the describe forward rates is present. To illustrated the development of the forward rates over the different tenors, these are illustrated in Figure 18 below. From the plot we see that the level of the forward rate is higher for shorter tenors than longer dated tenors.

| **Tenor** | 1Y | 2Y | 3Y | 5Y | 7Y | 10Y | 12Y | 15Y | 20Y | 30Y |
|---|---|---|---|---|---|---|---|---|---|---|
| **Forward rate** | 0.2938 | 0.2976 | 0.2996 | 0.2992 | 0.2943 | 0.2840 | 0.2763 | 0.2654 | 0.2517 | 0.2340 |

Table 7: Forward rates for ten years to expiry and various tenors. Data source Citi Velocity 21.02.2024

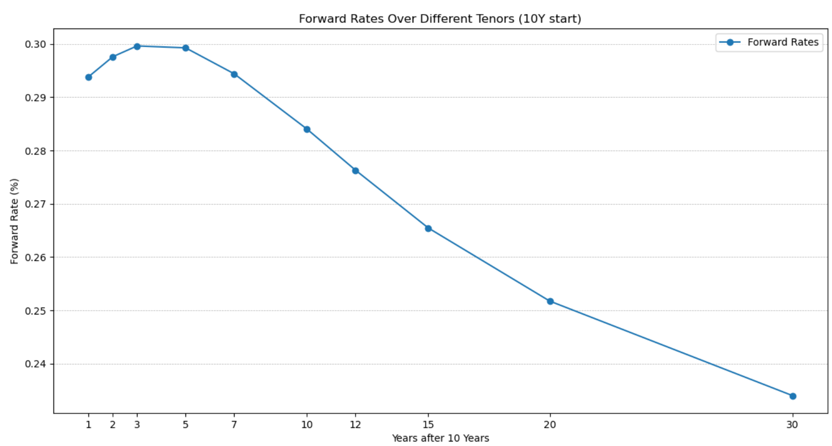



Figure 18: Forward rates for ten years to expiry and various tenors. Data source Citi Velocity 21.02.2024

So we have chosen the method where we chose a fixed $\beta$, so we chose the fixed $\beta$'a to be 0.5 and 1. And then we performed the described estimation for the parameters $\alpha$, $\rho$ and $\nu$. This estimation was performed across various combinations of expiry and tenors, resulting in two separate tables that display the estimated parameters when $\beta$ is fixed at either 0.5 or 1. The estimated values for $\alpha$, $\rho$, and $\nu$, with $\beta$ fixed at 0.5, are presented in Table 8. Similarly, the values for when $\beta$ is fixed at 1 are shown in Table 9.

Lets first comment on the estimated values in Table 8, where $\beta = 0.5$. The estimated parameters show generally moderate values for $\alpha$ across different tenor-expiry combinations, with a slight tendency to increase for longer expiries. The $\rho$ estimated values are predominantly positive, suggesting that higher rates correlate with higher volatility. Finally $\nu$ estimated values are high, indicating substantial volatility skew, which might suggest expectations of greater changes in rate movements over these periods. Then lets comment on the estimated values in Table 9, where $\beta = 1$. In this case, the estimated $\alpha$ values are relatively higher compared to $\beta = 0.5$, suggesting increased erratic behavior in volatility for these scenarios. The $\rho$ estimated values are also positive but exhibit a wider range across different tenors and expiries. The estimated $\nu$ values remain high, reinforcing the presence of a significant volatility skew.

| **Expiry x Tenor** | $\hat{\alpha}$ | $\hat{\rho}$ | $\hat{\nu}$ |
|---|---|---|---|
| 10Y x 1Y | 0.104213 | 0.369971 | 1.57081 |
| 10Y x 2Y | 0.103893 | 0.353521 | 1.56745 |
| 10Y x 3Y | 0.101618 | 0.335700 | 1.55828 |
| 10Y x 5Y | 0.098856 | 0.302105 | 1.53410 |
| 10Y x 7Y | 0.094310 | 0.275427 | 1.54121 |
| 10Y x 10Y | 0.088442 | 0.236242 | 1.54918 |
| 10Y x 12Y | 0.086733 | 0.245001 | 1.40264 |
| 10Y x 15Y | 0.082918 | 0.219553 | 1.50130 |
| 10Y x 20Y | 0.078033 | 0.204170 | 1.46493 |
| 10Y x 30Y | 0.076904 | 0.183612 | 1.40980 |

Table 8: $\hat{\alpha}$,$\hat{\rho}$ and $\hat{\nu}$ for fixed $\beta = 0.5$, using the SABR model.

| **Expiry x Tenor** | $\hat{\alpha}$ | $\hat{\rho}$ | $\hat{\nu}$ |
|---|---|---|---|
| 10Y x 1Y | 0.189190 | 0.311485 | 1.499733 |
| 10Y x 2Y | 0.187648 | 0.294726 | 1.501256 |
| 10Y x 3Y | 0.183161 | 0.277259 | 1.497992 |
| 10Y x 5Y | 0.178693 | 0.242966 | 1.483433 |
| 10Y x 7Y | 0.172151 | 0.217692 | 1.497899 |
| 10Y x 10Y | 0.164710 | 0.179948 | 1.515517 |
| 10Y x 12Y | 0.180309 | 0.176231 | 1.367529 |
| 10Y x 15Y | 0.15985 | 0.162386 | 1.472280 |
| 10Y x 20Y | 0.158105 | 0.145647 | 1.439637 |
| 10Y x 30Y | 0.158203 | 0.122134 | 1.389240 |

Table 9: $\hat{\alpha}$,$\hat{\rho}$ and $\hat{\nu}$ for fixed $\beta = 1$s, using the SABR model.

The analysis of the SABR model parameters, specifically at fixed $\beta$ values of 0.5 and 1, reveals distinctive volatility dynamics and correlation structures across various tenor-expiry combinations. At both $\beta$ values, the presence of a positive rho indicates that rate increases are likely correlated with heightened volatility. The notable distinctions at $\beta = 1$ are the higher alpha and nu values, suggesting a more erratic behavior and pronounced skewness in volatility. These characteristics are particularly significant for pricing and risk management in scenarios anticipating substantial rate dynamics. The insights garnered from this analysis are crucial for financial institutions and investors involved in hedging, pricing, or trading derivatives influenced by these model dynamics.

To end this section we will look at estimated parameters despited by the volatility smile. We chose to look a the volatility smile, for estimated values where $\beta$ was fixed to be 0.5. Since this is the most common choice for $\beta$. We despited the combination where the tenor is 10 years, i large format in Figure 19 below and all combined of the swaption volatility smile with expiry in 10 years, and the various tenors are despited in Figure 20a to Figure 20f below. From Figure 19 we se that estimated volatility smile fits the market data good, which make sense from how we estimated the parameters to find the volatility smile. We note that estimated values for $\nu$ in both Table 8 and Table 9, are relative high. But this align with the steepness or level we see the in the volatility smile in the market data. The volatility smile tends to be more convex we $\nu$ is large and the is clearly the case here. This tendency appearers also i most of the volatility smile despited in Figure 20a to Figure 20f.

Then we should ask yourself how does the steepness or level we see the in the volatility smile in the market data affects the price of a swaption. When we are determine swaption we should also consider with sensitivities there are related to the price of the swaption. The above analysis pointing at there could be a sensitive related to $\nu$. Therefor the next step is to look closer at the models sensitivities.

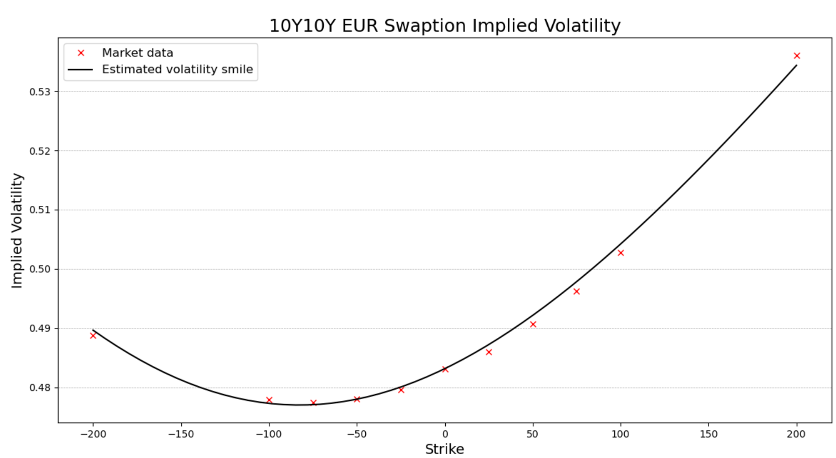
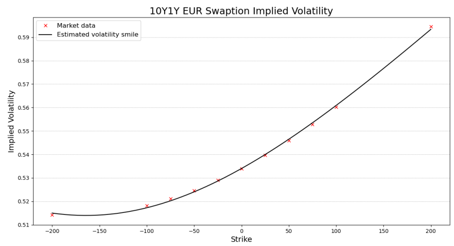
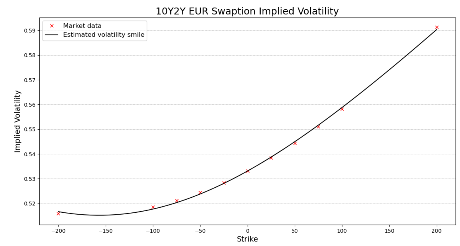


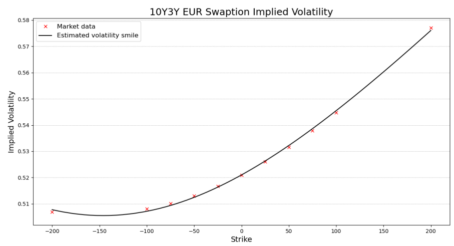

Figure 19



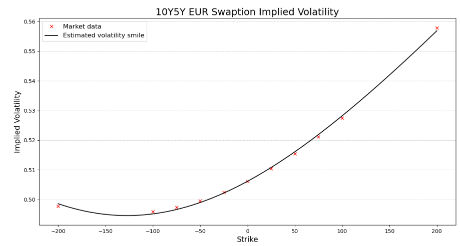

(a) EUR swaption 10Y1Y



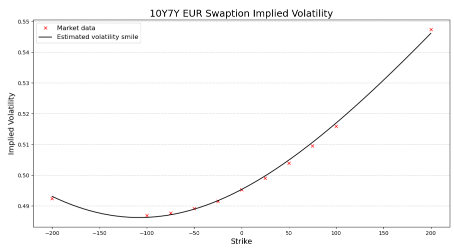

(b) EUR swaption 10Y2Y



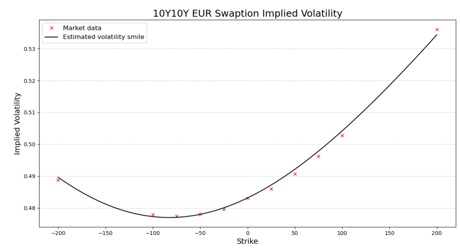

(c) EUR swaption 10Y3Y



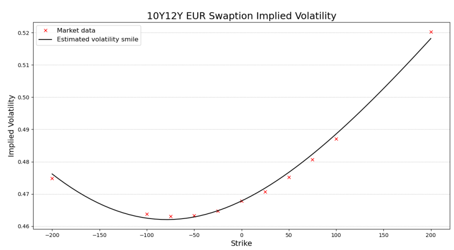

(d) EUR swaption 10Y5Y



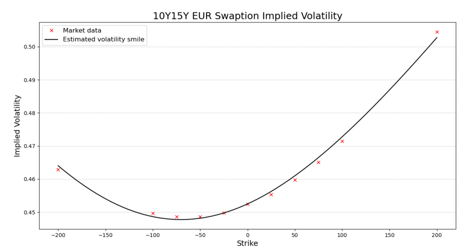

(e) EUR swaption 10Y7Y



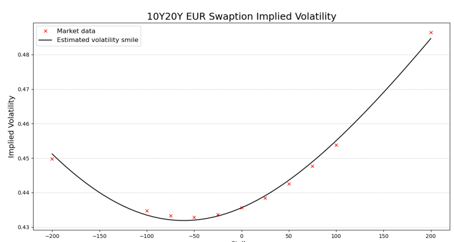

(f) EUR swaption 10Y10Y



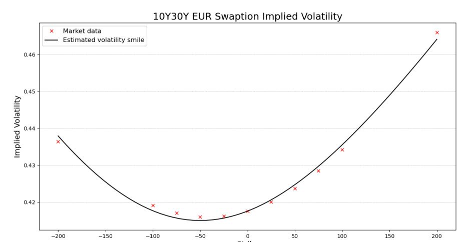

(g) EUR swaption 10Y12Y




(h) EUR swaption 10Y15Y




(i) EUR swaption 10Y20Y




(j) EUR swaption 10Y30Y

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