Homework 2 for CS-613

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1. Answer

Assume $x \in \mathbb{C}^n$ and $x = (x_1, x_2, ..., x_n)^T$. So its conjugate transpose is $x' = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$.
We also know that for $t, s \in (0, \infty)$,

$$k(t, s) = \frac{1}{t+s} = \int_0^\infty e^{-(t+s)u} du$$

For any $\omega_1, \omega_2, ..., \omega_n \in (0, \infty)$, assume $A = (k(\omega_j, \omega_k))_{j,k=1}^n$. Then it has

$$\begin{aligned}
x'Ax &= (\bar{x}_1, ..., \bar{x}_n) \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= (\bar{x}_1, ..., \bar{x}_n) \begin{bmatrix} \int_0^\infty e^{-(\omega_1+\omega_1)u} du & \cdots & \int_0^\infty e^{-(\omega_1+\omega_n)u} du \\ \vdots & \ddots & \vdots \\ \int_0^\infty e^{-(\omega_n+\omega_1)u} du & \cdots & \int_0^\infty e^{-(\omega_n+\omega_n)u} du \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= (\bar{x}_1, ..., \bar{x}_n) \begin{bmatrix} \int_0^\infty e^{-\omega_1 u} (x_1 e^{-\omega_1 u} + \cdots + x_n e^{-\omega_n u}) du \\ \vdots \\ \int_0^\infty e^{-\omega_n u} (x_1 e^{-\omega_1 u} + \cdots + x_n e^{-\omega_n u}) du \end{bmatrix} \\
&= \int_0^\infty (\bar{x}_1 e^{-\omega_1 u} + \cdots + \bar{x}_n e^{-\omega_n u})(x_1 e^{-\omega_1 u} + \cdots + x_n e^{-\omega_n u}) du \\
&= \int_0^\infty \overline{f(u)} f(u) du
\end{aligned}$$

Where $f(u) = x_1 e^{-\omega_2 u} + \cdots + x_n e^{-\omega_n u}$ and $\overline{f(u)}$ is the conjugate of $f(u)$. Since $f(u)\overline{f(u)} \geq 0$, $\int_0^\infty \overline{f(u)} f(u) du \geq 0$ and hence $x'Ax \geq 0$. Hence, A is positive and therefore, function $k(t,s)$ is positive definite on $(0, \infty)$.

2. Answer

The answer is basically the same as that in 1, except that matrix A is modified. Follow the same notation as in 1, I am going to show that A is still positive. Before that, recall that Taylor expansion of e^{ts} at 0 is $\sum\limits_{k=0}^{\infty} \frac{t^k s^k}{k!}$. So it has

$$
\begin{aligned}
x'Ax &= (\bar{x}_1, ..., \bar{x}_n) \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= (\bar{x}_1, ..., \bar{x}_n) \begin{bmatrix} e^{\omega_1 \omega_1} & \cdots & e^{\omega_1 \omega_n} \\ \vdots & \ddots & \vdots \\ e^{\omega_n \omega_1} & \cdots & e^{\omega_n \omega_n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= (\bar{x}_1, ..., \bar{x}_n) \begin{bmatrix} \sum\limits_{k=0}^{\infty} \frac{{\omega_1}^k {\omega_1}^k}{k!} & \cdots & \sum\limits_{k=0}^{\infty} \frac{{\omega_1}^k {\omega_n}^k}{k!} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k=0}^{\infty} \frac{{\omega_n}^k {\omega_1}^k}{k!} & \cdots & \sum\limits_{k=0}^{\infty} \frac{{\omega_n}^k {\omega_n}^k}{k!} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= \sum_{k=0}^{\infty} \left((\bar{x}_1, ..., \bar{x}_n) \begin{bmatrix} \frac{{\omega_1}^k {\omega_1}^k}{k!} & \cdots & \frac{{\omega_1}^k {\omega_n}^k}{k!} \\ \vdots & \ddots & \vdots \\ \frac{{\omega_n}^k {\omega_1}^k}{k!} & \cdots & \frac{{\omega_n}^k {\omega_n}^k}{k!} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right) \\
&= \sum_{k=0}^{\infty} \left((\bar{x}_1, ..., \bar{x}_n) \begin{bmatrix} \frac{{\omega_1}^k}{k!} (x_1 {\omega_1}^k + \cdots + x_n {\omega_n}^k) \\ \vdots \\ \frac{{\omega_n}^k}{k!} (x_1 {\omega_1}^k + \cdots + x_n {\omega_n}^k) \end{bmatrix} \right) \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{k!} (\bar{x}_1 {\omega_1}^k + \cdots + \bar{x}_n {\omega_n}^k)(x_1 {\omega_1}^k + \cdots + x_n {\omega_n}^k) \right) \\
&= \sum_{k=0}^{\infty} \left(\frac{\bar{g}_k g_k}{k!} \right)
\end{aligned}
$$

where $g_k = x_1 {\omega_1}^k + \cdots + x_n {\omega_n}^k$ and $\bar{g}$ is the conjugate of g. So $\bar{g}_k g_k \geq 0$. It means that $\sum_{k=0}^{\infty}(\frac{\bar{g}_k g_k}{k!}) \geq 0$ and hence $x'Ax \geq 0$. So A is positive and function $k(t,s) = e^{ts}$ is positive definite on the real line.

3. Answer

Below in Table 1 is a boolean function. Suppose the output is Y, then it has

$$Y = ABCD \vee AB\bar{C}\bar{D} \vee A\bar{B}\bar{C}\bar{D}$$

Figure 1 shows how a MLP can represent this boolean function.

Table 1: A boolean function

A	B	C	D	Output
1	1	1	1	1
1	1	0	0	1
1	0	0	0	1

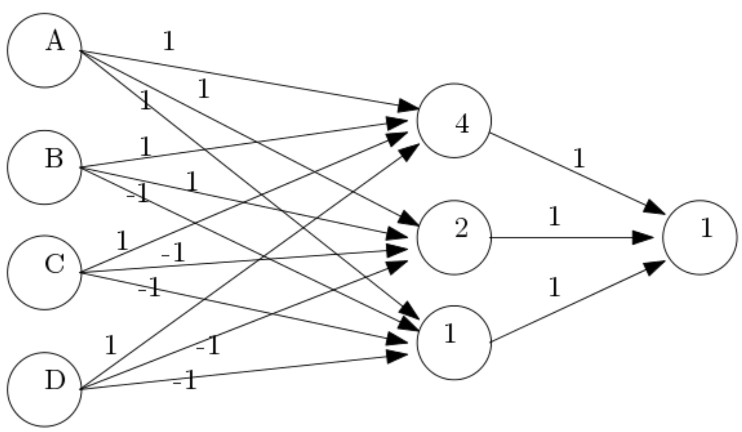

Figure 1: A MLP to represent the boolean function Y

4. Answer

The classical neuron doctrine, which was first confirmed by Santiago Felipe Ramon y Cajal (1852-1934), says that the nervous system was composed of discrete cellular units. Neurons are independent cells. They function independently within the nervous system. It is also a basic information processing unit. Different neurons communicate with each other through axons.

The classical programming takes input and the program and then, generates output. To let the process work, programmer has to define the rules in advance. In the artificial neuron network method, we give the network input and output and then, train it to figure out the best rule for the programmer. And that is implied by how neurons work. The neurons are not coded in advance. The brain is trained to learn new things. As the basic unit, neurons deal with the information and therefore form a learning process.

5. Answer

To simplify the notation, assume $X = h_1 \wedge h_2$, $Y = h_1 \vee h2$ and $Z = h_1 \triangle h_2$. I claim their relationships as follows.

Claim 1: $Y \geq_g X$.

For a ω such that $X(\omega) = 1$, it means $h_1(\omega) = h_2(\omega) = 1$. Hence, $Y(\omega) = 1$. Therefore, $Y \geq_g X$.

Claim 2: $Y \geq_g Z$.

For a ω such that $Z(\omega) = 1$, it means $h_1(\omega) = 1$ and $h_2(\omega) = 0$, or $h_1(\omega) = 0$ and $h_2(\omega) = 1$. In both cases, it has $Y(\omega) = 1$. Therefore, $Y \geq_g Z$.

6. Answer

See the attached file.