Homework 2 for CS-613

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1. Answer

Assume $x \in \mathbb{C}^n$ and $x = (x_1, x_2, ..., x_n)^T$. So its conjugate transpose is $x' = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$.
We also know that for $t, s \in (0, \infty)$,

$$k(t,s) = \frac{1}{t+s} = \int_0^\infty e^{-(t+s)u} du$$

For any $\omega_1, \omega_2, ..., \omega_n \in (0, \infty)$, assume $A = (k(\omega_j, \omega_k))_{j,k=1}^n$. Then it has

$$\begin{aligned}
x'Ax &= (\bar{x}_1, ..., \bar{x}_n) \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= (\bar{x}_1, ..., \bar{x}_n) \begin{bmatrix} \int_0^\infty e^{-(\omega_1+\omega_1)u} du & \cdots & \int_0^\infty e^{-(\omega_1+\omega_n)u} du \\ \vdots & \ddots & \vdots \\ \int_0^\infty e^{-(\omega_n+\omega_1)u} du & \cdots & \int_0^\infty e^{-(\omega_n+\omega_n)u} du \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= (\bar{x}_1, ..., \bar{x}_n) \begin{bmatrix} \int_0^\infty e^{-\omega_1 u}(x_1 e^{-\omega_1 u} + \cdots + x_n e^{-\omega_n u}) du \\ \vdots \\ \int_0^\infty e^{-\omega_n u}(x_1 e^{-\omega_1 u} + \cdots + x_n e^{-\omega_n u}) du \end{bmatrix} \\
&= \int_0^\infty (\bar{x}_1 e^{-\omega_1 u} + \cdots + \bar{x}_n e^{-\omega_n u})(x_1 e^{-\omega_1 u} + \cdots + x_n e^{-\omega_n u}) du \\
&= \int_0^\infty \overline{f(u)} f(u) du
\end{aligned}$$

Where $f(u) = x_1 e^{-\omega_2 u} + \cdots + x_n e^{-\omega_n u}$ and $\bar(f(u))$ is the conjugate of $f(u)$. Since $f(u)\overline{f(u)} \geq 0$, $\int_0^\infty \overline{f(u)} f(u) du \geq 0$ and hence $x'Ax \geq 0$. Hence, A is positive and therefore, function $k(t,s)$ is positive definite on $(0, \infty)$.

2. Answer

The answer is basically the same as that in 1, except that matrix A is modified. Follow the same notation as in 1, I am going to show that A is still positive.

$$
\begin{aligned}
x'Ax &= (\bar{x}_1, ..., \bar{x}_n) \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= (\bar{x}_1, ..., \bar{x}_n) \begin{bmatrix} e^{\omega_1+\omega_1} & \cdots & e^{\omega_1+\omega_n} \\ \vdots & \ddots & \vdots \\ e^{\omega_n+\omega_1} & \cdots & e^{\omega_n+\omega_n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= (\bar{x}_1, ..., \bar{x}_n) \begin{bmatrix} e^{\omega_1}(x_1 e^{\omega_1} + \cdots + x_n e^{\omega_n}) \\ \vdots \\ e^{\omega_n}(x_1 e^{\omega_1} + \cdots + x_n e^{\omega_n}) \end{bmatrix} \\
&= (\bar{x}_1 e^{\omega_1} + \cdots + \bar{x}_n e^{\omega_n})(x_1 e^{\omega_1} + \cdots + x_n e^{\omega_n}) \\
&= \overline{g(u)} g(u)
\end{aligned}
$$

Where $g(u) = x_1 e^{\omega_1} + \cdots + x_n e^{\omega_n}$ and $\overline{g(u)}$ is the conjugate of $g(u)$. Since for $i = 1, 2, .., n$, w_i are real numbers, $\overline{g(u)} g(u) \geq 0$ and hence $x'Ax \geq 0$. So A is positive and function $k(t,s) = e^{ts}$ is positive definite on real line.

3. Answer

Below in Table 1 is a boolean function. Suppose the output is Y, then it has

$$Y = ABCD \vee AB\bar{C}\bar{D} \vee A\bar{B}\bar{C}\bar{D}$$

Figure 1 shows how a MLP can represent this boolean function.

4. Answer

Table 1: A boolean function

A	B	C	D	Output
1	1	1	1	1
1	1	0	0	1
1	0	0	0	1

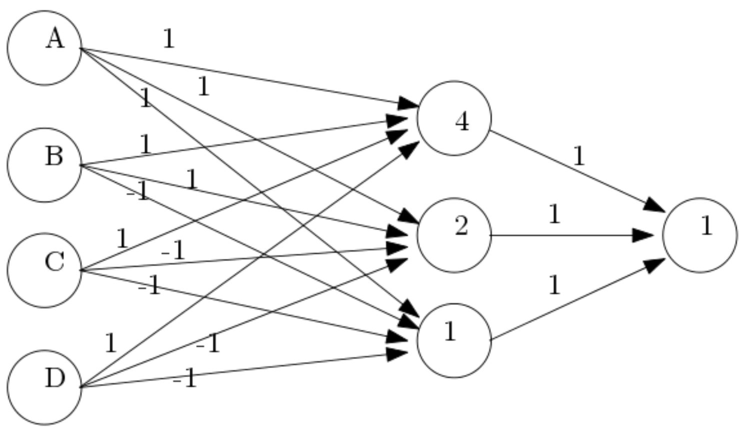

Figure 1: A MLP to represent the boolean function Y

5. Answer

To simplify the notation, assume $X = h_1 \wedge h_2$, $Y = h_1 \vee h2$ and $Z = h_1 \triangle h_2$. I claim their relationships as follows.

Claim 1: $Y \geq_g X$.

For a ω such that $X(\omega) = 1$, it means $h_1(\omega) = h_2(\omega) = 1$. Hence, $Y(\omega) = 1$. Therefore, $Y \geq_g X$.

Claim 2: $Y \geq_g Z$. For a ω such that $Z(\omega) = 1$, it means $h_1(\omega) = 1$ and $h_2(\omega) = 0$, or $h_1(\omega) = 0$ and $h_2(\omega) = 1$. In both cases, it has $Y(\omega) = 1$. Therefore, $Y \geq_g Z$.