Trig Identities

$tan(x) = \frac{sin(x)}{cos(x)}$, $csc(x) = \frac{1}{sin(x)}$, $sec(x) = \frac{1}{cos(x)}$
$sin^2(x) + cos^2(x) = 1$, $tan^2(x) + 1 = sec^2(x)$
$cot^2(x) + 1 = csc^2(x)$, $sin(2\theta) = 2sin(\theta)cos(\theta)$
$cos(2\theta) = cos^2(\theta) - sin^2(\theta)$
$tan(2\theta) = \frac{2tan(\theta)}{1-tan^2(\theta)}$, $sin^2(x) = \frac{1-cos(2x)}{2}$
$cos^2(x) = \frac{1+cos(2x)}{2}$

Derivatives

$d(sinx) = cosx$, $d(cosx) = -sinx$, $d(tanx) = sec^2x$
$d(secx) = secxtanx$, $d(cscx) = -cscxcotx$
$d(cotx) = -csc^2x$
$d(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$, $d(cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$
$d(tan^{-1}x) = \frac{1}{1+x^2}$, $d(sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$
$d(csc^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$, $d(cot^{-1}x) = \frac{-1}{1+x^2}$

Integrals

$\int sinx = -cosx + C$, $\int cosx = sinx + C$, $\int tanx = -ln|cosx| + C$, $\int secx = ln|secx + tanx| + C$, $\int cscx = -ln|cscx + cotx| + C$, $\int cotx = ln|sinx| + C$, $\int sec^2x = tanx + C$, $\int csc^2x = -cotx + C$, $\int secxtanx = secx + C$, $\int cscxcotx = -cscx + C$, $\int tan^2x = tanx - x + C$, $\int cot^2x = -cotx - x + C$, $\int tanx = -ln|cosx| + C$, $\int cotx = ln|sinx| + C$, $\int \frac{1}{a^2+x^2} = \frac{1}{a}tan^{-1}(\frac{x}{a}) + C$, $\int \frac{1}{\sqrt{a^2+x^2}} = ln|x + \sqrt{a^2+x^2}| + C$, $\int \frac{1}{\sqrt{a^2-x^2}} = sin^{-1}(\frac{x}{a}) + C$, $\int \frac{1}{x^2-a^2} = \frac{1}{2a}ln|\frac{x-a}{x+a}| + C$

Ch1: Basic Definitions

Ordinary: single independent variable. **Partial**: multiple independent variables. **Order**: highest derivative. **Linear diffeq**: additive combination of first powers: $a_n(x)\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)$.

Example of linear: $t^3\frac{dx}{dt} - x = t^3$. $\frac{dy}{dx} = sin(x)$ bc $0 * y$ **Nonlinear**: $\frac{d^2y}{dy^2} + y^3 = 0$ bc of y^3 term.

Another: $\frac{d^2y}{dx^2} - y\frac{dy}{dx} = cosx$ bc of ydy term.

Explicit Soln: function $\phi(x)$ when sub for y in eqn satisfies for $\forall x$ in interval. **Implicit Soln**: Verify by differentiate implicit soln wrt ind. var on both sides to check = orig.

Note: Diff of y wrt x becomes $\frac{dy}{dx}$

Existence & Uniqueness: 1. Find the form of the differential equation $\frac{dy}{dx} = f(x,y)$. 2. Identify the function $f(x,y)$ and the partial derivative $\frac{\partial y}{\partial x}$ such that both are continuous in the rectangle $(x, f(x))$. **Autonomous eqns**: $y' = f(y)$, rhs is fxn of dependent var only.

Method of Isoclines: $y' = f(x,y) = c$, solve for y to make isoclines (dotted lines), slope same along isocline.

Euler's Method: $y_{n+1} = y_n + hf(x_n, y_n)$, where h is step size ($\frac{dist}{numstep}$).

Ch2: Solving Linear 1st Order

Separable: $y' = f(x)g(y)$, sep. into $\frac{dy}{g(y)} = f(x)dx$, integrate both sides.

Linear $a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$. Special cases: $a_0(x) = 0$(no y term) then $y(x) = \int \frac{b(x)}{a_1(x)}dx + C$ $a_0(x) = a_1'(x)$ product rule $y(x) = \frac{1}{a_1(x)}(\int b(x)dx + C)$, **Standard Form**: $\frac{dy}{dx} + P(x)y = Q(x)$, **Integrating Factor**: $\mu(x) = exp(\int P(x)dx)$, Solve: $\mu(x)y(x) = \int \mu(x)Q(x)dx + C$, **Exact**: Form $M(x,y)dx + N(x,y)dy = 0$, check $M_y = N_x$ for exactness, Solving: 1. Text exactness, 2. $F(x,y) = \int M(x,y)dx + g(y)$, 3. $\frac{\partial F}{\partial y} = N(x,y)$ to get $g'(y)$, 4. $g(y) = \int g'(y)dy$, 5. sub $g(y)$ into $F(x,y) = C$, **Exactness Test Fail**: check $\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$ only depends on x or $\frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$ only depends on y, then $\mu(\cdot) = exp(\int(\cdot)d(\cdot))$, plug in $\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$, solve with exact. **Homogenous**: 1. Express dy/dx in terms of y/x, 2. Sub $v = y/x = f(x,v)$, 3. $x(dv/dx) + v = f(x,v)$ solve with separable, 4. Resub $v = y/x$. **Solving** $dy/dx = G(ax+by)$: 1. sub $z = ax + by$, 2. diff $dz/dx = a + b(dy/dx)$ then sub $dy/dx = (dz/dx - a)/b$, 3. solve separable 4. resub $ax + by = z$. **Bernoulli**: 1. $dy/dx + P(x)y = Q(x)y^n$, $v = y^{1-n}$, 2. $1/(1-n)dv/dx + P(x)v = Q(x)$, 3. solve linear w/ $\mu(x)$. **Linear Coeff**: Special cases: 1. $a_1b_2 = a_2b_1$ then $z = ax + by$, 2. $b_1 = a_2$ then exact, 3. $c_1 = c_2 = 0$ then homogenous. **Coeff Solve**: 1. solve system $a_1h + b_1k = -c$ and $a_2h + b_2k = -c$ for h,k, 2. sub in $x = u + h$ and $y = v + k$ for form $M(u,v)du + N(u,v)dv = 0$, 3. Rearr for dv/du, multiply by $(1/u)/(1/u)$ then sub $z = v/u$, 4. $u(dz/du) + z = step3$, 5. solve separable (partial frac) 6. resub z, u, v.

Ch3: 1st Order Applications

Mixing Problems: $dx/dt = r_{in} - r_{out}$, where r(t) can also be salt concentration, if so, then $dx/dt = r_{in}c_{in} - r_{out}x(t)/V$, where x(t) is [salt] **Malthusian/Exponential Model**: $dP/dt = kp, p(0) = p_0$, $p(t) = p_0 * exp(kt)$ **Logistic Model**: $dP/dt = k_1p - k_2\frac{p(p-1)}{2}$, $p(t) = \frac{p_0p_1}{p_0+(p_1-p_0)exp(-Ap_1t)}$ **Newton's Law of Cooling**: $T(t) = exp(-kt)\int exp(kt)(kM + H + U)dt + C * exp(-kt)$, where M is initial temp, H is heat (≥ 0), U is heating(+)/cooling rate(-). **Oscillating M Problem**: Given time constant (1/k) and M varies as sine wave w/ min/max time/temp given, find min/max time/temp of building. 1. Find period $\omega = 2\pi/24hr$, $T_{high} = M_0 + B, T_{low} = M_0 - B$, solve for $M_0 = B_0, B$ 2. $T(t) = B_0 - BF(t)$ $Cexp(-kt)$ dies off, $F(t) = \pm 1/\sqrt{1 + (k/\omega)^2}$, 3. Find T_{high} and T_{low} by plugging in F(t) w/ $/pm1$, 4. Find t_{high} and t_{low} by plugging in T_{high} and T_{low} into $T(t) = M_0 + (T_0 - M_0)exp(-kt)$.

Physics: $F(t,v) = m\frac{dv}{dt} = mg - bv, v(0) = v_0$
$v(t) = \frac{mg}{b} + (v_0 - \frac{mg}{b})exp(-bt/m)$
$x(t) = \frac{mg}{b}t + (v_0 - \frac{mg}{b})\frac{m}{b}(1 - exp(-bt/m))$

Parachute Problem: Find distance (d) fallen after given t_1 w/o parachute w/ $x_1(t_1), b_1$, calc $v(t)$ to use as v_0 in $v_2(t_2)$, distance left $d = h - x_1(t_1)$, Solve for t_2 in $x_2(t_2) = d$, total time $t_{total} = t_1 + t_2$. **RL Circuit**: $exp(Rt/L)I(t) = \int exp(Rt/L)(E(t)/L)dt + K$ to $I(t) = I_{max}(1 - exp(-Rt/L))$, Resistor Voltage: $E_R(t) = I(t)$. Inductor Voltage: $E_L(t) = L(dI/dt)$. Capacitor Voltage: $E_C(t) = q/C$. Capacitor Current: $R(dq/dt) + q/C = E(t)$. **RC Circuit**: $exp(t/RC)q(t) = \int exp(t/RC)(E(t)/R)dt + K$. If $E(t) = V = const$, then $q(t) = CV + (Q - CV)exp(-t/RC)$, Q is initial charge. If $Q = 0$, then $V(t) = V_{max}(1 - exp(-t/RC))$.

Ch4: Linear 2nd Order Applications

Const coeff, Homogenous ($= 0$): Form $ay'' + by' + cy = 0$ where $a,b,c \in \mathbb{R}$ and $a \neq 0$. Solve aux eqn $ar^2 + br + c = 0$ for r_1, r_2. General form: $y(t) = c_1y_1 + c_2y_2$, Cases: 1. $r_1 \neq r_2 \in \mathbb{R}, y_i = e^{r_it}$ 2. $r_1 = r_2 \in \mathbb{R}, y_1 = e^{rt}, y_2 = te^{rt}$ 3. $r \in \mathbb{I}, y_1 = e^{\alpha t}\cos\beta t, y_2 = e^{\alpha t}\sin\beta t$

Method of Undetermined Coeffs: If $f(t) = Ct^m$ then guess $y = A_mt^m + \cdots + A_1t + A_0$. If $f(t) = Ce^{rt}$ then guess $y = Ae^{rt}$. If $f(t) = C\{cos|sin\}(\beta t)$ then guess $y = Asin(\beta t) + Bcos(\beta t)$. If form $ay'' + by' + cy = Ct^me^{rt}$ then guess $y = t^s(A_m + \cdots + A_1t + A_0)e^{rt}$ where $r \in \mathbb{I} \rightarrow s = 0, r_1 \neq r_2 \in \mathbb{R} \rightarrow s = 1, r_1 = r_2 \in \mathbb{R} \rightarrow s = 2$. If form $ay'' + by' + cy = Ct^m\{cos|sin\}(\beta t)$ then guess $y = t^s(A_mt^m + \cdots + A_1t + A_0)e^{at}\cos(\beta t) + t^s(B_mt^m + \cdots + B_1t + B_0)e^{at}\sin(\beta t)$ where $s = 1$ if $\alpha + i\beta$ from $f(t)$ is a root of the aux eqn else $s = 0$. **solving**: 1. Get y_p guess with rules 2. find y, y', y'' and plug into original diffeq. Solve for coeffs by equating to f(t) then plug coeffs into y_p.

Variation of Parameters: $y_p = v_1y_1 + v_2y_2$, $v_1 = \int \frac{-f(t)y_2}{aW[y_1,y_2]}dt$ and $v_2 = \int \frac{f(t)y_1}{aW[y_1,y_2]}dt$. Solve: 1. find y_h with aux eqn to get y_1, y_2 2. Find $W[y_1,y_2] = y_1y_2' - y_2y_1'$ 3. Find v_1, v_2 4. Plug into y_p.

Const coeff, Non-homogenous ($\neq 0$): Form $ay'' + by' + cy \neq 0$. General soln: $y = y_p + c_1y_1 + c_2y_2$ where y_p is particular soln and y_1, y_2 are homogenous general soln. To find y_p, use method of undetermined coeffs if $f(t)$ is a polynomial, exponential, or sin/cos function. Otherwise use variation of parameters to find homogenous general soln then compute v1, v2 to find y_p. Use superposition if $f(t)$ is a sum of functions so solve for each function then add.

Cauchy-Euler Form: $at^2y'' + bty' + cy = 0$ then 1. solve $ar^2 + (b-a)r + c = 0$ 2. Calc y_1, y_2 for general soln. 3. Show $W[y_1,y_2] \neq 0$ for y_p. Cases: 1. $r_1 \neq r_2 \in \mathbb{R} \rightarrow y = t^r$ 2. $r_1 = r_2 \in \mathbb{R} \rightarrow y = t^r, t^r\ln t$ 3. $r \in \mathbb{I} \rightarrow y = t^\alpha\{cos|sin\}(\beta\ln t)$

Reduction of Order given y_1: Convert to std form $y'' + p(t)y' + q(t)y = 0$ then $y_2 = y_1\int\frac{exp(-\int p(t)dt)}{y_1^2}dt$

Variable coeff, Homogenous ($= 0$): Form $a_2y'' + a_1y' + a_0y = 0$. No general method but can if Cauchy-Euler or Reduction of order given y_1. General form $y = c_1y_1 + c_2y_2$.

Variable coeff, Non-homogenous ($\neq 0$): Form $a_2y'' + a_1y' + a_0y = f(t)$. To find y_p, 1. find homogenous general soln. 2. put into std form 3. find v_1, v_2 using variation of params. General form $y = y_p + c_1y_1 + c_2y_2$.

Spring Motion Eqns

Free Mech Vibrations ($F_{ext} = 0$): eqn of spring motion $my'' + by' + ky = 0$ where m is mass, b is damping, k is spring const, F_{ext} is external force.

Undamped ($b = 0$): Form $my'' + ky = 0$. $w = \sqrt{\frac{k}{m}}$ is natural freq. General soln $y = c_1\cos wt + c_2\sin wt$. Or $y = Asin(wt + \phi)$ where amplitude $A = \sqrt{c_1^2 + c_2^2}$, phase angle $\phi = \tan^{-1}(\frac{c_1}{c_2})$, and period wt. How long after release does mass pass through equilibrium pos? Steps: 1. Find y(t) and y'(t). 2. Solve IVP. 3. Convert to A, ϕ form. 4. $t = (n\pi - \phi)/w$.

Underdamped ($b < 4mk$): Roots: $\alpha + i\beta$ where $\alpha = -b/2m$ and $\beta = (1/2m)\sqrt{4mk - b^2}$. Gen Soln: $y = c_1 e^{\alpha t} cos(\beta t) + c_2 e^{\alpha t} sin(\beta t) = Ae^{\alpha t} sin(\beta t + \phi)$ when A, ϕ, same as undamped. Obs: y oscillates btwn $-Ae^{\alpha t}$ and $Ae^{\alpha t}$, with quasiperiod $2\pi/\beta$ as $t \to \infty$, $Ae^{\alpha t} \to 0$ approach equilibrium.

Overdamped ($b > 4mk$): Roots: $r_{1,2} = \frac{-b}{2m} \pm \frac{1}{2m}\sqrt{b^2 - 4mk}$. Gen Soln: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$. Obs: as $t \to \infty$, $y \to 0$ approach equilibrium. y(t) has 1 min/max or none (max at $t = 0$). $y' = 0$ iff $c_1 r_1 + c_2 r_2 exp(r_2 - r_1) = 0$.

Critically Damped ($b = 4mk$): Roots: $r_1 = r_2 = -b/2m$ Gen Soln: $y = (c_1 + c_2 t)exp(rt)$. Obs: Same as overdamped, but $y' = 0$ iff $c_2 + c_1 r + c_2 r = 0$.

Forced Mech Vibrations ($F_{ext} = F_0 cos(\gamma t)$): Gen Soln: $y = y_p + Ae^{\alpha t} sin(\beta t + \phi)$ where α, β from underdamped homogenous is transient part ($\to 0$).

Undamped: use cos/sin from methods of undetermined variables.

Underdamped: use $y_p = Ae^{\alpha t} cos(\gamma t) + Be^{\alpha t} sin(\gamma t)$ that matches format of F_{ext}. Then find $A = \frac{F_0(k - m\gamma^2)}{(k - m\gamma^2)^2 + b^2\gamma^2}, B = \frac{F_0 b\gamma}{(k - m\gamma^2)^2 + b^2\gamma^2}$. Or $y_p = (F_0/\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}) sin(\gamma t + \theta), \theta = \tan^{-1}(\frac{k - m\gamma^2}{b\gamma})$ is steady-state solution where period matches external sinusoidal force. **Frequency Gain/Factor**: $M(\gamma) = 1/\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}$. **Freq-Response Curve**: $M(0) = 1/k$ bc $F_{ext} = F_0$ constant force. As $\gamma \to \infty$, $M(\gamma) \to 0$. Graphing: overdamped $\gamma = 0$, underdamped $\gamma = 0, \gamma_r = \sqrt{k/m - b^2/2m^2}$. **Resonance Freq** = $\gamma_r/2\pi$.

Vertical Spring Mass System: If $F_{ext} = mg$, then $y_p = mg/k$. If $F_{ext} = mg + F_0 cos(\gamma t)$ then solve for $F_{ext} = F_0 cos(\gamma t)$ as above, $y_{actual} = y + mg/k$. Note: $k = F/x, m = weight/g, g = 32 ft/s^2$.