5a)

Show:

$$\sum_x P(x) \log \frac{P(x)}{Q(x)} \geq 0$$

$$\Rightarrow E\left(\log \frac{P(x)}{Q(x)}\right) \geq 0$$

$$= E\left(\log \frac{Q(x)}{P(x)}\right) \leq 0$$

$$= E\left(-\log \frac{Q(x)}{P(x)}\right) \geq 0$$

✓ Jensen's inequality

$$\geq -\log\left(E\left(\frac{Q(x)}{P(x)}\right)\right) \geq 0$$

$$= -\log\left(\sum_x \frac{P(x)\,Q(x)}{P(x)}\right) \geq 0$$

$$\cancel{\text{or } \log\left(\sum_x \frac{P(x)P(x)}{Q(x)}\right) \geq 0}$$

$$= -\log(1) \geq 0$$

$$\Rightarrow 0 = 0 \checkmark$$

$$\Rightarrow KL(P \| Q) \geq 0$$

Second claim: ~~strictly convex only when P=Q~~

Equal only if $X = E(x)$ but $\cancel{E\left(\frac{P(x)}{Q(x)}\right)}$ $E\left(\frac{Q(x)}{P(x)}\right) = \frac{Q(x)}{P(x)}$ only when $P=Q$

5 (i)

PS 3 *

Needs to equal:

$$\sum_x P(x) \log \frac{P(x)}{Q(x)} + \cancel{\sum_y P(y) \sum_x P(x|y) \log \frac{P(x|y)}{Q(x|y)}}$$

$$+ \sum_x P(x) \sum_y P(y|x) \log \frac{P(y|x)}{Q(y|x)}$$

$KL(P(X,Y) \| Q(X,Y)) =$

$$= \sum_{x,y} P(x,y) \cdot \log \frac{P(x,y)}{Q(x,y)}$$

$$= \sum_{x,y} P(x) P(y|x) \cdot \log \frac{P(x) \cdot P(y|x)}{Q(x) \cdot Q(y|x)}$$

$$\cancel{= \sum_{x,y} P(x) \cdot P(x|y) \cdot \frac{P(y)}{P(x)} \cdot \log \frac{P(x) \cdot P(x|y) \cdot P(y) \cdot Q(x)}{P(x) \cdot Q(x) \cdot Q(x|y) \cdot Q(y)}}$$

$$\cancel{= \sum_{x,y} P(x|y) \cdot P(y) \cdot \log \left(\frac{P(x|y) \cdot P(y)}{Q(x|y) \cdot Q(y)} \right)}$$

$$= \sum_{x,y} P(x) P(y|x) \cdot \log \frac{P(y|x)}{Q(y|x)} + \underbrace{\sum_{x,y} P(x) P(y|x) \cdot \log \frac{P(x)}{Q(x)}}_{\underbrace{\sum_x \sum_y P(x,y)}_{\sum_x P(x)}}$$

Done

5c) PS3 *

$$KL(\hat{P} \| P_\theta) = \sum_x \hat{P}(x) \cdot \log \frac{\hat{P}(x)}{P_\theta(x)}$$

$$= \cancel{\sum_x \hat{P}(x) \left(\log (\hat{P}(x)) \right.}$$

$$= \cancel{\sum_x \frac{1}{m} \sum_{i=1}^m 1\{x^{(i)} = x\}}$$

$$= \sum_x \hat{P}(x) \left(\log (\hat{P}(x)) - \underbrace{\log (P_\theta(x))}_{} \right)$$

$$= \sum_x \hat{P}(x) \log (\hat{P}(x)) - \sum_x \hat{P}(x) \log (P_\theta(x))$$

Goal: minize => Maximize

$$\Rightarrow \arg\max_\theta \sum_x \hat{P}(x) \log (P_\theta(x))$$

$$= \sum_x \frac{1}{m} \sum_{i=1}^m 1\{x^{(i)} = x\} \cdot \log (P_\theta(x))$$

$$= \frac{1}{m} \sum_{i=1}^m \underbrace{\left(\sum_x 1\{x = x^{(i)}\} \right)}_{=1} \cdot \log (P_\theta(x^{(i)}))$$

$$= \arg\max_\theta \frac{1}{m} \sum_{i=1}^m \log (P_\theta(x^{(i)})) = \arg\max_\theta \sum_{i=1}^m \log P_\theta(x^{(i)})$$