PS 2

1) Kernels

a) $K(x,z) = K_1(x,z) + K_2(x,z)$

$$\phi_1(x)^T \phi_1(z) \quad + \phi_2(x)^T \phi_2(z)$$

$$= \sum_{j=1}^{n} \phi_1(x)_j \, \phi_1(z) + \sum_{k=1}^{n} \phi_2(x)_k \, \phi_2(z)_k$$

$$= \sum_{j=1}^{n} \left[\phi_1(x)_j \times \phi_1(z)_j + \phi_2(x)_j * \phi_2(z)_j \right]$$

$$= \begin{pmatrix} | \\ \phi_1(x) \\ | \\ \phi_2(x) \\ | \end{pmatrix}^T \times \begin{pmatrix} | \\ \phi_1(z) \\ | \\ \phi_2(z) \end{pmatrix}$$

$\Rightarrow$ Yes, just concatenate the ϕ vectors !

b.) $K(x,z) = K_1(x,z) - K_2(x,z)$

Pick a kernel where

Say $\phi_1 = (x_1)$

$\Rightarrow \phi_2 = (2x_1)$

$\Rightarrow K_1(x,z) = x_1 z_1$

$K_2(x,z) = 4x_1 z_1$

$\Rightarrow K(x,z) = -3x_1 z_1$

$\Rightarrow$ not pos. semidef.

c) $K(x,z) = a\,K_1(x,z)$

Yes, it's ~~an~~ valid kernel.

Follows from a, where

$K(x,z) = K_1(x,z) + K_2(x,z)$

and $a\,K(x,z) = \sum_{i=1}^{a} K(x,z)$

=> We know that it must be pos. semidefinite

Can also show iteratively: $K(x,z)_{i+1} = K(x,z)_i + K(x,z)$

until $i+1 = a$

with $K(x,z)_0 = 0$

d) $K(x,z) = -a\,K(x,z)$

Not a kernel as $-a \times$ pos semidefinite = neg semidefinite

Also: $z^T K z \geq 0$ as we know

but $a\, z^T K z \leq 0$ => not a kernel

e) $K(x,z) = K_1(x,z)\, K_2(x,z)$

$$= \sum_{i=1}^{n} \left(\phi_1(x)_i\, \phi_1(z)_i \right) \sum_{j=1}^{n} \left(\phi_2(x)_j\, \phi_2(z)_j \right)$$

$$z^T K z = \sum_i \sum_j z_i K_{ij} z_j$$

$$= \sum_i \sum_j z_i \left(\phi_1(x^i)^T \phi_1(z^j) \right) \left(\phi_2(x^i)^T \phi_2(z^j) \right) z_j$$

$$= \sum_i \sum_j z_i \sum_k \phi_{1_k}(x^i)\, \phi_{1_k}(z^j) \sum_f \phi_{2_f}(x^i)\, \phi_{2_f}(z^j)\, z_j$$

$$= \sum_i z_i \sum_k \phi_k(x^i) \sum_f \rho_f(x^i) \sum_j z_j\, \phi_k(x^j)\, \rho_f(x^j)$$

$$= \sum_k \sum_f \left(\sum_i z_i\, \phi_k(x^i)\, \rho_f(x^i) \right) \left(\sum_j z_j\, \phi_k(x^j)\, \rho_f(x^j) \right)$$

$$= \sum_k \sum_f \left(\sum_i z_i\, \phi_k(x^i)\, \rho_f(x^i) \right)^2$$

$\Rightarrow$ pos semidefinite $\Rightarrow$ Kernel !

f) $K(x,z) = f(x)f(z)$

~~Not, because f(x) could be neg w/ f(z) being positive~~

~~$\Rightarrow$ not pos. semidef~~

Again: Kernel if:

$K(x,z) = \phi(x)^T \phi(z)$ with $\phi: R^n \to R^x$

~~the~~ $f(x)f(z)$ = Kernel, because $\phi(x) = f(x)$

and $f(x)^T = f(x)$

g) $K(x,z) = K_3(\phi(x), \phi(z))$

$K_3(x,z) = \psi(x)^T \psi(z)$

$\Rightarrow K(x,z) = \psi(\phi(x))^T \psi(\phi(z))$

Say $g(z) = \psi(\phi(z))$

$\Rightarrow K(x,z) = g(\cancel{x})^T g(z)$

$\Rightarrow$ Kernel

h) $K(x,z) = p(K_1(x,z)) = \left(\phi(x)^T \phi(z)\right)^p$

$= \sum_{i=0}^{n} a_i \left(\phi(x)^T \phi(z)\right)^n$

$= \sum_{i=0}^{n} \overset{\geq 0}{a_i} \Big(\sum_j \overbrace{\phi(x)_j \phi(z)_j}^{\geq 0} \Big)^n \Rightarrow K(x,z) \geq 0$

~~Not a Kernel~~

~~Example: $(x \cdot z)^3 = x^3 \cdot z^3$~~

Actually just follows from a), c) and e): Kernel!